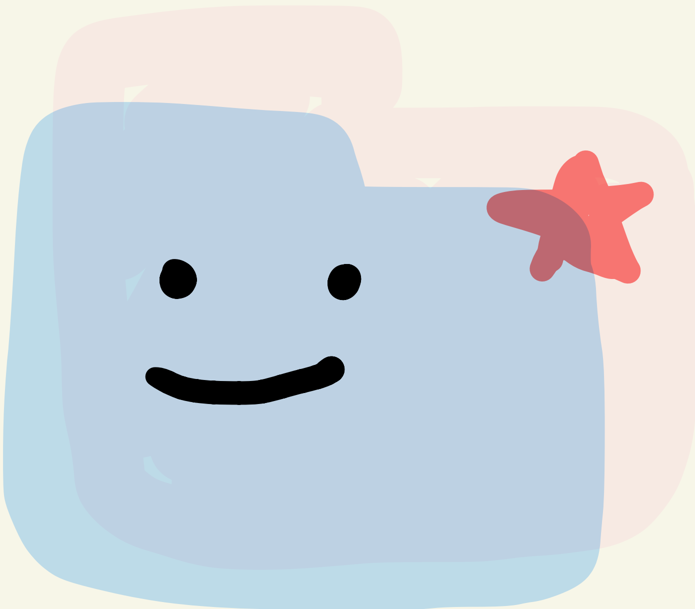
MH1811



MATH

Equations

Equation of a line: $ax + by = c$

Equation of a circle: $(x-a)^2 + (y-b)^2 = r^2$, $r > 0$ → radius

$(a,b) \rightarrow$ centre

Equation of an ellipse: $\alpha(x-a)^2 + \beta(y-b)^2 = r^2$, $\alpha > 0$, $\beta > 0$, $r > 0$

$\left(\frac{x-a}{k}\right)^2 + \left(\frac{y-b}{\ell}\right)^2 = 1$

Equation of a plane: $ax + by + cz = d$ / $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = d$

→ $\underset{\sim}{n}$ (normal vector)

Equation of a sphere: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$, $r > 0$ $(a, b, c) \Rightarrow$ center

Equation of an ellipsoid: $\alpha(x-a)^2 + \beta(y-b)^2 + \gamma(z-c)^2 = r^2$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $r > 0$

Domains

E.g. Find & sketch the domain of $g(x,y) = x\ln(2x-y-2)$ & evaluate $g(1,-2)$

$g(x,y) = x\ln(2x-y-2)$ defined when $2x-y-2 > 0$

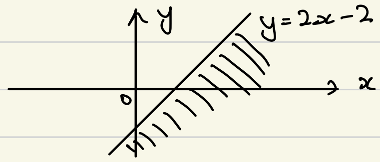
$\rightarrow y < 2x - 2$

$D = \{(x,y) : y < 2x - 2\}$

$g(1,-2) = 1 \times \ln(2-(-2)-2) = \ln 2$

$f(x,y) = \frac{\sqrt{y-x^2}}{x-2}$

$\hookrightarrow D = \{(x,y) : y \geq x^2 \text{ \& } x \neq 2\}$



Distance

E.g. Consider the distance function of a point $P(x,y)$ from the point $Q(-1,2)$ given by $d(x,y) = \sqrt{(x+1)^2 + (y-2)^2}$. Determine each of following level sets:

a) $d(x,y) = -2$

$\sqrt{\ } \geq 0 \Rightarrow$ no solution

$D = \emptyset$

b) $d(x,y) = 0$

$x + 1 = 0$, $y - 2 = 0$

$D = \{(-1, 2)\}$

c) $d(x,y) = 3$

$\sqrt{(x+1)^2 + (y-2)^2} = 3$

circle

2D graphs

↳ graph of 2-variable function $f(x,y)$ ⟶ 3D graph $z = f(x,y)$

↳ set of points (x,y) in $\mathbb{R}^2$ where $f(x,y) = k$ → level curve of f

E.g. Find the level curve of $f(x,y) = 100 - x^2 - y^2$

i) corresponds to $f(x,y) = 75$

ii) passes through $(10, 0)$

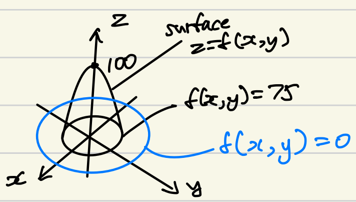
i) $100 - x^2 - y^2 = 75 \Rightarrow x^2 + y^2 = 25$

circle with radius 5 & center $(0,0)$

ii) $f(x,y) = f(10,0) \Rightarrow f(x,y) = 100 - 100 - 0 = 0$

$x^2 + y^2 = 10^2$

circle with radius 10 & center $(0,0)$.



3D graphs

E.g. Evaluate $F(10,5,9)$ when $F(x,y,z) = \max\{x-y,\ 3y-z,\ |x-2z|\}$.

$$F(10,5,9) = \max\{10-5,\ 3(5)-9,\ |10-2(9)|\}$$
$$= \max\{5, 6, 8\} = 8$$

E.g. Consider the function $f(x,y,z) = (x-1)^2 + y^2 + z^2$. Find level surface which passes through point $(1,-2,3)$.

$$f(1,-2,3) = 0^2 + (-2)^2 + 3^2 = 13$$

level surface set $\{(x,y,z) \in \mathbb{R}^3 \mid (x-1)^2 + y^2 + z^2 = 13\}$

sphere with radius $\sqrt{13}$ & center at $(1,0,0)$.

Limits

$\lim_{x \to a} f(x) = L$ / 2 variables $\rightarrow \lim_{(x,y)\to(a,b)} f(x,y) = L$

↳ if limit exist → it is unique

1) $\lim_{(x,y)\to(a,b)} x^2+y^3 = \lim_{(x,y)\to(a,b)} x^2 + \lim_{(x,y)\to(a,b)} y^3 = a^2+b^3$

2) $\lim_{(x,y)\to(a,b)} x^2y^3 = \left(\lim_{(x,y)\to(a,b)} x^2\right)\left(\lim_{(x,y)\to(a,b)} y^3\right) = a^2b^3$

3) $\lim_{(x,y)\to(3,1)} \frac{x^2y^3}{x^2+y^3} = \frac{\lim_{(x,y)\to(a,b)} x^2y^3}{\lim_{(x,y)\to(a,b)} x^2+y^3} = \frac{3^2(1)^3}{3^2+(1)^3} = \frac{9}{10}$

Continuity

↳ f is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

↳ f is continuous on D if it is continuous at each $(a,b) \in D$.

↳ polynomials & rational functions are continuous on their respective domains

$c_0 + c_1x + c_2y + c_3xy + \dots$; $\frac{p(x,y)}{q(x,y)}$, $q(x,y) \neq 0$

↳ If continuous → can just sub the value it is tending to into equation

Path Limits (use to verify if limit exist)

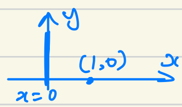
↳ express limit of $f(x,y)$ along path C as (x,y) approaches (a,b) as $\lim_{(x,y)\xrightarrow{C}(a,b)} f(x,y)$

↳ suppose C_1 & C_2 are paths passing through (a,b). $\lim_{(x,y)\to(a,b)}$ does not exist if:

a) $\lim_{(x,y)\xrightarrow{C_1}(a,b)} f(x,y)$ does not exist

b) $\lim_{(x,y)\xrightarrow{C_1}(a,b)} f(x,y) \neq \lim_{(x,y)\xrightarrow{C_2}(a,b)} f(x,y)$

$\lim_{(x,y)\to(1,0),\ x=0} \frac{x^2-x-y^2}{(x-1)+y^2}$ → not valid path → cannot use this value



E.g. Does $\lim_{(x,y)\to(1,0)} \frac{x^2-x-y^2}{(x-1)+y^2}$ exist?

$\lim_{(x,y)\to(1,0),\ y=0} \frac{x^2-x-0}{(x-1)+0} = \lim_{x\to1} \frac{x^2-x}{x-1} = \lim_{x\to1} x = 1$

$\lim_{(x,y)\to(1,0),\ x=1} \frac{1-1-y^2}{(1-1)+y^2} = \lim_{y\to0} \frac{-y^2}{y^2} = \lim_{y\to0} (-1) = -1$

$1 \neq -1 \rightarrow \lim_{(x,y)\to(1,0)} \frac{x^2-x-y^2}{(x-1)+y^2}$ does not exist.

E.g. $g(x,y) = \begin{cases} \frac{2x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Is g continuous on $(0,0)$?

$\lim_{(x,y)\to(0,0),\ y=0} g(x,y) \neq \lim_{(x,y)\to(0,0),\ x=y} g(x,y)$

$\lim_{x\to0} \frac{2x^2-x^2}{x^2+x^2}$] → replace $y=x$

Although $g(0,0) = 0 \rightarrow \lim_{(x,y)\to(0,0)} g(x,y)$ does not exist → g not continuous at $(0,0)$

$\star$ $\frac{\partial f}{\partial x}$ different from $\frac{df}{dx}$

Partial Derivatives

1. Partial derivative w.r.t x at Point $P(a,b)$

$$f_x(a,b) = \lim_{t \to a} \frac{f(t,b) - f(a,b)}{t-a}$$

can change (a,b) to (x,y) $f_x = \lim_{t \to x} \frac{f(t,y) - f(x,y)}{t-x}$

[keeping y unchanged] $\Rightarrow \frac{d}{dx} f(x,b)\big|_{x=a}$

E.g $f(x,y) = \begin{cases} \frac{x\sin(x+y)}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y = 0 \end{cases}$. Determine if $f_x(0,0)$ exist. If exist, what's its value? Is f increasing/decreasing along the x-direction?

$$f_x(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t-0}$$

$$= \lim_{t \to 0} \frac{\frac{t\sin(t)}{\cancel{t}}}{t} = \lim_{t \to 0} \frac{\sin t}{t} = 1 \quad , t \neq 0$$

Since $f_x(0,0) = 1 > 0$, function $f(x,y)$ is increasing (at rate 1) along the x-direction.

2. Partial derivative w.r.t y

$$f_y(x,y) = \lim_{t \to y} \frac{f(x,t) - f(x,y)}{t-y}$$

3. Partial derivative for 3-variable function

↳ Let $t = x+h \Rightarrow t \to x = h \to 0$

$$f_x(x,y,z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x,y,z)}{h}$$

[same for f_y & f_z]

Another way to evaluate partial derivative

↳ to evaluate f_x, we regard y as a constant & differentiate w.r.t x → same for y & z

E.g. If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$.

To find $f_x(2,1)$, we fix $y=1$ & evaluate.

$$f_x(2,1) = \frac{\partial}{\partial x}(f(x,1))\big|_{x=2}$$

$$= \frac{\partial}{\partial x}(x^3 + x^2 - 2)\big|_{x=2}$$

$$= (3x^2 + 2x)\big|_{x=2} = 16$$

E.g. If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right)$$

$$= \left[\cos\left(\frac{x}{1+y}\right)\right](-1)(1+y)^{-2} x(1)$$

$$= \frac{-x}{(1+y)^2} \cos\left(\frac{x}{y+1}\right)$$

E.g. $f(x,y,z) = x\sin y + e^{3x} z\sqrt{y^2+9}$

$$f_x(x,y,z) = \sin y + 3e^{3x} z\sqrt{y^2+9}$$

$$f_y(x,y,z) = x\cos y + e^{3x} z\left(\tfrac{1}{2}\right)(y^2+9)^{-\frac{1}{2}}(2y)$$

$$= x\cos y + \frac{e^{3x} zy}{\sqrt{y^2+9}}$$

$$f_z(x,y,z) = e^{3x}\sqrt{y^2+9}$$

Gradient Vector

↳ gradient vector of f → partial derivatives of f.

$$\nabla f(x,y,z) = (f_x, f_y, f_z) = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j} + \frac{\partial f}{\partial z}\underset{\sim}{k}$$

E.g. Let $F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$. Find the gradient vectors $\nabla F(x,y,z)$ & $\nabla F(-2,1,-3)$.

$$\nabla F(x,y,z) = \left(\frac{x}{2}, 2y, \frac{2}{9}z\right)$$

$$\nabla F(-2,1,-3) = \left(-1, 2, -\frac{2}{3}\right) = -\underset{\sim}{i} + 2\underset{\sim}{j} - \frac{2}{3}\underset{\sim}{k}$$

Second Derivatives

→ can use a Hessian matrix to show 2x2 matrix of 2nd order partial derivatives.

