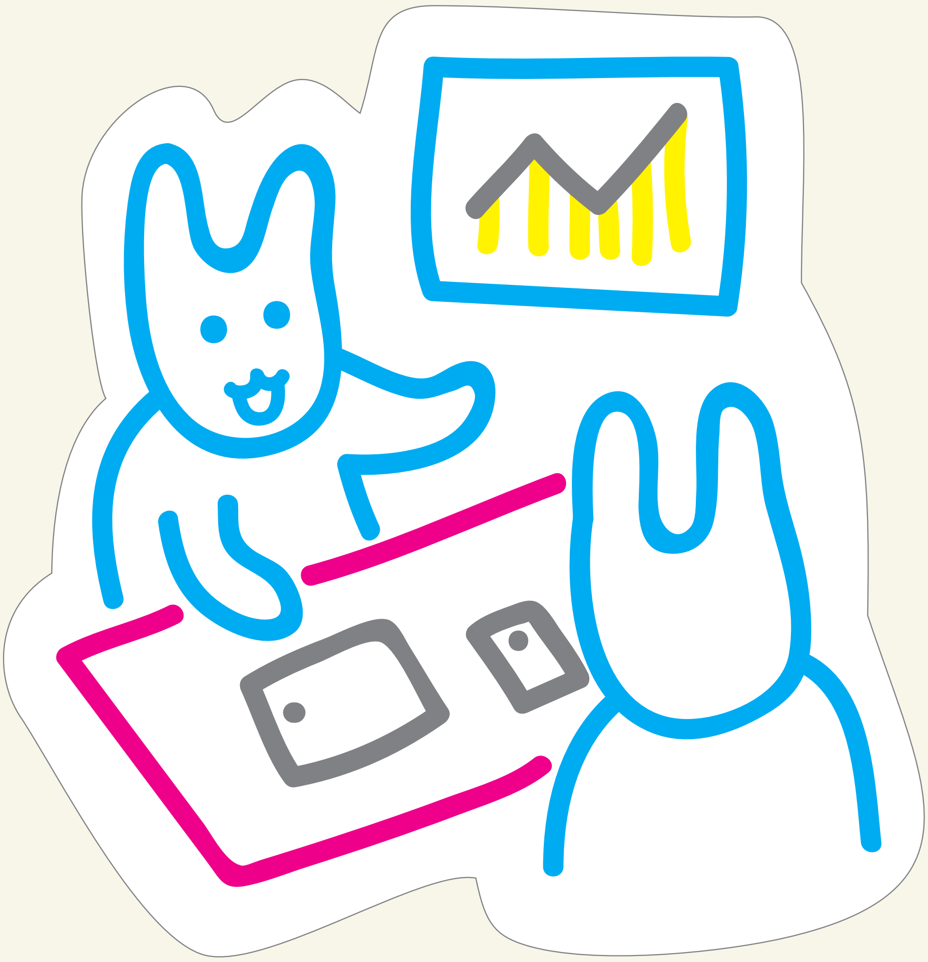
Engineering Math 1



Set Theory

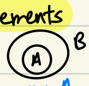
→ set is a collection of objects. Members in a set are called elements of A.

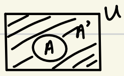
Patricia $\in A$ → Patricia is an element of A

Patricia $\notin A$ → Patricia is not an element of A

→ universal set, U → set of all objects

→ null set/empty set, $\emptyset$ → no elements

→ A is a subset of B, $A \subset B$ 

→ complement of A, $A' = U \setminus A$ → cut away A 

– intersection of A & B → $A \cap B = \{x : x \in A \ \& \ x \in B\}$ 

– union of A & B → $A \cup B = \{x : x \in A$ or $x \in B$ (or both)$\}$  $[P(A \cup B) = P(A) + P(B) - P(A \cap B)]$

– $A' \cup \emptyset = A'$, $A' \cap \emptyset = \emptyset$

A set of real no. b/w 0 & 1 inclusive:

$A = \{x : x \in \mathbb{R}, 0 \leq x \leq 1\}$

Sample Space & Events

→ Sample space, S → set of all possible random outcomes. A particular outcome (element of S) → sample point

– Discrete sample space → finite no. of outcomes / countable infinite no. of outcomes

↳ e.g. Toss a coin → $S = \{h, t\}$ (countable) (2 outcomes), no. of vehicles that pass through gantry → $S = \{0, 1, 2, 3, 4 \ldots\}$ (countable infinite) (know 5, 6, 7 will follow)

– Continuous sample space → noncountable outcomes

↳ e.g. new bulb put on life test, elapsed time at failure recorded → $S = \{t : t \in \mathbb{R}, t \geq 0\}$ (non-countable)

anything involving real no. cannot count $(2, \frac{1}{2}, \sqrt{3})$

→ Event is a subset of sample space S → represented by only one set

– Complementary Events → $A' = S \setminus A$

– Compound Events → A or B = $A \cup B$ (A will occur/B will occur (or both)), A and B = $A \cap B$

Mutually Exclusive Events

↳ Events that cannot happen at the same time → $A \cap B = \emptyset$ $[P(A \cup B \cup C) = P(A) + P(B) + P(C)]$

Probability

↳ Let A be an event & P(A) denotes probability of occurrence of event A. If $P(A) = 0$ → event A is certain not to occur. If $P(A) = 1$ → event is certain to occur

1. Relative Frequency Approach

↳ After n repetitions, an event A occurs k times → $P(A) = \frac{1}{n} \times k$

E.g. A die is tossed 100 times & comes up '1' 20 times. Dice tossed again, what is the probability that it will come up

a) '1' b) any one of $\{2, 3, 4, 5, 6\}$?

a) P(getting '1') $= \frac{20}{100} = 0.2$

b) P(getting any one of $\{2, 3, 4, 5, 6\}$) $= \frac{80}{100} = 0.8$

2. Equally Likely Outcomes Approach

↳ Sample space S contain n sample points, suppose each sample point in S is equally likely to occur. Let A be an event, $A \subset S$, there are k sample points in A. Then, $P(A) = \frac{k}{n}$

E.g. In a fruit basket, there are 5 apples, 3 bananas, 4 oranges & 6 pears. If one fruit is chosen at random, what is the probability that a) an apple, b) an orange or a banana c) a banana or pear is chosen? (every fruit is equally likely to be chosen)

a) P(an apple is chosen) $= \frac{5}{18}$

b) P(a banana or pear is chosen) $= \frac{4+3}{18} = \frac{7}{18}$

c) P(a banana or pear is chosen) $= \frac{3+6}{18} = \frac{1}{2}$

3. Finite Sample Space

Outcome	A_1	A_2	...	A_n
Probability	$P(A_1)$	$P(A_2)$	...	$P(A_n)$

} probability distribution

add tgt = 1

4. Independence of Events

→ Two events are independent if the occurence of one event does not change the probability of occurrence of the other event. If probability is affected, events are dependent

→ A & B are independent $\Longleftrightarrow P(A \wedge B) = P(A)P(B)$ → goes both ways → need $P(A)$, $P(B)$, $P(A \wedge B)$ to check independence

E.g. Let A = event that family has children of both sexes, B = event that family has at least one boy. Are A & B independent events if a family has two children?

Sample space $S = \{bb, bg, gb, gg\}$

$A = \{bg, gb\} \rightarrow P(A) = \frac{2}{4}$

$B = \{bb, bg, gb\} \rightarrow P(B) = \frac{3}{4}$

$A \wedge B = \{bg, gb\} \rightarrow P(A \wedge B) = \frac{2}{4}$

$P(A \wedge B) \neq P(A)P(B)$, A & B are dependent events

5. Conditional Probability

→ Suppose event B has occurred & we ask for probability of event A occurring → conditional probability of event A given B, $P(A|B) = \frac{P(A \wedge B)}{P(B)}$ → $P(A \wedge B) = P(A|B)P(B)$

→ condition that B has already happened

→ If A & B are independent → $P(A|B) = \frac{P(A \wedge B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A)$

E.g. In a country club, 75% of members play golf, 65% of members play tennis, 50% of members play both golf & tennis. A club member is selected at random → $P(G) = 0.75$, $P(T) = 0.65$, $P(G \wedge T) = 0.5$

Let G = {a member plays golf}, T = {a member plays tennis}.

a) If he plays golf (given), what is the probability that he also plays tennis?

$$P(T|G) = \frac{P(T \wedge G)}{P(G)} = \frac{0.5}{0.75} = \frac{2}{3}$$

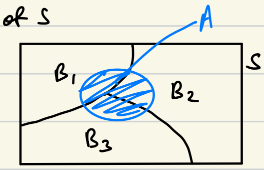
b) What is the probability that he plays golf or tennis?

$$P(G \text{ or } T) = P(G \cup T) = P(G) + P(T) - P(G \wedge T)$$

$$= 0.75 + 0.65 - 0.5 = 0.9$$

Partitions, Total Probability, Bayes Theorem

→ If $B_1, B_2, \ldots B_k$ are disjoint (no intersection b/w them) subsets of S & $B_1 \cup B_2 \cup B_3 \ldots \cup B_k = S$, then $\{B_1, B_2, \ldots, B_k\}$ is a partition of S



Let A be an event of S. $A = (A \wedge B_1) \cup (A \wedge B_2) \cup \ldots \cup (A \wedge B_k) \rightarrow (A \wedge B_i)$'s are mutually exclusive (no intersect)

$P(A) = P(A \wedge B_1) + P(A \wedge B_2) + \ldots P(A \wedge B_k)$

Using $P(A \wedge B_j) = P(A|B_j)P(B_j)$ (replace), $P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_k)P(B_k)$

1. Total Probability Theorem $\rightarrow P(A) = \sum_{j=1}^{k} P(A|B_j)P(B_j)$

E.g. Square box contains 3 red & 2 blue marbles, round box contains 2 red & 8 blue marbles. A fair coin is tossed. If coin is head, marble picked from square box, if tail, marble picked from round box. Find probability that red marble is chosen.

