Discussion Notes for Undergraduate Analysis (MATH 131A)

UCLA

Fall 2017

Created by

Howard Heaton

Purpose: This document is a compilation of notes generated for discussion in MATH 131A with reference credit due to Kenneth Ross's text *Elementary Analysis*. If the reader finds any errors/-typos, please feel free to email me at `heaton@math.ucla.edu` and I will address these and post an updated set of notes to my webpage.

Contents

Section 1: Introduction

These notes are provided to compliment the TA discussion sessions on Thursdays for MATH 131A. Typically, more detail is provided here than on the board during discussion. And, the solutions to problems provided here illustrate the level of rigor desired from students this quarter.

Below we provide examples following the content of the first chapter, *Introduction*, in our text. For the first example, we note here the principle of mathematical induction asserts that a list of statements $P_1, P_2, P_3, \ldots$ are true provided P_1 is true and P_{n+1} is true whenever P_n is true.

Example 1: Prove $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers n.

Proof:

We first prove a preliminary result. We claim

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

for each $n \in \mathbb{N}$. The base case holds since

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1. \tag{2}$$

Suppose now the statement holds for some $k \in \mathbb{N}$. Then

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1) + 2k + 2}{2} = \frac{(k+1)(k+2)}{2}, \tag{3}$$

and the inductive step holds. The claim follows by the principle of mathematical induction.

We now proceed by induction to prove the desired result, using the above preliminary result. First observe $1^3 = 1 = 1^2$, and so the base case holds. For the inductive step, suppose the equation is true for some $k \in \mathbb{N}$. Then, applying the inductive hypothesis and our above result, we see

$$\begin{aligned}
1^3 + 2^3 + \cdots k^3 + (k+1)^3 &= (1 + \cdots + k)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + k(k+1)^2 + (k+1)^2 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + 2\left(\frac{k(k+1)}{2}\right)(k+1) + (k+1)^2 \\
&= \left(\frac{k(k+1)}{2} + (k+1)\right)^2 \\
&= \left(1 + \cdots + k + (k+1)\right)^2.
\end{aligned} \tag{4}$$

This shows the statement holds for $k+1$ and, thus, closes the induction. Thus we conclude by the principle of mathematical induction that the statement holds for all $n \in \mathbb{N}$. ■

Example 2: Prove $11^n - 4^n$ is divisible by 7 for each $n \in \mathbb{N}$.

Proof:
We proceed by way of induction. In the base case we see $11^1 - 4^1 = 11 - 4 = 7 = 7 \cdot 1$, and so the claim holds when $n = 1$. For the inductive step, suppose $11^k - 4^k$ is divisible by 7 for some $k \in \mathbb{N}$. Then there is some $\alpha_k \in \mathbb{Z}$ such that $11^k - 4^k = 7\alpha_k$. This implies

$$11^{k+1} - 4^{k+1} = (7)11^k + (4)11^k - (4)4^k = (7)11^k + 4\left(11^k - 4^k\right) = 7\left(11^k + 4\alpha_k\right). \tag{5}$$

Because $11^k + 4\alpha_k \in \mathbb{Z}$, we see 7 divides $11^{k+1} - 4^{k+1}$ and we have closed the induction. The claim then follows by the principle of mathematical induction. ■

Example 3:

a) Show $|b| \leq a$ if and only if $-a \leq b \leq a$.

b) Prove $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Proof:

a) Recall for each $b \in \mathbb{R}$ we define

$$|b| := \begin{cases} b & \text{if } b \geq 0, \\ -b & \text{if } b \leq 0. \end{cases} \tag{6}$$

First suppose $|b| \leq a$, which implies $a \geq |b| \geq 0$. We consider the two possible cases. If $b \geq 0$, then

$$-a \leq 0 \leq b = |b| \leq a. \tag{7}$$

Similarly, if $b \leq 0$, we find

$$-a \leq -|b| = -(-b) = b \leq 0 \leq a. \tag{8}$$

Combining our two cases, we deduce $|b| \leq a$ implies $-a \leq b \leq a$.

Conversely, suppose $-a \leq b \leq a$. Then $-a \leq a$ or, equivalently, $0 \leq 2a$ and so $0 \leq a$. If $b \geq 0$, then $|b| = b \leq a$. If $b \leq 0$, then $-a \leq b = -(-b) = -|b|$. Thus $|b| \leq a$ in either case. This completes the proof.

b) We proceed by repeated application of the triangle inequality. Let $a, b \in \mathbb{R}$. Then

$$|a| = |(a - b) + b| \leq |a - b| + |b| \quad \Longrightarrow \quad |a| - |b| \leq |a - b|. \tag{9}$$

Similarly,

$$|b| = |(b - a) + a| \leq |b - a| + |a| = |a - b| + |a| \quad \Longrightarrow \quad |b| - |a| \leq |a - b|. \tag{10}$$

Multiplying the final inequality by -1 gives $-|a - b| \leq |a| - |b|$. Thus

$$-|a - b| \leq |a| - |b| \leq |a - b|. \tag{11}$$

By the result in a), we conclude $||a| - |b|| \leq |a - b|$. ∎

Example 4: Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$. Prove $\sup S \leq \inf T$.

Proof:

Let $s \in S$ and fix $t \in T$. Then, by hypothesis,

$$s \leq t \quad \forall\, s \in S. \tag{12}$$

This shows t is an upper bound for S. Since $\sup S$ is the least upper bound for S, it follows that $\sup S \leq t$. Now, because t was arbitrarily chosen, this shows

$$t \geq \sup S \quad \forall\, t \in T. \tag{13}$$

Whence $\sup S$ is a lower bound for T. Since $\inf T$ is the greatest lower bound for T, we conclude $\sup S \leq \inf T$, as desired. ■

Example 5: Show $\sup\{r \in \mathbb{Q} \ : \ r < a\} = a$ for each $a \in \mathbb{R}$.

Proof:

Define $S := \{r \in \mathbb{Q} \ : \ r < a\}$. Observe a is an upper bound for S, by definition. All we must show is that a is the least upper bound for S. By way of contradiction, suppose there is an upper bound $m \in \mathbb{R}$ for S satisfying $m < a$. Then, by the density of the rationals in $\mathbb{R}$, there is $r \in \mathbb{Q}$ such that $m < r < a$. Since $r < a$ and $r \in \mathbb{Q}$, we see $r \in S$. The fact $m < r$ for some $r \in S$ shows m cannot be an upper bound for S, a contradiction. This shows a must be the least upper bound for S, i.e., $a = \sup S$, and we are done. ■

Example 6: Let A and B be nonempty bounded subsets of $\mathbb{R}$, and define $A - B := \{a - b \ : \ a \in A,\ b \in B\}$. Prove $\inf(A - B) = \inf A - \sup B$.

Proof:

Pick any $a \in A$ and $b \in B$. Then, by definition of the infimum and supremum,

$$a - b \geq \inf A - b \geq \inf A - \sup B, \tag{14}$$

noting $b \leq \sup B$ implies $-b \geq -\sup B$. Since this inequality holds for arbitrary $(a - b) \in (A - B)$, we write

$$a - b \geq \inf A - \sup B \quad \forall\ (a - b) \in (A - B). \tag{15}$$

This shows the right hand side is a lower bound for $(A - B)$. Since $\inf(A - B)$ is the greatest lower bound for $(A - B)$, we deduce

$$\inf(A - B) \geq \inf A - \sup B. \tag{16}$$

Now pick any $b \in B$ and fix $a \in A$. Then $(a - b) \in (A - B)$ and, by definition of the infimum,

$$\inf(A - B) \leq a - b \quad \implies \quad b \leq a - \inf(A - B) \quad \forall\ b \in B. \tag{17}$$

This gives an upper bound for B. Since $\sup B$ is the least upper bound for B,

$$\sup B \leq a - \inf(A - B) \quad \implies \quad \inf(A - B) + \sup B \leq a. \tag{18}$$

Since this final inequality holds for arbitrary a, the left hand side provides a lower bound for A. Thus

$$\inf(A - B) + \sup B \leq \inf A \quad \implies \quad \inf(A - B) \leq \inf A - \sup B. \tag{19}$$

Combining (16) and (19), we conclude $\inf(A - B) = \inf A - \sup B$, as desired. ∎

Section 2: Sequences

Definition: A **sequence** is a function whose domain is a set of the form $\{n \in \mathbb{Z} : \ n \geq m\}$. We write the value of the function evaluated at n by s_n, e.g., and let $(s_n)_{n=m}^{\infty}$ denote the sequence. If the indices are understood or not important, we may more compactly write a sequence as (s_n). △

Definition: A sequence (s_n) of real number is said to **converge** to $s \in \mathbb{R}$, denoted $s_n \longrightarrow s$, provided for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $n > N$ implies $|s_n - s| < \varepsilon$. △

Remark 1: To show the a sequence (s_n) converges to a limit s, we often proceed in roughly the following manner. We say "Let $\varepsilon > 0$ be given." That is, we let someone hand us any arbitrary ε. Then we use previous results about limits to deduce new relationships (e.g., we can use the Archimedean property of $\mathbb{R}$, or the "squeeze" theorem). With this, we are looking for some $N \in \mathbb{N}$ that enables to obtain the inequality $|s_n - s| < \varepsilon$ whenever $n > N$. ◇

Example 7: Determine the limit of the sequence $a_n = n/(n^2+1)$ and prove this is the limit.

Solution:

We claim $\lim\limits_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$ be given. We must show there is $N \in \mathbb{N}$ such that

$$|a_n - 0| < \varepsilon \quad \forall n \geq N. \tag{20}$$

First observe

$$|a_n| = \left|\frac{n}{n^2+1}\right| = \frac{n}{n^2+1} = \frac{1}{n+1/n} \leq \frac{1}{n}. \tag{21}$$

Then by the Archimedean property of $\mathbb{R}$, we know there is $N \in \mathbb{N}$ such that $N > 1/\varepsilon$, which implies $1/N < \varepsilon$. Thus

$$|a_n - 0| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon \quad \forall\ n \geq N, \tag{22}$$

and we are done. □

Example 8: Let (s_n) be a convergent sequence, and suppose $\lim_{n\to\infty} s_n > a$. Prove there exists a number N such that $n > N$ implies $s_n > a$.

