Graph

Yue

Outline

- Chain
- Graph basic
- Connectivity

Chain & Antichain

Chain: a subset of comparable elements (a complete graph)

Antichain: a subset of incomparable elements

- **Maximal**: can’t be extended
- **Maximum**: max length

Height: maximum size of chain

Width: maximum size of antichain

Dilworth's Theorem

k: least integer that P is a union of k chains

m: size of largest antichain of P

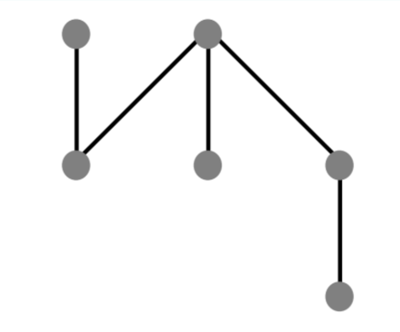
Dilworth Theorem: k=m

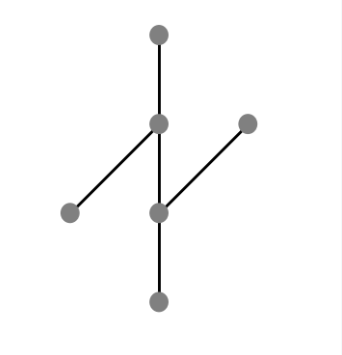
"dual":

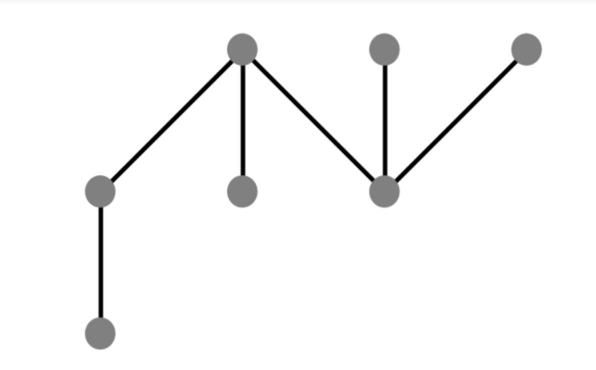
k: least integer that P is a union of k antichains

m: size of largest chain

Mirsky's Theorem: k=m







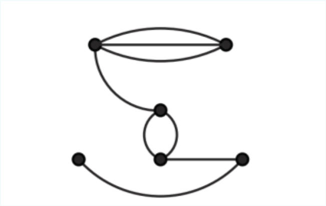
Example:

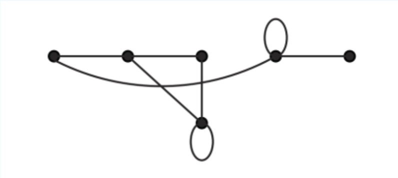
width of the graph on the right?

Given a finite poset, would removing a maximal chain decreases the width of the poset?

Basic Graph Definitions

- Loop, parallel, simple graph





- Isomorphism G ≅ H
  - Bijection from V(G) -> V(H) that keep the edges
  - Equivalence relation
- Complement: $uv \in E(\overline{G})$ iff $uv \notin E(G)$.
- Bipartite: union of two disjoint (possibly empty) independent sets
- Complete graph(K_n)/Clique: pairwise adjacent, simple graph
- Null graph: vertex set & edge set empty (not the complement of complete graph)
- Path(P_n): no repeat vertices
- Cycle graph(C_n): Path + $e_n = v_n v_1$

