Reynolds Averaging for compressible N/S Equations:

To derive the compressible RANS equations. $\overline{(\ )}$ - denote avg

we will use a process called Favre Averaging

to see why we'll first examine the cty equation.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

Let's apply standard Reynolds Averaging to ρ, u_j respectively

$$\rho = \bar{\rho} + \rho'$$
$$u_j = \bar{u}_j + u_j'$$

Inserting into the cty equation gives

$$\frac{\partial(\bar{\rho} + \rho')}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_j + \bar{\rho}u_j' + \rho'\bar{u}_j + \rho' u_j'\right) = 0 \qquad (1)$$

Take Re avg. as $\frac{1}{T}\int_{t}^{t+T} a(\vec{x},\tau)\,d\tau = \bar{a} \equiv \langle a \rangle$

Recall that (from turbulence course and) by definition

$\langle \bar{a} \rangle = \bar{a}$ $\quad \langle a' \rangle = 0$ $\quad$ b.c. $a = \bar{a} + a'$ and

if $\langle \bar{a} \rangle = \bar{a}$ then $\langle a' \rangle = 0$ thus we now avg

(1)

gives

$$\frac{\partial}{\partial t}\bar{\rho} + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_j + \bar{\rho}\cancelto{0}{\langle u_j' \rangle} + \langle \rho' \rangle \bar{u}_j + \langle \rho' u_j' \rangle\right) = 0$$

gives

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_j + \overline{\rho' u_j'}\right) = 0$$

Thus we see that we have an extra term requiring closure i.e. $\overline{\rho' u_j'}$. Rather than try to close this problem we will introduce a new average namely a mass averaging a.k.a. favre average defined as

$$\tilde{u}_j = \frac{1}{\bar{\rho}} \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \rho(\vec{x},\tau)\, u_j(\vec{x},\tau)\, d\tau \Rightarrow$$

$$\bar{\rho}\,\hat{u}_j = \overline{\rho u_j}$$

Remark: This is simply a way of defining an avg velocity from a Reynolds average mtm i.e. a way of extracting primitive avg variable from Re averaged conservative ones. But it is a mass average because $\rho(\vec{x},\tau)$ is in all integrals.

Before favre averaging the NS' equations Let's first examine the relationship between mass and Re avg variables.

Let's introduce a new favre decomposition name

$$u_j = \tilde{u}_j + u_j''$$ where $\tilde{u}_j$ - is mass averaged u_j and u_j'' is the fluction about $\tilde{u}_j$. Note u_j' is fluctuation about $\bar{u}_j$ - which is Re avg.

Now to form favre average multiply by ρ and time avers

$$\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \rho u_j \, d\tau = \lim_{T\to\infty} \frac{1}{T} \int \bar{\rho}\,\tilde{u}_j \, d\tau + \lim_{T\to\infty} \frac{1}{T} \int \rho u_j'' \, d\tau$$

$$\overline{\rho u_j} = \bar\rho \tilde u_j + \bar\rho \overline{u_j''} = \bar\rho \tilde u_j + \overline{\rho u_j''} \quad \text{by definition}$$

$$\bar\rho \tilde u_j = \overline{\rho u_j} \Rightarrow \text{this implies } \overline{\rho u_j''} = 0$$

If we were to simply Reynolds averge we would get

$$u_j'' = u_j - \tilde u_j$$

$$\text{if } \bar\rho \tilde u_j = \overline{\rho u_j} = \bar\rho \bar u_j + \overline{\rho' u_j'}$$

$$\Rightarrow \tilde u_j = \frac{\cancel{\bar\rho}}{\cancel{\bar\rho}} \bar u_j + \frac{\overline{\rho' u_j'}}{\bar\rho}$$

$$u_j'' = u_j - \bar u_j - \frac{\overline{\rho' u_j'}}{\bar\rho}$$

$$u_j'' = u_j' - \frac{\overline{\rho_j' u_j'}}{\bar\rho}$$

$$\langle u_j'' \rangle = 0 - \frac{\overline{\rho' u_j'}}{\bar\rho} \neq 0 \text{ i.e.}$$ Strict Reynolds average of u_j'' is not $\neq 0$

Also if we take expression for u_j'' and Favre avs. as follows

$$\lim_{T\to\infty} \frac{1}{T} \frac{1}{\bar\rho} \int_{t}^{t+T} \rho u_j'' = \overline{\rho u_j''} = \overline{\rho u_j'} + \overline{\rho' \cancel{u_j'}} - \cancel{\bar\rho} \frac{\overline{u_j' \cancel{\rho'}}}{\cancel{\bar\rho}} - \cancel{\rho'}^{0} \frac{\overline{u_j' \rho'}}{\bar\rho}$$

$$= \overline{\rho u_j''} = \bar\rho \cancel{\overline{u_j'}}^{0} + \overline{\rho' u_j'} - \overline{\bar\rho u_j'} - \cancel{\overline{\rho'}}^{0} \frac{\overline{\rho' u_j'}}{\bar\rho}$$

$= 0$ So no matter how you attack the u_j'' with favre avs it's $= 0$

Mathematically u_j'' has included the $\overline{\rho' u_j'}$ correlation and Favre averaging is mathematical simplification that removes this correlation ~~the~~ from mass averaged variable $\tilde{u}_j$ or any other primitive variable. This is not to say that density fluctuations have been removed from turbulence rather the correlation $\overline{\rho' u_j'}$ has been averaged out of average equations

Farvre Averaging of RANS Equations:

Let's introduce a set of variables.

$\bar{(\ )}$ - denotes standard reynolds avg.

$\tilde{(\ )}$ - denotes mass averaging.

$$\rho = \bar{\rho} + \rho'$$
$$u_j = \tilde{u}_j + u_j''$$
$$p = \bar{p} + p'$$
$$h = \tilde{h} + h''$$
$$e = \tilde{e} + e''$$
$$T = \tilde{T} + T''$$
$$q_j = \bar{q}_j + q_j'$$

The Instantaneous Equations are given as (slightly different from my previous writing but these form is better for the farvre avg.)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\tau_{ij})$$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \tfrac{1}{2} u_i u_i\right)\right] + \frac{\partial}{\partial x_j}\left[\rho u_j\left(h + \tfrac{1}{2} u_i u_i\right)\right] = \frac{\partial}{\partial x_j}(u_i \tau_{ij}) - \frac{\partial q_j}{\partial x_j}$$

$$\tau_{ij} = 2\mu S_{ij} - \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k} = 2\mu\left(S_{ij} - \frac{1}{2}\frac{\partial u_k}{\partial x_k}\right)$$

Cty: Equation

$$\frac{\partial}{\partial t}(\bar{\rho} + \rho') + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j + \bar{\rho}u_j'' + \rho'\tilde{u}_j + \rho' u_j'') = 0$$

time averaging gives

$$\frac{\partial}{\partial t}\bar{\rho} + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j + \overline{\bar{\rho}u_j''} + \cancelto{0}{\overline{\rho'}\tilde{u}_j} + \overline{\rho' u_j''}) = 0$$

Recognise $\overline{\bar{\rho}u_j''} + \overline{\rho' u_j''} = \overline{(\bar{\rho}+\rho')u_j''} = \overline{\rho u_j''} = 0$

Results in

$$\frac{\partial(\bar{\rho})}{\partial t} + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j) = 0 \qquad \text{RANS cty.}$$

Mtm. equation:

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_j + \bar{\rho}u_j'' + \rho'\tilde{u}_j + \rho' u_j'') + \frac{\partial}{\partial x_j}\left[(\bar{\rho}+\rho')(\tilde{u}_i\tilde{u}_j + \tilde{u}_i u_j'' + u_i''\tilde{u}_j + u_i'' u_j'')\right]$$

$$= -\frac{\partial(\bar{P}+P')}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\bar{\tau}_{ji} + \tau_{ji}'\right]$$

$\overline{\rho u_i''} = 0$ F-A RE stress

Averaging gives

$$\underbrace{\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_j)}_{\text{same as in Cty}} + \frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_i\tilde{u}_j + \cancelto{0}{\tilde{u}_i\overline{\rho u_j''}} + \cancelto{0}{\tilde{u}_j\overline{\rho u_i''}} + \overline{\rho u_i'' u_j''}\right]$$

$$= -\frac{\partial}{\partial x_i}\bar{P} + \frac{\partial}{\partial x_j}\left[\bar{\tau}_{ji}\right]$$

RANS Mtm:

$$\frac{\partial}{\partial t}(\bar\rho\tilde u_i) + \frac{\partial}{\partial x_j}\left[\bar\rho\tilde u_i\tilde u_j\right] + \frac{\partial(\bar P)}{\partial x_i} - \frac{\partial}{\partial x_j}\left[\bar\tau_{ji} - \overline{\rho u_j'' u_i''}\right] = 0$$

Favre-Avg Re stress

Energy:

$$\frac{\partial}{\partial t}\left[(\bar\rho+\rho')(\tilde e + e'' + \tfrac{1}{2}(\tilde u_i + u_i'')^2\right] + \frac{\partial}{\partial x_j}\left[(\bar\rho+\rho')(\tilde u_j + u_j'')(\hat h + h'' + \tfrac{1}{2}(\tilde u_i + u_i'')^2\right]$$

$$= \frac{\partial}{\partial x_j}\left[(\tilde u_i + u_i'')(\bar\tau_{ji} + \tau_{ji}')\right] - \frac{\partial \bar q_{Lj}}{\partial x_j} - \frac{\partial q_j}{\partial x_j}$$

1, 2, 3, 4

We will proceed term by term as numbered above.

Averaging gives

1: $\frac{\partial}{\partial t}\left[\bar\rho\tilde e + \cancel{\overline{\rho'}\tilde e}^{\,0} + \cancel{\overline{\rho e''}}^{\,0} + \frac{1}{2}\bar\rho\tilde u_i\tilde u_i + \cancel{\tfrac{1}{2}\overline{\rho'}\tilde u_i\tilde u_i}^{\,0} + \frac{1}{2}\overline{\rho u_i'' u_i''} + \cancel{\overline{\rho u_i''}}^{\,0}\tilde u_i\right]$

$\Rightarrow \frac{\partial}{\partial t}\left[\bar\rho\tilde e + \frac{1}{2}\overline{\rho u_i'' u_i''} + \bar\rho\frac{\tilde u_i\tilde u_i}{2}\right]$

2: $\frac{\partial}{\partial x_j}\left[(\bar\rho+\rho')\{\tilde u_j\tilde h + \tilde u_j h'' + \frac{1}{2}\tilde u_j(\tilde u_i\tilde u_i + 2\tilde u_i u_i' + u_i'' u_i'') + u_j''\tilde h + u_j'' h'' \right.$

$\left. + u_j''\tfrac{1}{2}(\tilde u\tilde u_i + 2\tilde u u_i'' + u_i'' u_i'')\}\right]$

$\frac{\partial}{\partial x_j}\left[\bar\rho\tilde u_j\tilde h + \cancel{\overline{\rho'}\tilde u_j\tilde h} + \tilde u_j\cancel{\overline{\rho h''}} + \frac{1}{2}\bar\rho\tilde u_j(\tilde u\tilde u_i) + \cancel{\tfrac{1}{2}\overline{\rho'}\tilde u_j(\tilde u_i\tilde u_i)}\right.$

$+ \cancel{\tilde u_j\tilde u_i\overline{\rho u_i''}}^{\,0} + \frac{1}{2}\tilde u_j\overline{\rho u_i'' u_i''} + \cancel{\overline{\rho u_j''}}^{\,0}\tilde h + \overline{\rho u_j'' h''} +$

$\left.\cancel{\tfrac{1}{2}\overline{\rho u''}\tilde u_i}^{\,0} + \overline{\rho u_j'' u_i''}\tilde u_i + \frac{1}{2}\overline{\rho u_i'' u_i'' u_j''}\right]$

$= \frac{\partial}{\partial x_j}\left[\bar\rho\tilde u_j(\tilde h + \frac{1}{2}\tilde u_i\tilde u_i) + \frac{1}{2}\tilde u_j\overline{\rho u_i'' u_i''} + \tilde u_i\overline{\rho u_j'' u_i''}\right.$

$\left. + \frac{1}{2}\overline{\rho u_i'' u_i'' u_j''} + \overline{\rho u_j'' h''}\right]$

3: $\frac{\partial}{\partial x_j}\left[\tilde u_i\bar\tau_{ji} + u_i''\bar\tau_{ji} + \tilde u_i\tau_{ji}' + u_i''\tau_{ji}'\right] =$

$\frac{\partial}{\partial x_j}\left[\tilde u_i\bar\tau_{ij} + \overline{u_i''\tau_{ji}}\right]$

4: $\frac{\partial \bar q_{Lj}}{\partial x_j}$

Piecing together (1) → (4) gives

RANS Energy.

$$\frac{\partial}{\partial t}\left[\bar{\rho}\left(\tilde{e}+\tfrac{1}{2}\bar{\rho}\tilde{u}_i\tilde{u}_i\right)+\tfrac{1}{2}\overline{\rho u_i''u_i''}\right]+\frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_j\left(\tilde{h}+\tfrac{1}{2}\tilde{u}_i\tilde{u}_i\right)\right]$$

$$+\frac{\partial}{\partial x_j}\left[q_{jL}+\overline{\rho u_j''h''}+\tfrac{1}{2}\tilde{u}_j\overline{\rho u_i''u_i''}+\tilde{u}_i\overline{\rho u_j''u_i''}+\tfrac{1}{2}\overline{\rho u_i''u_j''u_i''}\right]$$

$$-\frac{\partial}{\partial x_j}\left[\tilde{u}_i\bar{\tau}_{ji}+\overline{u_i''\tau_{ji}}\right]=0$$

Turbulent terms

$\tfrac{1}{2}\overline{\rho u_i''u_i''}=\bar{\rho}K$ or turbulent kinetic energy. (T.K.E.)

$\overline{\rho u_j''h''}=q_{T_j}$ is turbulent heat transport

$\overline{\tau_{ji}u_i''}$ – molecular diffusion of T.K.E.

$\tfrac{1}{2}\tilde{u}_j\overline{\rho u_i''u_i''}$ – conservation of T.K.E of dilatation

$\tilde{u}_i\overline{\rho u_j''u_i''}$ – Re stress ~~????~~ work.

$\tfrac{1}{2}\overline{\rho u_i''u_j''u_i''}$ – turbulent transport of K.E.

We shall re-write the above so that all work terms are lump and all heat and energy transfer are lumped. gives

$$\frac{\partial}{\partial t}\left[\bar{\rho}\left(\tilde{e}+\tfrac{1}{2}\tilde{u}_i\tilde{u}_i\right)+\tfrac{1}{2}\bar{\rho}K\right]+\frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_j\left(\tilde{h}+\tfrac{1}{2}\tilde{u}\tilde{u}_i\right)+q_{jL}+q_{jT}+\tilde{u}_j\bar{\rho}K\right.$$

$$\left.+\tfrac{1}{2}\overline{\rho u_i''u_j''u_i''}-\overline{u_i''\tau_{ji}}\right]-\frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(\bar{\tau}_{ji}+\overline{\rho u_j''u_i''}\right)\right]$$

One final rewrite to cast it in terms of $\widetilde{\rho E}$ gives

$$\widetilde{\rho E} = \bar{\rho}(\tilde{e} + \tfrac{1}{2}\tilde{U}_i \tilde{U}_i)$$

~ANS Energy.

$$\frac{\partial}{\partial t}\left[\widetilde{\rho E} + \bar{\rho} K\right] + \frac{\partial}{\partial x_j}\left[\tilde{U}_j(\widetilde{\rho E} + \bar{P}) + q_{jL} + q_{jT} + \tilde{U}_j\, \bar{\rho} K + \tfrac{1}{2}\overline{\rho U_i'' U_j'' U_i''} - \overline{U_i'' \tau_{ji}}\right]$$

$$- \frac{\partial}{\partial x_j}\left[\tilde{U}_i\left(\bar{\tau}_{ji} - \overline{\rho U_j'' U_i''}\right)\right] = 0$$

The closure problem:

As expected additional unknowns have been introduced. Turbulence course notes go into more detail and analysis of this. Here we will make use of the Boussinesq approximation namely.

$$-\overline{\rho U_i'' U_j''} \approx 2\mu_T\left(\tilde{S}_{ij} - \frac{1}{3}\frac{\partial \tilde{U}_k}{\partial x_k}\delta_{ij}\right) - \frac{2}{3}\bar{\rho} K \delta_{ij}$$

The last term is there to ensure $\mathrm{trace}(\overline{\rho U_i'' U_j''}) =$
$-2\bar{\rho}K$

The turbulent heat-flux is given as

$$q_{jT} = \overline{\rho U_j'' h''} \approx -\frac{\mu_T C_p}{Pr_T}\frac{\partial \tilde{T}}{\partial x_j} \approx \frac{\mu_T}{Pr_T}\frac{C_p}{C_v}\frac{\partial e}{\partial x} \approx \frac{\mu_T}{Pr_T}\gamma \cdot \frac{\partial e}{\partial x}$$

For most engineering applications up to supersonic range $\bar{\rho}K \ll \bar{P} \Rightarrow K \ll \hat{h}$ ~~But~~ this allows us to ignore

$$\tfrac{1}{2}\overline{\rho U_i'' U_j'' U_i''} - \overline{U_i'' \tau_{ji}}$$

In the case of Hypersonic flows this may not be a good approximation. Thus we use the following closure

$$\overline{t_{ji}u_i''} - \overline{\rho u_j'' \tfrac{1}{2} u_i'' u_i''} \approx \left(\mu + \frac{\mu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_j}$$

i.e.

$$\overline{\rho u_j'' \tfrac{1}{2} u_i'' u_i''} - \overline{t_{ji} u_i'} \approx -\left(\mu + \frac{\mu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_j}$$

Inserting these closures into our RANS equations gives

$$\frac{\partial}{\partial t}[\bar\rho] + \frac{\partial}{\partial x_j}[\bar\rho \tilde u_j] = 0$$

$$\frac{\partial}{\partial t}(\bar\rho u_i) + \frac{\partial}{\partial x_j}[\bar\rho \tilde u_i \tilde u_j] + \frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_j}\left[2(\mu+\mu_T)\left(\tilde S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - \frac{2}{3}\bar\rho K \delta_{ij}\right]$$

$$\frac{\partial}{\partial t}(\bar\rho \tilde E_t + \bar\rho K) + \frac{\partial}{\partial x_j}\left[\tilde u_j(\bar\rho \tilde E + \bar P) + \left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\gamma\frac{\partial e}{\partial x_j} + \tilde u_j \bar\rho K - \left(\mu + \frac{\mu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_j}\right]$$

$$- \frac{\partial}{\partial x_j}\left[\tilde u_i\left[2(\mu+\mu_T)\left(\tilde S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - \frac{2}{3}\bar\rho K\delta_{ij}\right]\right] = 0$$

For a first attempt we'll ignore T.K.E as stated earlier $K \ll \tilde h$ this gives the following system.

$$\frac{\partial}{\partial t}(\bar\rho) + \frac{\partial}{\partial x_j}[\bar\rho \tilde u_j] = 0$$

$$\frac{\partial}{\partial t}(\bar\rho \tilde u_i) + \frac{\partial}{\partial x_j}[\bar\rho \tilde u_i \tilde u_j + \bar P\delta_{ij}] - \frac{\partial}{\partial x_j}\left[2(\mu+\mu_T)\left(\tilde S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right] = 0$$

$$\frac{\partial}{\partial t}(\bar\rho \tilde E_t) + \frac{\partial}{\partial x_j}\left[\tilde u_j(\bar\rho \tilde E + \bar P) - \left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\gamma\frac{\partial e}{\partial x_j}\right] - \frac{\partial}{\partial x_j}\left[\tilde u_i\left(2(\mu+\mu_T)\left(\tilde S_{ij} - \frac{1}{3}\frac{\partial \tilde u_k}{\partial x_k}\delta_{ij}\right)\right)\right]$$

$$= 0$$

This last set of equations is what we will first attempt to solve.

With the final assumption all we really need to do is replace μ with $(\mu+\mu_T)$ and linearize μ_T w.r.t $\vec{q}$. We now have $\vec{q} = \begin{Bmatrix} \bar{\rho} \\ \bar{\rho}\tilde{u} \\ \bar{\rho}\tilde{v} \\ \bar{\rho}\tilde{E} \\ \tilde{\nu} \end{Bmatrix}$

where $\tilde{\nu}$ will be given by S. A Model equation

$$\frac{\partial}{\partial t}(\rho\tilde{\nu}) + \frac{\partial}{\partial x_j}(\rho\tilde{\nu}u_j) = C_{b1}\tilde{S}(\rho\tilde{\nu}) + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}\left((\mu+\rho\tilde{\nu})\frac{\partial\tilde{\nu}}{\partial x_j}\right) + C_{b2}\rho\frac{\partial\tilde{\nu}}{\partial x_j}\cdot\frac{\partial\tilde{\nu}}{\partial x_j}\right] - C_{w1}f_w\rho\frac{\tilde{\nu}^2}{d^2}$$

$$\hat{S} = \begin{cases} S+\bar{S} & \bar{S} \geq -C_{v2}S \\ S + \dfrac{S(C_{v2}^2 S + C_{v3}\bar{S})}{(C_{v3}-2C_{v2})S-\bar{S}} & \bar{S} < C_{v2}S \end{cases}$$

$$\bar{S} = \frac{\tilde{\nu} f_{v2}}{\kappa^2 d^2} \qquad f_{v2} = \frac{1-\chi}{1+\chi f_{v1}}$$

SA Non-dimensionalization:

$$\frac{\partial}{\partial t}(\rho\tilde{\nu}) + \frac{\partial}{\partial x_j}(\rho\tilde{\nu}u_j) = C_{b1}\tilde{S}(\rho\tilde{\nu}) + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}\left((\mu+\rho\tilde{\nu})\frac{\partial\tilde{\nu}}{\partial x_j}\right) + C_{b2}\rho\frac{\partial\tilde{\nu}_i}{\partial x_j}\cdot\frac{\partial\tilde{\nu}_i}{\partial x_j}\right]$$

$$- C_{w1}f_w\rho\left[\frac{\tilde{\nu}}{d}\right]^2$$

$\tilde{\nu}$ is a turbulent viscosity transport qty so it has viscosity units.

$$\bar{\tilde{\nu}} = \frac{\tilde{\nu}}{L_{ref}\cdot a_\infty}$$ – giving an inertia scaling to turbulent viscosity.

Scaling the equation.

$$\frac{a_\infty\partial}{\cancel{L_{ref}}\partial\tau}(\bar{\rho}\rho_\infty \cancel{L_{ref}}\cdot a_\infty\cdot\bar{\tilde{\nu}}) + \frac{1}{\cancel{L_{ref}}}\frac{\partial}{\partial\bar{x}_j}(\bar{\rho}\rho_\infty\bar{\tilde{\nu}}\cancel{L_{ref}}a_\infty a_\infty\bar{u}_j) = C_{b1}\frac{a_\infty}{L_{ref}}\bar{\tilde{S}}(\bar{\rho}\rho_\infty L_{ref}a_\infty\bar{\tilde{\nu}}) + \frac{1}{\sigma}\left[\frac{1}{L_{ref}}\frac{\partial}{\partial\bar{x}_j}\left((\bar{\mu}\mu_\infty + \rho_\infty\bar{\rho}L_{ref}a_\infty\bar{\tilde{\nu}})\frac{1}{\cancel{L_{ref}}}\frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\cancel{L_{ref}}\cdot a_\infty\right) + C_{b2}\frac{a_\infty^2 L_{ref}^2}{\cancel{L_{ref}^2}}\rho_\infty\frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\cdot\frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\right] - C_{w1}f_w\left[\frac{\bar{\tilde{\nu}}}{\bar{d}}\right]^2\frac{\cancel{L_{ref}^2}a_\infty^2}{\cancel{L_{ref}^2}}\rho_\infty$$

Simplification gives

$$a_\infty^2\rho_\infty\frac{\partial}{\partial\tau}(\bar{\rho}\bar{\tilde{\nu}}) + \rho_\infty a_\infty^2\frac{\partial}{\partial\bar{x}_j}(\bar{\rho}\bar{u}_j\bar{\tilde{\nu}}) = a_\infty^2\rho_\infty C_{b1}\bar{\tilde{S}}\bar{\rho}\bar{\tilde{\nu}} + \frac{1}{\sigma}\left[\frac{\partial}{\partial\bar{x}_j}\left(\left(\bar{\mu}\frac{\mu_\infty a_\infty}{L_{ref}} + \rho_\infty a_\infty^2\bar{\rho}\bar{\tilde{\nu}}\right)\frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\right) + \frac{\cancel{L_{ref}^2}a_\infty^2}{\cancel{L_{ref}^2}}\rho_\infty C_{b2}\frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\right]$$

$$- C_{w1}f_w\left[\frac{\bar{\tilde{\nu}}}{\bar{d}}\right]^2\rho_\infty a_\infty^2$$

$\div\, a_\infty^2\rho_\infty$ gives

$$\frac{\partial}{\partial\tau}(\bar{\rho}\bar{\tilde{\nu}}) + \frac{\partial}{\partial\bar{x}_j}(\bar{\rho}\bar{u}_j\cdot\bar{\tilde{\nu}}) = C_{b1}\bar{\tilde{S}}\bar{\rho}\bar{\tilde{\nu}} + \frac{1}{\sigma}\left[\frac{\partial}{\partial\bar{x}_j}\left(\left(\bar{\mu}\frac{M_\infty}{\rho_\infty a_\infty L_{ref}} + \bar{\rho}\bar{\tilde{\nu}}\right)\frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\right) \frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\cdot\frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\frac{\cancel{\rho_\infty a_\infty^2}}{\cancel{\rho_\infty a_\infty^2}}\right] - C_{w1}f_w\frac{\cancel{\rho_\infty a_\infty^2}}{\cancel{\rho_\infty a_\infty^2}}\left[\frac{\bar{\tilde{\nu}}}{\bar{d}}\right]^2$$

$$\frac{\mu_\infty M_\infty}{\rho_\infty U_\infty L_{ref}} = \frac{M_\infty}{Re_\infty}$$ yields.

$$\frac{\partial}{\partial\tau}(\bar{\rho}\bar{\tilde{\nu}}) + \frac{\partial}{\partial\bar{x}_j}(\bar{\rho}\bar{u}_j\bar{\tilde{\nu}}) = C_{b1}\bar{\tilde{S}}\bar{\rho}\bar{\tilde{\nu}} + \frac{1}{\sigma}\left[\frac{\partial}{\partial\bar{x}_j}\left(\left(\bar{\mu}\frac{M_\infty}{Re_\infty} + \bar{\rho}\bar{\tilde{\nu}}\right)\frac{\partial\bar{\tilde{\nu}}}{\partial\bar{x}_j}\right) +\right.$$

$$\rho \frac{\partial \bar{\nu}}{\partial \bar{x}_j} \cdot \frac{\partial \bar{\nu}}{\partial \bar{x}_j} \Big] - C_{w_1} f_{w_1} \rho \left[\frac{\bar{\nu}}{d}\right]^2$$

Dropping $(\bar{\ })$ notation for convience we now have

$$\boxed{\frac{\partial}{\partial t}(\rho \tilde{\nu}) + \frac{\partial}{\partial x_j}(\rho U_j \tilde{\nu}) = C_{b_1} \tilde{S} \rho \tilde{\nu} + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}\left(\left(\frac{M_\infty}{Re_\infty}\mu + \rho \tilde{\nu}\right)\frac{\partial \tilde{\nu}}{\partial x_j}\right) + \rho \frac{\partial \tilde{\nu}}{\partial x_j} \cdot \frac{\partial \tilde{\nu}}{\partial x_j}\right] - C_{w_1} f_{w_1} \rho \left[\frac{\tilde{\nu}}{d}\right]^2}$$

Closure term scaling.

Mtm Re stress term this is scaled by $\frac{\rho_\infty a_\infty^2}{L_{ref}}$

$$\frac{\partial}{\partial x_j}\left[2\mu_T\left(S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right]$$

$\downarrow$

$$\frac{L_{ref}}{\cancel{\rho_\infty a_\infty^2}}\left(\frac{1}{\cancel{L_{ref}}} \cdot \frac{\partial}{\partial \bar{x}_j}\left\{2 \cancel{a_\infty} \cancel{\rho_\infty} \cancel{L_{ref}} \left(\frac{\cancel{a_\infty}}{\cancel{L_{ref}}}\right)\right\}\right) \Rightarrow \text{just leave it}$$

N/D term is $\frac{\partial}{\partial \bar{x}_j}\left[2\bar{\mu}_T\left(\bar{S}_{ij} - \frac{1}{3}\frac{\partial \bar{u}_k}{\partial \bar{x}_k}\delta_{ij}\right)\right]$

Energy closures.

N/D factor is $\div \frac{\rho_\infty a_\infty^3}{L_{ref}}$

1) Heat transfer

$$\frac{\partial}{\partial x_j}\left(\frac{\mu_T}{Pr_T}\gamma\frac{\partial e}{\partial x_j}\right) = \frac{1}{L_{ref}} \cdot \frac{\partial}{\partial \bar{x}_j}\left(\frac{\bar{\mu}_T}{Pr_T}\gamma \cancel{\rho_\infty} \cancel{a_\infty} \cancel{L_{ref}} \frac{\cancel{a_\infty^2}}{\cancel{L_{ref}}}\frac{\partial \bar{e}}{\partial \bar{x}_j}\right)\frac{\cancel{L_{ref}}}{\cancel{\rho_\infty a_\infty^3}}$$

$$= \frac{\partial}{\partial \bar{x}_j}\left(\frac{\bar{\mu}_T}{Pr_T}\gamma\frac{\partial \bar{e}}{\partial \bar{x}_j}\right) \text{ - again just leave it.}$$

2). Re. stress work

$$\frac{1}{\cancel{L_{ref}}}\frac{\partial}{\partial \bar{x}_j}\left(U_i \cancel{a_\infty}\, 2\, \cancel{\rho_\infty} \cancel{a_\infty} \cancel{L_{ref}} \bar{\mu}_T\left(\frac{a_\infty}{L_{ref}}\right)\right) \cdot \frac{L_{ref}}{\cancel{\rho_\infty a_\infty^3}} = \frac{\partial}{\partial \bar{x}_j}\left(\bar{U}_i 2\bar{\mu}_T\left(S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right)$$

i.e. just leave it be.

