1. Vectors and Vector Spaces

Monday, January 2, 2023 5:35 AM

linear algebra $\simeq$ geometry in arbitrary dimension

Data?

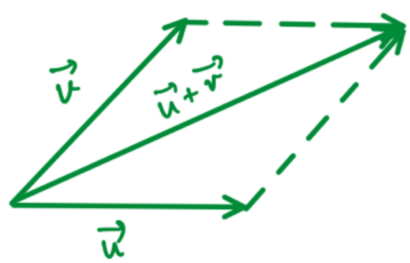
- Collection of data points
- To understand the structure of our data, we have to investigate the geometry of data points:
  are they divided into clusters?
  are they aligned?

Vector Spaces

Vectors:

2 fundamental operations:

1.) Add 2 vectors $\vec{u}$ & $\vec{v}$ to obtain another vector $\vec{u}+\vec{v}$



2) Multiply a vector $\vec{u}$ by a scalar λ to get another vector $\lambda.\vec{u}$.

Vector Spaces

A vector space consists of a set V (whose elements are called vectors) and two operations $+$ and $\cdot$ such that:

* The sum of two vectors is a vector:

$$\forall \vec{x}, \vec{y} \in V,$$
$$\vec{x} + \vec{y} \in V$$

* Multiplying a vector $\vec{x} \in V$ by a scalar $\lambda \in R$ gives $\lambda . \vec{x} \in V$

* Operations $+$ and $\cdot$ are "nice & compatible"

« Nice and compatible » ?

1. The vector sum is commutative and associative. For all $\vec{x}, \vec{y}, \vec{z} \in V$:

$$\vec{x} + \vec{y} = \vec{y} + \vec{x} \quad \text{and} \quad \vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}.$$

2. There exists a zero vector $\vec{0} \in V$ that verifies $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$.
3. For all $\vec{x} \in V$, there exists $\vec{y} \in V$ such that $\vec{x} + \vec{y} = \vec{0}$. Such $\vec{y}$ is called the additive inverse of $\vec{x}$ and is written $-\vec{x}$.
4. Identity element for scalar multiplication: $1 \cdot \vec{x} = \vec{x}$ for all $\vec{x} \in V$.
5. Distributivity: for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x}, \vec{y} \in V$,

$$\underbrace{(\alpha + \beta)}_{\text{Scalars}} \cdot \underbrace{\vec{x}}_{\text{vector}} = \underbrace{\alpha \cdot \vec{x} + \beta \cdot \vec{x}}_{\text{split}} \quad \text{and} \quad \alpha \cdot (\vec{x} + \vec{y}) = \alpha \cdot \vec{x} + \alpha \cdot \vec{y}.$$

6. Compatibility between scalar multiplication and the usual multiplication: for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x} \in V$, we have