→ $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$

$$H(f) = \begin{array}{c} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \end{array} \begin{array}{l} \to x \text{ row} \\ \to y \text{ row} \end{array}$$

→ Clairaut's Theorem → if $f(x,y)$ & partial derivatives (f_x, f_y, f_{xy} & f_{yx}) are defined throughout an open region containing (a,b) & are continuous at (a,b) → $f_{xy}(a,b) = f_{yx}(a,b)$

E.g. $f(x,y,z) = e^{xy}\ln z$. Find $\frac{\partial^2 f}{\partial z \partial y}$ & f_{zxy}.

$$\frac{\partial}{\partial z}\left(\frac{df}{dy}\right) = \frac{\partial}{\partial z}\left(x(\ln z)(e^{xy})\right) = \frac{x}{z}(e^{xy})$$

$$((f_z)_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{e^{xy}}{z}\right)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{y e^{xy}}{z}\right) = \frac{1(e^{xy}) + y(x)e^{xy}}{z}$$

$$= \frac{e^{xy}(1+xy)}{z}$$

Chain Rule

1. If $w = f(x,y)$ is differentiable & $x = x(t), y = y(t)$ differentiable function of t ⇒ $w = f(x(t), y(t))$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \quad \& \quad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

E.g. Suppose $z = x^2y + 3xy^4$, where $x = e^t$ & $y = \sin t$. Find $\frac{dz}{dt}$. What is the rate of change in z at $t=0$?

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$ → sub in $x = e^t$ & $y = \sin t$

$$= (2yx + 3y^4)(e^t) + (x^2 + 12xy^3)(\cos t) \quad \text{——— ①}$$

$$= [2(e^t)\sin t + 3(\sin^4 t)]e^t + (e^{2t} + 12(e^t)\sin^3 t)(\cos t) \quad \text{——— ②}$$

can sub $t=0$ into either ①/② — need sub $t=0$ into $x = e^t$ & $y = \sin t$

$$\left.\frac{dz}{dt}\right|_{t=0} = (2e^0\sin 0 + 3\sin^4 0)e^0 + (e^{2(0)} + 12e^0\sin^3 0)\cos 0 = 1$$

E.g. Radius of a right circular cone is increasing at a rate of 1.8cm/s while height is decreasing at rate of 2.5cm/s. At what rate is the volume of cone changing when radius is 24cm & height is 14cm?

$V = \frac{1}{3}\pi r^2 h$ — $r_t(t_0) = 1.8$

$\frac{dr}{dt} = 1.8$ & $\frac{dh}{dt} = -2.5$ — $r(t_0) = 24$ — $h(t), r(t) = ?$

By chain rule, $\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$ — $h(t_0) = 14$

$$= \frac{\pi}{3}\left(2rh\cdot\frac{dr}{dt} + r^2\cdot\frac{dh}{dt}\right)$$

when $r=24$ & $h=14$, $\frac{dV}{dt} = \frac{\pi}{3}(2(24)(14)\cdot(1.8) + (24)^2\cdot(-2.5))$

$$= -76.8\pi \text{ cm}^3/\text{s}$$

2. Suppose $z = f(x,y)$, $x = g(s,t)$ & $y = h(s,t)$ are differentiable. Then z has partial derivatives w.r.t s & t.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$ add in $y = h(s,t,u)$: $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$ → If $w = f(x,y,z)$ → $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$

E.g. If $z = e^x\sin y$, where $x = st^2$ & $y = s^2t$, find $\frac{\partial z}{\partial s}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= ((e^x)\sin y)t^2 + e^x(\cos y)(2st)$$

$$= t^2e^{st^2}\sin(s^2t) + 2st\,e^{st^2}\cos(s^2t)$$

E.g. If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$ & $z = r^2 s \sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r=2$, $s=1$, $t=0$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= 64(2) + 64(4) + 0 = 192$$

Implicit Differentiation

↳ Suppose z is defined implicitly as a function of x & y by an equation of form $F(x,y,z)=C$, C is a constant

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

(y independent of x)

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

– If $\frac{\partial F}{\partial z} \neq 0 \longrightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{-F_x}{F_z}$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \frac{-F_y}{F_z}$$

E.g. Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$.

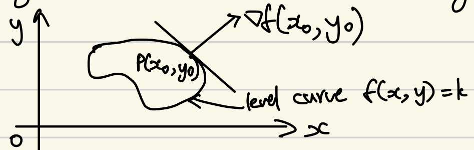
Let $F(x,y,z) = x^3+y^3+z^3+6xyz$. Then $F(x,y,z)=1$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$$

Gradient Vector Perpendicular to level curve

↳ gradient vector $\nabla f(P_0)$ is orthogonal (perpendicular) to the tangent of level curve $f(x,y)=k$ at P_0



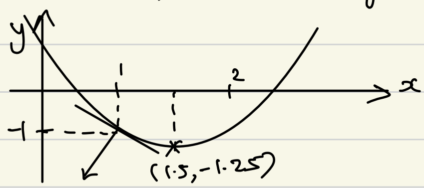
E.g. Let $f(x,y)=x(x-3)-y$. Sketch the level curve of f which passes through $P_0(1,-1)$ & indicate the gradient vector $\nabla f(P_0)$ on the sketch.

$$f(P_0) = f(1,-1) = -1$$

$$y = x(x-3)+1 = x^2-3x+1$$

$$\nabla f(x,y) = (2x-3, -1) \rightarrow \nabla f(P_0) = (-1,-1)$$

At $P_0(1,-1)$, $\nabla f(P_0) = (-1,-1) \perp$ to tangent of parabola $y = x(x-3)+1$.



→ $\nabla F(P_0) \perp$ level surface $F(x,y,z)=k$ at P_0

Tangent Planes to level surface

↳ tangent plane to level surface $F(x,y,z)=k$ at $P(x_0,y_0,z_0)$ → plane passes through P & has normal vector $\nabla F(x_0,y_0,z_0)$

$$\nabla F(x_0,y_0,z_0)\cdot(x-x_0, y-y_0, z-z_0)=0$$ [eqn of tangent plane]

$$\tilde{r} = (x_0,y_0,z_0) + t\nabla F(x_0,y_0,z_0)$$ [eqn of normal line]

E.g. Find the equation of the tangent plane & normal at the point $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$.

Let $F(x,y,z)=\frac{x^2}{4}+y^2+\frac{z^2}{9}$

Equation of tangent plane is: $\nabla F(-2,1,-3)\cdot(x-x_0, y-y_0, z-z_0)=0$

$\nabla F(-2,1,-3)=(\frac{x}{2}, 2y, \frac{2z}{9})=(-1, 2, \frac{-2}{3})$

$(-1,2,\frac{-2}{3})\cdot(x-(-2), y-1, z-(-3))=0$

$-x+2y-\frac{2}{3}z=6$

sub $(-2,1,-3)$ into $\frac{x^2}{4}+y^2+\frac{z^2}{9}=\frac{(-2)^2}{4}+1^2+\frac{(-3)^2}{9}=3$

Equation of normal line to point: $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$

$\underset{\sim}{r}=(x_0, y_0, z_0)+t\nabla F(x_0, y_0, z_0)$

$(x,y,z)=(-2,1,-3)+t(-1,2,\frac{-2}{3})$

Tangent Plane to Graph $z=f(x,y)$

↳ apply eqn of tangent plane to level surface $F(x,y,z)=k$ from preceding section to graph

$z=f(x,y) \rightarrow f(x,y)-z=0 \rightarrow F(x,y,z)=f(x,y)-z$

$(f_x, f_y, -1)\cdot(x-x_0, y-y_0, z-z_0)=0$ rearrange

$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$ [eqn of tangent plane]

E.g. Find the equation of the tangent plane to the surface $z=2x^2+y^2$ at the point $(1,1,3)$.

Let $f(x,y)=2x^2+y^2$.

$z=f(1,1)+f_x(1,1)(x-1)+f_y(1,1)(y-1)$ $\qquad f_x(x,y)=4x$

$z=3+4(1)(x-1)+2(1)(y-1)$ $\qquad f_y(x,y)=2y$

$\therefore z=4x+2y-3$

Linear Approximation

↳ normally approximated using tangent function

$$f(x,y)\approx f(a,b)+f_x(a,b)\cdot(x-a)+f_y(a,b)\cdot(y-b)$$

E.g. Use linear approximate the value $\sqrt{9(1.95)^2+(8.1)^2}$

Let $f(x,y)=\sqrt{9x^2+y^2} \longrightarrow$ 1.95 close to 2, 8.1 close to 8

$f(1.95,8.1)=f(2,8)\cdot(\overset{x-a}{1.95-2})+f_y(2,8)\cdot(8.1-8)$

$f_x(x,y)=\frac{\partial}{\partial x}\sqrt{9x^2+y^2}=\frac{9x}{\sqrt{9x^2+y^2}}$ $\qquad f_y(x,y)=\frac{\partial}{\partial y}\sqrt{9x^2+y^2}=\frac{y}{\sqrt{9x^2+y^2}}$

$f(1.95,8.1)\approx 10+\frac{9(2)}{\sqrt{9(2)^2+8^2}}(-0.05)+\frac{8}{\sqrt{9(2)^2+8^2}}(0.1)$

$=10+(-0.01)=9.99$

Total Differential

↳ To estimate the change of f (Δf),

3-variable function

$df=F_x(a,b,c)\cdot(x-a)+F_y(a,b,c)\cdot(y-b)+F_z(a,b,c)\cdot(z-c)$

actual change

$\Delta f\approx df=f_x(a,b)\Delta x+f_y(a,b)\Delta y$, Δx & Δy are small

after Δ $\qquad \Delta f=f(x_2,y_2)-f(x_1,y_1)$ original

E.g. A cylindrical can is designed to have a radius 3cm & a height of 15cm. But the radius & height are off by the amount not more than 0.04cm & 0.2cm respectively. Estimate the resulting Δ in volume of can.

↳ cannot compute Δ cause donno precise measurement.