→ B_1 & B_2 is disjoint subsets & union is S → marble pick from one of the boxes

Let A = {red marble chosen}, B_1 = {marble pick from square box}, B_2 = {marble picked from round box}

$P(B_1) = 0.5$, $P(B_2) = 0.5$, $P(A|B_1) = \frac{3}{3+2}$, $P(A|B_2) = \frac{2}{2+8}$

Using Total Probability Theorem,

$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$

$= 0.6 \times 0.5 + 0.2 \times 0.5 = 0.4$

2. Bayes Theorem $\longrightarrow P(B_r|A) = \dfrac{P(A|B_r)P(B_r)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)}$, $r = 1, 2, \ldots, k$ → total probability theorem

E.g. From above, suppose we don't know if the coin turns up head or tails, but we were told a red marble is chosen.

a) probability that red marble chosen from square box?

$P(B_1|A) = \dfrac{P(A|B_1)P(B_1)}{P(A)}$

$= \dfrac{(0.6)(0.5)}{0.4} = 0.75$

b) probability red marble chosen from round box?

$P(B_2|A) = \dfrac{P(A|B_2)P(B_2)}{P(A)}$

$= \dfrac{(0.2)(0.5)}{0.4} = 0.25$

$\sum_{j=1}^{k} P(B_j|A) = 1$

Random variable

↳ random variable (r.v.) is an uncertain quantity whose value depend on chance

1. Discrete random variable (values countable)

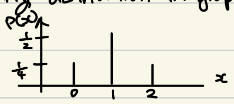
- → $P(k) \equiv P(X=k) \rightarrow$ probability mass function (pmf) of X, $0 \le P(x) \le 1$, $\sum_k P(k) = 1$
- ↳ Let r.v. X be the no. of heads obtained when two coins are tossed. The possible values for X are 0, 1, 2.

$S = \{hh, ht, th, tt\} \rightarrow P(X=0) = P(\{tt\}) = \frac{1}{4}$, $P(X=1) = \frac{1}{2}$, $P(X=2) = \frac{1}{4}$

Probability distribution table for discrete r.v. X:

x	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Probability distribution in graphic form:



2. Continuous random variable (take on any value in an interval of numbers)

- → probability density function (pdf), $f(x) \ge 0$ for all x, $P(a<X<b) = \int_a^b f(x)\,dx$ → area under curve, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

  $= P(a \le X \le b) = P(a \le X < b) = P(a < X \le b)$
- ↳ Let $f(x)$ be the probability density function of a random variable X given by $f(x) = \begin{cases} c(x^{-\frac{1}{2}}), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$, c is constant

a) Find c b) Find probability $P(X \le \frac{1}{4})$

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \longrightarrow \int_0^1 c(x^{-\frac{1}{2}})\,dx = 1$

$c\left[\dfrac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^1 = 1 \rightarrow c = \frac{1}{2}$ → from range $0 \le x \le 1$

b) $P(X \le \frac{1}{4}) = \int_0^{\frac{1}{4}} f(x)\,dx = \int_0^{\frac{1}{4}} c(x^{-\frac{1}{2}})\,dx$

$= c\left[\dfrac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{\frac{1}{4}} = \frac{1}{2}$

Distribution Functions

- → Distribution function F of X defined by $F(x) = P(X \le x)$
- → X discrete r.v. → $F(x) = \sum_{k \le x} P(X=k) \longrightarrow F(2) = P(X \le 2) = P(X=1) + P(X=2)$ for $x \in \{1, 2, 3, 4\}$
- ↳ X continuous r.v. → $F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy$, probability density function → $F'(x) = f(x)$

  ↳ $P(a<X<b) = \int_a^b f(x)\,dx = F(b) - F(a)$

E.g. Let X be a continuous r.v. having the probability density function $f(x) = \begin{cases} 0.75(1-x^2), & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

a) Find distribution function of X.

For $x < -1$, $F(x) = P(X \le x) = 0$

For $-1 \le x \le 1$, $F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-1}^{x} 0.75(1-y^2)\,dy$ → for $-1 \le x \le 1$ range

$= 0.5 + 0.75x - 0.25x^3$

For $x > 1$, $F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-1}^{1} f(y)\,dy = 1$ → entire area under probability density function

$$F(x) = \begin{cases} 0, & x < -1 \\ 0.5 + 0.75x - 0.25x^3, & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

b) Find $P(-0.25 \le X \le 2)$ → 2 is outside, just write 1 to be inside range

$P(-0.25 \le X \le 2) = \int_{-0.25}^{1} f(y)\,dy = 0.6836$ OR

$P(-0.25 \le X \le 2) = F(2) - F(-0.25) = 0.6836$

c) Find x such that $P(X \le x) = 0.95$ → not 0/1

$P(X \le x) = F(x) = 0.95 \rightarrow 0.5 + 0.75x - 0.25x^3 = 0.95$

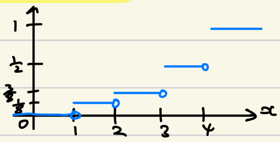
$\therefore x \approx 0.73$

E.g. The probability mass function of a random variable X is shown in table

x	1	2	3	4
$P(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$

Find the distribution function $F(x)$ & sketch graph of $F(x)$.

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{8}, & 1 \le x < 2 \rightarrow P(X=1) \\ \frac{3}{8}, & 2 \le x < 3 \rightarrow P(X=1)+P(X=2) \\ \frac{1}{2}, & 3 \le x < 4 \\ 1, & x \ge 4 \end{cases}$$



Expected Value

→ expected value (mean/average) of a r.v. $X \rightarrow E(X) / \mu$

$$E(X) = \begin{cases} \sum_x x P(x), & X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx, & X \text{ is continuous} \end{cases}$$

X becomes $g(x)$

$$E[g(X)] = \begin{cases} \sum_x g(x) P(x), & X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\,dx, & X \text{ is continuous} \end{cases}$$

→ Rules: $E(c) = c$, $E(cX) = cE(X)$, $E(X+Y) = E(X) + E(Y)$ ⟶ $E(aX + bY) = aE(X) + bE(Y)$

Variance & Standard Deviation

→ $Var(X)$ measures the spread of a random variable about its mean → the larger the variance, the further away from mean

$Var(X) = E(X^2) - [E(X)]^2$

→ Rules: $Var(c) = 0$, $Var(cX) = c^2 Var(X)$, $Var(X+Y) \neq Var(X) + Var(Y)$

→ $\sigma_x = \sqrt{Var(X)}$

$Y = aX + b$

$E(Y) = aE(X) + b$

$Var(Y) = a^2 Var(X)$ no more b!

E.g. $P(x) = \begin{cases} \frac{1}{3}, & x = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$. Find mean & variance of X

$E(X) = \sum_x x P(x) = (-1)(\frac{1}{3}) + 0(\frac{1}{3}) + 1(\frac{1}{3}) = 0$ → mean = 0

$E(X^2) = \sum_x x^2 P(x) = (-1)^2(\frac{1}{3}) + 0^2(\frac{1}{3}) + 1^2(\frac{1}{3}) = \frac{2}{3}$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{3} - 0 = \frac{2}{3}$ → variance $= \frac{2}{3}$

E.g. $f(x) = \begin{cases} cx, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$, where c is a constant. Find c & the mean & variance of X.

$\int_{-\infty}^{\infty} f(x)\,dx = 1 \rightarrow \int_0^1 cx\,dx = 1$

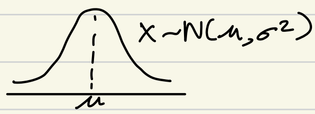
$c[\frac{x^2}{2}]_0^1 = 1 \rightarrow \frac{c}{2} = 1 \rightarrow c = 2$

$E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x(2x)\,dx = \frac{2}{3}$

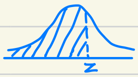
$E(X^2) = \int_0^1 \overbrace{x^2}^{g(x)} f(x)\,dx = \int_0^1 x^2(2x)\,dx = \frac{1}{2}$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}$

Normal Distribution

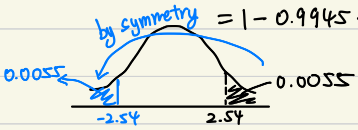


- → continuous r.v. density function has a symmetrical bell-shaped graph
  - $X\sim N(\mu,\sigma^2) \rightarrow X$ is normally distributed with mean μ & variance σ^2
- → $Z\sim N(0,1) \rightarrow$ standard normal r.v. $\rightarrow Z=\frac{X-\mu}{\sigma}$
- → Distribution function of $Z \rightarrow \phi(z)=P(Z\leq z)$ → given in table A1



E.g. $P(Z\geq 2.54)=1-P(Z<2.54)=1-\phi(2.54)$

by symmetry $=1-0.9945=0.0055$



E.g. Evaluating Arbitary Normal Probabilities → Let $X\sim N(40, 2.25)$. Find $P(39\leq X\leq 42)$ ($\mu = 40$, $\sigma^2 = 2.25$ → $\sigma=\sqrt{2.25}=1.5$)

$P(39\leq X\leq 42)=P\left(\frac{39-\mu}{\sigma}\leq Z\leq \frac{42-\mu}{\sigma}\right)$

$=P(-0.67\leq Z\leq 1.33)$

$=\phi(1.33)-\phi(-0.67)=0.9082-0.2514=0.6568$

E.g. A certain brand of battery last on the average 3 weeks, with a standard deviation of 0.5 week. Assume that the battery lives are normally distributed.

a) What percentage of batteries will last more than 4 weeks?

b) Suppose at least 80% of batteries will last at least α weeks. Find the largest possible α.

Let X = battery life. $X\sim N(3, 0.5^2)$

a) $P(X>4)=P\left(Z>\frac{4-3}{0.5}\right)$

$=P(Z>2)=1-P(Z\leq 2)$

$=1-\phi(2)=1-0.9772=0.0228$

∴ 2.28% of batteries will last more than 4 weeks

b) $P(X\geq \alpha)\geq 0.8$

$P(X\geq\alpha)=P\left(Z\geq\frac{\alpha-3}{0.5}\right)$

$=1-P\left(Z<\frac{\alpha-3}{0.5}\right)=1-\phi\left(\frac{\alpha-3}{0.5}\right)$

$1-\phi\left(\frac{\alpha-3}{0.5}\right)\geq 0.8 \rightarrow \phi\left(\frac{\alpha-3}{0.5}\right)\leq 0.2$

From table A1, we see that $\phi(-0.84)=0.2005$, $\phi(-0.85)=0.1977$ ← choose !