$$\alpha \cdot (\beta \cdot \vec{x}) = (\alpha\beta) \cdot \vec{x}.$$

1. Vector spaces 1.1 Definition & examples

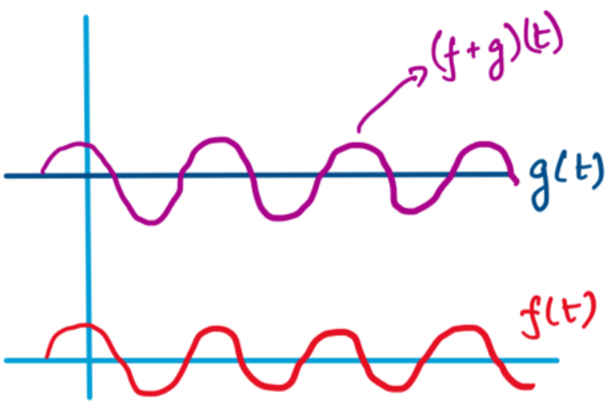
Example of Vector Spaces: Functions

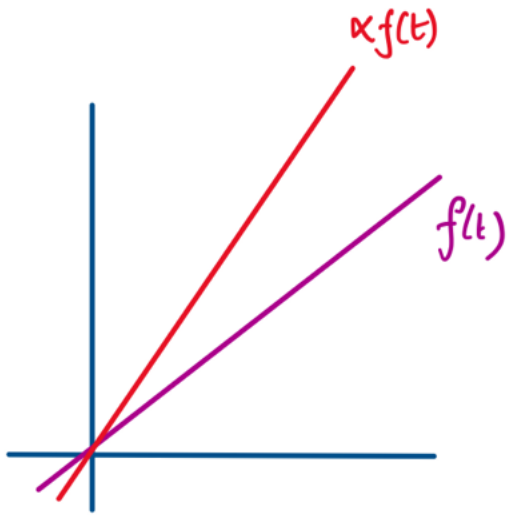
The set $V \stackrel{\text{def}}{=} \{f \mid f: \mathbb{R} \to \mathbb{R}\}$ of all functions from $\mathbb{R}$ to itself

endowed by $+$ & $\cdot$ defined by:

$$f+g: \mathbb{R} \to \mathbb{R} \quad \text{and} \quad \alpha \cdot f: \mathbb{R} \to \mathbb{R}$$
$$t \to f(t)+g(t) \qquad\qquad t \to \alpha f(t)$$

is a vector space.





Example 3: random variables

Imp

The set of random variables on a given probability space Ω is a vector space:
If X and Y are two random variables and $\alpha \in \mathbb{R}$, $X + Y$ and αX are also random variables.

Important to have this in mind when doing stats/probabilities!

Need?

* Geometric Intuition

Notion of length in $\mathbb{R}^n$ is deeply connected to notion of variance of r.v.

* Save time

Theorems that apply to vector spaces will be true for all examples seen above.

Subspaces

Definition:

We say that a non-empty subset S of a vector space V is a subspace if it is closed under addition and multiplication by a scalar, that is if:

1. for all $x, y \in S$ we have $x+y \in S$
2. for all $x \in S$ and all $\alpha \in \mathbb{R}$ we have $\alpha x \in S$.

Remark: a subspace is also a vector space.

Span & Linear Dependency

Linear combination

Let V be a vector space (think for instance $V = \mathbb{R}^n$).

Definition

We say that $y \in V$ is a *linear combination* of the vectors $x_1, \dots, x_k \in V$ if there exists $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that

$$y = \sum_{i=1}^{k} \alpha_i x_i = \alpha_1 x_1 + \dots + \alpha_k x_k.$$

Remarks

- A linear combination is always a finite sum. not an infinite number of $\alpha_k x_k$
- If S is a subspace of V, then any linear combination of vectors $x_1, \dots, x_k$ of S is also in S: If $x_1, \dots, x_k$ is in S then

$$\alpha_1 x_1 + \dots + \alpha_k x_k \in S, \quad \text{for all } \alpha_1, \dots, \alpha_k \in \mathbb{R}.$$

« Subspaces are closed under linear combinations. »

Exercise: Prove it !

If $x_1, \dots x_k \in S$

then

$\alpha x_1 \in S$

$\alpha_2 x_2 \in S$

$\vdots$

$\alpha_n x_n \in S$

If $\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n \in S$,

then so does

$\alpha_1 x_1 + \dots + \alpha_n x_n \in S$

as S is a vector space

$\boxed{\text{Hence subspaces are closed under linear combination}}$

?

Let $x_1, \dots, x_k \in S$

and let y_1 & y_2 be linear combinations of $x_1, \dots, x_k$

Then,

$$y_1 = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_k x_k$$

$$y_2 = \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k$$

$$\therefore y_1 + y_2 = (\alpha_1 + \beta_1) x_1 + (\alpha_2 + \beta_2) x_2 + \dots + (\alpha_k + \beta_k) x_k$$

Span

Definition

Let $x_1, \ldots, x_k$ be vectors of V. We define the *linear span* of $x_1, \ldots, x_k$ as the set of all linear combinations of these vectors:

$$\mathrm{Span}(x_1, \ldots, x_k) \stackrel{\text{def}}{=} \left\{ \alpha_1 x_1 + \cdots + \alpha_k x_k \,\middle|\, \alpha_1, \ldots, \alpha_k \in \mathbb{R} \right\}.$$

