1 Fundamental Theorem of Calculus II

As stated in $Calculus(Larson)$:

If a function f is continuous on the open interval I containing a, then for every x in the interval

$$\frac{d}{dx}\int_a^b [f(t)dt] = f(x) \tag{1}$$

2 Proof

Begin by defining F as

$$F(x) = \int_a^x f(t)dt \tag{2}$$

Then, you can write by the definition of the derivative

$$\begin{aligned}
F'(x) &= lim_{\Delta x \to 0}\frac{F(x+\Delta x) - F(x)}{\Delta x} \\
&= lim_{\Delta x \to 0}\frac{1}{\Delta x}[\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt] \\
&= lim_{\Delta x \to 0}\frac{1}{\Delta x}[\int_a^{x+\Delta x} f(t)dt + \int_x^a f(t)dt] \\
&= lim_{\Delta x \to 0}\frac{1}{\Delta x}[\int_x^{x+\Delta x} f(t)dt]
\end{aligned}$$

From the Mean Value Theorem for Integrals when $\Delta x > 0$, there exists a number c in the interval $[x, x + \Delta x]$ such that the integral in the expression above is equal to $f(c)\Delta x$. Because $x \geq c \geq x + \Delta x$, it follows that $c \to x$ as $\Delta x \to 0$.

$$F'(x) = lim_{\Delta x \to 0}[\frac{1}{\Delta x}f(c)\Delta x] = lim_{\Delta x \to 0}f(c) = f(x)$$

Using the area model for definite integrals, the approximation

$$f(x)\Delta x \approx \int_x^{x+\Delta x} f(t)dt \tag{3}$$

can be viewed as the area of the height of $f(x)$ and the width Δx is approximately equal to the area of the region between $[x, x + \Delta x]$.