$\frac{\alpha-3}{0.5}\leq -0.85 \rightarrow \alpha\leq 2.575$

∴ largest possible $\alpha=2.575$ weeks

Poisson Distribution

→ models the no. of random arrivals of event within a period of time/space ☆☆

$X\sim P(\lambda) \rightarrow P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$ → λ: average events expected to occur

E.g. At a housing estate, power failures occur randomly & independently of one another at the rate of 4.2 per year. Thus probability distribution of X (no. of power failures occuring during a given period of time) can be modelled by Poisson distribution. Find the probability of

a) No power failure during a given 4-month period

b) No more than 2 power failures occur in a given 6-month period

a) 4.2 per year → 4 months period → $\lambda=\frac{4.2}{3}=1.4$

Let X = no. of power failures during any 4-month period. $X\sim P(1.4)$

$P(X=0)=\frac{e^{-1.4}(1.4)^0}{0!}=e^{-1.4}=0.247$

b) 6-month period $\rightarrow \lambda = \frac{4.2}{2} = 2.1$

Let X = no. of power failures during any 6-month period. $X \sim P(2.1)$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{e^{-2.1}(2.1)^0}{0!} + \frac{e^{-2.1}(2.1)^1}{1!} + \frac{e^{-2.1}(2.1)^2}{2!}$

$= e^{-2.1}\left[1 + 2.1 + \frac{(2.1)^2}{2}\right] = 0.650$

Binomial Distribution

→ binomial r.v. X ~~~~is discrete~~~~ represents the no. of success in n trails, probability of success on a single trial equal to p

→ $X \sim B(n, p) \rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $E(X) = np$, $Var(X) = np(1-p)$

E.g. In a durian-tree orchard, it is found that 5% of the durians produced are rejected because they are tasteless. What is the probability that a sample of 12 durians contains a) exactly 2, b) at least 2 rejects?

Let X = no. of rejected durians in a sample of 12 durians, $p = 5\% = 0.05 \rightarrow X \sim B(12, 0.05)$

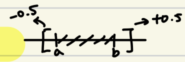
a) exactly 2 rejects $\rightarrow P(X=2) = \binom{12}{2}(0.05)^2(0.95)^{12-2} = 0.099$

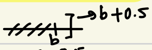
b) at least 2 rejects $\rightarrow P(X \geq 2) = 1 - P(X < 2)$

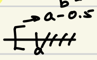
$= 1 - [P(X=0) + P(X=1)]$

$= 1 - (0.54 + 0.341) = 0.12$

→ Envelope principle (when Δ to normal)

1. $P_b(a \leq X \leq b) \approx P_n(a - 0.5 \leq X \leq b + 0.5)$ 

2. $P_b(X \leq b) \approx P_n(X \leq b + 0.5)$ 

3. $P_b(X \geq a) \approx P_n(X \geq a - 0.5)$ 

E.g. A fair coin is tossed 100 times. Find the probability that tails occur a) exactly 60 times b) less than 45 times

X = no. of tails (→ successes) in 100 tosses (→ binomial experiment), $n = 100$, $p = 0.5$

Since $np = 50 > 5$ & $n(1-p) = 50 > 5$, can use normal distribution to approximate the binomial probability,

$\mu = np = 50$, $\sigma = \sqrt{np(1-p)} = 5 \longrightarrow X \sim N(50, 5^2)$

a) $P_b(X=60) \approx P_n(60 - 0.5 \leq X \leq 60 + 0.5) = P_n(59.5 \leq X \leq 60.5)$

$= P_n\left(\frac{59.5-50}{5} \leq Z \leq \frac{60.5-50}{5}\right) = P_n(1.9 \leq Z \leq 2.1)$

$= \phi(2.1) - \phi(1.9) = 0.9821 - 0.9713 = 0.0108$ → same as $P_b(X=60) = \binom{100}{60}(0.5)^{60}(0.5)^{100-60}$

b) $P_b(X < 45) = P_b(X \leq 44)$

$\approx P_n(X \leq 44 + 0.5) = P_n(X \leq 44.5)$

$= P_n\left(Z \leq \frac{44.5-50}{5}\right)$

$= 0.1357$

Sample Mean & Sample Variance

→ population mean $\rightarrow \mu$, population variance $\rightarrow \sigma^2$, sample mean $\rightarrow \bar{x}$, sample variance $\rightarrow s^2$

→ $\bar{x} = \mu = \frac{\Sigma x}{n}$, $s^2 = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$ (→ sample variance), when $\bar{X}$ is approximately normal $\rightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

a) if $n < 30$, assume that X is approx normal $\rightarrow \bar{X}$ will also be approx normal

b) if $n \geq 30$, use Central Limit Theorem $\rightarrow \bar{X}$ approx normal

E.g. A study shows that the average no. of pens a university student brings to class is 1.75 & standard deviation is 0.65. In a sample of 50 university students, find the probability that the mean number of pens (→ sample mean) brought to class is more than 2.

Let X = the number of pens a university student brings to class. $\mu = 1.75$ & $\sigma = 0.65$

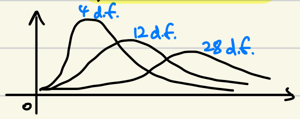
Since sample size $n = 50\ (\geq 30)$, by central limit theorem $\rightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$P(\bar{X} > 2) = P\left(Z > \frac{2-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = P\left(Z > \frac{2-1.75}{\frac{0.65}{\sqrt{50}}}\right) = 0.0033$

↳ suppose random samples of size n corresponding to some r.v. X drawn from a population whose size is much larger than n. $X \sim N(\mu, \sigma^2)$ same

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \rightarrow \text{chi-square with } (n-1) \text{ deg of freedom}$$

continuous probability distribution



As d.f. ↑, graph of χ^2 become more bell-like & more like normal distribution

E.g. Suppose time required for bus 179 to start from Interchange & arrive at EEE form a normal distribution with a standard deviation of 0.5min. If sample of 17 arrival time selected at random, find probability that sample variance is greater than 0.5.

Let X = time required for bus 179 to start from Interchange & arrive at EEE. $\sigma = 0.5$, $n=17$

to use this, x must be normal

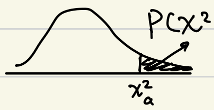
$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \rightarrow (n-1)=16$$

$$P(s^2 > 0.5) = P\left(\frac{(n-1)s^2}{\sigma^2} > \frac{(n-1)(0.5)}{\sigma^2}\right)$$

$$= P\left(\frac{(n-1)s^2}{\sigma^2} > \frac{(17-1)(0.5)}{(0.5)^2}\right) = P\left(\frac{(n-1)s^2}{\sigma^2} > 32\right)$$

$$= P(\chi^2 > 32) = 0.01$$

→ degree of freedom 16
→ value in table



$$P(\chi^2 > \chi^2_\alpha) = \alpha$$

32 ↑, 0.01 ↑

Confidence Interval for μ

↳ suppose r.v. X defined on population → mean μ. From random sample size n → value $\bar{x}$ of sample mean $\bar{X}$ can be obtained

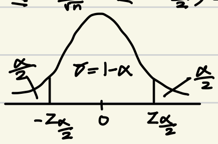
1. When σ is known → sample mean $\bar{X}$ need to be approximately normally distributed

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \rightarrow Z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

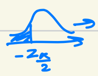
Let γ be confidence level → $\alpha = 1-\gamma$

$$P\left(-Z_{\frac{\alpha}{2}} \leq \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq Z_{\frac{\alpha}{2}}\right) = \gamma \rightarrow P\left(\bar{X} - \overbrace{Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}}^{\bar{x}-E} \leq \mu \leq \bar{X} + Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = \gamma$$

→ make subject



$$\phi\left(-Z_{\frac{\alpha}{2}}\right) = \frac{\alpha}{2}$$

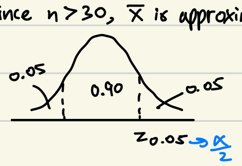


→ z table

E.g. The GMAT score can range from 200 to 800 with standard deviation of 100 (σ). NUS received more than 1000 GMAT scores. Director of admission ask that sample of 75 (n) scores to be chosen randomly to develop 90% (γ) confidence interval for mean (μ) GMAT received last year. Sample mean ($\bar{x}$) = 534. Construct desired interval estimate.

Let X = GMAT score. $\sigma = 100$, $n=75$, $\bar{x} = 534$, $\gamma = 0.90$

Since $n > 30$, $\bar{X}$ is approximately normally distributed by Central Limit Theorem



$$\phi(Z_{0.05}) = 0.9 + 0.05 = 0.95$$

$$Z_{0.05} = 1.65 \text{ or } 1.64$$

$$E = Z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}} = 1.65 \times \frac{100}{\sqrt{75}} = 18.93$$

determine confidence level → $[\bar{x}-E, \bar{x}+E] = [534-18.93, 534+18.93]$

$$= [515.07, 552.93]$$

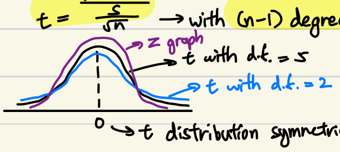
there is a 10% risk that interval (515 & 553) will not include μ

∴ 90% confidence interval for mean μ of X is [515.07, 552.93]

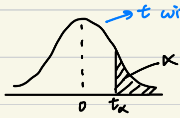
2. When σ is unknown → $\bar{X}$ is normally distributed

↳ suppose r.v. X has mean μ. Let $\bar{x}$ be sample mean of random samples of size n & let s be sample standard deviation.

$$t = \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} \rightarrow \text{with } (n-1) \text{ degrees of freedom}$$



→ t distribution symmetrical & has mean of 0



$P(t > t_\alpha) = \alpha$

↳ from stat table A3

$P(t > t_{0.05}) = 0.05$ ⇒ $t_{0.05} = 1.796$

$P(\bar{X} - \overbrace{t_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}}^{E} \le \mu \le \bar{X} + t_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}) = \gamma$ → t distribution with (n−1) d.f.

E.g. An experiment has been carried out to evaluate a new sliming process. 6 (n) participants go through the whole process over a period of 3 days. The weight loss (kg) of participants at end of experiments are 0.46, 0.61, 0.52, 0.48, 0.57, 0.54. Find 95% (γ) confidence interval for mean weight loss resulting from 3-day process.

