Engineering Math 2

Matrix Notation

$\circledast$ $\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\sin(x-y) = \sin x \cos y - \cos x \sin y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$
$\cos(x-y) = \cos x \cos y + \sin x \sin y$

E.g. system of linear eqns

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x \quad\quad - x_3 &= 2 \\ x_2 - 4x_3 &= 4 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \end{array}\right]$$

argumented matrix

coefficient matrix

Elementary Row Operations

↳ Three types of ERO (performaning any will not Δ solution of linear system):

1. Interchanging any two rows, $R_i \longleftrightarrow R_j$
2. Multiplying any row by a nonzero constant, $R_j \leftarrow \alpha R_j$, $\alpha \neq 0$
3. Adding multiple of one row to another, $R_j \leftarrow R_j + \beta R_i$

} 2 linear systems are equivalent if they have same soln

Gaussian Elimination Methods (algorithm to solve linear systems)

↳ convert $m \times n$ linear system into equivalent system in triangular form → use back substitution method to get ans

E.g. $x_1 - 2x_2 + x_3 = 0$, $x_1 - x_2 = 2$, $x_2 - 4x_3 = 4$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & -4 & 4 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -3 & 3 \end{array}\right]$$ back substitution

Back substitution, $x_3 = -1$, $x_2 = \frac{1}{2}(2 + 2x_3) = 0$, $x_1 = -x_3 + 2x_2 = 1$

↳ 3 solutions:

1. Unique solution → $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -3 & 3 \end{array}\right]$ → 3 unknowns → 3 non-zero eqn
2. Many solution → $\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 3 \\ 0 & 2 & 2 & -6 & -4 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$ → 4 unknowns → 3 non-zero eqn

} consistent soln

3. No solution → $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 3 & -4 & -8 \\ 0 & 0 & 0 & 37 \end{array}\right]$ → last row all 0 & still have no. 37 (not possible)

– Many solutions

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 3 \\ 0 & 2 & 2 & -6 & -4 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \longrightarrow \begin{aligned} x_1 - 2x_2 - x_3 + x_4 &= 3 \\ 2x_2 + 2x_3 - 6x_4 &= -4 \\ x_3 - 2x_4 &= 1 \end{aligned}$$

If we choose x_4 to be free variable → by back substitution:

$x_4 = t$

$x_3 = 1 + 2x_4 = 1 + 2t$

$x_2 = \frac{1}{2}(-4 + 6x_4 - 2x_3) = -3 + t$

$x_1 = 3 - x_4 + x_3 + 2x_2 = -2 + 3t$

} can be written as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

– Row Echelon (RE) form → nonzero term in each row → pivot → all entries in a column below a pivot are zeros

below must be 0; non 0; doesn't matter no.

$$\begin{bmatrix} \triangle & \square & \square & \square & \square \\ 0 & 0 & \triangle & \square & \square \\ 0 & 0 & 0 & \triangle & \square \end{bmatrix} \qquad \begin{bmatrix} 1 & \square & 0 & 0 & \square \\ 0 & 0 & 1 & 0 & \square \\ 0 & 0 & 0 & 1 & \square \end{bmatrix}$$

doesn't matter what no.

→ unique

– Reduced Row Echelon (RE) form → each pivot is 1 → all other entries in column are 0

↳ Gauss-Jordan Elimination → process of transforming matrix to RRE form

$$\left[\begin{array}{cccc|c} 1 & -3 & 4 & 2 & 5 \\ 0 & 1 & -2 & 1 & -3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$ → work forward to Δ others to RRE using ERO

Matrix Algebra

↳ properties of matrix addition & scalar multiplication (A, B, C → $m \times n$ matrices):

a) $A + B = B + A$ / $A + (B + C) = (A + B) + C$

b) $c(A + B) = cA + cB$ / $(c + d)A = cA + dA$

c) $c(dA) = (cd)A$

→ matrix multiplication (not commutative, $AB \neq BA$)

row, column: 2×5

$$\overset{3\times 3}{A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -3 \\ -4 & 6 & 2 \end{bmatrix}} \quad \& \quad \overset{3\times 2}{B = \begin{bmatrix} 3 & -2 \\ -1 & 4 \\ 1 & 0 \end{bmatrix}}$$

* column of A = row of B for multiplication to happen

$$AB = \begin{bmatrix} (1)(3)+(3)(-1)+(0)(1) & (1)(-2)+3(4)+0(0) \\ 6+(-1)+(-3) & -4+4+0 \\ -12+(-6)+2 & 8+24+0 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ 2 & 0 \\ -16 & 32 \end{bmatrix}$$

properties: a) $A(BC) = (AB)C$

b) $c(AB) = (cA)B = A(cB)$

c) $A(B+C) = AB + AC$ / $(B+C)A = BA + CA$

Transpose of a Matrix

↳ A is $m \times n$ matrix → transpose of A (A^T) → $n \times m$ matrix (dot product of 2 column vectors → $u \cdot v = u^T v$)

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$$

→ properties: a) $(A+B)^T = A^T + B^T$

b) $(AC)^T = C^T A^T$

c) $(A^T)^T = A$

d) $(cA)^T = cA^T$

e) $A^T = A$ → A is known as a symmetric matrix

Inverse of Square Matrix

↳ let A be $n \times n$ → inverse of A (A^{-1}) if exist → $AA^{-1} = I = A^{-1}A$ (identity matrix)

→ if inverse of A exist → A is invertible / non-singular

↳ A & B invertible → AB invertible → $(AB)^{-1} = B^{-1}A^{-1}$ → $B^{-1}A^{-1}AB = I$ → $(AB)^{-1} = B^{-1}A^{-1}$

– Inverse of 2x2 Matrix (by definition)

↳ inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exist if & only if $ad - bc \neq 0$ → $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

– Inverse of nxn matrix (by Augmented Matrix) → $[A|I] \sim \ldots \sim [I|A^{-1}]$

↳ E.g. Find A^{-1} if $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{I} & \multicolumn{3}{c}{A^{-1}} \\ 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

Matrix Form of Linear System

$$\left.\begin{aligned} x_1 - 6x_2 - 4x_3 &= -5 \\ 2x_1 - 10x_2 - 9x_3 &= -4 \\ -x_1 + 6x_2 + 5x_3 &= 3 \end{aligned}\right\} Ax+b \rightarrow A = \begin{bmatrix} 1 & -6 & -4 \\ 2 & -10 & -9 \\ -1 & 6 & 5 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ b = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

$Ax = b$ → $A^{-1}Ax = A^{-1}b$ → $x = A^{-1}b$ → get x

– Homogeneous linear system (system of $Ax = 0$) → vector $x = 0$ always a solution to system & is called trivial solution (If $x \neq 0$ → infinite no. of soln → non-trival soln)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow x = t\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

Determinant of Matrix

- 2×2 matrix $\rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  $\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- 3×3 matrix

  take away 1st row & 1st column

  $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{minor}}$$

  $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightarrow$ can expand along any row/column $\rightarrow$ expand those with more 0s

  ★ need follow sign

  $\underbrace{+\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{cofactor}}$

- $n \times n$ matrix

  ↳ expand along ith row of $A \rightarrow \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$ (go through column)

  ↳ expand along jth column of $A \rightarrow \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$

  minor → determinant of $(n-1)\times(n-1)$ matrix result from removing ith row & jth column

  → expression $(-1)^{i+j} M_{ij} \rightarrow$ cofactor

- triangular matrix

  ↳ if A is $n \times n$ matrix → det of A is pdt of terms on diagonal

  $$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 7 & 9 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \rightarrow ★ \det(A) = 1 \times 2 \times 8 \times 10 = 160$$

→ Find determinant using ERO:

a) if 2 rows/columns of A are interchanged ($R_i \leftrightarrow R_j$) to produce $B \rightarrow \det(B) = -\det(A)$

b) if multiple of one row/column of A added to another to produce B ($R_j \leftarrow R_j + 2R_i$) $\rightarrow \det(B) = \det(A)$

c) if row/column multiplied by real no. α to produce B ($R_j \leftarrow \alpha R_j$) $\rightarrow \det(B) = \alpha \det(A)$

E.g. $A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ 2 & 4 & -6 & 1 \\ 0 & 3 & 9 & 2 \\ -2 & -4 & 1 & -3 \end{bmatrix} \longrightarrow \det(A) \underset{R_4 \leftarrow R_4 + R_2}{=} \begin{vmatrix} 0 & 1 & 3 & -1 \\ 2 & 4 & -6 & 1 \\ 0 & 3 & 9 & 2 \\ 0 & 0 & -5 & -2 \end{vmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} (-1)\begin{vmatrix} 2 & 4 & -6 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 3 & 9 & 2 \\ 0 & 0 & -5 & -2 \end{vmatrix}$

$\underset{R_3 \leftarrow R_3 - 3R_2}{\sim} (-1)\begin{vmatrix} 2 & 4 & -6 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -5 & -2 \end{vmatrix} \underset{R_3 \leftrightarrow R_4}{\sim} (-1)(-1)\begin{vmatrix} 2 & 4 & -6 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & 5 \end{vmatrix}$

$\det(A) = (-1)(-1)(2)(1)(-5)(5) = -50$

★ properties of determinant (A & B are $n \times n$ matrices):

a) $\det(AB) = \det(A)\det(B)$

★ b) $\det(\alpha A) = \alpha^n \det(A)$ (real no.)

