4. Norms, Inner Products & Orthogonality

Friday, May 19, 2023 3:47 PM

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1. Norms

Euclidean Norm (aka l2 norm)

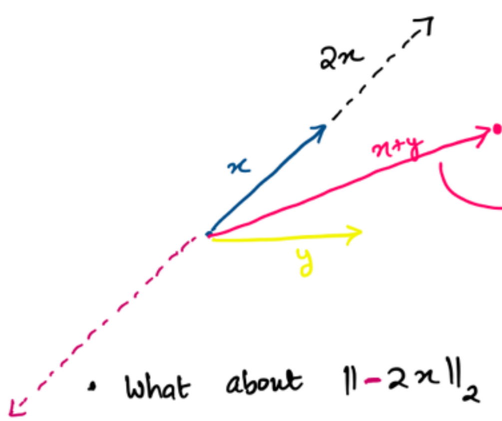
Definition: We define the Euclidean Norm of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as :

$$\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2} . \quad (\text{a.k.a } l2 \text{ norm})$$

Observations :

- What is going to be $\|2x\|_2$?

$$\boxed{\|2x\|_2 = 2\|x\|_2}$$



This distance is faster way to reach the point thus

- What about $\|-2x\|_2$?

$$\boxed{\|-2x\|_2 = 2\|x\|}$$

- What about $\|x+y\|$

$$\boxed{\|x+y\| \leq \|x\|_2 + \|y\|_2}$$

- What if $\|x\|_2 = 0$?

  then $x = 0$ (0 vector)

General Norms

Let V be a vector space

Definition: A norm $\|.\|$ on V is a function from V to $\mathbb{R}_{\geq 0}$ that verifies:

1. *Homogenity:* $\|\alpha v\| = |\alpha| \times \|v\|$ for all $\alpha \in \mathbb{R}$ and $v \in V$
2. *Positive Definiteness:* if $\|v\| = 0$ for some $v \in V$, then $v = 0$
3. *Triangular Inequality:* $\|u+v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

Other examples of norm:

- L1 norm: $\|x\|_1 \overset{\text{def}}{=} \sum_{i=1}^{n} |x_i| = |x_1| + \dots + |x_n|$
- Infinity norm: $\|x\|_\infty \overset{\text{def}}{=} \max(|x_1|, \dots, |x_n|)$

Differences between Norms?

Ex: Balls drawing.

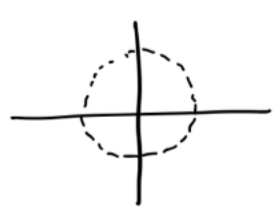
For each of the norms $\|.\|_2$, $\|.\|_1$, $\|.\|_\infty$, draw the «ball»

$$B = \{ x \in \mathbb{R}^2 \mid \|x\| \leq 1 \}$$

L2 Norm: ($x \in \mathbb{R}^2$)

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2} \leq 1$$

$$x_1^2 + x_2^2 \leq 1$$



Boundary:

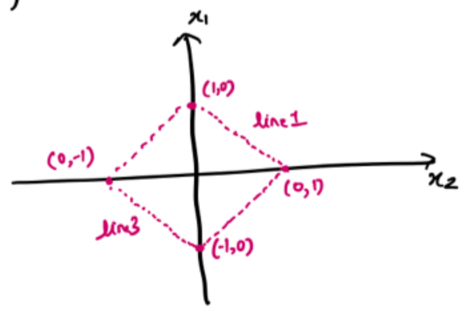
$\|x\|_2 = 1$

$x_1^2 + x_2^2 = 1$

↳ circle

↳ all vectors inside the circle are in the Ball.

