Set & Logic

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Outline

- Truth tree for first order logic
- Inference rule/Natural Deduction Rules
- Logic and type 
- Russel paradox

Anything marked with  is an exact topic

Truth tree for first order logic

- Simplify:

$$\neg[\exists x \in M : A(x)] \Leftrightarrow \forall x \in M : \neg A(x)$$
$$\neg[\forall x \in M : A(x)] \Leftrightarrow \exists x \in M : \neg A(x)$$

- How to use:

$\exists x,\ P(x)$: exist a that P(a) is true, while a is a new constant symbol here

$P(a)$ should use a new variable each time,

as we don't know what a is, only know a exists

$\forall x,\ \neg P(x)$: can choose arbitrary x. but…how to choose? $\neg P(b)$ Is true, but useless

$\neg P(a)$ "Delay" the choose! (create contradictory)

Example

- Prove: $\forall x \exists y (F(x) \to G(y)) \vdash \exists y \forall x (F(x) \to G(y))$

  note: $\forall x,\ P(x)$ can be reused

Inference rule

$$\frac{J_1 \quad \dots \quad J_k}{J}$$

Premises of the rule

Conclusion

$$J_1 \ \dots \ J_k \vdash J$$

premises are sufficient for the conclusion

it is not necessary that the premises hold

Examples: an inductive definition of nat(natural number)

$$\frac{}{zero \ \ \text{nat}}$$

$$\frac{a \ \ \text{nat}}{succ(a) \ \ \text{nat}}$$

Examples: a simultaneous inductive definition of even and odd

$$\frac{}{zero \ \ \text{even}}$$

$$\frac{b \ \ \text{even}}{succ(b) \ \ \text{odd}}$$

$$\frac{a \ \ \text{odd}}{succ(a) \ \ \text{even}}$$

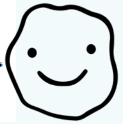
Inference rule

First-order propositional logic captures the essence of hypothetical reasoning by way of the hypothetical judgement $\Gamma \vdash A$

which species how to derive that a proposition A is true if we assume, without proof, the truth of **a finite set of propositions** Γ

We call the propositions in Γ the "hypotheses" or "assumptions".

$$\frac{\Gamma\,\Gamma_1 \vdash J_1 \quad \dots \quad \Gamma\,\Gamma_n \vdash J_n}{\Gamma \vdash J}.$$



Classical Propositional Logic

Assumption

$$\frac{A \in \Gamma}{\Gamma \vdash A}\ (\text{assumption})$$

Conjunctions

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge\text{-I})$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge\text{-E-L}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge\text{-E-R})$$

Absurdities

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\bot\text{-E})$$

$$\neg A \overset{abbr}{=} A \supset \bot$$

Disjunctions

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\text{-I-L}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee\text{-I-R})$$

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee\text{-E})$$

Implication

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ (\supset\text{-I})$$

$$\frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ (\supset\text{-E})$$

Axiom of the Excluded Middle

$$\frac{}{\Gamma \vdash A \vee \neg A}\ (\text{AEM}$$

only in classical logic,
not in constructive logic

Natural Deduction Rules

What is the small "a"?
Tags for assumptions!

Assumption

$$\frac{A \in \Gamma}{\Gamma \vdash A}\ \text{(assumption)}$$

Conjunctions

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{I}$$

$$\frac{A \wedge B}{A}\wedge\mathrm{E}_1 \qquad \frac{A \wedge B}{B}\wedge\mathrm{E}_2$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge\text{-I}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge\text{-E-L}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge\text{-E-R})$$

Absurdities

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\bot\text{-E}) \qquad \frac{\bot}{A}\bot\mathrm{E}$$

$$\neg A \overset{abbr}{=} A \supset \bot$$

Disjunctions

$$\frac{A}{A \vee B}\vee\mathrm{I}_1 \qquad \frac{B}{A \vee B}\vee\mathrm{I}_2$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\text{-I-L}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee\text{-I-R})$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}$$

$$\frac{A \vee B \quad \begin{matrix}[A]^a \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]^a \\ \vdots \\ C\end{matrix}}{C}\vee\mathrm{E},a$$

$$\frac{\begin{matrix}[A]^a \\ \vdots \\ \bot\end{matrix}}{\neg A}\neg\mathrm{I},a$$

$$\frac{\neg A \quad A}{\bot}\neg\mathrm{E}$$

Implication

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ (\supset\text{-I}) \qquad \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}\ (\supset\text{-E})$$

$$\frac{\begin{matrix}[A]^a \\ \vdots \\ B\end{matrix}}{A \to B}\to\mathrm{I},a \qquad \frac{A \to B \quad A}{B}\to\mathrm{E}$$