Double Counting

- Relation between Degree & Edge
- Handshaking lemma

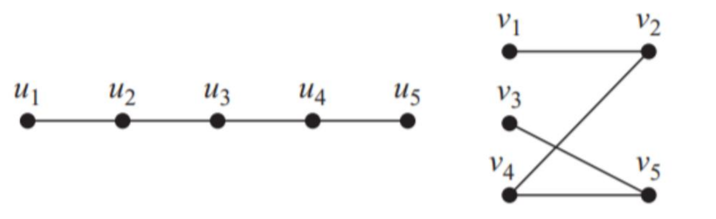
For all finite graph $G = (V, E)$,

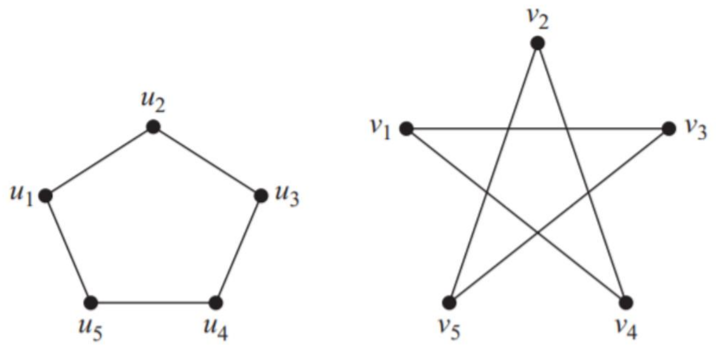
$$\sum_{v \in V} \deg(v) = 2|E|$$

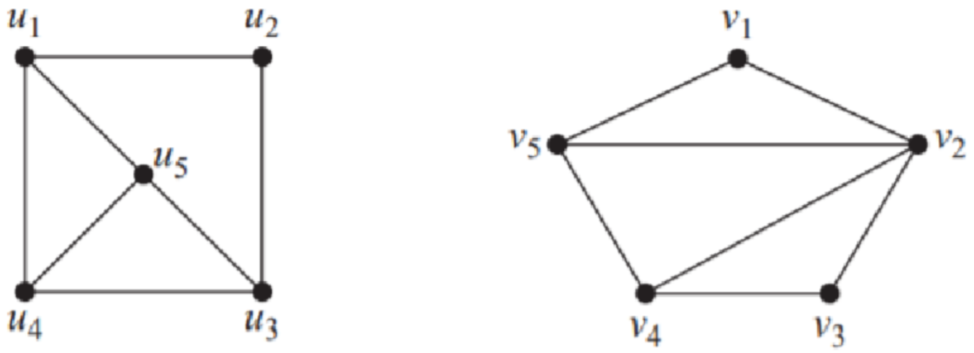
- Exercise:
  - In any graph with at least two nodes, there are at least two nodes of the same degree
  - Is it true that a finite graph having exactly two vertices of odd degree must contain a path from one to the other? Give a proof or a counterexample.
- Another example(extra content):

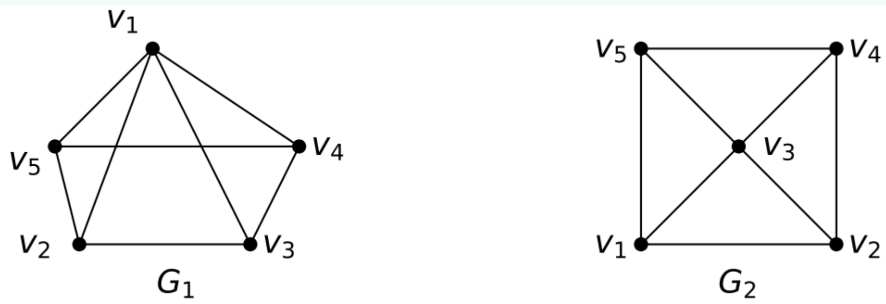
Mantel's Theorem: For a simple graph G containing **no triangles**, |V(G)|=n, what is the maximum number of edges G can have?

- Determine whether the given pairs of graphs are isomorphic









Exercise

3. The complement of a simple graph $G = (V, E)$ is given by $G^c = (V, E^c)$, where $E^c = V \times V \setminus E$, i.e., the complement has the same vertex set and an edge is in E^c if and only if it is not in E. A graph G is said to be *self-complementary* if G is isomorphic to G^c .

i) Show that a self-complementary graph must have either $4m$ or $4m + 1$ vertices, $m \in \mathbb{N}$.

ii) Find all self-complementary graphs with 8 or fewer vertices.

(Taken from Ve203 FA2020 Assignment10)

Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).

- Theorem: Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others)
- Theorem: Consider a 6-clique where every edge is colored red or blue. The graph contains a red triangle or a blue triangle
- Theorem (Ramsey's Theorem): For any natural number n, there is a smallest natural number R(n) such that if the edges of an R(n)-clique are colored red or blue, the resulting graph will contain either a red n-clique or a blue n-clique.
    - Out proof: n=3, R(n)<=6

Definition

Walk: a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, k-1$.

- Distinct Vertices => path
- $v_0 = v_n$ => closed

Length: number of edges

Connected: A graph G is connected if for all u, v ∈ V(G), there is a walk from u to v

(intuitively, one can pick up an entire graph by grabbing just one vertex)

G is **disconnected** iff there is a partition {X,Y } of V(G) such that no edge has an end in X and an end in Y

Each **maximal connected** piece of a graph is called a connected **component**

Theorem: If there is a walk from u to v, then there is a path from u to v.

Bridge

If the deletion of a edge/vertex v from G causes the number of components to increase, then v is called a cut edge/vertex

- *either* e *is a cut-edge and* $\mathrm{comp}(G - e) = \mathrm{comp}(G) + 1$;
- *or* e *is NOT a cut-edge and* $\mathrm{comp}(G - e) = \mathrm{comp}(G)$.

Exercise:

Prove that an edge e is a bridge of G if and only if e lies on no cycle of G

Connectivity κ(G) : delete at least κ(G) vertices can make the graph disconnected

If $\kappa(G) \geq k$, the graph is k-connected

- Every 2-connected graph contains at least one cycle.

$\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$.

Let G be a graph where $\delta(G) \geq k$.

- Prove that G has a path of length at least k.
- If $k \geq 2$, prove that G has a cycle of length at least $k + 1$

- Let G be a graph with v vertices that is not connected. What is the maximum number of edges of G?
- Let G be a graph of order n and size strictly less than $n - 1$. Prove that G is not connected.

End

~~QAQ~~Q&A

Reference

- *Combinatorics and Graph Theory* by Harris, Hirst and Mossinghoff
- Exercise from VE203 2022 spring TA Hamster
- Graph from VE203 2022 spring TA Yucheng Huang