2.2 Linear dependency

Definition

Vectors $x_1, \ldots x_k \in V$ are *linearly dependent* is there exists $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ **that are not all zero** such that

$$\alpha_1 x_1 + \cdots + \alpha_k x_k = 0.$$

They are said to be *linearly independent* otherwise. **Abuse of language:** Instead of saying «$x_1, \ldots, x_k$ are linearly dependent», we should say «the family $(x_1, \ldots, x_k)$ is linearly dependent».

Key observation: « $x_1, \ldots, x_k$ are linearly dependent » *is equivalent to* « one of the vectors $x_1, \ldots, x_k$ can be obtained as a linear combination of the others.»

Proof:

<< $x_1, \ldots, x_k$ >> are linearly dependent

Thus,

$$\alpha_1 x_1 + \ldots + \alpha_k x_k = 0$$

We get,

$$-\alpha_1 x_1 = \alpha_2 x_2 + \ldots + \alpha_k x_k$$

$$\therefore x_1 = \left(\frac{-\alpha_2}{\alpha_1}\right) x_2 + \ldots + \left(\frac{-\alpha_k}{\alpha_1}\right) x_k$$

$$\boxed{x_1 = \beta_2 x_2 + \ldots + \beta_k x_k}$$

$x_1, \dots, x_k$ are linearly dependent $\Longleftrightarrow$ one of them is a linear combination of the other.

Why ?

$x_1 \dots x_k$ linearly dependent $\Leftrightarrow$ one of them is a linear combinaison of the others.

① Assume $x_1, \dots x_k$ linearly dependent:

exists $\alpha_1 \dots \alpha_k \in \mathbb{R}$ not all 0 such that $\alpha_1 x_1 + \dots + \alpha_k x_k = 0$

$\alpha_i \neq 0 \Rightarrow \alpha_i x_i = -\alpha_1 x_1 - \dots - \alpha_k x_k$ (no term in x_i)

$\Rightarrow x_i = -\frac{\alpha_1}{\alpha_i} x_1 - \dots - \frac{\alpha_k}{\alpha_i} x_k$ since $\alpha_i \neq 0$

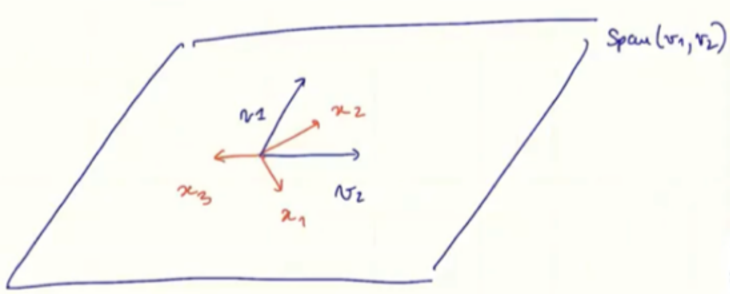
② Assume $x_i = \beta_1 x_1 + \dots + \beta_k x_k$ (no term i) $\beta_1 \dots \beta_k \in \mathbb{R}$

$\Rightarrow \beta_1 x_1 + \dots \underbrace{- x_i}_{\beta_i = -1 \neq 0} + \dots + \beta_k x_k = 0 \rightarrow$ linearly dependent.

A useful lemma

Lemma

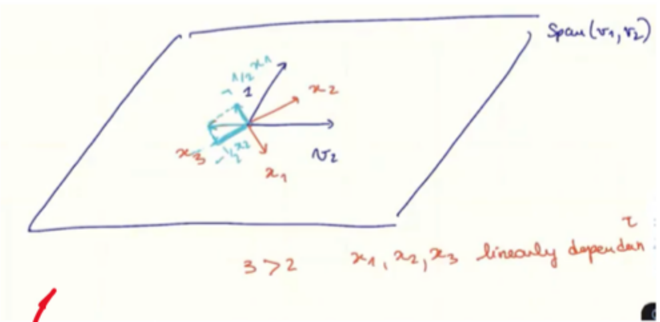
Let $v_1, \dots, v_n \in V$ and let $x_1, \dots, x_k \in \mathrm{Span}(v_1, \dots, v_n)$.
Then, if $k > n$, $x_1, \dots, x_k$ are linearly dependent.