★ c) $\det(A^T) = \det(A)$

d) if A has a row/column of all 0s, A has 2 equal row/column, A has a row/column multiple of another $\rightarrow \det(A) = 0$

e) square matrix is invertible if & only if $\det(A) \neq 0$

Adjoint Matrix & Inverse

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

↳ adjoint matrix

E.g. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (delete 1st row & 1st column)

$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} (+1)M_{11} & (-1)M_{12} \\ (-1)M_{21} & (+1)M_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

★ - $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$

Elementary Matrices

↳ any matrix that can be obtained from identity matrix by performing a single ERO

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_2(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{13}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}$$

$\det(E_{13}) = -1$ — $R_1 \leftrightarrow R_3$

$\det(E_2(\alpha)) = \alpha$ — $R_2 \leftarrow \alpha R_2, \alpha \neq 0$

$\det(E_{13}(\beta)) = 1$ — $R_3 \leftarrow R_3 + \beta R_1$

↳ $A \overset{R}{\sim} B \longleftrightarrow B = EA$ (elementary matrix). If we perform a series of ERO $\longrightarrow B = E_k \cdots E_2 E_1 A$

E.g. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \\ -1 & 1 & 1 \end{bmatrix} \underset{\sim}{E_1 : R_2 \leftarrow R_2 - 3R_1} [\cdot] \underset{\sim}{E_2 : R_3 \leftarrow R_3 + R_1} [\cdot] \underset{\sim}{E_3 : R_3 \leftarrow R_3 + 3R_2} B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 9 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \longleftarrow$ get from doing respective ERO on identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$B = E_3 E_2 E_1 A$

– $\det(E_3 E_2 E_1 A) = \det(E_3)\det(E_2)\det(E_1)\det(A) \longrightarrow \det(A) = \dfrac{\det(B)}{\det(E_3)\det(E_2)\det(E_1)}$

from identity matrix

– elementary matrices are invertible → when converting it to E^{-1}, just inverse the ERO → $R_1 \leftarrow R_1 - 2R_2$ to $R_1 \leftarrow R_1 + 2R_2$

LU Factorisation (Not unique)

↳ some cases, a $m \times n$ matrix A can be written as $A = LU$ → L is a lower triangular matrix & U is an upper triangular matrix → LU factorisation of A

↳ can solve linear system $Ax = b$ using forward & back substitution

$Ax = b \Rightarrow LUx = b \begin{cases} Ly = b \rightarrow \triangle \\ Ux = y \rightarrow \bigtriangledown \end{cases}$

E.g. Solve $Ax = b \longrightarrow \begin{bmatrix} 3 & 6 & -3 \\ 6 & 15 & -5 \\ -1 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix}$

$A = \begin{bmatrix} 3 & 6 & -3 \\ 6 & 15 & -5 \\ -1 & -2 & 6 \end{bmatrix} \underset{\sim}{E_1 : R_1 \leftarrow \tfrac{1}{3}R_1} \begin{bmatrix} 1 & 2 & -1 \\ 6 & 15 & -5 \\ -1 & -2 & 6 \end{bmatrix} \underset{E_3 : R_3 \leftarrow R_3 + R_1}{\overset{E_2 : R_2 \leftarrow R_2 - 6R_1}{\sim}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} = U$

$E_3 E_2 E_1 A = U \longrightarrow A = \underbrace{(E_3 E_2 E_1)^{-1}}_{L} U$

$L = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_3 \leftarrow R_3 - R_1}{\overset{R_1 \leftarrow 3R_1,\; R_2 \leftarrow R_2 + 6R_1}{\sim}} \begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

each row

$LUx = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{y} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix} \Rightarrow \begin{cases} Ly = b \\ Ux = y \end{cases}$

Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow Ly = b$

$\begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$

$Ux = y \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

↳ PLU (permutation) Factorisation → sometimes need to interchange rows to reduce A to LU → $A = (E_3 E_2 E_1)^{-1} U = \underset{E_1 : R_1 \leftrightarrow R_3}{E_1^{-1}} \underbrace{(E_2^{-1} E_3^{-1})}_{L} U = PLU$

Vectors in $\mathbb{R}^n$

↳ $\mathbb{R}^n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ → entries of vector → components of vector

↳ algebratic properties: a) commutative: $u + v = v + u$, associative: $(u + v) + w = u + (v + w)$

b) addictive identity: vector 0 satisfies $0 + u = u + 0 = u$

c) additive inverse: for every u, vector $-u$ satisfies $u + (-u) = 0$

– Linear Combination → to determine if a set of vectors is linear combi of another → just check if have solution

Spanning Set

↳ set of all linear combinations of $v_1, v_2, \dots v_k$ / span(S)

E.g. Let $u = \begin{bmatrix}1\\0\\3\end{bmatrix}$ & $v = \begin{bmatrix}-1\\1\\-3\end{bmatrix}$. What is span(u,v)?

Let $[x\ y\ z]^T$ be a vector in span(u,v),

$$c_1\begin{bmatrix}1\\0\\3\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\3\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix} \longrightarrow \left[\begin{array}{cc|c}1 & -1 & x\\0 & 1 & y\\3 & -3 & z\end{array}\right] \sim \left[\begin{array}{cc|c}1 & -1 & x\\0 & 1 & y\\0 & 0 & z-3x\end{array}\right] \rightarrow \text{must be 0 to have sol}^{\underline{n}}$$

∴ span(u,v) is plane $z - 3x = 0$ → vector does not span R^3 but instead a plane

Show span$\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right) = R^3$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & x\\1 & 0 & 1 & y\\1 & 2 & 0 & z\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 1 & x\\0 & -1 & 0 & y-x\\0 & 0 & 1 & z+y-2x\end{array}\right]$$ has solution for x, y, z → if given x, y, z can solve for c_1, c_2, c_3

Linear Independence

↳ set of vectors $S = \{v_1, v_2, v_3 \dots, v_m\}$ in R^n is linearly independent provided $c_1 = c_2 = \dots = c_m = 0$ when $c_1v_1 + c_2v_2 + \dots + c_mv_m = 0$, else linearly dependent

E.g. $\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right]$ → $c_1 = c_2 = c_3 = 0$, linearly independent

$\left[\begin{array}{cccc|c}1 & -1 & -2 & 2 & 0\\0 & 1 & 3 & 1 & 0\\0 & 0 & -7 & -7 & 0\end{array}\right]$ → $-7c_3 - -7c_4 = 0$ → many solution → linearly dependent

→ two vectors u & v are orthongonal/perpendicular if $u^Tv = 0$

→ linear system $Ax = b$ is consistent if & only if b can be expressed as linear combi of column vector of A

$Ax + b = [v_1\ v_2 \dots v_n]\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = b$

Null Space, Column Space, Row Space of Matrix

Let $A = \begin{bmatrix}-1 & 0 & 3 & 1\\2 & 3 & -3 & 1\\2 & -2 & -2 & -1\end{bmatrix}$,

→ column space → linear combi of column vectors [col(A) = span$\left(\begin{bmatrix}-1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\3\\-2\end{bmatrix}, \begin{bmatrix}3\\-3\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right)$]

→ row space → linear combi of row vectors [row(A) = span$\left(\begin{bmatrix}-1\\0\\3\\1\end{bmatrix}, \begin{bmatrix}2\\3\\-3\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\-2\\-1\end{bmatrix}\right)$]

↳ null space → set of vectors in R^n such that $Ax = 0$

$$\left[\begin{array}{cccc|c}-1 & 0 & 3 & 1 & 0\\2 & 3 & -3 & 1 & 0\\2 & -2 & -2 & -1 & 0\end{array}\right] \sim \begin{bmatrix}1 & 0 & -3 & -1\\0 & 1 & 1 & 1\\0 & 0 & 2 & 1\end{bmatrix} \longrightarrow 2x_3 = -x_4$$

Let $x_3 = \alpha$, $x_4 = -2\alpha$, $x_2 = -x_3 - x_4 = \alpha$, $x_1 = 3x_3 + x_4 = \alpha$, $x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \alpha\begin{bmatrix}1\\1\\1\\-2\end{bmatrix}$

E.g. Determine whether b is in col(A). → same as whether B can be written as linear combi of A

$c_1\begin{bmatrix}1\\-1\\2\end{bmatrix} + c_2\begin{bmatrix}-1\\2\\-2\end{bmatrix} + c_3\begin{bmatrix}-2\\3\\-2\end{bmatrix} = \begin{bmatrix}3\\3\\-2\end{bmatrix}$ has sol$^{\underline{n}}$, then b in col(A)

Subspaces

↳ subspace of R^n is any collection S of vectors in R^n such that the ① zero vector is in S, ② S is closed under linear combination

E.g. Show that $W = \left\{\begin{bmatrix}a\\a+1\end{bmatrix} \middle| a \in \mathbb{R}\right\}$ is not a subspace.

Consider two vectors in W, $u = \begin{bmatrix}a_1\\a_1+1\end{bmatrix}$, $v = \begin{bmatrix}a_2\\a_2+1\end{bmatrix}$,

since $u + v = \begin{bmatrix}a_1+a_2\\a_1+a_2+2\end{bmatrix}$ not in W → W is not a subspace

format different, should be 1

W is also not subspace since zero vector not in W → when sub $a=0$ $\begin{bmatrix}0\\1\end{bmatrix}$, not $\begin{bmatrix}0\\0\end{bmatrix}$

↳ every line & plane that does not pass through origin in R^n is not subspace in R^n

Basis

- B is a basis for V provided ① B is a linearly independent set of vectors in V, ② span(B) = V
- steps to finding a basis from a set of vectors:
  1. Form a matrix whose column/rows are vectors
  2. Reduce matrix to RE form
  3. Vectors that correspond to pivoting column/non-zero rows of RE matrix form basis for span(S)

E.g. Let $S = \left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}, \begin{smallmatrix}0\\1\\1\end{smallmatrix}, \begin{smallmatrix}1\\1\\2\end{smallmatrix}, \begin{smallmatrix}1\\2\\1\end{smallmatrix}, \begin{smallmatrix}-1\\1\\-2\end{smallmatrix}\right]$. Find basis for span of S.

$$\begin{bmatrix}1&0&1&1&-1\\0&1&1&2&1\\1&1&2&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&1&1&-1\\0&1&1&2&1\\0&0&0&-2&-2\end{bmatrix}$$

row space $A \sim \begin{bmatrix}1&2&3&4\\0&4&8&12\\0&0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{bmatrix}$ } non-zero rows

$\therefore$ basis is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$

Dimension, Rank & Nullity

- dimension → S is subspace of $\mathbb{R}^n$, no. of vectors in basis for S → dimension of S, dim(S)
- rank → dimension of no. of independent row/column, rank(A)

  properties of rank: a) rank(AB) ≤ rank(B)

  b) rank(AB) = rank(B), if A^{-1} exists, rank(AB) = rank(A), if B^{-1} exists

  c) rank(A+B) ≤ rank(A) + rank(B)

  d) rank(A) ≤ min(m, n), where A is m×n matrix　　rank(A) ≤ min(3, 5) = 3
- nullity → dimension of null space, no. of free variable if you solve $Ax = 0$ problem
- Rank Theorem → If A is m×n matrix → rank(A) + nullity(A) = n (column)

E.g. Find the rank & nullity of matrix $A = \begin{bmatrix}1&2&3&4\\5&6&7&8\\15&18&21&24\end{bmatrix}$

ERO → $A \sim \begin{bmatrix}1&2&3&4\\0&-4&-8&-12\\0&0&0&0\end{bmatrix}$, rank(A) = 2 → n = 4 = rank(A) + nullity(A) → nullity = 2

Eigenvalues & Eigenvectors

Let A be n×n matrix,

- number λ → eigenvalue of A if there exists a nonzero vector v in $\mathbb{R}^n$ such that $Av = \lambda v$
- every nonzero vector v satisfy the eqn → eigenvector of A corresponding to eigenvalue λ

Steps to finding eigenvalues & eigenvectors, e.g. for matrix $A = \begin{bmatrix}0&1\\1&0\end{bmatrix}$:

1. Find eigenvalues → use $\det(\lambda I - A) = 0$ (I → identity matrix)

$$\det(\lambda I - A) = \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix} - \begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}\lambda&-1\\-1&\lambda\end{bmatrix} \overset{ad-bc}{=} \lambda^2 - 1 = 0 \rightarrow \lambda = \pm 1$$

2. Find corresponding eigenvectors by subing in eigenvalues found earlier

sub $\lambda_1 = 1$, solve for v_1 → $(\lambda_1 I - A)v_1 = 0$ → $\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$ → $x_1 = x_2$ → $v_1 = \begin{bmatrix}1\\1\end{bmatrix}$ (assume $x=1$)

sub $\lambda_2 = -1$, solve for v_2 → $(\lambda_2 I - A)v_2 = 0$ → $\begin{bmatrix}-1&-1\\-1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$ → $x_1 = -x_2$ → $v_2 = \begin{bmatrix}-1\\1\end{bmatrix}$

Diagonalisation

- n×n matrix A is diagonalisable if & only if A has n linearly independent eigenvectors
- $A = PDP^{-1}$ (inverse should exist) → D is a diagonal matrix, diagonal entries of D are eigenvalues of A & column vector of P are eigenvectors

E.g. Diagonalise $A = \begin{bmatrix}1&0&0\\6&-2&0\\7&-4&2\end{bmatrix}$, eigenvalues & eigenvectors of A: $\lambda_1 = 1, v_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $\lambda_2 = -2, v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, $\lambda_3 = 2, v_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

$P = \begin{bmatrix}1&0&0\\2&1&0\\1&1&1\end{bmatrix}$, $D = \begin{bmatrix}1&0&0\\0&-2&0\\0&0&2\end{bmatrix}$ & $A = PDP^{-1}$

$P = \begin{bmatrix}1&1&1\\0&0&3\\0&0&9\end{bmatrix}$ → not diagonalisable (linearly dependent)

Application: System of Linear Differential Equations

E.g. Solve following differential equation $\frac{du}{dt} = Au$ where $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ & u is a vector of appropriate dimensions.