L1 Norm: $(x \in \mathbb{R}^2)$ $\qquad \|x\|_1 = |x_1| + |x_2| \leq 1$



Quadrant 1:
$x_1 > 0$
$x_2 > 0$
we get: $x_1 + x_2 \leq 1$
Consider inequality:
$x_1 + x_2 = 1$
$x_1 = -x_2 + 1$
line 1
passes through
$(1,0)$ $(0,1)$
$x_1\ x_2$ $\quad$ $x_1\ x_2$

Quadrant 3:
$x_1 < 0, \ x_2 < 0$
we get:
$-x_1 - x_2 \leq 1$
Consider:
$-x_1 - x_2 = 1$
line 3
passes through:
$(0,-1)$ & $(-1,0)$

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2. Inner Products

2.1 Euclidean dot Product

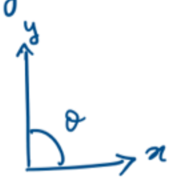
Definition:

We define the Euclidean dot product of two vectors x and y of $\mathbb{R}^n$ as:

$$x \cdot y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + \dots + x_n y_n$$

Remark:

- $x \cdot x = \sum x_i^2 = \left(\|x\|_2 \right)^2$
- $x \cdot y = \|x\|_2 \|y\|_2 \cos\theta$



if $\theta = \pi/2$
$x \cdot y = 0$

if $\theta < \pi/2$
$x \cdot y \geq 0$

if $\theta > \pi/2$
$x \cdot y \leq 0$

Inner Product

Let V be a vector space

Definition :

An inner product on V is a function $\langle \cdot , \cdot \rangle$ from $\overbrace{V \times V}^{2 \text{ inputs from } V}$ to $\mathbb{R}$ that verifies the following points :

1. Symmetry : $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$
2. Linearity : $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

   and

   $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$

   $\forall \; u, v, w \in V$ and $\alpha \in \mathbb{R}$
3. Positive Definiteness :

   $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

Link between Norm & Inner Products :

Norm induced by an inner product

Proposition : If $\langle \cdot , \cdot \rangle$ is an inner product on V then :

$$\|v\| \overset{\text{def}}{=} \sqrt{\langle v, v \rangle}$$

is a norm on V. We say that the norm $\| \cdot \|$ is induced by the inner product.

Cauchy- Schwarz Inequality

Theorem :

Let $\|\cdot\|$ be the norm induced by the inner product $\langle\cdot,\cdot\rangle$ on the vector space V. Then for all $x, y \in V$:

$$|\langle x,y\rangle| \leqslant \|x\| \, \|y\| \qquad (1)$$

Moreover, there is equality in (1) if and only if x and y are linearly dependent, i.e. $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$.

Example:

$V = \mathbb{R}^n$ Euclidean dot product $\longrightarrow \|\cdot\|_2$

$$|x \cdot y| = \big|\, \|x\|_2 \, \|y\|_2 \cos\theta \,\big| \leqslant \|x\|_2 \, \|y\|_2$$

Orthogonality

Let V be a vector space, and $\langle\cdot,\cdot\rangle$ be an inner product on V.

Definition:

- We say that vectors x and y are orthogonal if $\langle x,y\rangle = 0$.
  We write then $x \perp y$.
- We say that a vector x is orthogonal to a set of vectors A if x is orthogonal to all the vectors in A. We write then $x \perp A$.

Orthogonal and Orthonormal families

Definition:

We say that a family of vectors $(v_1, \dots, v_k)$ is:

- orthogonal if the vectors $v_1, \dots, v_k$ are pairwise orthogonal, i.e. $\langle v_i, v_j\rangle = 0$ for all $i \neq j$.
- orthonormal if it is orthogonal and if all the v_i have unit norm:
  $\|v_1\| = \|v_2\| = \dots = \|v_k\| = 1$

Coordinates in an orthonormal basis

Proposition

A vector space of finite dimension admits an orthonormal basis.

Proposition

Assume that $\dim(V) = n$ and let $(v_1, \dots, v_n)$ be an orthonormal basis of V. Then the coordinates of a vector $x \in V$ in the basis $(v_1, \dots, v_n)$ are $(\langle v_1, x\rangle, \dots, \langle v_n, x\rangle)$:

$$x = \langle v_1, x\rangle v_1 + \dots + \langle v_n, x\rangle v_n$$

Coordinates in Orthonormal basis:

Remark: Let x, y in V with coordinates $x = (\alpha_1, \dots, \alpha_n)$ and $y = (\beta_1, \dots, \beta_n)$ in an orthonormal basis $(v_1, \dots, v_n)$.

