MATH

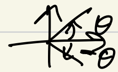
Complex Numbers

$i^2 = -1$

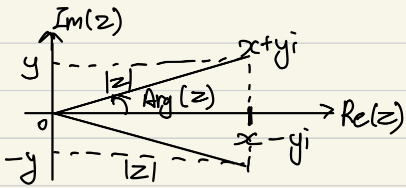
$z = x + yi$ $\qquad$ $\bar{z}/z^* = x - yi$ (conjugate of z)

$r = |z| = \sqrt{x^2+y^2}$

$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \theta$ $\qquad$ $-\pi < \theta \leq \pi$



$\arg(z) = \theta + 2k\pi$ for every integer k



- Polar form of z

$z = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$

- Exponential form of z

$z = re^{i\theta}$

Properties

$z \times z^* = |z|^2$ $\qquad$ $|z_1 \times z_2| = |z_1||z_2|$ $\qquad$ $(z^*)^* = z$

$z + z^* = 2\mathrm{Re}(z)$ $\qquad$ $z - z^* = 2i\,\mathrm{Im}(z)$

$(z+w)^* = z^* + w^*$ $\qquad$ $(zw)^* = z^* w^*$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ $\qquad$ ☆ $\arg(z^n) = n\arg z$

$\arg(z^*) = -\arg(z)$

- $z_1 \times z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$
- $\frac{z_1}{r_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$
- Every odd degree polynomial $p(x)$ with real coefficient has at least one real root.

  ↳ $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for every interger n.

Euler representation DeMoivre's Theorem is

$(e^{i\theta})^n = e^{i(n\theta)}$

Theorem (Distinct nth roots)

Distinct roots of $z = r(\cos\theta + i\sin\theta)$ are

$z_k = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$, $k = 0, 1, 2, 3, \ldots, n-1$

$z_k = \sqrt[n]{r}\left(e^{i\frac{\theta+2k\pi}{n}}\right)$, $k = 0, 1, 2, 3, \ldots, n-1$

E.g. Find all distinct 5th roots of $\sqrt{3}+i$

$\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$

$(2e^{i\frac{\pi}{6}})^{\frac{1}{5}} = 2^{\frac{1}{5}} e^{\frac{\frac{\pi}{6}+2k\pi}{5}i}$, $k = 0, 1, 2, 3 \ldots, n-1$

All distinct 5th of $\sqrt{3}+i$ are $z_0 = 2^{\frac{1}{5}}e^{\frac{\pi}{30}i}$, $z_1 = 2^{\frac{1}{5}}e^{\frac{13\pi}{30}i}$, $\ldots$ $z_4 = 2^{\frac{1}{5}}e^{\frac{49\pi}{30}i} = 2^{\frac{1}{5}}e^{-\frac{11\pi}{30}i}$

Corollary (roots of unity)

The n distinct nth roots of $\cos\theta + i\sin\theta$ are

$w_k = \text{cis}\left(\frac{\theta+2k\pi}{n}\right) = \cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)$, $k = 0, 1, 2 \ldots n-1$

The n distinct nth roots of unity are

$z_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, $k = 0, 1, 2, \ldots n-1$

E.g. Express $\sin 3\theta$ in terms of powers of $\sin\theta$

$\sin 3\theta = \text{Im}(\cos 3\theta + i\sin 3\theta)$ cos sin

$= \text{Im}(\cos\theta + i\sin\theta)^3 = \text{Im}(c + is)^3$

$= \text{Im}(c^3 + 3c^2 is + 3ci^2s^2 + i^3s^3)$

$= \text{Im}(c^3 - 3cs^2 + i(3c^2s - s^3))$

$= 3c^2s - s^3$

$= 3(1-s^2)s - s^3$ (use $c^2 + s^2 = 1$)

$= 3s - 4s^3 = 3\sin\theta - 4\sin^3\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$$\boxed{\begin{aligned} \cos(-\theta) &= \cos\theta \\ \sin(-\theta) &= -\sin\theta \\ \tan(-\theta) &= -\tan\theta \end{aligned}}$$

$- z^k = \cos k\theta + i\sin k\theta$

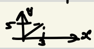
$\frac{1}{z^k} = \cos k\theta - i\sin k\theta$

Vectors

$\overrightarrow{PQ} = \underset{\sim}{a}$ $\quad \overrightarrow{QP} = -\underset{\sim}{a}$ $\quad |\overrightarrow{PQ}| = |\underset{\sim}{a}| \rightarrow$ magnitude of vector $\rightarrow \left|\begin{pmatrix}x\\y\\z\end{pmatrix}\right| = \sqrt{x^2+y^2+z^2}$

Unit vector → vector with magnitude = 1 $\longrightarrow \hat{\overrightarrow{PQ}} = \hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$

$\overrightarrow{OP} = \begin{pmatrix}3\\5\end{pmatrix}$ → from origin



Dot product (Scalar pdt of vectors)

★ $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$ $\qquad \begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} = (1)(2)+(2)(3) = 8$

↳ $\underset{\sim}{a}\cdot\underset{\sim}{a} = |\underset{\sim}{a}|^2$ $\qquad \underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{b}\cdot\underset{\sim}{a}$ (commutative)

↳ $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \rightarrow \underset{\sim}{a}\perp\underset{\sim}{b}$

Cross product

$\underset{\sim}{a}\times\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix}$$

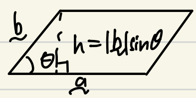
$\underset{\sim}{a}\times\underset{\sim}{b} = -(\underset{\sim}{b}\times\underset{\sim}{a})$ (anti-commutative)

↳ $\underset{\sim}{a}\times\underset{\sim}{b} = 0 \longrightarrow \underset{\sim}{a}=0 / \underset{\sim}{b}=0 / \underset{\sim}{a}\parallel\underset{\sim}{b}$

↳ $\underset{\sim}{a}\times\underset{\sim}{b} = \underset{\sim}{n} \longrightarrow \perp$ to both $\underset{\sim}{a}$ & $\underset{\sim}{b}$

Area of parallelogram → $|\underset{\sim}{a}\times\underset{\sim}{b}|$

Area of triangle → $\frac{1}{2}|\underset{\sim}{a}\times\underset{\sim}{b}|$

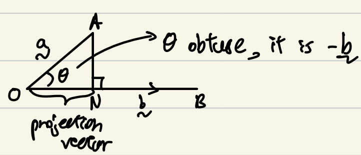


Projection vector & perpendicular distance from point to line

↳ Projection vector of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ $= (\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \overrightarrow{ON}$

↳ Length of projection $= |\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}|$

↳ $AN = |\underset{\sim}{a}|\sin\theta = |\underset{\sim}{a}\times\hat{\underset{\sim}{b}}|$



Scalar Triple product

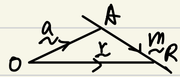
$\underset{\sim}{u}\cdot(\underset{\sim}{v}\times\underset{\sim}{w}) = \underset{\sim}{v}\cdot(\underset{\sim}{w}\times\underset{\sim}{u}) = \underset{\sim}{w}\cdot(\underset{\sim}{u}\times\underset{\sim}{v})$

↳ $\underset{\sim}{u}\cdot(\underset{\sim}{v}\times\underset{\sim}{w}) = 0$ → either one is 0 or 3 vectors are coplanar (lie on same plane)

↘ Non-planar vectors $\underset{\sim}{u}, \underset{\sim}{v}$ & $\underset{\sim}{w}$ → $|\underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w})|$ → volume of parallelepiped (3 dimentional figure formed by 6 parallelograms 

Vector equation of line

$\ell : r(\lambda) = \underset{\sim}{a} + \lambda \underset{\sim}{m}$ → direction of line



Cartesian & Parametric eqn of line

$\underset{\sim}{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 + tm_1 \\ a_2 + tm_2 \\ a_3 + tm_3 \end{pmatrix}$

$\frac{x - a_1}{m_1} = \frac{y - a_2}{m_2} = \frac{z - a_3}{m_3}$ (cartesian eqn)

$x = a_1 + tm_1$, $y = a_2 + tm_2$, $z = a_3 + tm_3$ (parametric eqn)

Angle b/w 2 lines

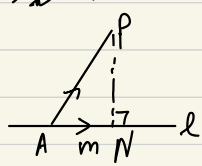
$\cos\theta = \frac{m_1 \cdot m_2}{|\underset{\sim}{m_1}||\underset{\sim}{m_2}|}$

Distance from a point to a line [$\ell : \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{m}, \lambda \in \mathbb{R}$ & $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{p} = \overrightarrow{OP}$]

a) Zero scalar product

$\overrightarrow{PN} = \overrightarrow{ON} - \overrightarrow{OP}$

$\overrightarrow{PN} \cdot \underset{\sim}{m} = 0 \longrightarrow (\underset{\sim}{a} + \lambda \underset{\sim}{m} - \underset{\sim}{p}) \cdot \underset{\sim}{m} = 0$



b) Projection method

$\overrightarrow{AN}$ = projection vector of $\overrightarrow{AP}$ on $\ell / \underset{\sim}{m}$

$\overrightarrow{AN} = (\overrightarrow{AP} \cdot \hat{\underset{\sim}{m}})\, \hat{\underset{\sim}{m}}$

$\overrightarrow{ON} = \underset{\sim}{a} + \overrightarrow{AN}$

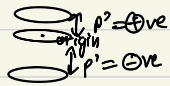
Vector equation of plane

$\Pi : \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$ → direction vector (cannot be // to each other)

