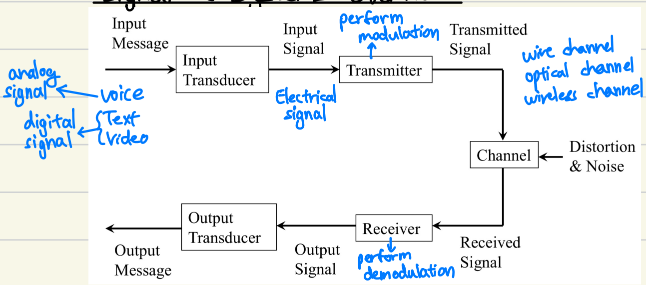
Signal & Systems

Signal & Systems Overview

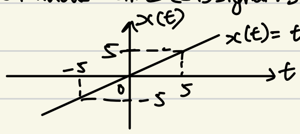


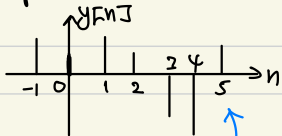
Classification of Signals

1. Continuous-Time vs Discrete-Time Signal

↳ continuous-time (CT) signal: signal $x(t)$ that is specified for all value of time t [e.g. $x(t) = A\cos(2\pi f_0 t)$ / $x(t) = t$]

☆ normal bracket

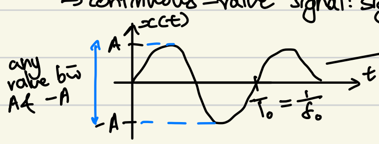




↳ discrete-time (DT) signal: signal $y[n]$ that is specified only for integer value of n

2. Continuous-Value vs Discrete-Value Signal

↳ continuous-value signal: signal $x(t)$ whose amplitude can take on any value → focus on vertical axis

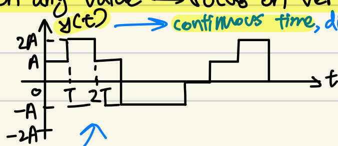


$x(t) = A\sin(2\pi f_0 t)$ → in second

amplitude

$\omega = 2\pi f_0$ → rad s⁻¹ → Hz ($T_0 = \frac{1}{f_0}$)

continuous time, continuous-value signal

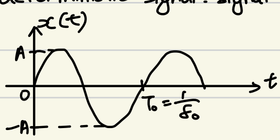


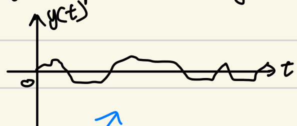
→ continuous time, discrete-value signal

↳ discrete-value signal: signal $y(t)$ whose amplitude can take on only a finite no. of values

3. Deterministic vs Random Signal

↳ deterministic signal: signal $x(t)$ that even if you reproduce the signal (take new data) → you will get the same graph



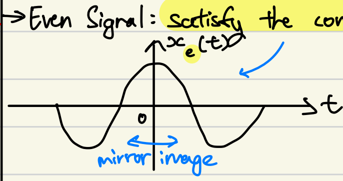


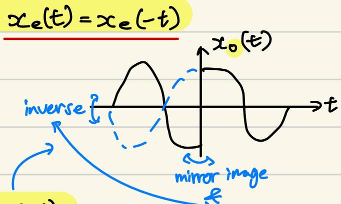
↳ random signal: signal $y(t)$ that is known only in terms of probabilistic description (e.g. noise)

- voice signal → continuous-time & continuous-value
- signal in binary data file → discrete-time & discrete value

4. Even vs Odd Signal

☆ ↳ Even Signal: satisfy the condition $x_e(t) = x_e(-t)$





☆ ↳ Odd Signal: satisfy condition $x_o(t) = -x_o(-t)$

☆ need to be satisfied ☆

↳ Any deterministic signal $x(t)$ can be decomposed into sum of an even & odd signal

- $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$ → replace t in $x(t)$ with $-t$
- $x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$

☆ $\cos(-t) = \cos(t)$, $\sin(-t) = -\sin(t)$

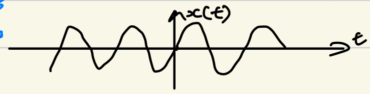
E.g. Show that signal $x(t) = A\sin(2\pi f_0 t)$ is an odd signal $x = -x(-t)$

$x(-t) = A\sin\{2\pi f_0(-t)\}$ prove / or / sketch

$= -A\sin(2\pi f_0 t)$

$= -x(t)$

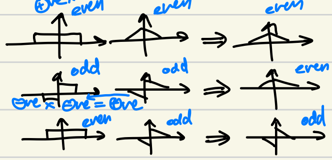
$x(t) = -x(-t)$



→ product of 2 signals

- even × even = even signal
- odd × odd = even signal
- even × odd = odd signal

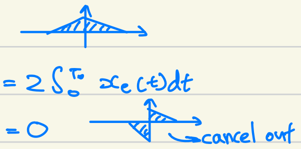
Ⓔve × Ⓞve → Ⓔve



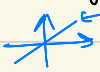
area under graph

$\int_{-T_0}^{T_0} x_e(t)\,dt = 2\int_0^{T_0} x_e(t)\,dt$

$\int_{-T_0}^{T_0} x_o(t)\,dt = 0$



E.g. Evaluate $\int_{-T_0}^{T_0} x(t)\,dt$ where $x(t) = t^3\cos^2(10t)$



$t^3\cos^2(10t)$ even or $x = -x(-t)$

$(\text{odd}^3) = \text{odd} \times (\text{even}^2) = \text{odd}$

$\therefore x(t)$ is an odd signal $\rightarrow \int_{-T_0}^{T_0} x(t)\,dt = 0$

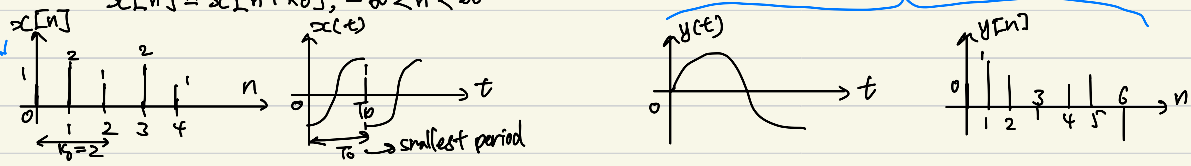
5. Periodic vs Aperiodic Signal

→ periodic signal: signal $x(t)$ with constant period $0 < T_0 < \infty$,

$x(t) = x(t + T_0)$ ↳ move waveform horizontally & can repeat

For discrete-time signal, constant period is an integer $0 < K_0 < \infty$,

$x[n] = x[n + K_0]$, $-\infty < n < \infty$



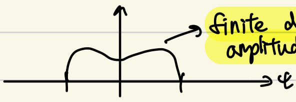
→ aperiodic signal: signal $y(t)$ / $y[n]$ does not satisfy above eqn

6. Energy-type vs Power-type Signal

→ energy-type signal: $0 < E_x < \infty$, $P_x = 0$

CT signal: $E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt$

DT signal: $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$



finite duration & finite amplitude → energy-type

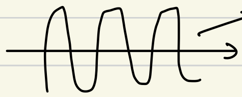
→ power-type signal: $E_x \rightarrow \infty$, $0 < P_x < \infty$

periodic signal { CT signal: $P_x = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} |x(t)|^2\,dt$

DT signal: $P_x = \frac{1}{K_0}\sum_{n=k}^{k+(K_0-1)} |x[n]|^2$ periods

aperiodic signal { CT signal: $P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,dt$

DT signal: $P_x = \lim_{K\to\infty} \frac{1}{2K+1}\sum_{n=-K}^{K} |x[n]|^2$



if amplitude bounded & periodic ↳ power-type

*Have chance that it may be neither energy/power-type signal

E.g. Determine the energy & power of signal $y(t) = e^{-|t|}$

$E_y = \int_{-\infty}^{\infty} |e^{-|t|}|^2\,dt$

$= 2\int_0^{\infty} |e^{-|t|}|^2\,dt$

$= 2\int_0^{\infty} e^{-2t}\,dt$

$= \cancel{2}(-\frac{1}{\cancel{2}})\int_0^{\infty} -2e^{-2t}\,dt$

$= -[e^{-2t}]_0^{\infty}$

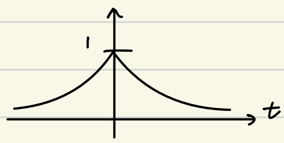
$= -(0-1) = 1$

$P_y = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |e^{-|t|}|^2\,dt$

$= \lim_{T\to\infty} \frac{1}{T} \times E_y$

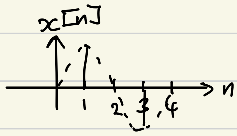
$= \lim_{T\to\infty} \frac{1}{T} \rightarrow \frac{1}{\infty}$

$= 0$



$\therefore y(t)$ is an energy-type signal → In general, energy-type signals are aperiodic signals

E.g. Determine the energy & power of discrete-time period signal $x[n] = A\sin(2\pi n/4)$



$E_x = \sum_{n=-\infty}^{\infty} |A\sin(\frac{2\pi n}{4})|^2$

$= \infty$

$P_x = \frac{1}{K_0} \sum_{n=k}^{k+K_0-1} |A\sin(\frac{2\pi n}{4})|^2$ period

$= \frac{1}{4} \sum_{n=0}^{3} A^2 |\sin(\frac{2\pi n}{4})|^2$

$= \frac{A^2}{4} [(\sin(0))^2 + (\sin(\frac{2\pi}{4}))^2 + (\sin(\frac{2\pi 2}{4}))^2 + (\sin(\frac{2\pi 3}{4}))^2]$

$= \frac{A^2}{4}(0 + 1^2 + 0^2 + 1^2)$

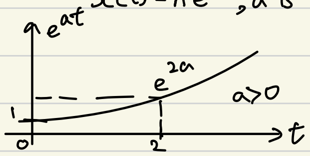
$= \frac{A^2}{2}$

$\therefore x[n]$ is a power-type signal

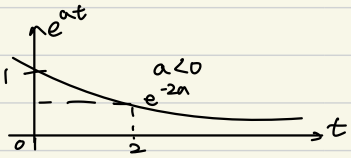
Elementary Signals

1. Exponential Signal

$x(t) = Ae^{at}$, a is a real no.



x(t) growing is a>0



x(t) decaying if a<0

2. Sinusoidal signal

↳ continuous-time: $x(t) = A\cos(2\pi f_0 t + \theta)$ OR $A\sin(2\pi f_0 t + \theta)$

frequency in Hz

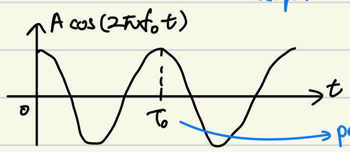
in second

amplitude

phase angle in radian

$\omega = 2\pi f$

(rad s^{-1}



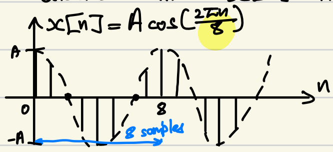
period ($T_0 = \frac{1}{f_0}$)

↳ discrete-time: $x[n] = A\cos(\frac{2\pi n}{K_0} + \theta)$ OR $A\sin(\frac{2\pi n}{K_0} + \theta)$

amplitude

phase angle in radians

integer defined as period

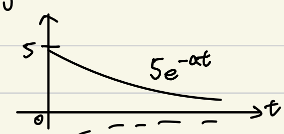


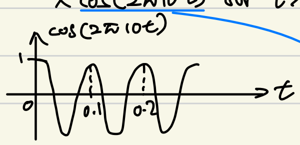
3. Complex exponential signal

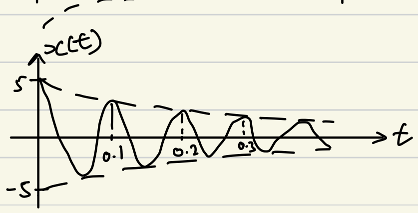
$Ae^{(j2\pi f_0 t)} = A\cos(2\pi f_0 t) + jA\sin(2\pi f_0 t)$