Nu

In short due to inertial scalings of turbulent viscosity
there are No scalings of turbulent closure terms.
just code them as they are written in dimensional form.

SA Non-dimensionalization

$$\frac{\partial}{\partial \tau}(\rho\tilde{\nu}) + \frac{\partial}{\partial x_j}(\rho\tilde{\nu}u_j) = c_{b1}\tilde{S}\rho\tilde{\nu} + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}\left\{(\mu+\rho\tilde{\nu})\frac{\partial\tilde{\nu}}{\partial x_j}\right\}\right.$$
$$\left. + \rho c_{b2}\frac{\partial\tilde{\nu}}{\partial x_j}\frac{\partial\tilde{\nu}}{\partial x_j}\right] - c_{w1}f_w\,\rho\frac{\tilde{\nu}^2}{d^2}$$

Introduce the scalings ∞

$$\bar{u}_j = \frac{u_j}{a_\infty}, \quad \bar{x}_j = \frac{x_j}{L_{ref}}, \quad \bar{\rho} = \frac{\rho}{\rho_\infty}, \quad \overline{\rho\tilde{\nu}} = \frac{\rho\tilde{\nu}}{\mu_\infty}, \quad \bar{\mu} = \frac{\mu}{\mu_\infty}$$
$$\bar{d} = \frac{d}{L_\infty}, \quad \bar{\mu}_T = \frac{\mu_T}{\mu_\infty} \qquad \tau = \frac{a_\infty}{L}t$$

Let us first examine the functions.

$$\mu_T = \rho\tilde{\nu}f_{v1}$$
$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}, \quad \chi = \frac{\tilde{\nu}}{\nu}$$

$$f_{v2} = 1 - \frac{\chi}{1+\chi f_{v1}}$$

$$\tilde{S} = \begin{cases} S + \hat{S} & \hat{S} \geq -c_{v2}S \\ S + \dfrac{S(c_{v2}^2 S + c_{v3}\hat{S})}{(c_{v3} - 2c_{v2})S - \hat{S}} & \hat{S} \leq -c_{v2}S \end{cases}$$

with $\hat{S} = \dfrac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}$

$$r = \frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2}$$

$$g = r + c_{w2}(r^6 - r)$$
$$f_w = g\left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right]^{1/6}$$

1). Scale the convection / Diffusion terms.

$$\frac{\partial}{\partial t}(\rho\tilde{\nu}) + \frac{\partial}{\partial x_j}(\rho\tilde{\nu}u_j) - \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}\left(\{\mu+\rho\tilde{\nu}\}\frac{\partial\tilde{\nu}}{\partial x_j}\right)\right] = (\ )$$

Introduce scaling

$$\frac{\mu_\infty a_\infty}{L_{ref}}\frac{\partial(\bar{\rho}\bar{\tilde{\nu}})}{\partial \tau} + \frac{a_\infty \mu_\infty}{L_{ref}}\frac{\partial}{\partial \bar{x}_j}(\bar{\rho}\bar{\tilde{\nu}}\bar{u}_j) - \frac{1}{\sigma}\left[\frac{\mu_\infty \mu_\infty}{L_{ref}^2 \rho_\infty}\frac{\partial}{\partial \bar{x}_j}\left(\{\bar{\mu}+\bar{\rho}\bar{\tilde{\nu}}\}\frac{\partial\bar{\tilde{\nu}}}{\partial \bar{x}_j}\right)\right] = (\ )$$

$\div$ by $\frac{\mu_\infty a_\infty}{L_{ref}}$ gives

$$\frac{\partial}{\partial \tau}(\bar{\rho}\bar{\tilde{\nu}}) + \frac{\partial}{\partial \bar{x}_j}(\bar{\rho}\bar{\tilde{\nu}}\bar{u}_j) - \frac{1}{\sigma}\frac{\mu_\infty^2}{L_{ref}^2}\frac{L_{ref}}{\rho_\infty \mu_\infty a_\infty}\left[\frac{\partial}{\partial \bar{x}_j}\left(\{\bar{\mu}+\bar{\rho}\bar{\tilde{\nu}}\}\frac{\partial\bar{\tilde{\nu}}}{\partial \bar{x}_j}\right)\right] = \frac{(\ )L_{ref}}{\mu_\infty a_\infty}$$

$$\downarrow$$

$$\frac{\mu_\infty}{\rho_\infty a_\infty L_{ref}} = \frac{M_\infty}{Re_\infty}$$

Gives

$$\frac{\partial}{\partial \tau}(\bar{\rho}\bar{\tilde{\nu}}) + \frac{\partial}{\partial \bar{x}_j}(\bar{\rho}\bar{\tilde{\nu}}\bar{u}_j) - \frac{\partial}{\partial \bar{x}_j}\left(\frac{1}{\sigma}\frac{M_\infty}{Re_\infty}\left[\bar{\mu}+\bar{\rho}\bar{\tilde{\nu}}\right]\frac{\partial\bar{\tilde{\nu}}}{\partial \bar{x}_j}\right) = \frac{(\ )L_{ref}}{\mu_\infty a_\infty}$$

2) Source terms.

$$\chi = \frac{\bar{\tilde{\nu}}\,\cancel{\frac{\mu_\infty}{\rho_\infty}}}{\bar{\nu}\,\cancel{\frac{\mu_\infty}{\rho_\infty}}} = \frac{\bar{\tilde{\nu}}}{\bar{\nu}} \Rightarrow$$ f_{v1} and f_{v2} have no scaling terms.

$$\hat{S} = \frac{\bar{\tilde{\nu}}\frac{\mu_\infty}{\rho_\infty} f_{v2}}{\kappa^2 \bar{d}^2 L_{ref}^2} = \frac{\bar{\tilde{\nu}} f_{v2}}{\kappa^2 \bar{d}^2}\frac{\mu_\infty}{\rho_\infty L_{ref}^2}$$ Let $\bar{\hat{S}} = \frac{\bar{\tilde{\nu}} f_{v2}}{\kappa^2 \bar{d}^2} \Rightarrow \hat{S} = \bar{\hat{S}}\cdot\frac{\mu_\infty}{\rho_\infty L_{ref}^2}$

$$S = \frac{a_\infty}{L_{ref}}\bar{S}$$

$$\tilde{S} = \begin{cases} \frac{a_\infty}{L_{ref}}\bar{S} + \bar{\hat{S}}\frac{\mu_\infty}{\rho_\infty L_{ref}^2} & \hat{S} \geq -C_{v2}S \\[2ex] \bar{S}\frac{a_\infty}{L_{ref}} + \dfrac{\bar{S}\frac{a_\infty}{L_{ref}}\left(C_{v2}^2\bar{S}\frac{a_\infty}{L_{ref}} + C_{v3}\bar{\hat{S}}\frac{\mu_\infty}{\rho_\infty L_{ref}^2}\right)}{(C_{v3}-2C_{v2})\frac{a_\infty}{L_{ref}}\bar{S} - \bar{\hat{S}}\frac{\mu_\infty}{\rho_\infty L_{ref}^2}} & \hat{S} < -C_{v2}S \end{cases}$$

Introduce $\bar{\bar{S}} = \tilde{S}\dfrac{L_{ref}}{a_\infty} \Rightarrow \tilde{S} = \bar{\bar{S}}\dfrac{a_\infty}{L_{ref}}$

gives.

$$\tilde{\bar{S}} = \begin{cases} \dfrac{a_\infty}{L_{ref}}\dfrac{L_{ref}}{a_\infty}\bar{S} + \dfrac{\mu_\infty L_{ref}}{a_\infty \rho_\infty L_{ref}^2}\hat{\bar{S}} & \dfrac{\mu_\infty}{\rho_\infty L_{ref}^2}\dfrac{L_{ref}}{a_\infty}\hat{\bar{S}} \geq -c_{v_2}\bar{S} \\ \bar{S}\dfrac{a_\infty}{L_{ref}}\dfrac{L_{ref}}{a_\infty} + \dfrac{\bar{S}\frac{a_\infty}{L_{ref}}\frac{L_{ref}}{a_\infty}\left(c_{v_2}^2\bar{S}\frac{a_\infty}{L_{ref}} + c_{v_3}\hat{\bar{S}}\frac{\mu_\infty}{\rho_\infty L_{ref}^2}\right)}{\frac{a_\infty}{L_{ref}}\left[(c_{v_3}-2c_{v_2})\bar{S} - \hat{\bar{S}}\frac{\mu_\infty}{\rho_\infty L_{ref}^2}\frac{L_{ref}}{a_\infty}\right]} & \hat{\bar{S}}\dfrac{\mu_\infty}{\rho_\infty L_{ref}^2}\dfrac{L_{ref}}{a_\infty} < -c_{v_2}\bar{S} \end{cases}$$

$$\tilde{\bar{S}} = \begin{cases} \bar{S} + \dfrac{\mu_\infty}{Re_\infty}\hat{\bar{S}} & \dfrac{\mu_\infty}{Re_\infty}\hat{\bar{S}} \geq -c_{v_2}\bar{S} \\ \bar{S} + \dfrac{\bar{S}\left(c_{v_2}^2\bar{S}\frac{a_\infty}{L_{ref}} + c_{v_3}\hat{\bar{S}}\frac{\mu_\infty}{\rho_\infty L_{ref}^2}\right)\frac{L_{ref}}{a_\infty}}{\left[(c_{v_3}-2c_{v_2})\bar{S} - \hat{\bar{S}}\frac{\mu_\infty}{Re_\infty}\right]} & \hat{\bar{S}}\dfrac{\mu_\infty}{Re_\infty} < -c_{v_2}\bar{S} \end{cases}$$

$$\tilde{\bar{S}} = \begin{cases} \bar{S} + \dfrac{\mu_\infty}{Re_\infty}\hat{\bar{S}} & \hat{\bar{S}}\dfrac{\mu_\infty}{Re_\infty} \geq -c_{v_2}\bar{S} \\ \bar{S} + \dfrac{\bar{S}\left(c_{v_2}^2\bar{S} + c_{v_3}\hat{\bar{S}}\frac{\mu_\infty}{Re_\infty}\right)}{\left((c_{v_3}-2c_{v_2})\bar{S} - \hat{\bar{S}}\frac{\mu_\infty}{Re_\infty}\right)} & \hat{\bar{S}}\dfrac{\mu_\infty}{Re_\infty} < -c_{v_2}\bar{S} \end{cases}$$

$$r = \frac{\bar{\tilde{\nu}}\,\mu_\infty}{\frac{a_\infty}{L_{ref}}\tilde{\bar{S}}\kappa^2 d^2 \rho_\infty L_{ref}^2} = \frac{\bar{\tilde{\nu}}}{\tilde{\bar{S}}\kappa^2 d^2}\frac{\mu_\infty}{\rho_\infty a_\infty L_{ref}} = \frac{\bar{\tilde{\nu}}}{\tilde{\bar{S}}\kappa^2 d^2}\frac{\mu_\infty}{Re_\infty}$$

Source term:

$$\frac{c_{b_1}\tilde{S}\left(\frac{a_\infty}{L_{ref}}\mu_\infty\right)\bar{\rho}\bar{\tilde{\nu}}}{\frac{\mu_\infty a_\infty}{L_{ref}}} + \frac{L_{ref}\rho_\infty}{\mu_\infty a_\infty}\frac{\mu_\infty}{\rho_\infty}\frac{\mu_\infty}{\rho_\infty}\frac{1}{L_{ref}^2}\ \bar{\rho}\, c_{b_2}\frac{\partial\tilde{\nu}}{\partial x_j}\cdot\frac{\partial\tilde{\nu}}{\partial x_j}$$

$$\frac{\mu_\infty}{a_\infty \rho_\infty L_{ref}}$$

$$-c_{w_1}f_w\,\rho_\infty\,\bar{\rho}\left(\frac{\bar{\tilde{\nu}}}{d}\right)^2\frac{\mu_\infty^2}{\rho_\infty^2 L_{ref}^2}\cdot\frac{L_{ref}}{\mu_\infty a_\infty}$$

$$\downarrow$$

$$\frac{\mu_\infty}{\rho_\infty a_\infty L_{ref}} \Rightarrow \frac{\mu_\infty}{Re_\infty}$$

The final Non-dimensional equation is

$$\frac{\partial}{\partial \tau}(\rho\tilde{\nu}) + \frac{\partial}{\partial x_j}(\rho\tilde{\nu}\bar{u}_j) - \frac{1}{\sigma}\frac{\partial}{\partial x_j}\left[\frac{M_\infty}{Re_\infty}(\bar{\mu}+\rho\tilde{\nu})\frac{\partial\tilde{\nu}}{\partial x_j}\right] = C_{b1}\tilde{S}\rho\tilde{\nu}$$
$$+ \frac{C_{b2}}{\sigma}\frac{M_\infty}{Re_\infty}\bar{\rho}\frac{\partial\tilde{\nu}}{\partial x_j}\frac{\partial\tilde{\nu}}{\partial x_j} - C_{w1} f_w \rho\left(\frac{\tilde{\nu}}{d}\right)^2\frac{M_\infty}{Re_\infty}$$

For Negative values the Model Reading.

$$\frac{\partial}{\partial\tau}(\bar\rho\tilde\nu) + \frac{\partial}{\partial x_j}(\rho\tilde\nu\bar u_j) - \frac1\sigma\frac{\partial}{\partial x_j}\left[\frac{M_\infty}{Re_\infty}\left(\bar\mu + \rho\tilde\nu + \frac12\rho\tilde\nu\left(\frac{\rho\tilde\nu}{\mu}\right)\right)\frac{\partial\tilde\nu}{\partial x_j}\right]$$
$$= C_{b1}\bar S\rho\tilde\nu\,\frac{a_\infty}{L_{ref}}\,\frac{\mu_\infty L_{ref}}{a_\infty\mu_\infty}\,g_n + \frac{C_{b2}}{\sigma}\bar\rho\frac{\partial\tilde\nu}{\partial x_j}\cdot\frac{\partial\tilde\nu}{\partial x_j}\frac{M_\infty}{Re_\infty} - \left[-C_{w1}\bar\rho\left(\frac{\tilde\nu}{d}\right)^2 \cancel{\rho_\infty}\frac{\mu_\infty^2}{\rho_\infty L_{ref}\,a_\infty\cancel{\mu_\infty}}\cancel{L_{ref}}\right]$$

$$= C_{b1}\bar S\bar\rho\tilde\nu g_n + \frac{C_{b2}}{\sigma}\bar\rho\frac{\partial\tilde\nu}{\partial x_j}\frac{\partial\tilde\nu}{\partial x_j} - \left[-C_{w1}\bar\rho\left(\frac{\tilde\nu}{d}\right)^2\frac{M_\infty}{Re_\infty}\right]$$

$$g_n = 1 - \frac{1000\left(\frac{\rho\tilde\nu\,\mu_\infty}{\bar\mu\,\mu_\infty}\right)^2}{1+\left(\frac{\rho\tilde\nu\,\mu_\infty}{\bar\mu\,\mu_\infty}\right)^2} = 1 - \frac{1000\,\chi^2}{1+\chi^2}$$

Summary on Non-dimensional Model.

$$\frac{\partial}{\partial\tau}(\rho\tilde\nu) + \frac{\partial}{\partial x_j}(\rho\tilde\nu\bar u_j) - \frac1\sigma\frac{\partial}{\partial x_j}\left[\frac{M_\infty}{Re_\infty}\begin{Bmatrix}\bar\mu + \rho\tilde\nu & \tilde\nu \geq 0 \\ \bar\mu + \rho\tilde\nu + \frac12\rho\tilde\nu\chi & \tilde\nu < 0\end{Bmatrix}\frac{\partial\tilde\nu}{\partial x_j}\right]$$

$$= \begin{cases} C_{b1}\tilde S\rho\tilde\nu & \tilde\nu \geq 0 \\ C_{b1}\bar S\rho\tilde\nu g_n & \tilde\nu < 0\end{cases} + \frac{C_{b2}}{\sigma}\bar\rho\frac{\partial\tilde\nu}{\partial x_j}\frac{\partial\tilde\nu}{\partial x_j}\frac{M_\infty}{Re_\infty}$$
$$- \begin{cases} C_{w1} f_w\bar\rho\left(\frac{\tilde\nu}{d}\right)^2\frac{M_\infty}{Re_\infty} & \tilde\nu \geq 0 \\ -C_{w1}\bar\rho\left(\frac{\tilde\nu}{d^2}\right)^2\frac{M_\infty}{Re_\infty} & \tilde\nu \leq 0\end{cases}$$

$$\tilde S = \begin{cases} \bar S + \frac{M_\infty}{Re_\infty}\frac{\tilde\nu f_{v2}}{\kappa^2 d^2} & \frac{\tilde\nu f_{v2}}{\kappa^2 d^2}\frac{M_\infty}{Re_\infty} > -C_{v2}\bar S \\ \bar S + \frac{\bar S\left(C_{v2}^2\bar S + C_{v3}\hat S\right)}{(C_{v3}-2C_{v2})\bar S - \hat S} & \hat S \leq -C_{v2}\bar S\end{cases} \qquad \hat S = \frac{\tilde\nu f_{v2}}{\kappa^2 d^2}\frac{M_\infty}{Re_\infty}$$

D.6. Discretization of S.A. equation:

$$\frac{\partial}{\partial t}(\rho\tilde{\nu}) + \vec{\nabla}\cdot(\rho\vec{u}\tilde{\nu}) = C_{b_1}\tilde{S}\rho\tilde{\nu} + \frac{1}{\sigma}\left[\vec{\nabla}\cdot\left(\left(\mu\frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right)\nabla\tilde{\nu}\right) + \rho\nabla\tilde{\nu}\cdot\nabla\tilde{\nu}\right]$$

$$- C_{w_1} f_w \rho\left[\frac{\tilde{\nu}}{d}\right]^2$$

Application of Method of Weighted Residuals

$$\int_{\Omega_K} \phi_i\frac{\partial}{\partial t}(\rho\tilde{\nu}) + \phi_i\vec{\nabla}\cdot(\rho\vec{u}\tilde{\nu}) - \phi_i C_{b_1}\tilde{S}\rho\tilde{\nu} - \frac{\phi_i}{\sigma}\left[\vec{\nabla}\cdot\left(\left(\mu\frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right)\vec{\nabla}\tilde{\nu}\right) + \rho\nabla\tilde{\nu}\cdot\nabla\tilde{\nu}\right]$$

$$+ \phi_i C_{w_1} f_w \rho\left[\frac{\tilde{\nu}}{d}\right]^2 d\Omega_K = 0.$$

I.B.P

$$\int_{\Omega_K} \phi_i\frac{\partial}{\partial t}(\rho\tilde{\nu})\, d\Omega_K - \int_{\Omega_K}\nabla\phi_i\cdot(\rho\vec{u}\tilde{\nu})\, d\Omega_K + \oint_{\partial\Omega_K}\phi_i(\rho\vec{u}\tilde{\nu})^{*}\cdot\vec{n}\, ds$$

$$+ \int_{\Omega_K} -\phi_i C_{b_1}\tilde{S}\rho\tilde{\nu} - \frac{\phi_i}{\sigma}\phi_i\nabla\tilde{\nu}\cdot\nabla\tilde{\nu} + \phi_i C_{w_1} f_w \rho\, d\Omega_K$$

$$+ \int_{\Omega_K}\frac{\nabla\phi_i}{\sigma}\cdot\left[\left(\mu\frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right)\vec{\nabla}\tilde{\nu}\right] d\Omega_K - \oint_{\partial\Omega_K}[\![\phi]\!]\left\{\left(\mu\frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right)\vec{\nabla}\tilde{\nu}\right\}\cdot\vec{n}$$

$$+ \left\{\left(\mu\frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right)\frac{\partial\tilde{\nu}}{\partial q}\cdot\nabla\phi_i\right\}\cdot\vec{n}[\![q]\!] - \left\{\left(\mu\frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right)\frac{\partial\tilde{\nu}}{\partial q}\right\}[\![q]\!]\, ds$$

$$\downarrow$$

$$\{G_{ii}\}_{[5,1:5]}.$$

Details (term by term)

1). Time derivative.

$$\int_{\Omega_K}\phi_i\frac{\partial}{\partial t}(\rho\tilde{\nu})\, d\Omega_K, \text{ by introducing } (\rho\tilde{\nu})_h \approx \sum_{j=1}^{N}\widehat{(\rho\tilde{\nu})}_j\,\phi_j$$

$$\frac{\partial}{\partial t}\left(\int_{\Omega_K}\phi_i\sum_{j=1}^{N}\phi_j\,\widehat{(\rho\tilde{\nu})}_j^{(t)}\, d\Omega_K\right) \text{ gives the usual mass matrix.}$$

$$[M]_{ij} = \int\phi_i\cdot\phi_j\, d\Omega_K \text{ gives}$$

$$[M]\frac{d\{\tilde{\nu}\}}{\partial t}$$

2) Combined (Convection and diffusion Volume terms).

$$-\int_{\Omega_K} \nabla\phi_i \cdot \rho \vec{U} \tilde{\nu} - \left[\frac{\nabla\phi_i}{\sigma} \cdot \left(\frac{\mu M_\infty}{Re_\infty} + \rho\tilde{\nu} \right) \nabla\tilde{\nu} \right] d\Omega_K$$ - simple.

3). <u>Convection surface term;</u>

$$+\oint_{\partial\Omega_K} (\rho\vec{U}\tilde{\nu}\cdot\vec{n})^* \, d\partial\Omega_K$$ - there is a seperate set of notes this term as it's interaction with the Basic RANS equations

4) Diffusion surface term;

? - There real question here is does the Model equation affect the original equations. Specifically in Symetry and penalty terms

The surface Diffusion term is given as.

$$-\oint -\frac{[\![\phi_i]\!]}{\sigma}\left\{\left(\mu \frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right)\nabla\tilde{\nu}\right\}\cdot\vec{n} + \left\{ A_v^T \cdot \nabla\phi_i \right\}\cdot\vec{n}\,[\![\vec{q}]\!]$$

$$-\nu\{G_{ii}\}[\![\vec{q}]\!]\,ds$$

$$A_v = \begin{bmatrix} [G_{11}] & [G_{12}] \\ [G_{22}] & [G_{22}] \end{bmatrix}, \quad A_v^T = \begin{bmatrix} [G_{11}] & [G_{21}] \\ [G_{12}] & [G_{22}] \end{bmatrix}.$$

To examine the turbulence Model's affect on the RANS equations we need to look at row 5, col 5 of G_{ij}. By examining the SA model equations and our approximate closures to RANS there is no dependance of $\vec{F}_v$ on $\nabla\tilde{\nu}$, but through μ_T ther is a dependance on $\tilde{\nu}$ via $\frac{\partial\mu_T}{\partial\vec{q}}$ for $\frac{\partial E_v}{\partial\vec{q}}, \frac{\partial F_v}{\partial\vec{q}}$ jacobians

Thus

$$G_{11} = \begin{bmatrix} \left[\;\;\right]_{4\times4} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \begin{matrix}-\frac{\tilde{\nu}}{\rho} & 0 & 0 & 0\end{matrix} & \frac{1}{\rho} \end{bmatrix} \quad G_{12} = \begin{bmatrix} \left[\;\;\right]_{4\times4} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \begin{matrix}0 & 0 & 0 & 0\end{matrix} & 0 \end{bmatrix}$$

$$\left(\frac{1}{Re_\infty} + \rho\tilde{\nu}\right)$$

$$G_{21} = \begin{bmatrix} \left[\;\;\right]_{4\times4} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \begin{matrix}0 & 0 & 0 & 0\end{matrix} & 0 \end{bmatrix} \quad G_{22} = \begin{bmatrix} \left[\;\;\right]_{4\times4} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \begin{matrix}-\frac{\tilde{\nu}}{\rho} & 0 & 0 & 0\end{matrix} & \frac{1}{\rho} \end{bmatrix}$$

Thus the symmetry term is very simply written as

$$\left\{ \frac{1}{\sigma}\left(\mu \frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right) \frac{\partial \tilde{\nu}}{\partial \vec{q}} \cdot \vec{\nabla}\phi_i \cdot \vec{n} \right\} [\![\vec{q}]\!]$$

note there is a term multiplying $[\![\rho]\!]$

It should also be noted that this will work until a T.K.E transport model is used and the $\overline{\rho u_j'' \frac{1}{2} u_i'' u_i''} - \overline{t_{ji} u_i''} \approx -\left(\mu + \frac{\mu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_j}$ in this case the energy equation will be complicated and we'll need to redo this ~~equation~~ derivation for the truly coupled case.

a simple version for implementation will give.

$$\left\{ \cdot \left(\mu \frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right) \left\lfloor -\frac{\tilde{\nu}}{\rho}, 0, 0, 0, \frac{1}{\rho} \right\rfloor \nabla\phi_i \cdot \vec{n} \right\} [\![\vec{q}]\!]$$

this is just a Dot product ~~gives~~. expanding the jump and average yields

for Left/Right (same sign)

$$\frac{1}{2}\left(2\left(\mu \frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right) \left\lfloor -\frac{\tilde{\nu}}{\rho}, 0, 0, 0, \frac{1}{\rho} \right\rfloor\right) \nabla\phi_i \cdot \vec{n} \cdot \left(\vec{q}_L - \vec{q}_R\right)$$

- gives a scalar for flux row 5.

Turbulence Model convection flux:

Let's assume we're going to use an upwind flux such a Roe's. (Note the following analysis is unnecessary for something like Rusanov, this flux is trivial).

The original eigen vectors of the flux jacobian are given as

$$\begin{Bmatrix} 1 \\ u - a n_x \\ v - n_y a \\ h - a\tilde{U} \end{Bmatrix}, \begin{Bmatrix} 1 \\ u \\ v \\ q^2/2 \end{Bmatrix}, \begin{Bmatrix} 0 \\ -n_y \\ n_x \\ U_t \end{Bmatrix}, \begin{Bmatrix} 1 \\ u + n_x a \\ v + n_y a \\ h + a\tilde{U} \end{Bmatrix}$$

where $\tilde{U} = \vec{U} \cdot \vec{n}$

for a single equation model we have 1 scalar transport equation, this eigen system is given as

$$\begin{Bmatrix} 1 \\ u - a n_x \\ v - n_y a \\ h - a\tilde{U} \\ \hat{\nu} \end{Bmatrix}, \begin{Bmatrix} 1 \\ u \\ v \\ q^2/2 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ -n_y \\ n_x \\ U_t \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ u + a n_x \\ v + a n_y \\ h + a\tilde{U} \\ \tilde{\nu} \end{Bmatrix}$$

Note we have only added an eigen vector of $\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$ and the $\lambda = \tilde{U} \mp c$, eigen vectors have added a row of $\tilde{\nu}$ at the bottom. the rest of the eigen structure is identical, Also $\lambda = \tilde{U}$ has multiplicity 3 now instead of 2.

The story is the same for 2 scalar, 3 ... just adding rows of S_2, S_3 .. onto $\lambda = \tilde{U} \mp c$ eigenvectors and eigen vectors of $\begin{Bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{Bmatrix}$ onto the system.

With this in mind we will do a Model Roe derivation a single equation Model.

Remark: this idea actually applies to any scalar transport equation. We just happen to be interested in turbulence at this point in time

Following the Method of Roe and Pike we seek to find a state about which to linearize the flow variables. Namely $\hat{\vec{q}}$ where $\hat{\vec{q}}$: $\hat{q}_L = \hat{\vec{q}} - O(\Delta^2)$, $\hat{q}_R = \hat{\vec{q}} + O(\Delta^2)$.

We begin by projection the jump in $\vec{q}$ onto the eigen structure.

$$\Delta\vec{q} = \vec{q}_L - \vec{q}_R = \sum_{i=1}^{5} \hat{\alpha}_i \{\hat{K}\}_i$$

giving use the following equation.

$$\Delta\rho = \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_5$$
$$\Delta(\rho u) = \hat{\alpha}_1(\hat{u} - n_x\hat{a}) + \hat{\alpha}_2\hat{u} - \hat{\alpha}_3 n_y + \hat{\alpha}_5(\hat{u} + n_x\hat{a})$$
$$\Delta(\rho v) = \hat{\alpha}_1(\hat{v} - n_y\hat{a}) + \hat{\alpha}_2\hat{v} + \hat{\alpha}_3 n_x + \hat{\alpha}_5(\hat{v} + n_y\hat{a})$$
$$\Delta(\rho E) = \hat{\alpha}_1(\hat{h} - a\hat{U}) + \hat{\alpha}_2(q^2/2) + \hat{\alpha}_3(u_t) + \hat{\alpha}_5(\hat{h} + \hat{a}\hat{U})$$
$$\Delta(\rho\tilde{\nu}) = \hat{\alpha}_1\hat{\tilde{\nu}} + \hat{\alpha}_4 + \hat{\alpha}_5\hat{\tilde{\nu}}$$

- We see that except for the fact that $\hat{\alpha}_4$ is now denoted $\hat{\alpha}_5$ our equations for $\Delta\vec{q}[1\to4]$ have not been altered. Further they do not involve the additional $\hat{\alpha}_4$ unknown. This implies that we can solve the first 4 equations for $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$ $\hat{\alpha}_5$.