$V=\pi r^2 h$ $\qquad r=3\text{cm}, h=15\text{cm}$

$dV=(V_r)(\Delta r)+(V_h)(\Delta h)=\pi(2rh(\Delta r)+r^2(\Delta h))$

$= \pi(2\times3\times15\times0.04+9\times0.2)$

$= 5.4\pi$

$\Delta V \approx dV = 5.4\pi \approx 17\text{cm}^3$

Directional Derivatives

↳ If f is a differentiable function of x & y, then f has a directional derivation in the direction of any unit vector $\underset{\sim}{u}=(u_1,u_2)$ &

$$D_u f(a,b) = \nabla f(a,b)\cdot \underset{\sim}{u} = f_x(a,b)u_1 + f_y(x,y)u_2.$$

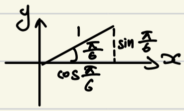
Let $f(x,y)=x^3-3xy+4y^2$

a) Find the directional derivative $D_u f(x,y)$ where u is the unit vector given by angle $\theta=\frac{\pi}{6}$

b) What is $D_u f(1,2)$?

a) $\nabla f = (3x^2-3y, -3x+8y)$

$\underset{\sim}{u} = \cos\frac{\pi}{6}\, i + \sin\frac{\pi}{6}\, j = (\frac{\sqrt3}{2}, \frac12)$



$D_u f(x,y) = \nabla f(x,y)\cdot \underset{\sim}{u} = (3x^2-3y, -3x+8y)\cdot(\frac{\sqrt3}{2},\frac12)$

$= \frac32\sqrt3 x^2 - \frac32 x + 4y - \frac32\sqrt3 y$

$\underset{\sim}{v} = 2i+5j$

$\underset{\sim}{u} = \frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|} = (\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}})$

sub in

b) $D_u f(1,2) = \frac32\sqrt3 - \frac32 + 4(2) - \frac32\sqrt3(2)$

$= \frac{13}{2} - \frac32\sqrt3$

Maximum & minimum rate of change

↳ Suppose f is a differentiable function of 2/3 variables.

↳ Max rate of Δ in f at P is $\|\nabla f(P)\|$ & it occurs in the direction $\underset{\sim}{u} = \frac{\nabla f(P)}{\|\nabla f(P)\|}$ → same direction as gradient vector $\nabla f(P)$

↳ Min rate: $-\|\nabla f(P)\|$ & occurs $\underset{\sim}{u} = \frac{-\nabla f(P)}{\|\nabla f(P)\|}$ → opposite direction of gradient vector $\nabla f(P)$

directional derivative

E.g. If $f(x,y)=xe^y$, find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q(\frac12,2)$. What is the maximum rate of change? In what direction does f have the maximum rate of change?

$\vec{PQ} = \vec{OQ} - \vec{OP} = (\frac12,2)-(2,0) = (-1.5,2)$

$\|\vec{PQ}\| = \sqrt{(-1.5)^2+2^2} = \sqrt{2.25+4} = 2.5$

$\vec{u} = \frac{\vec{PQ}}{\|\vec{PQ}\|} = (-\frac{1.5}{2.5}, \frac{2}{2.5}) = (-0.6, 0.8)$

$f(x,y)=xe^y \longrightarrow \nabla f = (e^y, xe^y)$

$\nabla f(2,0) = (e^0, 2e^0) = (1,2)$

$D_u f(2,0) = \nabla f(2,0)\cdot \underset{\sim}{u}$

$= (1,2)\cdot(-\frac35, \frac45) = 1$

Maximum rate of change in f at P is

$\|\nabla f(2,0)\| = \|(1,2)\| = \sqrt{1^2+2^2} = \sqrt5$

At P, the function f increases fastest in the direction of the gradient vector $\nabla f(2,0)=(1,2)$.

$\underset{\sim}{u} = \frac{1}{\|\nabla f(2,0)\|}\nabla f(2,0) = \frac{1}{\sqrt5}(1,2)$

Local Extrema Value at Interior Point

1. (a,b) singular point → either $f_x(a,b)$ / $f_y(a,b)$ / both does not exist
2. (a,b) stationary point → (a,b) is NOT a singular point & $\nabla f(a,b)=(0,0)$ → both $f_x(a,b)=0$ & $f_y(a,b)=0$

3. (a,b) critical point $\rightarrow$ (a,b) is stationary/singular point

$\rightarrow$ Point (a,b) is called a saddle point of f if it is a stationary point but neither local maximum nor local minimum $\rightarrow$ $f_x(a,b)$ & $f_y(a,b)=0$ but $f(a,b)$ is neither a local maximum/minimum

↳ local min/max $\rightarrow$ critical point (stationary/singular) $\rightarrow$ reverse not true

E.g. Find all critical points of $f(x,y)=e^{x^3\sqrt{y}-x^2}$

$$\nabla f(x,y)=e^{x\sqrt[3]{y}-x^2}\left(\sqrt[3]{y}-2x,\ \frac{x}{3\sqrt[3]{y^2}}\right) \text{ if } y\neq 0$$

Since $\nabla f(x,y)$ not defined when $y=0$, singular points are $(x,0)$, $x\in\mathbb{R}$

For stationary points, $\nabla f(x,y)=0$

$$\sqrt[3]{y}-2x=0 \quad \& \quad \frac{x}{3\sqrt[3]{y^2}}=0$$

$$\sqrt[3]{y}-2(0)=0 \qquad\qquad x=0$$

$$y=0, \text{ but } y\neq 0$$

$\therefore$ No stationary point $\rightarrow$ critical points are $(x,0)$, $x\in\mathbb{R}$

E.g. Find all stationary points of $f(x,y)=y^2-y^4-x^2$. Determine whether each of them is a local maximum/local minimum/a saddle point.

$$\nabla f(x,y)=(-2x,\ 2y-4y^3)$$

Since it is defined at every (x,y), there is no singular pt.

$$\nabla f(x,y)=(-2x,\ 2y-4y^3)=(0,0)$$

$$x=0 \quad \& \quad 2y(1-2y^2)=0$$

$$x=0 \quad \& \quad y=0 \text{ or } y=\pm\frac{1}{\sqrt{2}}$$

Stationary pts are $(0,0)$, $\left(0,\frac{1}{\sqrt{2}}\right)$, $\left(0,\frac{-1}{\sqrt{2}}\right)$

$$f(x,y)=-x^2-(y^4-y^2)$$

$$=\underbrace{-x^2}_{\leq 0}\underbrace{-(y^2-\tfrac{1}{2})^2}_{-\leq 0}+\frac{1}{4}$$

$\rightarrow x^2=0$ & $y^2-\frac{1}{2}=0$ to maximise it (if it is +ve $\rightarrow$ value will be smaller)

$\therefore$ $\left(0,\pm\frac{1}{\sqrt{2}}\right)$ are local maximum values $\rightarrow$ also can check using 2nd derivative test ☆

To show saddle pt $\rightarrow$ need show there are pts x & y close to 0 with both +ve & -ve of f

Along $x=0$, $f(x,y)=y^2(1-y^2)\geq 0=f(0,0)$ if $-1<y<1$

Along $y=0$, $f(x,y)=-x^2\leq 0=f(0,0)$ for $x<0$

$\therefore$ $(0,0)$ is a saddle point. $\rightarrow$ can use 2nd derivative also

Second Derivative Test for Local Extrema

☆ ↳ Suppose that (a,b) is a stationary point, $\nabla f(a,b)=0$

$$D=D(a,b)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix}=f_{xx}(a,b)\,f_{yy}(a,b)-[f_{xy}(a,b)]^2$$

1. If $D>0$ & $f_{xx}(a,b)>0$ $\rightarrow$ $f(a,b)$ is a local minimum
2. If $D>0$ & $f_{xx}(a,b)<0$ $\rightarrow$ $f(a,b)$ is a local maximum
3. If $D<0$ $\rightarrow$ $f(a,b)$ is saddle point

Lagrange Multiplier method

↳ To find maximum & minimum values of $f(x,y)$ subject to contraint $g(x,y)=k$ → find all values of x, y & λ → $\nabla f(x,y) = \lambda \nabla g(x,y)$

↳ Evaluate f at all points (x,y) → largest is max, smallest is min.

E.g. Find the extreme values of the function $f(x,y) = x^2+2y^2$ on the circle $x^2+y^2=1$

$x^2+y^2=1$ → bound & $f(x,y)=x^2+2y^2$ is continuous → max & min exist

Apply Lagrange Multiplier method → $g(x,y)=1$, $g(x,y)=x^2+y^2$ [constraint]

Solve $\nabla f = \lambda \nabla g$ → $f_x = \lambda g_x$ & $f_y = \lambda g_y$.

$$f_x = \lambda g_x \longrightarrow \overset{f_x}{2x} = \overset{g_x}{2x}\lambda \quad \text{—— ①}$$

$$f_y = \lambda g_y \longrightarrow \overset{f_y}{4y} = \overset{g_y}{2y}\lambda \quad \text{—— ②}$$

$$x^2+y^2=1 \text{ (constraint)} \quad \text{—— ③}$$

From ①, $x(1-\lambda)=0$ → $x=0$ / $\lambda=1$ → case ①

sub $x=0$ into ③, $y=1$ / $y=-1$

From ②, $\lambda=2$ → case ②

→ Evaluate these 4 points

If $\lambda=1$, two points ⇒ $(1,0)$ & $(-1,0)$. If $\lambda=2$, two points $(0,1)$ & $(0,-1)$

$$\underbrace{f(0,1)=2, \ f(0,-1)=2}_{\text{max}}, \ \underbrace{f(1,0)=1, \ f(-1,0)=1}_{\text{min}}$$

∴ Maximum value of f on circle $x^2+y^2=1$ is 2 & minimum is 1.

→ If boundary is $x^2+y^2 \le 1$ → $x^2+y^2<1$ OR $x^2+y^2=1$

To find all possible candidate for extreme value in interior of disk → solve $\nabla f(x,y) = (0,0)$, where (x,y) is inside unit circle. → $\nabla f(x,y) = (2x, 4y)$ → $2x=0$ & $4y=0$ → satisfy $x^2+y^2<1$

$f(0,0)=0$ → a point (minimum)

E.g. A rectangular box without a lid is to be made from $12m^2$ cardboard. Use Lagrange Multiplier method to find the dimension of the box with maximum volume.

Let x, y & z denote the length, width & height.

Volume of box: $V=xyz$

Area of the four sides & bottom: $2xz+2yz+xy=12$ → $g(x,y,z)$

$L(x,y,z,\lambda) = \underbrace{xyz}_{\to f(x,y,z)} - \lambda(2xz+2yz+xy-12)$ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$L_x = yz - \lambda(2z+y) = 0$ —— ①   times x —— ⑤

$L_y = xz - \lambda(2z+x) = 0$ —— ②   times y —— ⑥

$L_z = xy - \lambda(2x+2y) = 0$ —— ③   times z —— ⑦

$-L_\lambda = 2xz+2yz+xy-12=0$ —— ④

From ⑤ & ⑥,

$2xz+xy = 2yz+xy$ → $x=y$

From ⑥ & ⑦,

$2yz+xy = 2xz+2yz$ → $y=2z$

Solving, $x=y=2$ & $z=1$

$V(2,2,1)=4$ is maximum value. → only 1 point found → cannot be minimum

Iterated Integral (Fubini's Theorem)

↳ If f is continuous on a rectangle

$R=[a,b]\times[c,d]=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d\}$, then

$\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy$

E.g. Evaluate the double integral $\iint_R (x-3y^2)\,dA$, where $R=\{(x,y)\mid 0\le x\le 2,\ 1\le y\le 2\}$

$\iint_R (x-3y^2)\,dA=\int_0^2\int_1^2 (x-3y^2)\,dy\,dx$ OR $\int_1^2\int_0^2 (x-3y^2)\,dx\,dy$

$=\int_0^2 \left[xy-\frac{3y^3}{3}\right]_1^2 dx=\int_0^2 (2x-2^3)-(x-1)\,dx$

$=\int_0^2 x-7\,dx=\left[\frac{x^2}{2}-7x\right]_0^2$

$=-12$

E.g. Find the volume of the solid that lies above the square $R=\underbrace{[0,2]}_{x}\times\underbrace{[0,2]}_{y}$ & below the elliptic paraboloid $z=16-x^2-2y^2$.