Scalar product of plane (vector eqn)

point on plane ← $\underset{\sim}{r} \cdot \underset{\sim}{n} = p$ → normal vector to plane (cross pdt of 2 directional vectors on plane)

↳ $\underset{\sim}{r} \cdot \hat{\underset{\sim}{n}} = p'$ → ⊥ distance of plane from origin

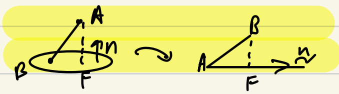


Cartesian equation of plane (scalar eqn)

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = p \rightarrow n_1 x + n_2 y + n_3 z = p$

Distance from any point to plane

Length of projection $\rightarrow |\vec{AF}| = |\vec{AB} \cdot \hat{\underset{\sim}{n}}|$



Angle b/w a line & plane

$\cos \phi = \left| \frac{\underset{\sim}{m} \cdot \underset{\sim}{n}}{|\underset{\sim}{m}||\underset{\sim}{n}|} \right| = \sin \theta \rightarrow \theta = 90° - \phi$

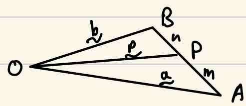
Angle b/w 2 planes

$\cos \theta = \left| \frac{\underset{\sim}{n_1} \cdot \underset{\sim}{n_2}}{|\underset{\sim}{n_1}||\underset{\sim}{n_2}|} \right|$

Ratio Theorem

↳ If P divides the line segment AB in the ratio m:n

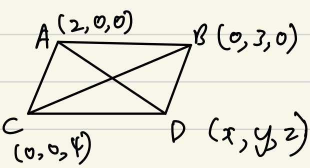
$\underset{\sim}{p} = \frac{1}{m+n}(n\underset{\sim}{a} + m\underset{\sim}{b})$



Parallelogram

Four points with adjacent sides AB & AC & vertices A(2,0,0), B(0,3,0) & C(0,0,4)

Find coordinate of D.



Let D be (x,y,z),

$\left(\frac{x+2}{2}, \frac{y}{2}, \frac{z}{2}\right) = \left(0, \frac{3}{2}, 2\right)$

$D = (x,y,z) = (-2,3,4)$

Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \rightarrow \underset{\text{row column}}{3 \times 4} \text{ matrix } A$$

2nd column

2nd row

- Capital letters (A, B, C) to denote matrices & lowercase letters a, b, c to denote numerical quantities

Types of Matrics

1. zero matrics → m×n matrix with zeros, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2. Square matrice → m×n matrix, m = n, $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$
3. Identity matrices → n×n square matrix, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
4. Diagonal matrices → n×n square matrix, all off-diagonal entries are 0, $A = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & g \end{pmatrix}$
5. Upper triangle matrices → n×n square matrix, all entries below main diagonal are 0, $A = \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & g \end{pmatrix}$
6. Lower triangle matrices → entries below main diagonal are 0, $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$

* Matrices of different sizes cannot be added / subtracted

- $(\alpha A)_{ij} = \alpha (A_{ij})$, α is a scalar
- Matrix Multiplication
  - will happen as long as no. of columns of A = no. of rows of B

$$A = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

$$A \times B = \begin{pmatrix} a\times 0 + d\times 1 & a\times 0 + d\times 2 \\ b\times 0 + e\times 1 & b\times 0 + e\times 2 \\ c\times 0 + f\times 1 & c\times 0 + f\times 2 \end{pmatrix} = \begin{pmatrix} d & 2d \\ e & 2e \\ f & 2f \end{pmatrix}$$

  - not commutative [AB ≠ BA]
  - If AB = 0, doesn't mean A=0 / B=0

Transpose

↳ For an $m \times n$ matrix A, the matrix A^T obtained by interchanging the rows & column of A → transpose of A

$$(A^T)_{ij} = A_{ji} \quad \rightarrow \quad A = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix}, \; A^T = \begin{pmatrix} 4 & 1 \\ 3 & 3 \\ 2 & 1 \end{pmatrix}$$

↳ $(A^T)^T = A \qquad (A \pm B)^T = A^T \pm B^T \qquad (\alpha A)^T = \alpha A^T$, where α is a scalar

↳ $(AB)^T = B^T A^T$

Invertible Matrices

↳ Let A be a $n \times n$ square matrix & if there is another square matrix B such that $AB = I_n$ & $BA = I_n$ → A is said to be invertible (non-singular), B is inverse of A. If no such B is found, A is not invertible (singular)

① Square matrix with row/column of zero → singular $\begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{pmatrix}$ ✗

② Square matrix with row/column which is a multiple of another row/column → singular

Identical rows → singular $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 7 & 9 \end{pmatrix}$ ✗ OR $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 5 \\ 1 & 2 & 6 \end{pmatrix}$ ✗

③ Square matrix in which row/column is a linear combination of other rows → singular

$$G = \begin{pmatrix} a & b & c \\ d & e & f \\ 2a-5d & 2b-5e & 2c-5f \end{pmatrix} \rightarrow R_3 = 2R_1 + (-5)R_2 \quad ✗$$

Inverse matrix

↳ Invertible matrix cannot have more than one inverses. Inverse is unique.

↳ Inverse of A is A^{-1}. $AA^{-1} = I$ & $A^{-1}A = I$. $(A^{-1})^{-1} = A$

① The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if & only if $ad - bc \neq 0$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

② Inverse for diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \longrightarrow D^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{pmatrix}$$

↳ $(A^n)^{-1} = (A^{-1})^n$, $A^r A^s = A^{r+s}$, $(A^r)^s = A^{rs}$

Cofactor of a Matrix (compute det(A) for $n \times n$, where $n \geq 3$)

↳ The (i, j)th cofactor of A, denoted by C_{ij} → product of number $(-1)^{i+j}$ & determinant M_{ij} of submatrix that remains after the ith-row & jth-column deleted from A

$$C_{ij} = (-1)^{i+j} M_{ij}$$ → (i, j)th minor of A

E.g. $A = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

$$C_{11} = (-1)^{1+1} \begin{pmatrix} \cancel{1} & \cancel{5} & \cancel{0} \\ \cancel{-3} & 2 & 1 \\ \cancel{1} & 2 & 1 \end{pmatrix} = (-1)^2 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}$$ → use $(ad-bc)$

↳ cancel row 1, column 1

$$= 0$$

Checkboard matrix $(-1)^{i+j}$

↳ 4 by 4 matrix → $\begin{bmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{bmatrix}$

Determinant

a) 2×2 matrices

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ → $\det(A) = ad - bc$

b) $n \times n$ matrixes, $n > 2$

↳ The determinant of an $n \times n$ matrix A can be found by summing the products of terms in the first row with the corresponding cofactors:

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$ → don't have to be just first row/column

$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$

c) 3×3 matrixes

copy first 2 rows

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Ans: $(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}+a_{21}+a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$

Useful things (Determinants)

a) Square matrix with a row of zeros → $\det\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = 0$

b) Two row / column one is multiple of other $\rightarrow \det\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 5 & 7 \end{pmatrix} = 0$

c) Triangular matrix $\rightarrow \det\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} = acf$

Determinant of Product

↳ For two $n \times n$ matrixes A & B, $\det(AB) = \det(A)\det(B)$

Determinant & Invertibility

↳ A is $n \times n$ square matrix $\rightarrow A$ is invertible if & only if $\det(A) \neq 0$

$\det(A^{-1}) = \frac{1}{\det(A)}$ [$\det(A) = 0 \rightarrow$ matrix is singular]

Cramer's Rule

If $Ax = b$ is a system of n linear equation in n unknowns, $\det(A) \neq 0$

$x_j = \frac{\det(A_j)}{\det(A)}$, $j = 1, 2, \ldots, n \rightarrow A_j$ is from matrix

$$\boxed{\begin{aligned} 2u - v + w &= 3 \\ u + v - 3w &= 5 \\ 5u - 4v + 9w &= 4 \end{aligned}} \Rightarrow \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -3 \\ 5 & -4 & 9 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$$

Linear ($Ax = b$) $\longrightarrow$ Matrix

E.g. Solve linear system.

$$\left.\begin{aligned} 7x_1 - 2x_2 &= 3 \\ 3x_1 + x_2 &= 5 \end{aligned}\right] \rightarrow \begin{pmatrix} 7 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$\det(A) = \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 7 - (-6) = 13 \neq 0 \rightarrow$ cramer's rule applies

$A_1 = \begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix} \rightarrow \det(A_1) = 13$, $x_1 = \frac{\det(A_1)}{\det(A)} = \frac{13}{13} = 1$

$A_2 = \begin{pmatrix} 7 & 3 \\ 3 & 5 \end{pmatrix} \rightarrow \det(A_2) = 26$, $x_2 = \frac{\det(A_2)}{\det(A)} = \frac{26}{13} = 2$

Limits & Continuity

↳ $\lim_{x \to a} f(x) = L \longrightarrow f(x)$ approaches the limit L as x tends to a

a) When $\lim_{x \to a} f(x)$ exist, real no. $L \to$ limit L is unique

b) No finite real no. $L \to \lim_{x \to a} f(x)$ does not exist

One sided limit

Left-hand limit: $\lim_{x \to 2^-} f(x) = -1 \longrightarrow x \to 2$ from the left

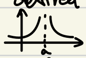
Right-hand limit: $\lim_{x \to 2^+} f(x) = 1 \longrightarrow x \to 2$ from the right

$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \to \lim_{x \to 2} f(x)$ does not exist

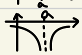
Infinite Limit (vertical asymptote)