Proof:

Set $s := \lim_{n\to\infty} s_n$ and let $\varepsilon := \frac{1}{2}(s-a)$. By hypothesis, $\varepsilon > 0$. Then because s_n converges, there is $N \in \mathbb{N}$ such that

$$|s_n - s| < \varepsilon \quad \forall\, n > N. \tag{23}$$

This implies $s_n - s > -\varepsilon \quad \forall\, n > N$, and so

$$s_n > s - \varepsilon = s - \frac{s-a}{2} = \frac{s+a}{2} > \frac{a+a}{2} = a \quad \forall\, n > N. \tag{24}$$

Thus we conclude $s_n > a$ for all $n > N$. ■

Remark 2: It is often helpful to make a small drawing to develop intuition for a problem. In the case of Example 8, we can draw a real line with s and a labeled, with $s > a$. Then we realize we want to choose ε small enough so that the lower bound on the interval $(s-\varepsilon, s+\varepsilon)$ is greater than a. It suffices, e.g., to choose $\varepsilon := \frac{1}{2}(s-a)$.

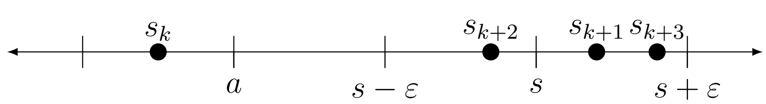

Figure 1: Illustration for Example 8 with iterates from a sample sequence (s_k). Note $s_k \in (s-\varepsilon, s+\varepsilon)$ for k sufficiently large.

◇

Example 9: Let $x_1 := 1$ and $x_{n+1} := 3x_n^2$ for $n \geq 1$. Prove the limit $\lim_{n\to\infty} x_n$ does not exist.

Proof:

We proceed by way of contraction. Suppose $\lim_{n\to\infty} x_n$ exists and set $x^* := \lim_{n\to\infty} x_n$. Then, using properties of convergent sequences,

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} 3x_n^2 = 3\left(\lim_{n\to\infty} x_n\right)^2 = 3(x^*)^2. \tag{25}$$

This shows $x^*(3x^* - 1) = 0$, and so either $x^* = 0$ or $x^* = 1/3$.

We next claim $x_n \geq 1$ for all $n \in \mathbb{N}$ and verify this by induction. The base case is given. Suppose now $x_k \geq 1$ for some $k \in \mathbb{N}$. Then

$$x_{k+1} = 3x_k^2 \geq 3(1)^2 = 3 \geq 1, \tag{26}$$

and we have closed the induction. Therefore, by the principle of mathematical induction, $x_n \geq 1$ for all $n \in \mathbb{N}$.

Then, for every $n \in \mathbb{N}$,

$$x_n - x^* \geq x_n - \frac{1}{3} \geq 1 - \frac{1}{3} > \frac{2}{3} \quad \Longrightarrow \quad |x_n - x^*| > \frac{2}{3}. \tag{27}$$

This shows there does not exist $N \in \mathbb{N}$ such that $|x_n - x^*| < 1/3$ for $n > N$. Hence the sequence (x_n) does not converge to x^*. This contradicts our initial assumption that (x_n) does converge to x^*. This implies the initial assertion that $\lim_{n\to\infty} x_n$ exists must be false. Therefore we conclude $\lim_{n\to\infty} x_n$ does not exist.

■

Remark 3: Let p be the statement defining x_n and let q be the statement that $\lim_{n\to\infty} x_n$ does not exist. In the above example, we show $p \Rightarrow q$. This is done by supposing p holds and $\neg q$ holds. From this, we arrive at a contradiction. This shows that if p holds, then $\neg q$ must be false. In other words, if p holds, then q holds. ◇

Example 10: Suppose there is N_0 such that $s_n \le t_n$ for all $n > N_0$. Prove that if $\lim_{n\to\infty} s_n$ and $\lim_{n\to\infty} t_n$ exist, then $\lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n$.

Proof:

Set $s := \lim_{n\to\infty} s_n$ and $t := \lim_{n\to\infty} t_n$ and suppose, by way of contradiction, that $s > t$. Then set $\varepsilon := \frac{1}{3}(s-t)$ and note $\varepsilon > 0$. By the convergence of (s_n) there is $N_1 \in \mathbb{N}$ such that

$$|s_n - s| < \varepsilon \quad \forall\, n > N_1. \tag{28}$$

Similarly, there is $N_2 \in \mathbb{N}$ such that

$$|t_n - t| < \varepsilon \quad \forall\, n > N_2. \tag{29}$$

Now define $N := \max\{N_0, N_1, N_2\}$. Then $s_n - s > -\varepsilon$ for all $n > N$, and so

$$s_n > s - \varepsilon = s - \frac{s-t}{3} = \frac{2s+t}{3} \quad \forall\, n > N. \tag{30}$$

Similarly,

$$t_n < t + \varepsilon = t + \frac{s-t}{3} = \frac{s+2t}{3} \quad \forall\, n > N. \tag{31}$$

Compiling these two results with the fact $s > t$, we deduce

$$s_n > \frac{2s+t}{3} > \frac{s+2t}{3} > t_n \quad \forall\, n > N. \tag{32}$$

However, this contradicts one of our initial hypotheses. Thus the assumption that $s > t$ must have been false, and we conclude $s \le t$, as desired. ■

Remark 4: For Example 10 we can draw a real line with s and t labeled, with $s > t$. Then we realize we want to choose ε small enough so that we can create intervals of length 2ε about s and t that do not overlap, e.g., by choosing $\varepsilon := \frac{1}{3}(s-t)$.

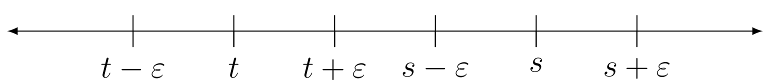

Figure 2: Illustration of the relevant quantities in Example 10.

◇

Example 11: Redo the previous problem by taking a direct approach.

Solution:

Set $a_n := t_n - s_n$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} t_n - s_n = \lim_{n\to\infty} t_n - \lim_{n\to\infty} s_n = t - s. \tag{33}$$

And, by hypothesis, $a_n = t_n - s_n \geq 0$ for all $n > N_0$. That is, $a_n \geq 0$ for all but finitely many n. By one of our homework exercises (Exercise 8.9a), this implies $\lim_{n\to\infty} a_n \geq 0$. Combining our results, we see $t - s = \lim_{n\to\infty} a_n \geq 0$, which implies $s \leq t$, as desired. $\square$

Example 12: Define $s_n := 1 + a + \cdots + a^n$ for $n \geq 0$. Prove $\lim_{n\to\infty} s_n = 1/(1-a)$ when $|a| < 1$.

Solution:

Let $\varepsilon > 0$ be given. We claim

$$s_n := \frac{1 - a^{n+1}}{1 - a}. \tag{34}$$

Indeed, $(1 - a) \neq 0$ by hypothesis, and

$$\begin{aligned} s_n(1-a) &= (1 + a + \cdots + a^n)(1-a) \\ &= (1 + a + \cdots + a^n) - \left(a + a^2 + a^{n+1}\right) \\ &= 1 + (a - a) + \left(a^2 - a^2\right) + \cdots + (a^n - a^n) - a^{n+1} \\ &= 1 - a^{n+1}. \end{aligned} \tag{35}$$

By a previous theorem in class, we know $\lim_{n\to\infty} a^n = 0$. This implies there is $N \in \mathbb{N}$ such that

$$|a^n - 0| < \varepsilon(1 - a) \quad \forall n > N. \tag{36}$$

Consequently,

$$\left|\frac{1 - a^{n+1}}{1 - a} - \frac{1}{1 - a}\right| = \frac{|a^n|}{1 - a} < \frac{\varepsilon(1 - a)}{1 - a} = \varepsilon \quad \forall\ n > N, \tag{37}$$

and the desired result follows. $\square$

Definition: We say a sequence (s_n) **diverges to** $+\infty$ provided for each $M > 0$ there is $N \in \mathbb{N}$ such that $n > N$ implies $s_n > M$. When this holds, we write $\lim_{n\to\infty} s_n = +\infty$. $\triangle$

Example 13: What is $\lim\limits_{n\to\infty}(1+a+a^2+\cdot+a^n)$ when $a\geq 1$?

Proof:

Let $s_n := 1+a+a^2+\cdots+a^n$ for $n\geq 0$. First, note $a^n\geq 1$ for $n\in\mathbb{N}$. The base case is given. Suppose $a^k\geq 1$ for some $k\in\mathbb{N}$. Then $a^{k+1}=aa^k\geq 1a^k=a^k\geq 1$, and we have closed the induction. Thus, the principle of mathematical induction implies $a^n\geq 1$ for $n\in\mathbb{N}$.

We claim $\lim\limits_{n\to\infty} s_n=+\infty$. Let $M>0$ be given. Then by the Archimedean property of $\mathbb{R}$ there is $N\in\mathbb{N}$ such that $N>M$. Consequently,

$$s_n = 1+a+a^2+\cdots+a^n \geq \underbrace{1+1+1^2+\cdots+1^n}_{n \text{ terms}} = n > N > M \quad \forall\, n>N. \tag{38}$$

Whence $\lim\limits_{n\to\infty} s_n=+\infty$. ■

Example 14: Give a formal proof that $\lim\limits_{n\to\infty} n^2=+\infty$ using only the definition of a sequence that diverges to $+\infty$.