$\iint_R (16-x^2-2y^2)\,dA=\int_0^2\int_0^2 (16-x^2-2y^2)\,dx\,dy$

$=\int_0^2\left[16x-\frac{x^3}{3}-2y^2x\right]_0^2 dy=\int_0^2\left(\frac{88}{3}-4y^2\right)dy$

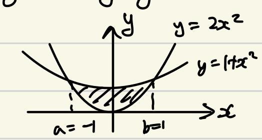
$=\left[\frac{88y}{3}-\frac{4y^3}{3}\right]_0^2=48$

Double Integral Over General Regions (Type I)

↳ If f is continuous on a region D such that $D=\{(x,y): a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$, then

$\iint_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

↳ Draw graph first to determine the space to integrate b/w.



$\iint_D (x+2y)\,dA$ bounded by $y=2x^2$ & $y=1+x^2$

↳ $\int_{-1}^1\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx$

Integrals Over Type II Regions

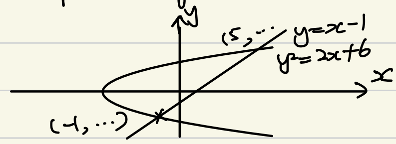
↳ If f is continuous on a region D such that $D=\{(x,y): c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$, then

$\iint_D f(x,y)\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

E.g. Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y=x-1$ & parabola $y^2=2x+6$.

$(x-1)^2=2x+6 \longrightarrow x^2-4x-5=0$

$(x-5)(x+1)=0$



$D=\{(x,y): -2\le y\le 4,\ \frac{y^2}{2}-3\le x\le y+1\}$

$\iint_D xy\,dA=\int_{-2}^4\int_{\frac{y^2}{2}-3}^{y+1} xy\,dx\,dy$

$=\int_{-2}^4\left[\frac{x^2y}{2}\right]_{\frac{y^2}{2}-3}^{y+1} dy$

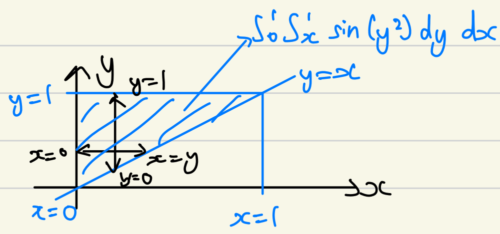
$=\int_{-2}^4 \frac{1}{8}y(-y^4+16y^2+8y-32)\,dy=36$

Changing the Order of Integration

E.g. Evaluate the iterated integral $\int_0^1\int_x^1 \underbrace{\sin(y^2)\,dy}_{\text{impossible}}\,dx$.

↳ Sketch region D & re-describe it,

$D=\{(x,y): 0\le y\le 1,\ 0\le x\le y\}$



Properties

1. If $f(x,y)\ge g(x,y)$ for all $(x,y)\in D \longrightarrow \iint_D f(x,y)\,dA\ge\iint_D g(x,y)\,dA$
2. If $m\le f(x,y)\le M$ for all (x,y) in $D \longrightarrow \iint_D m\,dA\le\iint_D f(x,y)\,dA\le\iint_D M\,dA$

Limit of a Sequence

- a sequence has a limit L, we write $\lim_{n\to\infty} a_n = L$ [as $n\to\infty$, $a_n \to L$] if for every $\varepsilon>0$ there is a corresponding integer N such that $|a_n - L| < \varepsilon$ for every $n>N$.
- if real number L exist → sequence converges to L, else the sequence diverges

E.g. Use definition to prove that $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Let $\varepsilon>0$ be an arbitrary positive real number. Our aim is to find an integer N such that for all $n>N$, $\left|\frac{n}{n+1} - 1\right| < \varepsilon$ $\quad a_n = \frac{n}{n+1}$, $L=1$

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$$

$$\frac{1}{n+1} < \varepsilon \longrightarrow n+1 > \frac{1}{\varepsilon} \longrightarrow n > \frac{1}{\varepsilon} - 1$$

$N(\varepsilon) > \frac{1}{\varepsilon} - 1$ → anything bigger will work

→ ceiling of $\frac{1}{\varepsilon}$

∴ We take N to be an integer such that $N(\varepsilon) = \lceil \frac{1}{\varepsilon} \rceil - 1$

For $n>N$, we have $\left|\frac{n}{n+1} - 1\right| < \varepsilon$

This proves that $\lim_{n\to\infty} \frac{n}{n+1} = 1$

Properties of Limit

a) $\lim_{n\to\infty}(C a_n) = C \lim_{n\to\infty} a_n$, C is a constant

b) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

c) $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

d) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, $\lim_{n\to\infty} b_n \neq 0$ & $b_n \neq 0$ for all n.

e) $\lim_{n\to\infty} \frac{1}{n^k} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^k = 0$ for all +ve integer k.

f) Suppose f is a continuous function → $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

- $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty}\frac{1}{n}\right) = 0$, $\lim_{n\to\infty} e^{\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{1}{n}} = 1$
- $\lim_{n\to\infty}\left(\frac{n^2}{n^2+3}\right)^3 = \left(\lim_{n\to\infty}\frac{n^2}{n^2+3}\right)^3 = 1$, $\lim_{n\to\infty}\sqrt{\frac{n^2}{n^2+3}} = \sqrt{\lim_{n\to\infty}\frac{n^2}{n^2+3}} = 1$

Subsequence

- subsequence is obtained from a sequence [even/odd subsequence]
- if $\lim_{n\to\infty} a_n = L$ → every subsequence of $\{a_n\}$ converges to L

E.g. Since $\lim_{n\to\infty}\frac{n}{n+1} = 1$, $\lim_{n\to\infty}\frac{2n}{2n+1} = \lim_{n\to\infty}\frac{(5n+1)}{(5n+1)+1} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{n!}{n!+1} = 1$

- can conclude $\lim_{n\to\infty} a_n$ does not exist if:
  - a) there is a divergent subsequence of $\{a_n\}$.
  - b) there are two convergent subsequent of $\{a_n\}$ with different limits.
- $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n-1} = L$ if & only if $\lim_{n\to\infty} a_n = L$

E.g. Determine the convergence of the following sequence $a_n = \begin{cases} \frac{2k}{2k+1} & \text{if } n=2k \\ e^{\frac{1}{(2k-1)}} & \text{if } n=2k-1 \end{cases}$

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty}\left(\frac{2n}{2n+1}\right) = 1 \quad \& \quad \lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} e^{\frac{1}{2n-1}} = 1$$

∴ Sequence $\{a_n\}$ converges & $\lim_{n\to\infty} a_n = 1$

Some tricks

1. Divide by highest power of n → $\lim_{n\to\infty}\frac{1+n}{2+n} = \lim_{n\to\infty}\frac{\frac{1}{n}+1}{\frac{2}{n}+1} = 1$ (converges)
2. $\lim_{n\to\infty}\left(\sqrt{n^2+2014} - \sqrt{n^2+1811n}\right) \cdot \frac{\sqrt{n^2+2014}+\sqrt{n^2+1811n}}{\sqrt{n^2+2014}+\sqrt{n^2+1811n}}$ (always do this first)

Squeeze Theorem

↳ Given 3 sequences $\{a_n\}$, $\{b_n\}$ & $\{c_n\}$, $a_n \le b_n \le c_n \longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \longrightarrow \lim_{n\to\infty} b_n = L$

E.g. Does $\lim_{n\to\infty} \frac{\cos n}{n}$ exist? If Yes, what is limit?

$n \ge 1 \longrightarrow -1 \le \cos n \le 1$

Multiply throughout by $\frac{1}{n}$, $\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ & $\lim_{n\to\infty} \frac{-1}{n} = 0$

By Squeeze Theorem, we conclude that $\lim_{n\to\infty} \frac{\cos n}{n} = 0$

E.g. Does $\lim_{n\to\infty} \frac{3^n}{n!}$ exist? If Yes, what is the limit?

$$\frac{3\times3\times3\times\ldots\times3}{1\times2\times3\times\ldots\times n} = \frac{3}{1}\times\frac{3}{2}\times\frac{3}{3}\times\underbrace{\frac{3}{4}}_{<1}\ldots\times\underbrace{\frac{3}{n-1}}_{\le 1}\frac{3}{n}$$
$$= \frac{3}{1}\times\frac{3}{2}\times\frac{3}{3}\,1\ldots1\times\left(\frac{3}{n}\right) = \frac{27}{2n}$$

$$0 < \frac{3^n}{n!} < \frac{27}{2n}$$

$\lim_{n\to\infty} 0 = 0$ & $\lim_{n\to\infty} \frac{27}{2n} = 0$ $\therefore$ By Squeeze Theorem, $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

L'Hospital's Rule

↳ Only can use if $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

↳ If can use, differentiate $f(x)$ & $g(x)$ respectively & use limit.

E.g. Determine whether $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = f(n)$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln(1+\frac{1}{n})} = e^{\lim_{n\to\infty} n\ln(1+\frac{1}{n})}$$

L'H Rule

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\cancel{(-1)n^{-2}}}{\cancel{(-1)n^{-2}}}$$

n^{-1}

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$$

$$\therefore \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1 = e$$

Monotone Convergence Theorem

↳ sequence $\{a_n\}$ is bounded if there are M & m such that $m \le a_n \le M$ for every n (lower bound: m; upper bound: M)

↳ a bounded sequence may not be convergent $[-1 \le (-1)^n \le 1]$

↳ sequence can be non-decreasing, non-increasing & monotonic [non-increasing & non-decreasing]

★ ↳ Every bounded monotonic sequence is convergent.

E.g. Consider the sequence $a_n = \frac{n}{n^2+1}$. Show that the sequence a_n is decreasing. Explain why sequence a_n is bounded. Is the sequence a_n convergent?

Let $f(x) = \frac{x}{x^2+1}$, then $a_n = f(n)$

$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1 \longrightarrow f$ decreasing on $(1, \infty)$

$\therefore \left\{\frac{n}{n^2+1}\right\}$ is decreasing

Since $\left\{\frac{n}{n^2+1}\right\}$ is decreasing, $\frac{1}{1^2+1}, \frac{2}{2^2+1}, \frac{3}{3^2+1} \ldots \frac{n}{n^2+1}$, for $n \ge 1$

$0 < \frac{n}{n^2+1} < \frac{1}{2} \longrightarrow$ sequence $\left\{\frac{n}{n^2+1}\right\}$ is bounded

Since $\left\{\frac{n}{n^2+1}\right\}$ is decreasing & bounded $\longrightarrow$ By Monotone Convergence Theorem, $\left\{\frac{n}{n^2+1}\right\}$ is convergent

Arithmetic Progression

↳ $U_n = a + (n-1)d$ (a term in AP; a: first number in series; $d = U_n - U_{n-1}$)

↳ $S_n = \frac{n}{2}[a + \underbrace{a+(n-1)d}_{U_n}]$ (S_n: sum)

Geometric Progression

- $U_n = ar^{n-1} \longrightarrow r = \frac{U_n}{U_{n-1}}$
- $S_n = \frac{a(1-r^n)}{1-r}$
- convergent if $|r|<1 \longrightarrow S_\infty = \frac{a}{1-r}$
- divergent if $|r| \geq 1$

E.g. Find the sum $\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{5^{n-2}}\right)$.

$$\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{5^{n-2}}\right) = \sum_{n=2}^{\infty}\left(\frac{2^n 2}{5^n 5^{-2}}\right) = \frac{2}{5^{-2}}\sum_{n=2}^{\infty}\left(\frac{2^n}{5^n}\right)$$

$$= 2(5^2)\sum_{n=2}^{\infty}\left(\frac{2}{5}\right)^n = 2(5^2)\sum_{n=2}^{\infty}\left(\frac{\frac{4}{25}}{1-\frac{2}{5}}\right)$$

$$= \frac{40}{3}$$

Telescoping Series

↳ series which can be expressed as $\sum_{n=1}^{\infty}(a_n - a_{n+m})$ for some $m \longrightarrow \lim_{N\to\infty}\sum_{n=1}^{N}(a_n - a_{n+m}), \lim_{N\to\infty} S_N$

E.g. Show that series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is convergent & find its sum.