- $$\begin{aligned} \langle x, y\rangle &= \langle \alpha_1 v_1 + \dots + \alpha_n v_n , \beta_1 v_1 + \dots + \beta_n v_n \rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \beta_j \underbrace{\langle v_i, v_j\rangle} \to \begin{cases} = 0 & \text{if } i \neq j \\ = 1 & \text{if } i = j \end{cases} \\ &= \sum_{i=1}^{n} \alpha_i \beta_i \end{aligned}$$
- $$\|x\| = \sqrt{\sum_{i=1}^{n} \alpha_i^2}$$

Pythagorean Theorem:

Theorem: Let $\|\cdot\|$ be a norm induced by $\langle \cdot, \cdot \rangle$. For all $x, y \in V$ we have

$$x \perp y \iff \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof:

(one side of proof)
(equivalence needs to be proved from both sides)

$$\begin{aligned} x \perp y \Rightarrow \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \end{aligned}$$

Since $x \perp y \Rightarrow \langle x, y\rangle = 0$

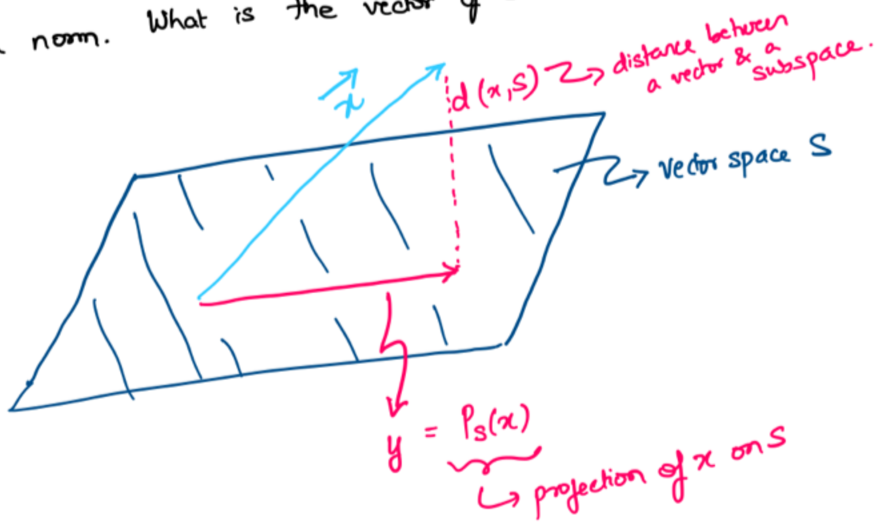
$$\boxed{\|x+y\|^2 = \|x\|^2 + \|y\|^2}$$

Picture this:

... dot product, and $\|\cdot\|$ the

Let $\langle \cdot , \cdot \rangle$ denote the Euclidean ...

Euclidean norm. What is the vector of S that is closest to x?



Orthogonal Projection

Let $\langle \cdot , \cdot \rangle$ denote the Euclidean dot product, and $\| \cdot \|$ the Euclidean norm.

Definition

Let S be a subspace of $\mathbb{R}^n$. The orthogonal projection of a vector x onto S is defined as the vector $P_S(x)$ in S that minimizes the distance to x:

$$P_S(x) \stackrel{\text{def}}{=} \underbrace{\arg\min_{y \in S} \|x - y\|}_{\text{return } y^*.}$$

"minimize $f(y) = \|x - y\|$ for y in S give me the y^* does that."

The distance of x to the subspace S is then defined as:

$$d(x, S) \stackrel{\text{def}}{=} \underbrace{\min_{y \in S} \|x - y\|}_{} = \|x - P_S(x)\|$$

returns the "distance" of x from y^* (y^* obtained from previous step)

Computing orthogonal projections

Proposition:

Let S be a subspace of $\mathbb{R}^n$ and let $(v_1, \dots, v_k)$ be an orthonormal ... $\mathbb{R}^n$

basis of S. Then for all $x \in \mathbb{R}$,

$$P_S(x) = \langle v_1, x\rangle v_1 + \cdots + \langle v_k, x\rangle v_k.$$