1 Fundamental Theorem of Calculus I

As stated in $Calculus(Larson)$:

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{1}$$

2 Proof

Write the difference $F(b) - F(a)$ in expanded form. Let Δ be any partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

Write the pairwise subtraction and addition of like terms.

$$F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \ldots - F(x_1) + F(x_1) - F(x_0)$$

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \tag{2}$$

By the Mean Value Theorem, there exists a number c_i in the ith subinterval.

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

Because $F'(c_i) = F(c_i)$, let $\Delta x_i = x_i - x_{i-1}$ to obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i \tag{3}$$

By continuously applying the Mean Value Theorem, you can find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$ for any partition.

$$F(b) - F(a) = \int_a^b f(x)dx$$