- Magnitude of complex signal: $|Ae^{(j2\pi f_0 t)}| = A$
- Sinusoidal signal:
  - ↳ $A\cos(2\pi f_0 t + \theta) = \mathcal{R}\{Ae^{(j2\pi f_0 t)} e^{(j\theta)}\}$ — real part
  - ↳ $A\sin(2\pi f_0 t + \theta) = \mathcal{I}\{Ae^{(j2\pi f_0 t)} e^{(j\theta)}\}$ — imaginary part

E.g. Sketch function $x(t) = 5e^{(-at)} \times \cos(2\pi 10t)$ for $t > 0$. Assume $a > 0$







$2\pi 10t = \omega t$

$2\pi 10 = 2\pi f$

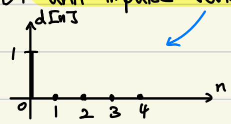
$f = 10$

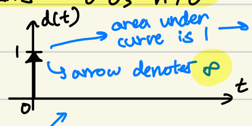
$T = \frac{1}{10} = 0.1$

Singularity Signals

1. Impulse function

→ DT unit impulse function $d[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$





area under curve is 1 → $\int_{-\infty}^{\infty} d(t)\,dt = 1$

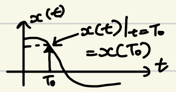
↳ arrow denotes ∞

→ CT unit impulse function $d(t) = \begin{cases} \infty, & t=0 \\ 0, & t\neq 0 \end{cases}$

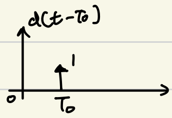
a) $\int_{-\infty}^{\infty} d(t)\,dt = 1 \rightarrow \int_{-\infty}^{\infty} 5\,d(t-2)\,dt = 5$ (doesn't matter; still 5)

b) $x(t) \times d(t-T_0) = x(T_0) \times d(t-T_0)$ (height; location)

c) $\int_{-\infty}^{\infty} x(t) \times d(t-T_0)\,dt = x(T_0)$

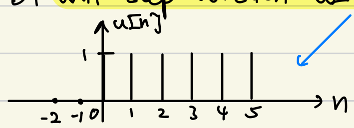


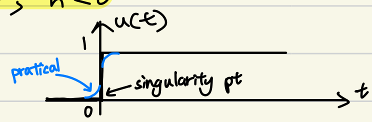
$x(t)|_{t=T_0} = x(T_0)$



2. Step function

→ DT unit step function $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n<0 \end{cases}$ (only interested in this)



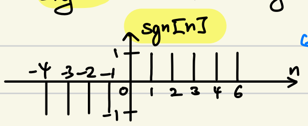


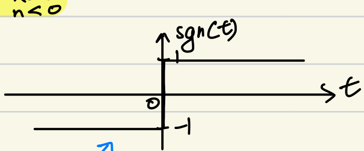
practical; singularity pt

→ CT step function $u(t) = \begin{cases} 1, & t\geq 0 \\ 0, & t<0 \end{cases}$

3. Signum function

→ DT signum function $\mathrm{sgn}[n] = \begin{cases} 1, & n>0 \\ 0, & n=0 \\ -1, & n<0 \end{cases}$

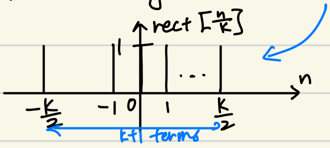




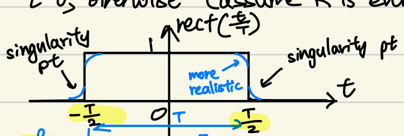
→ CT signum function $\mathrm{sgn}(t) = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases}$

4. Rectangle function

→ DT unit rectangle function $\mathrm{rect}\left[\frac{n}{K}\right] = \begin{cases} 1, & |n| \leq \frac{K}{2} \\ 0, & \text{otherwise} \end{cases}$ (assume K is even)



K+1 terms

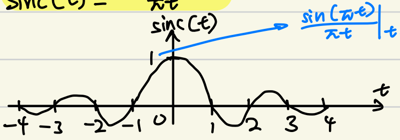


singularity pt; more realistic; singularity pt

→ CT unit rectangle function $\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$ (T: duration of pulse)

5. Sinc function

↳ $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

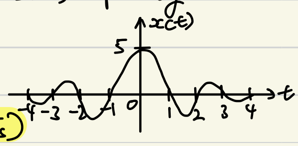


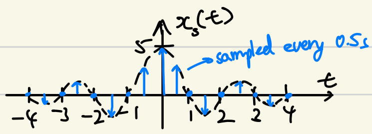
$\frac{\sin(\pi t)}{\pi t}\Big|_{t=0} = \frac{0}{0}$

$\frac{\pi \cos(\pi t)}{\pi}\Big|_{t=0} = 1$

E.g. Function $x(t) = 5\,\mathrm{sinc}(t)$ is sampled at every $T_s = 0.5s$ interval to produce the sampled signal $x_s(t)$. Sketch the waveforms for $x(t)$ & $x_s(t)$, respectively.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t) \times d(t-nT_s)$$
$$= \sum_{n=-\infty}^{\infty} x(nT_s) \times d(t-nT_s)$$
$$= \sum_{n=-\infty}^{\infty} 5 \times \mathrm{sinc}(nT_s) \times d(t-nT_s)$$



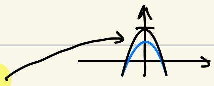


sampled every 0.5s

Operation on signals

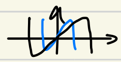
1. Time shifting
   - $x(t+a)$ → shift a unit in $\ominus$ve t-direction, $x(t-a)$ → shift a unit in $\oplus$ve t-direction
   - $x(t)+a$ → shift a unit in $\oplus$ve y-direction, $x(t)-a$ → shift a unit in $\ominus$ve y-direction

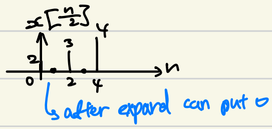
2. Amplitude scaling
   - $\frac{1}{2}x(t)$ → scale parallel to y-axis by factor $\frac{1}{2}$



   - $3x(t)$ → scale parallel to y-axis by factor 3,

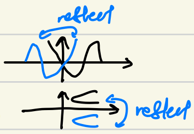
3. Time Scaling
   - $x(\frac{t}{2})$ → scale parallel to t-axis by factor 2 (Time expansion) →



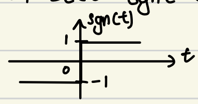


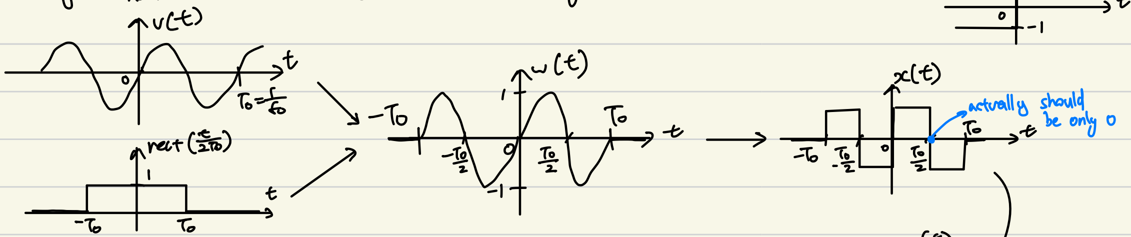
   - $x(3t)$ → scale parallel to x-axis by factor $\frac{1}{3}$ (Time compression)

4. Reflection
   - $x(-t)$ → reflect in y-axis
   - $-x(t)$ → reflect in x-axis



E.g. Assume $v(t) = \sin(2\pi f_0 t)$. Sketch $w(t) = v(t) \times \text{rect}(\frac{t}{2T_0})$, where $T_0 = \frac{1}{f_0}$, then sketch $x(t) = \text{sgn}(w(t))$, then sketch $y(t) = \sum_{n=-\infty}^{\infty} x(t) \times \delta(t - \frac{nT_0}{4})$ & evaluate $\int_{-\infty}^{\infty} y(t)\,dt$



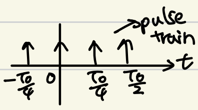


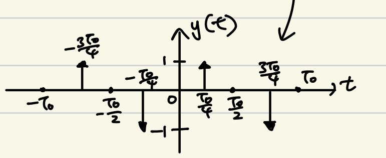
$$y(t) = \sum_{n=-\infty}^{\infty} x(t) \times \delta(t - \frac{nT_0}{4})$$

$\delta(t - n\frac{T_0}{4}) \to \delta(t), n=0$

$\delta(t - \frac{T_0}{4}), n=1$

$\delta(t - \frac{T_0}{2}), n=2$





$$\int_{-\infty}^{\infty} y(t)\,dt = \int_{-\infty}^{\infty} [\delta(t + \frac{3T_0}{4}) - \delta(t + \frac{T_0}{4}) + \delta(t + \frac{T_0}{4}) - \delta(t - \frac{3T_0}{4})]\,dt$$

$$= 0$$

Properties of Systems

↳ system refers to any physical device (communication channels, filters) that produces an output signal y(t) in response to an input signal x(t)

Input System Output

CT $x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$

DT $x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$

1. Stability

↳ system said to be bounded-input bounded-output (BIBO) stable if & only if every bounded input results in bounded output ↳ as long as not ∞

- BIBO stable system $\rightarrow y[n] = r^n x[n]u[n],\ |r| < 1$ ($\rightarrow 0.1$)
- BIBO unstable system $\rightarrow y[n] = r^n x[n]u[n],\ |r| > 1$ ($\rightarrow 10^n \rightarrow$ will go ∞)

2. Memory

→ system is said to possess memory if its output signal depends on past or future values of input signal or both $\rightarrow y[n] = x[n] + x[n-1] + x[n-2]$ → past value

↳ system is memoryless if its output signal only depend on present value of input signal $\rightarrow y(t) = x^2(t)$

3. Causality

→ system is causal if the present value of the output signal depends only on the present or past values or both of the input signal $\rightarrow y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$ (present → past)

↳ system is noncausal if present value of output signal depends on future values of input signal → it is not physically realisable in real time $\rightarrow y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$

4. Linearity

↳ system is linear if the principal of superposition holds → if input signal is $x_3(t) = a_1x_1(t) + a_2x_2(t)$, then output signal is $y_3(t) = a_1y_1(t) = a_2y_2(t)$ for any constants a_1 & a_2

5. Time Invariant

↳ system is time invariant if for any delayed $x(t-T)$, the output is delayed by the same amount $y(t-T)$

6. Linear Time-Invariant (LTI) Systems

↳ system is LTI if it satisfies both conditions of linear & time invariance

E.g. Determine the properties of system in terms of linearity & time invariance

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) = 3x\left(\tfrac{t}{2}\right)$$

$$x_1(t) \longrightarrow \boxed{H} \longrightarrow y_1(t) = 3x_1\left(\tfrac{t}{2}\right)$$

$$x_2(t) \longrightarrow \boxed{H} \longrightarrow y_2(t) = 3x_2\left(\tfrac{t}{2}\right)$$

$$x_3(t) = a_1x_1(t) + a_2x_2(t) \xrightarrow{\text{replace}} \boxed{H} \longrightarrow y_3(t) = 3a_1x_1\left(\tfrac{t}{2}\right) + 3a_2x_2\left(\tfrac{t}{2}\right)$$

$$= a_1y_1(t) + a_2y_2(t)$$

Explanation:

$$y_3(t) = 3x_3\left(\tfrac{t}{2}\right)$$
$$= 3\left[a_1x_1\left(\tfrac{t}{2}\right) + a_2x_2\left(\tfrac{t}{2}\right)\right]$$
$$= a_1\left[3x_1\left(\tfrac{t}{2}\right)\right] + a_2\left[3x_2\left(\tfrac{t}{2}\right)\right]$$
$$= a_1y_1(t) + a_2y_2(t)$$