– Solution for $\dot{x}(t) = \lambda x(t)$:

$$\frac{dx}{dt} = \lambda x \longrightarrow \int \frac{1}{x}\, dx = \int \lambda \, dt \longrightarrow \ln x = \lambda t + C_1 \longrightarrow x(t) = e^{\lambda t + C_1} = Ce^{\lambda t} \longrightarrow x(0) = C$$

$$x(t) = e^{\lambda t} x(0)$$

– Solution for diagonalising matrix A: $A = PDP^{-1}$

$$\frac{du}{dt} = Au = PDP^{-1}u$$

$$\text{let } w = P^{-1}u,\ \dot{w} = P^{-1}\dot{u} = \underbrace{P^{-1}P}_{\text{identity matrix}}DP^{-1}u \longrightarrow \dot{w} = Dw$$

$P^{-1}u$

same $\left(\frac{du}{dt} = PDP^{-1}u\right)$

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 w_1 \\ \lambda_2 w_2 \\ \lambda_3 w_2 \end{bmatrix} \longrightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & w_1(0) \\ e^{\lambda_2 t} & w_2(0) \\ e^{\lambda_3 t} & w_3(0) \end{bmatrix}$$

$$u = Pw = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} \begin{bmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{bmatrix}$$

$$u(t) = Pe^{Dt}P^{-1}u(0)$$

Step 1: Solve & get eigenvalues of A

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 1 \\ 0 & -1 & \lambda \end{bmatrix} \rightarrow \text{expand}$$

$$\det(\lambda I - A) = \lambda \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & \lambda \end{bmatrix}$$

$$= \lambda(\lambda^2 + 1) - (-\lambda - 0) = \lambda^3 + \lambda + \lambda = \lambda^3 + 2\lambda = \lambda(\lambda^2 + 2)$$

$$\lambda(\lambda^2 + 2) = 0 \longrightarrow \lambda_1 = 0,\ \lambda_{2,3} = \pm i\sqrt{2}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i\sqrt{2} & 0 \\ 0 & 0 & -i\sqrt{2} \end{bmatrix}$$

Step 2: Solve & get eginvectors → cannot be pivot as $\lambda = 0$

$$(\lambda I - A)v = \begin{bmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 1 \\ 0 & -1 & \lambda \end{bmatrix} v = 0$$

$$\begin{bmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 1 \\ 0 & -1 & \lambda \end{bmatrix} \sim \begin{bmatrix} -1 & \lambda & 1 \\ \lambda & 1 & 0 \\ 0 & -1 & \lambda \end{bmatrix} \underset{\sim}{R_2 \leftarrow R_2 + \lambda R_1} \begin{bmatrix} -1 & \lambda & 1 \\ 0 & 1+\lambda^2 & \lambda \\ 0 & -1 & \lambda \end{bmatrix} \underset{\sim}{R_3 \leftarrow R_3 + \frac{1}{1+\lambda^2} R_2} \begin{bmatrix} -1 & \lambda & 1 \\ 0 & 1+\lambda^2 & \lambda \\ 0 & 0 & \lambda + \frac{\lambda}{1+\lambda^2} \end{bmatrix},\ \lambda = 0, i\sqrt{2}, -i\sqrt{2}$$

For $\lambda_1 = 0$: → sub in

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \rightarrow x_1 \\ \rightarrow x_2 \\ \rightarrow \text{value of } x_3 \end{matrix}$$

choose $x_3 = 1$

For $\lambda_{2,3}$: → $\pm i\sqrt{2}$

$$\begin{bmatrix} -1 & \lambda_{2,3} & 1 \\ 0 & -1 & \lambda_{2,3} \\ 0 & 0 & 0 \end{bmatrix} v_{2,3} = 0 \longrightarrow v_{2,3} = \begin{bmatrix} -1 \\ \lambda_{2,3} \\ 1 \end{bmatrix} \begin{matrix} \rightarrow -x_1 - 2 + 1 = 0 \rightarrow x_1 = -1 \\ \rightarrow \text{either } \pm i\sqrt{2} \end{matrix}$$

$$P = [v_1, v_2, v_3] = \begin{bmatrix} 1 & -1 & -1 \\ 0 & i\sqrt{2} & -i\sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$$

Step 3: Solve based on $u = Pe^{Dt}P^{-1}u(0)$

$$u = Pe^{Dt}P^{-1}u(0) = [v_1\ v_2\ v_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\sqrt{2}t} & 0 \\ 0 & 0 & e^{-i\sqrt{2}t} \end{bmatrix} P^{-1}u(0)$$

$$= [v_1\ \ e^{i\sqrt{2}t}v_2\ \ e^{-i\sqrt{2}t}v_3] P^{-1}u(0)$$

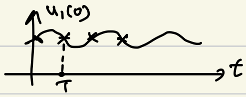
$$= c_1 v_1 + c_2 e^{i\sqrt{2}t} v_2 + c_3 e^{-i\sqrt{2}t} v_3$$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & i\sqrt{2} & -i\sqrt{2} & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \frac{1}{4}\begin{bmatrix} 2 & 0 & 2 \\ -1 & -i\sqrt{2} & 1 \\ -1 & i\sqrt{2} & 1 \end{bmatrix} \leftarrow \text{by EROs},\quad P^{-1}u(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

→ Find a time $u(t)$ is equal to initial value $u(0)$

$$\hookrightarrow e^{\pm i\sqrt{2}t} = \cos\sqrt{2}t \pm i\sin\sqrt{2}t$$

$$T = \frac{2\pi}{\omega} \longrightarrow T = \frac{2\pi}{\sqrt{2}} / \frac{\pi}{\sqrt{2}} \ \ldots$$



$$\sin \omega t \rightarrow \omega = 2\pi f \rightarrow T = \frac{2\pi}{\omega}$$

Application: Long Term Behaviour of Stable Dynamic System

E.g. Group of insurance plans allow 3 different options for participants, plan A, B & C. Percentage of participants enrolled in each is 25%, 30%, 45% respectively. 15% & 10% of participants who originally enrolled in plan A will switch to plan B & C. 25% & 30% from plan B to plan A & C. 20% & 40% from plan C to plan A & B. Construct model for this system & determine in long term, percentage of enrollment in each plan.

$$x_k = \begin{bmatrix} A \\ B \\ C \end{bmatrix}_k, \quad k = 0, 1, \dots$$

remaining percentage left

use to enroll in A, now enroll in B

$$x_{k+1} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}_{k+1} = \begin{bmatrix} 0.75 & 0.25 & 0.20 \\ 0.15 & 0.45 & 0.40 \\ 0.10 & 0.30 & 0.40 \end{bmatrix} x_k$$

$$x_{k+1} = T x_k$$

$$x = \lim_{k\to\infty} x_k \longrightarrow \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} T x_k$$

$$X = TX \longrightarrow X - TX = 0$$

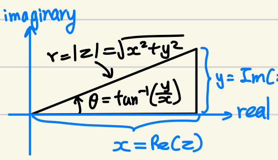
$$X(I - T) = 0$$

$$I - T = \begin{bmatrix} 0.25 & -0.25 & -0.2 \\ -0.15 & 0.55 & -0.4 \\ -0.10 & -0.3 & 0.6 \end{bmatrix} \overset{ERO}{\sim} \begin{bmatrix} 20 & 0 & -42 \\ 0 & -20 & 26 \\ 0 & 0 & 0 \end{bmatrix}$$

Assume $C = x_3 = 1$, $B = x_2 = 1.3$, $A = x_1 = 2.1$

$$x = \begin{bmatrix} 2.1 \\ 1.3 \\ 1.0 \end{bmatrix} \frac{1}{2.1 + 1.3 + 1.0} \times 100\% = \begin{bmatrix} 47.73\% \\ 29.54\% \\ 22.73\% \end{bmatrix}$$

Complex Numbers



- $i^2=-1 \longrightarrow i=\sqrt{-1}$
- rectangular, polar: $z=x+iy=re^{i\theta}=r(\cos\theta+i\sin\theta)$
- $\theta=Arg(z)+2n\pi,\ n=0,\pm1,\pm2$ → principal value of θ, $-\pi<\theta\leq\pi$

1. Euler's Formula

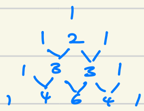
- $e^{i\theta}=\cos\theta+i\sin\theta,\ e^{-i\theta}=\cos\theta-i\sin\theta \Rightarrow \cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2},\ \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$
- easier to do in polar form:
  - $z_1z_2=r_1r_2\angle(\theta_1+\theta_2)$
  - $\frac{z_1}{z_2}=\frac{r_1\angle\theta_1}{r_2\angle\theta_2}=\frac{r_1}{r_2}\angle(\theta_1-\theta_2)$
- properties of complex conjugate: $z\bar{z}=x^2+y^2=|z|^2,\ \frac{z_1}{z_2}=\frac{z_1\bar{z}_2}{|z_2|^2},\ \overline{(z_1\pm z_2)}=\bar{z}_1\pm\bar{z}_2,\ \overline{z_1z_2}=\bar{z}_1\bar{z}_2,\ \overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}$

2. De Moivre's Formula

- $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

E.g. Express $\cos^4\theta$ in terms of multiples of θ.