Let f be a function defined on both sides of a, except a itself,

$\lim_{x \to a} f(x) = \infty$



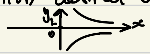
$\lim_{x \to a} f(x) = -\infty$



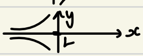
Limits at Infinity (Horizontal asymptote)

Let $f(x)$ be a function defined on some interval (a, ∞),

$\lim_{x \to \infty} f(x) = L$



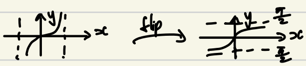
$\lim_{x \to -\infty} f(x) = L$



Impt: a) $\lim_{x \to \infty} \sin x$, $\lim_{x \to \infty} \cos x$, $\lim_{x \to \infty} \tan x$ does not exist

b) $\lim_{x \to \infty} e^x$, $\lim_{x \to \infty} \ln x = \infty$

c) $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$ $\qquad \lim_{x \to -\infty} \tan^{-1} x = \frac{-\pi}{2}$



Limit Laws

a) $\lim_{x \to a} C = C \to$ limit of constant is constant

b) $\lim_{x \to a} C f(x) = C \lim_{x \to a} f(x)$

c) $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

d) $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

e) $\lim_{x\to a} (f(x))^n = (\lim_{x\to a} f(x))^n$, n is +ve integer

f) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

Limit of a Polynomial

For a polynomial, can simply substitute the value of a into the polynomial p(x) to obtain the limit of p(x) at a,

$\lim_{x\to 1} (2x^5 - \pi x^3 - \sqrt{5}) = 2(1)^5 - \pi(1)^3 - \sqrt{5} = 2 - \pi - \sqrt{5}$

But may fail when $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9} \to \lim_{x\to 3} x^2-9=0$

Continuity

To check if function f is continuous at a point $x=c$/continuous from right/left

1. f(c) is defined → sub x as c & see if there is a no. [$f(1) = \frac{1-x^2}{1-x}$ → undefined]
2. $\lim_{x\to c} f(x)$ exist
3. f(c) & $\lim_{x\to c} f(x)$ are equal

If a curve is continuous, can substitute value of c directly into $f(x) \to \lim_{x\to c} f(x)$, continuity at $x=c$

as long as denominator ≠ 0

→ Polynomials, rational functions, $\sqrt[n]{x}$ (n-root functions)

→ Trigonometric functions: $\sin x$, $\cos x$, $\tan x$ (except asymptotes), e^x, $\ln x$ → continuous at every point

→ Hyperbolic functions: $\lim_{x\to c} \sinh x = \sinh c$, $\lim_{x\to c} \cosh x = \cosh c$, $\lim_{x\to c} \tanh x = \tanh c$

→ Inverse trigonometric function ($\cos^{-1} x$) & inverse hyperbolic functions are continuous at c ($\tanh^{-1} x$)

Properties (when f & g continuous at $x=c$): $f \pm g$, $f \cdot g$, $\frac{f}{g}$

Limit law for composite functions

Suppose $\lim_{x\to c} g(x) = b$ & f is a continuous function

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(b)$$

E.g. $\lim_{x\to c} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x\to c} g(x)}$

Eliminating Zero Denominators (Rationalisation)

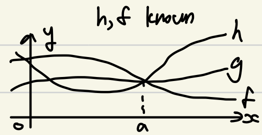
When denominator is 0 after subbing in value, need to try to factorise it by long division/smt

$$\lim_{x\to -3} \frac{x^3+4x^2+4x+3}{-x^3-2x^2+5x+6} = \lim_{x\to -3} \frac{\cancel{(x+3)}(x^2+x+1)}{\cancel{(x+3)}(-x^2+x+2)}$$
$$= \frac{9-3+1}{-9-3+2} = -\frac{7}{10}$$

$a^2-b^2=(a-b)(a+b)$

$$\frac{\sqrt{2-x}-1}{x-1}\left(\frac{\sqrt{2-x}+1}{\sqrt{2-x}+1}\right) = \frac{-(x-1)}{(x-1)(\sqrt{2-x}+1)}$$

Squeeze Theorem



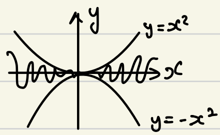
Suppose f, g & h are defined on an open interval I containing a.

If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, $f(x) \le g(x) \le h(x)$ on $I \to \lim_{x\to a} g(x) = L$ (squeeze theorem)

E.g. Evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right)$.

Construct 2 functions f & h with $g(x) = x^2\sin\left(\frac{1}{x^2}\right)$, $x \neq 0$

$$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$$



Multiply x^2 throughout

$$\underbrace{-x^2}_{f(x)} \le \underbrace{x^2\sin\left(\frac{1}{x^2}\right)}_{g(x)} \le \underbrace{x^2}_{h(x)}$$

$\lim_{x\to 0} -x^2 = 0$ & $\lim_{x\to 0} x^2 = 0$

By Squeeze Theorem, $\lim_{x\to 0} x^2\sin\left(\frac{1}{x^2}\right) = 0$

Limit Laws for Infinite Limits

$\infty + \infty = \infty$

$\infty \times \infty = \infty$

$\left.\begin{array}{l}\infty \times c = \infty \\ \infty \times -c = -\infty\end{array}\right] \to c$ is constant

$0 \times \infty =$ undefined

$\infty - \infty =$ undefined

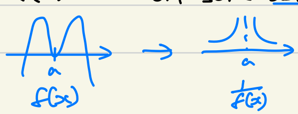
$\infty \div \infty =$ undefined

– Sometimes it helps to plot a graph

set of no. δ units away, omitting a

$\lim_{x\to a} f(x) = 0$. If $f(x) > 0$ on some deleted neighbourhood of a,

$\lim_{x\to a} \frac{1}{f(x)} = \infty$



a) $\lim_{x\to\infty} \sinh x = +\infty$ & $\lim_{x\to-\infty} \sinh x = -\infty$

b) $\lim_{x\to\infty} \cosh x = +\infty$ & $\lim_{x\to-\infty} \cosh x = +\infty$

c) $\lim_{x\to\infty} \tanh x = +1$ & $\lim_{x\to-\infty} \tanh x = -1$

Useful Limits

1. If n is a positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \& \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

2. If m & n are positive integers, then provided n is an odd integer

$$\lim_{x\to\infty} \frac{1}{x^{\frac{m}{n}}} = 0 \quad \& \quad \lim_{x\to-\infty} \frac{1}{x^{\frac{m}{n}}} = 0$$

E.g. Evaluating Limits at Infinity for Rational Functions

$$\lim_{x\to\infty} \frac{x+4}{x^2-6x+5} = \lim_{x\to\infty} \frac{x+4}{x^2-6x+5}\left(\frac{1/x^2}{1/x^2}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{4}{x^2}}{1-\frac{6}{x}+\frac{5}{x^2}}$$

$$= \frac{0+0}{1-0+0} = 0$$

$$\frac{\sqrt{2x^2+1}}{3x-5}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)$$

for $x>0$ $= \dfrac{\sqrt{2x^2+1}}{3x-5}\left(\dfrac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}}\right)$

for $x<0$ $= \dfrac{\sqrt{2x^2+1}}{3x-5}\left(\dfrac{-\frac{1}{\sqrt{x^2}}}{\frac{1}{x}}\right)$

Composite of Continuous Function

↳ $g \circ f$ only defined when $R_f \subseteq D_g$

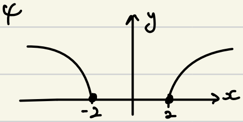
↳ f continuous on its domain D_f & g is continuous on range R_f of f. $g \circ f$ continuous on D_f.

E.g. Let $f(x) = \sqrt{x-4}$ on $[4, \infty)$ & $g(x) = x^2$. Find the composite function $f \circ g$ & its domain where $f \circ g$ is continuous on.

$f \circ g = f(x^2) = \sqrt{x^2-4}$, which provided $x^2 \geq 4$

$x \leq -2$ or $x \geq 2$

$D_{f\circ g} = (-\infty, -2] \cup \underbrace{[2, \infty)}_{\text{including } 2}.$



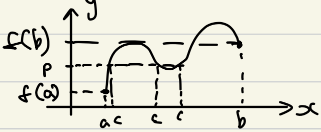
Inverse of Continuous Functions

↳ Horizontal line test → horizontal line intersects the graph $y=f(x)$ at most once → one-to-one function

↳ When f is one-to-one function → inverse f^{-1} is continuous

Intermediate Value Theorem

Suppose that f is continuous on the closed & bounded interval $[a, b]$ where $f(a) \neq f(b)$. Let p be any real number between $f(a)$ & $f(b)$. Then there exist a number c in the open interval (a, b) such that $f(c) = p$



1. Application: Find Roots

For polynomial $f(x) = x^3 - 5x^2 + 3$, do you think there is a root of f in the interval $(0, 1)$?

f is continuous $[0, 1]$

$\underbrace{f(0)}_{f(a)} = 3$, $\underbrace{f(1)}_{f(b)} = 1 - 5 + 3 = -1$

f cont on $[a, b]$
$f(a) \neq f(b)$
p is in b/w $f(a)$ & $f(b)$] → premise

∴ By IVT, $c \in (0, 1)$ such that $f(c) = p = 0$ → conclusion

2. Application: Intersection of Curves

Use IVT to explain why two curves $y = \cos x$ & $y = x^2$ intersect at some point with x-coordinate c where $c \in (0, \frac{\pi}{2})$

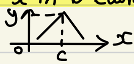
→ Curve $y = \cos x$ and $y = x^2$ intersect at some point with x-coordinate c means that $\cos c = c^2$. Shifting all expression to one side of the equation, the above equation is equivalent to $\cos c - c^2 = 0$. To find an appropriate function f, $f(c) = \cos c - c^2$ & 0 → intermediate values.

