Problem 103 UGC 2017 DEC

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Q. Suppose A, B, C are events in a common probability space with $P(A) = 0.2$, $P(B) = 0.2$, $P(C) = 0.3$, $P(A \cap B) = 0.1$, $P(A \cap C) = 0.1$, $P(B \cap C) = 0.1$ Which of the following are possible values of $P(A \cup B \cup C)$?

1) 0.5
2) 0.3
3) 0.4
4) 0.9

Solution:

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) \\ &\quad - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= 0.2 + 0.2 + 0.3 - 0.1 - 0.1 - 0.1 \\ &\quad + P(A \cap B \cap C) \\ &= 0.4 + P(A \cap B \cap C) \end{aligned}$$

Minimum value of $P(A \cap B \cap C)$ can be 0. So option 2, 0.3 is out of consideration.

Looking at $P(A \cap B) = 0.1$, $P(A \cap C) = 0.1$, $P(B \cap C) = 0.1$, maximum value of $P(A \cap B \cap C)$ can be 0.1. So option 4, 0.9 is out of consideration.

So option 1 and option 3 are the valid answers.

Proof

$A \cup B \cup C = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$$\begin{aligned} A &= A(B + B') \\ &= AB + AB' \end{aligned} \tag{1}$$

$$\begin{aligned} A + B &= A(B + B') + B \\ &= AB + AB' + B \\ &= B(A + 1) + AB' \\ &= B + AB' \end{aligned} \tag{2}$$

$$\begin{aligned} P(A + B) &= P(B) + P(AB') && \textit{(from (2))} \\ &= P(B) + P(A) - P(AB) && \textit{(from (1))} \end{aligned}$$

Now, imagine $X = B + C$,

$$\begin{aligned} P(A + X) &= P(A) + P(X) - P(XA) \\ &= P(A) + P(B + C) - P((B + C)A) \\ &= P(A) + P(B) + P(C) - P(BC) - P(AB + AC) \\ &= P(A) + P(B) + P(C) - P(BC) - P(AB) - P(AC) \\ &\quad - P(ABAC) \\ &= P(A) + P(B) + P(C) - P(BC) - P(AB) - P(AC) \\ &\quad - P(ABC) \end{aligned}$$