Space that is spanned by 2 vectors,

... must be linearly

if I have 3 vectors — they must be linearly dependent.



Note how x_3 can be written as linear combination of x_1 & x_2.

$\boxed{\text{Basis}}$

3.1 Basis definition

Definition

A family $(x_1, \ldots, x_n)$ of vectors of V is a basis of V if

1. $x_1, \ldots, x_n$ are linearly independent,
2. $\mathrm{Span}(x_1, \ldots, x_n) = V$.

This means that $(x_1, \ldots, x_n)$ is a basis of V if

1. None of the x_i is a linear combination of the others $(x_j)_{j \neq i}$.
2. Any vector of V can be expressed as a linear combination of $(x_1, \ldots, x_n)$.

Example: the canonical basis of $\mathbb{R}^n$

Let us define the vectors $e_1, \ldots, e_n \in \mathbb{R}^n$ by

$$e_1 = (1, 0, 0, \ldots, 0)$$

$$e_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$e_n = (0, 0, 0, \ldots, 1).$$

One can verify (homework!) that the family $(e_1, \ldots, e_n)$ is a basis of $\mathbb{R}^n$. This basis is called the "canonical basis" of $\mathbb{R}^n$.

3.2 Dimension

Theorem

Let V be a vector space.

- If V admits a basis $(v_1, \ldots, v_n)$, then every basis of V has also n vectors. We say that V has dimension n and write $\dim(V) = n$.
- Otherwise, we say that V has infinite dimension: $\dim(V) = +\infty$.

Example:

- $\mathbb{R}^2$ has dimension 2, because the canonical basis (e_1, e_2) is a basis of $\mathbb{R}^2$ with 2 vectors.
- $\{f \mid f : \mathbb{R} \to \mathbb{R}\}$ has infinite dimension.

The dimension is well defined!

Theorem

If V admits a basis $(v_1, \ldots, v_n)$, then every basis of V has also n vectors.

Proof. BY CONTRADICTION:

Assume
basis $(v_1, \ldots v_n)$ $k > n$ $\underline{\text{then}}$ for any i $x_i \in \underbrace{\mathrm{Span}(v_1, \ldots v_n)}_{= V}$
basis $(x_1, \ldots x_k)$

$\Rightarrow$ $x_1, \ldots x_k$ are linearly dependent (by the lemma)

CONTRADICTION

What about when $k < n$?

$\boxed{\text{Think.}}$

Properties of the dimension

Proposition

Let V be a vector space that has dimension $\dim(V) = n$. Then

1. Any family of vectors of V that spans V contains at least n vectors.

   i.e. if $x_1, \ldots, x_k \in V$ are such that $\mathrm{Span}(x_1, \ldots, x_k) = V$, then $k \geq n$.

2. Any family of vectors of V that are linearly independent contains at most n vectors.

   i.e. if $x_1, \ldots, x_k \in V$ are linearly independent, then $k \leq n$.

Properties of the dimension

Proposition

Let V be a vector space of dimension n and let $x_1, \ldots, x_n \in V$.

1. If $x_1, \ldots, x_n$ are linearly independent, then $(x_1, \ldots, x_n)$ is a basis of V.
2. If $\mathrm{Span}(x_1, \ldots, x_n) = V$, then $(x_1, \ldots, x_n)$ is a basis of V.

Very useful to show that a family of vector forms a basis:

Example: $x_1 = (12, 37)$ and $x_2 = (-9, 17)$ form a basis of $\mathbb{R}^2$.

How to prove that these 2 vectors form basis of R^2?

* Since R^2 needs 2 vectors — & we have these 2 vectors so they can be the basis.

* Now just need to prove that they are linearly independent.