Since $e^{i\theta}+e^{-i\theta}=2\cos\theta$,



$2^4\cos^4\theta=(e^{i\theta}+e^{-i\theta})^4$ → follow pascal's △: $(x+y)^4=1x^4y^0+4x^3y^1+6x^2y^2+4x^1y^3+1x^0y^4$

$=e^{i4\theta}+e^{-i4\theta}+4(e^{i2\theta}+e^{i2\theta})+6$

$=2\cos4\theta+4(2\cos2\theta)+6$

$\cos^4\theta=\frac{1}{8}[\cos4\theta+4\cos2\theta+3]$

Roots of Complex Numbers

- $w_k=\sqrt[n]{z}=\sqrt[n]{r}\angle\left(\frac{\theta+2k\pi}{n}\right),\ k=0,1,\dots,(n-1)$

E.g. Find values of $\sqrt[3]{-8i}$.

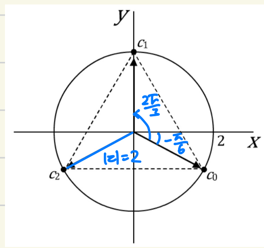
$-8i=8\angle\left(\frac{-\pi}{2}+2k\pi\right),\ k=0,\pm1,\pm2$

$w_k=\sqrt[3]{8}\angle\left(\frac{-\pi}{6}+\frac{2k\pi}{3}\right),\ k=0,1,2$

3 answer

values $=2\angle-\frac{\pi}{6},\ 2\angle-\frac{\pi}{6}+\frac{2\pi}{3},\ 2\angle-\frac{\pi}{6}+\frac{4\pi}{3},\ 2\angle-\frac{\pi}{6}+\frac{6\pi}{3}$

secondary rotation

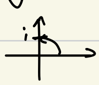


principal root $=2\angle\left(\frac{-\pi}{6}\right)=\sqrt{3}-i$

Complex Logarithm & General Power

- ① $y=z^c \longrightarrow y=e^{\ln z^c}=$ ② $e^{c\ln z}$, $z\neq0 \longrightarrow c\ln z=\ln e^{c\ln(re^{i\theta})}$ ③

E.g. Find all values of i^i & show that they are all real. Hence, find all z such that z^i is real.



$y=i^i=e^{i\ln i}$

$\ln y=\ln e^{i\ln(1e^{i(\frac{\pi}{2}+2k\pi)})}$

$\ln y=i\ln e^{i(\frac{\pi}{2}+2k\pi)}$

$\ln y=-(\frac{\pi}{2}+2k\pi)$

$y=e^{-(\frac{\pi}{2}+2k\pi)},\ k=0,\pm1,\pm2$, which is real-valued

Let $y=z^i$

$\ln y=i\ln z$

$=i\ln[re^{i(\theta+2k\pi)}],\ k=0,\pm1,\pm2\dots$

$=i(\ln r+i(\theta+2k\pi))$

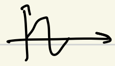
$=i\ln r-(\theta+2k\pi)$

$y=e^{i\ln r}\,e^{\underbrace{-(\theta+2k\pi)}_{\text{real}}}$

$e^{i \ln r} = \cos(\ln r) + i\sin(\ln r)$

For $y = z^i$ to be real, $e^{i\ln r}$ have to real, $\sin(\ln r) = 0$

$\ln r = \pm k\pi$



$r = e^{\pm k\pi}$

$z^i = (re^{i\theta})^i = (e^{\pm k\pi} e^{i\theta})^i$ is real

$\therefore$ Values of $z = e^{\pm k\pi} \cdot e^{i\theta}$, $k = 0, 1, 2 \ldots$

Limit of complex numbers

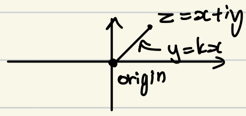
→ function near z_0, do they exist?

↳ function $f(z)$ said to have limit L as z approaches point z_0 if ① $f(z)$ exist in neighbourhood of z_0, ② $f(z)$ approaches same complex number L as $z \to z_0$ from all direction within its neighbourhood → As $z \to 0$, does function goes towards L?

E.g. Does limit of $f(z)$ at origin exist for $f(z) = \frac{x^2 y}{x^3 + y^3} + ixy$

For limit to exist, $\lim_{z\to z_0} f(z)$ need to be unique & independent of directions to which z approaches z_0.

Let direction that approach origin be $y = kx$, k constant → indicate direction



$\lim_{z\to 0} f(z) = \lim_{x\to 0, y = kx} f(z)$ ← replace y in $f(z)$

$= \lim_{x \to 0} \frac{kx^3}{x^3 + k^3 x^3} + ikx^2$

$= \frac{k}{1+k^3}$ → depend on k (direction x, y approach origin)

↳ if independent → have limit

$\therefore$ no limit

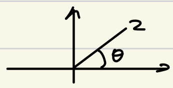
E.g. Does limit of $f(z)$ at origin exist for $f(z) = \frac{\bar{z}}{z} - \frac{z}{\bar{z}} - \frac{z^2}{\bar{z}^2}$?

Let $z = re^{i\theta}$

$\lim_{z\to 0} f(z) = \lim_{r\to 0} f(z)$

$= \lim_{r \to 0} \left[\frac{\cancel{r}e^{-i\theta}}{\cancel{r}e^{i\theta}} - \frac{\cancel{r}e^{i\theta}}{\cancel{r}e^{-i\theta}} - \frac{\cancel{r^2}e^{i2\theta}}{\cancel{r^2}e^{-i2\theta}} \right]$

$= e^{-i2\theta} - e^{i2\theta} - e^{i4\theta}$



$\therefore$ no limit → depend on θ

Continuity of complex numbers

↳ function $f(z)$ continuous at $z = z_0$ if ① have $f(z)$ at z_0, ② $\lim_{z\to z_0} f(z) = L$, ③ $f(z_0) = L$

E.g. Let $f(0) = 0$, for $z \neq 0$, $f(z) = \frac{\mathrm{Re}(z^2)}{|z^2|}$. Determine if $f(z)$ is continuous at origin. → $z = 0$

$\lim_{z\to 0} \frac{\mathrm{Re}(z^2)}{|z^2|} = \lim_{z\to 0} \frac{x^2 - y^2}{x^2 + y^2} = \begin{cases} \lim_{x\to 0, y = 0} \frac{x^2}{x^2} = 1 \\ \lim_{y \to 0, x = 0} \frac{-y^2}{y^2} = -1 \end{cases}$

$\therefore f$ not continuous at origin → ① $f(z)$ exist → $f(0) = 0$ → ② no limit → ②, ③ not fulfilled

E.g. For $f(z) = \begin{cases} \mathrm{Im}\left[\frac{z}{|z|}\right], & z \neq 0 \\ 0, & z = 0 \end{cases}$, determine if $f(z)$ is continuous at $z = 0$, $z = 5$ & $z = 5 + i$

For $z = 0$, $f(0) = 0$

$\lim_{z\to 0} f(z) = \lim_{r \to 0} \mathrm{Im}\left[\frac{\cancel{r}e^{i\theta}}{\cancel{r}}\right]$

$= \sin\theta$

$\therefore$ limit does not exist → $f(z)$ not continuous at $z = 0$

For $z = 5$, $f(5) = 0$

$\lim_{z\to 5} f(z) = \lim_{r\to 0} \mathrm{Im}\left[\frac{z_0 + re^{i\theta}}{|z_0 + \cancel{re^{i\theta}}|}\right]$ $=0$?

$= \lim_{r\to 0} \frac{r \sin\theta}{5}$

$= 0$

$\therefore \lim_{z\to 5} f(z) = f(5)$ → $f(z)$ at $z = 5$ is continuous

For $z=5+i$, $f(5+i) = \text{Im}\left[\frac{5+i}{|5+i|}\right] = \frac{1}{\sqrt{5^2+1^2}} = \frac{1}{\sqrt{26}}$

$\lim_{z\to z_0} f(z) = \lim_{r\to 0} \text{Im}\left[\frac{z_0+re^{i\theta}}{|z_0+re^{i\theta}|}\right]$

$= \lim_{r\to 0} \text{Im}\left[\frac{5+i+re^{i\theta}}{|5+i+\underbrace{re^{i\theta}}_{=0}|}\right]$

$= \frac{1}{|5+i|} = \frac{1}{\sqrt{26}}$

$\therefore \lim_{z\to z_0} f(z) = f(z_0)$, $z_0 = 5+i$, function is continuous at $z=5+i$

$\sin\theta = \cos(90°-\theta)$	$\cos(-\theta) = \cos\theta$
$\cos\theta = \sin(90°-\theta)$	$\sin(-\theta) = -\sin\theta$
$\tan\theta = \frac{1}{\tan(90°-\theta)}$	$\tan(-\theta) = -\tan\theta$

Derivates of complex functions

↳ $f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$

E.g. Discuss differentiability of $\bar{z}$.

Let $f(z) = \bar{z}$,

$f'(z) = \lim_{\Delta z\to 0} \frac{\overline{z+\Delta z}-\bar{z}}{\Delta z}$

Using property $\overline{z+\Delta z} = \bar{z}+\overline{\Delta z}$,

$f'(z) = \lim_{\Delta z\to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z\to 0} \frac{\Delta r e^{-i\theta}}{\Delta r e^{i\theta}}$

$= e^{-i2\theta}$

$\therefore$ limit depends on $\theta \to$ limit does not exist $\to f(z)=\bar{z}$ is not differentiable anywhere.

Analytic Functions

- function $f(z)$ said to be analytic in domain D if it is analytic at each point in D (neighbourhood of z)
- point $z=z_0$, where $f(z)$ is analytic $\to$ called singular point/singularity of $f(z)$
- analyticity implies differentiability & continuity
- Cauchy-Riemann (C-R) Equations (to test analyticity of complex function):

  complex function $f(z) = u(x,y)+iv(x,y)$ is analytic at pt z_0 for every point in neighbourhood of z_0

  1. u, v & their partial derivatives (u_x, u_y, V_x, V_y) exist & are continuous

  Cauchy-Riemann 2. $U_x = V_y$ & $V_x = -U_y$ satisfied

  U_x, V_x, U_y, V_y; $=\ominus$ve

  U_r, V_r, U_θ, V_θ; $=-\frac{1}{r}$, $=\frac{1}{r}$

  $\to$ When $z\neq 0$, C-R eqn in polar coordinates are: $U_r = \frac{1}{r}V_\theta$ & $V_r = -\frac{1}{r}U_\theta$

- if $f(z) = u(x,y)+iv(x,y)$ & $f'(z)$ exists:

  $f'(z) = U_x + iV_x = U_x - iU_y$

  $= V_y - iU_y = V_y + iV_x$

- some common & important functions:

  1. polynomials, $f(z) = C_0 + C_1 z + C_2 z^2 + \ldots + C_n z^n$ are analytic in entire complex plane
  2. rational functions, quotient of two polynomials, $f(z) = \frac{g(z)}{h(z)}$ are analytic except at points where $h(z)=0$
  3. partial fractions, $f(z) = \frac{c}{(z-z_0)^m}$, where c & z_0 are complex & m is integer are analytic except at z_0

– Steps to solve C-R eqn problems:

1. Decide form of z
2. Use form to simplify function to rectangular form
3. Compute partial derivative & check the 2 eqn
4. At which pt/region it is satisfied in
5. analytic $\to$ C-R true in neighbourhood, differentiability $\to$ C-R eqn true at point

E.g. Find $f'(z)$, derivative of $f(z)=2xy-ix^2$. State clearly point/points where $f'(z)$ exist.