Proof:

We say $\lim\limits_{n\to\infty} n^2=+\infty$ provided for all $M>0$ there exists N such that $n>N$ implies $n^2>M$. So, let $M>0$ be given. Then set $N:=\max\{1,M\}$. Then $n>N$ implies

$$n^2 = n\cdot n > N\cdot n > N^2 \geq 1N = N \geq M. \tag{39}$$

This completes the proof. ■

Example 15: Set $s_n := (3n+1)/(7n-4)$ for each $n \in \mathbb{N}$. Use the definition of a limit to prove (s_n) converges to $3/7$.

Proof:

Let $\varepsilon > 0$ be given. It suffices to show there is $N \in \mathbb{N}$ such that $n > N$ implies

$$\left| s_n - \frac{3}{7} \right| < \varepsilon. \tag{40}$$

Observe

$$\left| s_n - \frac{3}{7} \right| = \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| = \left| \frac{(21n+7)-(21n-12)}{(7n-4)7} \right| = \left| \frac{19}{49n-28} \right| = \frac{19}{49n-28} \le \frac{19}{n}, \tag{41}$$

noting $49n - 28 > n$ for $n \ge 1$. (This is quickly verified via an inductive argument.) By the Archimedean property of $\mathbb{R}$, there is $N \in \mathbb{N}$ such that $N\varepsilon > 19$, which implies $19/N < \varepsilon$. Whence $n > N$ implies

$$\left| s_n - \frac{3}{7} \right| \le \frac{19}{n} \le \frac{19}{N} < \varepsilon, \tag{42}$$

and we are done. ■

Remark 5: The above example does make use of the definition. However, we could just as well recall the limit of a quotient is the quotient of the limits, i.e.,

$$s_n = \frac{3n+1}{7n-4} = \frac{3+1/n}{7-4/n} \quad \Longrightarrow \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{3+1/n}{7-4/n} = \frac{\lim_{n\to\infty} 3+1/n}{\lim_{n\to\infty} 7-4/n} = \frac{3+0}{7-0} = \frac{3}{7}. \tag{43}$$

◇

Definition: We say a sequence (s_n) is a **Cauchy sequence** provided for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|s_m - s_n| < \varepsilon \quad \forall\, m, n > N. \tag{44}$$

△

Example 16: Let (s_n) be a sequence satisfying $|s_{n-1} - s_n| < 2^{-n}$ for $n \in \mathbb{N}$. Prove (s_n) converges.

Proof:
The sequence (s_n) converges if and only if it is Cauchy. Thus it suffices to show (s_n) is Cauchy, which we do as follows. Let $\varepsilon > 0$ be given. Pick any $m, n \in \mathbb{N}$. If $m = n$, then $|s_m - s_n| = 0 < \varepsilon$. Otherwise, without loss of generality, assume $m > n$. Then through repeated application of the triangle inequality we deduce

$$\begin{aligned} |s_m - s_n| &\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n| \\ &\leq \left(\frac{1}{2}\right)^{m-1} + \left(\frac{1}{2}\right)^{m-2} + \cdots + \left(\frac{1}{2}\right)^{n} \\ &= \sum_{i=n}^{m-1} \left(\frac{1}{2}\right)^{i}. \end{aligned} \tag{45}$$

However, we may factor out 2^{-n} and use the fact this is a geometric sum to write

$$\sum_{i=n}^{m-1} \left(\frac{1}{2}\right)^{i} = 2^{-n} \sum_{i=0}^{m-n-1} \left(\frac{1}{2}\right)^{i} = 2^{-n} \cdot \frac{1 - (1/2)^{m-n}}{1 - (1/2)} \leq 2^{-n} \cdot \frac{1}{1 - (1/2)} = 2^{1-n}, \tag{46}$$

noting $(1/2)^{m-n} < 1$. By the Archimedean property of $\mathbb{R}$ there is $N > 0$ such that $N > -\log_2(\varepsilon) + 1$, and so $2^{1-N} < \varepsilon$. Our results together imply

$$|s_m - s_n| \leq 2^{1-n} < 2^{1-N} < \varepsilon \quad \forall\, m, n \in \mathbb{N}. \tag{47}$$

Whence (s_n) is Cauchy, and we are done. ■

Remark 6: Could we complete the previous example with the same argument if we instead were given that $|s_{n+1} - s_n| < 1/n$? The answer in this case is *no*. The reason is that the sum $\sum_{n=1}^{N} 1/n$ diverges as $N \to \infty$. $\diamond$

Example 17: Let (a_n) and define $s_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i$. Prove if (s_n) converges, then (a_n) converges to 0.

Proof:
Let $\varepsilon > 0$ be given. We must show there is $N \in \mathbb{N}$ such that

$$|a_n - 0| < \varepsilon \quad \forall\ n > N. \tag{48}$$

Now since (s_n) converges, it is Cauchy. This implies there is $N^* > 0$ such that

$$|s_m - s_n| < \varepsilon \quad \forall\ n > N^*. \tag{49}$$

However, for $m > n$ we may write $s_m - s_n$ as

$$|s_m - s_n| = \left| \sum_{i=1}^{m} a_i - \sum_{i=1}^{n} a_i \right| = \left| \sum_{i=n+1}^{m} a_i \right| = |a_{n+1} + a_{n+2} + \cdots + a_m| \,. \tag{50}$$

In particular, taking $N = N^* + 1$, this implies

$$|a_n - 0| = |a_n| = |s_n - s_{n-1}| < \varepsilon \quad \forall\ n > N, \tag{51}$$

and we conclude (a_n) converges to 0. $\blacksquare$

Monotone Convergence Theorem: If a sequence is increasing/decreasing and bounded above/-below, then it converges. $\triangle$

Example 18: Let $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n$ for $n \geq 1$. Show $\lim_{n\to\infty} t_n$ exists.

Proof:

We proceed by applying the Monotone Convergence theorem. We claim $t_n > 0$ for each $n \in \mathbb{N}$. The base case is given. Inductively suppose $t_k > 0$. Then note

$$t_{k+1} = \left(1 - \frac{1}{4k^2}\right) t_k = \frac{4k^2 - 1}{4k^2} \cdot t_k \geq \frac{4-1}{4k^2} \cdot t_k = \frac{3t_k}{4k^2} > 0, \tag{52}$$

where the final inequality holds since $3t_k \geq 0$ by the inductive hypothesis and we note $4k^2 > 0$ for each k. This closes the induction, and so the claim follows by the principle of mathematical induction.

All that remains is to show (t_n) is decreasing. This may be directly seen from the recursive definition of t_{n+1} and the fact $t_n > 0$, i.e.,

$$t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n = t_n - \frac{t_n}{4n^2} \leq t_n = t_n. \tag{53}$$

The inequality follows from the fact $t_n > 0$ and $4n^2 > 0$, thereby making $t_n/(4n^2) > 0$. Therefore we conclude by the Monotone Convergence Theorem that (t_n) converges. $\blacksquare$

Example 19: Let (b_n) be a bounded nonnegative sequence and $r \in (0,1)$. For each $n \in \mathbb{N}$ define $s_n = b_0 + b_1 r + \cdots + b_n r^n$. Prove s_n converges.

Proof:
We proceed by applying the Monotone Convergence Theorem. We first show s_n is increasing. Indeed, observe

$$s_{n+1} = b_0 + b_1 r + \cdots + b_{n+1} r^{n+1} = s_n + b_{n+1} r^{n+1} \geq s_n, \tag{54}$$

noting $b_{n+1}, r^{n+1} \geq 0$ and so $b_{n+1} r^{n+1} \geq 0$. All that remains is to show (s_n) is bounded above. Let B denote the upper bound for (b_n). Then

$$s_n = b_0 + b_1 r + \cdots + b_n r^n \leq B\,(1 + r + \cdots + r^n) = B \cdot \frac{1 - r^{n+1}}{1 - r} \leq B \cdot \frac{1}{1 - r}. \tag{55}$$

This shows $s_n \leq B/(1-r)$ for every $n \in \mathbb{N}$. Thus we conclude by the Monotone Convergence Theorem the sequence (s_n) converges. ■

Remark 7: In the next example, we make use of factorials. We write $n! := n(n-1)(n-2)\cdots(2)(1)$ and $0! := 1$. From calculus class, we may recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{56}$$

In the next example we show the sum on the right hand side converges in the particular case where $x = 1$. $\diamond$

Example 20: Set $s_n = 1/0! + 1/1! + \cdots + 1/n!$ for $n \in \mathbb{N}$. Prove the sequence (s_n) converges.

Proof:

We proceed by applying the Monotone Convergence theorem. Observe s_n is increasing since

$$s_{n+1} = \sum_{k=0}^{n+1} \frac{1}{k!} = s_n + \frac{1}{(k+1)!} > s_n, \tag{57}$$

noting $(k+1)! > 0$ since it is the product of all positive numbers and so $1/(k+1)! > 0$. The remaining and more difficult task is to show (s_n) is bounded above. In order to do this, we find a bound for each term in s_n. In particular, we claim $n! \geq 2^n$ for $n \geq 2$. Indeed, in the base case $2! = 2 = 2^1$. Inductively, suppose $k! \geq 2^k$ for some $k \geq 2$. Then

$$(k+1)! = (k+1)k! \geq (k+1)2^k \geq 32^k > 2^{k+1}, \tag{58}$$

and we have closed the induction. The claim follows from the principle of mathematical induction. This shows

$$\frac{1}{k!} \leq \frac{1}{2^k} \quad \forall\ k \geq 2. \tag{59}$$

Hence

$$s_n = \frac{1}{1} + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 1 + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^n} = 1 + \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \tag{60}$$

However, the sum on the right hand side is a geometric sum. Thus we can rewrite this as

$$s_n = 1 + \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = 1 + \frac{1-(1/2)^{n+1}}{1-(1/2)} \leq 1 + \frac{1}{1-(1/2)} = 1 + 2 = 3. \tag{61}$$

This reveals $s_n \leq 3$ for each $n \in \mathbb{N}$ and we conclude (s_n) converges by the Monotone Convergence Theorem. $\blacksquare$

Example 21: Prove that if $\sum |a_n|$ converges and (b_n) is bounded, then $\sum a_n b_n$ converges.