Let X = weight loss resulting from 3-day process. σ unknown, $n=6$, $\gamma = 0.95$

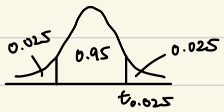
Assume that X is normally distributed, then $\bar{X}$ is normally distributed,

$\bar{x} = \frac{1}{6}(0.46+0.61+0.52+0.48+0.57+0.54) = 0.53$

$s^2 = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right) = \frac{1}{6-1}\left((0.46^2+0.61^2+0.52^2+0.48^2+0.57^2+0.54^2) - \frac{(0.46+0.61+0.52+0.48+0.57+0.54)^2}{6}\right)$

$= \frac{1}{5}(1.701 - 1.6854) = 0.00312$

$s = \sqrt{0.00312} = 0.0559$



6−1=5 degrees of freedom

$t_{0.025} = 2.571$

$E = t_{\frac{\alpha}{2}} \times \frac{s}{\sqrt{n}} = 2.571 \times \frac{0.0559}{\sqrt{6}} = 0.059$

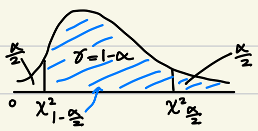
∴ 95% confidence interval of mean μ of X is $[\bar{x}-E, \bar{x}+E] = [0.471, 0.589]$

Confidence Interval for σ^2

↳ use chi-square → $\chi^2 \equiv \frac{(n-1)s^2}{\sigma^2}$, $(n-1)$ degrees of freedom

Let γ be the confidence level → $\alpha = 1-\gamma$

$P(\chi^2_{1-\frac{\alpha}{2}} < \frac{(n-1)s^2}{\sigma^2} \le \chi^2_{\frac{\alpha}{2}}) = \gamma \rightarrow P\left(\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}}\right) = \gamma$ make subject

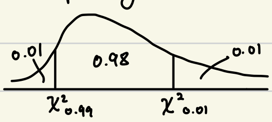


γ confidence interval for σ^2 → $\left[\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}}, \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}}\right]$

if want σ → square root

E.g. A pharmaceutical company interested in determining variance of potency measurements for a certain painkiller. 20 (n) painkillers produced a variance equal to 0.0018 (s^2). Construct a 98% (γ) confidence interval for the variance (σ^2) of potency measurements of painkillers.

Let X = potency measurement of painkiller & assume X is normally distributed.



χ^2 with 19 d.f.

$\chi^2_{\frac{\alpha}{2}} = \chi^2_{0.01} = 36.19$, $\chi^2_{1-\frac{\alpha}{2}} = \chi^2_{0.99} = 7.63$

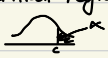
∴ 98% confidence interval is $\left[\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}}, \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}}\right] = \left[\frac{19(0.0018)}{36.19}, \frac{19(0.0018)}{7.63}\right] = [0.000945, 0.00448]$

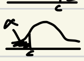
Hypothesis Testing

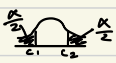
→ process of verifying the validity of the statistical hypothesis based on observations made from random samples in population

$H_0: \mu = ?$, $H_1: \mu \gtrless\neq ?$ (either one)

→ Significant level α determine critical region (α) → if test statistic falls within critical region → reject H_0 & accept H_1
& if test statistic does not fall within critical region → accept H_0 & reject H_1

a) $H_1: \mu > \mu_0$ → upper-tail test, $P(\bar{X} > c) = \alpha$ 

b) $H_1: \mu < \mu_0$ → lower-tail test, $P(\bar{X} < c) = \alpha$ 

c) $H_1: \mu \neq \mu_0$ → two-tail test, $P(\bar{X} < c_1) = P(\bar{X} > c_2) = \frac{\alpha}{2}$ 

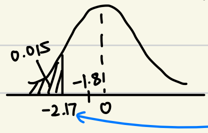
1. When σ known $\rightarrow z = \dfrac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}$

E.g. An engineer at a car service centre designed a new procedure to shorten a standard 5-minute checking time. A random sample of $\overset{n}{\text{36 cars}}$ required an average of $\overset{\mu}{\text{4.85 min}}$ for checking, standard deviation of checking time is $\overset{\sigma}{\text{0.5min}}$. Test whether the new procedure has reduced mean checking time to less than 5min using a 1.5% level of significance

X = checking time (min). $\sigma = 0.5$min, $\alpha = 0.015$. Since $n = 36 > 30 \rightarrow$ CLT $\bar{X}$ approximately normal

$H_0: \mu = 5$, $H_1: \mu < 5$ (lower-tail test)

$$z = \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{4.85-5}{\frac{0.5}{\sqrt{36}}} = -1.8$$



$P(z < c) = \alpha = 0.015$

$c = -2.17$

Since test statistic does not fall within critical region, no significant evidence against H_0 at 1.5% level. We will accept H_0 & reject H_1 $\rightarrow$ sample evidence not sufficient to indicate new procedure has reduced mean checking time to below 5 min.

2. When σ unknown $\rightarrow t = \dfrac{\bar{x}-\mu_0}{\frac{s}{\sqrt{n}}}$, d.f. $= n-1$

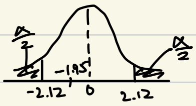
E.g. For one type of printer, the list price is $\overset{\mu}{\$750}$ & manufacturer wants to know whether current mean retail price differs from list price. $\overset{n}{\text{17 retailers}}$ are sampled & 17 prices give mean & s.d. of $\bar{x} = \$732$, $s = \$38$. Does this sample provide sufficient evidence to conclude that mean retail price differs from list price of \$750? Use $\alpha = 0.05$.

Let X = retail price of printer (\$).

Since $n = 17 < 30$, we assume X is normally distributed so $\bar{X}$ is normally distributed

$H_0: \mu = 750$, $H_1: \mu \neq 750$ (two-tail test)

$$t = \frac{\bar{x}-\mu_0}{\frac{s}{\sqrt{n}}} = \frac{732-750}{\frac{38}{\sqrt{17}}} = -1.95$$



$P(t < c_1) = P(t > c_2) = \frac{\alpha}{2} = 0.025$

$\therefore$ Accept H_0 & reject H_1 ...

Test Concerning the Population Variance σ^2

$\hookrightarrow H_0: \sigma^2 = \sigma_0^2$, $H_1: \sigma^2 \lessgtr\neq \sigma_0^2$, using chi-square r.v. $\rightarrow \hat{\chi}^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$

$\hat{\chi}^2 \rightarrow$ value of test statistic

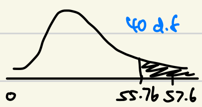
$s^2 \rightarrow$ sample variance

E.g. Over the years, weight of apple pies sold normally distributed with mean 75g & standard deviation of 8g. Recently, apple pies seem to get smaller. A sample of 41 apple pies has mean $\bar{x} = 73$g & s.d. $s = 9.6$g. Assuming that the weights are still normally distributed, test hypothesis the variation of the weights has increased, using $\alpha = 0.05$

Let X = the weight of an apple pie (g). $\alpha = 0.05$

$H_0: \sigma^2 = 64$, $H_1: \sigma^2 > 64$ (upper-tail test)

$$\hat{\chi}^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{40(9.6)^2}{64} = 57.6$$



$P(\chi^2 > c) = \alpha = 0.05$

$c = \chi^2_{0.05} = 55.76$

$\therefore$ Since test statistic fall within critical region $\rightarrow$ significant evidence against H_0 at 5% level $\rightarrow$ reject H_0 & accept H_1.

Conclude variation of weights has increased.

Numerical Methods

- generate 'approximate' solution to the required precision [e.g. $y(t) = \sin 2t + \cos 2t$, $y(0.1) = 1.17873$] → cannot find eqn of t → only can find $y(t)$ at specific values of t
- $\tilde{\alpha}$ be approximate value of the exact value α:
  1. Absolute error in $\tilde{\alpha}$: $\varepsilon = \alpha - \tilde{\alpha} \rightarrow |\varepsilon| < 0.5 \times 10^{-n}$ (correct to n decimal places)
  2. Relative error in $\tilde{\alpha}$: $r = \frac{\varepsilon}{\alpha} = \frac{\alpha - \tilde{\alpha}}{\alpha} \rightarrow |r| < 0.5 \times 10^{1-n}$ (correct to n significant digits)

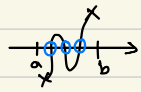
E.g. Consider $\alpha = 23.494$ & $\tilde{\alpha} = 23.491$. $\tilde{\alpha}$ is correct to 2 d.p & 4 s.f.

$|\varepsilon| = 23.494 - 23.491 = 0.003 < 0.5 \times 10^{-2} = 0.005$

$|r| = \frac{23.494 - 23.491}{23.494} \approx 0.00013 < 0.5 \times 10^{1-4} = 0.0005$

Solution of Nonlinear Equations

1. Solve by Iteration



- f continuous in interval $[a, b]$ & $f(a) \cdot f(b) < 0 \rightarrow$ f have at least a zero in $[a, b]$
- approx root $= \tilde{x}$, exact root $= x^*$, both in $[a, b]$
  - $|f'(x)| \geq m > 0$ for all $x \in [a, b] \rightarrow m = \min|f'(x)|$
  - absolute error $= |\tilde{x} - x^*| \leq \frac{|f(\tilde{x})|}{m} < 0.5 \times 10^{-n}$ (n d.p.)

2. Fixed-Point Iteration Method

- transform $f(x) = 0$ to $x = g(x) \rightarrow x_{n+1} = g(x_n)$, $n = 0, 1, 2 \ldots$
- x^* be solution of $x = g(x) \rightarrow |g'(x)| \leq L < 1$ for all $x \in$ interval containing $x^* \rightarrow L = \max|g'(x)|$

Steps of Fixed-Point Iteration Method

1. Find $J = [a, b]$ that contains root of $f(x) = 0$
2. Rewrite $f(x) = 0$ to $x = g(x)$
3. Check: $|g'(x)| \leq L < 1$, $x \in J$ → max $|g'(x)|$ → must fulfil
4. Start: $x_{n+1} = g(x_n)$, $n = 0, 1, 2 \ldots$ with any $x_0 \in J$
5. Only stop when 3 d.p. correct: $\frac{|f(x_n)|}{m} < 0.5 \times 10^{-3}$ → $m = \min|f'(x)|$

E.g. Set up an iteration process for $f(x) = x^2 - 3x + 1 = 0$ to 2 d.p.