∴ System is linear

$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) = 3x(\frac{t}{2})$

$x_1(t) = x(t-T) \longrightarrow \boxed{H} \longrightarrow y_1(t) = 3x(\frac{t}{2} - T) \neq y(t-T)$

Explanation:

For $x(t) \rightarrow y(t)$ & $x(t-T) \rightarrow y(t-T)$,

$y(t-T) = 3x(\frac{t-T}{2})$

$= 3x(\frac{t}{2} - \frac{T}{2})$ → different

Subbing in $x(\frac{t}{2}) = x(\frac{t}{2} - T)$, when t becomes $\frac{t}{2}$ into $y = 3x(\frac{t}{2})$,

$y_1(t) = 3x(\frac{t}{2} - T)$

Discrete-Time & Continuous-Time LTI Systems

↳ Any LTI system can be uniquely defined by its impulse response → output of any LTI system is the convolution of input signal & its impulse response

$\delta(t)/\delta[n] \longrightarrow \boxed{H} \longrightarrow h(t)/h[n]$ (input, system, output)

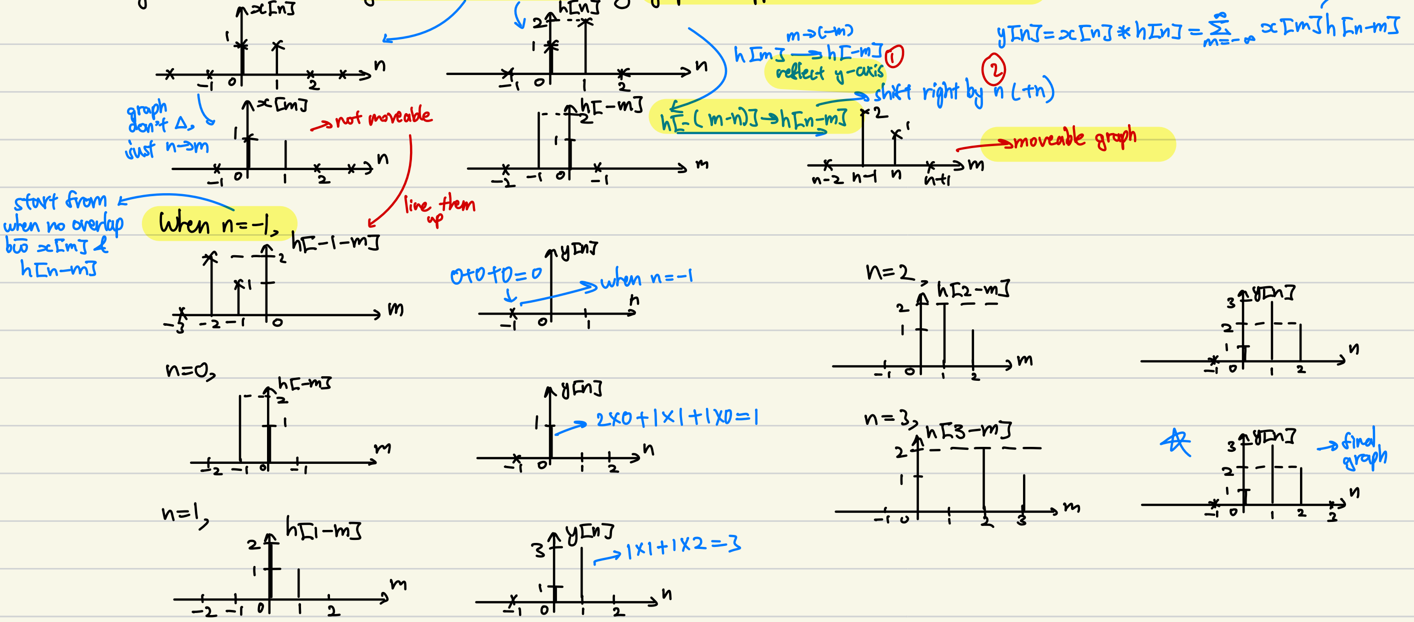
- Discrete time convolution sum

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t) = \sum_{m=-\infty}^{\infty} \underbrace{x[m]}_{①}\underbrace{h[n-m]}_{②}$ ☆

$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n] = \int_{-\infty}^{\infty} \underbrace{x(\tau)}_{①}\underbrace{h(t-\tau)}_{②} d\tau$

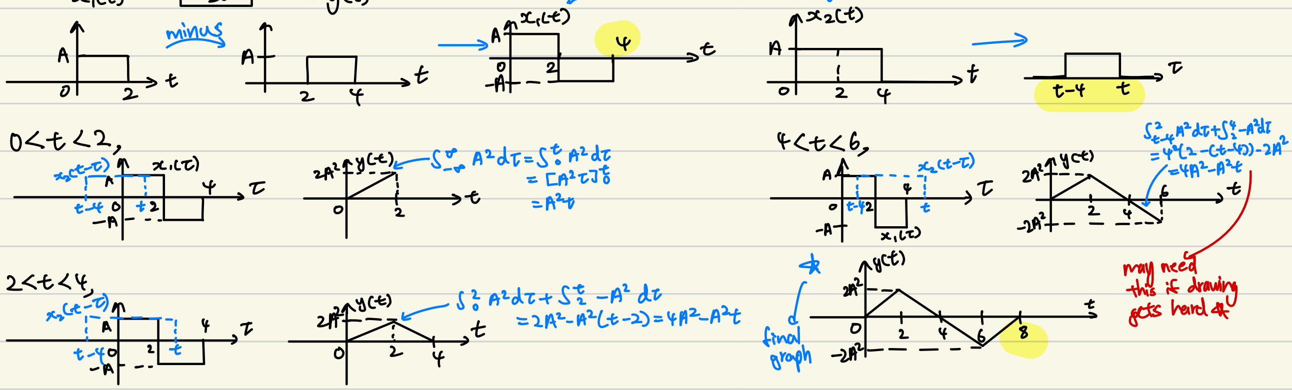
☆ → dummy variable

E.g. Sketch waveform $y[n] = x[n] * h[n]$ using graphical approach for convolution sum



E.g. Evaluate system output $y(t)$ where $x_1(t) = A\,\mathrm{rect}(\frac{t-1}{2}) - A\,\mathrm{rect}(\frac{t-3}{2})$, $x_2(t) = A\,\mathrm{rect}(\frac{t-2}{4})$.

$x_1(t) \longrightarrow \boxed{x_2(t)} \longrightarrow y(t)$



Convolution

1. Commutative
   ↳ $x_1[n] * x_2[n] = x_2[n] * x_1[n]$ (same for CT)

2. Distributive
   ↳ $x_1[n] * \{x_2[n] + x_3[n]\} = x_1[n] * x_2[n] + x_1[n] * x_3[n]$ (same for CT)

3. Associative
   ↳ $x_1[n] * \{x_2[n] * x_3[n]\} = \{x_1[n] * x_2[n]\} * x_3[n]$ (same for CT)

4. Convolution with Delta function
   → DT: $x[n] * \delta[n-k_0] = x[n-k_0]$
   → CT: $x(t) * \delta(t-T_0) = x(t-T_0)$
   → convolution become time shifting options

– Step response of LTI Systems
  → DT: $s[n] = u[n] * h[n] = \sum_{m=-\infty}^{n} h[m]$
  → CT: $s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$

E.g. Find step response of the one-stage RC filter, where impulse response is given by $h(t) = \frac{1}{RC} \times e^{(-\frac{t}{RC})} u(t)$

$s(t) = u(t) * h(t)$

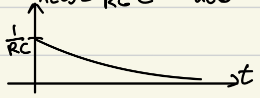
$= \int_{-\infty}^{t} h(\tau)\,d\tau = \int_{-\infty}^{t} \frac{1}{RC} e^{(-\frac{\tau}{RC})} u(\tau)\,d\tau$

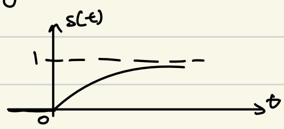
$= \frac{1}{RC} \int_{0}^{t} e^{(-\frac{\tau}{RC})}\,d\tau$

$= \begin{cases} 1 - e^{(-\frac{t}{RC})}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$u(\tau)$: unit step response start from 0

$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$





LTI System Properties

1. Memoryless LTI System (does not depend on past/future value)
   ↳ Memoryless if & only if impulse response given by:
   - DT system: $h[n] = c\delta[n]$
   - CT system: $h(t) = c\delta(t)$

   $y(t) = x(t) * \{5\delta(t)\}$
   $= 5x(t)$

2. Causal LTI System
   ↳ casual if & only if its impulse response satisfies:
   - DT system: $h[n] = 0$ for $n < 0$
   - CT system: $h(t) = 0$ for $t < 0$

   can have value after $n, t > 0$

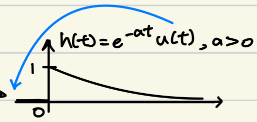
3. Stable LTI System
   ↳ LTI system BIBO stable if & only if its impulse response satisfy:
   - DT system: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$. e.g. $h[n] = p^n u[n]$, $|p| < 1$
   - CT system: $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$

   → converges

E.g. $h(t) = e^{(-at)} u(t)$, $a > 0$
→ not memoryless since $h(t) \neq c\delta(t)$
→ system causal since $h(t) = 0$ for $t < 0$
→ system is BIBO stable:

$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{0}^{\infty} e^{(-at)}\,dt$

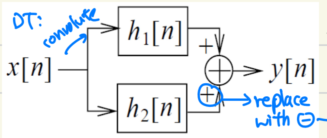
$= \frac{1}{a} < \infty$



System Interconnections

1. Parallel Connection

DT: convolute

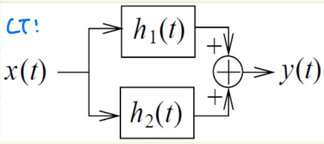


$\to x[n] \to \boxed{h_1[n]+h_2[n]} \to y[n]$

$\hookrightarrow y[n] = x[n] * \{h_1[n]+h_2[n]\}$

$\oplus \to$ replace with $\ominus \to x[n] \to \boxed{h_1[n]+(-h_2[n])} \to y[n]$

CT:



$\to x(t) \to \boxed{h_1(t)+h_2(t)} \to y(t)$

2. Cascade Connection

DT: $x[n] \to \boxed{h_1[n]} \to \boxed{h_2[n]} \to y[n]$

$\to x[n] \to \boxed{h_1[n] * h_2[n]} \to y[n]$

$\hookrightarrow y[n] = x[n] * \{h_1[n] * h_2[n]\}$

CT: $x(t) \to \boxed{h_1(t)} \to \boxed{h_2(t)} \to y(t)$

$\to x(t) \to \boxed{h_1(t) * h_2(t)} \to y(t)$

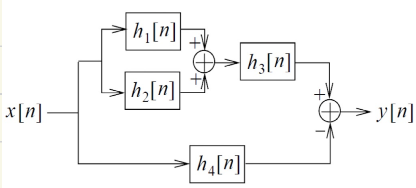
E.g. Determine the equivalent impulse response $h[n]$ of the overall system shown.