Why? because if $x_2 = 2x_1$ then x_2 is along x_1 and they would both line along a line that is they would span $\mathbb{R}$. (real line)

But if we prove that x_2 &

x_1 are L.I. then they would span $\mathbb{R}^2$.

Thus how do we know

$x_1 = (12, 37)$ & $x_2 = (-9, 17)$ are LI?

Because there is nothing we can multiply x_2 with that would result in x_1

i.e. there exists no α,

s.t. $\alpha x_1 = x_2$

Thus x_1 & x_2 are L.I.

Hence x_1 & x_2 can be basis of $\mathbb{R}^2$.

Example: $x_1 = (12, 37)$ and $x_2 = (-9, \cancel{17})$ form a basis of $\mathbb{R}^2$.

$-3/4$ $\quad -3/2$

there is no $\alpha \in \mathbb{R}$ such that $x_1 = \alpha x_2 \Rightarrow$ linearly independent family

$\oplus$ $\dim \mathbb{R}^2 = 2$ $\quad \Rightarrow (x_1, x_2)$ basis of V.

Geometry in $\mathbb{R}^n$

3.3 Coordinates of a vector in a basis

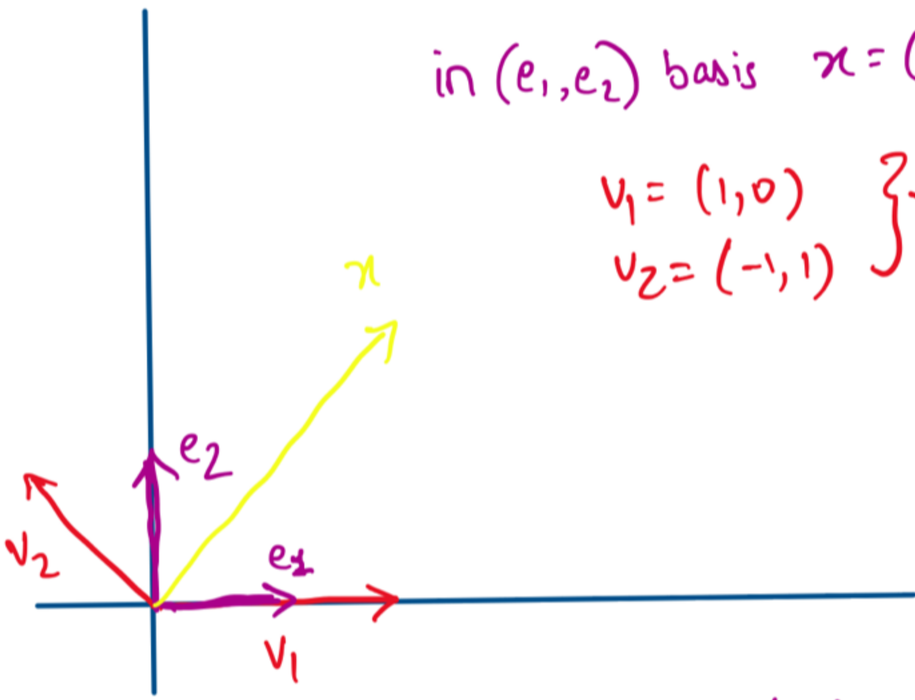
Definition & Theorem

If $(v_1, \ldots, v_n)$ is a basis of V, then for every $x \in V$ there exists a unique vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

We say that $(\alpha_1, \ldots, \alpha_n)$ are the coordinates of x in the basis

We say that $(\alpha_1, \dots, \alpha_n)$ are the coordinates of x in the basis $(v_1, \dots, v_n)$.



in (e_1, e_2) basis $x = (1,2)$

$v_1 = (1,0)$
$v_2 = (-1,1)$ $\Big\} \rightarrow$ Are they valid basis?

$\downarrow$

Yes! L.I.

What is $x = (1,2)$ in (v_1, v_2) basis?

$$x = 3v_1 + 2v_2$$

$\therefore x$ is $(3,2)$ in the basis (v_1, v_2)