Since $f(0) = 1 > 0$ & $f(1) = -1 < 0 \rightarrow J = [a, b] = [0, 1]$

put x as subject → $x = g_1(x) = \frac{1}{3}(x^2 + 1) \longrightarrow x_{n+1} = \frac{1}{3}(x_n^2 + 1)$

With $J = [0, 1]$, $L = \max|g_1'(x)| = \max\left|\frac{2x}{3}\right| = \frac{2}{3} < 1$ ✓

$f(x) = x^2 - 3x + 1 \longrightarrow m = \min|f'(x)| = \min|3 - 2x| = 1$

$\frac{|f(x_n)|}{m} = |f(x_n)| < 0.5 \times 10^{-2}$

Choose $x_0 = 1 \in J$

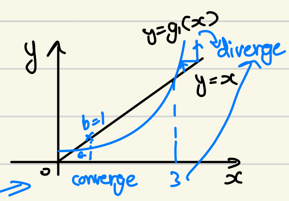
n	x_n	$\lvert f(x_n)\rvert/m$
0	1	
1	0.66666667	0.56
2	0.48148148	0.21
3	0.41060814	0.06
4	0.38953301	0.017
5	0.38391199	$0.00435 < 0.5 \times 10^{-2}$

$x_{n+1} = \frac{1}{3}(x_n^2 + 1)$

sub x_n into $f(x)$

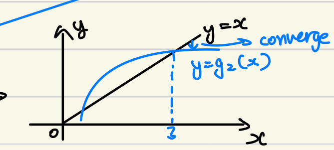
0.005

$\therefore x_5 = 0.38$ is correct to 2 d.p.



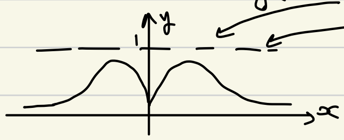
If choose $x_0 = 3 \notin J \rightarrow$ sequence will diverge

- $f(2) < 0$, $f(3) > 0 \rightarrow J = [a, b] = [2, 3]$
- $x^2 = 3x - 1 \rightarrow x = 3 - \frac{1}{x} \rightarrow x_{n+1} = 3 - \frac{1}{x_n}$ → larger



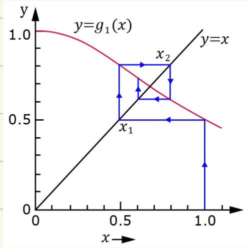
E.g. $f(x) = x^3 + x - 1 = 0 \longrightarrow g_1(x) = \frac{1}{1+x^2} \rightarrow |g_1'(x)| = \frac{2|x|}{(1+x^2)^2} < 1$

$x = 1 - x^3 \rightarrow$ no guarantee convergence



n	x_n		$\lvert f(x_n)\rvert/m$
0	0.7		
1	0.67114094	increase	0.027
2	0.68945064	decrease	0.017
3	0.67780886	increase	0.011
4	0.68520143		0.0069
5	0.68050311		$0.00437 < 0.5 \times 10^{-2}$

$\longrightarrow$



Polynomial Interpolation

– Newton's Divided Difference Interpolation

→ Given $(n+1)$ points (x_i, f_i), $i = 0, 1, \dots, n$, the Newton's divided difference interpolating polynomial $p_n(x)$ → by divided difference

$$p_n(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \dots (x - x_0)(x - x_1) \dots (x - x_{n-1}) f[x_0, x_1, \dots x_n]$$

$p_n(x)$ is better than Lagrange's representation when new data added

x_i	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
x_0	① $f[x_0]$			
		$\frac{②-①}{x_1 - x_0} = ⑤$		
x_1	② $f[x_1]$	$f[x_0, x_1]$		
		$\frac{③-②}{x_2 - x_1} = ⑥$	$\frac{⑥-⑤}{x_2 - x_0} = ⑧$	
x_2	③ $f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
		$\frac{④-③}{x_3 - x_2} = ⑦$	$\frac{⑦-⑥}{x_3 - x_1} = ⑨$	$\frac{⑨-⑧}{x_3 - x_0}$ → jump 3 steps
x_3	④ $f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$

→ Error inequality $\longrightarrow |f(x) - p_n(x)| \leq \frac{1}{(n+1)!} \left|\prod_{k=0}^{n} (x - x_k)\right| \times \max |f^{(n+1)}(\xi)|$ → upper estimate of error

E.g. With divided difference table, find the third degree interpolating polynomial $p_3(x)$ passing through the points $(-1, 6)$, $(0, 1)$, $(2, 3)$, $(5, 66)$.

x_i → $f[\cdot]$

x_i	$f[\cdot]$ 0	$f[\cdot,\cdot]$ 1	$f[\cdot,\cdot,\cdot]$ 2	$f[\cdot,\cdot,\cdot,\cdot]$ 3
-1	6			
		$\frac{1-6}{0-(-1)} = -5$		
0	1	-5		
		$\frac{3-1}{2-0} = 1$	$\frac{1-(-5)}{2-(-1)} = 2$	
2	3	1	2	
		$\frac{66-3}{5-2} = 21$	$\frac{21-1}{5-0} = 4$	$\frac{4-2}{5-(-1)} = \frac{1}{3}$
5	66	21	4	$\frac{1}{3}$

$$p_3(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]$$

$$= 6 + (x+1)(-5) + (x+1)(x-0)(2) + (x+1)(x-0)(x-2)\tfrac{1}{3}$$

$$= 6 - 5(x+1) + 2x(x+1) + \tfrac{1}{3} x (x+1)(x-2)$$

Numerical Methods for Differential Equations

1. Euler Method

→ only deal with ordinary differential equation (ODE) → an unknown function y (dependent variable) depend on 1 independent variable x

highest order of derivative

$y'' = \cos x$, $y(2) = 5$, $y'(2) = 0.3$ → initial conditions → conditions specified at same value of x ← Initial value problem (IVP) have to satisfy

$y(0) = 1$, $y(\pi) = 2$ → boundary conditions (BC) → conditions specified at boundary point of solution interval ← Boundary value problem (BVP) have to satisfy

→ $y_{n+1} = y_n + h f(x_n, y_n)$, $n = 0, 1, \dots$

E.g. Apply the Euler method to following IVP: $y' = x + y$, $y(0) = 0$. Choose $h = 0.2$ & compute $y_1, y_2 \dots, y_5$.

y_1 to y_5

$f(x, y) = x + y$, $x_0 = y_0 = 0$

$y_{n+1} = y_n + 0.2(x_n + y_n)$, $n = 0, 1, \dots$

h

$y_0 = 0$, $y_{n+1} = y_n + 0.2(x_n + y_n)$

n	x_n	y_n (8 d.p.)
0	0.0	0.000
1	0.2	0.000
2	0.4	0.040
3	0.6	0.128
4	0.8	0.274
$y_5 \to$ 5	1.0	0.489

$\downarrow th$ (between x_0 and x_1, and between x_1 and x_2)

→ Euler method is of **first order** ☆☆

2. Improved Euler Method

$y_{n+1} = y_n + \frac{1}{2}(a+b)$, $a = hf(x_n, y_n)$, $b = hf(x_{n+1}, y_n + a)$

E.g. Apply improved Euler method to example above → $h = 0.2$, $f(x,y) = x + y$, $x_0 = y_0 = 0$

$a = hf(x_n, y_n) = 0.2(x_n + y_n)$

$b = h\underbrace{f(x_{n+1}, y_n + a)}_{f(x,y)} = 0.2(x_{n+1} + y_n + a)$

$y_{n+1} = y_n + \frac{1}{2}(a+b)$, $n = 0, 1, \ldots$

x_n	y_n (8 d.p.)	a (8 d.p.)	b (8 d.p.)
0.0	0.0000	0.0	0.04
0.2	0.0200	0.044	0.0928
0.4	0.0884	0.2(0.4+0.0884)	...
0.6	0.2158	...	...
0.8	0.4153	...	...
1.0	0.7027	...	...

$\downarrow h$ (between x_n values)

→ Euler method is of second order

3. Runge-Kutta Method (fourth order)

$y_{n+1} = y_n + \frac{1}{6}(a + 2b + 2c + d)$

$a = hf(x_n, y_n)$ / $b = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}a)$ / $c = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}b)$ / $d = hf(x_n + h, y_n + c)$

E.g. Apply fourth order Runge-Kutta method to IVP $y' = (y - x - 1)^2 + 2$, $y(0) = 1$ for $0 \leq x \leq 0.4$ with $h = 0.2$

$f(x,y) = (y - x - 1)^2 + 2$, $x_0 = 0$, $y_0 = 1$

$a = hf(x_n, y_n) = 0.2[(y_n - x_n - 1)^2 + 2]$

$b = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}a) = 0.2[(y_n + 0.5a - x_n - 0.5h - 1)^2 + 2]$

$c = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}b) = 0.2[(y_n + 0.5b - x_n - 0.5h - 1)^2 + 2]$

$d = hf(x_n + h, y_n + c) = 0.2[(y_n + c - x_n - h - 1)^2 + 2]$

$y_{n+1} = y_n + \frac{1}{6}(a + 2b + 2c + d)$

n	x_n	y_n	a	b	c	d
0	0	1	0.4	0.402	0.4020402	0.40816405
1	0.2	1.402707408	...	...	...	...
2	0.4	1.822788917	...	...	...	...