$f(z)=2xy-ix^2 \rightarrow u(x,y)=2xy,\ V(x,y)=-x^2$

$u_x=2y \quad V_x=-2x$

$u_y=2x \quad V_y=0$

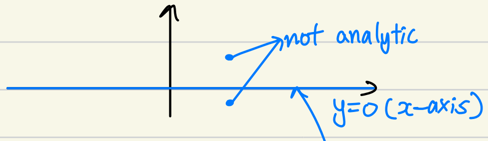
for C-R to be satified,

$\left.\begin{array}{l} u_x=V_y \\ V_x=-u_y \end{array}\right\} y=0 \ (x\text{-axis})$

$f'(z)$ exist only at x-axis

$f'(z)=u_x+iV_x=2y-i2x$

$=-i2x$



$\therefore$ not analytic cause not every point in neighbourhood, only differentiable on line

Complex Integration

- complex definite integral/line integral, $\int_C f(z)\,dz \rightarrow$ integration done along the curve C (in given direction). contour/path of integration
- If C is closed contour $\rightarrow$ complex line integral denoted by $\oint_C f(z)\,dz$
- contour/path of integration on complex plane $\rightarrow z(t)=x(t)+iy(t)$, $a\leq t\leq b$
- steps to solve contour integration:

1. $C: z(t)=x(t)+iy(t)$, $a\leq t\leq b$
2. $f(z)$ on C: $f[x(t)+iy(t)]$

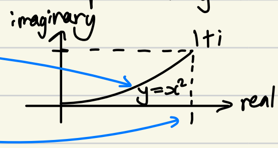
   $dz=\frac{dz}{dt}\times dt$
3. $\int_C f(z)\,dz=\int_a^b f(z(t))\,\frac{dz}{dt}\,dt$

E.g. Evaluate $\int_C f(z)\,dz$ where $f(z)=\mathrm{Re}[z]$, C the parabola $y=x^2$ from 0 to $1+i$

$f(z)=\mathrm{Re}[z]=\mathrm{Re}[x+iy]$

$z(t)=t+it^2 \quad y=t^2 \qquad 0\leq t\leq 1$ same as x

$\frac{dz}{dt}=1+i2t$

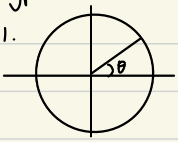


$\int_C f(z)\,dz=\int_0^1 t(1+i2t)\,dt=\int_0^1 t+i2t^2\,dt$

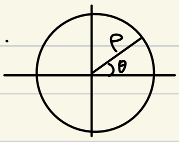
$=\left[\frac{t^2}{2}+\frac{i2t^3}{3}\right]_0^1$

$=\left(\frac{1}{2}+\frac{i2}{3}\right)=\frac{1}{2}+i\frac{2}{3}$

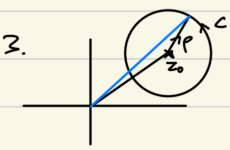
- types of closed contour around circle:

1. 

$z(\theta)=1e^{i\theta}$, $0\leq\theta\leq 2\pi$

2. 

$z(\theta)=\rho e^{i\theta}$, $0\leq\theta\leq 2\pi$

3. 

$z(\theta)=z_0+\rho e^{i\theta}$, $0\leq\theta\leq 2\pi$

E.g. Evaluate $\int_C (z-z_0)^m\,dz$, where C is a counter-clockwise circle of radius ρ with centre at z_0.

$z(\theta)=z_0+\rho e^{i\theta}$, $0\leq\theta\leq 2\pi$

$\frac{dz}{d\theta}=i\rho e^{i\theta}$

$(z-z_0)^m=\rho^m e^{im\theta}$

$\int_C (z-z_0)^m=\int_0^{2\pi}\rho^m e^{im\theta}(i\rho e^{i\theta})\,d\theta$

$=i\rho^{m+1}\int_0^{2\pi}e^{i(m+1)\theta}\,d\theta$

$=\begin{cases} 2\pi i, & m=1 \\ 0, & m\neq-1,\ m \text{ integer}\end{cases}$

when $m=-1$,

$I=i\rho^{-1+1}\int_0^{2\pi}e^0\,d\theta$

$=i\int_0^{2\pi}1\,d\theta$

$=i2\pi$

when $m\neq-1$:

$I=i\rho^{m+1}\int_0^{2\pi}e^{i(m+1)\theta}\,d\theta$

$=i\rho^{m+1}\left[\frac{e^{i(m+1)\theta}}{i(m+1)}\right]_0^{2\pi}$

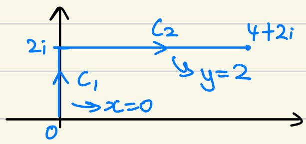
$=\frac{\rho^{m+1}}{m+1}\left[\underbrace{e^{i(m+1)2\pi}}_{1}-\underbrace{e^0}_{1}\right]=0$

$=0$

↳ three basic properties of complex line integrals:

1. linearity $\rightarrow \int_C [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$
2. subdivision of path $\rightarrow \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$
3. sense of integration $\rightarrow \int_{z_1}^{z_2} f(z)\,dz = -\int_{z_2}^{z_1} f(z)\,dz$

E.g. Evaluate $\int_C \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along the curve given by line $z=0$ to $z=2i$ & then line from $z=2i$ to $z=4+2i$.



Along $z=0$ to $z=2i$: $z(t) = 0 + it,\ 0 \leq t \leq 2$ & $d(z) = i\,dt$

Along $z=2i$ to $z=4+2i$: $z(t) = t + 2i,\ 0 \leq t \leq 4$ & $dz = dt$

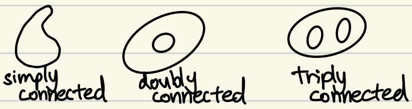
$$\int_C \bar{z}\,dz = \int_0^2 t\,dt + \int_0^4 (t-2i)\,dt$$
$$= 2 + \int_0^4 t\,dt - 2i\int_0^4 dt$$
$$= 2 + \left[\frac{t^2}{2}\right]_0^4 - 8i$$
$$= 2 + 8 - 8i = 10 - 8i$$

Simple Closed Path & Simply Connected Domain

→ if no singular pt in enclosed circle $\rightarrow \oint_C f(z) = 0$

→ simple closed path is a path that does not intersect/touch itself.

→ domain that is not simply connected is multiply connected.



↳ Cauchy's Integral Theorem

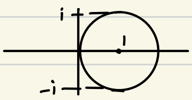
→ useful to evaluate integrals of $\oint_C \frac{f(z)}{(z-z_0)^m}\,dz$, where $m = 1, 2, 3 \ldots$

→ fully analytic as long as $z - z_0 \neq 0$
↳ point of singularity $(z - z_0 = 0)$

→ let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D & any simply closed path C in D that encloses $z_0 \rightarrow \oint_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0)$.
→ z at singularity

↳ General Case: $\oint_C \frac{f(z)}{(z-z_0)^m}\,dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(z_0)$ where $m = 1, 2, 3, \ldots$
→ if $m=2$ → need to differentiate twice

E.g. Integrate $\frac{1}{z^4-1}$ counterclockwise around circle $|z-1| = 1$



find singular point ← ① $z^4 - 1 = 0$

$z^2 = \pm 1$

$z = \pm 1, \pm i$

decide which point in circle ← ② $|z-1| \rightarrow$ path circle → encloses singular pt at $z=1$
→ radius = 1, centre at (1,0)

make into general form ← ③ $\oint_C \frac{1}{z^4-1}\,dz = \oint_C \frac{1}{(z-1)(z^3+z^2+z+1)}\,dz$ → factorise singular pt out & move rest to numerator

4 steps

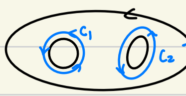
$$= \oint_C \frac{\frac{1}{z^3+z^2+z+1}}{z-1}\,dz$$
→ need be in form $z - z_0$ (~~$3z - i$~~) ✗ wrong → need factorise out

solve ← ④ $= 2\pi i \left(\frac{1}{z^3+z^2+z+1}\right)\Big|_{z=1}$ → sub singular pt

$$= \frac{\pi i}{2}$$

↳ Cauchy's Theorem for Multiply Connected Domains (more than 1 singular pt in circle)

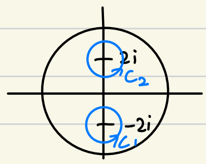
$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$



→ broken up into C_1 & C_2, each enclosing a singular pt

E.g. Evaluate $\oint_C \frac{5z}{z^2+4}\,dz$ where C is any closed path such that all the singularities lie inside c (CCW)

$z^2 + 4 = 0 \longrightarrow z = \pm 2i$



$$\oint_C \frac{5z}{z^2+4}\,dz = \oint_C \frac{5z}{(z-2i)(z+2i)}\,dz$$
$$= \oint_{C_1} \frac{\frac{5z}{z-2i}}{z+2i}\,dz + \oint_{C_2} \frac{\frac{5z}{z+2i}}{z-2i}\,dz$$
$$= 2\pi i \left(\frac{5z}{z-2i}\Big|_{z=-2i}\right) + 2\pi i \left(\frac{5z}{z+2i}\Big|_{z=2i}\right)$$
$$= 2\pi i\,\frac{5(-2i)}{-4i} + 2\pi i\,\frac{5(2i)}{4i}$$
$$= 10\pi i$$

Evaluation of Real Integrals

→ $z=e^{i\theta} \longrightarrow \cos\theta = \frac{z+\bar{z}}{2}$, $\sin\theta = \frac{z-\bar{z}}{2i}$, $dz = ie^{i\theta}\,d\theta$

→ convert $F(\cos\theta, \sin\theta)\,d\theta$ into $f(z) \longrightarrow \int_0^{2\pi} F(\cos\theta,\sin\theta)\,d\theta = \oint_C f(z)\frac{1}{iz}dz$

E.g. Evaluate $\int_0^{2\pi} \frac{d\theta}{5-3\sin\theta}$ using complex integration method.