Using the maple work sheet Roe-deriv_w scalar.mw it can very quickly be seen that $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{S}_8$ remain unchanged.

now that we have the wave strengths. it is necessary to find the Roe average state by solving.

$$\Delta F = \sum_{i=1}^{5} \hat{\alpha}(\tilde{w})_i \, \tilde{\lambda}_i(\tilde{w}) . \vec{K}_i$$

This gives the system of equations.

$$\left\{\begin{matrix} \Delta(\rho u n_x + \rho v n_y) \\ \Delta((\rho u^2 + p) n_x + \rho u v n_y) \\ \Delta(\rho u v n_x + (\rho v^2 + p) n_y) \\ \Delta(\rho H u n_x + \rho H v n_y) \\ \Delta(\rho S . (u n_x + v n_y)) \end{matrix}\right\} = \sum_i \hat{\alpha}_i(\tilde{w}) \tilde{\lambda}_i(\tilde{w}) \vec{K}_i(\tilde{w})$$

We now have to apply the Δ operator to $\Theta(\Delta^2)$.

$$\approx \left\{\begin{matrix} \Delta(\rho u) n_x + \Delta(\rho v) n_y \\ \Delta(\rho u^2) n_x + \frac{(\gamma-1)}{\gamma}\left(\Delta(\rho H) - \frac{1}{2}[\Delta(\rho u^2) + \Delta(\rho v^2)]\right) n_x + \Delta(\rho u v) n_y \\ \Delta(\rho u v) n_x + \Delta(\rho v^2) n_y + \left(\frac{\gamma-1}{\gamma}\right)\left[\Delta(\rho H) - \frac{1}{2}(\Delta(\rho u^2) + \Delta(\rho v^2))\right] n_y \\ \Delta(\rho U H) n_x + \Delta(\rho v H) n_y \\ \Delta(\rho u S) n_x + \Delta(\rho v S) n_y \end{matrix}\right\}$$

we now input these expressions into.

Remark here the in the computer have no $\tilde{\ }$ but all non superscript variables should be understood to represent the roe average. The code will also use capital letters for these variables

Unfortunately Maple does not seem up to the task of computing the Roe averages. It will have to be done by hand. However we know that

$$\tilde{S} = \frac{\sqrt{\rho_L} S_L + \sqrt{\rho_R} S_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \equiv b_L S_L + b_R S_R, \quad b_L = \frac{1}{\rho_L + \tilde{\rho}}, \quad b_R = \frac{1}{\sqrt{\rho_R} + \tilde{\rho}} \quad \tilde{\rho} = \sqrt{\rho_L \rho_R}$$

The Maple file has told use that the dissipation for the scalar transport does fit into our general framework for the Roe scheme

i.e.

$$F(S) = \frac{1}{2}\left[F_L(S) + F_R(S) - \left[|\bar{\lambda}_2|\,\Delta(\rho S) + \delta_1 \bar{S}\right]\right]$$

Nice and easy

Making this

$|\lambda_2| = \vec{U}\cdot\vec{n}$

Richard Gaffney's k-e-g Model Scalar Roe flux

For my purposes we have only to deal with the k-eqn dissip. term and verify that this is the same as my dissip. term for s.

For Rick

$\lambda_1 = |\vec{u}\cdot\vec{n} + a|$

$\lambda_2 = |\vec{u}\cdot\vec{n} - a|$

$\lambda_3 = |\vec{u}\cdot\vec{n}|$

$$D_K = \bar{K}\beta_1 + \beta_{N+6}$$

$$\beta_{N+6} = \bar{\lambda}_3 \bar{\rho}\,\Delta K$$

$$\beta_1 = \bar{\alpha}_1\bar{\lambda}_1 + \bar{\alpha}_2\bar{\lambda}_2 + \bar{\alpha}_3\bar{\lambda}_3$$

$$\bar{\alpha}_1 = \frac{1}{2}\left[\frac{\Delta p}{\bar{a}^2} + \frac{\bar{\rho}}{\bar{a}}\Delta(\vec{u}\cdot\vec{n})\right]$$

$$\bar{\alpha}_2 = \frac{1}{2}\left[\frac{\Delta p}{\bar{a}^2} - \frac{\bar{\rho}}{\bar{a}}\Delta(\vec{u}\cdot\vec{n})\right]$$

$$\bar{\alpha}_3 = \Delta\rho - \frac{\Delta p}{\bar{a}^2}$$

$$D_K = \bar{\lambda}_3\bar{\rho}\,\Delta K + \Delta\rho\,\bar{\lambda}_3\bar{K} + \bar{K}\left[\left(-\bar{\lambda}_3 + \frac{1}{2}(\bar{\lambda}_2 + \bar{\lambda}_1)\right)\frac{\Delta p}{\bar{a}^2} + \frac{1}{2}(\lambda_1 - \lambda_2)\frac{\bar{\rho}}{\bar{a}}\Delta(\vec{u}\cdot\vec{n})\right]$$

$$D_K = \bar{\lambda}_3\bar{\rho}\,\Delta K + \Delta\rho\,\bar{\lambda}_3\bar{K} + \bar{K}\left[\left(-\bar{\lambda}_3 + \frac{1}{2}(\bar{\lambda}_2 + \lambda_1)\right)\frac{\Delta p}{\bar{a}^2} + \frac{1}{2}(\bar{\lambda}_2 - \lambda_1)\frac{\bar{\rho}}{\bar{a}}\Delta\vec{u}\cdot\vec{n}\right]$$

Translating to my indices $\bar{\lambda}_3 \to |\lambda_2|$, $\lambda_1 \to |\lambda_3|$, $\lambda_2 \to |\lambda_1|$

$$D_K = |\bar{\lambda}_2|\left(\bar{\rho}\,\Delta K + \Delta\rho\,\bar{K}\right) + \bar{K}\left[\left(-|\bar{\lambda}_2| + \frac{1}{2}(|\lambda_1| + |\lambda_3|)\right)\frac{\Delta p}{\bar{a}^2} + \frac{1}{2}(|\lambda_1| - |\lambda_3|)\frac{\bar{\rho}}{\bar{a}}\Delta\vec{u}\cdot\vec{n}\right]$$

well to $O\Delta^2$ $\bar{\rho}\,\Delta K + \Delta\rho\,\bar{K} \approx \Delta(\rho K)$ thus this is equivalent to mine so we are ok.

$u\cdot\vec{n} - a$ | $\lambda = u\cdot\vec{n}$

$$[T] = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ u - an_x/\sqrt{n_x^2+n_y^2} & u & -n_y & 0 & u + an_x/\sqrt{n_x^2+n_y^2} \\ v - an_y/\sqrt{n_x^2+n_y^2} & v & n_x & 0 & v + an_y/\sqrt{n_x^2+n_y^2} \\ H - a\frac{u\cdot\vec{n}}{\sqrt{n_x^2+n_y^2}} & \frac{1}{2}(u^2+v^2) & u\tau & 0 & H + a\frac{u\cdot\vec{n}}{\sqrt{n_x^2+n_y^2}} \\ s & 0 & 0 & 1 & s \end{bmatrix}$$

$V_3 = V_3\big|_{Maple} \cdot n_x$

$v \cdot V_3\big|_{Maple}$

$$V_2 = V_2\big|_{maple} + V_3\big|_{maple}\cdot v = \begin{cases} [1] + [0] \\ \left[u + \frac{v\cancel{n_y}}{\cancel{n_x}}\right] - \left[\frac{v\cancel{n_y}}{\cancel{n_x}}\right] \\ [0] + [v] \\ \left[\frac{u^2}{2} + \cancel{\frac{v^2}{2}} - \frac{uv\cancel{n_y}}{\cancel{n_x}}\right] + \left[-\frac{u\cancel{n_y}v}{\cancel{n_x}} + \cancel{v^2}\right] = \frac{1}{2}(u^2+v^2) \\ [0] + [0] \end{cases}$$

$= \frac{1}{2}v^2$

$$= \begin{Bmatrix} 1 \\ u \\ v \\ \frac{1}{2}(u^2+v^2) \\ 0 \end{Bmatrix}$$

Eigenvectors <u>Not</u> assuming

$n_x^2 + n_y^2 = 1$

TOPS 35500

Matrix dissipation scheme w/ scalar:

Again as with the Roe scheme we need to examine the effect of an additional scalar equation of the matrix disp flux.

The dissipation vector is given by

$$[T][\Lambda][T]^{-1}\{\Delta U\}$$

where the eigenvalues and eigenvectors are evaluated at some intermediate state (I normally use the Roe state)

The new $[T]$ and $[T]^{-1}$ are given as

$$[T] = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ u-an_x & u & -n_y & 0 & u+an_x \\ v-an_y & v & n_x & 0 & v+an_y \\ H-a\vec{u}\cdot\vec{n} & \frac{q^2}{2} & -u_T & 0 & H+a\vec{u}\cdot\vec{n} \\ S & 0 & 0 & 1 & S \end{bmatrix},$$

$$[T]^{-1} = \begin{bmatrix} \frac{H(\gamma-1)-a^2+a\vec{u}\cdot\vec{n}}{2a^2} & \frac{-(1-\gamma)-an_x}{2a^2} & \frac{(1-\gamma)v-an_y}{2a^2} & \frac{(\gamma-1)}{2a^2} & 0 \\ \frac{(1-\gamma)H+2a^2}{a^2} & \frac{u(\gamma-1)}{a^2} & \frac{v(\gamma-1)}{a^2} & -\frac{(\gamma-1)}{a^2} & 0 \\ -vn_x+un_y & -n_y & n_x & 0 & 0 \\ \frac{s(1-\gamma)H+sa^2}{a^2} & \frac{su(\gamma-1)}{a^2} & \frac{sv(\gamma-1)}{a^2} & -\frac{s(\gamma-1)}{a^2} & 1 \\ \frac{H(\gamma-1)-a^2-a\vec{u}\cdot\vec{n}}{2a^2} & \frac{(1-\gamma)u+an_x}{2a^2} & \frac{(1-\gamma)v+an_y}{2a^2} & \frac{\gamma-1}{2a^2} & 0 \end{bmatrix}$$

We will now follow the original development.

Remark: Rows $1 \to 3$ of $[T]^{-1}$ are exactly the same as the pure Euler case. Row 4 is from our scalar equation and Row 5 is the Row 4 of Pure Euler.

Defining

$\{\Delta w\} = [T]^{-1}\{\Delta u\}$, we see that all scalar terms are confined to Δw_4, Further only Δw_4 has a contribut from Δu_5, i.e. ΔS.

Defining

$\{\Delta z\} = [\Lambda]\{\Delta w\}$ - trivial step.

Finall

$$\{\Delta d\} = [T] \cdot \begin{Bmatrix} \Delta w_1 \lambda_1 \\ \Delta w_2 \lambda_2 \\ \Delta w_3 \lambda_2 \\ \Delta w_4 \lambda_2 \\ \Delta w_5 \lambda_3 \end{Bmatrix}$$

Examing the $[T]$ Matrix we see that

$$\left\{\begin{aligned} d_1 &= f(\Delta w_1 \lambda_1,\ \Delta w_2 \lambda_2, \Delta w_3 \lambda_2,\ \Delta w_5, \lambda_3) \\ d_2 &= \text{"} \quad \text{"} \\ d_3 &= \text{"} \quad \text{"} \\ d_4 &= \text{"} \quad \text{"} \end{aligned}\right.$$

- same as befor

Remark None of the original convective flux has any dependance on the scalar. Thus.

$$d_5 = \bar{S}\,\Delta w_1 \bar{\lambda}_1 + \Delta w_4 \bar{\lambda}_2 + \bar{S}\,\Delta w_5 \bar{\lambda}_3$$

- all the $\bar{(\ )}$ qty's are evaluated at the Roe state.

Thus to form a flux for the scalar we simply define the dissipation using the standard qty's from our matrix dissipation routine and simply add Δw_4 -

Thus we should simply derive Δw because it's the only change from previous derivations.

$$\Delta w_4 = \frac{s[(1-\gamma)H + a^2]}{a^2}\Delta\rho + \frac{su(\gamma-1)}{a^2}\Delta(\rho u) + \frac{sv(\gamma-1)}{a^2}\Delta(\rho v)$$

$$- \frac{s(\gamma-1)}{a^2}\Delta(\rho E) + \Delta(\rho s) = \frac{s}{a^2}\Big[(-(\gamma-1)H + a^2)\Delta\rho + s(\gamma-1)\Delta\rho u + v(\gamma-1)\Delta(\rho v) - (\gamma-1)\Delta\rho E\Big] + \Delta\rho s$$

$$d_5 = s(u-a)\Delta w_1 + \overset{(\rho u s_x + \rho v s_y)}{u\Delta w_4} + s(u+c)\Delta w_5$$

It should be noted that all scalars will take this form, so we just need to call this routine with the correct # of times for N scalars the correct index the correct

Riemann Invariant B.C. for Scalars

As with all things that use characteristics we need to check the impact of our scalar on the Riemann Invariant B.C.

We started with the Euler Equations (normal to a surface) to which we will add a scalar convection equation. This gives the following system of 5 equations

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x'} + u \frac{\partial \rho}{\partial x'} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x'} + \frac{1}{\rho} \frac{\partial p}{\partial x'} = 0$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x'} = 0$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x'} + \rho c^2 \frac{\partial u}{\partial x'} = 0$$

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x'} = 0$$

Note u here is really $\vec{U} \cdot \vec{n}$, v is $U_{\vec{t}}$ and x' is a normal direvative.

The jacobian is now

$$\begin{bmatrix} u & \rho & 0 & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & u \end{bmatrix}$$

with eigen values

u
u
u
$u+c$
$u-c$

We now need to find the eigenvectors.

$[\lambda I - A]\{\vec{e}\} = 0$ gives

$$\lfloor \vec{e} \rfloor \cdot \begin{bmatrix} \lambda-u & \rho & 0 & 0 & 0 \\ 0 & \lambda-u & 0 & 1/\rho & 0 \\ 0 & 0 & \lambda-u & 0 & 0 \\ 0 & \rho c^2 & 0 & \lambda-u & 0 \\ 0 & 0 & 0 & 0 & \lambda-u \end{bmatrix} -$$

gives the following system of equations for the Left eigen vectors i.e. giving L^{-1} so we can directly compute S_w.

This yields the following equation.

① $\lambda l_1 = l_1 u$

② $l_1 \rho + \lambda l_2 + l_4 \rho c^2 = u l_2$

③ $\lambda l_3 = l_3 u$

④ $\frac{l_2}{\rho} + l_4 \lambda = l_4 u$

⑤ $\lambda l_5 = l_5 u$

For $\lambda = u$.

we get $l_1 = l_1 \Rightarrow l_1 = \alpha$ free parameter.

$l_3 = l_3 \Rightarrow l_3 = \beta$ free parameter,

$l_5 = l_5 \Rightarrow l_5 = \Gamma$ free parameter.

$\frac{l_2}{\rho} + l_4 u = \cancel{l_4} u \Rightarrow l_2 = 0$

$\alpha\rho - 0 + l_4 \rho c^2 = 0 \Rightarrow l_4 = \frac{-\alpha}{c^2}$

$$\vec{l}_1 = \begin{Bmatrix} \alpha \\ 0 \\ \beta \\ -\alpha/c^2 \\ \Gamma \end{Bmatrix}$$

Using the 5 distinct eigenvectors we get L^{-1} as

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & -1/c^2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{\rho c} & 0 \\ 0 & 1 & 0 & \frac{1}{\rho c} & 0 \end{bmatrix}$$

$$\delta w = L^{-1}\,\delta V$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1/c^2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/\rho c & 0 \\ 0 & 1 & 0 & \frac{1}{\rho c} & 0 \end{bmatrix} \begin{Bmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \\ \delta s \end{Bmatrix}$$

$$\delta w_1 = \delta\rho - \frac{\delta p}{c^2}$$

$$\delta w_2 = \delta v$$

$$\delta w_3 = \delta s$$

$$\delta w_4 = \delta u - \frac{\delta p}{\rho c}$$

$$\delta w_5 = \delta u + \frac{\delta p}{\rho c}$$

We see that we have recovered all the previous conditions unchange but that the scaler condition is the same as that on the tangential velocity.

$$w_3 = s$$

So at inflow $u \cdot \vec{n} < 0$ $\quad s = s_\infty$

at outflow $\vec{u} \cdot \vec{n} > 0$ $\quad s = s_{interior}$.

We will get 3 LI eigen Nectors by choosing.

$$\alpha\begin{Bmatrix}1\\0\\0\\-\frac{1}{c^2}\end{Bmatrix}+\beta\begin{Bmatrix}0\\0\\1\\0\\0\end{Bmatrix}+\Gamma\begin{Bmatrix}0\\0\\0\\0\\1\end{Bmatrix}.$$

For $\lambda=(U+C)$

$$(U+C)l_1=l_1U$$
$$l_1\rho+l_2(U+C)+l_4\rho c^2=Ul_4$$
$$(U+C)l_3=l_3$$
$$\frac{l_2}{\rho}+l_4(U+C)=l_4U$$
$$(U+C)l_5=l_5U$$

given

$$l_1C=0$$
$$l_3C=0$$
$$l_5C=0$$

$$l_2U+l_2C+l_4\rho c^2=Ul_4$$
$$l_4=-\frac{l_2}{\rho c}$$

$$\vec{l}_4=\begin{Bmatrix}0\\1\\0\\\frac{1}{\rho}c\\0\end{Bmatrix}$$

For $\lambda=(U-C)$

$$(U-C)l_1=l_1U \Rightarrow l_1=0$$
$$l_1\rho-l_2U+l_4\rho c^2=-(U-C)l_2 \Rightarrow l_4\rho c^2=+cl_2 \Rightarrow l_4=\frac{l_2}{\rho c}$$
$$(U-C)l_3=l_3U \Rightarrow l_3=0$$
$$l_4=-$$
$$\frac{l_2}{\rho}+l_4(U-C)=l_4U$$
$$(U-C)l_5=Ul_5 \Rightarrow l_5=0$$

$$\vec{l}_5=\begin{Bmatrix}0\\1\\0\\\frac{1}{\rho c}\\0\end{Bmatrix}$$

S.A. symmetry flux: Implementation details

We have written the S.A symmetry flux as

$$\left\{ \frac{1}{\sigma}\left(\mu \frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\right) \frac{\partial \tilde{\nu}}{\partial \vec{q}} \cdot \nabla\phi_i \cdot \vec{n}_x \right\} [\![\vec{q}]\!]$$

given

$$\frac{\partial \tilde{\nu}}{\partial \vec{q}} = \left(-\frac{\tilde{\nu}}{\rho}, 0, 0, 0, \frac{1}{\rho}\right)$$

Expanding the average operator as

$$\frac{1}{2}\left[\frac{1}{\sigma}\left(\mu_L \frac{M_\infty}{Re_\infty} + \rho_L \tilde{\nu}_L\right) \left\lfloor -\frac{\tilde{\nu}_L}{\rho_L}, 0, 0, 0, \frac{1}{\rho_L} \right\rfloor \cdot \left(\frac{\partial \phi_i^L}{\partial x} \cdot n_x + \frac{\partial \phi_i^L}{\partial y} \cdot n_y\right) + \frac{1}{\sigma}\left(\mu_R \frac{M_\infty}{Re_\infty} + \rho_R \tilde{\nu}_R\right) \left\lfloor \frac{\tilde{\nu}_L}{\rho_L}, 0, 0, 0, \frac{1}{\rho_R} \right\rfloor \cdot \left(\frac{\partial \phi_i^R}{\partial x} \cdot n_x + \frac{\partial \phi_i^R}{\partial y} \cdot n_y\right)\right] \cdot \begin{Bmatrix} \rho_L - \rho_R \\ \rho u_L - \rho u_R \\ \rho v_L - \rho v_R \\ \rho E_L - \rho E_R \\ \rho\tilde{\nu}_L - \rho\tilde{\nu}_R \end{Bmatrix} = \Delta\vec{q}$$

For "Left" only left piece is retained.
For "Right" only Right piece is retained.

Thus we'll write it down for "Left"s and understand "Right" is identical except that the L is replaced with a right.

Sym flux is given as

$$sf^L = \frac{1}{2}\left[\frac{1}{\sigma}\left(\mu_L \frac{M_\infty}{Re_\infty} + \rho_L \tilde{\nu}_L\right) \cdot \left(-\frac{\tilde{\nu}_L}{\rho_L}\Delta\rho + \frac{1}{\rho_L}\Delta(\rho\tilde{\nu})\right) \cdot \left(\frac{\partial \phi_i^L}{\partial x} n_x + \frac{\partial \phi_i^L}{\partial y} \cdot n_y\right)\right]$$

This is very simple.

S. A. penalty flux Implementation details:

For the penalty flux for S.A turb model we are only interested in the 5th Row of $\{G_{ic}\}$, which is given by.

$$pflux = pen \cdot \Big\{ \underbrace{\frac{1}{\sigma}\Big(\mu \frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\Big)\frac{\partial \tilde{\nu}}{\partial \vec{\xi}} + \frac{1}{\sigma}\Big(\mu \frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\Big)\frac{\partial \tilde{\nu}}{\partial \vec{\xi}}}_{G_{11}[5]} \Big\} \cdot \Delta\vec{\xi}\ \phi_i$$

Writing this out gives

$$pflux = pen \Big\{ \frac{2}{\sigma}\Big(\mu \frac{M_\infty}{Re_\infty} + \rho\tilde{\nu}\Big)\frac{\partial \tilde{\nu}}{\partial \vec{\xi}} \Big\} \Delta\vec{\xi} \cdot \phi_i$$

Expanding the average operator.

$$pflux = \frac{pen}{2}\Big[\frac{2}{\sigma}\Big(\mu_L \frac{M_\infty}{Re_\infty} + \rho_L\tilde{\nu}_L\Big)\frac{\partial \tilde{\nu}^L}{\partial \vec{\xi}} + \frac{2}{\sigma}\Big(\mu_R \frac{M_\infty}{Re_\infty} + \rho_R\tilde{\nu}_R\Big)\frac{\partial \tilde{\nu}^R}{\partial \vec{\xi}} \Big]$$

$$\Delta\vec{\xi}\ \phi_i$$

$$pflux = pen\Big[term_L\Big(-\frac{\tilde{\nu}_L}{\rho_L}\Delta\rho + \frac{\Delta(\rho\tilde{\nu})}{\rho_L}\Big) + term_R\Big(-\frac{\tilde{\nu}_R}{\rho_R}\Delta\rho + \frac{\Delta(\rho\tilde{\nu})}{\rho_R}\Big)\Big]$$

where

$$term_L = \frac{1}{\sigma}\Big(\mu_L \frac{M_\infty}{Re_\infty} + \rho_L\tilde{\nu}_L\Big)$$

$$term_R = \frac{1}{\sigma}\Big(\mu_R \frac{M_\infty}{Re_\infty} + \rho_R\tilde{\nu}_R\Big)$$

Non conservative SA diffusion terms:

Then non-conservative diffusion term is given as.

$$\left(\frac{\mu}{\rho} + \tilde{\nu}\right) \nabla \hat{\nu}$$

is $g(S) = \hat{\nu}$ then we have.

Then non-dimensional form is given as

$$\left(\frac{M_\infty}{Re}\frac{\mu}{\rho} + \tilde{\nu}\right) \nabla \tilde{\nu}$$

since there is no longer a $\frac{\partial \tilde{\nu}}{\partial \vec{q}}$ rather it is the following Identity. $[0, 0, 0, \ldots, 0, 1]$

giving

$$\left(\frac{M_\infty}{Re_\infty}\frac{\mu}{\rho} + \tilde{\nu}\right)[0, 0, 0, \ldots, 0, 1] \cdot \left(\frac{\partial \vec{q}_i}{\partial x} n_x + \frac{\partial \vec{q}}{\partial y} n_y\right)$$

This goes to

$$\frac{\left(\frac{M_\infty}{Re_\infty}\frac{\mu}{\rho} + \tilde{\nu}\right)}{\sigma}\left(\frac{\partial \tilde{\nu}}{\partial x} n_x + \frac{\partial \tilde{\nu}}{\partial y} n_y\right)$$

The non-conservative symmetry term looks like.

$$\frac{\left(\frac{M_\infty}{Re_\infty}\frac{\mu}{\rho} + \tilde{\nu}\right)}{\sigma}[0, 0, 0, \ldots, 0, 1] \left(\frac{\partial \phi_i}{\partial x} n_x + \frac{\partial \phi_i}{\partial y} n_y\right) \begin{Bmatrix} \Delta\rho \\ \vdots \\ \Delta\tilde{\nu} \end{Bmatrix}$$

Finally the penalty term looks like.

$$pen\left(\frac{\frac{M_\infty}{}\frac{\mu_L}{\rho_L} + \tilde{\nu}_L}{\sigma} + \frac{\frac{M_\infty}{Re_\infty}\frac{\mu_R}{\rho_R} + \tilde{\nu}_R}{\sigma}\right)\Delta\tilde{\nu}$$

Viscous fluxes and jacobians for RANS

For the RANS equations we have the following.

$$\frac{\partial}{\partial t}(\bar{\rho}) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_j) = 0$$

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_i) + \frac{\partial}{\partial x_j}\left[\bar{\rho}\tilde{u}_i\tilde{u}_j + \bar{p}\delta_{ij}\right] - \frac{\partial}{\partial x_j}\left[2(\mu+\mu_T)\left(\tilde{S}_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right]$$

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{E}) + \frac{\partial}{\partial x_j}\left[\tilde{u}_j(\bar{\rho}\hat{E} + \bar{p}) - \left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_t}\right)\gamma\frac{\partial e}{\partial x_j}\right] -$$
$$\frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(2(\mu+\mu_T)\left(\tilde{S}_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right)\right]$$

Thus we can define $\vec{E}_v$, $\vec{F}_v$

$$\vec{E}_v = \begin{cases} 0 \\ \left(\frac{\mu M_\infty}{Re_\infty} + \frac{\mu_T M_\infty}{Re_\infty}\right)\left(\frac{4}{3}\frac{\partial \bar{u}}{\partial x} - \frac{2}{3}\frac{\partial \bar{v}}{\partial y}\right) \\ \left(\frac{\mu M_\infty}{Re_\infty} + \frac{\mu_T M_\infty}{Re_\infty}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \gamma\frac{\partial \bar{e}}{\partial x}\left(\frac{\mu M_\infty}{Pr\, Re_\infty} + \frac{\mu_T}{Pr_T}\frac{M_\infty}{Re_\infty}\right) + \left(\mu\frac{M_\infty}{Re_\infty} + \mu_T\frac{M_\infty}{Re_\infty}\right)\left[\bar{u}\left\{\frac{4}{3}\frac{\partial \bar{u}}{\partial x} - \frac{2}{3}\frac{\partial \bar{v}}{\partial y}\right\} + \bar{v}\left\{\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right\}\right] \end{cases}$$

$$\vec{F}_v = \begin{cases} 0 \\ \mu\frac{M_\infty}{Re_\infty} + \mu_T\frac{M_\infty}{Re_\infty}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \left(\mu\frac{M_\infty}{Re_\infty} + \mu_T\frac{M_\infty}{Re_\infty}\right)\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right) \\ \gamma\frac{\partial \bar{e}}{\partial y}\left(\frac{\mu}{Pr}\frac{M_\infty}{Re_\infty} + \frac{\mu_T}{Pr_T}\frac{M_\infty}{Re_\infty}\right) + \left(\mu\frac{M_\infty}{Re_\infty} + \mu_T\frac{M_\infty}{Re_\infty}\right)\left[\bar{u}\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\} + \bar{v}\left\{\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right\}\right] \end{cases}$$

Jacobians G_{ij}

$$G_{11} \quad \frac{\partial E_v}{\partial \vec{q}_x} = \begin{bmatrix} \left(\mu\frac{M_\infty}{Re_\infty} + \frac{\mu_T M_\infty}{Re_\infty}\right)\left(\frac{4}{3}\left[\frac{\partial u}{\partial \vec{q}}\right]\right) \\ \left(\frac{\mu M_\infty}{Re_\infty} + \mu_T\frac{M_\infty}{Re_\infty}\right)\left(\left[\frac{\partial v}{\partial \vec{q}}\right]\right) \\ \gamma\left(\frac{\mu}{Pr}\cdot\frac{M_\infty}{Re_\infty} + \frac{\mu_T}{Pr_T}\frac{M_\infty}{Re_\infty}\right)\left[\frac{\partial e}{\partial \vec{q}}\right] + \left(\mu\frac{M_\infty}{Re_\infty} + \mu_T\frac{M_\infty}{Re_\infty}\right)\left[\bar{u}\left\{\frac{4}{3}\left[\frac{\partial u}{\partial \vec{q}}\right]\right\} + \bar{v}\left\{\left[\frac{\partial v}{\partial \vec{q}}\right]\right\}\right] \end{bmatrix}$$

G_{12}: $\dfrac{\partial E_v}{\partial \vec{q}_y}$

$$\begin{bmatrix} \lfloor 0 \rfloor \\ \left(\mu \frac{M_\infty}{Re_\infty} + \mu_T \frac{M_\infty}{Re_\infty}\right) \left\lfloor -\frac{2}{3} \frac{\partial v}{\partial \vec{q}} \right\rfloor \\ \left(\mu \frac{M_\infty}{Re_\infty} + \mu_T \frac{M_\infty}{Re_\infty}\right) \left\lfloor \frac{\partial u}{\partial \vec{q}} \right\rfloor \\ \left(\mu \frac{M_\infty}{Re_\infty} + \mu_T \frac{M_\infty}{Re_\infty}\right) \left\lfloor u\left\{ -\frac{2}{3} \frac{\partial v}{\partial \vec{q}} \right\} + v\left\{ \frac{\partial u}{\partial \vec{q}} \right\} \right\rfloor \end{bmatrix}$$