$h_1[n] = u[n],$

$h_2[n] = u[n+2] - u[n],$

$h_3[n] = \delta[n-2],$

$h_4[n] = \alpha^n u[n]$



$h[n] = \{h_1[n]+h_2[n]\} * h_3[n] - h_4[n]$

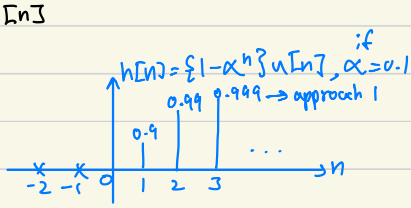
$= \{u[n] + u[n+2] - u[n]\}\, \delta[n-2] - \alpha^n u[n]$

$= u[n+2] * \delta[n-2] - \alpha^n u[n]$

$= u[n+2-2] - \alpha^n u[n]$

$= \{1-\alpha^n\} u[n]$

if $h[n] = \{1-\alpha^n\}u[n],\ \alpha = 0.1$



$\hookrightarrow$ not memoryless, $h[n] \neq c\delta[n]$

$\hookrightarrow$ casual, $h[n]=0,\ n<0$

$\hookrightarrow$ unstable, $\sum_{n=-\infty}^{\infty} |h[n]| \to \infty$

Autocorrelation

$\hookrightarrow$ correlation of a function with itself

a) For energy-type signal $x[n]$ or $x(t)$

- $\to$ DT signal: $R_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n+m]$ (conjugate) $\quad E_x = R_{xx}[0]$
- $\to$ CT signal: $R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t+\tau)\, dt \quad E_x = R_{xx}(0)$

b) For power-type signal $x[n]$ or $x(t)$

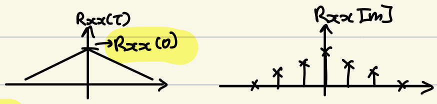
no. of terms it sum

- $\to$ DT signal: $R_{xx}[m] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]\, x^*[n+m] \quad P_x = R_{xx}[0]$
- $\to$ CT signal: $R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, x^*(t+\tau)\, dt \quad P_x = R_{xx}(0)$

— Properties of autocorrelation function

a) Peak of autocorrelation function occurs at zero shift

- $\to$ DT signal: $R_{xx}[0] \geq R_{xx}[m]$
- $\to$ CT signal: $R_{xx}(0) \geq R_{xx}(\tau)$



b) Autocorrelation functions are even functions

- $\to$ DT signal: $R_{xx}[m] = R_{xx}[-m]$
- $\to$ CT signal: $R_{xx}(\tau) = R_{xx}(-\tau)$

— Time shift does not change its autocorrelation function,

$\circledast$ autocorrelation functions of $x(t)$ & $x(t-T_1)$ are the same

E.g. Find the autocorrelation function & power of the sinusoidal signal $x(t) = A\sin(2\pi f_0 t)$ → power-type signal

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,x^*(t+\tau)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \sin(2\pi f_0 t) \times \sin(2\pi f_0(t+\tau))\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{A^2}{2}\left[\cos(2\pi f_0 \tau) - \cos(2\pi f_0(2t+\tau))\right]dt$$

$$= \frac{A^2}{2}\cos(2\pi f_0 \tau) \xleftarrow{\text{same ans}} y(t) = A\sin\{2\pi f_0(t-T_1)\}$$

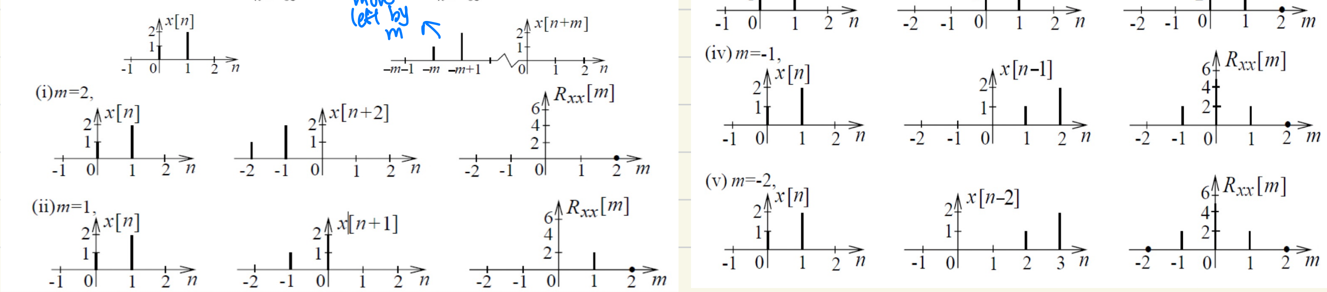
$$P_x = R_{xx}(0) = \frac{A^2}{2}$$

E.g. Find autocorrelation using graphical approach

Since $x[n]$ is an energy-type signal,

$$\mathcal{R}_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]x^*[n+m] = \sum_{n=-\infty}^{\infty} x[n]x[n+m]$$

move left by m



Cross Correlation Function

↳ cross correlation is correlation of two different function

a) energy-type signals $x[n]$ & $y[n]$ / $x(t)$ & $y(t)$

→ DT signal: $R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n]\,y^*[n+m]$

→ CT signal: $R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\,y^*(t+\tau)\,dt$

b) power-type signals $x[n]$ & $y[n]$ / $x(t)$ & $y(t)$

→ DT signal: $R_{xy}[m] = \lim_{K\to\infty} \frac{1}{2K+1}\sum_{n=-K}^{K} x[n]\,y^*[n+m]$

→ CT signal: $R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,y^*(t+\tau)\,dt$

E.g. Find the cross correlation function b/w 2 signals $x(t) = e^{j2\pi f_0 t}$ & $y(t) = e^{j2\pi 2 f_0 t}$

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,y^*(t+\tau)\,dt \quad \to \text{conjugate}$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi f_0 t}\, e^{-j2\pi 2 f_0(t+\tau)}\,dt$$

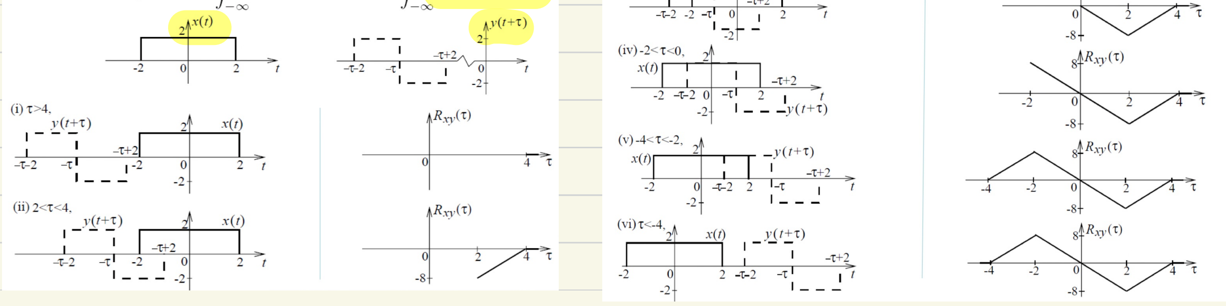
$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi f_0 t}\, e^{-j4\pi f_0 \tau}\,dt \quad \text{area cancel out}$$

$$= e^{-j4\pi f_0 \tau} \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [\cos(2\pi f_0 t) - j\sin(2\pi f_0 t)]\,dt$$

$= 0$ → uncorrelated / perpendicular

E.g. Find cross correlation function using graphical approach

$$\mathcal{R}_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y^*(t+\tau)dt = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt$$



Sinusoids (continuous-time sinusoids)

→ Have 3 representations of sinusoidal signal:

1. Formula: $x(t) = A\cos(2\pi f_0 t + \theta)$
2. Graph/plot
3. Line spectrum of amplitude & phase against frequency

→ Signal Characteristics of Sinusoids:

a) $E_x = 0$

b) Average value of $x(t) = 0$ since sinusoids are symmetric about horizontal axis → Area $= 0$

c) Average power $= \frac{A^2}{2}$ → phase shift doesn't matter → $P_x = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt$

d) RMS value $= \frac{A}{\sqrt{2}}$ → $\sqrt{P_x}$

1. $x(t) = A\cos(2\pi f_0 t + \theta)$, $\omega_0 = 2\pi f_0$ → radian frequency (rad/s) $[\sin(2\pi f_0 t + \theta) = \cos(2\pi f_0 t + \theta - \frac{\pi}{2})]$

→ A is amplitude (volt/ampere) → $A > 0$

→ f_0 is frequency ($Hz = \frac{\text{cycles}}{\text{second}}$) → $f_0 \geq 0$

→ θ is phase (in radians) → $\theta \in [-\pi, \pi]$

} standard form

2. Effect of Simple Signal Operations

Consider $x(t) = A\cos(2\pi f_0 t + \theta)$

a) Amplitude scaling: $y(t) = cx(t)$

→ $y(t) = cA\cos(2\pi f_0 t + \theta)$, $c \geq 0$

→ $y(t) = |c|A\cos(2\pi f_0 t + \theta - \pi)$, $c < 0$ → or $+\pi$ → whichever make $-\pi < \theta < \pi$

b) Time Scaling: $y(t) = x(at)$

→ $y(t) = A\cos(2\pi f_0 (at) + \theta) = A\cos(2\pi (af_0) t + \theta)$ → scale frequency

c) Time Shift: $y(t) = x(t - t_0)$

→ $y(t) = A\cos(2\pi f_0 (t - t_0) + \theta) = A\cos(2\pi f_0 t + \theta - 2\pi f_0 t_0)$ → phase shift

d) Squaring: $y(t) = x^2(t)$

→ $y(t) = A^2\cos^2(2\pi f_0 t + \theta) = \frac{A^2}{2} + \frac{A^2}{2}\cos[2\pi(2f_0)t + 2\theta]$ → $\cos^2\alpha = \frac{1}{2}[1 + \cos(2\alpha)]$ → double frequency

Multiplication of 2 sinusoids

→ Let $x_1(t) = A_1\cos(2\pi f_1 t + \theta_1)$ & $x_2(t) = A_2\cos(2\pi f_2 t + \theta_2)$

$y(t) = x_1(t)x_2(t)$ → $\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta)$

$= [A_1\cos(\underbrace{2\pi f_1 t + \theta_1}_{\alpha})][A_2\cos(\underbrace{2\pi f_2 t + \theta_2}_{\beta})]$

$= \frac{A_1 A_2}{2}\cos[2\pi(f_1 - f_2)t + (\theta_1 - \theta_2)] + \frac{A_1 A_2}{2}\cos[2\pi(f_1 + f_2)t + (\theta_1 + \theta_2)]$

Sum of Sinusoidal Signals of Same Frequency

1. Same frequency, same phase, different amplitudes ($\theta_1 = \theta_2 = \theta$, $A_1 \neq A_2$)

$y(t) = (A_1 + A_2)\cos(2\pi f_0 t + \theta)$

2. Same frequency, different phases, same amplitudes ($\theta_1 \neq \theta_2$, $A_1 = A_2 = A$)

$y(t) = 2A\cos\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(2\pi f_0 t + \frac{\theta_1 + \theta_2}{2}\right)$

3. Same frequency, different phases, different amplitudes ($\theta_1 \neq \theta_2$, $A_1 \neq A_2$)

$y(t) = A_1\cos(2\pi f_0 t + \theta_1) + A_2\cos(2\pi f_0 t + \theta_2)$

$= Re\{A_1 e^{j(2\pi f_0 t + \theta_1)} + A_2 e^{j(2\pi f_0 t + \theta_2)}\} = A\cos(2\pi f_0 t + \theta)$

$y(t) = 2\cos(5t + \frac{\pi}{4}) + 2\sqrt{2}\cos(5t - \frac{\pi}{2})$

$= Re\{2e^{j(5t + \frac{\pi}{4})} + 2\sqrt{2}e^{j(5t - \frac{\pi}{2})}\}$

$= Re\{[2e^{j\frac{\pi}{4}} + 2\sqrt{2}e^{-j\frac{\pi}{2}}]e^{j5t}\}$

VA fo cos & sin

$= 2\cos(5t - \frac{\pi}{4})$

- Relationship btw sinusodial signals & complex exponential signals:

a) $x(t) = A\cos(2\pi f_0 t + \theta) = \mathrm{Re}\{(Ae^{j\theta})e^{j2\pi f_0 t}\}$

b) $x(t) = \frac{1}{2}(Ae^{j\theta})e^{j2\pi f_0 t} + \frac{1}{2}(Ae^{-j\theta})e^{-j2\pi f_0 t}$ → $\mathrm{Re}(z) = \frac{1}{2}[z + z^*]$

Adding Sinusoids at Different Frequencies

↳ Adding sinusoids with different frequencies result in signal no longer sinusoidal → but is it periodic?