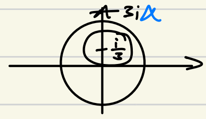
$$z = \cos\theta + i\sin\theta$$

$$\frac{1}{z} = \cos\theta - i\sin\theta \quad \rightarrow \frac{1}{z} = \frac{x-iy}{\cancel{x^2+y^2} \rightarrow 1}$$

$$\sin\theta = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$\int_0^{2\pi} \frac{1}{5-3\sin\theta}d\theta = \oint_C \frac{1}{5-3\left(\frac{z^2-1}{2i\cancel{z}}\right)}\left(\frac{1}{i\cancel{z}}\right)dz$$

$$= -2\oint_C \frac{1}{3z^2-10zi-3}dz$$



$$= -2\oint_C \frac{1}{(z-3i)(3z-i)}dz$$

$$= -2\oint_C \frac{\frac{1}{3(z-3i)}}{z-\frac{i}{3}}dz \quad \text{need a form } (z-z_0)$$

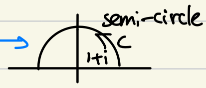
$$= -\frac{2}{3}\cdot 2\pi i\left(\frac{1}{z-3i}\bigg|_{z=\frac{i}{3}}\right) = \frac{\pi}{3}$$

Improper Integrals of Rational Functions

→ 1. $f(x) = \frac{p(x)}{q(x)}$ is a real function with no common factors b/w $p(x)$ & $q(x)$, $q(x) \neq 0$ for all x

→ 2. Degree of $q(x) \geq$ Degree of $p(x)+2$ [$f(x) = \frac{1}{1+x^4}$ satisfy condition, $f(x) = \frac{x^3}{1+x^4}$ does not]

E.g. Evaluate $\int_{-\infty}^{\infty} \frac{x}{(x^2-2x+2)^2}dx$ using complex integration method.



Δx to z ← ① $\int_{-\infty}^{\infty} \frac{x}{(x^2-2x+2)^2}dx = \oint_C \frac{z}{(z^2-2z+2)^2}dz$

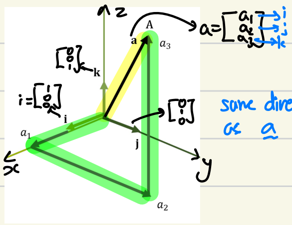
$$= \oint_C \frac{z}{[z-(1+i)]^2[z-(1-i)]^2}dz \rightarrow m=2 \text{ (differentiate twice)}$$

$$= \oint_C \frac{\frac{z}{[z-(1-i)]^2}}{[z-(1+i)]^2}dz$$

$$= 2\pi i\left(\frac{d}{dz}\frac{z}{[z-(1-i)]^2}\right)\bigg|_{z=1+i}$$

$$= 2\pi \cancel{i}\left(\frac{2}{8\cancel{i}}\right) = \frac{\pi}{2}$$ → if ans is complex no. → wrong → cause you are computing real integrals

Vector Fundamentals



vector, $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$

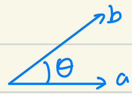
magnitude (length), $\|\underset{\sim}{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

same direction as $\underset{\sim}{a}$ ← unit vector, $\hat{\underset{\sim}{a}} = \frac{1}{\|\underset{\sim}{a}\|}\underset{\sim}{a} = \frac{1}{\sqrt{a_1^2+a_2^2+a_3^2}}(a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k})$

e.g. Find v if v has a length of 5 unit in direction $2\underset{\sim}{i}+3\underset{\sim}{j}-5\underset{\sim}{k}$

$\underset{\sim}{v} = 5\hat{\underset{\sim}{v}}$

1. Dot Product $[\underset{\sim}{a}\cdot\underset{\sim}{b} = \|\underset{\sim}{a}\|\|\underset{\sim}{b}\|\cos\theta = \underset{\sim}{b}\cdot\underset{\sim}{a}]$ → scalar (commutative)



↳ Properties: a) $\sqrt{\underset{\sim}{a}\cdot\underset{\sim}{a}} = \|\underset{\sim}{a}\|$

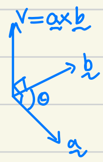
b) $\underset{\sim}{a} \,/\!/\, \underset{\sim}{b}$ parallel ⟶ $\underset{\sim}{a}\cdot\underset{\sim}{b} = \|\underset{\sim}{a}\|\|\underset{\sim}{b}\|$, $\cos(0) = 1$

→ different direction

c) $\underset{\sim}{a}$ & $\underset{\sim}{b}$ anti-parallel ⟶ $\underset{\sim}{a}\cdot\underset{\sim}{b} = -\|\underset{\sim}{a}\|\|\underset{\sim}{b}\|$, $\cos\pi = -1$

d) $\underset{\sim}{a} \perp \underset{\sim}{b}$ perpendicular/orthogonal ⟶ $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$, $\cos(\frac{\pi}{2}) = 0$ $(i\cdot j = 0)$

2. Cross Product $[\underset{\sim}{a}\times\underset{\sim}{b} = \|\underset{\sim}{a}\|\|\underset{\sim}{b}\|\sin\theta\,\hat{\underset{\sim}{n}}]$ → unit vector ⊥ to both $\underset{\sim}{a}$ & $\underset{\sim}{b}$, also acts as normal



↳ Properties: a) $\underset{\sim}{a}\times\underset{\sim}{b} = -\underset{\sim}{b}\times\underset{\sim}{a}$

b) $(\underset{\sim}{a}\times\underset{\sim}{b})\times\underset{\sim}{c} \neq \underset{\sim}{a}\times(\underset{\sim}{b}\times\underset{\sim}{c})$

c) $\underset{\sim}{i}\times\underset{\sim}{i} = 0$ (//), $\underset{\sim}{i}\times\underset{\sim}{j} = \underset{\sim}{k}$

↳ Method to calculate: $(3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})\times(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = -3\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k}$

→ expand along this

→ like matrix

3. Scalar Triple Product

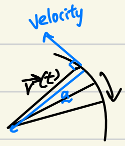
→ implied to do this first

$$\underset{\sim}{a}\cdot(\underset{\sim}{b}\times\underset{\sim}{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -\underset{\sim}{c}\cdot\underset{\sim}{b}\times\underset{\sim}{a}$$

↳ $\underset{\sim}{a}\cdot(\underset{\sim}{b}\times\underset{\sim}{c}) = \underset{\sim}{b}\cdot(\underset{\sim}{c}\times\underset{\sim}{a}) + \underset{\sim}{c}\cdot(\underset{\sim}{a}\times\underset{\sim}{b})$

Vector Differentiation

↳ particle moving and $r = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ changes with time t.



– position vector, $r(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$

– velocity vector, $v(t) = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

– acceleration vector, $a(t) = \frac{d^2x}{dt^2}\underset{\sim}{i} + \frac{d^2y}{dt^2}\underset{\sim}{j} + \frac{d^2z}{dt^2}\underset{\sim}{k}$

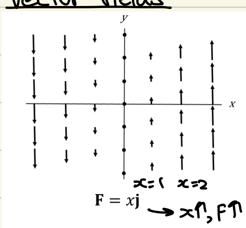
E.g. Find velocity, speed & acceleration for helical path given by: $r(t) = a\cos t\,\underset{\sim}{i} + a\sin t\,\underset{\sim}{j} + bt\,\underset{\sim}{k}$, a, b constants

velocity $= -a\sin t\,\underset{\sim}{i} + a\cos t\,\underset{\sim}{j} + b\underset{\sim}{k}$

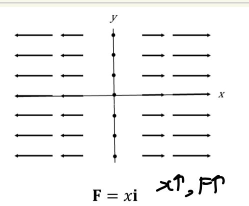
speed, $\|\underset{\sim}{v}\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2+b^2}$

acceleration $= -a\cos t\,\underset{\sim}{i} - a\sin t\,\underset{\sim}{j}$

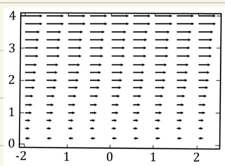
Vector fields



Vertical flow field

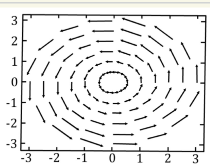


Horizontal flow field



$F(x, y) = \frac{1}{5}\sqrt{y}i$

Horizontal flow with magnitudes as a function of the vertical axis



$F(x, y) = -yi + xj$

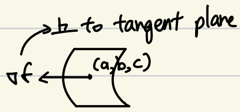
2D vortex

☆ Del Operator (∇)

→ f is a scalar & ∇f is a vector called gradient of f (grad f)

$\nabla f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j} + \frac{\partial f}{\partial z}\underset{\sim}{k}$ → partial differentiation ☆

☆ → properties: a) $\nabla f(a, b, c)$ is in direction of max gradient



b) $\|\nabla f(a,b,c)\|$ is max gradient

c) $\nabla f(a,b,c)$ is ⊥ to level surface of f at (a,b,c)

→ direction derivative of f in direction $u = \nabla f \cdot \hat{\underset{\sim}{u}}$ ($\hat{u}$ → unit vector) ($\underset{\sim}{u}$ → directional derivative (magnitude of rate of change of f in particular direction))

E.g. At point $(1,0)$, consider direction derivatives along $u_1 = (i + j)$ with $\nabla f(x,y) = 2x\,\underset{\sim}{i} + 2y\,\underset{\sim}{j}$

$$\nabla f \cdot \hat{\underset{\sim}{u}}_1 = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Bigg|_{\substack{x=1 \\ y=0}} = \sqrt{2}$$

E.g. For $f(x,y) = x^2 e^y$, find the directional derivative at $(-2,0,0)$ in direction $-\underset{\sim}{j}$ $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$. At point $(-2,0,0)$, what is the maximum directional derivative?

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j} + \frac{\partial f}{\partial z}\underset{\sim}{k}$$

$$= 2xe^y\,\underset{\sim}{i} + x^2e^y\,\underset{\sim}{j} + 0\underset{\sim}{k}$$

$$\text{directional derivative} = \nabla f \cdot (-\hat{j}) = \begin{pmatrix} 2xe^y \\ x^2e^y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$= -x^2e^y$$

$$\text{At } (-2,0,0),\ D_{-\hat{j}}f = -x^2e^y \Big|_{\substack{x=-2 \\ y=0 \\ z=0}}$$

$$= -4$$ → function decrease in $\downarrow^{y}$ direction $(-j)$

$$\text{Max } D_f = \|\nabla f\| = \left\| \begin{pmatrix} 2xe^y \\ x^2e^y \\ 0 \end{pmatrix} \right\|_{\substack{x=-2 \\ y=0 \\ z=0}}$$

$$= \left\| \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} \right\| = \sqrt{4^2+4^2+0^2} = 5.6569$$

Divergence (with Del Operator)

$$\nabla = \left(\frac{\partial}{\partial x}\underset{\sim}{i} + \frac{\partial}{\partial y}\underset{\sim}{j} + \frac{\partial}{\partial z}\underset{\sim}{k}\right)$$

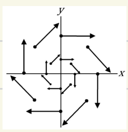
→ $\nabla \cdot F$ → divergence of F, div F (scalar) → tells extent to which field explodes/diverges

→ If $F = F_1\underset{\sim}{i} + F_2\underset{\sim}{j} + F_3\underset{\sim}{k}$ → $\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

→ Consider movement of fluid, velocity at any point $= v(x,y,z)$, net rate of gain in fluid per volume → $\nabla \cdot \underset{\sim}{v}$

- If $\nabla \cdot \underset{\sim}{v} = 0$ → rate of fluid flowing in & out same
- If $\nabla \cdot \underset{\sim}{v} > 0$ → more fluid flowing out than in → existance of 'source'
- If $\nabla \cdot \underset{\sim}{v} < 0$ → more fluid flowing in than out → existance of 'sink'
- If $\nabla \cdot \underset{\sim}{v} = 0$ → fluid incompressible → solenoidal/divergence free

E.g. $F = y\underset{\sim}{i} - x\underset{\sim}{j}$. Find divergence.