Proof:

Recall a series converges if and only if it satisfies the Cauchy criterion, i.e., for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $m, n > N$ imply $|s_n - s_m| < \varepsilon$, where (s_n) is the sequence of partial sums. So, it suffices to show the given series satisfies the Cauchy criterion. Let $\varepsilon > 0$ be given. Because (b_n) is bounded, there is $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. This implies

$$\left|\sum_{k=1}^{n} b_k a_n - \sum_{j=1}^{m} b_j a_j\right| = \left|\sum_{k=m+1}^{n} b_k a_k\right| \leq \sum_{k=m+1}^{n} |b_k a_k| \leq \sum_{k=m+1}^{n} M|a_k| = M \sum_{k=m+1}^{n} |a_k|. \tag{62}$$

Note the sum is of nonnegative terms, and so

$$\left|\sum_{k=1}^{n} b_k a_n - \sum_{j=1}^{m} b_j a_j\right| \leq M \left|\sum_{k=m+1}^{n} |a_k|\right| = M \left|\sum_{k=1}^{n} |a_k| - \sum_{j=1}^{m} |a_j|\right|. \tag{63}$$

Now because $\sum |a_n|$ converges, it satisfies the Cauchy criterion. Whence there is $N \in \mathbb{N}$ such that

$$\left|\sum_{k=1}^{n} |a_k| - \sum_{j=1}^{m} |a_j|\right| < \frac{\varepsilon}{M} \quad \forall\, n, m > N. \tag{64}$$

Together (63) and (64) imply

$$\left|\sum_{k=1}^{n} b_k a_n - \sum_{j=1}^{m} b_j a_j\right| < M\left(\frac{\varepsilon}{M}\right) = \varepsilon \quad \forall\, n, m > N, \tag{65}$$

and we are done. ■

Example 22: Let $s_N = \sum_{n=0}^{N} \frac{1}{3^n}$. Let (a_n) be a nonnegative sequence such that $a_n \leq s_n$ for each $n \in \mathbb{N}$. Prove (a_n) has a convergent subsequence.

Proof:

The Bolzano-Weierstrass Theorem asserts every bounded sequence has a convergent subsequence. So, it suffices to show (a_n) is bounded. Observe

$$a_n \leq s_n = \sum_{j=0}^{n} \frac{1}{3^j} = \sum_{j=0}^{n} \left(\frac{1}{3}\right)^n = \frac{1-(1/3)^{n+1}}{1-(1/3)} \leq \frac{1}{1-(1/3)} = \frac{3}{2}, \tag{66}$$

and so $a_n \in [0, 3/2]$ for each $n \in \mathbb{N}$. Whence (a_n) is bounded and, therefore, has a convergent subsequence. ■

Example 23: Let (t_n) be bounded and (s_n) converge to 0. Prove $\lim\limits_{n\to\infty} s_n t_n = 0$.

Proof:

Because (t_n) is bounded, there is $M > 0$ such that $|t_n| \leq M$ for all $n \in \mathbb{N}$. Now let $\varepsilon > 0$ be given. By the convergence of (s_n), there is $N \in \mathbb{N}$ such that $n > N$ implies $|s_n| = |s_n - 0| < \varepsilon/M$. Then we see $n > N$ implies

$$|s_n t_n - 0| = |s_n||t_n| \leq M|s_n| < M\left(\frac{\varepsilon}{M}\right) = \varepsilon. \tag{67}$$

This shows $\lim\limits_{n\to\infty} s_n t_n = 0$, and we are done. ■

Section 3: Continuous Functions

Remark 8: Roughly, the approach for $\varepsilon - \delta$ arguments in proving continuity of a function f may be listed as follows:

1. Let $\varepsilon > 0$ be given and $\overline{x} \in \text{dom}(f)$.
2. "Play" with $|f(x) - f(\overline{x})|$ and try to bound this by some function of $|x - \overline{x}|$ and $\overline{x}$.
3. Pick some $\delta > 0$, possibly in terms of ε and $\overline{x}$ such that the above bound makes $|f(x) - f(\overline{x})| < \varepsilon$ when $|x - \overline{x}| < \delta$.

$\diamond$

We next give two equivalent definitions of continuity.

Definition: Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is **continuous** at $\overline{x} \in \text{dom}(f)$ provided for each $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - \overline{x}| < \delta$ imply $|f(x) - f(\overline{x})| < \varepsilon$. $\triangle$

Definition: Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is **continuous** at $\overline{x} \in \text{dom}(f)$ provided for each sequence (x_n) with $x_n \in \text{dom}(f)$ for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(\overline{x}). \tag{68}$$

$\triangle$

Definition: If a function f is continuous at every $\overline{x} \in \text{dom}(f)$, then we say f is continuous. $\triangle$

Example 24: Prove $|x|$ is continuous on $\mathbb{R}$.

Proof:

Let $\varepsilon > 0$ be given. Pick any $\overline{x} \in \mathbb{R}$. It suffices to show there is a $\delta > 0$ such that

$$|x - \overline{x}| < \delta \quad \Longrightarrow \quad ||x| - |\overline{x}|| < \varepsilon. \tag{69}$$

We break this problem into two cases. First suppose $\overline{x} = 0$. Taking $\delta := \varepsilon$, we deduce $|x| = |x - 0| = |x - \overline{x}| < \delta$ implies

$$||x| - |\overline{x}|| = ||x| - |0|| = ||x| - 0| = |x| < \delta = \varepsilon. \tag{70}$$

Now suppose $\overline{x} \neq 0$. Then choosing $\delta := \min\{\varepsilon,\ |\overline{x}|/2\}$ reveals

$$|x - \overline{x}| < \delta \quad \Longrightarrow \quad -\frac{|\overline{x}|}{2} < x - \overline{x} < \frac{|\overline{x}|}{2} \quad \Longrightarrow \quad \begin{cases} x > |\overline{x}|/2 & \text{if } \overline{x} > 0, \\ x < -|\overline{x}|/2 & \text{if } \overline{x} < 0. \end{cases} \tag{71}$$

In particular, this shows $|x - \overline{x}| < \delta$ implies x has the same sign as $\overline{x}$. And when x and $\overline{x}$ have the same sign, $||x| - |\overline{x}|| = |x - \overline{x}|$ since the constant sign term ± 1 can be factored out. Thus

$$|x - \overline{x}| < \delta \quad \Longrightarrow \quad ||x| - |\overline{x}|| = |x - \overline{x}| < \varepsilon. \tag{72}$$

This verifies $|x|$ is continuous at $\overline{x}$. Because $\overline{x}$ was arbitrarily chosen in $\mathbb{R}$, we deduce this holds for every $\overline{x} \in \mathbb{R}$. Hence we conclude $|x|$ is continuous on $\mathbb{R}$. ■

Remark 9: Next we try and use the limit definition to show f is continuous. However, we see our limit argument does not work at $\overline{x} = 1$. There we use the $\varepsilon - \delta$ approach. ◇

Example 25: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x < 1, \\ 2x - 1 & \text{if } x \geq 1. \end{cases} \tag{73}$$

Prove f is continuous.

Proof:

We break this proof into different possible cases. First suppose $\overline{x} \neq 1$ and let (x_n) converge to $\overline{x}$. Then there exists $N \in \mathbb{N}$ such that

$$|x_n - \overline{x}| < \frac{|\overline{x} - 1|}{2} \quad \forall\, n > N. \tag{74}$$

If $\overline{x} > 1$, then for $n > N$ this implies

$$x_n - \overline{x} > \frac{1 - \overline{x}}{2} \quad \Longrightarrow \quad x_n > \frac{1 + \overline{x}}{2} > \frac{1 + 1}{2} = 1, \tag{75}$$

and so $f(x_n) = 2x_n - 1$ for $n > N$. Whence

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 2x_n - 1 = 2\overline{x} - 1 = f(\overline{x}). \tag{76}$$

Alternatively, if $\overline{x} < 1$, then (74) implies for $n > N$

$$x_n - \overline{x} < \frac{1 - \overline{x}}{2} \quad \Longrightarrow \quad x_n < \frac{1 + \overline{x}}{2} < \frac{1 + 1}{2} = 1, \tag{77}$$

and so $f(x_n) = x_n$ for $n > N$, which yields

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = \overline{x} = f(\overline{x}). \tag{78}$$

Combining (76) and (78), we deduce f is continuous at $\overline{x} \neq 1$.

Now suppose $\overline{x} = 1$ and let $\varepsilon > 0$ be given. To show f is continuous at $\overline{x}$, it suffices to find $\delta > 0$ such that $|x - \overline{x}| < \delta$ implies $|f(x) - f(\overline{x})| < \varepsilon$. Note $f(\overline{x}) = 2\overline{x} - 1 = 2 \cdot 1 - 1 = 1 = \overline{x}$. Set $\delta := \varepsilon/2$. If $|x - \overline{x}| < \delta$ and $x < 1$, then

$$|f(x) - f(\overline{x})| = |x - \overline{x}| < \delta < \varepsilon. \tag{79}$$

And if $|x - \overline{x}| < \delta$ and $x > 1$, then

$$|f(x) - f(\overline{x})| = |(2x-1) - (2\overline{x}-1)| = 2\,|x - \overline{x}| < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon. \tag{80}$$

This verifies f is continuous at $\overline{x} = 1$. Then because f is continuous at every $\overline{x} \in \mathbb{R}$, we conclude f is continuous. ■

Remark 10: The above argument is how this problem was presented in discussion. However, we could do the entire thing using an $\varepsilon - \delta$ argument. Because f is piecewise linear we could pick δ to be ε divided by the steepest slope of f (i.e., $\delta = \varepsilon/2$). But, note this approach would require checking four cases, depending on whether $\overline{x} < 1$ or $\overline{x} \geq 1$ and whether $x < 1$ or $x \geq 1$. ◇

Remark 11: Note the following example illustrates a good tool to keep in your "bag of tricks". Here note how we choose δ in such a way as to put an upper bound on $|x + \overline{x}|$ and still maintain keeping δ sufficiently small. ◇

Example 26: Prove $f(x) = x^2$ is continuous.