$x_1(t) = A_1\cos(2\pi f_1 t + \theta_1)$, $x_2(t) = A_2\cos(2\pi f_2 t + \theta_2)$

If $y(t)$ is periodic with period T_0, then:

equal to 1

$y(t+T_0) = A_1\cos[2\pi f_1 t + 2\pi \underbrace{f_1 T_0}_{=m} + \theta_1] + A_2\cos[2\pi f_2 t + 2\pi \underbrace{f_2 T_0}_{=k} + \theta_2] = y(t)$

↳ Ratio of 2 periods & 2 frequencies must be rational number, $\frac{T_1}{T_2} = \frac{k}{m}$, $\frac{f_1}{f_2} = \frac{m}{k}$

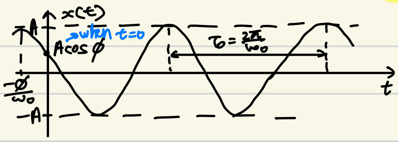
→ least common multiple → highest common factor

$T_0 = LCM(T_1, T_2)$ $f_0 = HCF(f_1, f_2)$

E.g. $T_0 = LCM(2,3) = 6$ $f_0 = HCF(\frac{1}{2}, \frac{1}{3}) = HCF(\frac{3}{6}, \frac{2}{6}) = \frac{HCF(3,2)}{6} = \frac{1}{6}$

Line Spectra & Fourier Series [for periodic signal]

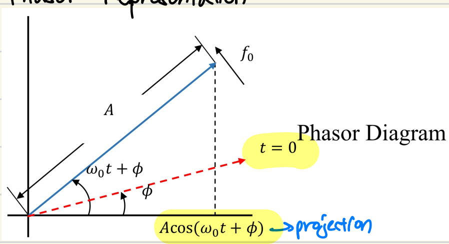
↳ Consider sinusoidal AC waveform $x(t) = A\cos(\omega_0 t + \phi)$



$2\pi f_0 t + \phi = 0 \rightarrow \cos(0) = 1 \rightarrow$ max

$t = \frac{-\phi}{2\pi f_0} = \frac{-\phi}{\omega_0}$

→ Phasor Representation

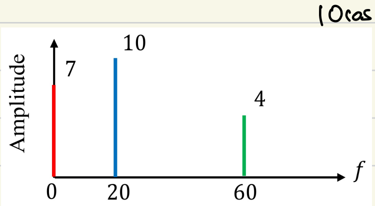


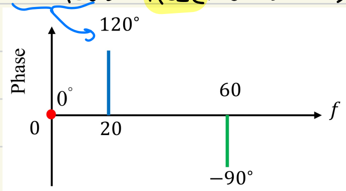
1. One-sided (Positive Frequency) Line Spectra

E.g. Show line spectra of signal $x(t) = 7 - 10\cos(40\pi t - 60°) + 4\sin(120\pi t)$

$2\pi f_1 \rightarrow f_1 = 20Hz$ $\quad$ $2\pi f_2 \rightarrow f_2 = 60Hz$

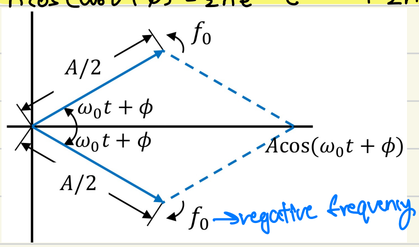
$10\cos(40\pi - 60° + 180°)$ $\quad$ $4\cos(120\pi t - 90°)$

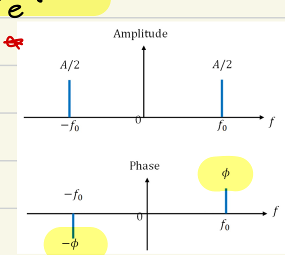




2. Two-sided Line Spectrum

$A\cos(\omega_0 t + \phi) = \frac{1}{2}Ae^{j\phi}e^{j\omega_0 t} + \frac{1}{2}Ae^{-j\phi}e^{-j\omega_0 t}$ complex conjugate





Periodic Signal Representation by Fourier Series

→ signal $x(t)$ is periodic if $x(t) = x(t + T_0)$, $T_0 \neq 0$ → Fourier Series Representation

↳ consider $x'(t) = a_0 + \sum_{n=1}^{k}[a_n\cos(2\pi n f_0 t) + b_n\sin(2\pi n f_0 t)]$ → original cosine function ⇒ $\cos(2\pi n f_0 t)$

$x'(t + T_0) = x'(t)$ ← $\cos[2\pi n f_0(t + T_0)] = \cos[2\pi n f_0 t + 2\pi n f_0 T_0]$

1. Trigonometric Form

fundamental frequency $\quad$ nf_0 = nth harmonic frequency, $n=2 \rightarrow$ 2nd harmonic

$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(2\pi n f_0 t) + b_n\sin(2\pi n f_0 t)]$,

$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$

$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(2\pi n f_0 t)\,dt$, $n = 1, 2, 3 \ldots$

$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(2\pi n f_0 t)\,dt$, $n = 1, 2, 3 \ldots$

→ $\int_{T_0}(1)\cos(2\pi m f_0 t)\,dt = 0$, for all $m \neq 0$, $\int_{T_0}(1)\sin(2\pi m f_0 t)\,dt = 0$ for all m

→ $\int_{T_0}[\sin(2\pi m f_0 t)][\cos(2\pi k f_0 t)]\,dt = 0$, for all m & k

→ $\int_{T_0}[\cos(2\pi m f_0 t)][\cos(2\pi k f_0 t)]\,dt = \begin{cases} 0, & m \neq k \\ \frac{T_0}{2}, & m = k \end{cases}$

↳ $\int_{T_0}[\sin(2\pi m f_0 t)][\sin(2\pi k f_0 t)]\,dt = \begin{cases} 0, & m \neq k \\ \frac{T_0}{2}, & m = k \end{cases}$

2. Amplitude–Phase Form

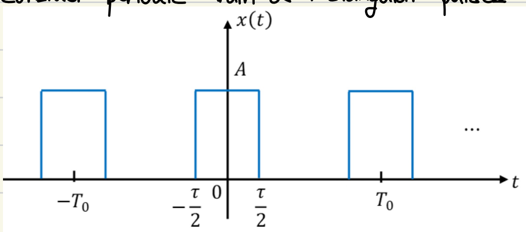
$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_0 t + \theta_n)$

$a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)$

$\Big\} \; A_0 = a_0, \; A_n = \sqrt{a_n^2 + b_n^2}, \; \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$

↳ n^{th} harmonic $A_n \cos(2\pi n f_0 t + \theta_n)$ has amplitude A_n & phase θ_n

E.g. Consider periodic train of rectangular pulses $x(t) = \begin{cases} A, & |t| \le \frac{\tau}{2} \\ 0, & |t| > \frac{\tau}{2} \end{cases}$, for $-\frac{T_0}{2} < t < \frac{T_0}{2}$



$$a_0 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,dt = \frac{1}{T_0} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\,dt = \frac{A\tau}{T_0}$$

$$a_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(2\pi n f_0 t)\,dt = \frac{2}{T_0} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A \cos(2\pi n f_0 t)\,dt = \frac{2A}{\pi n} \sin\left(\frac{n\pi\tau}{T_0}\right), \text{ for } n = 1, 2, 3 \ldots$$

$$b_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(2\pi n f_0 t)\,dt = \frac{2}{T_0} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A \sin(2\pi n f_0 t)\,dt = 0, \text{ for } n = 1, 2, 3 \ldots$$

even function

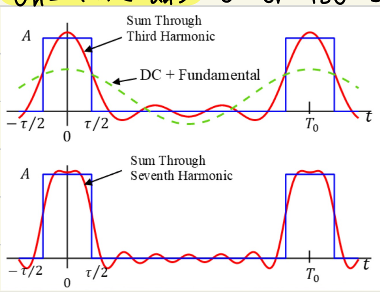
$A_0 = a_0 = \frac{A\tau}{T_0}$

$A_n = \sqrt{a_n^2 + b_n^2} = \frac{2A}{\pi n} \sin\left(\frac{n\pi\tau}{T_0}\right)$

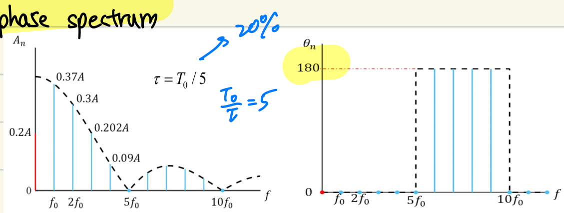
$\theta_n = \tan\left(\frac{-b_n}{a_n}\right) = 0^\circ$ or 180°

for $\frac{\tau}{T_0} = \frac{1}{4}$

$x(t) = \frac{A}{4} + \frac{\sqrt{2}A}{\pi} \cos(2\pi f_0 t) + \frac{A}{\pi} \cos(4\pi f_0 t) + \frac{\sqrt{2}A}{3\pi} \cos(6\pi f_0 t) + \ldots$



→ plot of A_n vs f yields one-sided discrete amplitude spectrum & plot of θ_n vs f yields one-sided discrete phase spectrum



20%

$\frac{T_0}{\tau} = 5$

3. Complex Exponential Form

$x(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$, $\omega_0 = \frac{2\pi}{T_0}$, $C_0 = a_0 = A_0$ → D.C.

$C_n = \frac{1}{T_0} \int_{T_0} x(t) e^{(-j2\pi n f_0 t)}\,dt$

$\left. \begin{array}{l} C_n = \frac{a_n - jb_n}{2} \\ C_n = \frac{A_n e^{j\theta_n}}{2} \end{array} \right\}$ AC

E.g. Consider the rectangular pulses above,

$$C_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{(-j2\pi n f_0 t)}\,dt$$

$$= \frac{1}{T_0} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A e^{(-j2\pi n f_0 t)}\,dt$$

$$= \frac{A}{-j2\pi n f_0 T_0} \left[e^{-j\pi n f_0 \tau} - e^{j\pi n f_0 \tau} \right]$$

$-2j\sin(\pi n f_0 \tau)$

$$= \frac{A}{T_0} \frac{\sin(\pi n f_0 \tau)}{\pi n f_0} = \frac{A\tau}{T_0} \frac{\sin(\pi n f_0 \tau)}{\pi n f_0 \tau}$$

$\frac{\tau}{T_0}$ = Duty cycle

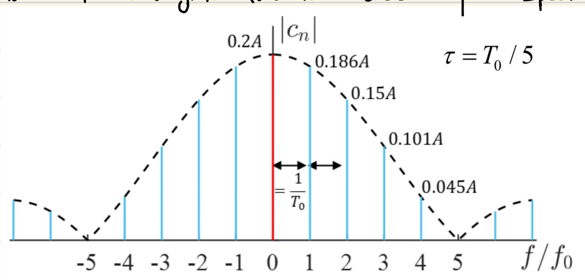
$C_n = |C_n| e^{j\phi_n}$

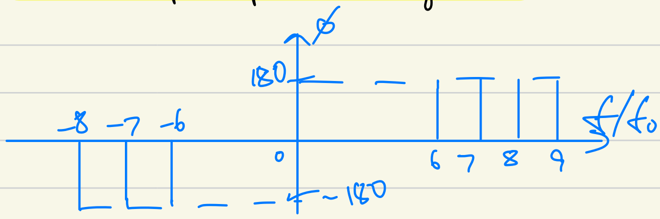
$C_n^* = C_{-n} = |C_{-n}| e^{j\phi_{-n}}$

Using $\text{sinc}(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda}$, $C_n = \frac{A\tau}{T_0}\,\text{sinc}(nf_0\tau)$