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_N = \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \cancel{\left(\frac{1}{2} - \frac{1}{3}\right)} + \cancel{\left(\frac{1}{3} - \frac{1}{4}\right)} + \cancel{\ldots} + \cancel{\left(\frac{1}{N-1} - \frac{1}{N}\right)} + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)$$

$$= 1 - \frac{1}{N+1}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1 - \frac{1}{N+1}\right) = 1 - \lim_{N\to\infty}\frac{1}{N+1}$$

$$= 1 - 0 = 1 \longrightarrow \text{convergent}$$

→ For $N > m > 0$, we have $S_N = \sum_{n=1}^{N}(a_n - a_{n+m}) = \left(\sum_{n=1}^{m} a_n\right) - \left(\sum_{n=N+1}^{N+m} a_n\right)$ [Sum of first m terms − Sum of last m terms]

E.g. $\sum_{n=1}^{\infty}\left(\cos\frac{\pi}{n} - \cos\frac{\pi}{n+2}\right) \longrightarrow a_n = \cos\left(\frac{\pi}{n}\right)$ & $m=2$

For $N>2$, $S_N = \left(\sum_{n=1}^{2}\cos\frac{\pi}{n}\right) - \left(\sum_{n=N+1}^{N+2}\cos\frac{\pi}{n}\right)$

$\lim_{n\to\infty}\cos\frac{\pi}{n} = \cos 0 = 1$

$$\sum_{n=1}^{\infty}\left(\cos\frac{\pi}{n} - \cos\frac{\pi}{n+2}\right) = \left(\sum_{n=1}^{2}\cos\frac{\pi}{n}\right) - \left(\lim_{N\to\infty}\sum_{n=N+1}^{N+2}\cos\frac{\pi}{n}\right)$$

$$= \left(\cos\pi + \cos\frac{\pi}{2}\right) - \left(\lim_{N\to\infty}\left(\underbrace{\cos\frac{\pi}{N+1}}_{=1} + \underbrace{\cos\frac{\pi}{N+2}}_{=1}\right)\right)$$

$$= -1 - 2(1) = -3$$

∴ Series converges to -3

Harmonic Series

- Harmonic Series: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$
- Although $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0 \longrightarrow$ harmonic series is divergent

Prove: $S_1 = 1 \qquad S_2 = 1 + \frac{1}{2} \qquad S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2\left(\frac{1}{2}\right)$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right)$$

$$S_{2^n} \geq 1 + \frac{n}{2} \longrightarrow \lim_{n\to\infty} S_n = \infty \longrightarrow \text{divergent}$$

Properties

a) Every non-zero constant multiple of a divergent series is divergent

b) If Σa_n converges & Σb_n diverges $\longrightarrow \Sigma(a_n \pm b_n)$ diverges

Test for Divergence

- If series $\sum_{n=0}^{\infty} a_n$ converges $\longrightarrow \lim_{n\to\infty} a_n = 0$ ($\not\Leftarrow$)
- If $\lim_{n\to\infty} a_n$ does not exist / $\lim_{n\to\infty} a_n \neq 0 \longrightarrow$ series $\sum_{n=0}^{\infty} a_n$ diverges

Alternating Series

↳ series that alternate positive & negative

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots \quad (b_n > 0) \rightarrow$ satisfy $\lim_{n\to\infty} b_n = 0$ (zero limit) & $b_{n+1} < b_n$ (non-increasing) → Alternating series test

↳ series convergence → $0 \leq S_{2m} \leq b_1$ (Monotone Convergence Theorem → $\lim_{m\to\infty} S_{2m}$ exist)

E.g. Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ convergent?

Let $b_n = \frac{n}{n^2+1} > 0$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+(\frac{1}{n^2})} = 0$ (Zero Limit)

To show that $b_n = \frac{n}{n^2+1}$ is decreasing,

differentiate <0

$f'(n) = \frac{(n^2+1) - n(2n)}{(n^2+1)^2} = \frac{1-n^2}{(n^2+1)^2} < 0, \; n > 1 \rightarrow$ decreasing

∴ By alternating series test, alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is convergent

Integral test

↳ $\{a_n\}$ a sequence of positive terms → suppose $a_n = f(n)$ → f is a continuous, positive, decreasing function of x →

use integral test → series $\sum_{N}^{\infty} a_n$ & $\int_N^{\infty} f(x)\,dx$ both converge/diverge.

↳ $\int_N^{\infty} f(x) = \lim_{M\to\infty} \int_N^M f(x)\,dx$ may/may not exist

E.g. Use integral test to determine the convergence of series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

Let $f(x) = \frac{1}{1+x^2} \rightarrow f$ continuous, positive, decreasing function on $[1,\infty)$

$\int_1^{\infty} \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \int_1^M \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} [\tan^{-1} x]_1^M$

$= \lim_{M\to\infty} (\tan^{-1} M - \frac{\pi}{4}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

Since $\int_1^{\infty} \frac{1}{1+x^2}\,dx$ converges → $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges

P-series

↳ $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots + \frac{1}{n^p} + \dots$ converges if $p > 1$ & diverges if $p \leq 1$ → $\frac{1}{3}$

The Comparison Test

↳ Suppose Σa_n & Σb_n be series with ⊕ve terms

a) If $a_n \leq b_n$ & Σb_n is convergent, then Σa_n is also convergent

b) If $a_n \geq b_n$ & Σb_n is divergent, then Σa_n is also divergent

E.g. Is series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ convergent?

$\frac{5}{2n^2+4n+3} \leq \frac{5}{2n^2} \rightarrow \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

E.g. Does series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge?

$\frac{\ln n}{n} > \frac{1}{n}$ ↳ diverge for $n > 2 \rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverge

The limit comparison test

↳ Suppose that $a_n > 0$ & $b_n > 0$ for all $n \geq N$, where N is some integer

a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \rightarrow$ both Σa_n & Σb_n converge/diverge

b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \rightarrow \Sigma b_n$ converge $\rightarrow \Sigma a_n$ converge

c) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \rightarrow \Sigma b_n$ diverge $\rightarrow \Sigma a_n$ diverge

E.g. Is series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ convergent?

$a_n = \frac{5}{2n^2+4n+3} \qquad b_n = \frac{1}{n^2}$ → difference in degree btw numerator & denominator

$\lim_{n\to\infty} \frac{5}{2n^2+4n+3} \div \frac{1}{n^2} = \frac{5}{2 - \frac{4}{n} + \frac{3}{n^2}} = \frac{5}{2-0+0} > 0$

∴ Since b_n converge → a_n also converge

Ratio Test

a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1 \longrightarrow$ series $\sum a_n$ is absolutely convergent

b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1 / = \infty \longrightarrow$ series $\sum a_n$ diverges

c) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \rightarrow$ inconclusive

↳ If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ doesn't exist $\rightarrow$ Ratio Test can't be applied

E.g. Discuss convergence of the following series: $\sum_{n=1}^{\infty}\frac{(3n)!}{(2n)!\,n!}$

$$a_n = \frac{(3n)!}{(2n)!\,n!} \qquad a_{n+1} = \frac{(3(n+1))!}{(2(n+1))!\,(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(3(n+1))!}{(2(n+1))!\,(n+1)!} \times \frac{(2n)!\,n!}{(3n)!}$$

$$\frac{(3(n+1))!}{3n!} = \frac{\cancel{1\cdot2\cdot3\cdots3n}\cdot(3n+1)\cdot(3n+2)\cdot(3n+3)}{\cancel{1\cdot2\cdot3\cdots3n}} \qquad \frac{(2n)!}{(2(n+1))!} = \frac{\cancel{1\cdot2\cdot3\cdot4\cdots2n}}{\cancel{1\cdot2\cdot3}\cdots(2n+1)(2n+2)}$$

$$\frac{a_{n+1}}{a_n} = \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(2n+1)(2n+2)} \div \frac{n}{n}$$

$$\sum_{n=1}^{\infty}\frac{(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}{(2+\frac{1}{n})(2+\frac{2}{n})(1+\frac{1}{n})} = \frac{(3)(3)(3)}{(2)(2)(1)} = 6.75 > 1$$

$\therefore$ By ratio test $\longrightarrow$ series $\sum_{n=1}^{\infty}\frac{(3n)!}{(2n)!\,n!}$ diverges

Root Test

a) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1 \rightarrow$ series $\sum a_n$ converges absolutely

b) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1 / = \infty \rightarrow$ series $\sum a_n$ diverges

c) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1 \rightarrow$ root test is inconclusive

E.g. Is the series $\sum_{n=1}^{\infty}(-1)^n\frac{n}{3^n}$ absolutely convergent?

$$|a_n| = \frac{n}{3^n}$$

$$\sqrt[n]{|a_n|} = \left(\frac{n}{3^n}\right)^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{3}$$

$$\sum_{n=1}^{\infty}\frac{n^{\frac{1}{n}}}{3} = \frac{n^0}{3} = \frac{1}{3} < 1$$

$\therefore$ It is convergent

Power Series

↳ power series about $x=a$ is a series of the form

$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$, center is a

↳ For $\sum_{n=0}^{\infty} x^n$, converges to $\frac{1}{1-x}$ if $|x|<1$ $(-1<x<1)$, diverges if $|x|\geq 1$

1. Approximation

E.g. Use first four terms of power series for $\frac{1}{1-x}$ to estimate, a) $\frac{1}{0.99}$ b) $\frac{1}{1.1}$

a) $\frac{1}{0.99} = \frac{1}{1-0.01} = 1 + 0.01 + 0.01^2 + 0.01^3 + 0.01^4 + \ldots$

$= 1.010101$

b) $\frac{1}{1.1} = \frac{1}{1-(-0.1)} = 1 + (-0.1) + (-0.1)^2 + (-0.1)^3$

$= 0.909$

2. Obtaining power series

E.g. Use power series of $\frac{1}{1-x}$ to obtain power series of a) $\frac{1}{1+x^3}$ & b) $\frac{1}{2-x}$

a) $\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = 1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + \ldots$

$= 1 - x^3 + x^6 - x^9 + \ldots = \sum_{n=0}^{\infty}(-1)^n x^{3n}$

b) $\frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right) = \left(\frac{1}{2}\right)\left(1 + \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \ldots\right)$

$= \left(\frac{1}{2}\right)\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \ldots\right)$

$= \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$

Absolute Convergence

↳ If $\sum |a_n|$ converges & $\sum a_n$ converges → series is absolutely convergent

E.g. Is series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5}$ absolutely convergent?

$a_n = \frac{(-1)^{n-1}}{n^5}$ $\quad |a_n| = \frac{1}{n^5}$

$\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges as $p = 5 > 1$

E.g. Is series $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ convergent?

$a_n = \frac{\cos n}{n^3}$ $\quad |a_n| = \frac{1}{n^3}$ $\quad |\cos n| \leq 1$

Since $p = 3 > 1$, $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ converges

Conditional Convergence

↳ $\sum a_n$ conditionally convergent → $\sum a_n$ converges but $\sum |a_n|$ diverges.