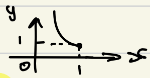
$$f(x) = \cos x - x^2, \text{ where } x \in [0, \tfrac{\pi}{2}]$$

f is continuous on $[0, \frac{\pi}{2}]$, values $f(0) = 1 > 0$, $f(\frac{\pi}{2}) = -(\frac{\pi}{2})^2 < 0$

∴ By IVT, there is a real number $c \in (0, \frac{\pi}{2})$ at which $f(c) = 0$, curves intersect at $x = c$

Extreme Value Theorem (Global Maximum/Minimum)

- Global maximum at c if $f(c) \geq f(x)$ for all x in D (domain of f). The number $f(c)$ is called the maximum value of f on D.  [extreme values of f]
- Global minimum 
- Global maximum & minimum guaranteed when both continuity & closed & bounded condition satisfied

E.g. $f(x) = \frac{1}{x}$, $x \in (0, 1]$ 

- No global max but global min is $1 = f(1)$

E.g. Determine the range of function f defined by $f(x) = 3x^2$, $x \in [-2, 1]$

The function $f(x) = 3x^2$ is continuous on $[-2, 1]$.

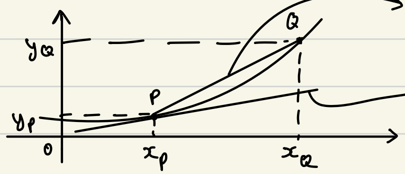
For $-2 \leq x \leq 1$, $f(x) = 3x^2 \leq 3(-2)^2 = 12 = f(-2)$

$$f(x) = 3x^2 \geq 3(0)^2 = f(0)$$

$$f(0) \leq f(x) \leq f(-2)$$

$\therefore$ Global maximum of f on $[-2, 1]$ is $f(-2) = 12$ & global minimum is $f(0) = 0$. Range of f is $[0, 12]$

Differentiation



chord/segment (straight line passing any two points of curve)

tangent line (straight line touches curve once)

- As point Q approaches P, line through PQ approaches tangent

$$\text{Slope of tangent at } P = \lim_{x \to x_P} \frac{f(x) - f(x_P)}{x - x_P}, \text{ if lim exist}$$

- Derivative of function f at a number c, [differentiable] Definition of derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}, \text{ if lim exist}$$

↳ $\Delta x = x - c$ for change in x & $\Delta x \to 0$ whenever $x \to c$,

$$f'(c) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

↳ use $h = x - c$ instead of Δx to denote the change

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$ To get $f'(x)$

E.g. a) Is $f(x) = x^2$ differentiable at $x = 1$?

b) Find $f'(x)$ & the domain of f'.

a) Check whether $\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$ exist.

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - (1)^2}{x - 1}$$
$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$
$$= \lim_{x \to 1} (x+1) = 2$$

$f(x) = |x|$ (differentiable at 0?)

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

have to consider $x \to 0^+$ & $x \to 0^-$

Since limit exist, f is differentiable at $x = 1$ & $f'(1) = 2$

b) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h \to 0} (2x + h) = 2x$$

$\therefore f'(x) = 2x$ for $x \in \mathbb{R}$

- $f(x)=x^n$, $f'(x)=nx^{n-1}$
- $(\alpha f)'(c)=\alpha \cdot f'(c)$

Differentiability & Continuity

necessary condition for f to be differentiable

↳ If a function f is differentiable at $x=c$, then f is continuous at $x=c$, but not vice versa. (useful on piecewise function f)

↳ If f not continuous at $x=c$, then f not differentiable at $x=c$

E.g. Let $f(x)=\begin{cases} x^3+2 & \text{if } x>1 \\ 3x & \text{if } x\le 1 \end{cases}$, find $f'(x)$.

$f(1)=3$, x^3+2 is differentiable

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 3$

$\lim_{x\to 1} f(x) = 3 = f(1)$

For $x>1$, $f'(x)=3x^2$ | $x<1$, $f'(x)=3$

Since f cont. at $x=1$, $\lim_{x\to 1^+} f'(x) = 3 = \lim_{x\to 1^-} f'(x) \to f$ is differentiable at $x=1$ & $f'(1)=3$

$$f'(x)=\begin{cases} 3x^2 & \text{if } x>1 \\ 3 & \text{if } x\le 1 \end{cases}$$

① f is diff on both pieces

② f is cont at c

③ $\lim_{x\to c^-} f'(x) = \lim_{x\to c^+} f'(x) = L$

$\therefore f$ is diff at c & $f'(c)=L$

- sum & difference rule: $(f\pm g)'(c)=f'(c)\pm g'(c)$
- product rule: $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$
- quotient rule: $\left(\frac{f}{g}\right)'(c)=\frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}$
- reciprocal rule: $\left(\frac{1}{g}\right)'(c)=\frac{-g'(c)}{(g(c))^2}$

Chain Rule

↳ Let f & g be functions such that $g(x)$ is differentiable at $x=c$ & $f(u)$ differentiable at $u=g(c)$. $f\circ g$ is differentiable at $x=c$

$$(f\circ g)'(c)=f'(g(c))g'(c)$$

$$\frac{df}{dx}=\frac{df}{du}\times\frac{du}{dx}$$ (Leibniz notation)

Trigonometric Functions Derivation

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\sec x) = \sec x \tan x$
5. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
6. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Exponential & Logarithmic Function Derivation

1. $\frac{d}{dx}(e^x) = e^x$
2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}(a^x) = a^x \ln a$, $a > 0$ & $a \neq 1$

$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, $a > 0$ & $a \neq 1$

$\log_a x = \frac{\ln x}{\ln a}$

Parametric Differentiation

$y = u(t)$ $x = v(t)$ (Parametric graph)

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Implicit Differentiation

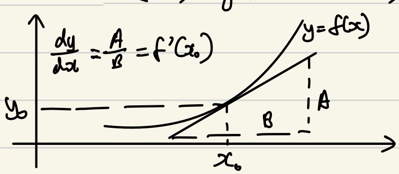
↳ function differentiate w.r.t x:

$y \longrightarrow \frac{dy}{dx} / y'$ $y^2 \longrightarrow 2y\left(\frac{dy}{dx}\right)$

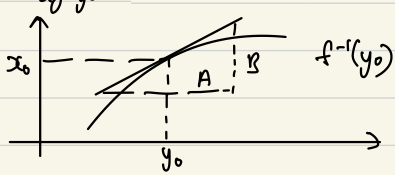
Derivative of Inverse Function

↳ f is increasing & continuous on interval (a,b) & $f'(x_0) > 0$ for some $x_0 \in (a,b)$, then f^{-1} is differentiable at the point $y_0 = f(x_0)$ &

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \rightarrow \left.\frac{dy}{dx}\right|_{x_0} = \frac{1}{\left.\frac{dx}{dy}\right|_{y_0}}$$



$\longrightarrow$



$f^{-1}(y_0) = \frac{B}{A}$

☆ E.g. let $f(x) = \cos x$, where $x \in (0, \pi]$. Find $(f^{-1})'(0)$

$$(f^{-1})'(0) = \frac{1}{-\sin(\cos^{-1}(0))} = \frac{1}{-\sin(\frac{\pi}{2})} = -1$$

Inverse Trigonometric Function Derivative

1. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, $-1<x<1$
2. $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$, $-1<x<1$
3. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$, $x\in\mathbb{R}$
4. $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$, $x\in\mathbb{R}$
5. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$, $x<-1$ or $x>1$
6. $\frac{d}{dx}(\mathrm{cosec}^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$, $x<-1$ or $x>1$

Rate of change

Average rate: $\frac{\Delta x}{\Delta t}$

Instantaneous rate: $\left.\frac{dx}{dt}\right|_{x=5}$

Linearisation

↳ To approximate $f(x)$ near $x=a$ by a linear function $L(x)$ through $x=a$

$$L(x) = \underbrace{f'(a)}_{\text{gradient}}(x-a) + \underbrace{f(a)}_{\text{y-intercept}}$$

E.g. Use the linearisation of $f(x)=\sqrt{x}$ at $a=4$ to approximate the value of $\sqrt{4.001}$

let $f(x)=\sqrt{x}$, $f'(x)=\frac{1}{2\sqrt{x}}$

Linearisation of f at $a=4$ is:

$$L(x) = f(4) + f'(4)(x-4) = \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4)$$
$$= 2+\frac{1}{4}(x-4)$$

Thus $\sqrt{4.001} = f(4.001) \approx L(4.001) = 2+\frac{1}{4}(4.001-4)$
$$= 2.00025$$

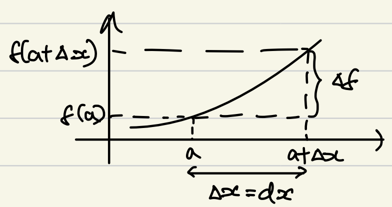
Estimation of Change using Differentials

↳ Consider a differentiable function f, suppose the value of x changes from $x=a$ to $x=a+dx$ $(\Delta x = dx)$

$\Delta f = f(a+\Delta x) - f(a)$

$\frac{\Delta f}{\Delta x} \approx f'(a)$

$\therefore \Delta f \approx f'(a)\Delta x$



E.g. The radius of a circular disk is given as 24cm with a maximum error in measurement of 0.2cm.

a) Use the differentials to estimate the maximum error in the calculated area of the disk.

b) What is the relative error? What is the percentage error?