Proof:

Let $\varepsilon > 0$ be given and pick any $\overline{x} \in \mathbb{R}$. To verify f is continuous at $\overline{x}$, it suffices to show there is $\delta > 0$ such that

$$|x - \overline{x}| < \delta \quad \Longrightarrow \quad |f(x) - f(\overline{x})| < \varepsilon. \tag{81}$$

Observe through factoring and the triangle inequality we obtain

$$|f(x) - f(\overline{x})| = |x^2 - \overline{x}^2| = |x - \overline{x}||x + \overline{x}| = |x - \overline{x}||x - \overline{x} + 2\overline{x}| \leq |x - \overline{x}|\left[|x - \overline{x}| + 2|\overline{x}|\right]. \tag{82}$$

Now pick $\delta := \min\{1, \varepsilon/(2|\overline{x}|+1)\}$. Then $|x - \overline{x}| < \delta$ implies

$$|f(x) - f(\overline{x})| \leq |x - \overline{x}|\left[|x - \overline{x}| + 2|\overline{x}|\right] < \delta\left[1 + 2|\overline{x}|\right] \leq \frac{\varepsilon}{2|\overline{x}|+1}\left[1 + 2|\overline{x}|\right] = \varepsilon. \tag{83}$$

This shows f is continuous at $\overline{x}$. Since $\overline{x}$ was arbitrarily chosen, this holds for all $\overline{x} \in \mathbb{R}$. Thus we conclude f is continuous. ■

Remark 12: We next show an argument for proving a function is discontinuous at a point $\overline{x} \in \mathrm{dom}(f)$. In essence, it suffices to construct a sequence (x_n) contained in $\mathrm{dom}(f)$ and converging to $\overline{x}$ such that $\lim_{n\to\infty} f(x_n) \neq f(\overline{x})$. $\diamond$

Example 27: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases} \tag{84}$$

Prove f is discontinuous everywhere.

Proof:

Let $\overline{x} \in \mathbb{R}$ be irrational so that $f(\overline{x}) = 0$. To show f is not continuous at $\overline{x}$, it suffices to construct a sequence (x_n) converging to $\overline{x}$ such that $\lim_{n\to\infty} f(x_n) \neq f(\overline{x})$. By the density of the rationals in $\mathbb{R}$, there is $x_1 \in \mathbb{Q}$ such that $|\overline{x} - x_1| < 1/1$. Similarly, there is $x_2 \in \mathbb{Q}$ such that $|\overline{x} - x_2| < 1/2$. Continuing in an inductive fashion, we see there is a sequence (x_n) of rational numbers such that $|\overline{x} - x_n| < 1/n$. We claim $x_n \longrightarrow \overline{x}$. Indeed, let $\varepsilon > 0$ be given. Then by the Archimedean property of $\mathbb{R}$ there is $N \in \mathbb{N}$ such that $N\varepsilon > 1$, which implies $1/N < \varepsilon$. Then

$$|\overline{x} - x_n| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon \quad \forall\, n > N, \tag{85}$$

and so $x_n \longrightarrow \overline{x}$. However,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1 \neq 0 = f(\overline{x}). \tag{86}$$

This shows f is not continuous at any irrational $\overline{x}$. Analogous argument holds for each $\overline{x} \in \mathbb{Q}$ since the irrationals are also dense in $\mathbb{R}$. Whence f is not continuous anywhere. $\blacksquare$

Remark 13: Sometimes when constructing a sequence we don't give an exact formula for the sequence elements. Instead, it suffices to show for each $n \in \mathbb{N}$, there is some x_n that satisfies a property (related to the integer n). And because n was arbitrary, this holds for each $n \in \mathbb{N}$. Then we simply say something like "Let (x_n) be a sequence with iterates satisfying this property." $\diamond$

Example 28: Suppose $f : [0,1] \to \mathbb{R}$ is continuous, $f(0) > 0$ and $f(1) = 0$. Prove there is $x_0 \in (0,1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for all $x \in [0, x_0)$, i.e., x_0 is the smallest point in $[0,1]$ at which f attains the value 0.

Proof:

By way of contradiction, suppose there is not a smallest point $x_0 \in [0,1]$ such that $f(x) > 0$ for $x \in [0, x_0)$. We will construct a sequence (x_n) converging to zero for which $\lim_{n\to\infty} f(x_n) = 0$. By our assumption, there is $x_1 \in [0,1]$ such that $f(x_1) = 0$. And, continuing inductively, for each $k \in \mathbb{N}$, there is $x_k \in (0, 1/k]$ such that $f(x_k) = 0$. Let (x_n) be a sequence generated by this choice for each x_k and $\varepsilon > 0$ be given. Then by the Archimedean property of $\mathbb{R}$ there is $N \in \mathbb{N}$ such that $1/N < \varepsilon$ and so

$$|x_n - 0| = |x_n| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon \quad \forall\, n > N. \tag{87}$$

This shows $x_n \longrightarrow 0$. By the continuity of f, we then see

$$0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(0), \tag{88}$$

a contradiction. Thus the initial assumption was false and the result follows. ■

Remark 14: Recall f is continuous at $\overline{x}$ provided every sequence (x_n) converging to $\overline{x}$ yields

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(\overline{x}). \tag{89}$$

So, to show a function is not continuous at $\overline{x}$, it suffices to find a single sequence (x_n) converging to $\overline{x}$ such that (89) does not hold. $\diamond$

Example 29: Prove $x = \cos(x)$ for some $x \in (0, \pi/2)$.

Solution:

Define the function $f(x) = x - \cos(x)$. Since polynomials are continuous and cosine is continuous and sums of continuous functions are continuous, we know f is also continuous. Also,

$$f(0) = 0 - \cos(0) = 0 - 1 = -1 < 0 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 0 > 0. \tag{90}$$

Then the intermediate value theorem asserts there is $x^* \in (0, \pi/2)$ such that $f(x^*) = 0$, which implies $x^* = \cos(x^*)$. This completes the proof. $\square$

Example 30: Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and that its image consists entirely of rational numbers. Prove f is a constant function.

Proof:

By hypothesis, for each $x \in [0, 1]$, $f(x) \in \mathbb{Q}$. Now, by way of contradiction, suppose f is not constant. Then there are $x, y \in [0, 1]$ with $x \neq y$ such that $f(x) \neq f(y)$. By the density of the irrationals in $\mathbb{R}$, there is some $z \in \mathbb{I}$ such that z is between $f(x)$ and $f(y)$. Because f is continuous, the intermediate value theorem asserts there is some c between x and y such that $f(c) = z \in I$, a contradiction. Thus f must be constant. $\blacksquare$

Remark 15: Note in the above example we use the phrase "c between x and y". This particular wording is important because we do not know if $x < y$ or $y > x$. $\diamond$

Example 31: Suppose the function $f : [a, b] \to \mathbb{R}$ is continuous. For some $k \in \mathbb{N}$, let $x_1, \ldots, x_k$ be points in $[a, b]$. Prove there is a point $z \in [a, b]$ at which

$$f(z) = \frac{f(x_1) + \cdots + f(x_k)}{k}. \tag{91}$$

Proof:
Let $x_1, \ldots, x_k \in [a, b]$ be given. Let $j \in \{1, \ldots, k\}$ be an index such that

$$f(x_j) = \max\{f(x_1), \ldots, f(x_k)\}. \tag{92}$$

Then observe

$$\frac{f(x_1) + \cdots + f(x_k)}{k} \le \frac{f(x_j) + \cdots f(x_j)}{k} = f(x_j), \tag{93}$$

noting there were k terms in the numerator. By similar argument, we see there is an index $\ell \in \{1, \ldots, k\}$ such that

$$f(x_\ell) = \frac{f(x_\ell) + \cdots + f(x_\ell)}{k} \le \frac{f(x_1) + \cdots + f(x_k)}{k}. \tag{94}$$

If the inequality in either (93) or (94) is an equality, then we may take $z = x_j$ or $z = x_\ell$, respectively. If this is not the case, then

$$f(x_\ell) < \frac{f(x_1) + \cdots + f(x_k)}{k} < f(x_j), \tag{95}$$

and the intermediate value theorem implies there is z strictly between x_ℓ and x_j such that

$$f(z) = \frac{f(x_1) + \cdots + f(x_k)}{k}. \tag{96}$$

Because z is between x_ℓ and x_j and $x_\ell, x_j \in [a, b]$, we deduce $z \in [a, b]$. This completes the proof. ■

Example 32: Prove

$$f(x) := \frac{3x^2}{5x^2 - x} \tag{97}$$

is uniformly continuous on $[5, \infty)$.

Proof:

Let $\varepsilon > 0$ be given. We must show there is $\delta > 0$ such that, for $x, y \in [5, \infty)$,

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon. \tag{98}$$

Observe

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{3x^2}{5x^2 - x} - \frac{3y^2}{5y^2 - y} \right| \\ &= \left| \frac{3x}{5x - 1} - \frac{3y}{5y - 1} \right| \\ &= \left| \frac{3x(5y-1) - 3y(5x-1)}{(5x-1)(5y-1)} \right| \\ &\leq 3 \left| (5xy - x) - (5xy - y) \right| \\ &= 3 \left| y - x \right|. \end{aligned} \tag{99}$$

The inequality follows from the fact $(5x - 1) > 1$ for $x \in [5, \infty)$. The above shows that if $x, y \in [5, \infty)$ and $|x - y| < \varepsilon/3$, then

$$|f(x) - f(y)| \leq 3|x - y| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon. \tag{100}$$

Hence (98) holds taking $\delta = \varepsilon/3$, and we are done. ■

Example 33: Let

$$f(x) := \frac{x^2 - 4}{x - 2}. \tag{101}$$

Prove $\lim_{x \to 2^+} f(x) = 4$.

