

1 Fundamental Theorem of Calculus II

If a function f is continuous on the open interval I containing a , then for every x in the interval

$$\frac{d}{dx} \int_a^b [f(t)dt] = f(x) \quad (1)$$

2 Proof

Begin by defining F as

$$F(x) = \int_a^x f(t)dt \quad (2)$$

Then, you can write by the definition of the derivative

$$\begin{aligned} F'(x) &= \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt \right] \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\int_a^{x+\Delta x} f(t)dt + \int_x^a f(t)dt \right] \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\int_x^{x+\Delta x} f(t)dt \right] \end{aligned}$$

From the Mean Value Theorem for Integrals when $\Delta x > 0$, there exists a number c in the interval $[x, x + \Delta x]$ such that the integral in the expression above is equal to $f(c)\Delta x$. Because $x \leq c \leq x + \Delta x$, it follows that $c \rightarrow x$ as $\Delta x \rightarrow 0$.

$$F'(x) = \lim_{\Delta x \rightarrow 0} \left[\frac{1}{\Delta x} f(c)\Delta x \right] = \lim_{\Delta x \rightarrow 0} f(c) = f(x)$$

Using the area model for definite integrals, the approximation

$$f(x)\Delta x \approx \int_x^{x+\Delta x} f(t)dt \quad (3)$$

can be viewed as the area of the height of $f(x)$ and the width Δx is approximately equal to the area of the region between $[x, x + \Delta x]$.