- $F = a$ (→ constant) → $\nabla \cdot F = 0$

$$\nabla \cdot F = \frac{\partial F}{\partial x}\underset{\sim}{i} + \frac{\partial F}{\partial y}\underset{\sim}{j}$$

$$= \frac{\partial y}{\partial x} + \frac{\partial(-x)}{\partial y} = 0$$

Curl (with Del Operator)

→ $\nabla \times F$ → curl of F (vector) → related to 'circulation' at each point of vector field

→ If $F = F_1\underset{\sim}{i} + F_2\underset{\sim}{j} + F_3\underset{\sim}{k}$, (→ expand along this line)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underset{\sim}{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\underset{\sim}{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underset{\sim}{k}$$

(-ve)

→ If $F = 2y^2\underset{\sim}{i}$, F have component in $\underset{\sim}{i}$ direction & $2y^2$ in $\underset{\sim}{j}$ direction ($\underset{\sim}{j}$ ⊥ $\underset{\sim}{i}$) → curl exist & equal to $-4y\underset{\sim}{k}$ ($\underset{\sim}{k}$ ⊥ to both $\underset{\sim}{i}$ & $\underset{\sim}{j}$)

E.g. Determine curl $(xy^2z\underset{\sim}{i} + 2x^3y\underset{\sim}{j} + 4x^2y^2\underset{\sim}{k})$ at point $(1,1,-1)$

$$F = (xy^2z\underset{\sim}{i} + 2x^3y\underset{\sim}{j} + 4x^2y^2\underset{\sim}{k}$$

$$\nabla \times F = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & 2x^3y & 4x^2y^2 \end{vmatrix}$$

$$= \begin{pmatrix} 8x^2y \\ xy^2 - 8xy^2 \\ 6x^2y - 2xy^2 \end{pmatrix} \Bigg|_{\substack{x=1 \\ y=1 \\ z=-1}} = \begin{pmatrix} 8 \\ -7 \\ 8 \end{pmatrix}$$

– Laplacian (with del operator) → $\text{div}(\nabla f) = \nabla\cdot(\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z}$

– Useful formulas: a) $\nabla(fg) = f\nabla g + g\nabla f$ → similar to pdt rule

b) $\nabla\cdot(fv) = f\nabla\cdot v + v\cdot\nabla f$, $\nabla\times(fv) = f\nabla\times v + \nabla f\times v$

c) $\nabla\times(\nabla f) = 0$ ← curl (grad f) $=0$

d) $\nabla\cdot(\nabla\times v) = 0$ ← div(curl $\vec{F}$) $=0$

Vector Line Integrals

→ operation of integrating vector field along a curve in space

→ steps to solve question:

1. parameterise path → $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ → change x, y, z to in terms of t $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$
2. change dr to dt through $\frac{dr}{dt} = a \rightarrow dr = a\,dt$
3. put $x(t), y(t), z(t)$ into F
4. $\int_C F\cdot d\underset{\sim}{r} = \int_{t_1}^{t_2} f(t)\,dt$ → normally in terms of θ

E.g. If $F = (3x^2+6y)\underset{\sim}{i} - 14yz\underset{\sim}{j} + 20xz^2\underset{\sim}{k}$, evaluate $\int_C F\cdot d\underset{\sim}{r}$ from $(0,0,0)$ to $(1,1,1)$ along paths C:

a) $x=t,\ y=t^2,\ z=t^3$. b) straight lines from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ & to $(1,1,1)$

a) $\underset{\sim}{r} = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ $0 \le t \le 1$

$$\vec{F} = \begin{pmatrix} 3x^2+6y \\ -14yz \\ 20xz^2 \end{pmatrix} = \begin{pmatrix} 3t^2+6t^2 \\ -14(t^2)(t^3) \\ 20t(t^6) \end{pmatrix} = \begin{pmatrix} 9t^2 \\ -14t^5 \\ 20t^7 \end{pmatrix}$$

$$\frac{dr}{dt} = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$$

$$\begin{aligned}
\int_C F\cdot d\underset{\sim}{r} &= \int_{t=0}^{t=1} \begin{pmatrix} 9t^2 \\ -14t^5 \\ 20t^7 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} dt \\
&= \int_{t=0}^{t=1} (9t^2 - 28t^6 + 60t^9)\,dt \\
&= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1 \\
&= 3 - 4 + 6 = 5
\end{aligned}$$

b) $C_1: (0,0,0) \longrightarrow (1,0,0)$

$C_2: (1,0,0) \longrightarrow (1,1,0)$

$C_3: (1,1,0) \longrightarrow (1,1,1)$

$$\int_C F\cdot d\underset{\sim}{r} = \int_{C_1} F\cdot d\underset{\sim}{r} + \int_{C_2} F\cdot d\underset{\sim}{r} + \int_{C_3} F\cdot d\underset{\sim}{r}$$

$$\begin{aligned}
\int_{C_1} F\cdot d\underset{\sim}{r} &= \int_{x=0}^{x=1} \left[\begin{pmatrix} 3x^2+6y \\ -14yz \\ 20xz^2 \end{pmatrix}\cdot(dx\,\underset{\sim}{i} + \underbrace{dy}_{=0}\,\underset{\sim}{j} + \underbrace{dz}_{=0}\,\underset{\sim}{k})\right] \\
&= \int_0^1 (3x^2 + \underbrace{6y}_{=0})\,dx \\
&= \left[\frac{3x^3}{3}\right]_0^1 = 1
\end{aligned}$$

$$\begin{aligned}
\int_{C_2} F\cdot d\underset{\sim}{r} &= \int_{y=0}^{y=1} F\cdot(\overbrace{dx}^{=0}\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + \overbrace{dz}^{=0}\,\underset{\sim}{k}) \\
&= \int_0^1 -14y\overbrace{z}^{=0}\,dy = 0 \quad \text{→ base on coordinate } (1,1,0)
\end{aligned}$$

$$\begin{aligned}
\int_{C_3} F\cdot d\underset{\sim}{r} &= \int_{z=0}^{z=1} F\cdot(\overbrace{dx}^{=0}\,\underset{\sim}{i} + \overbrace{dy}^{=0}\,\underset{\sim}{j} + dz\,\underset{\sim}{k}) \\
&= \int_0^1 20\overbrace{x}^{=1}z^2\,dz \,\Big|_{x=1,\ y=1} \\
&= \left[\frac{20z^3}{3}\right]_0^1 = \frac{20}{3}
\end{aligned}$$

$$\int_C \vec{F}\cdot d\underset{\sim}{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

Conservative Fields

→ Example above, work done for $(0,0,0)$ to $(1,1,1)$ is different (→ 5 & $\frac{23}{3}$) → work done is dependent on path C

→ vector field F is conservative (work done independent of path taken) if there is a scalar field V such that

to deduce: $F = \nabla V$ / $\nabla\times F = 0$ → to show F is conservative

→ Suppose there exist a scalar field V, $F = \nabla V = \frac{\partial V}{\partial x}\underset{\sim}{i} + \frac{\partial V}{\partial y}\underset{\sim}{j} + \frac{\partial V}{\partial z}\underset{\sim}{k}$

$\int_A^B F\cdot dx = \int_A^B \left(\frac{\partial V}{\partial x} i + \frac{\partial V}{\partial y} j + \frac{\partial V}{\partial z} k\right)\cdot(dx\, i + dy\, j + dz\, k)$

$= \int_A^B \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\right)$

$= \int_A^B dV = V(B) - V(A)$

Since $dV = \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\right)$, line integral independent of path.

☆ V is scalar potential function of F (conservative field).

↳ Non-conservative fields called dissipative, expressed as $\oint_C F\cdot dx \neq 0$

E.g. Show that $F = (2xy + z^3)i + x^2 j + 3xz^2 k$. Find the scalar potential field. Find the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$= (0-0)i + (3z^2 - 3z^2)j + (2x - 2x)k$

$= 0$ (F is conservative)

$$F = \nabla V = \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy + z^3 \\ x^2 \\ 3xz^2 \end{pmatrix}$$

$\frac{\partial V}{\partial x} = 2xy + z^3 \longrightarrow V(x,y,z) = x^2y + xz^3 + g_1(y,z)$

$\frac{\partial V}{\partial y} = x^2 \longrightarrow V(x,y,z) = x^2y + g_2(x,z)$

$\frac{\partial V}{\partial z} = 3xz^2 \longrightarrow V(x,y,z) = xz^3 + g_3(x,y)$

$V(x,y,z) = x^2y + xz^3 + c$

Work done $= \int_C \vec{F}\cdot dx = V(3,1,4) - V(1,-2,1)$

$= [3^2(1) + 3(4)^3] - [(1)^2(-2) + (1)(1)^3]$

$= 202$

Vector Surface Integrals

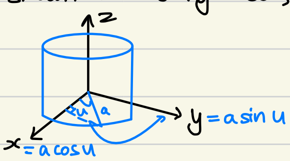
① ↳ If surface specified by $z = f(x,y)$ / $f(x,y,z) = 0 \longrightarrow \nabla f$ is a normal vector

② ↳ If surface specified parametrically $(x = x(u,v),\ y = y(u,v),\ z = z(u,v)) \longrightarrow$ normal vector $= r_u \times r_v$ (partial derivative of $r(u,v)$)

(Cylinder) E.g. The surface S of circular cylinder can be shown as: $x^2 + y^2 = a^2$, $-1 \le z \le 1$

look from z-axis (circle) ← $r = x i + y j + z k$

$x = a\cos u$, $y = a\sin u$, $z = v$, $0 \le u \le 2\pi$, $-1 \le v \le 1$



$r(u,v) = a\cos u\, i + a\sin u\, j + v k$

$r_u = \frac{\partial r}{\partial u} = -a\sin u\, i + a\cos u\, j$

$r_v = \frac{\partial r}{\partial v} = k$

$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$

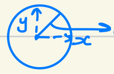
$= a\sin u\, j + a\cos u\, i = (x i + y j)$ same magnitude not impt

check: $\nabla f = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j\right)(x^2 + y^2) = 2(x i + y j)$

☆ plane, $x + y + z = 1 \rightarrow r(u,v) = u i + v j + (1-u-v)k$

☆ paraboloid, $z = 1 - (x^2 + y^2) \rightarrow \begin{cases} x = \sqrt{1-v}\cos u \\ y = \sqrt{1-v}\sin u \\ z = v,\ 0 \le v \le 1 \end{cases}$

circle: $x^2 + y^2 = a^2 \rightarrow$



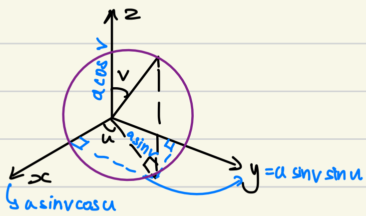
$\rightarrow \sqrt{x^2 + y^2} = \sqrt{a^2} = a$

(Sphere) Eg. Position vector r of point of sphere given by: $r = x i + y j + z k$. $x^2 + y^2 + z^2 = a^2$.