Axiom of the Excluded Middle

$$\frac{}{\Gamma \vdash A \vee \neg A}\ \text{(AEM}$$

only in classical logic,
not in constructive logic

$$\frac{\begin{matrix}[\neg A]^a \\ \vdots \\ \bot\end{matrix}}{A}\mathrm{DN},a$$

Interesting fact: these two can not derive each other

Example

- Prove: $((A \wedge B) \to C) \to (A \to B \to C)$ and $(A \to B \to C) \to ((A \wedge B) \to C)$

Example

- Prove: $\neg p \vee p$

This is the axiom in classical logic

Axiom of the Excluded Middle

$$\frac{}{\Gamma \vdash A \vee \neg A}\ (\mathrm{AEM}$$

only in classical logic,
not in constructive logic

While here we use

$$\frac{\begin{array}{c}[\neg A]^a \\ \vdots \\ \bot\end{array}}{A}\ \mathrm{DN},a$$

Namely, $\neg\neg A$ and A are equivalent

you can't prove this without this axiom

Example

- Prove: $\vdash (A \to B) \to (\neg A \vee B)$

Hint: use $\neg p \vee p$

Logic and type

expression

$\Gamma \vdash e : \tau$ ⟸ type

Γ is a mapping from variables to types. We will write it as a sequence of typing assumptions, written

$x_1 : \tau_1, \ldots, x_n : \tau_n$

		Structural Form	Concrete Form	
Typ τ	::=	Num	Num	
	|	Bool	Bool	
	|	Arrow$(\tau_{\text{in}}, \tau_{\text{out}})$	$\tau_{\text{in}} \to \tau_{\text{out}}$	(right assoc., precedence 1)

numbers

$$\frac{}{\Gamma \vdash \underline{n} : \mathsf{Num}}\ (\text{T-NumLiteral})$$

$$\frac{\Gamma \vdash e : \mathsf{Num}}{\Gamma \vdash -e : \mathsf{Num}}\ (\text{T-Neg})$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Num} \qquad \Gamma \vdash e_2 : \mathsf{Num}}{\Gamma \vdash e_1 + e_2 : \mathsf{Num}}\ (\text{T-Plus})$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Num} \qquad \Gamma \vdash e_2 : \mathsf{Num}}{\Gamma \vdash e_1 - e_2 : \mathsf{Num}}\ (\text{T-Minus})$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Num} \qquad \Gamma \vdash e_2 : \mathsf{Num}}{\Gamma \vdash e_1 * e_2 : \mathsf{Num}}\ (\text{T-Times})$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Num} \qquad \Gamma \vdash e_2 : \mathsf{Num}}{\Gamma \vdash e_1 > e_2 : \mathsf{Bool}}\ (\text{T-Gt})$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Num} \qquad \Gamma \vdash e_2 : \mathsf{Num}}{\Gamma \vdash e_1 < e_2 : \mathsf{Bool}}\ (\text{T-Lt})$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Num} \qquad \Gamma \vdash e_2 : \mathsf{Num}}{\Gamma \vdash e_1 =? e_2 : \mathsf{Bool}}\ (\text{T-Eq})$$

booleans

$$\frac{}{\Gamma \vdash \mathsf{True} : \mathsf{Bool}}\ (\text{T-True})$$

$$\frac{}{\Gamma \vdash \mathsf{False} : \mathsf{Bool}}\ (\text{T-False})$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 : \tau}\ (\text{T-If})$$

variables + functions

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}\ (\text{T-Var})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \mathsf{let}\ x\ :\ \tau_1\ \mathsf{be}\ e_1\ \mathsf{in}\ e_2 : \tau_2}\ (\text{T-LetAnn})$$

$$\frac{\Gamma, x : \tau_{\text{in}} \vdash e : \tau_{\text{out}}}{\Gamma \vdash \mathsf{fun}\ (x : \tau_{\text{in}}) \to e : \tau_{\text{in}} \to \tau_{\text{out}}}\ (\text{T-Fun})$$

$$\frac{\Gamma \vdash e_1 : \tau_{\text{in}} \to \tau_{\text{out}} \qquad \Gamma \vdash e_2 : \tau_{\text{in}}}{\Gamma \vdash e_1\ e_2 : \tau_{\text{out}}}\ (\text{T-Ap})$$

Logic and type

1. Product types $A \times B$ correspond to conjunction $A \wedge B$
2. Sum types $A + B$ correspond to disjunction $A \vee B$
3. Arrow types $A \to B$ correspond to implication $A \supset B$
4. The unit type 1 corresponds to the tautological proposition, $\top$