Let the radius be r cm. Then the area of the circular disk is $A(r) = \pi r^2$

$A'(r) = 2\pi r$, differential of A at $r = r_0 \rightarrow dA = A'(r_0)\,dr$

a) To use differentials to esmitate the maximum error in the calculated area of disk, note that the differential of $A(r)$ at $r_0 = 24$ is given by $A'(24)dr$, with $dr = 0.2$

$\therefore$ Maximum error (ΔA) estimated is

$$dA = A'(24)\,dr = 2\pi(24)(0.2) = 9.6\pi$$

b) The relative error is estimated to be

$$\frac{dA}{A(24)} = \frac{2\pi(24)(0.2)}{\pi(24)^2} = \frac{1}{60} \approx 0.01667$$

The estimated percentage error is 1.667 %

Newton's method

↳ Let x_0 be an approximate value of root, x_1 be x-intercept of tangent to the curve of f at x_0

$$y = f'(x_0)(x - x_0) + f(x_0)$$

When $y = 0, x = x_1$, $0 = f'(x_0)(x_1 - x_0) + f(x_0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \longrightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

E.g. Use Newton's method to solve $x^3 - x + 1 = 0$

Let $f(x) = x^3 - x + 1$, which is continuous & differentiable with $f'(x) = 3x^2 - 1$

Note that $f(-1) = 1 > 0$ & $f(-2) = -5 < 0$

By intermediate value theorem, $f(c) = 0$ for some $c \in (-2, -1)$.

We may choose either $x_0 = -1$ or $x_0 = -2$

Set $x_0 = -1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n + 1}{3x_n^2 - 1}$$

$$x_1 = x_0 - \frac{x_0^3 - x_0 + 1}{3x_0^2 - 1} = -1 - \frac{(-1)^3 - (-1) + 1}{3(-1)^2 - 1}$$

$$= -1 - \tfrac{1}{2} = -1.5$$

$$x_2 = -1.5 - \frac{(-1.5)^3 - (-1.5) + 1}{3(-1.5)^2 + 1} = -1.34883 \quad \ldots$$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1	1	2	-1.5
1	-1.5	-0.87	5.75	-1.34783
2	-1.34783	-0.10068	4.449905	-1.32520
3	-1.32520	-0.002058	4.264635	-1.324718
4	-1.324718	-9.2×10^{-7}	4.264633	-1.324718

stabilise at 5 sf

$\therefore$ Approximated root to 5 dp for $x^3 - x + 1 = 0$ is

$$x_4 = x_5 = -1.324718$$

Approximating reciprocals using Newton's method

↳ Reciprocal → e.g. $\frac{2}{5} \leftrightarrow \frac{5}{2}$ / $6 \leftrightarrow \frac{1}{6}$

E.g. Use Newton's method to find an approximate value the reciprocal of α where $\alpha \neq 0$. Given $\alpha \neq 0$, estimate x where $x = \frac{1}{\alpha}$

$$x = \frac{1}{\alpha} \rightarrow \frac{1}{x} = \alpha \rightarrow \frac{1}{x} - \alpha = 0$$

$f(x) = \frac{1}{x} - \alpha$ → Root of $f(x) = 0$ is $\frac{1}{\alpha}$ $\qquad f'(x) = -1(x^{-2})$

By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\frac{1}{x_n} - \alpha}{-\frac{1}{x_n^2}}$$

$$= x_n + x_n^2\left(\frac{1}{x_n} - \alpha\right)$$

$$= x_n + x_n(1 - \alpha x_n)$$

$$= x_n(2 - \alpha x_n)$$

Approximating square roots using Newton's method

E.g. Use Newton's method to find an iteration to approximate the positive square root of α, where $\alpha > 0$ Note $x = \sqrt{\alpha} \rightarrow x^2 = \alpha \Rightarrow x^2 - \alpha = 0$

$f(x) = x^2 - \alpha$, $f'(x) = 2x$

By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^2 - \alpha}{2x_n}$$
$$= \frac{x_n}{2} + \frac{\alpha}{2x_n}$$
$$= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

Approximate $\sqrt{3}$, $\alpha = 3$, $x_0 = 2$ → $\sqrt{3} \approx 2$

$x_1 = \frac{1}{2}\left(2 + \frac{3}{2}\right) = \frac{7}{4} = 1.75$

$x_2 = \frac{1}{2}\left(\frac{7}{4} + \frac{3}{7/4}\right) = \frac{97}{56} \approx 1.7321429$

Critical points

1. A point c where $f'(c) = 0$ is called a stationary point.
2. A point where $f'(c)$ fails to exist is called a singular point

Closed interval method

↳ 3 step procedure can be used to find global maximum/absolute minimum of a continuous function f on a closed & bounded interval $[a, b]$

1. Determine all critical points of f in (a,b) & find corresponding f-values
2. Compute $f(a)$ & $f(b)$
3. The largest value of f from step ① & ② is global maximum of f on $[a,b]$

E.g. Find the global maximum & global minimum of $f(t) = \sqrt[3]{t}\,(8-t)$ on $[-1, 8]$.

Note that f is continuous on $[-1, 8]$ → product of continuous functions $\sqrt[3]{t}$ & $8-t$.

By the Extreme Value Theorem, f has a global maximum & a global minimum on $[-1, 8]$

To find critical points of f, we have to find c at which $f'(c)$ does not exist/$f'(c)=0$

For $-1 < t < 0$ or $0 < t < 8$

$$f'(t) = \frac{1}{3}t^{-\frac{2}{3}}(8-t) - \sqrt[3]{t} = \frac{8-4t}{3(\sqrt[3]{t})^2}$$

step 1 → [Singular pt: $t=0$ is a singular point of f, since f is not differentiable at $t=0$.
Stationary pt: $f'(t)=0 \rightarrow 8-4t=0 \rightarrow t=2$

step 2 → [End points: $t=-1$, $t=8$

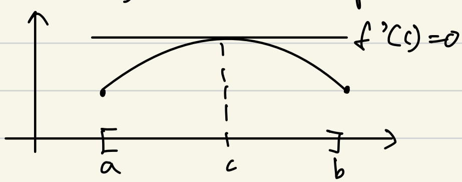
step 3 → [Comparing values of f:

$$f(2)=\sqrt[3]{2}\,(6),\quad f(0)=0,\quad f(-1)=-9,\quad f(8)=0$$

$\therefore$ Global maximum of f on $[-1,8]$ is $f(2)=\sqrt[3]{2}\,(6)$. Global minimum of f on $[-1,8]$ is $f(-1)=-9$

Rolle's Theorem

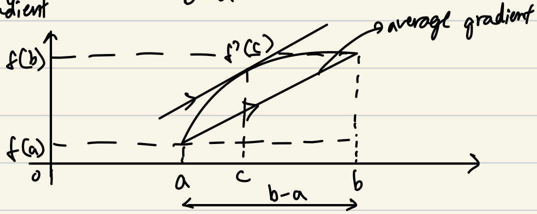
↳ Let f be continuous on the closed interval $[a,b]$ & differentiable on the open interval (a,b). If $f(a)=f(b)$, then there is a point c in (a,b) such that $f'(c)=0$



Mean Value Theorem (MVT)

↳ Let f be continuous on the closed interval $[a,b]$ & differentiable on the open interval (a,b). Then there is (at least one point) c in (a,b) such that

$$\underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{average gradient}} = \underbrace{f'(c)}_{\text{instantaneous gradient}}$$



E.g. (MVT in approximation). Use mean value theorem to estimate $\sqrt[3]{65}$. Note that $64 < 65 < 125$, where $\sqrt[3]{64} = 4$ & $\sqrt[3]{125} = 5$ → suggest we consider $f(x) = \sqrt[3]{x}$, where $x \in [64, 65]$.

Function $f(x) = \sqrt[3]{x}$ continuous on $[64, 65]$, differentiable on $(64, 65)$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}, \quad x \in (64, 65)$$

By MVT, there is $x_0 \in (64, 65)$ such that

$$\frac{f(65) - f(64)}{65 - 64} = f'(x_0)$$

$$\sqrt[3]{65} - 4 = \frac{1}{3} x_0^{-\frac{2}{3}} = \frac{1}{\frac{1}{3} x_0^{-\frac{2}{3}}}$$

Since $64 < x_0 < 65$, $3(64^{\frac{2}{3}}) < 3x_0^{\frac{2}{3}} < 3(65^{\frac{2}{3}})$ → estimate this

$$\frac{1}{3x_0^{\frac{2}{3}}} < \frac{1}{3(64^{\frac{2}{3}})} = \frac{1}{3(4^2)} = \frac{1}{48}$$

$$\sqrt[3]{64} < \sqrt[3]{65} = 4 + \frac{1}{3x_0^{\frac{2}{3}}} < 4 + \frac{1}{48}$$

$$4 < \sqrt[3]{65} < 4 + \frac{1}{48}$$

∴ Take no. in $(4, 4 + \frac{1}{48})$ as an approximation of $\sqrt[3]{65}$.