→ plot of $|C_n|$ vs f yields the two-sided discrete amplitude spectrum → $|C_{-n}| = |C_n|$ → amplitude spectrum symmetric

→ plot of ϕ_n vs f yields two-sided discrete phase spectrum → $\phi_{-n} = -\phi_n$ → phase spectrum is antisymmetric





Symmetrical Signals of Fourier Series

↳ properties of functions:

a) Even function × Even function = Even

b) Even function × Odd function = Odd

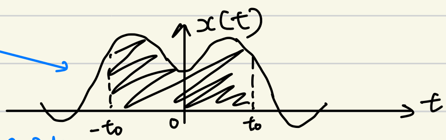
c) Odd function × Odd function = Even

1. Even Function

- → $x(t) = x(-t)$, $\int_{-t_0}^{t_0} x(t)\,dt = 2\int_0^{t_0} x(t)$
- → $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)$ (Fourier cosine series)
- → $a_0 = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} x(t)\,dt$
- → $a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} x(t)\cos n\omega_0 t\,dt$, $b_n = 0$

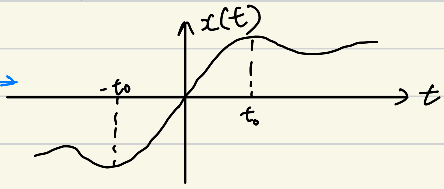
$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(2\pi n f_0 t)\,dt$

$= 2\times\frac{2}{T_0}\int_0^{\frac{T_0}{2}} x(t)\cos(2\pi f_0 t)\,dt$



2. Odd Function

- → $x(t) = -x(-t)$, $\int_{-t_0}^{t_0} x(t)\,dt = 0$
- → $x(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$ (Fourier Sine Series)
- → $b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt$, $a_0 = 0$, $a_n = 0$

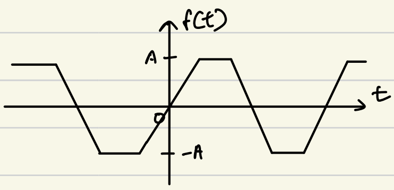


3. Half-wave symmetry

- → $x(t) = -x(t \pm \frac{T}{2})$ → shift by half a period
- → $a_0 = 0$
- → $a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt$
- → $b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt$

$\}$ n is odd

- → $a_n = b_n = 0$, n is even



Effect of Time Operations on Fourier Series

1. Time Reversal

- → if signal $x(t)$ is time reversed, $x(-t)$ → $x(-t) \longleftrightarrow C_{-n} = C_n^*$ → if $x(t)$ is a real valued signal ↳ all imaginary component = 0
- ↳ since $C_n = |C_n|\angle\phi_n$ → $C_n^* = |C_n|\angle(-\phi_n)$ → for rectangular pulses, $\phi_n = 180°$, $-\phi_n = -180° = 180°$ → $x(t) = x(-t)$

↳ above (complex exponential form)

2. Time Shifting

- → if signal $x(t)$ shift by an amount of τ → $x(t-\tau) \longleftrightarrow C_n e^{-j2\pi n f_0 \tau}$
- ↳ for rectangular pulses, $x_1(t) \longleftrightarrow C_n e^{-j\pi n f_0 \tau} = \frac{A\tau}{T_0}\,\text{sinc}(nf_0\tau)\,e^{-j2\pi n f_0 \tau}$

↳ above (complex exponential form)

3. Time Scaling

→ if signal $x(t)$ scaled in time by factor of $a>0 \rightarrow x(at) \leftrightarrow c_n$ (unchanged)

→ $x(at)$ has same fourier coefficients c_n, but fundamental frequency changed from f_0 to af_0

Parseval's Power Theorem

→ relates the average power of a periodic signal to its Fourier Series coefficients

→ $P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$

Fourier Transform [for non-periodic signals]

normally we table

→ $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$ → fourier transform of $x(t)$

→ $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} dt$ → inverse fourier transform of $X(f)$

$x(t)$ & $X(f)$ form a fourier transform pair

→ $X(f)$ is a complex function of f → $X(f) = |X(f)| e^{j\theta f}$

continuous amplitude spectrum of $x(t)$

continuous phase spectrum of $x(t)$

→ For real-valued of $x(t)$: $X(-f) = X^*(f)$

$|X(-f)| = |X(f)|$ → even function of f

$\theta(-f) = -\theta(f)$ → odd function of f

→ $X(0) = \int_{-\infty}^{\infty} x(t) dt$

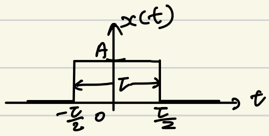
E.g. Consider a rectangular pulse of duration τ & amplitude A. $x(t) = A\,\mathrm{rect}(\frac{t}{\tau})$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

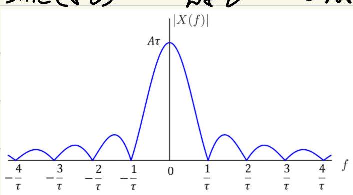
$$= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A e^{-j2\pi f t} dt = A\left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

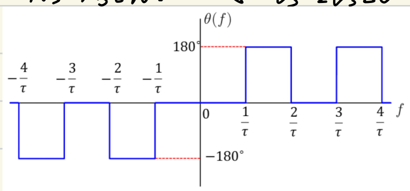
$$= A\left[\frac{-j2\sin(\pi f \tau)}{-j2\pi f}\right] = \frac{A\tau}{\pi f \tau}\sin(\pi f \tau)$$

$$= A\tau\,\mathrm{sinc}(f\tau)$$



$\mathrm{sinc}(f\tau) = \frac{\sin(\pi f \tau)}{\pi f \tau}$ → $\pi f \tau = \pi, 2\pi, 3\pi \ldots$ → $f = \frac{1}{\tau}, \frac{1}{2\tau}, \frac{1}{3\tau}$





→ significant portion of spectrum in range $|f| < \frac{1}{\tau}$ → as $\tau\downarrow$ → $f\uparrow$ (reciprocal spreading)

Properties of Fourier Transform

1. Linearity

→ $ax_1(t) + bx_2(t) \overset{F}{\longleftrightarrow} aX_1(f) + bX_2(f)$

2. Time Scaling

→ for $a>0$ → $F[x(at)] = \frac{1}{a}X(\frac{f}{a})$

→ for $a<0$ → $F[x(at)] = \frac{1}{|a|}X(\frac{f}{a})$

3. Duality (conversion) & Conjugate Functions

→ $x(t) \overset{F}{\longleftrightarrow} X(f)$ & $X(t) \longleftrightarrow x(-f)$ & $x^*(t) \longleftrightarrow X^*(-f)$

4. Time shifting

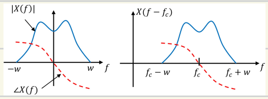
→ $x(t-t_0) \overset{F}{\longleftrightarrow} X(f)e^{-j2\pi f t_0}$

→ amplitude spectrum of delayed signal same as original one, phase spectrum is $-2\pi f t_0 + \angle X(f)$

slope of $-2\pi t_0$ (linear phase)

5. Frequency Shifting carrier frequency

↳ $x(t)e^{j2\pi f_c t} \overset{F}{\longleftrightarrow} X(f-f_c)$



6. Modulation Theorem

↳ $x(t)\cos(2\pi f_c t+\phi) \overset{F}{\longleftrightarrow} \frac{1}{2}X(f-f_c)e^{j\phi}+\frac{1}{2}X(f+f_c)e^{-j\phi}$

↳ multiplying signal by sinusoid translates its spectrum up & down in frequency by f_c.

7. Area Under $x(t)$

↳ $\int_{-\infty}^{\infty} x(t)\,dt = X(0)$

not in formula

E.g. Find $\int_{-\infty}^{\infty}\frac{\sin t}{t}\,dt$.

→ fourier transform pairs

$\frac{\sin t}{t} = \text{sinc}\left(\frac{t}{\pi}\right) \longleftrightarrow \pi\,\text{rect}(\pi f)$ $\frac{1}{\frac{1}{\pi}}\text{rect}\left(\frac{f}{\frac{1}{\pi}}\right)$

$\int_{-\infty}^{\infty}\text{sinc}\left(\frac{t}{\pi}\right)dt = \pi\,\text{rect}(0) = \pi$

8. Area Under $X(f)$

↳ $x(0)=\int_{-\infty}^{\infty}X(f)\,df$

9. Differentiation in time domain

↳ $\frac{d^n}{dt^n}x(t) \longleftrightarrow (j2\pi f)^n X(f)$

10. Integration in time domain

↳ $\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{1}{j2\pi f}X(f)+\frac{1}{2}X(0)\delta(f)$

11. Multiplication in time domain

↳ $x_1(t)x_2(t) \longleftrightarrow \int_{-\infty}^{\infty}X_1(\lambda)X_2(f-\lambda)\,d\lambda$

E.g. $x(t)=\text{sinc}(2Wt)\,\text{rect}\left(\frac{t}{T}\right)$

$\text{sinc}(2Wt)\longleftrightarrow \frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$, $\text{rect}\left(\frac{t}{T}\right)\longleftrightarrow T\,\text{sinc}(fT)$

$X(f)=\frac{T}{2W}\int_{-\infty}^{\infty}\text{rect}\left(\frac{\lambda}{2W}\right)\text{sinc}[(f-\lambda)T]\,d\lambda$

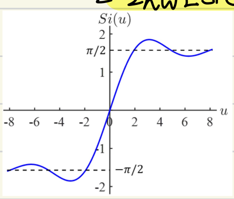
$=\frac{T}{2W}\int_{-W}^{W}\text{sinc}[(f-\lambda)T]\,d\lambda$

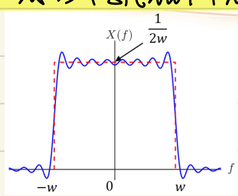
$=\frac{T}{2W}\int_{-W}^{W}\frac{\sin[\pi(f-\lambda)T]}{\pi(f-\lambda)T}\,d\lambda$ $\alpha=\pi(f-\lambda)T \to d\alpha=-\pi T\,d\lambda$

$X(f)=-\frac{1}{2W\pi}\int_{\pi(f+W)T}^{\pi(f-W)T}\frac{\sin\alpha}{\alpha}\,d\alpha$ $Si(u)=\int_0^u\frac{\sin x}{x}\,dx$ (sine integral)

(Cosine integral) $Ci(x)=-\int_x^{\infty}\frac{\cos t}{t}\,dt$

$=\frac{1}{2\pi W}[Si(\pi WT-\pi fT)+Si(\pi WT+\pi fT)]$





12. Convolution in time domain

↳ $\int_{-\infty}^{\infty}x_1(\tau)x_2(t-\tau)\,d\tau \longleftrightarrow X_1(f)X_2(f)$

13. Rayleigh's Theorem → Energy Spectral Density

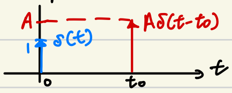
↳ $\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_{-\infty}^{\infty}|X(f)|^2\,df$

Dirac Delta Function/Unit Impulse

→ $\delta(t) = \begin{cases} \infty; & t=0 \\ 0; & t \neq 0 \end{cases}$

→ $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

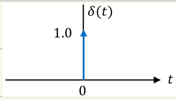
Functions approaching $\delta(t)$ → $\delta_\varepsilon(t) = \frac{1}{\varepsilon}\,\mathrm{rect}\left(\frac{t}{\varepsilon}\right) / \delta_\varepsilon(t) = \frac{1}{\varepsilon}\,\mathrm{sinc}\left(\frac{t}{\varepsilon}\right) \rightarrow \lim_{\varepsilon \to 0} \delta_\varepsilon(t) \rightarrow \delta(t)$



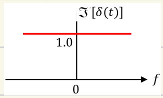
Dirichlet's Conditions for $x(t)$ to be Fourier transformable:
i) $x(t)$ single-valued, with finite no. of maxima & minima & finite no. of discontinuities in any finite time interval
ii) $x(t)$ is absolutely integrable → $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$