☆☆ Systems & Higher Order Equations

- → $\tilde{u} = (u_1, u_2, \ldots, u_m)$, $\tilde{u} + \tilde{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_m + v_m)$, $\tilde{u} = \tilde{v} \to (u_1 = v_1, u_2 = v_2 \ldots)$, $\frac{1}{2}\tilde{u} = (\frac{1}{2}u_1, \frac{1}{2}u_2, \ldots \frac{1}{2}u_m)$
- → put ~ on all variables except x & h → independent variables
- → $\tilde{f}(x, \tilde{y}) = (f_1(x, \tilde{y}), f_2(x, \tilde{y}), \ldots, f_m(x, \tilde{y}))$ → $f_m(x, \tilde{y}) = f_m(x, y_1, y_2, \ldots, y_m)$
- → $\tilde{y}(x_0) = (y_1(x_0), y_2(x_0), \ldots, y_m(x_0))$
- → $\tilde{y}_0 = (y_{1,0}, y_{2,0}, \ldots, y_{m,0})$

Euler Method for Systems

↳ $\tilde{y}_{n+1} = \tilde{y}_n + h\tilde{f}(x_n, \tilde{y}_n) \longrightarrow \tilde{y}_n = (y_{1,n}, y_{2,n} \ldots, y_{m,n})$, $\tilde{y}_{n+1} = (y_{1,n+1}, y_{2,n+1}, \ldots y_{m,n+1})$

dependent variable
which grid point

Improved Euler Method for Systems

- $\tilde{y}_{n+1} = \tilde{y}_n + \frac{1}{2}(\tilde{a} + \tilde{b})$
- $\tilde{a} = h\tilde{f}(x_n, \tilde{y}_n)$
- $\tilde{b} = h\tilde{f}(x_{n+1}, \tilde{y}_n + \tilde{a})$ → $(y_{1,n} + a_1, y_{2,n} + a_2, \ldots y_{m,n} + a_m)$

Runge-Kutta Method for Systems

- $\tilde{y}_{n+1} = \tilde{y}_n + \frac{1}{6}(\tilde{a} + 2\tilde{b} + 2\tilde{c} + \tilde{d}) \longrightarrow y_{i,n+1} = y_{i,n} + \frac{1}{6}(a_i + 2b_i + 2c_i + d_i)$
- $\tilde{a} = h\tilde{f}(x_n, \tilde{y}_n) \longrightarrow a_i = hf_i(x_n, y_{1,n}, y_{2,n}, \ldots, y_{m,n})$
- $\tilde{b} = h\tilde{f}(x_n + \frac{1}{2}h, \tilde{y}_n + \frac{1}{2}\tilde{a}) \longrightarrow b_i = hf_i(x_n + \frac{1}{2}h, y_{1,n} + \frac{1}{2}a_1, y_{2,n} + \frac{1}{2}a_2, \ldots, y_{m,n} + \frac{1}{2}a_m)$
- $\tilde{c} = h\tilde{f}(x_n + \frac{1}{2}h, \tilde{y}_n + \frac{1}{2}\tilde{b}) \longrightarrow c_i = hf_i(x_n + \frac{1}{2}h, y_{1,n} + \frac{1}{2}b_1, y_{2,n} + \frac{1}{2}b_2 \ldots, y_{m,n} + \frac{1}{2}b_m)$
- $\tilde{d} = h\tilde{f}(x_n + h, \tilde{y}_n + \tilde{c}) \longrightarrow d_i = hf_i(x_n + h, y_{1,n} + c_1, y_{2,n} + c_2, \ldots, y_{m,n} + c_m)$

Eg. Solve the system $\frac{dy}{dx} = 2z - 1$, $y(0) = 0.5$; $\frac{dz}{dx} = 2y + 1$, $z(0) = 3.5$ to obtain approximate values of $y(0.2)$ & $z(0.2)$ to 3 d.p. Use $h = 0.1$

$f_1(x, y, z) = 2z - 1$, $f_2(x, y, z) = 2y + 1$, $x_0 = 0$, $y_0 = 0.5$, $z_0 = 3.5$

$m = 2$ since $y_1 \xrightarrow{\text{like}} y$, $y_2 \xrightarrow{\text{like}} z$

From $f_1(x, y, z) = 2z - 1$,

$a_1 = 0.1(2z_n - 1)$

$b_1 = 0.1[2(z_n + 0.5a_2) - 1] = 0.1(2z_n + a_2 - 1)$

$c_1 = 0.1[2(z_n + 0.5b_2) - 1] = 0.1(2z_n + b_2 - 1)$

$d_1 = 0.1[2(z_n + c_2) - 1]$

$y_{n+1} = y_n + \frac{1}{6}(a_1 + 2b_1 + 2c_1 + d_1)$

From $f_2(x, y, z) = 2y + 1$,

$a_2 = 0.1(2y_n + 1)$

$b_2 = 0.1[2(y_n + 0.5a_1) + 1] = 0.1(2y_n + a_1 + 1)$

$c_2 = 0.1[2(y_n + 0.5b_1) + 1] = 0.1(2y_n + b_1 + 1)$

$d_2 = 0.1[2(y_n + c_1) + 1]$ → $hf_2(x_n + h, \tilde{y}_n + \tilde{c}) = hf_2(x_n + h, y_n + c_1, z_n + c_2)$

$z_{n+1} = z_n + \frac{1}{6}(a_2 + 2b_2 + 2c_2 + d_2)$

n	x_n	y_n ✓	z_n ✓	a_1 ✓	a_2 ✓
0	0	0.5	3.5	0.6	0.2
1	0.1	1.124	3.762	0.6524	0.3248
2	0.2	1.813	4.154		

✓ → all need 8 d.p.

b_1 ✓	b_2 ✓	c_1 ✓	c_2 ✓	d_1 ✓	d_2 ✓
0.62	0.26	0.626	0.262	0.6524	0.3253
0.6849	0.3900	0.6914	0.3933	0.7311	0.4631

$\therefore y(0.2) \approx 1.813$ & $z(0.2) \approx 4.154$

Higher Order ODE

↳ Given mth order differential eqn → rewritten as system of m first order d.e. & solve

$\hookrightarrow$ set $y_1 = y,\ y_2 = y',\ y_3 = y'' \cdots y_m = y^{(m-1)} \longrightarrow y_1' = y_2,\ y_2' = (y')' = y_3 \ \cdots$

E.g. Consider IVP $y'' = -y$, $y(0) = 0$, $y'(0) = 1$. Using $h = 0.1$, find an approximate value of $y(0.2)$ to 3 d.p.

First convert the second order ODE to 2 first order ODE, let $y_1 = y$, $y_2 = y'$

$$y_1' = y' = y_2$$
$$y_2' = (y')' = -y_1$$
$$y_1(0) = 0,\ y_2(0) = 1$$

$f_1(x, y_1, y_2) = y_2$, $f_2(x, y_1, y_2) = -y_1$, $x_0 = 0$, $y_{1,0} = 0$, $y_{2,0} = 1$

method 1 $\rightarrow$ By Euler method, $\tilde{y}_{n+1} = \tilde{y}_n + h\tilde{f}(x_n, \tilde{y}_n)$

$$y_{n+1} = y_{1,n+1} = y_{1,n} + h f_1(x_n, y_{1,n}, y_{2,n})$$
$$= y_{1,n} + h y_{2,n}$$
$$y'_{n+1} = y_{2,n+1} = y_{2,n} + h f(x_n, y_{1,n}, y_{2,n})$$
$$= y_{2,n} - h y_{1,n}$$

n	x_n	$y_n = y_{1,n}$	$y'_n = y_{2,n}$
0	0	0 $\rightarrow y_{1,0} = 0$	1
1	0.1	0.1 $\rightarrow y_{1,1} = y_{1,0} + h(y_{2,0})$	1
2	0.2	0.2	0.99

$\therefore y(0.2) \approx 0.2$, $y'(0.2) \approx 0.99$

method 2 $\rightarrow$ By Runge-Kuttar method ...

Laplace Transform

↳ $f(t)$ is a given function with variable t for all $t \geq 0$. Laplace transform of $f(t)$ is:

$s = \sigma + j\omega$ (complex variable)

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

↳ $f(t)$ is the "inverse" of $F(s)$ → $f(t) = \mathcal{L}^{-1}[F(s)]$ for $t > 0$

$$\begin{array}{ccc} \frac{d}{dt}, \int & & +, - \\ f(t) & \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} & F(s) \\ \text{time domain} & & \text{s-domain} \end{array}$$

E.g. Find Laplace Transform of $\mathcal{L}(u(t)) = F(s)$

$$\mathcal{L}(u(t)) = \int_0^\infty u(t)\, e^{-st}\, dt$$

→ unit step function → from $x = 0 \to \infty = 1$

$$= \int_0^\infty 1\, e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{e^{-\infty}}{-s} - \frac{e^{-0}}{-s}$$

$$= \frac{1}{-s}(0 - 1) = \frac{1}{s}$$

→ to find Laplace Transform of unit ramp $f(t) = t\,u(t)$ need integration by parts → $\mathcal{L}[t] = \left[uv - \int v \frac{du}{dt}\, dt\right]_0^\infty$

Linearity of Laplace Transforms

↳ $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$

E.g. Find $\mathcal{L}(\sin t)$.

* $e^{j\theta} = \cos\theta + j\sin\theta$, $e^{-j\theta} = \cos\theta - j\sin\theta$ [Euler's Formula]

$$\sin t = \frac{e^{jt} - e^{-jt}}{2j}$$

$$\mathcal{L}(\sin t) = \frac{1}{2j}\mathcal{L}(e^{jt}) - \frac{1}{2j}\mathcal{L}(e^{-jt})$$

$$= \frac{1}{2j}\frac{1}{s-j} - \frac{1}{2j}\frac{1}{s+j}$$

$$= \frac{1}{2j}\frac{2j}{s^2+1} = \frac{1}{s^2+1}$$

$\mathcal{L}(\sin \omega_0 t) = \frac{\omega_0}{s^2 + \omega_0^2}$ (in table)

Shift/Exponential Property of Laplace Transforms

↳ function $u(t)$ shifted in t by an amount a, the result gave rise to an extra term e^{-as} in laplace domain

shift in t-domain

↳ $\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}\, dt = \underbrace{\int_0^a 0 \cdot e^{-st}\, dt}_{} + \int_a^\infty 1 \cdot e^{-st}\, dt = \frac{1}{s} \times e^{-as}$ (e^{-as})

↳ If $\mathcal{L}[f(t)u(t)] = \frac{s}{s^2+9}$, delay time by 5 units → * $\mathcal{L}[f(t-5)u(t-5)] = \frac{s}{s^2+9} \times e^{-5s}$ / * $\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$

* ↳ A pure shift in one domain (s/t) gives rise to multiplication of an exponential term (s/t) in other domain & vice versa

shift in s-domain ↳ $\mathcal{L}[f(t)] = F(s) \to$ * $\mathcal{L}[f(t)\, e^{-at}] = F(s+a)$ → all s will have a shift / $\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} = e^{-3t}\, t\, u(t)$

E.g. Find Laplace Transform of $e^{-(t-2)}\cos(t-2)u(t-2)$.