Proof:

Let $\varepsilon > 0$ be given. We must show there is $\delta > 0$ such that

$$x \in (2, 2+\delta) \quad \implies \quad |f(x) - 4| < \varepsilon. \tag{102}$$

For $x > 2$, observe

$$|f(x) - 4| = \left| \frac{(x+2)(x-2)}{x-2} - 4 \right| = |(x+2) - 4| = |x - 2| \tag{103}$$

Taking $\delta = \varepsilon$, we thus see

$$x \in (2, 2+\delta) \quad \implies \quad |f(x) - 4| = |x - 2| < \delta = \varepsilon, \tag{104}$$

and so the result follows. ■

Section 4: Differentiation

We now begin to dive into the more familiar concepts from calculus. Below we provide the definition of derivative, and the we provide examples on the following pages illustrating the use of this definition.

Definition: We say a function f is differentiable at a in its domain provided the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{105}$$

exists and is finite. When this is the case, we write

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{106}$$

$\triangle$

Before continuing, we note it will be useful to remember the following definition.

Definition: Let $f : S \to \mathbb{R}$ for a subset $S \subset \mathbb{R}$, and $a, L \in \mathbb{R}$. Then we say $\lim_{x \to a} f(x) = L$ provided for each $\varepsilon > 0$, there is a $\delta > 0$ such that $x \in S$ and $|x - a| < \delta$ imply $|f(x) - L| < \varepsilon$. $\triangle$

Example 34: Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has the property that

$$-x^2 \le f(x) \le x^2 \tag{107}$$

for each $x \in \mathbb{R}$. Prove f is differentiable at $x = 0$ and $f'(0) = 0$.

Proof:
We must show

$$0 = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}. \tag{108}$$

Let $\varepsilon > 0$ be given. It suffices to show there is $\delta > 0$ such that, for $x \neq 0$,

$$|x - 0| < \delta \quad \Longrightarrow \quad \left| \frac{f(x) - f(0)}{x - 0} - 0 \right| < \varepsilon. \tag{109}$$

First note, by hypothesis, $|f(0)| \le 0^2 = 0$, which implies $f(0) = 0$. Thus, for $x \neq 0$, we see

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{f(x) - 0}{x - 0} \right| = \frac{|f(x)|}{|x|} \le \frac{|x|^2}{|x|} = |x| = |x - 0|. \tag{110}$$

Taking $\delta = \varepsilon$ yields that $|x - 0| < \delta$ with $x \neq 0$ implies

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| \le |x - 0| < \delta = \varepsilon, \tag{111}$$

and so (109) holds. This completes the proof. ■

Example 35: Let the function $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = 0$. Prove

$$\lim_{x\to 0} \frac{f(x^2) - f(0)}{x} = 0. \tag{112}$$

Proof:
Let $\varepsilon > 0$ be given. We must show there is a $\delta > 0$ such that, for $x \neq 0$,

$$|x - 0| < \delta \quad \Longrightarrow \quad \left|\frac{f(x^2) - f(0)}{x} - 0\right| < \varepsilon. \tag{113}$$

By hypothesis, there is a $\delta^* > 0$ such that, for $x \neq 0$,

$$|x - 0| < \delta^* \quad \Longrightarrow \quad \left|\frac{f(x) - f(0)}{x - 0} - f'(0)\right| < 1, \tag{114}$$

where $f'(0)$ exists and is finite. Then, for $|x - 0| < \delta^*$ and $x \neq 0$, this with the triangle inequality implies

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x) - f(0)}{x - 0} - f'(0) + f'(0)\right| \leq \left|\frac{f(x) - f(0)}{x - 0} - f'(0)\right| + \left|f'(0)\right| < 1 + |f'(0)|. \tag{115}$$

Then

$$|x - 0| < 1 \quad \Longrightarrow \quad |x^2 - 0| = |x|^2 < |x| \cdot 1 < 1. \tag{116}$$

Letting $\delta := \min\{1, \delta^*, \varepsilon/(|f'(0)| + 1)\}$, this shows $|x^2 - 0| < \delta$ whenever $|x - 0| < \delta$. Consequently, using (115), when $x \neq 0$ and $|x - 0| < \delta$ we see

$$\left|\frac{f(x^2) - f(0)}{x - 0}\right| = |x| \left|\frac{f(x^2) - f(0)}{x^2 - 0}\right| \leq |x| \left[1 + |f'(0)|\right] < \left(\frac{\varepsilon}{1 + |f'(0)|}\right) \left[1 + |f'(0)|\right] = \varepsilon. \tag{117}$$

This verifies (113) and completes the proof. ■

Example 36: Suppose the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0. Prove

$$\lim_{x\to x_0} \frac{xf(x_0) - x_0 f(x)}{x - x_0} = f(x_0) - x_0 f'(x_0). \tag{118}$$

Proof:
First note, for $x \neq x_0$,

$$\frac{xf(x_0) - x_0 f(x)}{x - x_0} = \frac{xf(x_0) - x_0 f(x_0) + x_0 f(x_0) - x_0 f(x)}{x - x_0} = f(x_0) + x_0 \left(\frac{f(x_0) - f(x)}{x - x_0}\right). \tag{119}$$

Using the definition of derivative, this implies

$$\begin{aligned} \lim_{x\to x_0} \frac{xf(x_0) - x_0 f(x)}{x - x_0} &= \lim_{x\to x_0} f(x_0) + x_0 \left(\frac{f(x_0) - f(x)}{x - x_0}\right) \\ &= f(x_0) + \lim_{x\to x_0} x_0 \left(\frac{f(x_0) - f(x)}{x - x_0}\right) \\ &= f(x_0) - x_0 \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= f(x_0) - x_0 f'(x_0), \end{aligned} \tag{120}$$

and we are done. ∎

Example 37: Let the function $h : \mathbb{R} \to \mathbb{R}$ be bounded and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = 1 + 4x + x^2 h(x) \quad \forall\, x \in \mathbb{R}. \tag{121}$$

Prove $f(0) = 1$ and $f'(0) = 4$.

Proof:

By hypothesis, h is bounded, i.e., there is $M > 0$ such that $|h(x)| \leq M$ for all $x \in \mathbb{R}$. Consequently,

$$\begin{aligned} |f(x) - 1| &= \left|\big(1 + 4x + x^2 h(x)\big) - 1\right| \\ &= \left|4x + x^2 h(x)\right| \\ &= |x|\,[4 + |x||h(x)|] \\ &\leq |x|\,[4 + |x|M]\,. \end{aligned} \tag{122}$$

This implies

$$|f(0) - 1| \leq |0|\,[4 + |0|M] = 0 \cdot (4 + 0) = 0, \tag{123}$$

and so $f(0) = 1$.

Let $\varepsilon > 0$ be given. To show $f'(0) = 4$, it suffices to verify there is a $\delta > 0$ such that, for $x \neq 0$,

$$|x - 0| < \delta \quad \Longrightarrow \quad \left|\frac{f(x) - f(0)}{x - 0} - 4\right| < \varepsilon. \tag{124}$$

Note

$$\left|\frac{f(x) - f(0)}{x - 0} - 4\right| = \left|\frac{4x + x^2 h(x)}{x - 0} - 4\right| = \left|\frac{x^2 h(x)}{x}\right| = |xh(x)| \leq |x|M \tag{125}$$

whenever $x \neq 0$. Thus, taking $\delta = \varepsilon/M$, we see whenever $x \neq 0$

$$|x - 0| < \delta \quad \Longrightarrow \quad \left|\frac{f(x) - f(0)}{x - 0} - 4\right| \leq |x - 0|M < \delta M = \left(\frac{\varepsilon}{M}\right) M = \varepsilon. \tag{126}$$

This shows (124) holds and completes the proof. ■

Example 38: For real numbers a and b, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 5x^2 & \text{if } x \leq 1, \\ a + bx & \text{if } x > 1. \end{cases} \tag{127}$$

For what values of a and b is f differentiable at $x = 1$? (You do not need to verify f is differentiable at $x = 1$ once these values are derived.)

Proof:
Assume f is differentiable at $x = 1$. We must find a and b. Recall if f is differentiable at 1, then f is continuous at 1. This implies, for any sequence (x_n) such that $x_n \longrightarrow 1$, $\lim\limits_{n\to\infty} f(x_n) = f(1) = 5$. In particular, for $x_n = 1 + 1/n$, we see

$$5 = f(1) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} a + bx_n = a + b(1) = a + b. \tag{128}$$

Similarly, by hypothesis $f'(1)$ exists and

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}. \tag{129}$$

Observe, for $x_n = 1 + 1/n$,

$$\frac{f(x_n) - f(1)}{x_n - 1} = \frac{(a + bx_n) - 5}{(1 + 1/n) - 1} = \frac{a + b(1 + 1/n) - 5}{1/n} = \frac{0 + b/n}{1/n} = b, \tag{130}$$

noting $a + b - 5 = 0$. Thus $f'(1) = b$. Similarly, for $y_n = 1 - 1/n$, we see

$$\frac{f(y_n) - f(1)}{y_n - 1} = \frac{5y_n^2 - 5}{y_n - 1} = 5\left(\frac{(y_n - 1)(y_n + 1)}{y_n - 1}\right) = 5\,(y_n + 1) = 5\,(2 - 1/n) = 10 - 5/n. \tag{131}$$

In the limit as $n \longrightarrow \infty$, we see $f'(1) = \lim\limits_{n\to\infty} (10 + 5/n) = 10$. Combining our results, we see $b = f'(1) = 10$. Then $a = 5 - b = 5 - 10 = -5$. We conclude $\boxed{a = -5 \text{ and } b = 10}$. ■

4.1 – Mean Value Theorem:

Example 39: Show $\sin(x) \leq x$ for $x \geq 0$.