Properties of $\delta(t)$

1. Sifting/Sampling Property: $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$
2. Replication Property: $x(t) * \delta(t-t_0) = x(t-t_0)$
3. $x(t)\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$
4. $F(\delta(t)) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\,dt = 1$ →



→

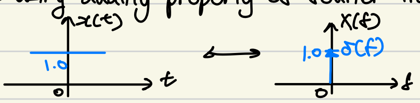


5. $\delta(at) = \frac{1}{|a|}\delta(t)$

Applications of Delta Function

1. DC signal

↳ using duality property of fourier transform pair: $\delta(t) \longleftrightarrow 1$, $1 \longleftrightarrow \delta(f)$

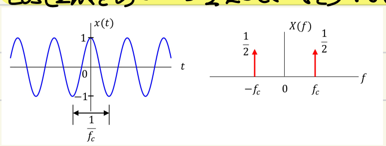


2. Complex Exponential Function

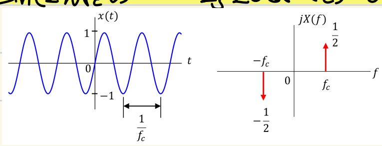
↳ $e^{j2\pi f_c t} \longleftrightarrow \delta(f-f_c)$

3. Sinusoidal Functions

→ $\cos(2\pi f_c t) \longleftrightarrow \frac{1}{2}[\delta(f-f_c) + \delta(f+f_c)]$, $\cos(2\pi f_c t) = \frac{1}{2}(e^{j2\pi f_c t} + e^{-j2\pi f_c t})$



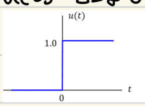
→ $\sin(2\pi f_c t) \longleftrightarrow \frac{1}{2j}[\delta(f-f_c) - \delta(f+f_c)]$, $\sin(2\pi f_c t) = \frac{1}{2j}(e^{j2\pi f_c t} - e^{-j2\pi f_c t})$

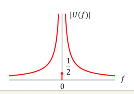


4. Unit Step Function

↳ $\int_{-\infty}^{t} \delta(\tau)\,d\tau \longleftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$ (integration property)

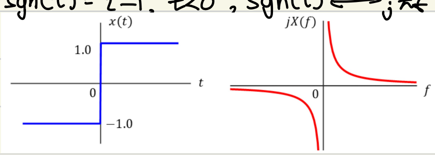
e.g. $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \longleftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$





5. Sigma Function

↳ $\mathrm{sgn}(t) = \begin{cases} 1; & t>0 \\ -1; & t<0 \end{cases}$, $\mathrm{sgn}(t) \xleftrightarrow{F} \frac{1}{j\pi f}$



Fourier Transform of Periodic Signals

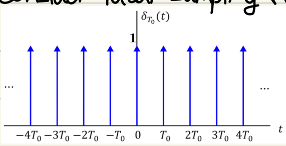
→ $x'(t)$ is called generating function & is fourier transformable → $x'(t) = \begin{cases} x(t), & -\frac{T_0}{2} \le t \le \frac{T_0}{2} \\ 0, & \text{elsewhere} \end{cases}$

→ $c_n = \frac{1}{T_0}\int_{-\infty}^{\infty} x'(t) e^{-j2\pi n f_0 t} dt = \frac{1}{T_0} X'(nf_0)$

→ $F[x(t)] = \frac{1}{T}\sum_{n=-\infty}^{\infty} X'(nf_0)\delta(f-nf_0)$, $X'(f)$ is fourier transform of $x'(t)$

→ Fourier Transform of periodic signal consists of delta functions occuring at integer multiples of fundamental frequency f_0 including origin.

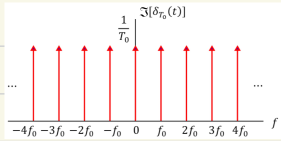
E.g. Consider ideal sampling function $\delta_{T_0}(t) = \sum_{m=-\infty}^{\infty}\delta(t-mT_0)$.



generating function $x'(t)$ for $\delta_{T_0}(t) \to \delta(t)$

$X'(f) = 1$ & $X'(nf_0) = 1$

$\sum_{m=-\infty}^{\infty}\delta(t-mT_0) \leftrightarrow \frac{1}{T_0}\sum_{m=-\infty}^{\infty}\delta(f-mf_0)$



Frequency Domain Analysis of LTI Systems

→ Consider LTI System with impulse responce $h(t)$ & subject to input $x(t) = e^{j2\pi ft}$ & produces output $y(t)$:

$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j2\pi f(t-\tau)} d\tau$

$H(f) = \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f\tau} d\tau$

} $y(t) = H(f) e^{j2\pi ft}$ → transfer function/frequency response of system

$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} df$ } $H(f)$ & $h(t)$ form fourier transform pair

→ $H(f) = |H(f)| e^{j\beta(f)}$ [complex quantity] ($|H(f)|$ → amplitude response, $\beta(f)$ → phase response)

$|H(f)| = |H(-f)|$ → even function of f

$\beta(f) = -\beta(-f)$ → odd function of f

} linear system with real valued $h(t)$

Distortionless Transmission

→ output of LTI system is a delayed & scaled version of input

$y(t) = Kx(t-t_0) \longrightarrow Y(f) = KX(f)e^{-j2\pi ft_0}$

$H(f) = \frac{Y(f)}{X(f)} = Ke^{-j2\pi ft_0} = Ke^{j(-2\pi ft_0 \pm n\pi)}$

→ $|H(f)| = K$ → constant for all f

→ $\beta(f) = -2\pi ft_0 \pm n\pi$, $n = 0, \pm1, \pm2 \ldots$ → phase responce linear with frequency

$\beta(f)$, π, $-\pi$, f, ← slope $= -2\pi t_0$

Ideal Low-Pass Filters

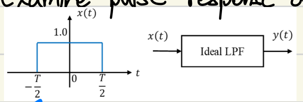
→ filter is a frequency selective device use to limit spectrum of signal to some frequency (passband/stopband)

$H(f) = \begin{cases} e^{-j2\pi ft_0}, & |f| \le B \\ 0, & |f| > B \end{cases}$

$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} df = \int_{-B}^{B} e^{j2\pi f(t-t_0)} df$

$= 2B\,\mathrm{sinc}[2B(t-t_0)]$

E.g. Examine pulse responce of an ideal low-pass filter.



$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

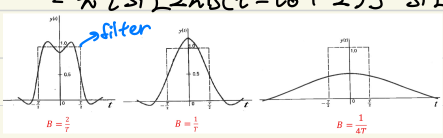
$= 2B\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin[2\pi B(t-t_0-\tau)]}{2\pi B(t-t_0-\tau)} d\tau$

Let $\lambda = 2\pi B(t-t_0-\tau) \to d\lambda = -2\pi B d\tau$

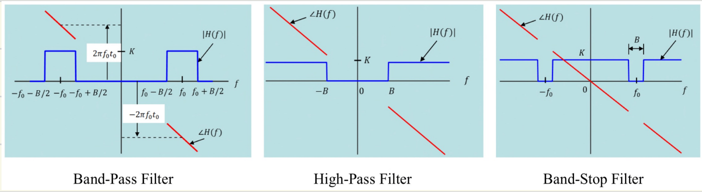
$y(t) = \frac{1}{\pi}\int_{2\pi B(t-t_0-\frac{T}{2})}^{2\pi B(t-t_0+\frac{T}{2})} \frac{\sin\lambda}{\lambda} d\lambda$

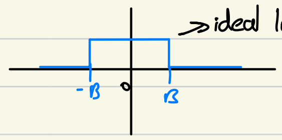
$= \frac{1}{\pi}\{Si[2\pi B(t-t_0+\frac{T}{2})] - Si[2\pi B(t-t_0-\frac{T}{2})]\}$

assume $t_0 = 0$ {



→ when $B < \frac{1}{T}$, output is distorted version of input





→ ideal low-pass filter is noncausal

Spectral Density

↳ for calculating the energy/power of signal by its Fourier Transform

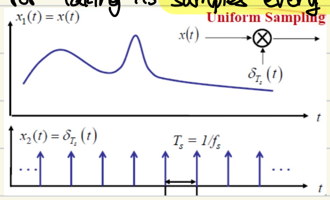
1. Energy Spectral Density of $x(t)$ in Joules/Hz
   ↳ $E = \int_{-\infty}^{\infty} \Psi(f)\, df \rightarrow \Psi(f) = |X(f)|^2$

2. Power Spectral Density of periodic signal consists of a succession of weighted delta functions
   ↳ $P = \int_{-\infty}^{\infty} S_p(f)\, df \rightarrow S_p(f) = \frac{1}{T_0^2}\sum_{n=-\infty}^{\infty} |X'(\frac{n}{T_0})|^2 \delta(f - \frac{n}{T_0})$

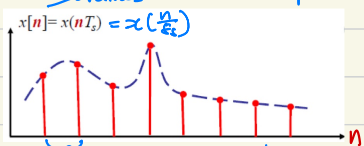
3. Relation among input & output energy spectral densities
   ↳ $\Psi_y(t) = |H(f)|^2 \Psi_x(f)$

Sampling & Aliasing

→ continuous-time signals are difficult to handle because computers only handle samples/data.
→ process of converting continuous-time signals to discrete-time signal is called sampling/campling process
→ system which performs sampling operation is called continuous-to-discrete (C-to-D) converter.
↳ To conduct uniform sampling process, the continuous-time signal $x(t)$ multiplied with $\delta_{T_s}(t)$ impulse train signal for taking its samples every T_s second



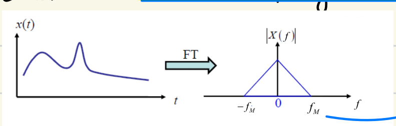
→ values $x[n]$ are samples of $x(t)$



T_s → sampling interval

Low-pass Sampling Theorem

→ consider a band-limited signal $x(t)$ with Fourier Transform $X(f)$



↳ if bandwidth of signal is finite
   > $|X(f)| = 0$, for $|f| > f_M$

→ f_M denotes bandwidth of $x(t)$

→ $f_M = \infty$ → signal $x(t)$ is called non-band-limited signal. E.g. rectangular pulse $\xrightarrow{\text{fourier transform}}$ sinc → infinite bandwidth
   → to become band-limited → pass through low-pass filter

→ Two fundamental properties for Low Pass Sampling Theorem:
   1. $x_1(t)x_2(t) \longleftrightarrow X_1(f) * X_2(f)$
   2. $\delta_{T_s}(t) \longleftrightarrow \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f - \frac{n}{T_s})$

↳ $x_s(t) = x(nT_s) = x(t)\delta_{T_s}(t)$

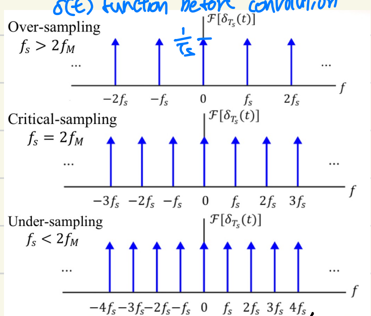
$X_s(f) = X(f) * [\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f - \frac{n}{T_s})]$

☆ $X_s(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s})$ $f - nf_s$

→ must sample $f_s \geq 2f_m$ (sample frequency)

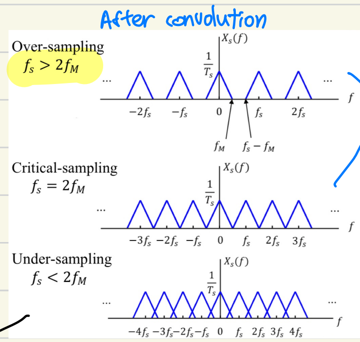
→ baseband spectrum → $-\frac{1}{2} \leq \frac{f_m}{f_s} \leq \frac{1}{2}$

δ(t) function before convolution



results in less samples being generated with aliasing

After convolution



→ can use low-pass to recover original signal (without aliasing)

$X_s(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s})$

Nyquist Sampling Rate

↳ Nyquist Rate = 2 × maximum frequency → $f_N = 2f_m$

E.g. $x(t) = 10\cos(426\pi t)$

$\omega_m = 2\pi f_m = 426\pi \rightarrow f_m = 213\text{Hz}$ → if have 2 f_m → take $\max\{a, b\}$

$f_N = 2f_m = 426\text{Hz}$

Sampling Sinusoidal Signals

→ subscripts: a → analog (continuous-time); d → digital (discrete-time)

f_a = analog frequency of $x(t) = f_o$

f_d = normalised digital frequency of $x[n]$

$\frac{1}{f_d}$ = no. of samples/period

not necessarily periodic signal (depend on f_a & f_s)

→ $x[n] = A\cos(2\pi f_d n + \theta)$ → $f_d = \frac{f_a}{f_s}$ (cycles/samples) $\omega_d = 2\pi f_d$ (radians/sample)

→ if $x[n]$ is periodic with period N_o (samples/cycle) → $2\pi f_d N_o = m2\pi$, m is int.