Simpler transform without shift → $\mathcal{L}[e^{-t}\cos(t)u(t)]$

$$\mathcal{L}\{e^{-(t-2)}\cos(t-2)u(t-2)\} = e^{-2s}\,\mathcal{L}\{e^{-t}\cos(t)\, u(t)\}$$

$\mathcal{L}[\cos(t)u(t)] = \frac{s}{s^2+1}$ → from table

$$\mathcal{L}[e^{-t}\cos(t)u(t)] = \frac{s+1}{(s+1)^2+1}$$

$$\mathcal{L}\{e^{-(t-2)}\cos(t-2)u(t-2)\} = e^{-2s}\frac{s+1}{(s+1)^2+1}$$

Inverse Laplace Transforms

1. Use partial fractions

E.g. Given $F(s) = \frac{s+1}{s^2(s^2+9)}$. Find $f(t) = \mathcal{L}^{-1}[F(s)]$.

$$\frac{s+1}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cx+D}{s^2+9} = \frac{As(s^2+9) + B(s^2+9) + Cs^3 + Ds^2}{s^2(s^2+9)}$$

$A = \frac{1}{9}$, $B = \frac{1}{9}$, $C = -\frac{1}{9}$, $D = -\frac{1}{9}$ → if sub $s=0$ → only get B, not A

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{9}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{1}{9}\frac{s+1}{s^2+9}\right\}$$

$$= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{1}{27}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \frac{1}{9}u(t) + \frac{1}{9}t\,u(t) - \frac{1}{9}\cos 3t\, u(t) - \frac{1}{27}\sin 3t\, u(t)$$

2. Complete the square

E.g. Given $F(s) = \frac{1}{s^2-2s+5}$, find $f(t) = L^{-1}[F(s)]$.

$$s^2-2s+5 = s^2-2(1s)+1^2-1^2+5 = (s-1)^2+4$$

$$L^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+2^2}\right\} = \frac{1}{2}e^{t}\sin 2t\, u(t) \quad \frac{2}{(s-1)^2+2^2}$$

Properties of Laplace Transforms (more)

1. Differentiation of $f(t)$

$$L[f^n(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - sf^{n-2}(0) - f^{n-1}(0).$$ E.g. $L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0)$

E.g. $L\{\sin t\} = \frac{1}{s^2+1}$. Use this result to find $L(\cos t)$

$$L\{\tfrac{d}{dt}\sin t\} = s\frac{1}{s^2+1} - \sin(0) = \frac{s}{s^2+1} = L\{\cos t\}$$

2. Integration of $f(t)$

$L\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}L[f(t)]$ / $L^{-1}\{\frac{1}{s}F(s)\} = \int_0^t f(\tau)d\tau$

E.g. Find the inverse transform of $G(s) = \frac{1}{s(s^2+\omega^2)}$ using integral property

$$L^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = L^{-1}\left\{\frac{1}{\omega}\frac{\omega}{s^2+\omega^2}\right\}$$

can use partial fractions as well

$$= \frac{1}{\omega}\sin(\omega t)$$

$$L^{-1}\left\{\frac{1}{s}\frac{1}{s^2+\omega^2}\right\} = \int_0^t \frac{1}{\omega}\sin(\omega\tau)d\tau$$

$$= \frac{1}{\omega}\left[-\frac{\cos(\omega\tau)}{\omega}\right]_0^t$$

$$= \frac{1}{\omega^2}(1-\cos(\omega t))$$

3. Differentiation of $F(s)$

$-F'(s) = L[tf(t)]$ / $-tf(t) = L^{-1}[F'(s)]$ $\longrightarrow$ $\frac{(-1)^n d^n}{ds^n}F(s) = L[t^nf(t)]$

E.g. Find Laplace Transform of $t\sin 2t$

$$L[\sin(2t)] = \frac{2}{s^2+2^2} = F(s)$$

$$L[t\sin(2t)] = -\frac{d}{ds}F(s) = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= \frac{4s}{[s^2+4]^2}$$ differentiate normally

4. Integration of $F(s)$

$\int_s^\infty F(u)du = L\left[\frac{f(t)}{t}\right]$ / $L^{-1}\left[\int_s^\infty F(u)du\right] = \frac{f(t)}{t}$

Convolution

→ input-output relationship of linear system in time domain → $y(t) = h(t) * x(t)$ → $y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$ (output: $y(t)$, input: $x(t)$)

→ input-output relationship of same linear system in Laplace domain → $Y(s) = H(s)X(s)$

E.g. Convolve $x(t) = e^{-t}u(t)$ with $h(t) = e^{-2t}u(t)$

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$$

$$= \int_0^t e^{-2\tau}\, 1\, e^{-(t-\tau)}u(t-\tau)d\tau$$

$$= e^{-t}\int_0^t e^{-2\tau}e^{\tau}\, 1\, d\tau = e^{-t}\int_0^t e^{-\tau}d\tau$$

$$= e^{-t}[-e^{-\tau}]_0^t = e^{-t}(-e^{-t}+1)$$

impulse function

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

transfer function

$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$

- $f(t) * g(t) \longleftrightarrow F(s)G(s)$ / $f(t)g(t) \longleftrightarrow F(s) * G(s)$
- Impt properties: commutative, distributive, associative law. $f * 0 = 0$

Solving Differential Equations by Laplace Transform

E.g. Solve the initial value problem: $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=0$, $y(0)=3$, $y'(0)=1$

Let $L\{y(t)\}=Y(s)$,

$[s^2Y(s)-sy(0)-y'(0)]+4[sY(s)-y(0)]+3Y(s)=0$

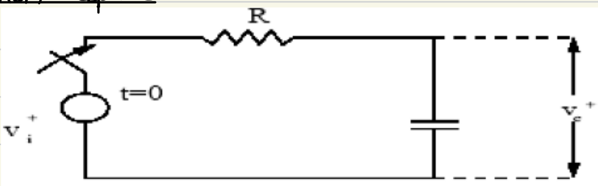
$Y(s)=\frac{3s+13}{s^2+4s+3}$ → make $Y(s)$ the subject

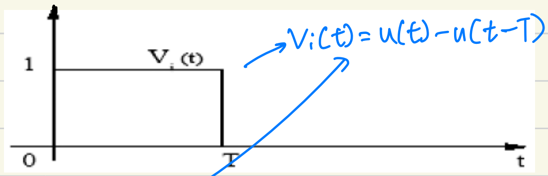
$Y(s)=\frac{3s+13}{(s+3)(s+1)}=\frac{-2}{s+3}+\frac{5}{s+1}$

$y(t)=(-2e^{-3t}+5e^{-t})u(t)$ → system stable since $\lim_{t\to\infty} e^{-3t}\to 0$ & $\lim_{t\to\infty} e^{-t}\to 0$

Analysis of System Response

E.g. Find $v_c(t)$





$V_i(t)=u(t)-u(t-T)$

$C=\frac{q}{v_c}\to q=Cv_c$, $i=\frac{dq}{dt}=C\frac{dv_c}{dt}$

Using Kirchoff's Law,

$Ri+v_c(t)=v_i(t)=u(t)-u(t-T)$

$RC\frac{dv_c(t)}{dt}+v_c(t)=u(t)-u(t-T)$

Let $L[v_c(t)]=V_c(s)$, taking laplace transform,

$RC[sV_c(s)-v_c(0^+)]+V_c(s)=\frac{1}{s}[1-e^{-Ts}]$

With initially uncharged capacitor, $v_c(0^+)=0$,

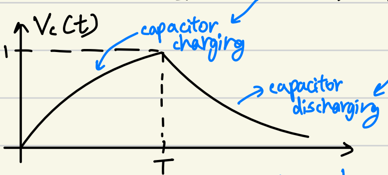
make subject → $V_c(s)=\frac{1}{s(RCs+1)}[1-e^{-Ts}]$

$\frac{1}{s(RCs+1)}=\frac{1}{RCs(s+\frac{1}{RC})}=\frac{1}{RC}\left[\frac{A}{s}+\frac{B}{s+\frac{1}{RC}}\right]$

$\therefore A=RC \quad B=-RC$ → $L^{-1}\left[\frac{1}{s(RCs+1)}\right]=L^{-1}\left[\frac{1}{s}-\frac{1}{s+\frac{1}{RC}}\right]=1-e^{-\frac{t}{RC}}$

$L^{-1}[V_c(s)]=L^{-1}\left[\frac{1}{s(RCs+1)}-\frac{1}{s(RCs+1)}e^{-Ts}\right]$

$=[1-e^{-\frac{t}{RC}}]u(t)-[1-e^{-\frac{(t-T)}{RC}}]u(t-T)$

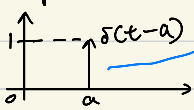


- Transfer function of this linear system → $H(s)=\frac{Y(s)}{X(s)}$ → all initial conditions are set to 0

$X(s)\to\boxed{H(s)}\to Y(s)$

$(RCs+1)V_{out}(s)=V_{in}(s)\to\frac{V_{out}(s)}{V_{in}(s)}=\frac{1}{RCs+1}=H(s)$

Impulse function – $\delta(t)$



$L\{\delta(t-a)\}=e^{-as}$ → area of pulse = 1

$L\{\delta(t)\}=1$

E.g. A linear dynamic system is described by $\frac{dy(t)}{dt}+3y(t)=x(t)$. Find i) impulse response $h(t)$ & transfer function of this system. ii) If $x(t)=u(t)-u(t-2)$, what is the output response $y(t)$ for $t>2$? Find this by convolution operation & by use of the Convolution Theorem.