Proof:
Define $f(x) = x - \sin(x)$. Because f is the sum of differentiable functions, it is itself differentiable. In particular, $f'(x) = 1 - \cos(x) \geq 0$, where the final inequality holds since $-1 \leq \cos(x) \leq 1$. Now observe $f(0) = 0 - \sin(0) = 0$ and so the result holds at $x = 0$. For $x > 0$ the mean value theorem asserts there is $\xi \in (0, x)$ such that

$$f(x) - f(0) = f'(\xi)(x - 0) \quad \Longrightarrow \quad f(x) = f'(\xi)x \geq 0x = 0. \tag{132}$$

Thus for $x > 0$ we see

$$x - \sin(x) = f(x) \geq 0 \quad \Longrightarrow \quad x \geq \sin(x). \tag{133}$$

This completes the proof. ■

Example 40: Suppose f is differentiable on $\mathbb{R}$ and $5 \leq f'(x) \leq 10$ for each $x \in \mathbb{R}$. Also assume $f(0) = 0$. Prove $5x \leq f(x) \leq 10x$ for each $x \geq 0$.

Proof:
First note the result holds for $x = 0$ since

$$(5)0 = 0 \leq 0 = f(0) \leq 0 = (10)0. \tag{134}$$

Then for $x > 0$ the mean value theorem asserts there is $\xi \in (0, x)$ such that

$$f(x) - f(0) = f'(\xi)(x - 0) \quad \Longrightarrow \quad f(x) = f'(\xi)x. \tag{135}$$

Then, using our given inequality and the fact $x > 0$, we see

$$5x \leq f'(\xi)x = f(x) \leq 10x, \tag{136}$$

which completes the proof. ■

Example 41: Suppose f is differentiable and $\alpha := \sup\{|f'(x)| \ : \ x \in \mathbb{R}\} < 1$. Let $x_0 \in \mathbb{R}$ and define $x_n = f(x_{n-1})$ for $n \in \mathbb{N}$. Prove (x_n) converges to a fixed point of f.

Proof:
Observe for each $n \in \mathbb{N}$, the mean value theorem asserts there is ξ_n between x_n and x_{n-1} such that

$$f(x_n) - f(x_{n-1}) = f'(\xi_n)(x_n - x_{n-1}). \tag{137}$$

This implies

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| = |f'(\xi_n)||x_n - x_{n-1}| \leq \alpha |x_n - x_{n-1}|. \tag{138}$$

We claim for each $n \in \mathbb{N}$

$$|x_{n+1} - x_n| \leq \alpha^n |x_1 - x_0|. \tag{139}$$

The case for $n = 1$ is given. Now suppose (139) holds for some $k \in \mathbb{N}$. Then (138) implies

$$|x_{k+2} - x_{k+1}| \leq \alpha |x_{k+1} - x_k| \leq \alpha \left(\alpha^k |x_1 - x_0|\right) = \alpha^{k+1} |x_1 - x_0|, \tag{140}$$

which closes the induction. Thus (139) holds by the principle of mathematical induction.

Now for $m > n$ observe

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\ &\leq \alpha^{m-1}|x_1 - x_0| + \cdots + \alpha^n |x_1 - x_0| \\ &= \alpha^n |x_1 - x_0| \sum_{k=0}^{m-n-1} \alpha^k \\ &= \alpha^n |x_1 - x_0| \frac{1 - \alpha^{m-n}}{1 - \alpha} \\ &\leq \alpha^n \frac{|x_1 - x_0|}{1 - \alpha}. \end{aligned} \tag{141}$$

The first inequality holds by repeated application of the triangle inequality. The second equality holds by (139). The fourth equality holds since the sum is geometric, and the final inequality holds since $\alpha \in (0,1)$. From results earlier this quarter, we know[1]

$$\lim_{n\to\infty} \frac{|x_1 - x_0|}{1-\alpha} \cdot \alpha^n = 0. \tag{142}$$

Let $\varepsilon > 0$ be given. Then there is $N \in \mathbb{N}$ such that

$$|x_m - x_n| \leq \frac{|x_1 - x_0|}{1-\alpha}\alpha^n < \varepsilon \quad \forall\, m, n > N. \tag{143}$$

This shows (x_n) is Cauchy, and therefore the sequence converges to some limit $p \in \mathbb{R}$. Moreover,

$$p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(p), \tag{144}$$

where the final equality holds because f is continuous, which follows from the fact f is differentiable. Thus we conclude (x_n) converges to a fixed point p of f, as desired. $\blacksquare$

[1]This follows from the fact the sequence is monotonically decreasing and is bounded below by zero.

Section 5: Integration

Example 42: Without using Corollary 32.10, prove

$$\int_a^b x \ \mathrm{d}x = \frac{b^2 - a^2}{2}. \tag{145}$$

Proof:
Let $P_N := \{a = x_0 < \cdots < x_n = b\}$ be a regular partition of $[a, b]$ so that

$$x_n = \left(\frac{b-a}{N}\right) n + a \quad \text{for } n = 0, \ldots, N. \tag{146}$$

Then observe

$$\begin{aligned} U(f, P_N) &= \sum_{n=1}^{N} x_n (x_n - x_{n-1}) \\ &= \sum_{n=1}^{N} \left(a + \left[\frac{b-a}{N}\right] n\right) \left(\frac{b-a}{N}\right) \\ &= \frac{b-a}{N} \sum_{n=1}^{N} \left(a + \left[\frac{b-a}{N}\right] n\right) \\ &= \frac{b-a}{N} \left(aN + \left[\frac{b-a}{N}\right] \frac{N(N+1)}{2}\right) \\ &= (b-a) \left(a + (b-a)\, \frac{1 + 1/N}{2}\right) \\ &= \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2N}. \end{aligned} \tag{147}$$

Consequently,

$$\lim_{N\to\infty} U(f, P_N) = \lim_{N\to\infty} \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2N} = \frac{b^2 - a^2}{2}. \tag{148}$$

This implies $U(f,P) \leq (b^2-a^2)/2$. In similar fashion, we see

$$\begin{aligned} L(f,P_N) &= \sum_{n=1}^{N} x_{n-1}(x_n - x_{n-1}) \\ &= \sum_{n=1}^{N} \left(x_n - \frac{b-a}{N} \right)(x_n - x_{n-1}) \\ &= \left[\sum_{n=1}^{N} x_n(x_n - x_{n-1}) \right] - \frac{b-a}{N} \sum_{n=1}^{N} (x_n - x_{n-1}) \\ &= U(f,P_N) - \frac{b-a}{N}(b-a) \\ &= \frac{b^2-a^2}{2} - \frac{(b-a)^2}{2N}. \end{aligned} \tag{149}$$

Again taking the limit as $N \longrightarrow \infty$, we deduce $L(f) \geq (b^2-a^2)/2$, which implies

$$\frac{b^2-a^2}{2} \leq L(f) \leq U(f) \leq \frac{b^2-a^2}{2}. \tag{150}$$

This shows $L(f) = U(f) = (b^2-a^2)/2$ and the result follows. ■

Example 43: Suppose $f : [a, b] \to \mathbb{R}$ is nonnegative and $f(x) > 0$ at finitely many points. Prove

$$\int_a^b f \ \mathrm{d}x = 0. \tag{151}$$

Proof:

By a theorem from our text, a bounded function f on $[a, b]$ is integrable if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\mathrm{mesh}(P) < \delta \quad \Longrightarrow \quad U(f, P) - L(f, P) < \varepsilon. \tag{152}$$

We shall use this theorem to prove the result. Let $\varepsilon > 0$ be given. Pick S to be the collection of $x \in [a, b]$ such that $f(x) > 0$, and set L to be the number of elements of S. Then set

$$B := \max_{x \in [a,b]} f(x) = \max_{x \in S} f(x), \tag{153}$$

which is well defined since S is finite. Let $P = \{a = t_1 < \cdots < t_N = b\}$ be any partition of $[a, b]$ with $\mathrm{mesh}(P) < \delta := \varepsilon/BL$. Then there are at most L partitions for which f attains a nonzero value. And, on each of these intervals $0 \leq f < B$, which implies

$$\sum_{k=1}^{N} M(f, [t_{k-1}, t_k]) \leq LB. \tag{154}$$

Consequently,

$$U(f, P) = \sum_{k=1}^{N} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \leq \delta \sum_{k=1}^{L} M(f, [t_{k-1}, t_k]) \leq \delta LB < \left(\frac{\varepsilon}{LB}\right) LB = \varepsilon. \tag{155}$$

This implies $U(f, P) < \varepsilon$. Now note

$$L(f, P) = \sum_{k=1}^{N} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \geq \sum_{k=1}^{N} 0(t_k - t_{k-1}) = 0, \tag{156}$$

and so

$$U(f, P) - L(f, P) < \varepsilon - 0 = \varepsilon. \tag{157}$$

This verifies (152), which implies f is integrable on $[a, b]$. Moreover,

$$0 \leq L(f, P) \leq L(f) \leq U(f) \leq U(f, P) < \varepsilon \quad \Longrightarrow \quad 0 \leq L(f) < \varepsilon. \tag{158}$$

Because $\varepsilon > 0$ was arbitrary, it follows that $L(f) = 0$, which completes the proof. ■

Remark 16: Suppose the function $f : [a, b] \to \mathbb{R}$ is integrable. Then for each $[c, d] \subseteq [a, b]$ we define

$$\int_d^c f := -\int_c^d f \quad \text{and} \quad \int_c^c f := 0. \tag{159}$$

This provides a convenient way to express integrals over the "negative direction" of intervals. ⋄

Example 44: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) := \int_{x-2}^{x+3} f(t)\ \mathrm{d}t. \tag{160}$$

Prove F is differentiable and find F'.

