

Problem 103 UGC 2017 DEC

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Q. Suppose A, B, C are events in a common probability space with $P(A) = 0.2$, $P(B) = 0.2$, $P(C) = 0.3$, $P(AB) = 0.1$, $P(AC) = 0.1$, $P(BC) = 0.1$. Which of the following are possible values of $P(A + B + C)$?

- 1) 0.5
- 2) 0.3
- 3) 0.4
- 4) 0.9

Solution:

$$\begin{aligned}
 P(A + B + C) &= P(A) + P(B) + P(C) - P(AB) \\
 &\quad - P(AC) - P(BC) + P(ABC) \\
 &= 0.2 + 0.2 + 0.3 - 0.1 - 0.1 - 0.1 \\
 &\quad + P(ABC) \\
 &= 0.4 + P(ABC)
 \end{aligned}$$

Minimum value of $P(ABC)$ can be 0. So option 2, 0.3 is out of consideration.

Looking at $P(AB) = 0.1$, $P(AC) = 0.1$, $P(BC) = 0.1$, maximum value of $P(ABC)$ can be 0.1. So option 4, 0.9 is out of consideration.

So option 1 and option 3 are the valid answers.

Proof

$$P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

$$\begin{aligned}
 A &= A(B + B') \\
 &= AB + AB'
 \end{aligned} \tag{1}$$

$$\begin{aligned}
 A + B &= A(B + B') + B \\
 &= AB + AB' + B \\
 &= B(A + 1) + AB' \\
 &= B + AB'
 \end{aligned} \tag{2}$$

$$\begin{aligned}
 P(A + B) &= P(B) + P(AB') && \text{(from (2))} \\
 &= P(B) + P(A) - P(AB) && \text{(from (1))}
 \end{aligned}$$

Now, imagine $X = B + C$,

$$\begin{aligned}
 P(A + X) &= P(A) + P(X) - P(XA) \\
 &= P(A) + P(B + C) - P((B + C)A) \\
 &= P(A) + P(B) + P(C) - P(BC) - P(AB + AC) \\
 &= P(A) + P(B) + P(C) - P(BC) - P(AB) - P(AC) \\
 &\quad - P(ABAC) \\
 &= P(A) + P(B) + P(C) - P(BC) - P(AB) - P(AC) \\
 &\quad - P(ABC)
 \end{aligned}$$