L'Hospital's Rule

↳ $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ → If get indeterminate form $\left(\frac{0}{0} / \frac{\infty}{\infty}\right)$ ★ only ★ → use L'Hospital's Rule → $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ → differentiated

E.g. Evaluate $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$ → $\frac{\infty}{\infty}$ form → LHR applies

$$\begin{aligned}\lim_{x\to\infty} \frac{e^x}{x^2} &= \lim_{x\to\infty} \frac{e^x}{2x} \\ &= \lim_{x\to\infty} \frac{e^x}{2} \\ &= \infty\end{aligned}$$

→ may have to use L'Hospital Rule repeatedly

Other indeterminate form

E.g. Evaluate $\displaystyle\lim_{x\to 0^+} (x^x)$ → indeterminate form of 0^0 → cannot use LHR

$$x^x = e^{\ln(x^x)} = e^{x \ln x}$$ → need to Δ it up a bit

$$\lim_{x\to 0^+} (x^x) = \lim_{x\to 0^+} e^{x \ln x}$$

Since e^x is continuous, can interchange order of taking limit & e^x,

$$\lim_{x\to0^+} e^{x\ln x} = e^{\lim_{x\to0^+}(x\ln x)}$$
$$= e^0$$
$$= 1$$

change ↳ $x\ln x \to 0\cdot\infty$ cannot ✗

$\frac{\ln x}{\frac{1}{x}} \to \frac{0}{0}$

$$\lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to0^+}(-x) = 0$$

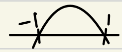
First derivative

1. If $f'(x)>0$ on (a,b), then f is increasing on $[a,b]$.
2. If $f'(x)<0$ on (a,b), then f is decreasing on $[a,b]$.

E.g. Find interval where f is defined by $f(x)=2+3x-x^3$ is increasing.

$f(x)=2+3x-x^3$ continuous on $\mathbb{R}$

$f'(x)=3-3x^2=3(1-x)(1+x)$ on $\mathbb{R}$



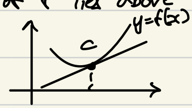
$\therefore f'(x)>0$ for $x\in(-1,1)$ & $f'(x)<0$ for $x\in(-\infty,-1)\cup(1,\infty)$ → f is increasing on $[-1,1]$

→ If f is ↑/↓ on (a,b), then f is one-to-one on (a,b).

Second Derivatives

↳ Suppose f is differentiable.

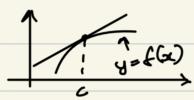
1. The graph of a function f concaves upward at a point c if the graph of f lies above its tangent at c → $f''(x)>0$ for all x in (a,b)



$\underbrace{f(x)\geq f(c)+f'(c)(x-c)}_{\text{eqn of line}}$ for x in a neighbourhood of c

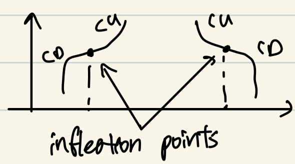
2. The graph of a function f concaves downward at point c if the graph of f lies below its tangent at c → $f''(x)<0$ for all x in (a,b)

$f(x)\leq f(c)+f'(c)(x-c)$ for x in a neighbourhood of c



Inflection Points

↳ f is continuous & curve changes concavity → from upwards to downwards or vice versa

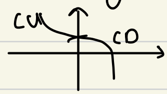


E.g. (Concavity Test: Examples) Let $f(x) = 2+3x-x^3$. Find the intervals where the graph concaves upwards, downwards & the points of inflection.

$f(x) = 2+3-x^3$, $f'(x) = 3-3x^2$, $f''(x) = -6x$ at every $x \in \mathbb{R}$

$f''(x) > 0 \rightarrow x < 0$

$f''(x) < 0 \rightarrow x > 0$



$\therefore$ Graph of f concave downward on $(0, \infty)$, concave upward $(-\infty, 0) \rightarrow$ change in concavity at $x=0 \rightarrow x=0$ is point of inflection.

Nature of extrema

1. First derivative check

x	$x-0.05$	x	$x+0.05$
$f'(x)$	$f'(x-0.05)$	0	$f'(x+0.05)$
line of tangent	\	—	/

$\rightarrow$ only one to test inflection pt

$\rightarrow$ look at shape & see if min/max

2. Second derivative check

$f''(x) > 0 \rightarrow$ global minimum

$f''(x) < 0 \rightarrow$ global maximum

$\rightarrow$ provided $f'(x) = 0$

Local/Global max/min

* Global maximum/min is also a local maximum/min

E.g. Let $f(x) = (x-1)^{\frac{2}{3}}$. Find & classify all critical points of f on $\mathbb{R}$.

$$f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}}$$

which is undefined at $x=1 \rightarrow$ singular point at $x=1$.

Since $f'(x) < 0$ for $x < 1$ & $f'(x) > 0$ for $x > 1 \rightarrow$ first derivative test tells us that $f(1) = 0$ is a local minimum for f.

Integration

$\frac{dF}{dx} = f(x) \rightarrow F$ = antiderivative of f → could have more than 1, not unique

$\hookrightarrow f$ = derivative of F } process of finding F → integration

$\int \underbrace{f(x)}_{\text{integrand}}\, dx$ integral $\qquad \frac{d}{dx}\left(\int f(x)\, dx\right) = f(x)$

E.g. Prove that $\int \frac{1}{x}\, dx = \ln|x| + C$

Need to show that the derivative of RHS = integrand

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Let $F(x) = \ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$ $\qquad$ ln 0 not defined

$$\frac{dF(x)}{dx} = \begin{cases} \frac{1}{x} & x > 0 \\ \frac{1}{(-x)}(-1) & x < 0 \end{cases} = \begin{cases} \frac{1}{x} & x > 0 \\ \frac{1}{x} & x < 0 \end{cases} = \frac{1}{x} = f(x)$$

$\therefore \frac{1}{x}$ is derivative of $\ln|x| \Rightarrow \ln|x|$ is antiderivative of $\frac{1}{x}$

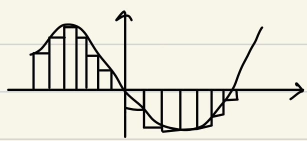
$\therefore \int \frac{1}{x}\, dx = \ln|x| + C$

Riemann Sum

With $x_k^* \in [x_{k-1}, x_k]$, the finite sum

$$\sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*)$$

is called a Riemann sum of f on $[a,b]$



☆ proof: To find the area under a curve, we divide the interval $[a,b]$ into n equal subintervals

width of each subinterval is $\Delta x = x_k - x_{k-1} = \frac{b-a}{n}$

In each kth subinterval $[x_{k-1}, x_k]$, we choose a point x_k^* & evaluate the value $f(x_k^*)$. Area of k-th rectangle with height $f(x_k^*)$.

- f is a function on $[a,b] \rightarrow x_k = a + k\left(\frac{b-a}{n}\right)$
- Area under curve → sum of ▯ $\rightarrow \sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*)$

→ impt

$$\sum_{k=1}^{n} \frac{b-a}{n} f\left(a + k\left(\frac{b-a}{n}\right)\right)$$

E.g. Riemann sum of $f(x) = x^2$ on $[1, 3]$

For $k = 1, 2, 3 \dots, n$, note that

$$x_k = 1 + k\left(\frac{3-1}{n}\right) = 1 + \frac{2k}{n}$$

Suppose we take $x_k^* = x_k$, the right end point of the kth subinterval. We have the following Rieman sum $f(x)$ on $[1, 3]$

$$\sum_{k=1}^{n} \left(\frac{2}{n}\right) f(x_k^*) = \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(1 + \frac{2k}{n}\right)^2$$

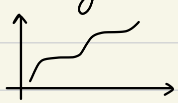
Definite Integrals

The definite integral of f from a to b is defined as

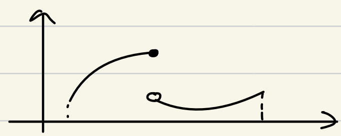
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*)}_{\text{Riemann's sum}} \qquad b > a$$

If f is ⓐ continuous, ⓑ monotonic or ⓒ piecewise continuous with finite number of jump discontinuities on $[a, b]$, then $\int_a^b f(x)\,dx$ exist

ⓑ either increasing or decreasing



ⓒ



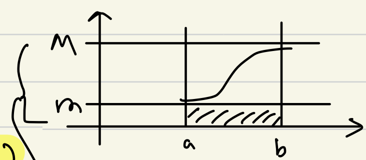
properties: 1. $\int_a^b c\,dx = c\,(b-a) \rightarrow c$ is a constant

2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Order Preserving Property

Suppose the following integrals exist and $a < b$

1. $f(x) \geq 0$ on $[a, b]$
2. $f(x) \geq g(x)$ on $[a, b]$
3. $m \leq f(x) \leq M$ on $[a, b]$



$$m\,(b-a) \leq \int_a^b f(x)\,dx \leq M\,(b-a)$$

small rectangle

big rectangle

☆ E.g. Estimate the value of the integral $\int_1^2 \frac{1}{x}\,dx$ without evaluating it