G_{21}: $\dfrac{\partial F_v}{\partial \vec{q}_x}$

$$\begin{bmatrix} \lfloor 0 \rfloor \\ \left(\mu \frac{M_\infty}{Re_\infty} + \mu_T \frac{M_\infty}{Re}\right) \left\lfloor \frac{\partial v}{\partial \vec{q}} \right\rfloor \\ \left(\mu \frac{M_\infty}{Re_\infty} + \mu_T \frac{M_\infty}{Re_\infty}\right) \left\lfloor -\frac{2}{3} \frac{\partial u}{\partial \vec{q}} \right\rfloor \\ \left(\mu \frac{M_\infty}{Re_\infty} + \mu_T \frac{M_\infty}{Re_\infty}\right) \left\lfloor u\left\{ \frac{\partial v}{\partial \vec{q}} \right\} + v\left\{ -\frac{2}{3} \frac{\partial u}{\partial \vec{q}} \right\} \right\rfloor \end{bmatrix}$$

G_{22}: $\dfrac{\partial F_v}{\partial \vec{q}_y}$

$$\begin{bmatrix} \lfloor 0 \rfloor \\ \left(\mu \frac{M_\infty}{Re_\infty} + \mu_T \frac{M_\infty}{Re_\infty}\right) \left\lfloor \frac{\partial u}{\partial \vec{q}} \right\rfloor \\ \left(\mu \frac{M_\infty}{Re_\infty} + \mu_T \frac{M_\infty}{Re_\infty}\right) \left\lfloor \frac{4}{3} \frac{\partial v}{\partial \vec{q}} \right\rfloor \\ \gamma \frac{\partial e}{\partial \vec{q}} \left(\frac{\mu M_\infty}{Pr\, Re_\infty} + \frac{\mu_T}{Pr_T} \frac{M_\infty}{Re_\infty}\right) + \left(\mu \frac{M_\infty}{Re_\infty} + \mu_T \frac{M_\infty}{Re_\infty}\right) \left\lfloor u \frac{\partial u}{\partial \vec{q}} + \frac{4}{3} v \frac{\partial v}{\partial \vec{q}} \right\rfloor \end{bmatrix}$$

RANS Penalty flux:

As with the Laminar soln. we will write our penalty flux in the following manner, Albiet with a few extra definitions.

$$\text{pflux} = \left\{ \begin{array}{l} 0 \\ \tilde{A}_2 \Delta q_1 + \tilde{B}_2 \Delta q_2 \\ \tilde{A}_3 \Delta q_1 + \tilde{C}_3 \Delta q_3 \\ \tilde{A}_4 \Delta q_1 + \tilde{B}_4 \Delta q_2 + \tilde{C}_4 \Delta q_3 + \tilde{D}_4 \Delta q_4 \end{array} \right\}$$

Let us denote.

$$\alpha_{L/R} = \left(\mu_{L/R} \frac{M_\infty}{Re_\infty} + \mu_{T L/R} \frac{M_\infty}{Re_\infty} \right) \text{ and } \beta_{L/R} = \left(\frac{\mu_{L/R}}{Pr} \frac{M_\infty}{Re_\infty} + \frac{\mu_{T L/R}}{Pr_T} \frac{M_\infty}{Re_\infty} \right)$$

This gives according to MAPLE sheet penalty_and_Symflux.mw

$$\tilde{A}_1 = 0$$

$$\tilde{A}_2 = -\frac{7}{6} \left(\frac{\alpha_L U_L}{\rho_L} + \frac{\alpha_R U_R}{\rho_R} \right)$$

$$\tilde{A}_3 = -\frac{7}{6} \left(\frac{\alpha_L V_L}{\rho_L} + \alpha_R \frac{V_R}{\rho_R} \right)$$

$$\tilde{A}_4 = \frac{1}{\rho_L} \left[\gamma \beta_L (U_L^2 + V_L^2 - E_L) - \frac{7}{6} \alpha_L (U_L^2 + V_L^2) \right] + \frac{1}{\rho_R} \left[\gamma \beta_R (U_R^2 + V_R^2 - E_R) - \frac{7}{6} \alpha_R (U_R^2 + V_R^2) \right]$$

$$\tilde{B}_1 = 0$$

$$\tilde{B}_2 = \frac{7}{6} \left(\frac{\alpha_L}{\rho_L} + \frac{\alpha_R}{\rho_R} \right)$$

$$\tilde{B}_3 = 0$$

$$\tilde{B}_4 = -\frac{\gamma}{\rho_L} \beta_L U_L + \frac{7}{6} \frac{\alpha_L U_L}{\rho_L} - \frac{\gamma}{\rho_R} \beta_R U_R + \frac{7}{6} \frac{\alpha_R U_R}{\rho_R}$$

$$\tilde{B}_4 = \frac{U_L}{\rho_L} \left(-\gamma \beta_L + \frac{7}{6} \alpha_L \right) + \frac{U_R}{\rho_R} \left(-\gamma \beta_R + \frac{7}{6} \alpha_R \right)$$

$$\tilde{C}_1 = 0$$
$$\tilde{C}_2 = 0$$
$$\tilde{C}_3 = \frac{7}{6}\left(\frac{\alpha_L}{\rho_L} + \frac{\alpha_R}{\rho_R}\right)$$
$$\tilde{C}_4 = -\frac{\gamma}{\rho_L}\beta_L V_L + \frac{7}{6}\frac{\alpha_L V_L}{\rho_L} - \frac{\gamma \beta_R V_R}{\rho_R} + \frac{7}{6}\frac{\alpha_R V_R}{\rho_R} = \frac{V_L}{\rho_L}\left(-\gamma\beta_L + \frac{7}{6}\alpha_L\right) + \frac{V_R}{\rho_R}\left(-\gamma\beta_R + \frac{7}{6}\alpha_R\right.$$
$$\tilde{D}_1 = 0$$
$$\tilde{D}_2 = 0$$
$$\tilde{D}_3 = 0$$
$$\tilde{D}_4 = \frac{\gamma}{\rho_L}\beta_L + \gamma\frac{\beta_R}{\rho_R} = \gamma\left(\frac{\beta_L}{\rho_L} + \frac{\beta_R}{\rho_R}\right)$$

Boundary Penalty Flux:

With the new method of applying Boundary conditions we need to evaluate the flux as

$$\text{ptflux} = \sigma_{ii}(\vec{q}_B) \cdot (\vec{q}_L - \vec{q}_B)$$

Thus we will redefine our A, B, C, D terms as follows for a boundary flux routine.

$$\text{ptflux} = \left\{ \begin{array}{l} 0 \\ A_2\, dq_1 + B_2\, dq_2 \\ A_3\, dq_1 + C_3\, dq_3 \\ A_4\, dq_1 + B_4\, dq_2 + C_4\, dq_3 + D_4\, dq_4 \end{array} \right\}$$

$$A_2 = -\frac{7}{3}\frac{\alpha_R u_R}{\rho_R}$$

$$B_2 = \frac{7}{3}\frac{\alpha_R}{\rho_R}$$

$$A_3 = -\frac{7}{3}\frac{\alpha_R v_R}{\rho_R}$$

$$C_3 = \frac{7}{3}\frac{\alpha_R}{\rho_R}$$

$$A_4 = \frac{2\gamma \beta_R (u_R^2 - v_R^2 - E_R)}{\rho_R} - \frac{7}{3}\left(\frac{u_R^2}{\rho_R} + \frac{v_R^2}{\rho_R}\right)$$

$$B_4 = \frac{-2\gamma \beta_R u_R}{\rho_R} + \frac{7}{3}\frac{\alpha_R u_R}{\rho_R} = \frac{u_R}{\rho_R}\left(-2\gamma\beta_R + \frac{7}{3}\alpha_R\right)$$

$$C_4 = -\frac{2\gamma \beta_R v_R}{\rho_R} + \frac{7}{3}\frac{\alpha_R v_R}{\rho_R} = \frac{v_R}{\rho_R}\left(-2\gamma\beta_R + \frac{7}{3}\alpha_R\right)$$

$$D_4 = \frac{2\gamma \beta_R}{\rho_R}$$

RANS Symmetry Flux

As with the Laminar case we will write the symmetry flux as. Albeit with a few extra definitions

$$Sflux = \left(\left[[G_{11}^L]\frac{\partial\phi_i^L}{\partial x} + [G_{21}]^L\frac{\partial\phi_i}{\partial y}\right]n_x + \left[[G_{12}^L]\frac{\partial\phi_i^L}{\partial x} + [G_{22}]\frac{\partial\phi_i}{\partial y}\right]n_y\right)\cdot\{\Delta\vec{q}\}$$

This can we written as a vector of the form
NOTE: the ½ is Not present and is applied extension for B.C. enforcement.

$$Sflux = \begin{Bmatrix} 0 \\ A_2\Delta q_1 + B_2\Delta q_2 + C_2\Delta q_3 \\ A_3\Delta q_1 + B_3\Delta q_2 + C_3\Delta q_3 \\ A_4\Delta q_1 + B_4\Delta q_2 + C_4\Delta q_3 + D_4\Delta q_4 \end{Bmatrix}$$

Defining $\alpha_{L/R} = \left(\mu_{L/R}\frac{M_\infty}{Re_\infty} + \mu_{T L/R}\right)$, $\beta_{L/R} = \left(\frac{\mu_{L/R}}{Pr}\frac{M_\infty}{Re_\infty} + \frac{\mu_{T L/R}}{Pr_T}\right)$

$$A_2 = -\frac{n_x}{\rho_L}\left(\frac{4}{3}\alpha_L U_L\cdot\frac{\partial\phi_i}{\partial x} + \alpha_L V_L\cdot\frac{\partial\phi_i}{\partial y}\right) + \frac{n_y}{\rho_L}\left(\frac{2}{3}\alpha_L V_L\frac{\partial\phi_i}{\partial x} - \alpha_L U_L\cdot\frac{\partial\phi_i}{\partial y}\right)$$

$$= -\frac{\alpha_L}{\rho_L}\left[n_x\left(\frac{4}{3}U_L\frac{\partial\phi_i}{\partial x} + V_L\frac{\partial\phi_i}{\partial y}\right) + n_y\left(\frac{2}{3}V_L\frac{\partial\phi_i}{\partial x} - U_L\frac{\partial\phi_i}{\partial y}\right)\right]$$

$$A_3 = \frac{\alpha_L}{\rho_L}\left[n_x\left(\frac{2}{3}U_L\cdot\frac{\partial\phi_i}{\partial y} - V_L\cdot\frac{\partial\phi_i}{\partial x}\right) - n_y\left(U_L\cdot\frac{\partial\phi_i}{\partial x} + \frac{4}{3}V_L\cdot\frac{\partial\phi_i}{\partial y}\right)\right]$$

$$A_4 = \frac{n_x}{\rho_L}\left(\left[\gamma\beta_L(U_L^2+V_L^2-E_L) + \alpha_L\left(-\frac{4}{3}U_L^2 - V_L^2\right)\right]\frac{\partial\phi_i}{\partial x} - \frac{\alpha_L U_L V_L}{3}\frac{\partial\phi_i}{\partial y}\right) +$$

$$\frac{n_y}{\rho_L}\left(\left[\gamma\beta_L(U_L^2+V_L^2-E_L) + \alpha_L\left(-U_L^2 - \frac{4}{3}V_L^2\right)\right]\cdot\frac{\partial\phi_i}{\partial y} - \frac{\alpha_L U_L V_L}{3}\cdot\frac{\partial\phi_i}{\partial x}\right)$$

$$= \frac{\gamma\beta_L(U_L^2+V_L^2-E_L)}{\rho_L}\left(n_x\frac{\partial\phi_i}{\partial x} + n_y\frac{\partial\phi_i}{\partial y}\right) - \frac{\alpha_L U_L V_L}{3\rho_L}\left(n_x\frac{\partial\phi_i}{\partial y} + n_y\frac{\partial\phi_i}{\partial x}\right)$$

$$\frac{\alpha_L}{\rho_L}\left[n_x\left(-\frac{4}{3}U_L^2 - V_L^2\right)\frac{\partial\phi_i}{\partial x} + n_y\left(-U_L^2 - \frac{4}{3}V_L^2\right)\frac{\partial\phi_i}{\partial y}\right]$$

$$A_4 = \frac{\gamma \beta_L (U_L^2+V_L^2-E_L)}{\rho_L}\left(\frac{\partial \phi_i}{\partial x}\cdot n_x + \frac{\partial \phi_i}{\partial y}\cdot n_y\right) + \frac{\alpha_L}{\rho_L}\left[-\frac{U_L V_L}{3}\left(\frac{\partial \phi_i}{\partial y}\cdot n_x + \frac{\partial \phi_i}{\partial x}\cdot n_y\right)\right.$$

$$\left. U_L^2\left(-\tfrac{4}{3} n_x \frac{\partial \phi_i}{\partial x} - n_y \frac{\partial \phi_i}{\partial y}\right) + V_L^2\left(-n_x \frac{\partial \phi_i}{\partial x} - \tfrac{4}{3} n_y \frac{\partial \phi_i}{\partial y}\right)\right]$$

Note: A_4 has been greatly simplified.

$B_1 = 0$

$$B_2 = \frac{\alpha_L}{\rho_L}\left(\tfrac{4}{3} n_x \frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial y}\cdot n_y\right)$$

$$B_3 = \frac{\alpha_L}{\rho_L}\left(n_y \frac{\partial \phi_i}{\partial x} - \tfrac{2}{3} n_x \frac{\partial \phi_i}{\partial y}\right)$$

$$B_4 = \frac{n_x}{\rho_L}\left(\left[-\gamma \beta_L U_L + \tfrac{4}{3}\alpha_L U_L\right]\frac{\partial \phi_i}{\partial x} - \tfrac{2}{3}\alpha_L V_L \frac{\partial \phi_i}{\partial y}\right) +$$

$$\frac{n_y}{\rho_L}\left(\alpha_L V_L \frac{\partial \phi_i}{\partial x} + \left[-\gamma \beta_L U_L + \alpha_L U_L\right]\frac{\partial \phi_i}{\partial y}\right)$$

$$B_4 = \frac{-\gamma \beta_L U_L}{\rho_L}\left(n_x \frac{\partial \phi_i}{\partial x} + n_y \frac{\partial \phi_i}{\partial y}\right) + \frac{\alpha_L}{\rho_L}\left[V_L\left(-\tfrac{2}{3} n_x \frac{\partial \phi_i}{\partial y} + n_y \frac{\partial \phi_i}{\partial x}\right) + \right.$$

$$\left. U_L\left(\tfrac{4}{3}\frac{\partial \phi_i}{\partial x}\cdot n_x + \frac{\partial \phi_i}{\partial y}\cdot n_y\right)\right]$$

$C_1 = 0$

$$C_2 = \frac{\alpha_L}{\rho_L}\left(n_x \frac{\partial \phi_i}{\partial y} - \tfrac{2}{3}\frac{\partial \phi_i}{\partial x}\cdot n_y\right)$$

$$C_3 = \frac{\alpha_L}{\rho_L}\left(n_x \frac{\partial \phi_i}{\partial x} + \tfrac{4}{3} n_y \frac{\partial \phi_i}{\partial y}\right)$$

$$C_4 = -\frac{\gamma \beta_L V_L}{\rho_L}\left(n_x \frac{\partial \phi_i}{\partial x} + n_y \frac{\partial \phi_i}{\partial y}\right) + \frac{\alpha_L}{\rho_L}\left[V_L\left(\frac{\partial \phi_i}{\partial x}\cdot n_x + \tfrac{4}{3}\frac{\partial \phi_i}{\partial y}\cdot n_y\right) + \right.$$

$$\left. U_L\left(\frac{\partial \phi_i}{\partial y}\cdot n_x - \tfrac{2}{3} n_y \frac{\partial \phi_i}{\partial x}\right)\right]$$

$$D_4 = \frac{\gamma \beta_L}{\rho_L}\left(\frac{\partial \phi_i}{\partial x}\cdot n_x + n_y \frac{\partial \phi_i}{\partial y}\right)$$

Note that these terms have been re-written to make linearization a bit more straight forward.

S.A Diffusion term Linearization:

The diffusion term (volume integral) is written as.

$$F_{v_x} = \frac{1}{\sigma}\left(\mu \cdot \frac{M_\infty}{Re_\infty} + (\rho\tilde{\nu})\frac{M_\infty}{Re_\infty}\right)\left(-\frac{\tilde{\nu}}{\rho}\frac{d\rho}{dx} + \frac{1}{\rho}\frac{d(\rho\tilde{\nu})}{dx}\right)$$

$$F_{v_y} = \frac{1}{\sigma}\left(\mu \frac{M_\infty}{Re_\infty} + (\rho\tilde{\nu})\frac{M_\infty}{Re_\infty}\right)\left(-\frac{\tilde{\nu}}{\rho}\frac{d\rho}{dy} + \frac{1}{\rho}\frac{d(\rho\tilde{\nu})}{dy}\right)$$

We need. $G_{11}, G_{12}, G_{21}, G_{22}, \frac{dF_{vx}}{d\vec{q}}, \frac{dF_{vy}}{d\vec{q}}$. We also note that the above applies to both volume and surface integrals b/c the surface is simply the average of the volume and surface.

$$G_{11} = \frac{1}{\sigma}\left(\mu \frac{M_\infty}{Re_\infty} + (\rho\tilde{\nu})\frac{M_\infty}{Re_\infty}\right)\left\lfloor -\frac{\tilde{\nu}}{\rho}, 0, 0, 0, \frac{1}{\rho}\right\rfloor$$

$$G_{12} = \lfloor 0 \rfloor$$

$$G_{21} = \lfloor 0 \rfloor$$

$$G_{22} = \frac{1}{\sigma}\left(\mu \frac{M_\infty}{Re_\infty} + (\rho\tilde{\nu})\frac{M_\infty}{Re_\infty}\right)\left\lfloor -\frac{\tilde{\nu}}{\rho}, 0, 0, 0, \frac{1}{\rho}\right\rfloor$$

$$\frac{\partial F_{vx}}{\partial \vec{q}} = \frac{1}{\sigma}\left(\frac{d\mu}{d\vec{q}} \cdot \frac{M_\infty}{Re_\infty} + \lfloor 0,0,0,0,1\rfloor \cdot \frac{M_\infty}{Re_\infty}\right)\left(-\frac{\tilde{\nu}}{\rho}\frac{d\rho}{dx} + \frac{1}{\rho}\frac{d(\rho\tilde{\nu})}{dx}\right) +$$
$$\frac{1}{\sigma}\left(\mu \frac{M_\infty}{Re_\infty} + (\rho\tilde{\nu})\frac{M_\infty}{Re_\infty}\right)\left(\left\lfloor 2\frac{\tilde{\nu}}{\rho^2}, 0, 0, 0, \frac{-1}{\rho^2}\right\rfloor \frac{d\rho}{dx} - \frac{1}{\rho^2}\lfloor 1,0,0,0,0\rfloor \frac{d(\rho\tilde{\nu})}{dx}\right)$$

$$\frac{dF_{vy}}{d\vec{q}} = \frac{1}{\sigma}\left(\frac{d\mu}{d\vec{q}} \cdot \frac{M_\infty}{Re_\infty} + \lfloor 0,0,0,0,1\rfloor \cdot \frac{M_\infty}{Re_\infty}\right)\left(-\frac{\tilde{\nu}}{\rho}\frac{d\rho}{dy} + \frac{1}{\rho}\frac{d(\rho\tilde{\nu})}{dy}\right) +$$
$$\frac{1}{\sigma}\left(\mu \frac{M_\infty}{Re_\infty} + (\rho\tilde{\nu})\frac{M_\infty}{Re_\infty}\right)\left(\left\lfloor 2\frac{\tilde{\nu}}{\rho^2}, 0, 0, 0, \frac{-1}{\rho^2}\right\rfloor \frac{\partial\rho}{dy} - \frac{1}{\rho^2}\lfloor 1,0,0,0,0\rfloor \frac{d(\rho\tilde{\nu})}{dy}\right)$$

Shock Capturing with Artificial Viscosity:

Consider a general system of Hyperbolic conservation laws given as

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0 \quad (1)$$

If at any point in the solution the characteristics cross a "shock" will form. When (1) is not purely hyperbolic

shocks begin thin layers across which the solution varies rapidly but smoothly. Thus we will add an artificial model for a diffusion type term that will provide this smoothing of the solution. This is the idea behind artificial or shock_viscosity. We will use the term shock-viscosity so as to distinguish between this and artificial dissipation profided by Riemann solvers or other types of flux flns.

We will begin by using a simple diffusion model that is based on the Laplacian of the solution giving the modified equation in (1) as.

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = \nabla \cdot (r \nabla U) \quad (2), \quad \text{where } r = \varepsilon \cdot \lambda_{max}, \ \lambda_{max} = |U n| + c$$

where ε is the viscosity parameter. To keep shocks "sharp" or as "sharp" as possible ε should scale with $O(h/p)$ which is a measure of the available resolution of the D.G. scheme at the shock. We want the shock thickness to be $O(\varepsilon)$ which implies $O(\varepsilon) = O(\frac{h}{p}) \Rightarrow$
$\varepsilon \sim h/p$

When we are away from the shock we do not want to have any shock viscosity. Thus we want $\varepsilon = 0$ in smooth regions. Thus we need a way to sense where we have discontinuities. For this we will use Persson and Paire's resolution dector.

$$S_e = \frac{\langle v-\hat{v}, v-\hat{v} \rangle_{L_2}}{\langle v, v \rangle_{L_2}} \quad (3) \qquad \hat{v} = \sum_{j=1}^{N_p-1} \bar{v}_j \phi_j$$

We have used this in the past for ~~de~~ smoothness detection " .

When adding shock-viscosity we do not want to just ad-hocly add stuff. The more non-smooth the soln the more we want to add. Thus we will use the value of S_e to guide when to add viscosity. Following Persson and Pairre, we will do the following.

define $\quad s_e = \log_{10}(S_e)$

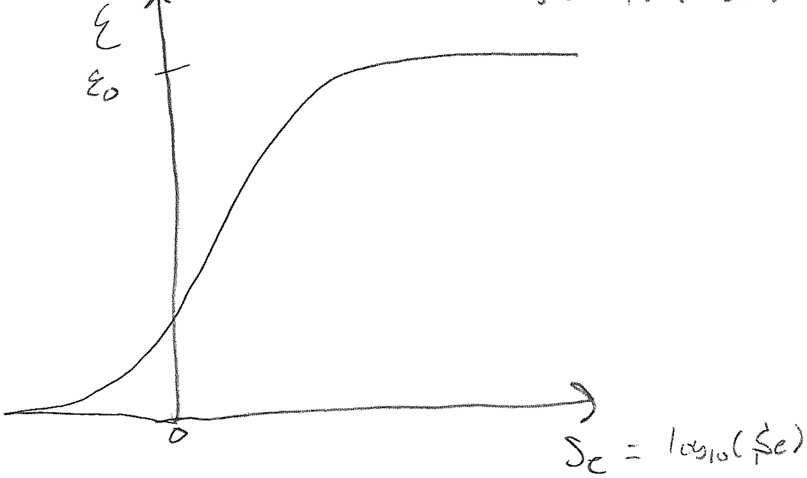
We will then use the following smooth function

$$\varepsilon = \begin{cases} 0 & \text{if } s_e < s_0 - K \\ \frac{\varepsilon_0}{2}\left(1+\sin\frac{\pi(s_e - s_0)}{2K}\right) & s_0 - K \le s_e \le s_0 + K \\ \varepsilon_0 & \text{if } s_e > s_0 + K \end{cases}$$

This function is a smooth f'n of the solution through the value s_e

The value of S_e will be done with $v = \rho \equiv$ density because this has worked well in the past.

We just do a quick plot of ε vs. S_e to verify using $K = 4$, $\varepsilon_0 = 1$. which came from Prof. Andy Shelton (A.E., Auburn) who has used this with considerable success.



We originally wanted $\varepsilon \sim O(\frac{h}{p})$, and $S_0 \sim O(\frac{1}{p^4})$ which gives

$$\varepsilon_0 = C_\varepsilon \frac{h}{p} \quad \text{and} \quad S_0 = \log_{10}\left(\frac{C_{S_0}}{p^4}\right) \quad \text{with } C_\varepsilon = \tfrac{1}{2},\ C_{S_0} = 1.0$$

With these values we can get an idea of how this function behaves. This value of artificial viscosity will be a constant each cell.

A better way to compute this is to follow Andy's code

$$\varepsilon = \frac{\varepsilon_0}{2}\left(1 + \sin\left(\tfrac{1}{2}\pi z\right)\right)$$

with $z = \min\left(1, \max\left(-1, \frac{S_e - S_0}{K}\right)\right)$.

The smoothness of the fn should make it easy to amenable to Linearization.

The hardest thing to linearize is $S_e = \log_{10}(\mathcal{S}_e)$

$$\mathcal{S}_e = \frac{\int_{\Omega_K} (\upsilon - \hat{\upsilon})^2 \, d\Omega_K}{\int_{\Omega_K} \upsilon^2 \, d\Omega_K}$$

$$\frac{\partial \mathcal{S}_e}{\partial \bar{\upsilon}_j} = \frac{\partial \mathcal{S}_e}{\partial \upsilon} \cdot \frac{\partial \upsilon}{\partial \bar{\upsilon}_j} + \frac{\partial \mathcal{S}_e}{\partial \hat{\upsilon}} \cdot \frac{\partial \hat{\upsilon}}{\partial \bar{\upsilon}_j}$$

$$\frac{\partial \mathcal{S}}{\partial \upsilon} = \frac{\int_{\Omega_K} 2(\upsilon - \hat{\upsilon}) \, d\Omega_K}{\upsilon^2} - 2 \cdot \frac{\int_{\Omega_K} (\upsilon - \hat{\upsilon})^2 d\Omega_K \cdot \int \upsilon \, d\Omega_K}{\left(\int_{\Omega_K} \upsilon \, d\Omega_K\right)^2}$$

$$\frac{\partial \mathcal{S}}{\partial \hat{\upsilon}} = \frac{-\int_{\Omega_K} 2(\upsilon - \hat{\upsilon}) \, d\Omega_K}{\int_{\Omega_K} \upsilon^2 \, d\Omega_K}$$

We will compute these and store them to facilitate the jacobian construction.

Now that we have a form for ε we can proceede to discretize our new equation.

$$\frac{\partial \upsilon_h}{\partial t} + \nabla \cdot F(\upsilon_h) - \nabla(\varepsilon \nabla \upsilon_h) = 0$$

D.G. discretization:

$$\int_{\Omega_K} \phi_i \frac{\partial \upsilon_h}{\partial t} + \phi_i \nabla \cdot \vec{F}(u_h) - \phi_i \nabla(\varepsilon \nabla u_h) \, d\Omega_K = 0$$

I.B.P.

$$\int \phi_i \frac{\partial u_h}{\partial t} \, d\Omega_K + \int_{\Omega_K} -\nabla\phi_i \cdot \vec{F}(u_h) + \nabla\phi_i \cdot \varepsilon \nabla u_h \, d\Omega_K + \oint \phi_i F^*(u_h) \cdot \vec{n} - (\phi_i \nabla u_h \cdot \vec{n})^* \, dS = 0$$

For the actual Diffusion term, we need to have the appropriate scaling of the diffusion terms.

We have so far derived and equation for $\varepsilon|_e$ in each element.

$$\varepsilon|_e = \begin{cases} 0 & S_e \leq S_0 - K \\ \frac{\varepsilon_0}{2}\left(1 + \sin\left(\frac{\pi(S_e - S_0)}{2K}\right)\right) & S_0 - K < S_e < S_0 + K \\ \varepsilon_0 & S_e \geq S_0 + K \end{cases}$$

with $\varepsilon_0 \sim \frac{h}{p}$
$S_0 \sim \log_{10}\left(\frac{1}{p^4}\right)$

Thus it is easily seen that $\varepsilon|_e$ has units of L i.e. $\varepsilon|_e \sim L$. Thus the statement in persson06 paper that $D = \varepsilon \nabla \vec{q}$ is incorrect. It has units of $\vec{q}$ which is not consistent with the rest of the equation.

To fix this consider diffusion terms in Riemman interface fluxes. i.e. $d \sim |\lambda|(\vec{q}_L - \vec{q}_R)$ thus we see that all we need to do is define dissipation as

$$D \sim \varepsilon|\lambda| \nabla \vec{q}$$

To determine $|\lambda|$ we will need something that is consistent with the volume and surface terms.

Here we define $|\lambda| = |U| + |V| + c$, which is a more conservative estimate than $|U\cdot\vec{n}| + c$ which is the standard flux function. Note here that $\vec{n}$ is a unit vector and thus there is no inconsistency in units between $|U\cdot\vec{n}|+c$ and $|U|+|V|+c$.

Thus for the Artificial viscosity system we will preserve total enthalpy by taking the following

For $\frac{\partial \vec{q}}{\partial t} + \nabla\cdot(\vec{F}_c) - \nabla(\nu\nabla w) = 0$ where

$$w = \begin{Bmatrix} \rho \\ \rho U \\ \rho V \\ \rho H \end{Bmatrix}$$ and $\nu = \varepsilon\cdot|\lambda|$ defined above.

Thus the D.G discretization becomes

$$\int \frac{\partial \vec{q}}{\partial t}\phi_i \, d\Omega - \int \nabla\phi_i\cdot\left(\vec{F}_c - \varepsilon|\lambda|\nabla w\right) d\Omega + \oint \phi_i\left(H_c^{*}\cdot\vec{n}\right) - \overset{\text{Avg.}}{\left\{\varepsilon|\lambda|\,\nabla\vec{w}\cdot n_x\right\}}\phi_i$$

$$- \left\{ \begin{bmatrix} [G_{11}] & [G_{12}] \\ [G_{21}] & [G_{22}] \end{bmatrix} \begin{Bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{Bmatrix} \cdot \vec{n}\cdot(\vec{w}_L - w_R) \right\} + pen\left\{ [G_{ii}]\cdot \right\} (\vec{w}_L - \vec{w}_R)\phi_i \, dS$$

where $H_c^{*}\cdot\vec{n}$ should be an enthalpy preserving flux, and ε is defined by the function in the previous section.