↳ If both $\sum |a_n|$ & $\sum a_n$ diverge → $\sum a_n$ divergent

E.g. Does the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n^2}{n^3+n+1}$ converge absolutely, converge conditionally / diverge?

$a_n = (-1)^{n-1} \frac{2n^2}{n^3+n+1}$ $\quad |a_n| = \frac{2n^2}{n^3+n+1}$ $\quad \frac{n^2}{n^3} = \frac{1}{n} = b_n$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{2n^3}{n^3+n+1} = \lim_{n\to\infty} \frac{2}{1+\frac{1}{n^2}+\frac{1}{n^3}}$$

$$= 2 > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges → $|a_n|$ diverges, a_n does not converge absolutely.

Test if a_n converges / diverges → alternating series test → ① zero limit ② and

$$\lim_{n\to\infty} \frac{2n^2}{n^3+n+1} = \lim_{n\to\infty} \frac{\frac{2}{n}}{1+\frac{1}{n^2}+\frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \times \frac{2}{1+\frac{1}{n^2}+\frac{1}{n^3}} = 0 \cdot \frac{2}{1+0+0} = 0 \text{ (Zero limit)}$$

$$f(x) = \frac{2x^2}{x^3+x+1}$$

$$f'(x) = \frac{4x(x^3+x+1) - 2x^2(3x^2+1)}{(x^3+x+1)^2}$$

$$= \frac{4x^4+4x^2+4x-6x^4-2x^2}{(x^3+x+1)^2} = \frac{-2x^4+2x^2+4x}{(x^3+x+1)^2}$$

$$= 2\,\frac{-x^4+x^2+x}{\underbrace{(x^3+x+1)^2}_{>0}} \quad (2 > 0)$$

$-x^4+x^2+x < 0$ for $f'(x) < 0$

↳ for $x > 3 \rightarrow x^4 > 3x^3 > 3x^2 = x^2+2x^2 > x^2+2x$

Since $x^4 > x^2+x$, $-x^4+x^2+x < 0$ → decreasing

∴ a_n converges & $|a_n|$ diverges → converges conditionally

Radius of Convergence, Interval of Convergence

↳ For given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only 3 possibilities:

set of all values for which the power series converges

a) Power series converges absolutely for all x → radius of convergence $R=\infty$

b) Power series converges only at 1 point $x=a$ & diverges elsewhere → radius of convergence $R=0$

c) Ǝ ve real no. R → power series converges if $|x-a|<R$ & diverges if $|x-a|>R$

Step 1. Use appropriate test to find values of x

Step 2. If case (c) → remains to determine convergence at end-pt where $x=a\pm R$ → sub this into series & apply test

E.g. For what values of x does the power series $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$ converge?

Root Test: $\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{x^n}{n2^n}\right|} = \frac{|x|}{\sqrt[n]{n}\cdot 2}$

$\lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n}\cdot 2} = \frac{|x|}{n^{\frac{1}{n}}\cdot 2}$ → $n^{\frac{1}{\infty}} = n^0 = 1$

$= \frac{|x|}{2}$

converges abs if $\frac{|x|}{2}<1$ ⟶ $|x|<2$

diverges if $\frac{|x|}{2}>1$ ⟶ $|x|>2$

} Radius of convergence, $R=2$

inconclusive if $|x|=2$ → need to consider this next ($x=2$ & $x=-2$)

When $x=2$ $\sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ (divergent)

when $x=-2$, $\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} (-1)^n \left(\frac{\cancel{2^n}}{n\cancel{2^n}}\right)$ (converges)

∴ $R=2$ & interval of convergence is $[-2,2)$

E.g. For what values of x does power series $\sum_{n=1}^{\infty} (n!)(x-1)^n$ converge?

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!(x-1)^{n+1}}{n!(n-1)^n}\right| = (n+1)\cdot|x-1|$

As $n\to\infty$, $(n+1)\cdot|x-1| \to \infty$

∴ Diverges for all except when $x=1$ $\{1\}$ ⇒ $R=0$

interval of convergence

Term-by-term Differentiation

↳ Suppose $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $-R<x-a<R$ → $\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

$f(x)$

Eg. Let $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $-1<x<1$. Find power series $f'(x)$ & function $\frac{1}{(1-x)^3}$

$1+x+x^2+x^3+\ldots$ $\frac{d}{dx}$ term by term

a) $f'(x) = \frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+5x^4+\ldots$

$= \sum_{n=0}^{\infty} (n+1)x^n$

b) $f''(x) = \frac{2}{(1-x)^3} = 2+3\times 2x+4\times 3x^2+5\times 4x^3+\ldots = \sum_{n=0}^{\infty} (n+1)(n+2)x^n$

$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=0}^{\infty} (n+1)(n+2)x^n$

Term-by-term integration

↳ $\int \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} + C$

E.g. Determine power series of $\ln(1+x)$ for $-1<x<1$

$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1+(-x)+(-x)^2+(-x)^3+\ldots$

$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = [\ln(1+t)]_0^x$

$= \ln(1+x) - \ln 1 = \ln(1+x)$

$\int_0^x 1-t+t^2-t^3+t^4-\ldots dt = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\ldots$

$= \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

$\int_0^x t^n dt = \left[\frac{t^{n+1}}{n+1}\right]_0^x$

$= \frac{x^{n+1}}{n+1} - \frac{0^{n+1}}{0}$

Taylor & Maclaurin Series

↳ function f has power series representation at $x=a$. $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \longrightarrow c_n=\frac{f^{(n)}(a)}{n!}$

1. Taylor series: $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\ldots\frac{f^{(n)}(a)}{n!}(x-a)^n$ coefficient $c_n=\frac{f^{(n)}(a)}{n!}$

2. Maclaurin Series $(a=0)$: $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f^{(2)}(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n$

E.g. Find Taylor series generated by $f(x)=\frac{1}{x}$ at $a=3$ ☆

To determine c_n, $f(x)=\frac{1}{x}$, $f'(x)=\frac{-1}{x^2}$, $f^{(2)}x=\frac{(-1)(-2)}{x^3}=\frac{(-1)^2 2!}{x^3}$

$f^{(3)}(x)=\frac{(-1)^3 3!}{x^4} \rightarrow f^{(n)}(x)=\frac{(-1)^n n!}{x^{n+1}}$

$f^{(n)}(3)=\frac{(-1)^n n!}{3^{n+1}} \longrightarrow c_n=\frac{(-1)^n \cancel{n!}}{(3^{n+1})\cancel{n!}}$

$\frac{(-1)^n}{3^{n+1}}(x-a)^n = \frac{1}{3}-\frac{1}{3^2}(x-3)+\frac{1}{3^3}(x-3)^2+\ldots+\frac{(-1)^n}{3^{n+1}}(x-3)^n+\ldots$

↳ Geometric series → 1st term $=\frac{1}{3}$, common ratio, $r=\frac{-(x-3)}{3}$

$\text{Sum}=\frac{\frac{1}{3}}{1-\left(\frac{-(x-3)}{3}\right)}=\frac{1}{x}=f(x) \longrightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-3)^n$

Taylor Polynomials

↳ $T_n(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\ldots+\underbrace{\frac{f^{(k)}(a)}{k!}}_{\text{coefficient}}(x-a)^k+\ldots$

E.g. Taylor Polynomials $T_3(x)$ & $T_4(x)$ for $f(x)=3\cos 2x$ at $a=\pi$

		$n!$	c_n
$f=3\cos 2x=3$	÷	1	3
$f'=-6\sin 2x=0$	÷	1	0
$f''=-12\cos 2x=-12$	÷	2	-6
$f'''=24\sin 2x=0$	÷	6	0
$f^{IV}=48\cos 2x=48$	÷	24	2

$T_3(x)=3-6(x-\pi)^2$

$T_4(x)=3-6(x-\pi)^2+2(x-\pi)^4$

Taylor's Remainder/Formula

↳ Taylor remainder of order n, $R_n(x)=f(x)-T_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ (c → value b/w a & x)

E.g. Find the Maclaurin series of $\sin x$ & show that it converges to $\sin x$ at every x

$f(x)$	$f'(x)$	$f''(x)$	$f^{(3)}(x)$	$f^{(4)}(x)$	$f^{(5)}(x)$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$
0	1	0	-1	0	1
	x		$-\frac{x^3}{3!}$		$+\frac{x^5}{5!}$

↳ $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$

$f(x)=\sin x$ $\quad \sin x = T_n(x)+R_n(x)$, need to show $\lim_{n\to\infty}R_n(x)=0$

$|f^{(2k)}(x)|=|\sin x|\leq 1$ & $|f^{(2k-1)}x|=|\cos x|\leq 1 \longrightarrow |f^{(n+1)}(c)|\leq 1$

$|R_n(x)|=|f^{(n+1)}(c)|\cdot\left|\frac{x^{n+1}}{(n+1)!}\right|\leq\frac{|x|^{n+1}}{(n+1)!}\xrightarrow{n\to\infty}0$

$|R_n(x)|\to 0$

∴ Maclaurin series of $\sin x$ converges to $\sin x$

E.g. Use Taylor Remainder to prove that $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$f(x)=e^x \quad f^{(n)}(0)=1$

$|f(x)-T_n|=\left|\frac{e^c\cdot x^{n+1}}{(n+1)!}\right|=\frac{e^c}{(n+1)!}\cdot|x|^{n+1}$

$0\leq|f(x)-T_n(x)|\leq\frac{e^{|x|}\cdot|x|^{n+1}}{(n+1)!}\xrightarrow{n\to\infty}0$

Summary of Sequence & Series

1. Sequence $[a_1, a_2 \dots \{a_n\}]$

- → Squeeze Theorem
- → If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$
- → If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

} to find value of $\lim_{n\to\infty} a_n$, but don't tell you if it converges

Monotone Convergence Theorem

↳ every bounded monotonic sequence converges

★ If the sequence converges, then $\lim_{n\to\infty} a_n$ exist

★ If lim exist → sequence converges

↳ series → may not converge (Harmonic series)

2. Series $(\sum a_n)$

↳ Geometric, telescoping, harmonic, p-series

$\sum_{n=0}^{\infty} ar^n$ $\quad \sum_{n=1}^{\infty}(a_n - a_{n+m})$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^p}$

(m=1) ↳ $\sum_{n=1}^{k} (a_n - a_{n+1}) = a_1 - \cancel{a_2} + \cancel{a_2} - \cancel{a_3} + \dots$

$= a_1 - a_{k+1}$

Geometric → converges, if $|r| < 1$

↳ diverges, if $|r| \geq 1$

Telescoping → $\lim_{k\to\infty} \sum_{n=1}^{k} (a_n - a_{n+m}) = \lim_{k\to\infty}(a_1 - a_{k+m})$ → value (converge) / → ∞ (diverge)

Harmonic → always divergent

p-series → $p > 1$ → converges; → $p \leq 1$ → diverges

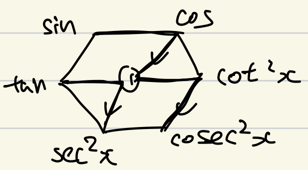
Test to check series converges/not:

1. Alternating Series Test $(-1)^n$
2. Comparison Test
3. Limit Comparison Test
4. Ratio Test → there is $n!$
   $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ $\quad \frac{(n+1)!}{n!}$
5. Root Test (n^n)
   $\lim_{n\to\infty} \sqrt[n]{|a_n|}$
6. Integral Test ✗ (not normally used) → but still imp!