Proof:

Pick any $x^* \in \mathbb{R}$ and let $c \in (x^* - 2, x^* + 3)$. Then Theorem 33.6 asserts

$$F(x^*) = \int_{c}^{x^*+3} f(t)\ \mathrm{d}t + \int_{x^*-2}^{c} f(t)\ \mathrm{d}t = \int_{c}^{x^*+3} f(t)\ \mathrm{d}t - \int_{c}^{x^*-2} f(t)\ \mathrm{d}t, \tag{161}$$

where the second equality follows by definition. Because f is integrable on $[c, x^* + 3]$ and on $[x^* - 2, c]$ and continuous, the second fundamental theorem (Theorem 34.3) asserts F is differentiable and, combined with the fact the derivative of a sum is the sum of the derivatives,

$$F'(x^*) = f(x^* + 3) - f(x^* - 2). \tag{162}$$

Because x^* was chosen arbitrarily, this holds for all of $\mathbb{R}$ and so F is differentiable with derivative F' given by (162). ■

Example 45: Suppose the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable. If the function $G : [0, \infty) \to \mathbb{R}$ is defined by

$$G(x) := \int_{-2x}^{2x} (f(t) + f(-t)) \text{ d}t, \tag{163}$$

then find $G''(x)$ for $x > 0$.

Solution:

In what follows, assume $x > 0$. Then define $q : \mathbb{R} \to \mathbb{R}$ by $q(t) := f(t) + f(-t)$ and note

$$q(-t) = f(-t) + f(-(-t)) = f(-t) + f(t) = q(t), \tag{164}$$

i.e., q is even. Taking $u(t) := -t$, this implies

$$\int_{-2x}^{0} q(t) \text{ d}t = \int_{-2x}^{0} q(-t) \text{ d}t = -\int_{-2x}^{0} -q(-t) \text{ d}t = -\int_{2x}^{0} q(u) \text{ d}u = \int_{0}^{2x} q(u) \text{ d}u. \tag{165}$$

The first equality holds by (164). The second inequality holds by the linearity of the integral (Theorem 33.3). The third equality holds by the change of variables theorem (Theorem 34.4), and the final equality follows from the definition in Remark 16 above. Together Theorem 33.6 and (165) imply

$$G(x) = \int_{-2x}^{0} q(t) \text{ d}t + \int_{0}^{2x} q(t) \text{ d}t = 2\int_{0}^{2x} q(t) \text{ d}t. \tag{166}$$

Letting $h(x) := 2x$ and

$$F(x) := 2\int_{0}^{x} q(t) \text{ d}t, \tag{167}$$

we see $G(x) = F(h(x))$, and so application of the chain rule yields

$$G'(x) = F'(h(x))h'(x) = 2q(h(x))h'(x) = 4q(2x). \tag{168}$$

Because q is differentiable, we may differentiate G once more, which gives

$$G''(x) = 4q'(2x) \cdot 2 = 8q'(2x). \tag{169}$$

$\square$

Example 46: Using the material in §33 and §34 of our text, prove if $f : [a,b] \to \mathbb{R}$ is continuous and

$$0 = \int_a^b f(x)^2 \ \mathrm{d}x, \tag{170}$$

then f is identically 0 in $[a,b]$.

Proof:

First note $f(x)^2 = |f(x)|^2 \geq 0$, and so f^2 is nonnegative. Consequently, Theorem 33.4 implies f^2 is identically 0 in $[a,b]$. But, for each $x \in [a,b]$, $0 = f(x)^2 = |f(x)|^2$ implies $|f(x)| = 0$. Thus we conclude $f = 0$ in $[a,b]$, as desired. ■

Example 47: Suppose $f[a,b] \to \mathbb{R}$ is continuous and

$$0 = \int_a^b f(x)g(x)\ \mathrm{d}x \tag{171}$$

for each continuous function $g : [a,b] \to \mathbb{R}$. Prove f is identically 0 in $[a,b]$.

Proof:
By way of contradiction, suppose f is not identically zero. Then let $g = f$ and observe $fg = f^2$ is nonnegative. Also let $x_0 \in (a,b)$ be such that $f(x_0) \neq 0$, which implies $f^2(x_0) > 0$. Because f^2 is continuous, there is $\delta^* > 0$ such that for $x \in [a,b]$,

$$|x - x_0| < \delta^* \quad \Longrightarrow \quad |f^2(x) - f^2(x_0)| < \frac{|f^2(x_0)|}{2} \quad \Longrightarrow \quad |f^2(x)| \geq \frac{|f^2(x_0)|}{2}, \tag{172}$$

where the final inequality holds by applying the triangle inequality. Now define the quantities $\delta := \min\{\delta^*,\ (b-a)/2\}$, $b^* := \min\{b, x_0 + \delta\}$, and $a^* := \max\{a, x_0 - \delta\}$. Pick any partition P of $[a,b]$ and set $P^* := P \cup \{a^*, b^*\}$. Then

$$L(f^2) \geq L(f^2, P^*) \geq m\left(f^2, [a^*, b^*]\right)(b^* - a^*) \geq m\left(f^2, [a^*, b^*]\right)\delta \geq \frac{|f^2(x_0)|}{2}\delta > 0. \tag{173}$$

The first inequality follows by definition of the lower Darboux integral. The second inequality follows from the fact f^2 is nonnegative. The third inequality holds by choice of δ, and the following inequality follows from (172). From (173), we deduce

$$0 = \int_a^b f(x)^2\ \mathrm{d}x = L(f) > 0, \tag{174}$$

a contradiction. Hence the initial assumption must have been false and the result follows. ■

Example 48: Prove if $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and g is nonnegative, then there is $x^* \in (a, b)$ such that

$$\int_a^b f(x)g(x)\ \mathrm{d}x = f(x^*) \int_a^b g(x)\ \mathrm{d}x. \tag{175}$$

Proof:

If g is identically zero, then the result holds for every $x^* \in (a, b)$. Now suppose g is not identically zero so that Theorem 33.4 implies

$$\int_a^b g(x)\ \mathrm{d}x > 0. \tag{176}$$

Since f is continuous on a closed bounded interval, the extreme value theorem asserts f attains its supremum and infimum on $[a, b]$, denoted M and m, respectively. This implies $m \leq f(x) \leq M$ for each $x \in [a, b]$. Consequently, the linearity of the integral (Theorem 33.3) and Theorem 33.4 imply

$$m \int_a^b g(x)\ \mathrm{d}x = \int_a^b mg(x)\ \mathrm{d}x \leq \int_a^b f(x)g(x)\ \mathrm{d}x \leq \int_a^b Mg(x)\ \mathrm{d}x = M \int_a^b g(x)\ \mathrm{d}x. \tag{177}$$

Because f attains m and M, there are $x_m, x_M \in [a, b]$ such that $f(x_m) = m$ and $f(x_M) = M$. Due to (176), we may divide (177) by the integral of g over $[a, b]$ to discover

$$f(x_m) = m \leq \frac{\int_a^b f(x)g(x)\ \mathrm{d}x}{\int_a^b g(x)\ \mathrm{d}x} \leq M = f(x_M). \tag{178}$$

Since f is continuous, the intermediate value theorem then asserts there is x^* between x_m and x_M and thus in (a, b) such that

$$f(x^*) = \frac{\int_a^b f(x)g(x)\ \mathrm{d}x}{\int_a^b g(x)\ \mathrm{d}x}. \tag{179}$$

Multiplying each side of (179) by the integral of g over $[a, b]$ yields the desired result. ■

Example 49: Evaluate the limit

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right], \tag{180}$$

without using Corollary 32.10. You may use the fact the derivative of $\ln(x)$ is $1/x$.

Proof:

First observe

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}. \tag{181}$$

Next define the function $f(x) := 1/x$ and the partition $P_n = \{1 = t_0 < \cdots < t_n = 2\}$ of the interval $[1, 2]$ such that $t_k = 1 + k/n$ for $k = 0, \ldots, n$. Because f is decreasing, we see

$$L(f,P_n)=\sum_{k=1}^{n} m\left(f,[t_{k-1},t_k]\right)[t_k-t_{k-1}]=\sum_{k=1}^{n} f(t_k)\cdot\frac{1}{n}=\sum_{k=1}^{n}\frac{1}{1+k/n}\cdot\frac{1}{n}=\sum_{k=1}^{n}\frac{1}{k+n}. \tag{182}$$

Since the right hand side of (181) equals the right hand side of (182), it suffices to evaluate the limit of $L(f, P_n)$ as $n \longrightarrow \infty$. An analogous result to (182) holds for $U(f, P_n)$, replacing k with $k - 1$. Thus

$$\begin{aligned} U(f,P_n)-L(f,P_n) &= \sum_{k=1}^{n}\frac{1}{n+(k-1)}-\frac{1}{n+k} \\ &= \sum_{k=1}^{n}\frac{1}{n^2-n+k(k-1)} \\ &\le \sum_{k=1}^{n}\frac{1}{n^2-n} \\ &= \frac{1}{n-1}. \end{aligned} \tag{183}$$

Note the inequality above follows from the fact $k(k - 1) \geq 0$ for each k. Recalling for each partition P_n we have $U(f, P_n) \geq U(f) \geq L(f) \geq L(f, P_n)$, we see

$$0 \le U(f)-L(f,P_n) \le U(f,P_n)-L(f,P_n) \le \frac{1}{n-1}. \tag{184}$$

Letting $n \longrightarrow \infty$, the right hand side of (184) converges to 0. Hence

$$\lim_{n\to\infty} L(f, P_n) = U(f) = \int_1^2 f(x)\ \mathrm{d}x, \tag{185}$$

where the second equality holds since f is integrable due to the fact it is continuous on $[1, 2]$. Then because $f(x)$ is the derivative of $\ln(x)$ the fundamental theorem of calculus asserts

$$\int_1^2 f(x)\ \mathrm{d}x = \int_1^2 \frac{\mathrm{d}x}{x} = \ln(2) - \cancelto{0}{\ln(1)} = \ln(2). \tag{186}$$

Whence we conclude the limit evaluates to $\ln(2)$. ■

REMARK 17: Suppose we repeated the above Example making use of Corollary 32.10. Then we could note

$$\lim_{n\to\infty} \mathrm{mesh}(P_n) = \lim_{n\to\infty} \frac{1}{n} = 0, \tag{187}$$

and so we immediately know the sequence (S_n) of partial sums defined by $S_n = L(f, P_n)$ converges to $L(f)$. Using this fact, we could jump straight from (182) to (185). ◇