As usual we will now explicitly write out each term here we omit the convective fluxes because those have been previously documented.

1). Volume term.

$$\int_\Omega \nabla\phi_i(\vec{F}_c - \varepsilon|\lambda|\nabla\vec{w})\,d\Omega = \int_\Omega \nabla\phi_i(\vec{F}_c - (E_{av}, F_{av}))\,d\Omega$$

which implies.

$$E_{av} = \varepsilon|\lambda|\frac{\partial\vec{w}}{\partial x}, \quad F_{av} = \varepsilon|\lambda|\frac{\partial\vec{w}}{\partial y}$$

with $\frac{\partial\vec{w}}{\partial x} = \frac{\partial w}{\partial \vec{q}}\cdot\frac{\partial\vec{q}}{\partial x}$, $\frac{\partial\vec{w}}{\partial y} = \frac{\partial\vec{w}}{\partial\vec{q}}\cdot\frac{\partial\vec{q}}{\partial y} \Rightarrow$ $E_{av} = \varepsilon|\lambda|\frac{\partial\vec{w}}{\partial\vec{q}}\cdot\frac{\partial\vec{q}}{\partial x}$, $F_{av} = \varepsilon|\lambda|\frac{\partial\vec{w}}{\partial\vec{q}}\cdot\frac{\partial\vec{q}}{\partial y}$

Thus we make our usual definitions for $G_{11}^{av}, G_{12}^{av}, G_{21}^{av}, G_{22}^{av}$

$$\underbrace{\frac{\partial E_{av}}{\partial q_x} = \varepsilon|\lambda|\frac{\partial\vec{w}}{\partial\vec{q}}}_{G_{11}}, \quad \underbrace{\frac{\partial E_{av}}{\partial q_y} = [0]}_{G_{12}}, \quad \underbrace{\frac{\partial F_{av}}{\partial q_x} = [0]}_{G_{21}}, \quad \underbrace{\frac{\partial F_{av}}{\partial q_y} = \varepsilon|\lambda|\frac{\partial\vec{w}}{\partial\vec{q}}}_{G_{22}}$$

with $w = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho H \end{Bmatrix}$ we get

$$\frac{\partial w}{\partial\vec{q}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\partial(\rho H)}{\partial\rho} & \frac{\partial(\rho H)}{\partial(\rho u)} & \frac{\partial(\rho H)}{\partial(\rho v)} & \frac{\partial(\rho H)}{\partial(\rho E)} \end{bmatrix}$$

given that $H = E + \frac{p}{\rho}$

$$\rho H = \rho E + p = \rho E + (\gamma-1)(\rho E) - \tfrac{1}{2}\frac{(\gamma-1)}{\rho}\left(\frac{(\rho u)^2}{\cancel{\rho}} + \frac{(\rho v)^2}{\cancel{\rho}}\right)$$

$$\frac{\partial(\rho H)}{\partial\rho} = +\tfrac{1}{2}\frac{(\gamma-1)}{\rho^2}\left((\rho u)^2 + (\rho v)^2\right) = \tfrac{1}{2}(\gamma-1)(u^2+v^2)$$

$$\frac{\partial}{\partial\rho u} = -\tfrac{1}{2}\frac{(\gamma-1)}{\cancel{\rho}}(\cancel{2}\rho u = -(\gamma-1)u, \quad \frac{\partial\rho H}{\partial\rho v} = -\tfrac{1}{2}(\gamma-1)v, \quad \frac{\partial\rho H}{\partial\rho E} = 1 + (\gamma-1) = \gamma$$

gives.

$$\frac{\partial w}{\partial \vec{q}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{(\gamma-1)}{2}(u^2+v^2) & -u(\gamma-1) & -v(\gamma-1) & \gamma \end{bmatrix}$$

gives

$$G_{11} = G_{22} = \varepsilon|\lambda| \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{(\gamma-1)}{2}(u^2+v^2) & -u(\gamma-1) & -v(\gamma-1) & \gamma \end{bmatrix}$$

Surface terms:

a) $\{E_{av} + F_{av}\}$ - straight forward

b) Symmetry term: For a general G_{ij} is given as

$$\frac{1}{2}\left[\left([G_{11}]\frac{\partial \phi_i}{\partial x} + [G_{21}]\frac{\partial \phi_i}{\partial y}\right)n_x \left([G_{12}]\frac{\partial \phi_i}{\partial x} + [G_{22}]\frac{\partial \phi_i}{\partial y}\right) n_y\right](\Delta \vec{q}_i)$$

For A.V. $G_{21} = G_{12} = [0]$ gives

$$\frac{1}{2}\left([G_{11}]\frac{\partial \phi_i}{\partial x}\cdot n_x + [G_{22}]\frac{\partial \phi_i}{\partial y}\cdot n_y\right)(\Delta \vec{q})$$, now $G_{11} = G_{22}$ gives

Using $[G_{11}] = [G_{22}] \equiv [G]$ we have.

$$\frac{1}{2}\left([G]\cdot\left(\frac{\partial \phi_i}{\partial x}\cdot n_x + \frac{\partial \phi_i}{\partial y}\cdot n_y\right)\cdot(\Delta \vec{q})\right)$$

Explicitly writing out the flux

$$Sflux = \frac{1}{2}\begin{Bmatrix} \varepsilon|\lambda|\Delta\rho \\ \varepsilon|\lambda|\Delta\rho u \\ \varepsilon|\lambda|\Delta\rho v \\ \varepsilon|\lambda|\left[\frac{(\gamma-1)}{2}(u^2+v^2)\Delta\rho - u(\gamma-1)\Delta\rho u - v(\gamma-1)\Delta\rho v + \gamma\Delta\rho E\right] \end{Bmatrix}\left(\frac{\partial \phi_i}{\partial x}\cdot n_x + \frac{\partial \phi_i}{\partial y}\cdot n_y\right).$$

$$= \frac{1}{2}\begin{Bmatrix} \varepsilon|\lambda|\Delta\rho \\ \varepsilon|\lambda|\Delta\rho u \\ \varepsilon|\lambda|\Delta\rho v \\ (A_4\Delta\rho + B_4\Delta\rho u + C_4\Delta\rho v + D_4\Delta\rho E) \end{Bmatrix}$$

with
$A_4 = \varepsilon|\lambda|\frac{(\gamma-1)}{2}(\vec{q}^2/2)$
$B_4 = \varepsilon|\lambda|(-u(\gamma-1))$
$C_4 = \varepsilon|\lambda|(-v(\gamma-1))$
$D_4 = \varepsilon|\lambda|\gamma$

c) Penalty flux

$$\text{pen}\{G_{ii}\}(\Delta\vec{q}) = \frac{1}{2}\left[G_{11}^L + G_{11}^R + G_{22}^L + G_{22}^R\right](\Delta\vec{q}) =$$

$$= \text{pen}\left(\varepsilon_L|\lambda_L|\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma-1}{2}(u_L^2+v_L^2) & -u_L(\gamma-1) & -v_L(\gamma-1) & \gamma \end{bmatrix} + \varepsilon_R|\lambda_R|\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma-1}{2}(u_R^2+v_R^2) & -u_R(\gamma-1) & -v_R(\gamma-1) & \gamma \end{bmatrix}\right)\begin{Bmatrix} \Delta\rho \\ \Delta\rho u \\ \Delta\rho v \\ \Delta\rho E \end{Bmatrix}$$

We can write the flux down explicitly.

$$= \text{pen}\begin{Bmatrix} (\varepsilon_L|\lambda_L| + \varepsilon_R|\lambda_R|)\,\Delta\rho \\ (\varepsilon_L|\lambda_L| + \varepsilon_R|\lambda_R|)\,\Delta\rho u \\ (\varepsilon_L|\lambda| + \varepsilon_R|\lambda_R|)\,\Delta\rho v \\ A_4\Delta\rho + B_4\Delta\rho u + C_4\Delta\rho v + D_4\Delta\rho E \end{Bmatrix}$$

with

$$A_4 = \varepsilon_L|\lambda_L|\left(\frac{\gamma-1}{2}(u_L^2+v_L^2)\right) + \varepsilon_R|\lambda_R|\left(\frac{\gamma-1}{2}(u_R^2+v_R^2)\right)$$

$$B_4 = -\left[\varepsilon_L|\lambda_L|\,u_L(\gamma-1) + \varepsilon_R|\lambda_R|\,u_R(\gamma-1)\right]$$

$$C_4 = -\left[\varepsilon_L|\lambda_L|\,v_L(\gamma+1) + \varepsilon_R|\lambda_R|\,v_R(\gamma-1)\right]$$

$$D_4 = (\varepsilon_L|\lambda_L| + \varepsilon_R|\lambda_R|)\gamma$$

Implementation of Indicator Linearization:

$$S_e = \int \frac{(U-\hat{U})^2 \, d\Omega_k}{\int U^2 \, d\Omega_k}$$

Assuming that U is some field variable then we only need to store 1 row that is nmode's long.

Given, expansions of the form $U = \sum_{j=1}^{N_p} \bar{U}_j \varphi_j$, $\hat{U} = \sum_{j=1}^{N_{p-1}} \bar{U}_j \varphi_j$

Then
$$\frac{\partial S_e}{\partial \bar{U}_j} = \frac{\int 2(U-\hat{U}) \frac{\partial U}{\partial \bar{U}_j} d\Omega_k}{\int (U)^2 d\Omega_k} - \frac{\int (U-\hat{U})^2 d\Omega_k \int 2U \frac{\partial U}{\partial \bar{U}_j} d\Omega_k}{\left[\int (U)^2 d\Omega_k\right]^2}$$
$$- \frac{\int 2(U-\hat{U}) \frac{\partial \hat{U}}{\partial \bar{U}_j} d\Omega_k}{\int (U)^2 d\Omega_k}$$

Recognizing that for $j \geq N_{p-1}$ $\frac{\partial \hat{U}}{\partial \bar{U}_j} = 0$ b/c it's not in the expansion then we 2 possibilities.

for $j \leq N_{p-1}$ we know that $\frac{\partial U}{\partial \bar{U}_j} = \frac{\partial \hat{U}}{\partial \bar{U}_j}$; same terms

thus
$$\frac{\partial S_e}{\partial \bar{U}_j} = - \frac{\int (U-\hat{U})^2 d\Omega_k \int 2U \varphi_j d\Omega_k}{\left[\int (U)^2 d\Omega_k\right]^2}$$

for $j > N_{p-1}$ $\frac{\partial \hat{U}}{\partial \bar{U}_j} = 0$ giving

$$\frac{\partial S_e}{\partial \bar{U}_j} = \frac{\int 2(U-\hat{U}) \varphi_j \, d\Omega_k}{\int (U)^2 d\Omega_k} - \frac{\int (U-\hat{U})^2 d\Omega_k \int 2U \varphi_j \, d\Omega_k}{\left[\int (U)^2 d\Omega_k\right]^2}$$

In the code we make the following definition

numer $= \int (U-\hat{U})^2 d\Omega_{\ell\ell}$

denom $= \int (U)^2 d\Omega_{\ell\ell}$

numer1(j) $= \int \partial U \phi_j d\Omega_{\ell\ell}$

numer2(j) $= \int \partial (U-\hat{U}) \phi_j d\Omega_{\ell\ell}$

We do the following loops

for j = 1, N_{p-1}

$$\frac{\partial S}{\partial \bar{U}_j} = -\frac{\text{numer}}{(\text{denom})^2} \cdot \text{numer1(j)}$$

end

for j = $N_{p-1}+1$, N_p

$$\frac{\partial S}{\partial \bar{U}_j} = \frac{\text{numer2(j)}}{\text{denom}} - \frac{\text{numer} \cdot \text{numer1(j)}}{(\text{denom})^2}$$

Linearization of Shock-Viscosity

- All terms in the Laplacian shock viscosity are formed as $f(\nabla\vec{q}, \vec{q}, \varepsilon)$. Now ε is a constant over each cell but also has dependance on the nodes in it's cell by relation $\varepsilon = \varepsilon(S_e)$, where

$$S' = \frac{\langle \rho - \rho_h, \rho - \rho_h \rangle}{\langle \rho, \rho \rangle}$$

Thus we will linear all Laplacian Artificial diffusion terms as

$$\frac{\partial f}{\partial \hat{q}_j} = \frac{\partial f}{\partial q_x}\frac{\partial \phi_j}{\partial x} + \frac{\partial f}{\partial q_y}\frac{\partial \phi_j}{\partial y} + \frac{\partial f}{\partial q}\phi_j + \frac{\partial f}{\partial \varepsilon}\cdot\frac{\partial \varepsilon}{\partial S_e}\cdot\frac{\partial S_e}{\partial S'_e}\frac{\partial S'}{\partial \hat{q}_j}$$

Further given as $S_e = \log_{10}(S'_e) = \dfrac{\ln(S'_e)}{\ln(10)}$

$$= \frac{\partial f}{\partial q_x}\cdot\frac{\partial \phi_j}{\partial x} + \frac{\partial f}{\partial q_y}\frac{\partial \phi_j}{\partial y} + \frac{\partial f}{\partial q}\phi_j + \frac{\partial f}{\partial \varepsilon}\cdot\frac{\partial \varepsilon}{\partial S_e}\cdot\frac{1}{S'_e \ln(10)}\cdot\frac{\partial S'}{\partial \hat{q}_j}$$

Where the term $\dfrac{\partial S'}{\partial \hat{q}_j}$ was shown in the previous page

<u>Remark:</u> For sflux, pflux the terms $\dfrac{\partial f}{\partial q_x}$ and $\dfrac{\partial f}{\partial q_y}$ are simply zero (0) because these fluxes are functions of the state variables and ε only.

U^+, y^+ - Non-dimensionalization

From turbulence theory $U_\tau = \sqrt{\frac{\tau_w}{\rho}}$, $U^+ = \frac{U}{U_\tau}$

Using our non-dimensional notation

$$\bar{U}_\tau = U_\tau / a_\infty \quad , \quad \bar{\tau}_w = \frac{L_\infty \tau_w}{a_\infty \mu_\infty} \quad \bar{\rho} = \rho/\rho_\infty$$

gives

$$a_\infty \bar{U}_\tau = \sqrt{\frac{a_\infty}{\rho_\infty L_\infty} \mu_\infty \frac{\bar{\tau}_w}{\bar{\rho}}}$$

$$\bar{U}_\tau = \frac{1}{a_\infty} \sqrt{\frac{a_\infty}{\rho_\infty L_\infty} \mu_\infty \frac{\bar{\tau}_w}{\bar{\rho}}}$$

$$\bar{U}_\tau = \sqrt{\frac{a_\infty \mu_\infty}{\rho_\infty L_\infty a_\infty^2} \frac{\bar{\tau}_w}{\bar{\rho}}} = \sqrt{\frac{M_\infty}{Re_\infty} \frac{\bar{\tau}_w}{\bar{\rho}}}$$

$$\bar{U}_\tau = \sqrt{\frac{M_\infty}{Re_\infty}} \sqrt{\frac{\bar{\tau}_w}{\bar{\rho}}}$$

$$U^+ = \frac{\bar{U}}{\bar{U}_\tau} = \frac{\bar{U}}{\sqrt{\frac{\bar{\tau}_w}{\bar{\rho}}}} \sqrt{\frac{Re_\infty}{M_\infty}}$$

$$y^+ = \frac{\bar{y}\, \bar{U}_\tau\, a_\infty \cdot L_\infty}{\bar{\nu} \cdot \nu_\infty}$$

$$y^+ = \frac{L_\infty a_\infty}{\nu_\infty} \cdot \frac{\bar{y}\, \bar{U}_\tau}{\bar{\nu}} = \frac{L_\infty U_\infty \rho_\infty}{M_\infty \mu_\infty} \frac{\bar{y}\, \bar{U}_\tau}{\bar{\nu}} = \frac{Re_\infty}{M_\infty} \frac{\bar{y}\, \bar{U}_\tau}{\bar{\nu}}$$

$$U^+ = \frac{\bar{U}}{\bar{U}_\tau} \quad , \quad y^+ = \frac{Re_\infty}{M_\infty} \frac{\bar{y}\, \bar{U}_\tau}{\bar{\nu}} \qquad \bar{U}_\tau = \sqrt{\frac{M_\infty}{Re_\infty} \frac{\bar{\tau}_w}{\bar{\rho}}}$$

Spalart - Allmaras Model Modifications:

The following are modifications are designed to improve the behavior of S.A. turbulence model. The analysis was done by Todd Oliver (MIT) and David Darmofal (MIT) in T. Oliver's PhD Thesis. Here we simply seek to understand it fully by redoing the analysis using the thesis to compare results.

1) C_1 - Continous production term this was given to the MIT group in personal communication with Steve Allmaras at Boeing.

$$\tilde{S} = S + \bar{S} \qquad \text{where } \bar{S} = \frac{\tilde{\nu}^2 f_{v_2}}{\kappa^2 d^2}$$

is replaced by

$$\tilde{S} = \begin{cases} S + \bar{S} & \bar{S} \geq -c_{v_2} S \\ S + \dfrac{S\,(c_{v_2}^2 S + c_{v_3}\bar{S})}{(c_{v_3} - 2c_{v_2})S - \bar{S}} & \bar{S} < -c_{v_2} S \end{cases}$$

To study this. pull out a factor of S

$$\tilde{S} = S \begin{cases} 1 + \bar{S}/S & \bar{S}/S \geq -c_{v_2} \\ 1 + \dfrac{\cancel{S}\,(c_{v_2}^2 + c_{v_3}\,\bar{S}/S)}{\cancel{S}\,[(c_{v_3} - 2c_{v_2})1 - \bar{S}/S]} & \bar{S}/S < -c_{v_2} \end{cases}$$

$$\Rightarrow \frac{\tilde{S}}{S} = \begin{cases} 1 + \bar{S}/S & \bar{S}/S \geq -c_{v_2} \\ 1 + \dfrac{(c_{v_2}^2 + c_{v_3}\,\bar{S}/S)}{[(c_{v_3} - 2c_{v_2}) - \bar{S}/S]} & \bar{S}/S < -c_{v_2} \end{cases}$$

This type of manipulation is reasonable since $\bar{S}$ is the only thing that can be negative. Thus if we analyze $\tilde{S}/S$ vs. $\bar{S}/S$ we'll see behavior of $\tilde{S}$ as fn of $\tilde{\nu}$ in $\bar{S}$.

$$\frac{\tilde{S}}{S} = \begin{cases} 1 + \bar{S}/S & \bar{S}/S \geq -CV_2 \\ 1 + \dfrac{(CV_2^2 + CV_3\, \bar{S}/S)}{(CV_3 - 2CV_2) - \bar{S}/S} & \bar{S}/S < -CV_2 \end{cases}$$

Defining $\Gamma = \bar{S}/S$ we have

$$\tilde{S}/S = \begin{cases} 1 + \Gamma & \Gamma \geq -CV_2 \\ 1 + \dfrac{(CV_2^2 + CV_3 \Gamma)}{(CV_3 - 2CV_2) - \Gamma} & \Gamma < -CV_2 \end{cases}$$

Further we can define Γ as fn of $X = \frac{\tilde{\nu}}{\nu}$

plot this.

$$\bar{S} = \frac{\nu X f_{v2}(f_{v1}(X), X)}{k^2 d^2} \quad \text{this has units } \frac{1}{s}$$

$$\frac{\bar{S}}{S} = \frac{\nu X f_{v_2}}{k^2 d^2 S} \qquad \frac{\nu}{k^2 d^2 S} \text{ is non-dimensional} \qquad \frac{\cancel{m^2}}{\cancel{s}\, \cancel{m^2}}\frac{1}{s} = 1,$$

since K is a fixed non-dimensional const. the parameter here known as $\sigma = \frac{\nu}{d^2 S}$

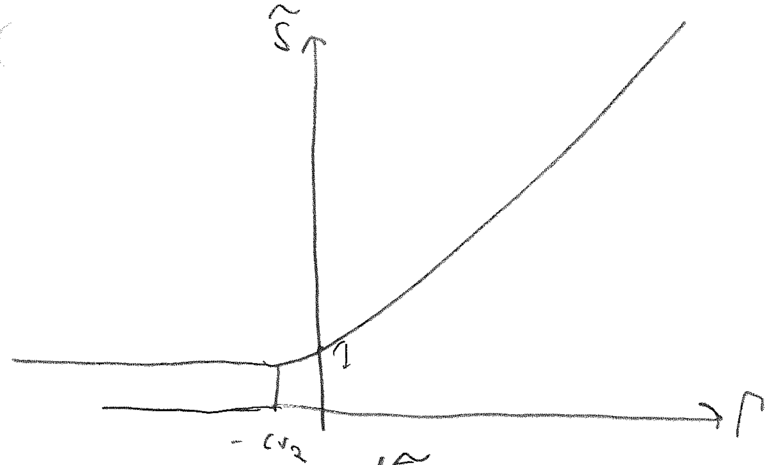
Thus $\Gamma(X, \sigma)$ is a family of curves in X defined by σ values. $\sigma \in [0, C]$ where C is a finite Large value.

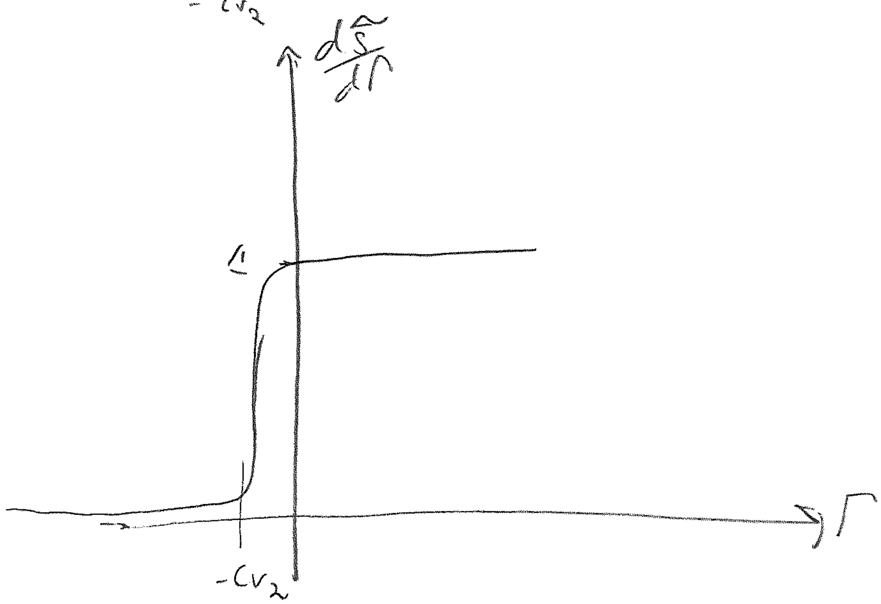
$$\Gamma = \sigma X f_{v_2}(f_{v_1}(X), X)/k^2$$

This gives $$\frac{\tilde{S}}{S} = \begin{cases} 1 + \Gamma & \Gamma \geq -CV_2 \\ 1 + \dfrac{(CV_2^2 + CV_3 \Gamma)}{(CV_3 - 2CV_2) - \Gamma} & \Gamma < -CV_2 \end{cases}$$

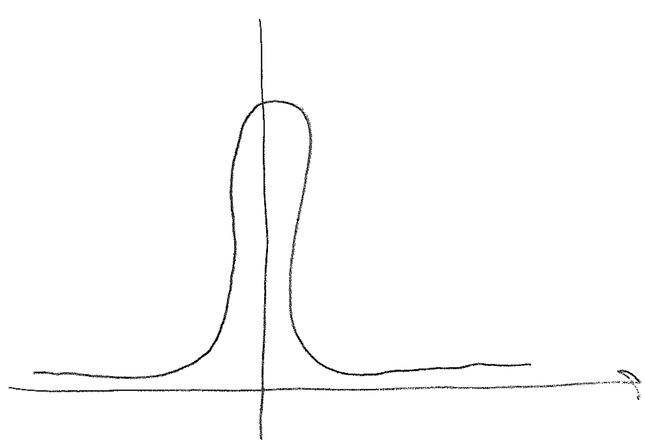
where $\Gamma = \sigma X f_{v_2}/k^2$, $f_{v_1} = \dfrac{X^3}{X^3 + C_{v_1}^3}$, $f_{v_2} = 1 - \dfrac{X}{1 + X f_{v_1}}$

Using the Maple sheet sa_source.mw we have the following plots





More Importantly $\tilde{S}$ vs χ is given by



giving a product that for $-\tilde{\nu}$ will always be negative thus allowing other terms to compensate.

$$\frac{P}{S\nu} = \chi \tilde{S} C_{b1}$$

Now that we have a gaurentee of $\tilde{S} > 0$ we can move onto fixing all the terms P, D, Diffusion etc to ensure bounded turbulent variable energy $\tilde{e}_\nu = \frac{1}{2}\tilde{\nu}^2$

Beginning with.

$$\frac{\partial}{\partial t}(\rho\tilde{\nu}) + \frac{\partial}{\partial x_j}(\rho u_j \tilde{\nu}) = Cb_1 \tilde{S}\rho\tilde{\nu} + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}\left((\mu+\rho\tilde{\nu})\frac{\partial\tilde{\nu}}{\partial x_j}\right) + Cb_2\rho\frac{\partial\tilde{\nu}}{\partial x_j}\frac{\partial\tilde{\nu}}{\partial x_j}\right]$$
$$- Cw_1 fw\left(\rho\frac{\tilde{\nu}^2}{d^2}\right)$$

Multiply by $\tilde{\nu}$ gives

$$\frac{\partial}{\partial t}(\rho \tfrac{1}{2}\tilde{\nu}^2) + \frac{\partial}{\partial x_j}(\rho u_j \tfrac{1}{2}\tilde{\nu}^2) = Cb_1\tilde{S}\rho\tilde{\nu}^2 + \frac{1}{\sigma}\left[\tilde{\nu}\frac{\partial}{\partial x_j}\left((\mu+\rho\tilde{\nu})\frac{\partial\tilde{\nu}}{\partial x_j}\right) + Cb_2\rho\tilde{\nu}\frac{\partial\tilde{\nu}}{\partial x_j}\frac{\partial\tilde{\nu}}{\partial x_j}\right]$$
$$- Cw fw\ \rho\left(\frac{\tilde{\nu}^2}{d^2}\right)\tilde{\nu} \quad (2)$$

We need to work out diffusion:

$\tilde{\nu}\frac{\partial}{\partial x_j}\left[(\mu+\rho\tilde{\nu})\frac{\partial\tilde{\nu}}{\partial x_j}\right]$ Define $\mu+\rho\tilde{\nu} = \eta$, $\tilde{\nu}\frac{\partial}{\partial x_j}\left(\eta\frac{\partial\tilde{\nu}}{\partial x_j}\right)$

If we had $\frac{\partial}{\partial x_j}\left(\tilde{\nu}\eta\frac{\partial\tilde{\nu}}{\partial x_j}\right)$ if we be

$$\frac{\partial}{\partial x_j}\left(\tilde{\nu}\eta\frac{\partial\tilde{\nu}}{\partial x_j}\right) = \eta\frac{\partial\tilde{\nu}}{\partial x_j}\cdot\frac{\partial\tilde{\nu}}{\partial x_j} + \tilde{\nu}\frac{\partial}{\partial x_j}\left(\eta\frac{\partial\tilde{\nu}}{\partial x_j}\right) \Rightarrow$$

$$\tilde{\nu}\frac{\partial}{\partial x_j}\left(\eta\frac{\partial\tilde{\nu}}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\tilde{\nu}\eta\frac{\partial\tilde{\nu}}{\partial x_j}\right) - \eta\frac{\partial\tilde{\nu}}{\partial x_j}\cdot\frac{\partial\tilde{\nu}}{\partial x_j} \quad \text{Further}$$

$$\tilde{\nu}\frac{\partial\tilde{\nu}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\tilde{\nu}^2\right) \quad \text{gives}$$

$$\tilde{\nu}\frac{\partial}{\partial x_j}\left(\eta\frac{\partial\tilde{\nu}}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\eta\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\tilde{\nu}^2\right)\right) - \eta\frac{\partial\tilde{\nu}}{\partial x_j}\frac{\partial\tilde{\nu}}{\partial x_j}$$

Inseration into (2) gives with $\tilde{e}_v = \frac{1}{2}\tilde{\nu}^2$

$$\frac{\partial}{\partial t}(\rho\tilde{e}_v) + \frac{\partial}{\partial x_j}(\rho u_j \tilde{e}_v) = \tilde{\nu}(P - D) + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}\left(\eta \frac{\partial \tilde{e}_v}{\partial x_j}\right)(\rho\tilde{\nu}c_{b_2} - \eta)\frac{\partial\tilde{\nu}}{\partial x_j}\cdot\frac{\partial\tilde{\nu}}{\partial x_j}\right]$$

with $P = \rho\tilde{\nu}\cdot\tilde{S}c_{b_1}$, $D = c_{w_1} f_{w_1} \rho\left(\frac{\tilde{\nu}^2}{d^2}\right)$ (3)

If we define the integrated energy.

$$E_{\tilde{\nu}}(t) = \int_{\Omega} \rho\tilde{e}_v(x,t)\, d\vec{x} \qquad \Omega \subset R^n$$ further we can split this domain to areas where the $\tilde{\nu}$ is < 0 and greater then zero i.e.

$$\Omega^+ = \{\vec{x} \in \Omega \mid \tilde{\nu} > 0\}$$
$$\Omega^- = \{\vec{x} \in \Omega \mid \tilde{\nu} < 0\}$$

$$E^+_{\tilde{\nu}} = \int_{\Omega^+} \rho\tilde{e}_v(x,t)\, d\vec{x}$$

$$E^-_{\tilde{\nu}} = \int_{\Omega^-} \rho\tilde{e}_v(x,t)\, d\vec{x}.$$

In regions where $\tilde{\nu} < 0$ the R.H.S of (3) should dissipate $\tilde{e}_\nu$ Given an initial value at $t = t_0$ one can term in $E_{\tilde{\nu}}$ will grow in time by examining

$$\frac{dE_{\tilde{\nu}}}{dt} = \frac{d}{dt}\int_{\Omega_t^-} \rho e_{\tilde{\nu}}(x,t)\, dx$$

Using Liebniz theorem for a moving Boundary we have.

$$\frac{dE_{\tilde{\nu}}}{dt} = \int_{\Omega_t^-} \frac{\partial}{\partial t}(\rho e_{\tilde{\nu}})\, dx + \int_{\partial\Omega^-} \rho e_{\tilde{\nu}}\, \vec{v}\cdot\vec{n}\, ds$$

More over following Oliver we consider the case when

$\partial\Omega^- \cap \partial\Omega = 0$ i.e. all the faces of Ω^- are Not boundary faces of Ω.