Result:

1. Absolutely convergent ($\sum a_n$ ✓, $\sum |a_n|$ ✓ converges)
2. Conditionally convergent ($\sum a_n$ ✓, $\sum |a_n|$ ✗)
3. Divergent ($\sum a_n$ ✗, $\sum |a_n|$ ✗)

Differentiation



$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $|x| < 1$ $\qquad \frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, $|x| < 1$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$

Rules:

1. $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
2. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos x^2 - \sin^2 x$

$= 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

Integration

$\int \cos x \, dx = \sin x + C$

$\int \sin x \, dx = -\cos x + C$

$\int \sec^2 x \, dx = \tan x + C$

$\int -\operatorname{cosec}^2 x \, dx = -\cot x + C$

$\int \sec x \tan x \, dx = \sec x + C$

$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$, $|x| < |a|$

Rules:

1. By substitution let $t = \tan\frac{x}{2}$ $\quad \frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$
   ↳ Replace $dt \to \left(\frac{dx}{dt}\right)dt \to dx$
2. By parts → $\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx$

Differential equation

↳ Order of differential equation → highest derivative occuring in a differential eqn. [$\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^5$ → order = 3]

↳ Types of differential equation (D.E):

1) 1st order D.E → separate & linear differential eqn

   ↳ variations: Homogenous eqn, Bernoulli eqn & exact eqn

2) 2nd order D.E with constant coefficient

   ↳ variations: Homogenous D.E & non-homogeneous D.E.

First-order D.E

↳ 1st order D.E. separable if can separate expressions involving x & y

$$p(y)\,y' = q(x)$$

E.g. Solve $(1+y^2)\frac{dy}{dx} = x\cos x$

$$\int (1+y^2)\,dy = \int x \cos x \,dx$$

let $u = x$ let $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1$ $V = \sin x$

$$y + \frac{y^3}{3} = uv - \int v\left(\frac{du}{dx}\right)dx$$

$$y + \frac{y^3}{3} = x\sin x - \int \sin x \,dx$$

$$y + \frac{y^3}{3} = x\sin x + \cos x + C$$

Initial Value Problem

↳ Sometimes just interested in value of y at some value of x (not all)

E.g. Solve initial value problem $3y^2\frac{dy}{dx} = \cos x$, $y\left(\frac{\pi}{2}\right) = -4$

$$\int 3y^2\,dy = \int \cos x\,dx$$

$$\frac{\cancel{3}y^3}{\cancel{3}} = \sin x + c$$

sub $y=-4$, $x=\frac{\pi}{2}$ into eqn, $(-4)^3 = 1 + c$ → $c = -65$

$$y^3 = \sin x - 65 \longrightarrow y = \sqrt[3]{\sin x - 65}$$

E.g. Solve initial value problem $y' = -2(y-1)^2 x$, $y(179) = 1$. How about $y(1) = 3$

$$\int \frac{1}{(y-1)^2}\,dy = \int -2x\,dx$$

$$\cancel{-}(y-1)^{-1} = \frac{\cancel{-2}x^2}{\cancel{2}} + c$$

If $y=1$,

$0 = $ LHS   RHS $= -2(1-1)^2 x = 0$

$$\frac{1}{y-1} = x^2 + C$$

$$y = \frac{1}{x^2+c} + 1 \text{ if } y \neq 1 \text{ or } y = 1$$

when $y(179) = 1$,

$$y = \frac{1}{x^2+c} + 1$$

$$1 = \frac{1}{179+c} + 1 \text{ ✗ (impossible)}$$

$\therefore y(179) = 1$

$y = 1$ so $y(179) = 1$ gg to happen

When $y(1) = 3$

$y = 1$ ✗ (not gg to happen)

$$3 = \frac{1}{1^2+c} + 1$$

$$c = -\frac{1}{2}$$

$$\therefore y = 1 + \frac{1}{x^2 - 0.5}$$

First-Order Linear Differential equation

↳ linear in y', y & not linear in x & can be written in form

$$a(x)\frac{dy}{dx}+b(x)y=r(x)$$

↳ $\underbrace{x^2}_{a(x)}\left(\frac{dy}{dx}\right)+\underbrace{x}_{b(x)}y=\underbrace{x^2e^x}_{r(x)}, x>0$ (linear), $(x+y)\frac{dy}{dx}+xy=e^x, x>0$ (not linear), $\sqrt[3]{y}$ / $\sin y$ (not linear)

↳ 1st order D.E in standard form if coefficient $\frac{dy}{dx}$ is 1

To solve $a(x)y'+b(x)y=r(x)$

① Convert to standard form, $y'+p(x)y=q(x)$

② Compute $I(x)=e^{\int p(x)\,dx}$

③ General solution is $y=\frac{1}{I(x)}\int I(x)q(x)\,dx$

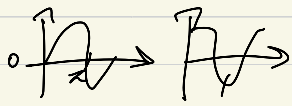
E.g. Solve the initial value problem $x\frac{dy}{dx}+2y=\cos x$, $y(\pi)=2022$

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x}\cos x$$

$$I(x)=e^{\int\frac{2}{x}dx}=e^{2\ln x}=x^2$$

$$y=\frac{1}{x^2}\int x^2\left(\frac{1}{x}\cos x\right)dx$$

$$x^2y=\int x\cos x\,dx=x\sin x+\cos x+c$$



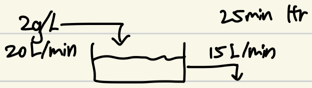
$$y=\frac{x\sin x+\cos x+c}{x^2}$$

$$2022=\frac{0-1+c}{\pi^2}\longrightarrow c=1+2022\pi^2$$

$$y=\frac{1}{x^2}(x\sin x+\cos x+2022\pi^2+1)$$

Applications

E.g. A tank initially contains 380L of brine in which 220g of salt are dissolved. A brine containing 2g/L of salt runs into the tank at the rate of 20L/min. Mixture flows out of tank at rate 15L/min. Determine the amount of salt in the tank 25min after the process starts



$y(t)=$ mass of salt in g

$2\frac{g}{L}\times20\frac{L}{min}=40\frac{g}{min}$ (salt in)

$\frac{mass}{volume}\times15\frac{L}{min}=\frac{y}{380+5(t)}\times15$

volume $=380+20(t)-15(t)$

$=380+5(t)$

$\frac{dy}{dt}=$ salt in − salt out.

$y(0)=220$

$$=40-\frac{15y}{380+5t}$$

$$y'+\frac{15}{380+5t}y-40=0$$

$I(x)=e^{\int\frac{\cancel{15}\,3}{\cancel{380+5t}\,76}dt}$

$=e^{\int\frac{3}{76+t}dt}=e^{\ln(76+t)^3}$

$=(76+t)^3$

$$y=\frac{1}{(76+t)^3}\int(76+t)^3(40)\,dt$$

$$(76+t)^3y=\frac{(76+t)^4}{\cancel{4}}(\cancel{40})^{10}+c$$

$$y=\frac{10(76+t)^4+c}{(76+t)^3}$$

sub $y(0)=220$,

$$220=\frac{10(76)^4+c}{76^3}$$

$$c=220\times76^3-10(76)^4=76^3(-540)$$

$$y(25)=\frac{10(76+25)^4-(540\times76^3)}{(76+25)^3}\approx780g$$

Reduction to 1st order D.E: missing y

↳ suppose y is missing from D.E. → we treat y' as a function of x & use substitution ($z=y'$ & $z'=y''$) E.g. $\frac{d^2y}{dx^2}=\frac{1}{x}\left(\frac{dy}{dx}+x^2\cos x\right)$

E.g. Find general solution to $\frac{d^2y}{dx^2}=\frac{1}{x}\left(\frac{dy}{dx}+x^2\cos x\right), x>0$ ($\frac{d^2y}{dx^2}\to z'$, $\frac{dy}{dx}\to z$)

$$\frac{dz}{dx}=\frac{1}{x}(z+x^2\cos x)\to\underbrace{\frac{dz}{dx}-\frac{1}{x}z}_{\times\frac{1}{x}}=\underbrace{x\cos x}_{\times\frac{1}{x}}$$

$\left(\frac{z}{x}\right)' = \frac{z'x - z(1)}{x^2} = \frac{1}{x}z' - \frac{1}{x^2}z$

$$\left(\frac{1}{x}\cdot z\right)' = \cos x$$
$$\frac{z}{x} = \int \cos x = \sin x + c_1$$
$$y' = z = x\sin x + c_1 x$$
$$y = \int x\sin x + c_1 x \, dx$$
$$= x(-\cos x) - \int 1(-\cos x)dx + \frac{c_1 x^2}{2} + c_2$$
$$= -x\cos x + \sin x + c_1\frac{x^2}{2} + c_2$$

$$I(x) = e^{\int -\frac{1}{x}dx} = e^{\ln(x)^{-1}} = \frac{1}{x}$$

Reduction to First Order D.E: missing x

↳ Suppose x is missing from D.E. → let $z = y' = \frac{dy}{dx}$ & $y'' = z\frac{dz}{dy}$

E.g. Find general solution to $\frac{d^2y}{dx^2} = -\frac{2}{1-y}\left(\frac{dy}{dx}\right)^2$.

$$z\cdot\frac{dz}{dy} = -\frac{2}{1-y}(z)^2$$
$$\frac{dz}{dy} + \frac{2}{1-y}z = 0 \text{ (linear)}$$
$$\left(z\cdot\frac{1}{(1-y)^2}\right)' = 0$$
$$\frac{z}{(1-y)^2} = c_1 \longrightarrow \frac{\frac{dy}{dx}}{(1-y)^2} = c_1$$
$$\int\frac{dy}{(1-y)^2} = \int c_1\,dx$$
$$\frac{1}{1-y} = c_1 x + c_2$$
$$1 - y = \frac{1}{c_1 x + c_2}$$
$$y = 1 - \frac{1}{c_1 x + c_2}$$

$$e^{\int\frac{2}{1-y}dy} = e^{-2\ln|1-y|} = e^{\ln\frac{1}{(\ln|1-y|)^2}} = e^{\ln\frac{1}{(1-y)^2}} = \frac{1}{(1-y)^2}$$

Homogeneous Function

- $f(x,y) = \frac{y}{x}$ & $f(x,y) = \frac{x}{y}$
- To check sub x & y with tx & ty respectively
  $$f(x,y) = \frac{2xy - y^2}{x^2 + y^2}$$
  $$f(tx,ty) = \frac{2(tx)(ty) - (ty)^2}{(tx)^2 + (ty)^2} = \frac{t^2(2xy - y^2)}{t^2(x^2+y^2)} = f(x,y) \longrightarrow f \text{ is homogeneous}$$
- change variable $V = \frac{y}{x}$ & $Vx = y \rightarrow \frac{dy}{dx} = \frac{dV}{dx}\cdot x + V$

E.g. Find the general solution to $\frac{dy}{dx} = \frac{4x+y}{x-4y}$

1. Check if homogenous → $\frac{4(tx)+(ty)}{tx-4(ty)} = \frac{\cancel{t}(4x+y)}{\cancel{t}(x-4y)} = \frac{dy}{dx}$
2. Replace $\frac{y}{x} = V$ $\quad \frac{4+\frac{y}{x}}{1-\frac{4y}{x}} = \frac{4+V}{1-4V}$

$$\underbrace{\frac{dV}{dx}\cdot x + V}_{\frac{dy}{dx}} = \frac{4+V}{1-4V}$$
$$x\cdot\frac{dV}{dx} = \frac{4+V}{1-4V} - V = \frac{4+4V^2}{1-4V} = \frac{4(1+V^2)}{1-4V}$$
$$\int\frac{1-4V}{1+V^2}dV = \int\frac{4}{x}dx$$
$$\int\frac{1}{1+V^2}dV - \int\frac{4V}{1+V^2}dV = 4\ln|x| + c$$
$$\tan^{-1}(V) - 2\ln|1+V^2| = 4\ln|x| + c$$
$$\tan^{-1}\left(\frac{y}{x}\right) - 2\ln\left(1 + \left(\frac{y}{x}\right)^2\right) = 4\ln|x| + c$$