On interval $[1,2]$, function $f(x)=\frac{1}{x}$ is decreasing → largest value occurs at left endpoint, smallest value → right endpoint

$$\frac{1}{2} \le f(x) \le 1 \quad \text{for } x \in [1,2]$$

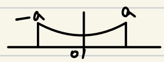
By the Order-preserving, we have

$$\frac{1}{2}(2-1) \le \int_1^2 f(x)\,dx \le 1\,(2-1)$$

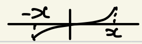
which means

$$\frac{1}{2} \le \int_1^2 \frac{1}{x}\,dx \le 1$$

- f is an even continuous function → $\int_{-a}^{a} x^2\,dx = 2\int_0^a x^2\,dx$



- f is an odd continuous function → $\int_{-a}^{a} f(x)\,dx = 0$

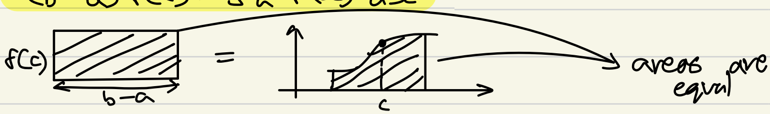


Mean Value via Definite Integral

If f is continuous on $[a,b]$, then the mean value of f on $[a,b]$ is

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{[mean/average value of } f\text{]}$$

$$(b-a)\,f(c) = \int_a^b f(x)\,dx$$



The First Fundamental Theorem of Calculus

If f is continuous on $[a,b]$, then function $F(x)$ defined by

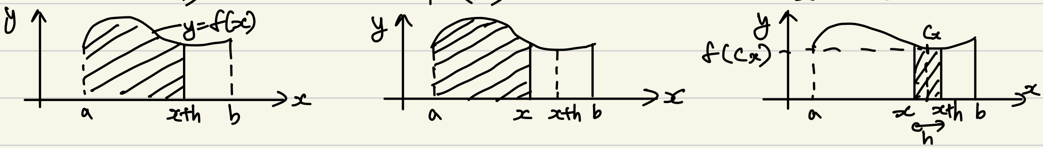
$$F(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

is continuous on $[a,b]$ & differentiable on (a,b) & $F'(x) = f(x)$

$$\underbrace{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right)}_{\frac{d}{dx}F(x)} = f(x)$$

Prove $\overbrace{\frac{d}{dx}}^{\lim_{h\to 0}\left(\frac{F(x+h)-F(x)}{h}\right)} F(x) = f(x)$

$$F(x+h) \quad - \quad F(x) \quad = \quad \int_x^{x+h} f(t)\,dt$$



By MVT for integrals, $C_x \in (x, x+h)$ such that $\qquad x < C_x < x+h$

$$f(C_x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt = f(C_x) \times h$$

$$\frac{F(x+h)-F(x)}{h} = f(C_x)$$

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(C_x) = f(x)$$

$$\therefore F'(x) = f(x)$$

E.g. Consider $g(x) = \int_1^x \frac{\sin t}{t}\,dt,\ 1 \le x \le b$

By the Fundamental Theorem of Calculus, the function $g(x) = \int_1^x \frac{\sin t}{t}\,dt$ is continuous on $[1, b]$ & differentiable on $(1, b)$. Its derivative is given by

$$g'(x) = \frac{d}{dx}\left(\int_1^x \frac{\sin t}{t}\,dt\right) = \frac{\sin x}{x} \text{ by FTC (1)}$$

1. Chain Rule $\left[\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)\cdot f(u(x))\right]$

E.g. Let $u = \sin x$

$$\frac{d}{dx}\int_1^{\sin x} \ln(t^2+1)\,dt = \frac{d}{dx}\int_1^u \ln(t^2+1)\,dt$$

$$= \frac{d}{du}\left(\int_1^u \ln(t^2+1)\,dt\right)\frac{du}{dx} \qquad u = \sin x,\ \frac{du}{dx} = \cos x$$

$$= (\ln(u^2+1))\cos x$$

$$= (\cos x)\ln(\sin^2 x + 1)$$

The Second Fundamental Theorem of Calculus

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a), \text{ where } G \text{ is any antiderivative of } f \text{ (e.g. } G' = f)$$

E.g. Evaluate the integral $\int_{-\pi}^{\pi} f(x)\,dx$ where

$$f(x) = \begin{cases} x & \text{if } -\pi \leq x \leq 0 \\ \sin x & \text{if } 0 < x \leq \pi \end{cases}$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= \int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} \sin x\,dx$$

$$= \left[\frac{x^2}{2}\right]_{-\pi}^{0} + \left[-\cos x\right]_{0}^{\pi} = \frac{-\pi^2}{2} + 2$$

2. Substitution Rule $\left[\int f(u(x))\overbrace{u'(x)\,dx}^{du} = \int f(u)\,du\right]$

E.g. Evaluate $\int \sin^3 x \cos x\,dx$

$\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$

same

$$\frac{d}{dx}(\sin x) = \cos x \rightarrow u = \sin x$$

$$\int \underbrace{\sin^3 x}_{u^3}\, \underbrace{\cos x\,dx}_{du} = \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{\sin^4 x}{4} + C$$

3. Integration by Parts $\left[\int u(x)\overbrace{v'(x)\,dx}^{dv} = u(x)v(x) - \int v(x)\overbrace{u'(x)\,dx}^{du}\right]$

E.g. Evaluate $\int \tan^{-1} x\,dx$

$$\int 1 \cdot \tan^{-1} x\,dx = uv - \int u'v$$

$v' = 1 \quad u = \tan^{-1} x$

$v = x \quad u' = \frac{1}{1+x^2}$

$$= x\tan^{-1}x - \int \frac{1}{1+x^2}\cdot x\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1}x + \frac{1}{2}\ln(1+x^2) + C$$

Reduction formula

$I_n = \int x^n e^x\,dx$, prove that $I_n = x^n e^x - nI_{n-1}$

For $n \geq 1$, use integration by parts

$$I_n = \int x^n e^x\,dx = x^n e^x - \int n(x^{n-1})e^x\,dx$$

$u(x) = x^n \quad v'(x) = e^x$

$u'(x) = nx^{n-1} \quad v(x) = e^x$

$$= x^n e^x - n\int x^{n-1} e^x\,dx$$

$$= x^n e^x - nI_{n-1}$$

Partial Fractions

↳ split into partial fractions before integrating

1. $\frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$
2. $\frac{x^2-3x-13}{(x^2+4)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$
3. $\frac{7x+4}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$
4. $\frac{4x^3+10x+4}{2x^2+x} = 2x-1+\frac{11x+4}{2x^2+x}$ ←

$$\begin{array}{r} 2x-1 \\ 2x^2+x \overline{)\,4x^3+0x^2+10x+4} \\ - \quad\quad\quad\quad \\ \hline 11x+4 \end{array}$$

do long division

5. If ax^2+bx+c cannot be reduced to product of linear factors → irreducible quadratic functions → $(Ax+B)^2+D^2$ [complete square]

Integration formulas

1. $\int f'(x)(f(x))^n\,dx = \frac{(f(x))^{n+1}}{n+1} + C$
2. $\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$
3. $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$ $\quad\quad [\int a^x\,dx = \frac{a^x}{\ln a} + C,\ a>0]$
4. $\int \cos x\,dx = \sin x + C$
5. a) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

   b) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\ |x|<|a|$

Eg. Evaluate $\int \frac{1}{x^2+4x+5}\,dx$

$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx$ $\quad\quad x^2+4x+5 = (x+2)^2+1$

let $u(x) = x+2,\quad u'(x) = 1$

$\int \frac{1}{u^2+1}\,du = \tan^{-1}u + C$

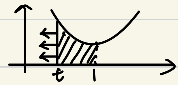
$= \tan^{-1}(x+2) + C$

Improper Integrals (1): Unbounded Integrand

Consider the integral $\int_{-1}^{1} \frac{1}{x^2}\,dx$ which is not defined on $[-1,1]$, and $\frac{1}{x^2}$ is not bounded on $[-1,1]$ since $\lim_{x\to 0}\frac{1}{x^2} = \infty$

↑ singular point.

E.g. Evaluate $\int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx$



Note that $\lim_{x\to 0^+} \frac{1}{\sqrt{2x-x^2}} = \infty$. The integrand has a singular point at 0.

$$\int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx = \lim_{t\to 0^+}\left(\int_t^1 \frac{1}{\sqrt{2x-x^2}}\,dx\right)$$

$$= \lim_{t\to 0^+}\left(\int_t^1 \frac{1}{\sqrt{1-(x-1)^2}}\,dx\right)$$

$$= \lim_{t\to 0^+}\left(-\sin^{-1}(t-1)\right) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Improper integral converges to $\frac{\pi}{2}$.

Improper Integral (2): Unbounded Interval

Consider the integral $\int_1^\infty \frac{x}{1+x^2}\,dx$ & $\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$. Both intervals $[1,\infty)$ or $(-\infty,-1]$ are not bounded.

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

E.g. Evaluate $\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$

$$\int_{-\infty}^{-1} \frac{1}{x^3}\,dx = \lim_{t\to-\infty}\int_t^{-1} \frac{1}{x^3}\,dx$$

$$= \lim_{t\to-\infty}\left[-\frac{1}{2x^2}\right]_t^{-1}$$

$$= \lim_{t\to-\infty}\left(-\frac{1}{2} - \left(-\frac{1}{2t^2}\right)\right) = -\frac{1}{2}$$

$\therefore$ The improper integral $\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$ converges to $-\frac{1}{2}$.

Comparison Test for Integrals

Suppose f & g are continuous functions such that $f(x) \geq g(x) \geq 0$, for $x \geq a$.

1) If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges
2) If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

E.g. Determine whether $\int_2^\infty \frac{1}{\ln x}\,dx$ converges or diverges

For $x>2$, $\ln x < x \rightarrow \underbrace{\frac{1}{\ln x}}_{f(x)} > \underbrace{\frac{1}{x}}_{g(x)} \geq 0$

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\left(\int_2^t \frac{1}{x}\,dx\right) = \lim_{t\to\infty}(\ln t - \ln 2) = \infty$$

By comparison theorem, we conclude that $\int_2^\infty \frac{1}{\ln x}\,dx$ also diverges.