Intersection i.e what is common.

Assuming $\hat{v}$ continuous the $\hat{v}|_{\partial\Omega^-} = 0 \Rightarrow e\tilde{v}|_{\partial\Omega^-} = 0$

gives.

$$\int_{\partial\Omega^-} \rho \tilde{e}r \cdot \vec{n} \cdot \vec{v} \, ds = 0 \quad \text{gives.}$$

$$\frac{dE\tilde{v}}{dt} = \int_{\Omega^-} \frac{\partial}{\partial t}(\rho e \tilde{v}) \, d\vec{x} \quad \text{using ③ we have}$$

$$\frac{dE\tilde{v}}{dt} = \int_{\Omega^-} -\frac{\partial}{\partial x_j}(\rho u_j \tilde{e}r) + \frac{1}{\sigma}\frac{\partial}{\partial x_j}\left(\eta \frac{\partial \hat{e}r}{\partial x_j}\right) d\vec{x} +$$

$$\int_{\Omega^-} \frac{(\rho \tilde{v} C6_2 - \eta)}{\sigma} \frac{\partial \hat{v}}{\partial x_j} \cdot \frac{\partial \tilde{v}}{\partial x_j} + \tilde{v}(P - D) \, d\vec{x}.$$

Application of divergence theorem to first integral

$$\int_{\Omega^-} -\frac{\partial}{\partial x_j}(\rho u_j e \tilde{v}) + \frac{1}{\sigma}\frac{\partial}{\partial x_j}\left(\eta \frac{\partial \hat{e}r}{\partial x_j}\right) d\vec{x} \quad \text{gives}$$

$$\int_{\partial\Omega^-} \vec{v} \cdot \vec{n}\left[\rho u_j e \tilde{v} + \frac{1}{\sigma}\eta \frac{\partial \hat{e}r}{\partial x_j}\right] dx \to 0.$$

gives

$$\frac{dE\tilde{v}}{dt} = \int_{\Omega^-} \frac{(C6_2 \, \rho \tilde{v} - \eta)}{\sigma} \frac{\partial \tilde{v}}{\partial x_j} \cdot \frac{\partial \tilde{x}}{\partial x_j} + \tilde{v}(P - D) \, d\vec{x}$$

Essentially for continous $\tilde{\nu}$ on Ω^- inside Ω we can Bound $\tilde{E}_v$ entirely with source term, we ~~[illegible]~~ enter $\frac{d\tilde{E}_v}{dt} < 0$ with source. If $\frac{d\tilde{E}_v}{dt} \leq 0$ the $\tilde{E}_v$ is bounded by $\tilde{E}_v|_{t=t_0}$. This will bound solutions in regions of $-\tilde{\nu}$.

Let's examine each term for $\tilde{\nu} < 0$.

1) $\frac{(cb_2 \rho\tilde{\nu} - \eta)}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_j} \cdot \frac{\partial \tilde{\nu}}{\partial x_j}$ this is > 0 when

$(cb_2 \rho\tilde{\nu} - \eta) > 0$ ie. $\eta > cb_2 \rho \tilde{\nu}$

$$\frac{\mu + \cancel{\rho\tilde{\nu}}}{\rho\tilde{\nu}} > cb_2 \cancel{\rho\tilde{\nu}}$$

$$\frac{1}{\chi} + 1 > cb_2 \rightarrow \quad 1 + \chi > \chi cb_2$$

$$1 > (cb_2 - 1)\chi$$

$$\chi < \frac{1}{(cb_2 - 1)}$$

$$\chi < \frac{1}{(.622 - 1)} \simeq -2.6455$$

2) $\tilde{\nu}P = cb_1 \tilde{S} \tilde{\nu}^2$ for an $\tilde{S} > 0$ $\tilde{\nu}P > 0$ always

thats why we need $\tilde{S} > 0$ $\forall \tilde{\nu} \in [-\infty, 0]$

3) $-\tilde{\nu}D = -cw_1 f_w \rho \left(\frac{\tilde{\nu}^3}{d^2}\right)$ since f_w is a fn of $\tilde{\nu}, \tilde{S}$

the sign is not uniquely determined by $\tilde{\nu}$

$$f_w = g\left[\frac{1 + cw_3^6}{g^6 + cw_3^6}\right]^{1/6} \quad g = r + cw_2(r^6 - r)$$

$$r = \frac{\tilde{\nu}}{\tilde{S} k^2 d^2}$$

Now r's sign for $\tilde{S} > 0$ is ~~a~~ determined by $\tilde{r}$, so if we can find f_{w1} positive in $r < 0$ we will have.

$$-c_{w1} f_{wp}\left(\frac{\tilde{r}^3}{d^2}\right) > 0 \quad \text{which is bad.}$$

Again using maple we can plot f_w vs r. for $r < 0$ we have one sign change $r \approx 1.18$ according to oliver, it looks to be so on my Maple graph. thus for this case we would have $-\tilde{r} D > 0$ again Bad.

We Need to ensure the following.

$$(c_{b2}\,\rho\tilde{r} - \eta)\frac{\partial \tilde{r}}{\partial x_j}\cdot\frac{\partial \tilde{r}}{\partial x_j} < 0 \quad \forall\, \tilde{r} \in [-\infty, 0]$$

$$\tilde{r}(P - D) < 0 \quad \forall\, \tilde{r} \in [-\infty, 0]$$

1). To ensure the diffusion term is stable we have from Oliver's PhD thesis

$$\eta = \begin{cases} \mu(1+\chi), & \chi \geq 0 \\ \mu(1+\chi+\tfrac{1}{2}\chi^2) & \chi < 0 \end{cases}$$

For analo $(c_{b2}\rho\tilde{r} - \eta)$ can be non-dim by μ giving $c_{b2}\chi - \frac{\eta}{\mu}$

Function is C_1 - continuous

$@\ \chi = 0 \quad \mu = \mu$ ✓

$$\frac{d\eta}{d\chi} = \begin{cases} \mu & \chi \geq 0 \\ \mu(1+\chi) & \chi < 0 \end{cases} \qquad @\ \chi = 0 \quad \mu = \mu$$

C_1 ✓

6). similarly for D we have to do f_w, g, r 1st.

$$r = \frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2} = \frac{\chi \cdot \left(\frac{\nu}{d^2 \cdot S}\right)}{(\tilde{S}/S)} \to \sigma \text{ from page 1 of this}$$

$$r = \frac{\chi\sigma}{\kappa^2(\tilde{S}/S)}$$

Thus now, g, f_w are defined entirely in terms of χ.

$$\frac{D}{\mu S} = \begin{cases} c_{w_1} f_w \frac{\rho}{S}\left(\frac{\tilde{\nu}^2}{d^2}\right) & \chi \geq 0 \\ -\frac{c_{w_1}}{\mu}\frac{\rho}{S}\left(\frac{\tilde{\nu}^2}{d^2}\right) & \chi \leq 0 \end{cases} = \begin{cases} c_{w_1} f_w \left(\frac{\rho\tilde{\nu}}{\mu}\right)\frac{\nu \cdot \chi}{d^2 S} & \chi \geq 0 \\ -c_{w_1}\left(\frac{\rho\tilde{\nu}}{\mu}\right)\frac{\nu \cdot \chi}{d^2 S} & \chi \leq 0 \end{cases}$$

$$\frac{D}{\mu S} = \begin{cases} c_{w_1} f_w \chi^2 \sigma & \chi \geq 0 \\ -c_{w_1} \chi^2 \sigma & \chi < 0 \end{cases}$$

For energy we need $\hat{\nu} P$ or ~~this~~

Thus it is sufficient to ensure positivity of $\frac{P}{\mu S}$ move over show $(P-D) \geq 0$ for $\hat{\nu} < 0$, $(P-D) \leq 0$ for $\hat{\nu} > 0$

In summary we have a set of modifications to the S.A. Model. Plots in sa_source.mw will show that they do have the properties they say

for $\tilde{\nu} > 0$ $P-D < 0$ ✓

for $\tilde{\nu} < 0$ $(P-D) > 0$ ✓

for all $\tilde{\nu}$: $(c_{b2}\rho\tilde{\nu} - \eta) > 0$ ✓

For P we have

$$P = \begin{cases} c_{b1} \tilde{S} \rho \tilde{r} & X \geq 0 \\ c_{b1} S' \rho \tilde{r} g_n, & X < 0, \end{cases} \quad \text{where } g_n = 1 - \frac{1000 X^2}{1+X^2}$$

The function g_n is chosen such that P has a continous first derivative, at $\tilde{r} = 0$.

For the destruction term we have

$$D = \begin{cases} c_{w1} f_w \rho\left(\frac{\tilde{r}^2}{d^2}\right), & X \geq 0 \\ -c_{w1} \rho\left(\frac{\tilde{r}^2}{d^2}\right), & X < 0 \end{cases}$$

clearly $D < 0$ for $\tilde{r} < 0$ thus $\tilde{r} D > 0$

for $\tilde{r} < 0$ thus $-\tilde{r} D < 0$ for $\tilde{r} < 0$.

for $\tilde{r} > 0$ $-\tilde{r} D < 0$ also.

We Need to plot the $\tilde{r}(P-D)$ term is som non-dimensional sence. So we divided $(c_{b2} \rho \tilde{r} \cdots r)$ by μ Let's do this to P and D as.

Also $\tilde{S}/S \Rightarrow \div S$ thus.

a). Begin with P.

- We have a function for $\tilde{S}/S$ thus,

$$\frac{P}{\mu S} = \begin{cases} c_{b1}\left(\frac{\tilde{S}}{S}\right) \frac{\rho \tilde{r}}{\mu} & X \geq 0 \\ c_{b1} \frac{\tilde{S}}{S} \frac{\rho \tilde{r}}{\mu} g_n(X) & X < 0 \end{cases} = \begin{cases} c_{b1}\left(\frac{\tilde{S}}{S}\right) X & X \geq 0 \\ c_{b1}\left(\frac{\tilde{S}}{S}\right) X\, g_n(X) & X < 0 \end{cases}$$

$= \quad c_{b1}$

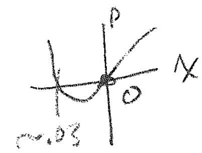
Thus in summary we have

$$\eta = \begin{cases} \mu(1+\chi) & \chi \geq 0 \\ \mu(1+\chi+\frac{\chi^2}{2}) & \chi < 0 \end{cases}$$

$$P = \begin{cases} \rho\tilde{\nu}\, c_{b1}\hat{S} & \chi \geq 0 \\ \rho\tilde{\nu}\, c_{b1} S g_n & \chi < 0 \end{cases}$$

Note for small $\tilde{\nu} < 0$ $P < 0$

$0 > \tilde{\nu} > -.03$ ∴



$$g_n = 1 - (1000\,\chi^2)/(1+\chi^2)$$

$$D = \begin{cases} c_{w1} f_w\, \rho\left(\frac{\tilde{\nu}}{d}\right)^2 & \chi \geq 0 \\ -c_{w1}\, \rho\left(\frac{\tilde{\nu}}{d}\right)^2, & \chi < 0 \end{cases}$$

For the diffusion term it's nice to avoid things like χ so

$$\eta = \begin{cases} (\mu + \rho\tilde{\nu}) & \chi \geq 0 \\ \left(\mu + \rho\tilde{\nu} + \frac{1}{2}\frac{(\rho\tilde{\nu})^2}{\mu}\right) & \chi < 0 \end{cases} = \mu + \rho\tilde{\nu} + \tfrac{1}{2}(\rho\tilde{\nu})\chi$$

For clarity we really only apply the η as

$$\frac{\partial \rho\tilde{\nu}}{\partial t} + \frac{\partial}{\partial x_j}(\rho\tilde{\nu} u_j) = P + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_j}\left(\eta \frac{\partial \tilde{\nu}}{\partial x_j}\right) + c_{b2}\,\rho\, \frac{\partial \tilde{\nu}}{\partial x_j}\cdot\frac{\partial \tilde{\nu}}{\partial x_i}\right] - D$$

where η, P, D are given as above.

Local CFL Reduction:

As a measure to improve turbulence model robustness, we will modify the local CFL # based on the element-wise resolution indicator. Based on the following.

Remark: This is an adoption of the smooth formula for $\min(\max(f, CFL_{min}), CFL_{max})$. And is just the formula for Persson and Peraire's Shock-viscosity Reversed. We have also modified the S_o term.

current global #

$$CFL = \begin{cases} CFL_c & S_e < S_o - K \\ \alpha\left[1-\sin\left(\frac{\pi(S_e - S_o)}{2K}\right)\right] + CFL_{min} & S_o - K \leq S_e \leq S_o + K \\ CFL_{min} & S_e > S_o + K. \end{cases}$$

with $\alpha = \left(\frac{CFL_c - CFL_{min}}{2}\right)$

$$S_o = \log_{10}\left(\frac{.75}{p^2}\right)$$

$$S_e = \log_{10}(\acute{S}_K)$$

where $\acute{S}_K$ is the resolution indicator value for each cell. $K \in \mathcal{T}_h$

Boundary conditions on gradients:

Following Todd Oliver's PhD thesis we will present the Boundary conditions on the gradients for an inviscid wall.

1). For scalar quantities such as $\rho, \rho E$ we have the following state vector.

$$\begin{Bmatrix} \rho^b \\ \rho u^b \\ \rho v^b \\ \rho E^b \\ \rho\tilde{\nu}^b \end{Bmatrix} = \begin{Bmatrix} \rho^i \\ \rho u^i - (\rho u^i n_x + \rho v^i n_y) n_x \\ \rho v^i - (\rho u^i n_x + \rho v^i n_y) n_y \\ \rho E^i \\ \rho\tilde{\nu}^i \end{Bmatrix}$$

This $\Rightarrow$ that $\frac{\partial \rho^b}{\partial n} = 0 = \frac{\partial \rho^b}{\partial x} n_x + \frac{\partial \rho^b}{\partial y} \cdot n_y$

but we don't know $\frac{\partial \rho^b}{\partial \vec{\tau}}$ thus set $\frac{\partial \rho^b}{\partial \vec{\tau}} = \frac{\partial \rho^i}{\partial \tau}$ gives

$$\frac{\partial \rho^i}{\partial \tau} = -\frac{\partial \rho^i}{\partial x} \cdot n_y + \frac{\partial \rho^i}{\partial y} \cdot n_x$$

$$\begin{bmatrix} A \end{bmatrix} \begin{Bmatrix} \frac{\partial \rho^b}{\partial x} \\ \frac{\partial \rho^b}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \rho^b}{\partial n} \\ \frac{\partial \rho^b}{\partial \tau} \end{Bmatrix}$$

$$\begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix} \begin{Bmatrix} \frac{\partial \rho^b}{\partial x} \\ \frac{\partial \rho^b}{\partial y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\frac{\partial \rho^i}{\partial x} n_y + \frac{\partial \rho^i}{\partial y} \cdot n_x \end{Bmatrix}$$

Thus we can solve this for $\frac{\partial \rho^b}{\partial x}, \frac{\partial \rho^b}{\partial y}$

$$\begin{Bmatrix} \frac{\partial \rho^b}{\partial x} \\ \frac{\partial \rho^b}{\partial y} \end{Bmatrix} = \begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} \begin{Bmatrix} 0 \\ -\frac{\partial \rho^i}{\partial x} n_y + \frac{\partial \rho^i}{\partial y} \cdot n_x \end{Bmatrix}$$

$$\frac{\partial \rho^b}{\partial x} = \left(-\frac{\partial \rho^i}{\partial x} n_y^2 - \frac{\partial \rho^i}{\partial y} n_x n_y \right)$$

$$\frac{\partial \rho^b}{\partial y} = \left(-\frac{\partial \rho^i}{\partial x} n_y n_x + \frac{\partial \rho^i}{\partial y} n_x^2 \right)$$

The same can be done for any scalar such as ρE, $\rho\tilde{n}$

For the mtm we have

$$\frac{\partial \rho u_\tau^b}{\partial n} = 0 \quad \text{specified}$$

$$\left.\begin{aligned} \frac{\partial \rho u_n^b}{\partial n} &= \frac{\partial(\rho u_n^i)}{\partial n} \\ \frac{\partial \rho u_\tau^b}{\partial \tau} &= \frac{\partial \rho u_\tau^i}{\partial \tau} \\ \frac{\partial \rho u_n}{\partial \tau} &= \frac{\partial \rho u_n^i}{\partial \tau} \end{aligned}\right\} \quad \text{taken from interior}$$

We can write the derivatives as

$$\frac{\partial u_1}{\partial x} = n_x n_x \frac{\partial u_1}{\partial x} + \cancel{n_x n_y \frac{\partial u_1}{\partial y}} + n_y^2 \frac{\partial u_1}{\partial x} - \cancel{n_y n_x \frac{\partial u_1}{\partial y}} = n_x \frac{\partial u_1}{\partial n} - n_y \frac{\partial u_1}{\partial \tau}$$

$$\frac{\partial u_1}{\partial y} = \cancel{n_y n_x \frac{\partial u_1}{\partial x}} + n_y^2 \frac{\partial u_1}{\partial y} + \cancel{-\frac{\partial u_1}{\partial x} n_y n_x} + n_x^2 \frac{\partial u_1}{\partial y} = n_y \frac{\partial u_1}{\partial n} + n_x \frac{\partial u_1}{\partial \tau}$$

$$\frac{\partial u_2}{\partial x} = n_x \frac{\partial u_2}{\partial n} - n_y \frac{\partial u_2}{\partial \tau}$$

$$\frac{\partial u_2}{\partial y} = n_y \frac{\partial u_2}{\partial n} + n_x \frac{\partial u_2}{\partial \tau}$$

$$\begin{Bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \\ \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial y} \end{Bmatrix}^b = \begin{bmatrix} n_x & 0 & -n_y & 0 \\ 0 & n_x & 0 & -n_y \\ n_y & 0 & n_x & 0 \\ 0 & n_y & 0 & n_x \end{bmatrix} \begin{Bmatrix} \frac{\partial u_1}{\partial n} \\ \frac{\partial u_2}{\partial n} \\ \frac{\partial u_1}{\partial \tau} \\ \frac{\partial u_2}{\partial \tau} \end{Bmatrix}^b$$

This is the same as todd oliver's thesis but corrected entries of LHS vector and RHS vector.

Further we can write,

$$\rho u_1 = n_x \rho u_n - n_y \rho u_\tau$$

$$\rho u_2 = n_y \rho u_n + n_x \rho u_\tau$$

Inserting this into RHS of above.

$$\begin{Bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \\ \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial y} \end{Bmatrix}^b = \begin{bmatrix} n_x & 0 & -n_y & 0 \\ 0 & n_x & 0 & -n_y \\ n_y & 0 & n_x & 0 \\ 0 & n_y & 0 & n_x \end{bmatrix} \begin{Bmatrix} n_x \frac{\partial \rho u_n}{\partial \vec{n}} - n_y \frac{\partial \rho u_\tau}{\partial \vec{n}} \\ n_y \frac{\partial \rho u_n}{\partial \vec{n}} + n_x \frac{\partial \rho u_\tau}{\partial \vec{n}} \\ n_x \frac{\partial \rho u_n}{\partial \vec{\tau}} - n_y \frac{\partial \rho u_\tau}{\partial \vec{\tau}} \\ n_y \frac{\partial \rho u_n}{\partial \vec{\tau}} + n_x \frac{\partial \rho u_\tau}{\partial \vec{\tau}} \end{Bmatrix}^b$$

$$= \begin{Bmatrix} n_x^2 \frac{\partial \rho u_n}{\partial \vec{n}} - n_x n_y \frac{\partial \rho u_\tau}{\partial \vec{n}} - n_y n_x \frac{\partial \rho u_n}{\partial \vec{\tau}} + n_y^2 \frac{\partial \rho u_\tau}{\partial \vec{\tau}} \\ n_x n_y \frac{\partial \rho u_n}{\partial \vec{n}} + n_x^2 \frac{\partial \rho u_\tau}{\partial \vec{n}} - n_y^2 \frac{\partial \rho u_n}{\partial \vec{\tau}} - n_x n_y \frac{\partial \rho u_\tau}{\partial \vec{\tau}} \\ n_y n_x \frac{\partial \rho u_n}{\partial \vec{n}} - n_y^2 \frac{\partial \rho u_\tau}{\partial \vec{n}} + n_x^2 \frac{\partial \rho u_n}{\partial \vec{\tau}} - n_x n_y \frac{\partial \rho u_\tau}{\partial \vec{\tau}} \\ n_y^2 \frac{\partial \rho u_n}{\partial \vec{n}} + n_y n_x \frac{\partial \rho u_\tau}{\partial \vec{n}} + n_x n_y \frac{\partial \rho u_n}{\partial \vec{\tau}} + n_x^2 \frac{\partial \rho u_\tau}{\partial \vec{\tau}} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \\ \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial y} \end{Bmatrix} = \begin{bmatrix} n_x^2 & -n_x n_y & -n_x n_y & n_y^2 \\ n_x n_y & n_x^2 & -n_y^2 & -n_x n_y \\ n_x n_y & -n_y^2 & n_x^2 & -n_x n_y \\ n_y^2 & n_x n_y & n_x n_y & n_x^2 \end{bmatrix} \begin{Bmatrix} \frac{\partial \rho u_n}{\partial n} \\ \frac{\partial \rho u_\tau}{\partial n} \to 0 \\ \frac{\partial \rho u_n}{\partial \tau} \\ \frac{\partial \rho u_\tau}{\partial \tau} \end{Bmatrix}^b$$

This is correct Matrix same as T. O. but with the correct RHS vector the thesis has an error.

Using 1 more expansion gives

$$= \begin{bmatrix} n_x^2 & -n_x n_y & -n_x n_y & n_y^2 \\ n_x n_y & n_x^2 & -n_y^2 & -n_x n_y \\ n_x n_y & -n_y^2 & n_x^2 & -n_x n_y \\ n_y^2 & n_x n_y & n_x n_y & n_x^2 \end{bmatrix} \begin{Bmatrix} \left(n_x^2 \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial x} n_x n_y + \frac{\partial \rho u}{\partial y} n_x n_y + \frac{\partial \rho v}{\partial y} n_y^2 \right) \\ 0 \\ \frac{\partial \rho u}{\partial x} n_y^2 - \frac{\partial \rho v}{\partial x} n_x n_y - \frac{\partial \rho u}{\partial y} n_y n_x + \frac{\partial \rho v}{\partial y} n_x^2 \\ n_x^2 \frac{\partial \rho u}{\partial x} - \frac{\partial \rho v}{\partial x} n_x n_y - n_y n_x \frac{\partial \rho u}{\partial y} + n_y^2 \frac{\partial \rho v}{\partial y} \end{Bmatrix}$$

We'll Do the rest in maple.

Log formulation of the S.A. equation.

First we will use the Non-conservative equation with the "original" diffusion term. We'll take the log form of this and then we'll make it conservative again.

Our starting point is

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = \frac{1}{\sigma}\left[\nabla(\tilde{\nu}\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu})^2\right] + \tilde{S}\tilde{\nu}\cdot c_{b1} - c_{w1} f_w \left(\frac{\tilde{\nu}}{d}\right)^2$$

$w = \ln(\tilde{\nu}) \Rightarrow$

$$\cancel{\tilde{\nu}}\frac{\partial w}{\partial t} + u_j \cancel{\tilde{\nu}}\frac{\partial w}{\partial x_j} = \frac{1}{\sigma}\left[\frac{\nabla(\tilde{\nu}^2\nabla w)}{\tilde{\nu}} + c_{b2}\tilde{\nu}(\nabla w)^2\right] + \tilde{S}\cancel{\tilde{\nu}}\cdot c_{b1} - c_{w1} \frac{f_w \left(\frac{\tilde{\nu}^2}{d^2}\right)}{\tilde{\nu}}$$

$$\frac{\partial w}{\partial t} + u_j \frac{\partial w}{\partial x_j} = \frac{1}{\sigma}\left[\frac{2\cancel{\tilde{\nu}}\nabla\tilde{\nu}\nabla w}{\cancel{\tilde{\nu}}} + \frac{\tilde{\nu}^{\cancel{2}}}{\cancel{\tilde{\nu}}}\nabla^2 w + c_{b2}\tilde{\nu}(\nabla w)^2\right] + \tilde{S} c_{b1} - c_{w1} f_w \frac{\tilde{\nu}}{d^2}$$

$$\frac{\partial w}{\partial t} + u_j \frac{\partial w}{\partial x_j} = \frac{1}{\sigma}\left[2\cdot\nabla\tilde{\nu}\cdot\nabla w + \tilde{\nu}\nabla^2 w + c_{b2}\tilde{\nu}(\nabla w)^2\right] + \tilde{S} c_{b1} - c_{w1} f_w \frac{\tilde{\nu}}{d^2}$$

Diffusion term

$$\underbrace{\nabla\tilde{\nu}\nabla w}_{\tilde{\nu}(\nabla w)^2} + \underbrace{\nabla\tilde{\nu}\nabla w + \tilde{\nu}\nabla^2 w}_{\nabla(\tilde{\nu}\nabla w)} + c_{b2}\tilde{\nu}(\nabla w)^2$$

gives

$$\frac{\partial w}{\partial t} + u_j \frac{\partial w}{\partial x_j} = \frac{1}{\sigma}\left[\nabla(\tilde{\nu}\nabla w) + (c_{b2}+1)\tilde{\nu}(\nabla w)^2\right] + \tilde{S} c_{b1} - c_{w1} f_w \frac{\tilde{\nu}}{d^2}$$

From here we can derive a conservative scheme consistent with a compressible mass equation.

$$\rho \frac{\partial \omega}{\partial t} + \rho v_j \frac{\partial \omega}{\partial x_j} = \frac{1}{\sigma}\left[\nabla(\rho \tilde{\nu} \nabla \omega) + [1+c_{b2}]\rho \tilde{\nu} (\nabla \omega)^2 \right] + \tilde{S}\rho c_{b1} - c_{w1} f_w \frac{\rho \tilde{\nu}}{d^2}$$

$$\rho \frac{\partial \omega}{\partial t} + \rho v_j \frac{\partial \omega}{\partial x_j} = \frac{\partial(\rho\omega)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j \omega) - \omega \frac{\partial \rho}{\partial t} + \omega \frac{\partial \rho v_j}{\partial x_j}$$

gives

$$\frac{\partial \rho \omega}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j \omega) = \frac{1}{\sigma}\left[\nabla(\rho \tilde{\nu} \nabla \omega) + [1+c_{b2}]\rho \tilde{\nu} (\nabla\omega)^2\right] + \tilde{S}\rho c_{b1} - c_{w1} f_w \frac{\rho \tilde{\nu}}{d^2}$$

If we add a ν to the $\tilde{\nu}$ coefficients

we get

① $$\frac{\partial(\rho\omega)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j \omega) = \frac{1}{\sigma}\left[\nabla((\mu+\rho\tilde{\nu})\nabla\omega) + [\mu+\rho\tilde{\nu} + c_{b2}\rho\tilde{\nu}](\nabla\omega)^2\right] + \tilde{S}\rho c_{b1} - c_{w1} f_w \frac{\rho\tilde{\nu}}{d^2}$$

If we followed the $(\nu+\tilde{\nu})\nabla^2$ we could h

② $$\frac{\partial(\rho\omega)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j \omega) = \frac{1}{\sigma}\left[\nabla((\mu+\rho\tilde{\nu})\nabla\omega) + (\mu+\rho\tilde{\nu})(1+c_{b2})(\nabla\omega)^2\right] + \tilde{S}\rho c_{b1} - c_{w1} f_w \frac{\rho\tilde{\nu}}{d^2}$$

Thus we have a new conserved variable

$\rho\omega$ $\quad \tilde{\nu} = e^{\omega} = e^{\left(\frac{\rho\omega}{\rho}\right)}$, $\quad \nabla\omega = \frac{\partial\omega}{\partial q}\cdot\left(\frac{\partial q}{\partial x}\ \frac{\partial q}{\partial y}\right)$

$$\frac{\partial \omega}{\partial q} = \left\{ \begin{matrix} -\frac{\rho\omega}{\rho^2} = -\frac{\omega}{\rho} \\ 0 \\ 0 \\ 0 \\ \frac{1}{\rho} \end{matrix} \right\}$$

We'll start by using form 1.

Newton update limiting:

We want to determine an update parameter such that the norm of the updated solution is $\leq$ to a fraction $\beta \in (0,1]$ of the "old" solution.

$$\alpha = \left\{ \alpha \in (0,1] : \frac{\|\vec{\omega} + \alpha\delta\vec{\omega}\| - \|\vec{\omega}\|}{\|\vec{\omega}\|} \leq \beta \right\}$$

For certain norms we can come up with nice expressions for α.

a). L_2 norm squared.