Let $L\{y(t)\}=Y(s)$ & $L\{x(t)\}=X(s)$ → get $H(s)$ & take inverse to get $h(t)$

i) $L\{\frac{dy(t)}{dt}+3y(t)=x(t)\}\to[sY(s)-y(0)]+3Y(s)=X(s)$

All initial condition $=0$, $sY(s)+3Y(s)=X(s)$

$$\frac{Y(s)}{X(s)} = \frac{1}{s+3} = H(s)$$

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\{\frac{1}{s+3}\} = e^{-3t}$$

ii) $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

$= \int_{-\infty}^{\infty} [u(\tau)-u(\tau-2)][e^{-3(t-\tau)}u(t-\tau)]d\tau$

$= \int_{-\infty}^{t} [u(\tau)-u(\tau-2)]\ e^{-3(t-\tau)}d\tau$

$= \int_{0}^{t} [1-u(\tau-2)]e^{-3(t-\tau)}d\tau$

$= e^{-3t}\int_0^2 e^{3\tau}d\tau = \frac{1}{3}e^{-3t}[e^6-1]$

Partial Differential equation

- → eqn involving one/more partial derivatives of an unknown function of two/more independent variables
- → $F(x,y\ldots u, u_x, u_y, \ldots u_{xx}, u_{yy})=0$, u is function to be solved. $f=f(x,y,z) \rightarrow f_x = \frac{\partial f}{\partial x}$ (partial differentiate w.r.t x)
- → Important properties in DE:

a) Ordinary differential eqn (ODE)/partial differential eqn (PDE)

- → $u''+3uu^2+2u=3x$ → ODE (u only one independent variable) → $u_x+u_y = u_xu_y$ → got x, got y
- → $2u_{xx}+3\sin x\, u_x - 7u = xt$ → PDE (u is a function of x & t → 2 independent variable)

b) Order of Differential Equation (by highest derivative in DE)

- → $u_{xx}+uu_{xy}+x^2u_{yy}=0$ → 2nd order
- → $u_x + u_xu_y = xy$ → 1st order

c) Linear/non-linear DE in $u(x,y)$

- → $u_x+2u_y=x^3y^2$ → linear in u (no u in coefficient)
- → $u_{xxx}+uu_{xy}+\sin u = y$ → non-linear in u (have u in coefficient)

d) Homogenous & Non-homogenous Eqn

eqn = 0

- → $ax^2+by+c=0$ → homogenous in a,b,c & x^2, y, c, but not a, b / x, y, c (x^2)
- → $u_{xx}+uu_{xy}+x^2u_{yy}=0$ (homogenous → don't have a term without u)

e) Invariant / Variant PDE

- → If all system coefficient are constants → system is invariant (system will behave the same forever
- → $2u_{xx}+5u_t-7u=xt$ → invariant system (2, 5 & 7 constants)
- → $2u_{xx}+3xu_t-7u=xt$ → variant system ($3x$)
  - ↳ variant in x, invariant in t

Solve linear PDEs

Method 1: Direct Integration

↳ When pde/ode only one differentiation term → can be solved by direct integration if separable

E.g. Consider the first order PDE $u_x = y\sin x$

$\frac{\partial u}{\partial x} = y\sin x \rightarrow \partial u = y\sin \partial x$

Integrating on both sides, $\int \partial u = \int y \sin \partial x$ [separable]

General solution → $u = y\int \sin x\, dx = -y\cos x + f(y)$ (constant) since y is considered constant of integration w.r.t x, arbitrary constant must include y terms

- General solution of nth order PDE → n independent arbitrary functions
- General solution of nth order ODE → n independent arbitrary constants

E.g. Consider the second order PDE $u_{xy}=x^2+y$.
2 independent arbitrary functions

Integrating w.r.t y,
$u_x(x,y)=x^2y+\frac{y^2}{2}+f(x)$, $f(x)$ is an arbitrary function of y

Integrating again w.r.t x,
General solution
$u(x,y)=\frac{x^3y}{3}+\frac{xy^2}{2}+g(x)+h(y)$, where $g(x)$ & $h(y)$ are arbitrary functions

– Some well-known Second-order PDEs:

1. The one-dimensional wave equation $\longrightarrow \frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$ $[u_{tt}=c^2u_{xx}]$
2. The one-dimensional heat/diffusion equation $\longrightarrow u_t=a^2u_{xx}$

– Solutions for 1st & 2nd order ODEs by Laplace Transform:

1. $y'+py=0 \Rightarrow y(t)=ke^{-pt}$ $(t>0)$, k constant → 1st order ODE
2. $y''+p^2y=0 \Rightarrow y(t)=A\cos pt+B\sin pt$ $(t>0)$
3. $y''-p^2y=0 \Rightarrow y(t)=Ae^{-pt}+Be^{pt}$ $(t>0)$ (real no.)
4. $y''=0 \Rightarrow y(t)=at+b$ (a & b are constants)

(2.–4.) 2nd order ODE

Method 2: Separation of Variables Method (limited to homogeneous boundary conditions)

1. Apply separation of variables assumption
   assume $u(x,t)=F(x)G(t)$ → lead two ODEs, one in x, one in t.
2. Satisfy boundary conditions in terms of the new ODE in x → leads to a set of solutions of $F(x)$.
3. Solve entire solution combining results in step 2 & applying initial conditions to solve for $G(t)$.

E.g. Given a 1-D heat eqn $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$, $0<x<L$, $t>0$ subject to boundary conditions at two ends: $u(0,t)=0$, $u(L,t)=0$, $\forall t\geq 0$ & one initial condition: $u(x,0)=f(x)$, $0<x<L$. Solve for $u(t,x)$.

1. Need to compute u_t & u_{xx} for $F(x)$ & $G(t)$.

$$u_x(x,t)=\frac{d(F(x)G(t))}{dx}=\left[\frac{dF(x)}{dx}\right]G(t)=F'(x)G(t)$$

$$(u_x)_x=u_{xx}=\frac{d[F'(x)G(t)]}{dx}=F''(x)G(t)$$

$$u_t(x,t)=F(x)\dot{G}(t)$$
use different symbol to denote derivaties w.r.t t.

Sub in original statement, $F\dot{G}=a^2F''G$

Dividing by $a^2FG \rightarrow \frac{\dot{G}(t)}{a^2G(t)}=\frac{F''(x)}{F(x)}=\mu$ → arbitrary constant (only happens when all t on one side & all x on other)

$F''-\mu F=0$ & $\dot{G}-\mu a^2G=0$

2. Satisfying Boundary Conditions → given in terms of $u(x,t)$ → convert to in terms of $F(x)$

$u(0,t)=F(0)G(t)=0$, $u(L,t)=F(L)G(t)=0$ (x,t)

If $G(t)=0 \longrightarrow u(x,t)=F(x)G(t)=F(x)(0)=0$ (trivial soln, no interest)

$G(t)\neq 0 \rightarrow F(0)=0$, $F(L)=0$

familiar → $F''-\mu F=0$, $F(0)=0$, $F(L)=0$ → type of solution of $F(x)$ depend on μ. ($\mu>0$, $\mu=0$, $\mu<0$)

Case 1: $\mu=0 \rightarrow F''=0$

integrate

$F(x)=c_1x+c_2$ (general solution) sub in

$F(0)=0\rightarrow c_2=0$

$F(x)=c_1x \rightarrow F(L)=0 \rightarrow c_1=0$ (trivial)

Case 2: μ positive → set $\mu=p^2 \rightarrow F''-p^2F=0$

$F(x)=Ae^{px}+Be^{-px}$ (general soln) ← sub in initial conditions

$A+B=0$ & $Ae^{pL}+Be^{-pL}=0$

$A=-B$ & $-Be^{pL}+Be^{-pL}=0$

$B=0$ & $A=-B=0$ (trivial)

Case 3: μ negative $\rightarrow$ let $\mu=-p^2 \rightarrow F''+p^2F=0$

$F(x)=A\cos px+B\sin px$ (general solⁿ)

$F(0)=0 \rightarrow A=0 \rightarrow F(x)=B\sin px$

$F(L)=B\sin pL=0 \rightarrow B\neq 0,\ \sin pL=0 \rightarrow pL=n\pi \rightarrow p=\frac{n\pi}{L}$

solⁿ $\downarrow$ $F_n(x)=\sin\frac{n\pi}{L}x,\ \mu=-p^2=-(\frac{n\pi}{L})^2,\ n=1,2,3\ldots$

$\dot{G}+(\frac{n\pi a}{L})^2G=0$

Let $\lambda_n=\frac{an\pi}{L}$, $\dot{G}+\lambda_n^2G=0 \rightarrow \frac{dG}{dt}+\lambda^2G=0$

$G_n(t)=B_ne^{-\lambda_n^2t}$, $n=1,2,\ldots$, where B_n's are constant

$F_n(x)=\sin\frac{n\pi}{L}x$

$u_n(x,t)=F_n(x)G_n(t)=B_n(\sin\frac{n\pi x}{L})e^{-\lambda_n^2t}$, $n=1,2,\ldots$

3. Solution of Entire Problem

$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_ne^{-\lambda_n^2t}\sin\frac{n\pi x}{L}$

$U(x,0)=f(x)$.

Setting $t=0$, $u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}=f(x)$

$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

Method 3: Solving Linear PDEs by Laplace Transform [linear, invariant system]

Given $u(x,t)=t\cos 2x$

Taking L.T. w.r.t t: $L[t\cos 2x]=\cos\frac{2x}{s^2}=U(x,s)$ } don't forget to state which domain variable you are transforming

OR Taking L.T. w.r.t x: $L[t\cos 2x]=t\frac{s}{s^2+4}=V(s,t)$

$L[\frac{\partial u(x,t)}{\partial t}]=sU(x,s)-u(x,0)=U_x(x,s)$ $\rightarrow$ LT does not affect x

$L[\frac{\partial^2u(x,t)}{\partial t^2}]=s^2U(x,s)-su(x,0)-\frac{\partial u(x,0)}{\partial t}$ $=\frac{\partial^2U(x,s)}{\partial x^2}=U_{xx}(x,s)$

E.g. Solve first order PDE $\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial t}=0$, $x>0$, $t>0$ subject to $u(x,0)=0,\ x\geq 0$ (initial condition); $u(0,t)=t,\ t\geq 0$ (boundary condition)

Take Laplace transform w.r.t t: Let $U(x,s)=L[u(x,t)]$

$L[u_x]+x(sL[u]-u(x,0))=0$

With initial condition $u(x,0)=0$, $U_x+xsU=0 \rightarrow$ ODE with x as independent variable, s is constant

$\ln u=-\frac{sx^2}{2}+c \rightarrow u(x,s)=e^{-\frac{sx^2}{2}} \rightarrow U(x,s)=C(s)e^{-\frac{sx^2}{2}}$

Transforming boundary conditions $u(0,t)=t \rightarrow U(0,s)=\frac{1}{s^2}$

$c(s)=\frac{1}{s^2} \rightarrow U(x,s)=\frac{1}{s^2}e^{-\frac{sx^2}{2}}$

shifting theorem $e^{-as}F(s)$ Δ $U(x,s)$ to $u(x,t)$ $\rightarrow L^{-1}[\frac{1}{s^2}]=t$

$u(x,t)=(t-\frac{x^2}{2})H(t-\frac{x^2}{2})=\begin{cases}0, & \text{if } t<\frac{x^2}{2}\\ t-\frac{x^2}{2}, & \text{if } t\geq\frac{x^2}{2}\end{cases}$ $\rightarrow$ unit step function