- E.g. $\vdash (\text{fun } x \to x.0) : (A \wedge B) \supset A$

$\vdash (\text{fun } f \to \text{fun } g \to \text{fun } a \to g(f(a))) : (A \supset B) \supset (B \supset C) \supset (A \supset C)$

- Remember we have just proved that $((A \wedge B) \to C) \leftrightarrow (A \to B \to C)$
- function $f(x, y) = x + y$ has type Num -> Num-> Num

fun x -> fun y -> x+y & fun (x, y) -> x+y

Russel paradox

Axiom of Extensionality: For two sets A, B,

$$A = B \iff \forall x (x \in A \iff x \in B)$$

Axiom of comprehension: any assertion $\phi(x)$ depending on a variable x, exist unique set A that

$$\forall x (x \in A \iff \phi(x))$$

the set A is denoted

$$A := \{x | \phi(x)\}$$

Russel paradox: Naive Set Theory is inconsistent (self-contradictory).

$$A := \{x | x \notin x\}$$

Russel paradox

Russel paradox: Naive Set Theory is inconsistent (self-contradictory).
Another proof:

Cantor's Theorem

Let X be a set, $f : X \to P(X)$ be a function. Then f is not surjective,
i.e., there is an $A \in P(X)$ such that for all $x \in X, f(x) \neq A$

idea: We want to find a subset A ⊆ X which does not equal any f (x),
Diagonalization

Proof of Cantor's Theorem:

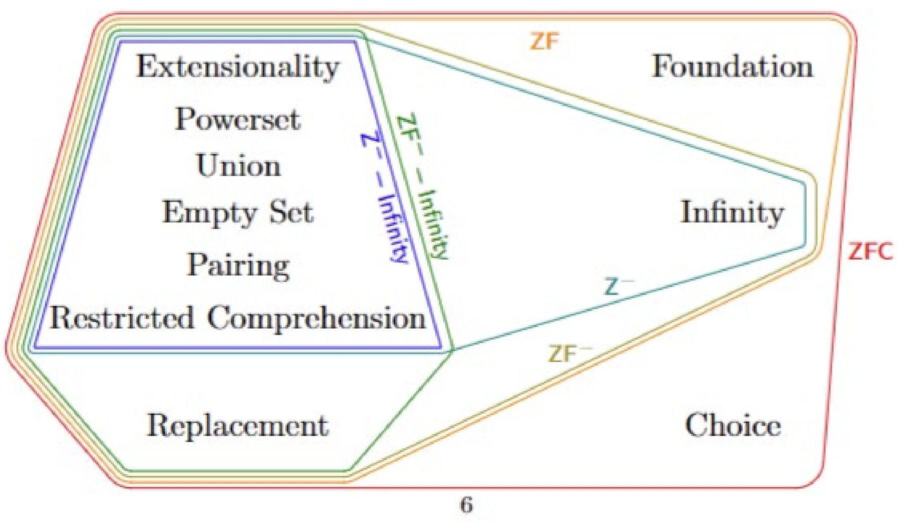
Proof of Russel paradox : let $V = \{x \mid true\}$ be the set of all sets. Note that $V = P(V)$ (since all objects
are sets). Thus $id : V \to V = P(V)$ is a surjection, contradicting Cantor's theorem

Russel paradox

Russel paradox: Naive Set Theory is inconsistent (self-contradictory).

Solution:
The most common way: restrict the Axiom of Comprehension so that only “sufficiently small” classes form sets. => The **theory ZF−-infinity**
A **class** is an informal collection {x | φ(x) } defined by a property φ(x)

Axiom: (Powerset). For any set X, the class P(X) = {A | A ⊆ X} is a set.
Axiom: (Union). For any set A, $\bigcup A = \{x \mid \exists A \in A\ (x \in A)\}$ is a set.
Axiom: (Finite Sets). For any $x_1, \ldots, x_n$, $\{x_1, \ldots, x_n\}$ is a set.
Axiom: (Empty Set). ∅ = {x | false} is a set.
Axiom: (Pairing). For any x, y, {x, y} = {z | x = z or y = z} is a set.
Axiom: (Restricted Comprehension/Separation). Any class contained in a set is a set



Gödel’s incompleteness theorem. find a complete and consistent set of axioms for all mathematics is impossible

End

~~QAQ~~Q&A

Reference

- Umich MATH 582 notes
- Umich EECS490 HW6
- *Practical Foundation for Programming Language*, Robert Harper