→ $f_d = \frac{f_a}{f_s} = \frac{m}{N_o}$ = rational no.

E.g. a) Consider an analogue sinusodial signal $x(t) = 10\cos(426\pi t)$. If $f_s = 1\text{kHz}$ & applied to $x(t)$, what is the resulted discrete-time sinusoidal signal $x[n]$?

$T_s = \frac{1}{f_s} = \frac{1}{1000}$

$x[n] = x(t)|_{t=nT_s} = 10\cos(2\pi \cdot 213 \cdot nT_s)$

$= 10\cos(2\pi \cdot \frac{213}{1000} \cdot n) = 10\cos(\frac{213}{500} n\pi)$

$f_d = \frac{213}{1000}$ digital frequency of $x[n]$

b) If same $f_s = 1\text{kHz}$ is used to recover analog signal $x(t)$ from $x[n]$, what is resulted signal $x(t)$?

Since $f_s = 1000\text{Hz}$, $f_d = \frac{213}{1000} = \frac{f_a}{f_s} \rightarrow f_a = 213\text{Hz}$

c) If f_s used is 200Hz instead, redo problem.

$T_s = \frac{1}{f_s} = \frac{1}{200}$

$x[n] = x(t)|_{t=nT_s} = 10\cos(2\pi \cdot \frac{213}{200} \cdot n)$ → $-\frac{1}{2} < f_d < \frac{1}{2}$

$= 10\cos[2\pi \cdot (1 + \frac{13}{200}) \cdot n] = 10\cos[2\pi n + 2\pi n(\frac{13}{200})]$ → same cos waveform

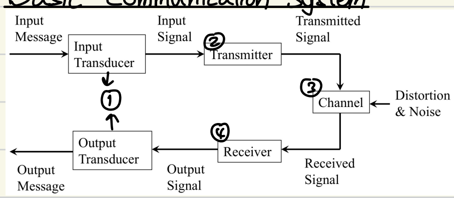
$= 10\cos[2\pi n(\frac{13}{200})] = 10\cos(\frac{13}{100}\pi n)$

$f_s = 200\text{Hz}$, $f_d = \frac{13}{200} = \frac{f_a}{f_s} \rightarrow f_a = 13\text{Hz}$ (aliased frequency) → original frequency 213Hz cannot be recovered due to undersampling

Concept of Digital Frequency

Variables	Unit	Relationship	Range
ω_a	rad/sec	$\omega_a = 2\pi f_a$	$-\infty < \omega_a < \infty$
f_a	Cycles/sec	$f_a = f_d f_s$	$-\infty < f_a < \infty$
ω_d	rad/sample	$\omega_d = 2\pi f_d$	$-\pi \le \omega_d \le \pi$
f_d	Cycles/sample	$f_d = f_a / f_s$	$-\frac{1}{2} \le f_d \le \frac{1}{2}$

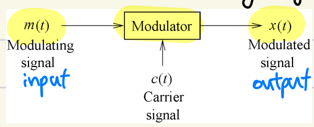
Basic Communication System



1. Input/Output Transducer → to convert message into an electrical signal & vice versa
2. Transmitter → convert electrical signal into form suitable for transmission through physical channel (modulation)
3. Channel → physical medium used to send signal from transmitter to receiver
4. Reciever → recover message signal from received signal (demodulation)

Modulation

→ process in which modulator systematically alters a carrier signal in accordance with modulating signal which represents the message



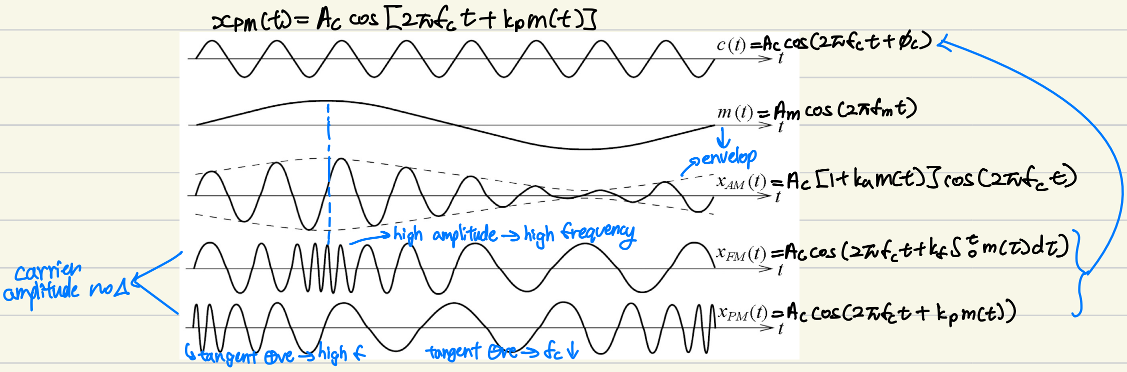
→ modulating signal, $m(t)$: message/information signal to be sent

→ modulated signal, $x(t)$: resultant signal of modulating process

→ carrier signal, $c(t)$: sinusoidal wave given by $c(t) = A_c \cos(2\pi f_c t + \phi_c)$ → carrier amplitude → carrier frequency → carrier phase angle

→ types of modulation:

1. Amplitude Modulation (AM): carrier amplitude varied with message signal
   $x_{AM}(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$
2. Frequency Modulation (FM): carrier frequency varied with message signal
   $x_{FM}(t) = A_c \cos[2\pi f_c t + k_f \int_0^t m(\tau)d\tau]$
3. Phase Modulation (PM): carrier phase angle is varied with message signal
   $x_{PM}(t) = A_c \cos[2\pi f_c t + k_p m(t)]$



↳ main purpose of modulation:

a) to shift spectral content of message signal to operating frequency band

b) to allow efficient transmission → size of antenna ≈ $\frac{\lambda}{10}$, $c = f_c \lambda$ → speed of light (3×10^8)

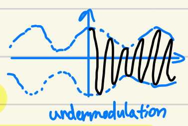
c) to permit use of multiplexing

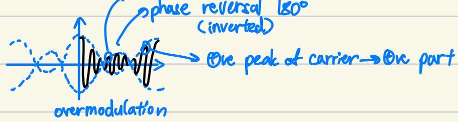
d) to provide better utilisation of radio frequency spectrum

Amplitude Modulation

$x_{AM}(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$ → must always be 1 → carrier frequency → modulating signal

constant → amplitude sensitivity factor of modulator





→ maximum absolute value of $k_a m(t)$ → modulation index μ

→ under-modulation → $|k_a m(t)| \leq 1$, for all t ✓, over-modulation → $|k_a m(t)| > 1$, for some t ✗

1. Time-Domain Description

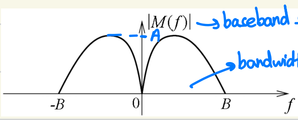
E.g. $m(t) = A_m \cos(2\pi f_m t)$. For under-modulation case, let A_{max} & A_{min} denote max & min values of positive envelope of modulate wave, $\frac{A_{max}}{A_{min}} = \frac{A_c(1+\mu)}{A_c(1-\mu)}$

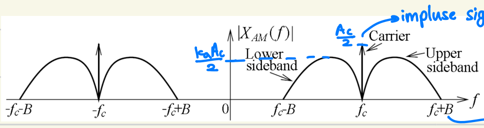
solving for $\mu \rightarrow \mu = \frac{A_{max} - A_{min}}{A_{max} + A_{min}} \rightarrow$ graph above

$\rightarrow x_{AM}(t) = A_c[1 + k_a m(t)] \cos(2\pi f_c t)$

$= A_c \cos(2\pi f_c t) + k_a A_c m(t) \cos(2\pi f_c t)$

$\rightarrow$ Performing Fourier Transform, $X_{AM}(f) = \frac{A_c}{2}[\delta(f-f_c) + \delta(f+f_c)] + \frac{k_a A_c}{2}[M(f-f_c) + M(f+f_c)]$ $\rightarrow$ $M(f)$ is FT of $m(t)$





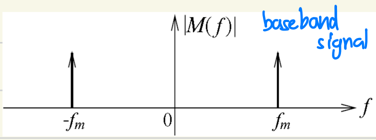
$\rightarrow$ no distortion, $f_c > B$ (in practise, $f_c >> B$)

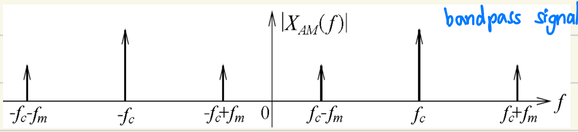
$\rightarrow$ bandwidth $= 2B$

2. Frequency-Domain Description

E.g. $m(t) = A_m \cos(2\pi f_m t) \rightarrow x_{AM}(t) = A_c[1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t)$ ($\mu \rightarrow k_a A_m$)

$X_{AM}(f) = \frac{A_c}{2}[\delta(f-f_c) + \delta(f+f_c)] + \frac{\mu A_c}{4}[\delta(f-f_c-f_m) + \delta(f-f_c+f_m)] + \frac{\mu A_c}{4}[\delta(f+f_c+f_m) + \delta(f+f_c-f_m)]$





$\rightarrow$ carrier power $\rightarrow P_c = \frac{A_c^2}{2}$

$\rightarrow$ sideband power $\rightarrow P_s = \frac{\mu^2 A_c^2}{4}$

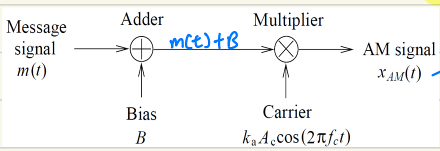
$\rightarrow$ power efficiency $\rightarrow \frac{P_s}{P_s + P_c} / \frac{\mu^2}{2 + \mu^2} \rightarrow$ since max value for μ for under-modulation is $\mu = 1$, maximum power efficiency for full AM is $\frac{1}{2+1} = 33.33\%$

Generation of AM Signal

$\hookrightarrow x_{AM}(t) = A_c[1 + k_a m(t)] \cos(2\pi f_c t)$

$= k_a[m(t) + B] A_c \cos(2\pi f_c t)$

$\hookrightarrow$ constant bias $\rightarrow B = \frac{1}{k_a}$

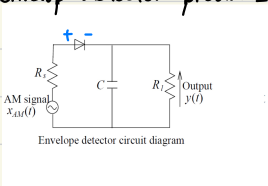


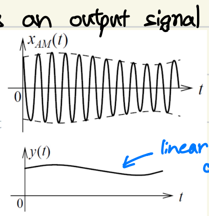
Demodulation of AM signal

$\hookrightarrow$ envelop detector commonly used to demodulate full AM signal, contains diode & resistor-capacitor filter

$\hookrightarrow$ envelop detector produces an output signal that follows envelope of input signal waveform

Received AM signal $\rightarrow$





$\leftarrow$ linear function of $m(t)$