$x = a\cos u \sin v$; $0 \le u \le 2\pi$

$y = a\sin u \sin v$; $0 \le v \le \pi$

$z = a\cos v$



If $r(u,v) = a\cos u \sin v\, i + a\sin u \sin v\, j + a\cos v\, k$,

$r_u = \frac{\partial r}{\partial u} = -a\sin u \sin v\, i + a\cos u \sin v\, j$

$r_v = \frac{\partial r}{\partial u} = a\cos u \cos v\, i + a\sin u \cos v\, j - a\sin v\, k$

$$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ & r_u & \\ & r_v & \end{vmatrix}$$

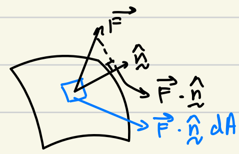
$$= -a\sin v\,(a\cos u \sin v\, i + a\sin u \sin v\, j + a\cos v\, k)$$

normal vector pointing inwards (+ve → take −N) $= \underline{-a\sin v}\,(x i + y j + z k)$

$\nabla f = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(x^2+y^2+z^2)$

outward direction

$= 2(x i + y j + z k)$



Flux (with surface integrals)

↳ $\iint_S F \cdot dA = \iint_S F \cdot \hat{n}\, dA$ → area of small patch in surface S → $\hat{n}\,dA = N\,du\,dv$

$= \iint_R F(u,v) \cdot N\,du\,dv$ → depends on scenario

(Surface) E.g. Compute flux of vector field $F = i + xy\,j$ across surface: $x = u+v,\ y = u-v,\ z = u^2,\ 0 \le u \le 1,\ 0 \le v \le 1$

$$\iint_S F \cdot N\,du\,dv = \int_0^1 \int_0^1 (i + xy\,j) \cdot (r_u \times r_v)\,du\,dv$$

$$r(u,v) = (u+v)i + (u-v)j + u^2 k$$

$$r_u = i + j + 2u k,\quad r_v = i - j$$

$$F(u,v) = i + (u^2 - v^2) j$$

$$F \cdot r_u \times r_v = \begin{vmatrix} 1 & u^2-v^2 & 0 \\ 1 & 1 & 2u \\ 1 & -1 & 0 \end{vmatrix} = 2u^3 - 2uv^2 + 2u$$

$$\int_0^1 \int_0^1 (2u^3 - 2uv^2 + 2u)\,du\,dv = \int_0^1 \left[\tfrac{1}{2} - v^2 + 1\right] dv = \tfrac{7}{6}$$

(Sphere) E.g. Find flux of vector field $F(x,y,z) = z\,k$ across the outward-oriented sphere $x^2+y^2+z^2 = a^2$

$\oiint_S \vec{F} \cdot d\vec{A} \longrightarrow d\vec{A} = \vec{N}\,du\,dv$ → $\vec{r_u} \times \vec{r_v}$

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cos u \sin v \\ a \sin u \sin v \\ a\cos v \end{pmatrix} \quad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le \pi \end{matrix}$$

$$r_u = \begin{pmatrix} -a\sin u \sin v \\ a\cos u \sin v \\ 0 \end{pmatrix} \qquad r_v = \begin{pmatrix} a\cos u \cos v \\ a \sin u \cos v \\ -a\sin v \end{pmatrix}$$

$$\vec{N} = r_u \times r_v = -\begin{pmatrix} a^2 \cos u \sin^2 v \\ a^2 \sin u \sin^2 v \\ a^2 \sin v \cos v \end{pmatrix} = -a\sin v \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -a\sin v\, x$$

$$\oiint_S \vec{F} \cdot d\vec{A} = \iint a\sin v \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} du\,dv = \iint a\sin v\,(z^2)\,du\,dv$$

$$= \int_{v=0}^{v=\pi} \int_{u=0}^{u=2\pi} a\sin v \cdot a^2 \cos^2 v\,du\,dv$$

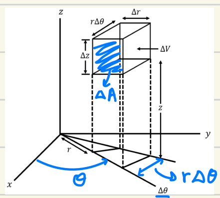
$$= a^3 \int_{v=0}^{\pi} \sin v \cos^2 v\,du \cdot \int_{u=0}^{2\pi} du$$

$$= a^3 \left[-\frac{\cos^3 v}{3}\right]_0^{\pi} [u]_0^{2\pi} = a^3 \left[\tfrac{2}{3}\right][2\pi] = \tfrac{4}{3}\pi a^3$$

Volume Integrals

↳ $\iiint_V f\,dV = \iiint f(x,y,z)\,dx\,dy\,dz$

↳ cartesian coordinates (x,y,z) can be converted into:



– cylindrical coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$ ⟶ $dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$ → make it a length measure

– spherical coordinates $\begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases}$, $0 \le \rho < \infty,\ 0 \le \theta < 2\pi,\ 0 \le \phi < \pi$ ⟶ $dV = (d\rho)(\rho\,d\phi)(\rho\sin\phi\,d\theta)$

E.g. Let $f = z^2(x^2+y^2+z^2)^{\frac{1}{2}}$. Find volume integral of f over the hemisphere give by: $x^2+y^2+z^2 = 4$

$$\iiint f\,dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$$\iiint f\,dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 (\rho\cos\phi)^2\,\rho\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^5 \cos^2\phi \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left[\tfrac{1}{6}\rho^6\right]_0^2 \cos^2\phi\sin\phi\,d\phi\,d\theta$$

$$= \frac{32}{3} \int_0^{2\pi} \left[-\tfrac{1}{3}\cos^3\phi\right]_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{32}{9} \int_0^{2\pi} d\theta$$

$$= \frac{64}{9}\pi$$

Gauss/Divergence Theorem

↳ $\oiint_S \vec{F}\cdot\hat{n}\,dA = \iiint \nabla\cdot F\,dV$ → flux of vector field across closed surface with outward orientation = triple integral of divergence over region enclosed by surface

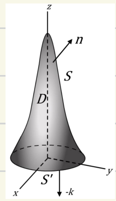
E.g. Let $F = e^y\cos z\,\mathbf{i} + \sqrt{x^3+1}\,\sin z\,\mathbf{j} + (x^2+y^2+3)\,\mathbf{k}$ & S expressed as: $z = (1-x^2-y^2)e^{(1-x^2-3y^2)}$; $z \geq 0$.

Calculate $\iint_S \vec{F}\cdot\hat{n}\,dA$.

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\cdot\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = 0$$

Direct evaluation of surface integral not feasible → $\iint_S \vec{F}\cdot\hat{n}\,dA + \iint_{S'} \vec{F}\cdot\hat{n}\,dA = \oiint_{S+S'} \vec{F}\cdot\hat{n}\,dA = \iiint_V \nabla\cdot F\,dV = 0$

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \underbrace{\iiint \nabla\cdot F\,dV}_{=0} - \iint_{S'} \vec{F}\cdot\vec{n}\,dA$$



S' is a disk, given by eqn: $x^2+y^2 \leq 1$, $z=0$ $\quad \begin{cases} x = v\cos u \\ y = v\sin u,\ 0 \leq v \leq 1 \end{cases}$

$r(u,v) = v\cos u\,\mathbf{i} + v\sin u\,\mathbf{j}$

$\vdots$

$F.N = -v(x^2+y^2+3)$

$$-\iint_{S'} \vec{F}\cdot\hat{n}\,dA = -\iint_{S'} -v(x^2+y^2+3)\,du\,dv$$
$$= \int_0^1\int_0^{2\pi} v[v^2+3]\,du\,dv$$
$$= \tfrac{7}{2}\pi = \iint_S F\cdot\hat{n}\,dA$$

Stokes Theorem (can only be applied to open surfaces in 3D)

no sphere

↳ for a given surface, bounded be contour C, $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times F\cdot\hat{n}\,dA$ → direction must align with closed path

E.g. If vector field is $F(x,y,z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$, calculate $\iint_S (\nabla\times F)\cdot\hat{n}\,dA$ across surface of S which is the portion of paraboloid $z = 4-x^2-y^2$, $z\geq 0$, with upward orientation & $C: x^2+y^2=4$ forms boundary of S on xy plane.

Let $x = 2\sin v\cos u$, $\quad 0 \leq u \leq 2\pi$

$y = 2\sin v\cos u$, $\quad 0 \leq v \leq \frac{\pi}{2}$

$z = 4 - 4\sin^2 v$

$\vec{r}(u,v) = 2\sin v\cos u\,\mathbf{i} + 2\sin v\sin u\,\mathbf{j} + (4-4\sin^2 v)\,\mathbf{k}$

$\vec{r}_u = -2\sin v\sin u\,\mathbf{i} + 2\sin v\cos u\,\mathbf{j}$

$\vec{r}_v = 2\cos v\cos u\,\mathbf{i} + 2\cos v\sin u\,\mathbf{j} - 8\sin v\cos v\,\mathbf{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

$$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin v\sin u & 2\sin v\cos u & 0 \\ 2\cos v\cos u & 2\cos v\sin u & -8\sin v\cos v \end{vmatrix} = -16\sin^2 v\cos v\cos u\,\mathbf{i} - 16\sin^2 v\cos v\sin u\,\mathbf{j} - 4\sin v\cos v\,\mathbf{k}$$

$$(\nabla\times\vec{F})\cdot\vec{n}\,dA = (5\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot(16\sin^2 v\cos v\cos u\,\mathbf{i} + 16\sin^2 v\cos v\sin u\,\mathbf{j} + 4\sin v\cos v\,\mathbf{k})\,du\,dv$$
$$= (80\sin^2 v\cos v\cos u + 32\sin^2 v\cos v\sin u + 12\sin v\cos v)\,du\,dv$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dA = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 80\sin^2 v\cos v\cos u + 32\sin^2 v\cos v\sin u + 12\sin v\cos v)\,du\,dv$$
$$= \int_0^{2\pi} (\cos u(\tfrac{80}{3}\sin^3 v) + \sin u(\tfrac{32}{3}\sin^3 v) + \tfrac{12}{2}\sin^2 v)\Big|_0^{\frac{\pi}{2}}\,du$$
$$= \int_0^{2\pi} \tfrac{80}{3}\cos u + \tfrac{32}{3}\sin u + 6\,du = 6\times 2\pi = 12\pi$$

Green's Theorem

↳ 2D case of Stokes theorem → $\oint_C (F_1\,dx + F_2\,dy) = \iint_S \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] dA$