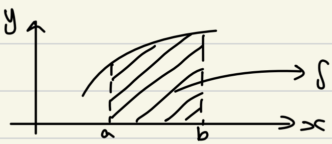
p-integral → useful in comparison test

a) Unbounded integral: $\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \frac{a^{1-p}}{1-p} & \text{for } p<1 \text{ (diverges)} \\ \infty & \text{for } p \geq 1 \text{ (converges)} \end{cases}$

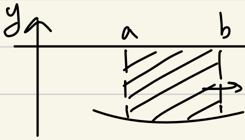
b) Unbounded interval: $\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{a^{1-p}}{1-p} & \text{for } p<1 \text{ (diverges)} \\ \infty & \text{for } p \geq 1 \text{ (converges)} \end{cases}$

Area under curve

1. About x-axis

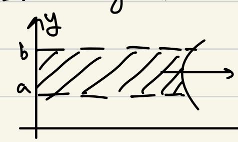


$\int_a^b y\,dx$

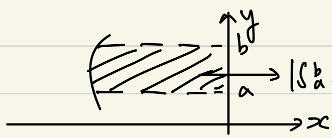


$\left|\int_a^b y\,dx\right| = -\int_a^b y\,dx$

2. About y-axis



$\int_a^b x\,dy$



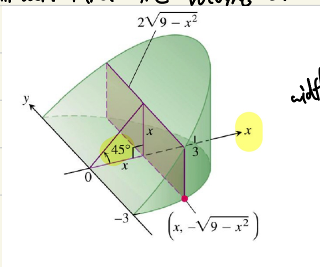
$\left|\int_a^b x\,dy\right| = -\int_a^b x\,dy$

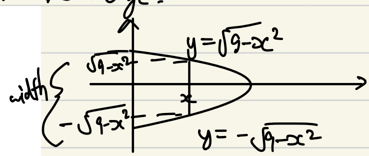
Volume using cross-sections

Volume of each typical slice at x with thickness dx given by $A(x)\,dx$.

$\therefore$ Total volume of solid is $V = \int_a^b A(x)\,dx$

E.g. A curved wedge is cut from a circular cylinder of radius 3 by 2 planes. One plane is $\perp$ to the axis of cylinder. The second plane crosses the first plane at a 45° at the centre of the cylinder. Find the volume of the wedge.





As typical cross section is a rectangle with width $2\sqrt{9-x^2}$ & height x, $0 \le x \le 3$

The cross sectional area is given by $A(x) = 2x\sqrt{9-x^2}$.

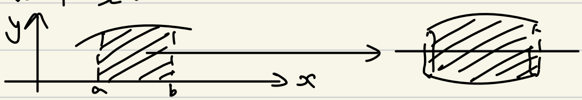
Volume required is $V = \int_0^3 2x\sqrt{9-x^2}\,dx$

$= \left[-\frac{2}{3}(9-x^2)^{\frac{3}{2}}\right]_0^3 = 18$

Volume of Solid of Revolution

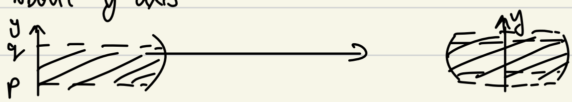
↳ solids obtained by resolving a region about a line

1. About x-axis



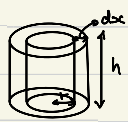
Volume $= \pi \int_a^b y^2\,dx$

2. About y-axis



Volume $= \pi \int_p^q x^2\,dy$

Cylindrical Shell Method → useful when $\int f(x)\,dx$ cannot be easily found



Volume of cylindrical shell with radius r, height h, thickness dx

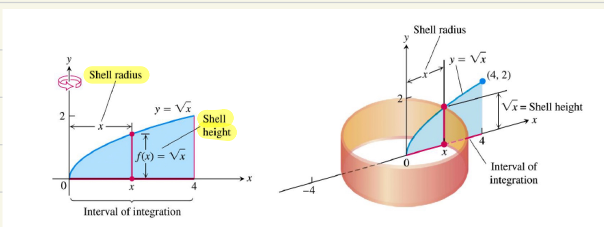
$V = 2\pi \int_a^b (\text{shell radius})(\text{shell height})\,dx$

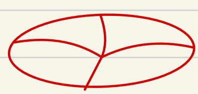
Eg. The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of solid.

A typical shell has height $\sqrt{x}$ and radius x for $0 \le x \le 4$.

Volume of typical shell $= 2\pi x\sqrt{x}\,dx$ → when $y = \sqrt{x}$, $x = x$

$\therefore V = 2\pi \int_0^4 x^{\frac{3}{2}}\,dx = \frac{128}{5}\pi$



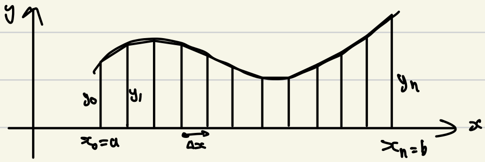


Numerical Integration

↳ Approximate area under graph with various methods

1. Trapezoidal Rule (Approximate via linear functions)

↳ Assume $f(x) > 0$ on $[a,b] \to \int_a^b f(x)\,dx$ is area under curve $f(x)$ on $[a,b]$



First subdivide $[a,b]$ into n subintervals of equal length $\Delta x \to \Delta x = \frac{b-a}{n}$

Area of trapezium $= \frac{1}{2}(y_0 + y_1)\Delta x$

$A_k = \frac{\Delta x}{2}(y_{k-1} + y_k)$, $k = 1, 2, \ldots, n$ $[y_{k-1} = f(x_{k-1})]$

$\therefore \int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$ → no. of partition involved

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$, $y_k = f(x_k)$ for $k = 0, 1, 2, \ldots, n$

Eg. Use trapezoidal rule to approximate $\int_1^2 x^2\,dx$ by T_4

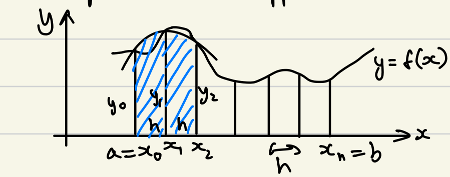
Partition $[1, 2]$ into 4 subintervals $\longrightarrow \Delta x = \frac{(2-1)}{4} = \frac{1}{4}$

k	x_k	$y_k = f(x_k) = x_k^2$
0	1	1
1	$\frac{5}{4}$	$\frac{25}{16}$
2	$\frac{6}{4} = \frac{3}{2}$	$\frac{36}{16}$
3	$\frac{7}{4}$	$\frac{49}{16}$
4	$\frac{8}{4} = 2$	4

$$\int_1^2 x^2\,dx \approx T_4 = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4)$$
$$= \frac{\frac{1}{4}}{2}\left(1 + \frac{25}{8} + \frac{36}{8} + \frac{49}{8} + 4\right)$$
$$= \frac{75}{32} = 2.34375$$

Relative error $= \left|\frac{T_4 - I}{I}\right| = \frac{\frac{75}{32} - \frac{7}{3}}{\frac{7}{3}} \approx 0.446\%$ → exact value of $\int_1^2 x^2\,dx$

2. Simpson's Rule (approximation via quadratic functions)

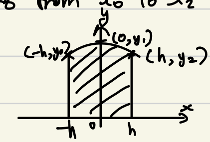


Partition interval $[a,b]$ into n subintervals of equal length $\rightarrow h = \Delta x = \frac{b-a}{n} \rightarrow n$ (even number)

↳ Approximate curve $y=f(x)$ by a parabola $y = Q(x) \rightarrow Ax^2+Bx+C$

Area under $Q(x) = \int_{-h}^{h} (Ax^2+Bx+C)$ → same as from x_0 to x_2

$= \frac{h}{3}(2Ah^2+6C)$



Express Area in terms of y_0, y_1 & y_2:

$$\begin{cases} y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C \\ y_1 = A(0)^2 + B(0) + C = C \\ y_2 = A(h)^2 + B(h) + C = Ah^2 + Bh + C \end{cases}$$

$\rightarrow 2Ah^2 = y_0 + y_2 - 2y_1$, $6C = 6y_1$

Area $= \frac{h}{3}(y_0 + 4y_1 + y_2)$

y_k is odd y_k is even

1 4 2 4 2 ... 2 4 1

$\therefore \int_a^b f(x)\,dx \approx S_n = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 2y_{n-2} + 4y_{n-1} + y_n)$

where n is even, $x_k = a + k\Delta x$, where $k = 0, 1, 2, \dots, n$, $\Delta x = \frac{b-a}{n}$, $y_k = f(x_k)$

E.g. Use Simpson's Rule, with $n=4$, to approximate $\int_1^2 \sin(\pi x^2)\,dx$.

With $n=4$, we have $h = \frac{1}{4}$:

k	x_k	$y_k = f(x_k) = \sin(\pi x_k^2)$
0	1	$y_0 = 0$
1	$\frac{5}{4}$	$y_1 = \sin\left(\frac{25\pi}{16}\right) \approx -0.980785280$
2	$\frac{3}{2}$	$y_2 = \sin\left(\frac{36\pi}{16}\right) \approx 0.707106781$
3	$\frac{7}{4}$	$y_3 = \sin\left(\frac{49\pi}{16}\right) \approx -0.1950903220$
4	2	$y_4 = \sin(4\pi) = 0$

$\int_1^2 \sin(\pi x^2)\,dx \approx S_4 = \frac{\frac{1}{4}}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$

$\approx \frac{1}{12}(0 + 4y_1 + 2y_2 + 4y_3 + 0) \approx -0.274107 40383$