$\|\vec{\omega}\| \to \|\vec{\omega}\|_2^2$ Remark: This will be valid wether a continuous or discrete L_2 norm is used.

gives

$$\frac{\|\vec{\omega} + \alpha\delta\vec{\omega}\|_2^2 - \|\vec{\omega}\|_2^2}{\|\vec{\omega}\|_2^2} \leq \beta \quad \text{becomes}$$

$$\frac{\langle \vec{\omega} + \alpha\delta\vec{\omega}, \vec{\omega} + \alpha\delta\vec{\omega} \rangle - \langle \vec{\omega}, \vec{\omega} \rangle}{\langle \vec{\omega}, \vec{\omega} \rangle} \leq \beta$$

$$\frac{\langle \vec{\omega}, \vec{\omega} \rangle + 2\alpha\langle \vec{\omega}, \delta\vec{\omega} \rangle + \alpha^2\langle \delta\vec{\omega}, \delta\vec{\omega} \rangle - \langle \vec{\omega}, \vec{\omega} \rangle}{\langle \vec{\omega}, \vec{\omega} \rangle} \leq \beta$$

This is a quadratic for α.

$$\alpha^2\langle \delta\vec{\omega}, \delta\vec{\omega} \rangle + 2\alpha\langle \vec{\omega}, \delta\vec{\omega} \rangle - \beta\langle \vec{\omega}, \vec{\omega} \rangle \leq 0$$

If we solve for the $=0$ case and set the constraint on $\alpha \in (0,1]$ then we'll satisfy the $\leq \beta$ criteria.

Using the quadratic formula

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $b = 2\langle \vec{\omega}, \delta\vec{\omega} \rangle$, $a = \langle \delta\vec{\omega}, \delta\vec{\omega} \rangle$, $c = \langle \vec{\omega}, \vec{\omega} \rangle \beta$

$$\alpha = \frac{-\cancel{2}\langle \vec{\omega}, \delta\vec{\omega} \rangle \pm \sqrt{\cancel{4}(\langle \omega, \delta\vec{\omega} \rangle)^2 + \cancel{4}\langle \delta\vec{\omega}, \delta\vec{\omega} \rangle \langle \vec{\omega}, \vec{\omega} \rangle \beta}}{\cancel{2}\langle \vec{\omega}, \vec{\omega} \rangle}$$

$$\alpha = \frac{-\langle \vec{\omega}, \delta\vec{\omega} \rangle \pm \sqrt{\langle \omega, \delta\vec{\omega} \rangle^2 + \beta \langle \delta\vec{\omega}, \delta\vec{\omega} \rangle \langle \omega, \vec{\omega} \rangle}}{\omega\langle \vec{\omega}, \vec{\omega} \rangle}$$

Which Branch. $\langle \delta\vec{\omega}, \delta\vec{\omega} \rangle > 0$, $\langle \vec{\omega}, \vec{\omega} \rangle > 0$, $\langle \omega, \delta\vec{\omega} \rangle^2 > 0$

Thus if $b > 0$, $\langle \omega, \delta\vec{\omega} \rangle > 0$ gives

$\frac{-|b| \pm \sqrt{b^2 + ac}}{a}$ since $b^2 + ac > 0$ then the α

with < 0 for $\frac{-|b| - \sqrt{b^2+ac}}{a}$, so we should pick the

positive branch to ensure $\alpha > 0$

For $b < 0$, $\langle \omega, \delta\vec{\omega} \rangle < 0$ give

$\frac{|b| \pm \sqrt{b^2 + ac}}{a}$, here the largest α is $\frac{|b| + \sqrt{b^2 + ac}}{a}$

Further $b^2 + ac$ will alway $> |b|$ thus we again should pick the positive Branch to ensure $\alpha > 0$ too

In conclusion we pick the positive Branch of the soln

$$\boxed{\alpha = \frac{-b + \sqrt{b^2 + ac}}{a} = \frac{-\langle \delta\vec{\omega}, \vec{\omega} \rangle + \sqrt{\langle \delta\vec{\omega}, \vec{\omega} \rangle^2 + \langle \delta\vec{\omega}, \delta\vec{\omega} \rangle \langle \vec{\omega}, \vec{\omega} \rangle \beta}}{\langle \vec{\omega}, \vec{\omega} \rangle}}$$

Vorticity Based Artificial Viscosity

In an attempt to increase Boundary Layer Edge smoothness of turbulence Models an extra viscosity coefficient has been derived. Rather then the approach of Persson and Peraire where this is cell based and a function of the turbulence variables I propose the following. Since we know that turbulenent production will be present when there is vorticity then we will construct a smooth function of vorticity namely a gaussian distribution

$$\mu_{av} = \frac{A}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{S-S_0}{2\sigma^2}\right)}$$

where S is the magnitude of the vorticity. S_0 is a point about which the distribution is Gaussian. The parameters A and σ are set as follows.

If we want viscosity of Magnitude M and width w. then

$$M = \frac{A}{\sqrt{2\pi\sigma^2}} \Rightarrow A = M\sqrt{2\pi}\,\sigma,$$

and $\sigma = \sqrt{(w/4)}$. The width is only approximate. If we have a smooth function of vorticity then this will be a smooth function in space as well as in Vorticity. Further since it is not a fn of turbulence model quantities it does not play a role in a decoupled Linearization

Log formulation of ω - equation:

Begin with non-conservative form of the ω- equation (Non-dim.)

$$\cancel{\bar\rho}\frac{\partial\bar\omega}{\partial t}+\cancel{\bar\rho}\bar U_j\frac{\partial\bar\omega}{\partial x_j}=\alpha\frac{\bar\omega}{K}\frac{\tilde\tau_{ij}}{\bar\rho}\frac{\partial\bar U_i}{\partial x_j}-\beta\cancel{\bar\rho}\,\bar\omega^2+\frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[\left[\bar\nu+\sigma\bar\nu_T\right]\frac{\partial\bar\omega}{\partial x_j}\right]\frac{1}{\cancel{\bar\rho}}$$

$\bar\rho\,\overline{u_i'u_j'}$

$\overline{u_i'u_j'}$

$$\frac{\partial\bar\omega}{\partial t}+\bar U_j\frac{\partial\bar\omega}{\partial x_j}=\alpha\frac{\bar\omega}{K}\vec{\bar\tau}_{ij}\frac{\partial\bar U_i}{\partial x_j}-\beta\bar\omega^2+\frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[(\bar\nu+\sigma\bar\nu_T)\frac{\partial\bar\omega}{\partial x_j}\right]$$

Now $\frac{\partial}{\partial()}\ln(\omega)=\frac{1}{\omega}\frac{\partial\omega}{\partial()}\Rightarrow\frac{\partial\omega}{\partial()}=\frac{\partial}{\partial()}(\ln(\omega))\cdot\omega$, thus let $\tilde\omega=\ln(\omega)$

and drop $\bar{()}$ notation for convience. These are all non-dimensional variables.

$$\frac{\partial\omega}{\partial t}=\omega\frac{\partial\tilde\omega}{\partial()}\quad\text{thus we have}$$

$$\cancel{\omega}\frac{\partial\tilde\omega}{\partial t}+\cancel{\omega}\bar U_j\frac{\partial\tilde\omega}{\partial x_j}=\alpha\frac{\cancel{\omega}}{K}\tau_{ij}\frac{\partial u_i}{\partial x_j}-\beta\omega^{\cancel{2}}+\frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_T)\omega\frac{\partial\tilde\omega}{\partial x_j}\right]\frac{1}{\omega}$$

Thus the diffusion term needs work

$$\frac{1}{\omega}\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_T)\omega\frac{\partial\tilde\omega}{\partial x_j}\right]=\frac{1}{\omega}\left\{\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_T)\omega\right]\cdot\frac{\partial\tilde\omega}{\partial x_j}+\left[\nu+\sigma\nu_T\right]\omega\frac{\partial^2\tilde\omega}{\partial x_j^2}\right\}$$

$$=\frac{1}{\cancel{\omega}}\left\{\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_T)\right]\cancel{\omega}\frac{\partial\tilde\omega}{\partial x_j}+\left[\nu+\sigma\nu_T\right]\cancel{\omega}\frac{\partial\tilde\omega}{\partial x_j}\frac{\partial\tilde\omega}{\partial x_j}+\left[\nu+\sigma\nu_T\right]\cancel{\omega}\frac{\partial^2\tilde\omega}{\partial x_j^2}\right\}$$

$$\Downarrow$$

$$\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_T)\frac{\partial\tilde\omega}{\partial x_j}\right]$$

$$=\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_T)\frac{\partial\tilde\omega}{\partial x_j}\right]+\left[\nu+\sigma\nu_T\right]\frac{\partial\tilde\omega}{\partial x_j}\cdot\frac{\partial\tilde\omega}{\partial x_j}$$

Thus the Non-conservative log form. is

$$\frac{\partial\tilde\omega}{\partial t}+U_j\frac{\partial}{\partial x_j}(\tilde\omega_j)=\frac{M_\infty}{Re_\infty}\frac{\alpha}{K}\tau_{ij}\frac{\partial u_i}{\partial x_j}-\beta e^{\tilde\omega}+\frac{M_\infty}{Re_\infty}\left[\nu+\sigma\nu_T\right]\frac{\partial\tilde\omega}{\partial x_j}\cdot\frac{\partial\tilde\omega}{\partial x_j}+$$

$$\frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[(\nu+\sigma\nu_T)\frac{\partial\tilde\omega}{\partial x_j}\right]$$

Again Multiply by density and add mass to LHS $\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)$

Gives

$$\rho\,\overline{u_i' u_j'}$$

$$\frac{\partial}{\partial t}(\rho\tilde{\omega}) + \frac{\partial}{\partial x_j}(\rho\tilde{\omega}u_j) = \frac{\alpha}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta\rho e^{\tilde{\omega}} + \frac{M_\infty}{Re_\infty}(\mu+\sigma\mu_T)\frac{\partial\tilde{\omega}}{\partial x_j}\cdot\frac{\partial\tilde{\omega}}{\partial x_j}$$

$$\frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[(\mu+\sigma\mu_T)\frac{\partial\tilde{\omega}}{\partial x_j}\right]$$

With the Reynolds stress tensor $\tau_{ij} = 2\mu_T\left[S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right] - \frac{2}{3}\rho k\delta_{ij}$

If the Bar Notation is dropped then the K-ω system is now

$$\boxed{\begin{aligned}
&\frac{\partial\rho k}{\partial t} + \frac{\partial}{\partial x_j}(\rho k u_j) = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta^*\rho k e^{\tilde{\omega}} + \frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[(\mu+\sigma^*\mu_T)\frac{\partial k}{\partial x_j}\right]\\
&\frac{\partial\rho\tilde{\omega}}{\partial t} + \frac{\partial}{\partial x_j}(\rho\tilde{\omega}u_j) = \frac{\alpha}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta\rho e^{\tilde{\omega}} + \frac{M_\infty}{Re_\infty}(\mu+\sigma\mu_T)\frac{\partial\tilde{\omega}}{\partial x_j}\cdot\frac{\partial\tilde{\omega}}{\partial x_j} + \frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[(\mu+\sigma\mu_T)\frac{\partial\tilde{\omega}}{\partial x_j}\right]\\
&\text{with}\\
&\tau_{ij} = 2\mu_T\left[S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right] - \frac{2}{3}\rho k\delta_{ij}
\end{aligned}}$$

K-ω Turbulence Model Non-dimensionalization.

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho k u_j) = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta^* \rho k \omega + \frac{\partial}{\partial x_j}\left((\mu + \sigma^* \mu_T)\frac{\partial k}{\partial x_j}\right)$$

$$\frac{\partial}{\partial t}(\rho \omega) + \frac{\partial}{\partial x_j}(\rho \omega u_j) = \frac{\alpha\omega}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta \rho \omega^2 + \frac{\partial}{\partial x_j}\left((\mu + \sigma \mu_T)\frac{\partial \omega}{\partial x_j}\right)$$

$$\tau_{ij} = 2\mu_T S_{ij} - \frac{2}{3}\mu_T \frac{\partial u_k}{\partial x_k}\delta_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

Introduce the following scaling's, from CFL3D manual.

$$\bar{k} = \frac{k}{a_\infty^2} \quad \bar{\omega} = \frac{\omega\,\mu_\infty}{\rho_\infty a_\infty^2}, \quad \tau = \frac{t\,a_\infty}{L} \quad \bar{u}_j = \frac{u_j}{a_\infty}, \quad \bar{x}_j = \frac{x_j}{L}, \quad \bar{\rho} = \frac{\rho}{\rho_\infty}$$

$$\bar{\mu}_T = \frac{\mu_T}{\mu_\infty}$$

K-equation.

$$\frac{\rho_\infty a_\infty^3}{L}\frac{\partial(\bar{\rho}\bar{k})}{\partial \tau} + \frac{\rho_\infty a_\infty^3}{L}\frac{\partial}{\partial \bar{x}_j}(\bar{\rho}\bar{k}\bar{u}_j) = \left[2\mu_\infty \frac{a_\infty}{L}\bar{\mu}_T S_{ij} - \frac{2}{3}\bar{\mu}_T\mu_\infty \frac{a_\infty}{L}\frac{\partial \bar{u}_k \delta_{ij}}{\partial \bar{x}_k} - \frac{2}{3}\rho_\infty a_\infty^2 \bar{\rho}\bar{k}\delta_{ij}\right]\frac{\partial \bar{u}_i}{\partial \bar{x}_j}\frac{a_\infty}{L}$$

$$- \beta^* \frac{\rho_\infty^2 a_\infty^4}{\mu_\infty}\bar{\rho}\bar{k}\bar{\omega} + \frac{\partial}{\partial \bar{x}_j}\left((\bar{\mu} + \sigma^* \bar{\mu}_T)\frac{\partial \bar{k}}{\partial \bar{x}_j}\right)\frac{\mu_\infty}{L^2}a_\infty^2$$

$\div$ by $\frac{\rho_\infty a_\infty^3}{L}$ gives

$$\frac{\partial}{\partial \tau}(\bar{\rho}\bar{k}) + \frac{\partial}{\partial \bar{x}_j}(\bar{\rho}\bar{k}\bar{u}_j) = \left[\frac{\mu_\infty a_\infty^2}{L^2}\frac{L}{\rho_\infty a_\infty^3}\left(2\bar{\mu}_T S_{ij} - \frac{2}{3}\bar{\mu}_T\frac{\partial \bar{u}_k}{\partial \bar{x}_k}\delta_{ij}\right) - \frac{\rho_\infty a_\infty^3}{\rho_\infty a_\infty^3 L}\frac{2}{3}\bar{\rho}\bar{k}S_{ij}\right]\frac{\partial \bar{u}_i}{\partial \bar{x}_j}$$

$$- \beta^* \frac{\rho_\infty^2 a_\infty^4}{\mu_\infty}\frac{L}{\rho_\infty a_\infty^3}\bar{\rho}\bar{k}\bar{\omega} + \frac{\partial}{\partial \bar{x}_j}\left((\bar{\mu} + \sigma^* \bar{\mu}_T)\frac{\partial \bar{k}}{\partial \bar{x}_j}\right)\frac{\mu_\infty a_\infty^2}{L^2}\frac{L}{\rho_\infty a_\infty^3}$$

$$\frac{\mu_\infty}{\rho_\infty a_\infty L} = \frac{M_\infty}{Re}, \quad \frac{\rho_\infty U_\infty L}{\mu_\infty M_\infty} = \frac{Re}{M_\infty} \quad \text{gives}$$

$$\frac{\partial}{\partial \tau}(\bar{\rho}\bar{k}) + \frac{\partial}{\partial \bar{x}_j}(\bar{\rho}\bar{k}\bar{u}_j) = \left[\frac{M_\infty}{Re}\mu_T 2 S_{ij} - \frac{M_\infty}{Re}\bar{\mu}_T\frac{2}{3}\frac{\partial \bar{u}_k}{\partial \bar{x}_k}\delta_{ij} - \frac{2}{3}\bar{\rho}\bar{k}\delta_{ij}\right]\frac{\partial \bar{u}_i}{\partial \bar{x}_j} - \beta^* \frac{Re}{M_\infty}\bar{\rho}\bar{k}\bar{\omega}$$

$$+ \frac{\partial}{\partial \bar{x}_j}\left((\bar{\mu} + \sigma^* \bar{\mu}_T)\frac{\partial \bar{k}}{\partial \bar{x}_j}\right)\frac{M_\infty}{Re}$$

$$\boxed{\frac{\partial}{\partial \tau}(\bar{\rho}\bar{k}) + \frac{\partial}{\partial \bar{x}_j}(\bar{\rho}\bar{k}\bar{u}_j) = \left[\mu_T\frac{M_\infty}{Re}\left(2S_{ij} - \frac{2}{3}\frac{\partial \bar{u}_k}{\partial \bar{x}_k}\delta_{ij}\right) - \frac{2}{3}\bar{\rho}\bar{k}\delta_{ij}\right]\frac{\partial \bar{u}_i}{\partial \bar{x}_j} - \beta^* \frac{Re}{M_\infty}\bar{\rho}\bar{k}\bar{\omega} + \frac{M_\infty}{Re}\frac{\partial}{\partial \bar{x}_j}\left((\bar{\mu} + \sigma^* \bar{\mu}_T)\frac{\partial \bar{k}}{\partial \bar{x}_j}\right)}$$

ω - equation.

Let $\omega_\infty = \frac{\rho_\infty a_\infty^2}{\mu_\infty}$, then

$$\frac{\rho_\infty a_\infty \omega_\infty}{L}\frac{\partial}{\partial \tau}(\bar\rho\bar\omega) + \frac{\rho_\infty a_\infty \omega_\infty}{L}\frac{\partial}{\partial x_j}(\bar\rho\bar\omega\bar v_j) = \alpha \frac{\omega_\infty}{a_\infty^2}\frac{\bar\omega}{\bar K}\left[\mu_\infty\bar\mu_T\left(2S_{ij} - \frac{2}{3}\frac{\partial \bar u_k}{\partial x_k}\delta_{ij}\right)\frac{a_\infty}{L} - \frac{2}{3}\rho_\infty a_\infty^2 \bar\rho \bar K\delta_{ij}\right]$$

$$\frac{\partial \bar u_i}{\partial x_j}\frac{a_\infty}{L} - \beta\rho_\infty\omega_\infty^2\bar\rho\bar\omega^2 + \frac{\partial}{\partial x_j}\left((\bar\mu+\sigma\bar\mu_T)\frac{\partial\bar\omega}{\partial x_j}\right)\frac{\omega_\infty\mu_\infty}{L^2}$$

$\div$ by $\frac{\rho_\infty a_\infty \omega_\infty}{L}$ gives $\frac{\mu_\infty}{L\rho_\infty a_\infty}$

$$\frac{\partial}{\partial \tau}(\bar\rho\bar\omega) + \frac{\partial}{\partial x_j}(\bar\rho\bar\omega\bar v_j) = \alpha\frac{\bar\omega}{\bar K}\left[\frac{\mu_\infty \cancel{a_\infty}}{L}\frac{L}{\cancel{K}\rho_\infty a_\infty^2}\bar\mu_T\left(2S_{ij} - \frac{2}{3}\frac{\partial \bar u_k}{\partial x_k}\delta_{ij}\right) - \frac{\cancel{\rho_\infty a_\infty^2}}{K}\frac{L}{\cancel{\rho a_\infty^2}}\right]\frac{\partial \bar u_i}{\partial x_j}$$

$$-\beta\frac{\cancel{\rho_\infty}\omega_\infty^{\cancel{2}}L}{\cancel{\rho_\infty}a_\infty\cancel{\omega_\infty}} + \frac{\partial}{\partial x_j}\left((\bar\mu+\sigma\bar\mu_T)\frac{\partial\bar\omega}{\partial x_j}\right)\frac{\mu_\infty\cancel{\omega_\infty}}{L^{\cancel{2}}}\frac{\cancel{L}}{\rho_\infty a_\infty\cancel{\omega_\infty}}$$

Again $\frac{\mu_\infty}{\rho_\infty a_\infty L} = \frac{M_\infty}{Re}$

$$\omega_\infty\frac{L}{a_\infty} = \frac{\rho_\infty a_\infty^{\cancel{2}}}{\mu_\infty}\frac{L}{\cancel{a_\infty}} = \frac{\rho_\infty a_\infty L}{\mu_\infty} = \frac{Re}{M_\infty}$$

Gives.

$$\boxed{\frac{\partial}{\partial \tau}(\bar\rho\bar\omega) + \frac{\partial}{\partial x_j}(\bar\rho\bar\omega\bar v_j) = \alpha\frac{\bar\omega}{\bar K}\left[\frac{M_\infty}{Re}\bar\mu_T\left(2S_{ij} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - \frac{2}{3}\bar\rho\bar K\delta_{ij}\right] - \beta\frac{Re}{M_\infty}\bar\rho\bar\omega^2 \quad \frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left((\bar\mu+\sigma\bar\mu_T)\frac{\partial\bar\omega}{\partial x_j}\right)}$$

The corresponding Log - form is

$$\frac{\partial}{\partial \tau}(\rho\tilde\omega) + \frac{\partial}{\partial x_j}(\rho\tilde\omega v_j) = \frac{\alpha}{K}\left[\frac{M_\infty}{Re}\bar\mu_T\left(2S_{ij} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - \frac{2}{3}\bar\rho K\delta_{ij}\right] - \beta\frac{Re}{M_\infty}\bar\rho e^{\tilde\omega} +$$

$$\frac{M_\infty}{Re_\infty}(\bar\mu+\sigma\bar\mu_T)\frac{\partial\tilde\omega}{\partial x_j}\cdot\frac{\partial\tilde\omega}{\partial x_j} + \frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left((\mu+\sigma\mu_T)\frac{\partial\tilde\omega}{\partial x_j}\right)$$

$$\mu_T = \rho\frac{K}{\omega} \Rightarrow \mu_\infty\bar\mu_T = \frac{\bar\rho\,\cancel{\rho_\infty a_\infty^2}\bar K}{\cancel{\rho_\infty a_\infty^2}\bar\omega}\mu_\infty \Rightarrow \bar\mu_T = \bar\rho\frac{\bar K}{\bar\omega}$$

$$\bar\mu_T = \bar\rho\frac{\bar K}{\bar\omega}$$

K-ω Diffusion terms (D.G. Discretization); 2D

$$\frac{\partial K}{\partial x} = -\frac{K}{\rho}\frac{\partial \rho}{\partial x} + \frac{\partial(\rho K)}{\partial x}\cdot\frac{1}{\rho}$$

$$\frac{\partial \tilde\omega}{\partial x} = -\frac{\tilde\omega}{\rho}\frac{\partial \rho}{\partial x} + \frac{\partial(\rho\tilde\omega)}{\partial x}\frac{1}{\rho}$$

$$\frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[(\mu + \sigma^* \mu_T)\frac{\partial K}{\partial x_j}\right]:$$

$$\frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[(\mu + \sigma \mu_T)\frac{\partial \tilde\omega}{\partial x_j}\right]$$

Begin with K- equation

$$\int_\Omega \phi_i \frac{M_\infty}{Re_\infty}\frac{\partial}{\partial x_j}\left[(\mu+\sigma^*\mu_T)\frac{\partial K}{\partial x_j}\right]d\Omega = -\int_\Omega \frac{M_\infty}{Re_\infty}(\mu+\sigma^*\mu_T)\frac{\partial K}{\partial x}\cdot\frac{\partial \phi_i}{\partial x} + \frac{\partial K}{\partial y}\frac{\partial \phi_i}{\partial y}\, d\Omega$$

L/R - Left or Right

$$+ \oint \frac{M_\infty}{Re_\infty}(\mu+\sigma^*\mu_T)\frac{\partial K}{\partial x}\cdot n_x + \frac{\partial K}{\partial y}\cdot n_y\, ds + \oint \frac{1}{2}\frac{M_\infty}{Re_\infty}(\mu+\sigma^*\mu_T)\begin{bmatrix}[G_{11}]^{L/R} & [G_{12}]^{L/R}\\ [G_{21}]^{L/R} & [G_{22}]^{L/R}\end{bmatrix}\cdot\begin{pmatrix}\frac{\partial \phi_i}{\partial x}\\ \frac{\partial \phi_i}{\partial y}\end{pmatrix}\begin{Bmatrix}n_x\\ n_y\end{Bmatrix}\cdot \Delta\vec{q}$$

$$-\oint \frac{1}{2}\left([G_{11}^L]+[G_{22}^L] + [G_{11}^R]+[G_{22}^R]\right)pen\cdot \Delta\vec{q}\, ds$$

Thus we have 4 terms

1). Volume: $-\int_\Omega \frac{M_\infty}{Re_\infty}(\mu+\sigma^*\mu_T)\left(\frac{\partial K}{\partial x}\cdot\frac{\partial \phi_i}{\partial x} + \frac{\partial K}{\partial y}\cdot\frac{\partial \phi_i}{\partial y}\right)d\Omega$ - Straight forward

average

2) Surface: $\oint \frac{M_\infty}{Re_\infty}\left\{(\mu+\sigma^*\mu_T)\left(\frac{\partial K}{\partial x}\cdot n_x + \frac{\partial K}{\partial y}\cdot n_y\right)\right\}ds$ — straight forward.

3). Symmetric: $\oint \frac{1}{2}\frac{M_\infty}{Re_\infty}(\mu+\sigma^*\mu_T)\begin{bmatrix}[G_{11}]^{L/R} & [G_{12}]^{L/R}\\ [G_{21}]^{L/R} & [G_{22}]^{L/R}\end{bmatrix}\cdot\begin{pmatrix}\frac{\partial \phi_i}{\partial x}\\ \frac{\partial \phi_i}{\partial y}\end{pmatrix}\begin{Bmatrix}n_x\\ n_y\end{Bmatrix}\cdot\Delta\vec{q}\, ds$

Let $\vec{F}_x^v = \frac{M_\infty}{Re}(\mu+\sigma^*\mu_T)\frac{\partial K}{\partial x}$, $\quad \vec{F}_y^v = \frac{M_\infty}{Re}(\mu+\sigma^*\mu_T)\frac{\partial K}{\partial y}$

$$G_{11} = \frac{\partial \vec{F}_x^v}{\partial\left(\frac{\partial \vec{q}}{\partial x}\right)}, \quad G_{12} = \frac{\partial \vec{F}_x^v}{\partial\left(\frac{\partial \vec{q}}{\partial y}\right)}$$

$$G_{21} = \frac{\partial \vec{F}_y^v}{\partial\left(\frac{\partial \vec{q}}{\partial x}\right)}, \quad G_{22} = \frac{\partial \vec{F}_y^v}{\partial\left(\frac{\partial \vec{q}}{\partial y}\right)}$$

For these Recall $\frac{\partial K}{\partial x} = \frac{\partial K}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial x}$

$$\frac{\partial K}{\partial \vec{q}} = \left\lfloor -\tfrac{K}{\rho}, 0, 0, 0, \tfrac{1}{\rho}, 0\right\rfloor$$

Clearly the G_{ij}'s are Row Vectors with

$$G_{11} = \frac{M_\infty}{Re_\infty}(\mu+\sigma^*\mu_T)\left\lfloor -\tfrac{K}{\rho}, 0, 0, 0, \tfrac{1}{\rho}, 0\right\rfloor, \quad G_{12} = \lfloor 0\rfloor$$

$$G_{21} = \lfloor 0\rfloor, \quad G_{22} = \frac{M_\infty}{Re_\infty}(\mu+\sigma^*\mu_T)\left\lfloor -\tfrac{K}{\rho}, 0, 0, 0, \tfrac{1}{\rho}, 0\right\rfloor$$

Since $G_{11} = G_{22}$, $G_{12} = G_{21} = 0$ the symmetry term is simplified

$$\oint \frac{1}{2} \frac{M_\infty}{Re_\infty} (\mu + \sigma^* \mu_T) \left(\left[-\frac{K}{\rho}, 0, 0, 0, \frac{1}{\rho}, 0 \right] \frac{\partial Q_i}{\partial x} n_x + \left[-\frac{K}{\rho}, 0, 0, 0, \frac{1}{\rho}, 0 \right] \frac{\partial Q_i}{\partial y} n_y \right) \cdot \Delta \vec{\delta} \, ds$$

$$= \oint \frac{1}{2} \frac{M_\infty}{Re_\infty} (\mu + \sigma^* \mu_T) \left(-\frac{K}{\rho} \Delta \rho + \frac{\Delta \rho K}{\rho} \right) \left(\frac{\partial Q_i}{\partial x} \cdot n_x + \frac{\partial Q_i}{\partial y} \cdot n_y \right) ds$$

4). The penalty term, making use of the facts from the symmetry term:

$$\oint \frac{1}{2} pen \left[G_{11}^L + G_{22}^L + G_{11}^R + G_{22}^R \right] \Delta \delta \, ds = \oint pen \left[termL \left(-\frac{K^L}{\rho_L} \Delta \rho + \frac{\Delta \rho K}{\rho_L} \right) + termr \left(-\frac{K^R}{\rho_R} \Delta \rho + \frac{\Delta \rho K}{\rho_R} \right) \right] ds$$

where $termL = \frac{M_\infty}{Re_\infty} (\mu_L + \sigma^* \mu_T^L)$

where $termr = \frac{M_\infty}{Re_\infty} (\mu_R + \sigma^* \mu_T^R)$

For the ω equation

1) Volume: $\int \frac{M_\infty}{Re_\infty} (\mu + \sigma \mu_T) \left(\frac{\partial \tilde{\omega}}{\partial x} \frac{\partial \omega}{\partial x} + \frac{\partial \tilde{\omega}}{\partial y} \frac{\partial \omega}{\partial y} \right) d\Omega$ – simple

2) Surface: $\oint \frac{M_\infty}{Re_\infty} \left\{ (\mu + \sigma \mu_T) \left(\frac{\partial \tilde{\omega}}{\partial x} \cdot n_x + \frac{\partial \tilde{\omega}}{\partial y} \cdot n_y \right) \right\} ds$ – simple.