Bernoulli Differential Equations

- non-linear differential equation that can be reduced to a linear differential eqn
  $$\frac{dy}{dx} + p(x)y = q(x)y^\alpha \rightarrow \alpha \neq 0, \alpha \neq 1$$
- 1. Divide Bernoulli D.E by y^α & let $u = y^{1-\alpha}$ → $\frac{du}{dx} = (1-\alpha)y^{-\alpha}\frac{dy}{dx}$ $\left[y^{-\alpha}\frac{dy}{dx} = \frac{1}{1-\alpha}\frac{du}{dx}\right]$ – ①
- 2. Sub ① & $u = y^{1-\alpha}$ into eqn & rewrite in standard form

E.g. Solve $\frac{dy}{dx}+\frac{3}{x}y = \frac{12y^{\frac{2}{3}}}{\sqrt{1+x^2}}$, $x>0$ $\quad \rightarrow \alpha = \frac{2}{3}$

$$y^{-\frac{2}{3}}\left(\frac{dy}{dx}\right)+\frac{3}{x}y^{\frac{1}{3}} = \frac{12}{\sqrt{1+x^2}}$$

let $u=y^{\frac{1}{3}}$, $\frac{du}{dx}=\frac{1}{3}y^{-\frac{2}{3}}\left(\frac{dy}{dx}\right) \rightarrow y^{-\frac{2}{3}}\left(\frac{dy}{dx}\right)= 3\frac{du}{dx}$

$$\cancel{3}\left(\frac{du}{dx}\right)+\frac{\cancel{3}}{x}u = \frac{\cancel{12}\ 4}{\sqrt{1+x^2}}$$

$$I(x)=e^{\int\frac{1}{x}dx} = e^{\ln(x)} = x$$

$$\frac{d}{dx}(u\cdot x) = \frac{4x}{\sqrt{1+x^2}}$$

$$u\cdot x = \int\frac{4x}{\sqrt{1+x^2}}dx = 4\sqrt{1+x^2}+C$$

$$u = \frac{4\sqrt{1+x^2}}{x}+\frac{C}{x}$$

$$y = \left(\frac{4\sqrt{1+x^2}}{x}+\frac{C}{x}\right)^3$$

Exact Differential Equations

↳ D.E. $M(x,y)dx+N(x,y)dy=0$ is said to be exact in domain D if there is a function $u(x,y)=C$ such that $\frac{\partial u}{\partial x}=M$ & $\frac{\partial u}{\partial y}=N$ for all $(x,y)\in D$

potential function for D.E.

E.g. Solve equation $\underbrace{2x\sin y}_{M}\,dx+\underbrace{x^2\cos y}_{N}\,dy=0$

$$u_x = 2x\sin y \rightarrow u(x,y)=x^2\sin y = C$$

$$u_y = x^2\cos y \xrightarrow{\text{integrate}} u(x,y)=x^2\sin y = C$$

$$\sin y = \frac{C}{x^2}$$

$$y=\sin^{-1}\left(\frac{C}{x^2}\right)$$

Test for Exactness

↳ Suppose M, N & M_x & N_x are continuous in a simply connected domain D.

D.E. $M(x,y)dx+N(x,y)dy=0$ is exact for all $(x,y)\in D$ if & only if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \rightarrow M_y = N_x$

↳ If $M_y\neq N_x \rightarrow$ D.E. is not exact

E.g. Find general solution to $\underbrace{\boxed{2xe^y}}_{M}\,dx+\underbrace{\boxed{(x^2e^y+\cos y)}}_{N}\,dy=0$

$$M_y = 2xe^y \qquad N_x = 2xe^y$$

$$\underbrace{2xe^y}_{\frac{\partial u}{\partial x}}dx+\underbrace{(x^2e^y+\cos y)}_{\frac{\partial u}{\partial y}}dy=0$$

$u=\int 2xe^y dx = x^2e^y+\underbrace{c(y)}_{\text{function of } y}$

$$\frac{\partial u}{\partial y}=x^2\cdot e^y+c'(y)$$

$$x^2\cdot e^y+c'(y)=x^2e^y+\cos y$$

$$c(y)=\int\cos y\,dy = \sin y$$

$u(x,y)=\underline{x^2e^y+\sin y = C}$ (General solution)

Transform to exact D.E.

↳ Smts can multiply D.E with non-zero function $\underbrace{I(x,y)}_{\text{integrating factor}}$ to obtain exact eqn

$I(x,y)M(x,y)dx+I(x,y)N(x,y)dy=0 \longrightarrow \frac{\partial}{\partial y}(I(x,y)M(x,y)) = \frac{\partial}{\partial x}(I(x,y)N(x,y))$

E.g. Consider D.E. $(3y^2+5x^2y)dx+(3xy+2x^3)dy=0$. Verify that $I=x^2y$ is an integrating factor & proceed to solve D.E.

$$(x^2y)(3y^2+5x^2y)dx+(x^2y)(3xy+2x^3)dy=0$$

$$\underbrace{(3x^2y^3+5x^4y^2)}_{M}dx+\underbrace{(3x^3y^2+2x^5y)}_{N}dy=0$$

$$M_y = 9x^2y^2+10x^4y = N_x = 9x^2y^2+10x^4y$$

$u=\int 3x^2y^3+5x^4y^2dx = \frac{\cancel{3}x^3}{\cancel{3}}y^3+\frac{\cancel{5}x^5}{\cancel{5}}y^2+\underbrace{c(y)}_{\text{some function of } y}$

$$\frac{\partial u}{\partial y}=3x^3y^2+2x^5y+c'(y)=N \rightarrow c'(y)=0$$

$$x^3y^3+x^5y^2=C$$

Homogenous L.D.E with constant coefficient

↳ For homogenous linear D.E, $ay''+by'+cy=0$, $a\neq 0$, where a,b,c are constant

$a\lambda^2+b\lambda+c=0$ (characteristic eqn)

↳ general solution depends on the type of roots of eqn

a) $b^2-4ac>0 \rightarrow$ eqn has 2 distinct real roots $(y_h=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x})$

b) $b^2-4ac=0 \rightarrow$ eqn has repeating real root $(y_h=C_1e^{\lambda_1 x}+C_2 x e^{\lambda_1 x})$

c) $b^2-4ac<0 \rightarrow$ eqn has 2 complex conjugate root $(y_h(x)=C_1e^{\alpha x}\cos\beta x+C_2e^{\alpha x}\sin\beta x)$

$\lambda=-3\pm 4i$

E.g. General solution to $y''-y'-2y=0$.

$\lambda^2-\lambda-2=0$

$D=\underbrace{1+4\times 2}_{b^2-4ac}=9>0$ $\qquad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\lambda_{1,2}=\frac{1\pm\sqrt{1-4(1)(-2)}}{2}=2,-1$

$y=C_1e^{2x}+C_2e^{-x}$

E.g. General solution to $y''+4y'+4y=0$

$\lambda^2+4\lambda+4=0$ $\qquad D=0$

$(\lambda+2)^2=0 \rightarrow \lambda_{1,2}=-2$

$y=e^{-2x}(C_1x+C_2)$

E.g. General solution to $y''+6y'+25y=0$

$\lambda^2+6\lambda+25=0$ $\qquad D=36-100=-84<0$

$\lambda_{1,2}=\frac{-6\pm\sqrt{6^2-4(1)(25)}}{2}=-3\pm 4i$

$y=e^{-3x}(C_1\cos(4x)+C_2\sin(4x))$

Non-homogeneous Linear D.E.

↳ $ay''+by'+cy=F(x)$ on interval I. General solution: $y(x)=y_h(x)+y_p(x)$, $y_h(x)\rightarrow$ general soln for $ay''+by'+cy=0$ homogenous eqn. $y_p(x)\rightarrow$ particular solution of $F(x)$

1. Method of Undetermined Coefficients

E.g. Solve the non-homogenous D.E. $y''-y'-2y=10\sin x$

Homogenous D.E: $y''-y'-2y=0$

$\lambda_{1,2}=2,-1 \rightarrow y_h=C_1e^{2x}+C_2e^{-x}$

For $10\sin x \rightarrow$ assume $y_p=a\cos x+b\sin x=y$

$y'=-a\sin x+b\cos x$ $\quad$ sub into D.E

$y''=-a\cos x-b\sin x$

$-a\cos x-b\sin x-(-a\sin x+b\cos x)-2(a\cos x+b\sin x)=10\sin x$

$\overbrace{(-a-b-2a)}^{0}\cos x+\overbrace{(-b+a-2b)}^{10}\sin x=10\sin x$

↳ $-a-b-2a=0$ & $-b+a-2b=10$

$a=1$ & $b=-3$

$y=\cos x-3\sin x+C_1e^{-x}+C_2e^{2x}$

a) $y''-y'-2y=x^2 \longrightarrow y_p=ax^2+bx+c$

b) $y''-y'-2y=3e^{2x} \longrightarrow y_p=axe^{2x} \longrightarrow$ if sub this $=0$ then $y_p=ax^2e^{2x}$

c) $y''-y'-2y=x^2+\sin x \rightarrow y_p=ax^2+bx+c+d\cos x+e\sin x$

2. Method of Variation-of-Parameters

↳ For $ay''+by'+cy=F(x)$

↳ Replace general solution of eqn $y_h=C_1y_1(x)+C_2y_2(x)$ & determine $y_p=C_1(x)y_1(x)+C_2(x)y_2(x)$

Determine $y_p=C_1(x)y_1+C_2(x)y_2$ with formulae C_1' & C_2'

$C_1'=\frac{-(\frac{F}{a})y_2}{y_1y_2'-y_1'y_2}$, $\quad C_2'=\frac{(\frac{F}{a})y_1}{y_1y_2'-y_1'y_2}$

↳ $W=y_1y_2'-y_1'y_2$ (use this for easier computation)

E.g. Solve $y'' + y = \sec x = \frac{1}{\cos x}$ $\qquad a = 1$

$c_1 = -\frac{1}{a}\int \frac{F \cdot y_2}{w} dx \qquad C_2 = \frac{1}{a}\int \frac{F y_1}{w} dx$

$y'' + y = 0 \longrightarrow \lambda^2 + 1 = 0 \longrightarrow \lambda = \pm i$

$y_h = \underline{e^{\overset{1}{0x}}} (c_1 \overbrace{\cos x}^{y_1} + c_2 \overbrace{\sin x}^{y_2})$

$w = \cos x(\cos x) - (\sin x)(-\sin x) = \cos^2 x + \sin^2 x = 1$

$c_1(x) = -\frac{1}{1}\int \frac{\frac{1}{\cos x} \cdot \sin x}{1} dx = \ln|\cos x|$

$c_2(x) = \frac{1}{1}\int \frac{\frac{1}{\cos x} \cdot \cos x}{1} dx = x$

$y_p = \ln|\cos x| \cdot \cos x + x \sin x$

$y(x) = \underline{\alpha_1 \cos x + \alpha_2 \sin x} + \cos x \ln|\cos x| + x \sin x$, α_1 & α_2 constants