3) Symmetry: $\oint \frac{M_\infty}{Re_\infty} \frac{1}{2} (\mu + \sigma \mu_T) \left[\left[-\frac{\tilde{\omega}}{\rho}, 0, 0, 0, 0, \frac{1}{\rho} \right] \frac{\partial Q_i}{\partial x} n_x + \left[-\frac{\tilde{\omega}}{\rho}, 0, 0, 0, 0, \frac{1}{\rho} \right] \frac{\partial Q_i}{\partial y} n_y \right] \Delta \vec{\delta} \, ds$

$$= \oint \frac{M_\infty}{Re_\infty} \frac{1}{2} (\mu + \sigma \mu_T) \left(-\frac{\tilde{\omega}}{\rho} \Delta \rho + \frac{\Delta \rho \tilde{\omega}}{\rho} \right) \left(\frac{\partial Q_i}{\partial x} \cdot n_x + \frac{\partial Q_i}{\partial y} \cdot n_y \right) ds$$

4). Penalty:

$$\oint pen \left[termL \left(-\frac{\omega^L}{\rho_L} \Delta \rho + \frac{\Delta \rho \tilde{\omega}}{\rho_L} \right) + termR \left(-\frac{\omega^R}{\rho_R} \Delta \rho + \frac{\Delta \rho \tilde{\omega}}{\rho_R} \right) \right] ds$$

Remark: These diffusion terms follow the form of the S.A. & just twice. they are NOT "coupled" in that there is NO cross diffusion term.

k-ω Source term:

1). k – equation

The production term needs a bit of expansion for 2-D flows:

$$\left[\frac{M_\infty}{Re}\mu_T\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) - \frac{M_\infty}{Re}\mu_T\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} - \frac{2}{3}\bar{\rho}k\,\delta_{ij}\right]\frac{\partial u_i}{\partial x_j}$$

Expanding for 2-D.

$$\frac{M_\infty}{Re}\mu_T\Bigg[\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial x} +$$
$$\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y}\Bigg] - \frac{2}{3}\rho k\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

This can be further simplified as

$$\frac{M_\infty}{Re}\mu_T\left[\frac{4}{3}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{2}{3}\frac{\partial v}{\partial y}\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 + \frac{4}{3}\left(\frac{\partial v}{\partial y}\right)^2 - \frac{2}{3}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right] - \frac{2}{3}\rho k\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

K-ω Source term:

There are a couple of things to comment on for the K-ω source terms.

given as

$$K\text{-src} \equiv \frac{M_\infty}{Re_\infty} \tau_{ij} \frac{\partial u_i}{\partial x_j} - \beta^* \rho k e^{\tilde{\omega}}$$

$$\omega\text{-src} \equiv \frac{M_\infty}{Re_\infty} \frac{\alpha}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j} - \beta \rho e^{\tilde{\omega}} + \frac{M_\infty}{Re_\infty} (\mu + \sigma \mu_T) \frac{\partial \tilde{\omega}}{\partial x_j} \cdot \frac{\partial \tilde{\omega}}{\partial x_j}$$

$$\tau_{ij} = 2\mu_T \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right] - \frac{2}{3} \rho k \delta_{ij}$$

$$\tau_{ij} = \mu_T \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu_T \frac{\partial u_k}{\partial x_k} \delta_{ij} - \frac{2}{3} \rho k \delta_{ij}$$

The Product

$$\tau_{ij} \frac{\partial u_i}{\partial x_j} = \left[\mu_T \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu_T \frac{\partial u_k}{\partial x_k} \delta_{ij} - \frac{2}{3} \rho k \delta_{ij} \right] \frac{\partial u_i}{\partial x_j}$$

$$\tau_{ij} = \begin{bmatrix} \frac{4}{3}\mu_T \frac{\partial u}{\partial x} - \frac{2}{3}\rho k & \mu_T\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \mu_T\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{4}{3}\mu_T \frac{\partial v}{\partial y} - \frac{2}{3}\rho k \end{bmatrix}$$

$$\Rightarrow \tau_{ij} \frac{\partial u_i}{\partial x_j} = \left(\frac{4}{3} \mu_T \frac{\partial u}{\partial x} - \frac{2}{3} \rho k \right) \frac{\partial u}{\partial x} + \mu_T \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \cdot \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \mu_T \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\left(\frac{4}{3} \mu_T \frac{\partial v}{\partial y} - \frac{2}{3} \rho k \right) \frac{\partial v}{\partial y}$$

$$\tau_{ij} \frac{\partial u_i}{\partial x_j} = \boxed{\left(\frac{4}{3} \mu_T \frac{\partial u}{\partial x} - \frac{2}{3} \rho k \right) \frac{\partial u}{\partial x} + \mu_T \left(\frac{\partial u}{\partial y} \right)^2 + 2\mu_T \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) + \mu_T \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{4}{3} \mu_T \frac{\partial v}{\partial y} - \frac{2}{3} \rho k \right) \frac{\partial v}{\partial y}}$$

$$\omega = e^{\tilde{\omega}}$$

$$\boxed{\frac{\partial \omega}{\partial x_j} = \frac{\partial \omega}{\partial \tilde{\omega}} \cdot \frac{\partial \tilde{\omega}}{\partial x_j} = e^{\tilde{\omega}} \cdot \frac{\partial \tilde{\omega}}{\partial x_j}} = \omega \frac{\partial \tilde{\omega}}{\partial x_j}$$

$$\Omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \qquad S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right)$$

The Model also requires the computation of

$\Omega_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j}\right)$, $S_{ki} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)$

$\Omega_{ij}\Omega_{jk}S_{ki} = \Omega_{ij} M_{ji}$, with $M_{ji} = \Omega_{jk} S_{ki}$

$$M_{11} = \tfrac{1}{2}\left(\cancel{\tfrac{\partial u}{\partial x} - \tfrac{\partial u}{\partial x}}\right)\tfrac{1}{2}\left(\tfrac{\partial u}{\partial x} + \tfrac{\partial u}{\partial x}\right) + \tfrac{1}{2}\left(\tfrac{\partial u}{\partial y} - \tfrac{\partial v}{\partial x}\right)\tfrac{1}{2}\left(\tfrac{\partial v}{\partial x} + \tfrac{\partial u}{\partial y}\right)$$

$$M_{12} = \tfrac{1}{2}\left(\cancel{\tfrac{\partial u}{\partial x} - \tfrac{\partial u}{\partial x}}\right)\tfrac{1}{2}\left(\tfrac{\partial u}{\partial x} + \tfrac{\partial u}{\partial x}\right) + \tfrac{1}{2}\left(\tfrac{\partial u}{\partial y} - \tfrac{\partial v}{\partial x}\right)\tfrac{1}{2}\left(\tfrac{\partial v}{\partial y} + \tfrac{\partial v}{\partial y}\right)$$

$$M_{21} = \tfrac{1}{2}\left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}\right)\tfrac{1}{2}\left(\tfrac{\partial u}{\partial x} + \tfrac{\partial u}{\partial x}\right) + \tfrac{1}{2}\left(\cancel{\tfrac{\partial v}{\partial y} - \tfrac{\partial v}{\partial y}}\right)\tfrac{1}{2}\left(\tfrac{\partial v}{\partial y} + \tfrac{\partial v}{\partial y}\right)$$

$$M_{22} = \tfrac{1}{2}\left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}\right)\tfrac{1}{2}\left(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\right) + \tfrac{1}{2}\left(\cancel{\tfrac{\partial v}{\partial y} - \tfrac{\partial v}{\partial y}}\right)\tfrac{1}{2}\left(\tfrac{\partial v}{\partial y} + \tfrac{\partial v}{\partial y}\right)$$

$$M_{11} = \tfrac{1}{4}\left[\left(\tfrac{\partial u}{\partial y}\right)^2 + \tfrac{\partial v}{\partial x}\tfrac{\partial u}{\partial y} - \left(\tfrac{\partial v}{\partial x}\right)^2 - \tfrac{\partial v}{\partial x}\tfrac{\partial u}{\partial y}\right] = \tfrac{1}{4}\left[\left(\tfrac{\partial u}{\partial y}\right)^2 - \left(\tfrac{\partial v}{\partial x}\right)^2\right]$$

$$M_{12} = \tfrac{1}{2}\left[\left(\tfrac{\partial u}{\partial y} - \tfrac{\partial v}{\partial x}\right)\left(\tfrac{\partial v}{\partial y}\right)\right]$$

$$M_{21} = \tfrac{1}{2}\left[\left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}\right)\tfrac{\partial u}{\partial x}\right]$$

$$M_{22} = \tfrac{1}{4}\left[\left(\tfrac{\partial v}{\partial x}\right)^2 - \left(\tfrac{\partial u}{\partial y}\right)^2\right]$$

$$\Omega_{ij} M_{ji} = \Omega_{11} M_{11} + \Omega_{12} M_{21} + \Omega_{21} M_{12} + \Omega_{22} M_{22}$$

$$\Omega_{11} = \tfrac{1}{2}\left(\tfrac{\partial u}{\partial x} - \tfrac{\partial u}{\partial x}\right) = 0$$

$$\Omega_{12} = \tfrac{1}{2}\left(\tfrac{\partial u}{\partial y} - \tfrac{\partial v}{\partial x}\right)$$

$$\Omega_{21} = \tfrac{1}{2}\left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}\right)$$

$$\Omega_{22} = \tfrac{1}{2}\left(\tfrac{\partial v}{\partial y} - \tfrac{\partial v}{\partial y}\right) = 0$$

$$\Omega_{ij} M_{ij} = \tfrac{1}{2}\left(\tfrac{\partial u}{\partial y} - \tfrac{\partial v}{\partial x}\right)\tfrac{1}{2}\left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}\right)\left(\tfrac{\partial u}{\partial x}\right) + \tfrac{1}{2}\left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}\right)\tfrac{1}{2}\left(\tfrac{\partial u}{\partial y} - \tfrac{\partial v}{\partial x}\right)\left(\tfrac{\partial v}{\partial y}\right)$$

$$\boxed{\Omega_{ij}\Omega_{jk}S_{ki} = \tfrac{1}{4}\left[\left(\tfrac{\partial u}{\partial y} - \tfrac{\partial v}{\partial x}\right)\left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}\right)\tfrac{\partial u}{\partial x} + \left(\tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}\right)\left(\tfrac{\partial u}{\partial y} - \tfrac{\partial v}{\partial x}\right)\tfrac{\partial v}{\partial y}\right]}$$

Remark: This is **NOT** zero as per the Wilcox Book, ~~th~~ for some reason he insists on his analysis being done for incompressible flows for which this is true for comp it is not. These turbulence modeling people are kinda dumb about this.

$\underline{K\text{-}\omega}$:

Boundary Conditions

1) Free stream

$$K_\infty = \frac{3}{2}(.01)^2 \|\vec{U}_\infty\|^2 = \quad a_\infty^2 \bar{K}_\infty = \frac{3}{2}(.01)^2 a_\infty^2 M_\infty^2$$

$$\boxed{\bar{K}_\infty = \frac{3}{2}(.01)^2 M_\infty^2}$$

$$\boxed{\bar{\mu}_{T\infty} = .01} \qquad \omega_\infty = \frac{K_\infty}{\mu_{T\infty}}$$

$$\bar{\omega}_\infty \frac{\rho_\infty a_\infty^2}{\mu_\infty} = \frac{\rho_\infty a_\infty^2 \bar{\rho} \bar{K}_\infty}{\mu_\infty \bar{\mu}_T}$$

$$\boxed{\bar{\omega}_\infty = \bar{\rho} \frac{\bar{K}_\infty}{\bar{\mu}_T}}$$

2) Solid Wall

$$\bar{K}_\infty = 0$$

$$\omega_w = 60 \frac{\mu}{\rho \beta y_1^2}$$

$$\frac{\rho_\infty a_\infty^2}{\mu_\infty} \bar{\omega}_w = \frac{60 \bar{\mu} \mu_\infty}{\rho_\infty \bar{\rho} \beta L^2 \bar{y}_1^2}$$

$$\bar{\omega}_w = \frac{60 \bar{\mu}}{\bar{\rho} \beta y_1^2} \left(\frac{\mu_\infty}{\rho_\infty a_\infty L} \right)^2 = \boxed{\frac{60 \bar{\mu}}{\bar{\rho} \beta \bar{y}_1^2} \left(\frac{M_\infty}{Re} \right)^2 = \bar{\omega}_w}$$

K-ω Realizability Constraint on $\tilde{\omega}$

A given by R. Hartmann et al the Realizability constraints are

$$e^{\tilde{\omega}_{r0}} - \frac{3}{2} C_\mu S_{ii} \geq 0 \quad \text{for } i = 1, 2 \ldots d.$$

$$\left(e^{\tilde{\omega}_{r0}}\right)^2 - \frac{3}{2} C_\mu (S_{ii} + S_{jj}) e^{\tilde{\omega}_{r0}} + \frac{9}{4} C_\mu^2 (S_{ii} S_{jj} - S_{ij}^2) \geq 0 \quad \text{for } i,j = 1,2 \ldots d \quad i \neq j$$

with $S_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3} \nabla \cdot \bar{u}\, \delta_{ij}$

Remark: Here S_{ij} is $2 \cdot$ the S_{ij} I normally write down.

1) Non-dimensionalization.

Let ω_{r0} denote $e^{\tilde{\omega}_{r0}}$ thus $e^{\tilde{\omega}_{r0}}$ has the units of the ω variable thus we non-dimensionalization

$$\bar{\omega}_{r0} = \frac{\omega_{r0}\, \mu_\infty}{\rho_\infty a_\infty^2} = \frac{e^{\tilde{\omega}_{r0}}\, \mu_\infty}{\rho_\infty a_\infty^2}$$

Introducing this into the constraints gives

$$\frac{\rho_\infty a_\infty^2}{\mu_\infty} \bar{\omega}_{r0} - \frac{3}{2} C_\mu \frac{a_\infty}{L} \bar{S}_{ii} \geq 0 \quad i = 1, 2 \ldots d.$$

$$\bar{\omega}_{r0} - \frac{3}{2} C_\mu \frac{\mu_\infty}{\rho_\infty a_\infty L} \bar{S}_{ii} \geq 0 \quad i = 1, 2 \ldots d$$

$$\boxed{\bar{\omega}_{r0} - \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty} \bar{S}_{ii} \geq 0} \quad (1)$$

$$\frac{\rho_\infty^2 a_\infty^4}{\mu_\infty^2} \bar{\omega}_{r0}^2 - \frac{3}{2} C_\mu \frac{a_\infty}{L} (\bar{S}_{ii} + \bar{S}_{jj}) \frac{\rho_\infty a_\infty^2}{\mu_\infty} \bar{\omega}_{r0} + \frac{9}{4} C_\mu^2 \frac{a_\infty^2}{L^2} (\bar{S}_{ii}\bar{S}_{jj} - \bar{S}_{ij}^2) \geq 0 \quad i,j = 1,2 \ldots d \quad i \neq j$$

$$\bar{\omega}_{r0}^2 - \frac{3}{2} C_\mu (\bar{S}_{ii} + \bar{S}_{jj}) \bar{\omega}_{r0} \frac{\rho_\infty a_\infty^3}{L \mu_\infty} \frac{\mu_\infty^2}{\rho_\infty^2 a_\infty^4} + \frac{9}{4} C_\mu^2 (\bar{S}_{ii}\bar{S}_{jj} - \bar{S}_{ij}^2) \frac{a_\infty^2}{L^2} \frac{\mu_\infty^2}{\rho_\infty^2 a_\infty^4} \geq 0$$

gives

$$\bar{\omega}_{r0}^2 - \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty} (\bar{S}_{ii} + \bar{S}_{jj}) \bar{\omega}_{r0} + \frac{9}{4} C_\mu^2 \left(\frac{M_\infty}{Re_\infty}\right)^2 (\bar{S}_{ii}\bar{S}_{jj} - \bar{S}_{ij}^2) \geq 0$$

$$\boxed{\bar{\omega}_{r_0}^2 - \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty}\left(\bar{S}_{ii} + \bar{S}_{jj}\right)\bar{\omega}_{r_0} + \frac{9}{4} C_\mu^2 \left(\frac{M_\infty}{Re_\infty}\right)^2 \left(\bar{S}_{ii}\bar{S}_{jj} - S_{ij}^2\right) \geq 0 \quad \text{for } i,j = 1,2,3 \quad i \neq j} \quad (2)$$

Expand (1) and (2) Drop $\bar{(\,)}$ for convience

First expand (1) $\bar{S}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \nabla \cdot \vec{u}\, \delta_{ij}$

$i=1$: $\bar{\omega}_{r_0} - \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty} S_{11} \geq 0$

$i=2$: $\bar{\omega}_{r_0} - \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty} S_{22} \geq 0$

$S_{11} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} - \frac{2}{3} \nabla \cdot \vec{u}$, $\quad S_{22} = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} - \frac{2}{3} \nabla \cdot \vec{u}$

gives

$$\boxed{\begin{aligned} i=1:&\quad \bar{\omega}_{r_0} = \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty}\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{u}\right) \\ i=2:&\quad \bar{\omega}_{r_0} = \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty}\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\vec{u}\right) \end{aligned}}$$

Expand (2)

$i=1, j=2$: $\bar{\omega}_{r_0}^2 - \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty}\left(S_{11} + S_{22}\right)\bar{\omega}_{r_0} + \frac{9}{4} C_\mu^2 \left(\frac{M_\infty}{Re_\infty}\right)^2 \left(S_{11}S_{22} - S_{12}^2\right) \geq 0$

$i=2, j=1$. $\bar{\omega}_{r_0}^2 - \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty}\left(S_{22} + S_{11}\right)\bar{\omega}_{r_0} + \frac{9}{4} C_\mu^2 \left(\frac{M_\infty}{Re_\infty}\right)^2 \left(S_{22}S_{11} - S_{21}^2\right) \geq 0$

$S_{ij} = S_{ji}$ symmetric thus the above two equations are actually the same. thus.

$$\bar{\omega}_{r_0}^2 - \frac{3}{2} C_\mu \frac{M_\infty}{Re_\infty}\left(2\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial y} - \frac{4}{3}\nabla\cdot\vec{u}\right) + \frac{9}{4} C_\mu^2 \left(\frac{M_\infty}{Re_\infty}\right)^2 \left[\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{u}\right)\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\vec{u}\right) - S_{12}^2\right]$$

Using Maple the roots of the quadratic are

$$w_{rot\pm} = \frac{1}{2}\left[3\frac{\partial u}{\partial x} + 3\frac{\partial v}{\partial y} - 2\nabla\cdot\vec{u} \pm 3\sqrt{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + S_{12}^2}\right]$$

We need the larger of the 2 which is clearly the + branch the realizability is thus the third and final condition on

$$w_{r_0} = \left[\frac{3}{2}\frac{\partial u}{\partial x} + \frac{3}{2}\frac{\partial v}{\partial y} - \nabla\cdot\vec{u} \pm \frac{3}{2}\sqrt{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + S_{12}^2}\right]$$

$$\boxed{w_{r0} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + 3\sqrt{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + S_{12}^2}\right]}$$

$$S_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$ - here

Dual Consistency Analysis: Linear Problems (This is easier to build basics)

1). How do we do this

First form a weak form of the PDE. call $B(U_h, V_h)$ where $U_h \sim$ is approximate solution $U_h \in V_h \subset V$ and V_h the usual test function $V_h \in V_h$. The weak form can be written as,

$$B(U_h, V_h) = f_h \quad \text{for} \quad LU = f \quad \text{where } f_h \text{ is}$$

an appropriate projection of the f term into V_h.

Now we'll some information on the continuous Adjoint both for analysis and for a weak "descrete" form of the Adjoint equation

Continuous Adjoint. equation for linear operator L.

Consider $LU = f$.

The continuous Adjoint operator $L^*\psi$, where ψ is the Adjoint variable obeys the identity.

$$\langle LU, \psi \rangle = \langle U, L^*\psi \rangle \quad \text{where } \langle \ \rangle \text{ is the inner product}$$

Given a functional $J(U) = \langle U, g \rangle$.

The particular Adjoint variable corresponding to the functional $J(U)$ will be given by

$$L^*\psi = g$$

where L^* is the continuous Adjoint operator
ψ is the Adjoint variable
g is the fn that "samples" U to define $J(U)$.

With this in hand we note that the functional $J(U)$ can be defined in two ways.

$$J(U) = \langle U, g \rangle = \langle U, L^*\psi \rangle = \langle LU, \psi \rangle = \boxed{\langle f, \psi \rangle = \langle U, g \rangle}$$

using duality and definition of the Adjoint.

Thus really the choice of $L^*\psi = g$ is such that both the operator and the functional J satisfy the same duality statement which requires $L^*\psi = g$ to compute as above.

To do dual consistency analysis we will require the use of the continuous Adjoint of a given operator L.

Adjoint of Discretized weak forms

Again consider that the weak form of the PDE with approximate solution $u_h \in V_h$ given by

$$B(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

The above is said to be consistent if substitution of the approximate solution by the exact solution satisfies

$$B(u, v_h) = F(v_h) \quad \forall v_h \in V$$ where u is the exact solution to $Lu = f$.

Note that $B(u, v_h) = F(u_h)$ represents the discretized weak form acting on the exact solution u. This will be clear with an example.

In what follows I cannot prove. I've it written in many places, Hopefully Dimitri will be able to help us cover why. Regardless a schemes is said to be adjoint/Dual consistent if this operator $B(\ )$ also satisfies

$$B(w, \psi) = \langle w, g \rangle \equiv J(w) \quad \forall w \in V$$ w is just a test fn for the Adjoint problem just written differently to be clear.

Remark: Exactly why this should work I do not know. But that's it. It is really not that complicated.

As a simple example Let's follow R. Hartmann (SIAM, 2007) and analyze a linear wave equation. The analysis is modified by using a standard numerical flux for the boundary terms.

Ex. 1. Linear wave equation

$\nabla\cdot(\vec{b}u) + cu = f \quad \vec{x}\in\Omega$ with $u = a$ on Γ_{inflow}.

Discretize using DG.

$$\int_\Omega v_h \nabla\cdot(\vec{b}u_h) + c u_h v_h \, d\Omega = \int_\Omega v_h f \, d\Omega$$

Discretizers Ω into $\mathcal{T}_h$ simplex elements.

$$\sum_{e\in\mathcal{T}_h}\int_{\Omega_h} v_h\left(\nabla\cdot(\vec{b}u_h) + c u_h\right) d\Omega_h = \sum_{e\in\mathcal{T}_h}\int_{\Omega_h} v_h f \, d\Omega_h$$

Integrating by parts.

$$B(u_h,v_h) = -\sum_{e\in\mathcal{T}_h}\int_{\Omega_h}\nabla v_h\cdot\vec{b}u_h - v_h c u_h \, d\Omega_h + \sum_{e\in\mathcal{T}_h}\oint_{\partial e\setminus\Gamma} H(u^+,u^-,\vec{n})\cdot v_h^+ \, ds$$

$$+\sum_{b\in B_h}\oint_{\Gamma_{in}} \vec{b}\cdot\vec{n}\, a\, v_h^+ \, ds + \sum_{b\in B_h}\int_{\Gamma_{out}} \vec{b}\cdot\vec{n}\, u_h v_h^+ \, ds = \sum_{e\in\mathcal{T}_h}\int_{\Omega_h} v_h f \, d\Omega_h$$

Where $(\,)^+$ denotes value inside e and $(\,)^-$ denotes value outside e, and $\vec{n}$ points out of e.

<u>Consistency of the Primal Scheme</u>

Integrating by parts again yield.

$$= \sum_{e\in\mathcal{T}_h}\int_{\Omega_h} v_h\left(\nabla\cdot(\vec{b}u_h) + c u_h\right) d\Omega_h + \sum_{e\in\mathcal{T}_h}\oint_{\partial e\setminus\Gamma} H(u_h^+,u_h^-,\vec{n})\cdot v_h^+ - \vec{b}\cdot\vec{n}\, u_h v_h^+ \, ds$$

$$-\sum_{b\in B_h}\oint_{\Gamma_{in}}\left(\vec{b}\cdot\vec{n}\, a v_h^+ - \vec{b}\cdot\vec{n}\, u_h v_h^+\right) ds + \sum_{b\in B_h}\oint_{\Gamma_{out}}\left(\cancel{\vec{b}\cdot\vec{n}\, u_h v_h - \vec{b}\cdot\vec{n}\, u_h v_h}\right) ds = \sum_{e\in\mathcal{T}_h}\int v_h f \, d\Omega_h$$

Let $u_h \to u$ in the above.

$$= \sum_{e\in\mathcal{T}_h}\int_{\Omega_h} v_h\left(\nabla\cdot(\vec{b}u) + c u\right) d\Omega_h + \sum_{e\in\mathcal{T}_h}\oint_{\partial e\setminus\Gamma} H(u,u,\vec{n})\cdot v_h^+ - \vec{b}\cdot\vec{n}\, u v_h^+ \, ds$$

$$-\sum_{b\in B_h}\oint_{\Gamma_{in}}\left(\vec{b}\cdot\vec{n}\, a v_h - \vec{b}\cdot\vec{n}\, u v_h^+\right) ds = \sum_{e\in\mathcal{T}_h}\int v_h f \, d\Omega$$

With the use of the following identities

$H(u, u, \vec{n}) = \vec{b}\cdot\vec{n}\, u$ the scheme is shown be consistent.

$u|_{\Gamma in} = a$

by Definition of PDE

$$B(u, v_h) = \sum_{e \in \mathcal{E}_h} \int_{\Omega_h} v_h [\underbrace{\nabla\cdot(\vec{b}u) + cu - f}_{\to 0}]\, d\Omega_h + \sum_{e\in\mathcal{E}_h} \oint_{\partial e \setminus \Gamma} \underbrace{(\vec{b}\cdot\vec{n}u - \vec{b}\cdot\vec{n}u)}_{\to 0} v_h^+ ds$$

$$- \sum_{b \in B_h} \oint_{\Gamma in} \vec{b}\cdot\vec{n}\,\underbrace{(u|_{\Gamma in} - a)}_{=a \;\to 0} v_h^+ ds = 0$$

Thus the scheme is consistent.

b). Adjoint consistency:

i). Continuous Adjoint,

$\nabla\cdot(\vec{b}u) + cu = f \quad \vec{x}\in\Omega \qquad Bu = a \quad \vec{x}\in\Gamma in,$

$\qquad\qquad\qquad\qquad\qquad\qquad Bu = 0 \quad \vec{x}\in\Gamma out.$

The continuous Adjoint is found via the identity

$$\langle Lu, \psi\rangle + \langle Bu, C^*\psi\rangle_\Gamma = \langle u, L^*\psi\rangle + \langle Cu, B^*\psi\rangle_\Gamma$$

$$\int_\Omega (\nabla\cdot(\vec{b}u) + cu)\psi\, d\Omega = \int_\Omega -\nabla\psi\cdot\vec{b}u + c\psi u\, d\Omega - \oint_{\Gamma in} \psi\cdot\vec{b}\cdot\vec{n}\, a\, ds + \oint_{\Gamma out} \psi\cdot\vec{b}\cdot\vec{n}\, u\, ds \Rightarrow$$

$L^* \equiv -\nabla\psi(\vec{b}u) + c\psi$

The surface terms are Harder

$\oint_{\Gamma in} \psi\cdot\vec{b}\cdot\vec{n}\, a\, ds = \int_{\Gamma in} \psi\vec{b}\cdot\vec{n}\, Bu\, ds \Rightarrow C^* = \psi,\ Bu = a.$

$\oint_{\Gamma out} \psi\cdot\vec{b}\cdot\vec{n}\, u\, ds \Rightarrow B^*\psi = \psi\cdot\vec{b}\cdot\vec{n},\ Cu = u$

Therefor

Γin: $Bu = a,\ C^*\psi = \psi,\ Cu = 0,\ B^*\psi = 0$

Γout: $Bu = 0,\ C^*\psi = 0,\ Cu = u,\ B^*\psi = \psi(\vec{b}\cdot\vec{n}) = j_\Gamma$

$$\boxed{\text{where } J(u) = \int_\Omega u\, j_\Omega\, d\Omega + \oint_\Gamma u\, j_\Gamma\, d\Gamma}$$

Thus the continuous Adjoint equation is

$-\nabla\psi\cdot\vec{b} + c\psi = j_\Omega \quad x\in\Omega$ subject to $B^*\psi = j_\Gamma \quad x\in\Gamma out.$

ii). Discrete Adjoint Consistency:

The Discrete Adjoint is given by

$$B(\omega,\psi_h) \equiv \sum_{e\in\mathcal{T}_h}\int_{\Omega_h} \omega(-\nabla\psi_h\cdot\vec{b} + c\psi_h)\,d\Omega_h + \sum_{e\in\mathcal{T}_h}\oint_{\partial e\setminus\Gamma} H(\omega,\omega,\vec{n})\,\psi_h^{+}\,ds$$

$$-\sum_{\sigma\in B_h}\int_{\Gamma_{in}} \vec{b}\cdot\vec{n}\,\omega\,\psi_h\,ds + \sum_{\sigma\in B_h}\int_{\Gamma_{out}} \vec{b}\cdot\vec{n}\,\omega\,\psi_h\,ds = \sum_{e\in\mathcal{T}_h}\int_{\Omega_h}\omega\, j_\Omega\,d\Omega_h$$

Some manipulation of the surface term.

1). Recall $H(\omega,\omega,\vec{n}) = \vec{b}\cdot\vec{n}\,\omega$

2). For every outflow face of an element $\sum_{e\in\mathcal{T}_h}\oint_{\partial e_{out}} \vec{b}\cdot\vec{n}\,u_h^{+}v_h^{-} =$
there is an inflow face on the other element $-\sum_{e\in\mathcal{T}_h}\oint_{\partial e_{in}} \vec{b}\cdot\vec{n}\,u_h^{-}\,v_h^{+}\,ds$

thus the interior faces can be represented by

$-\sum_{e\in\mathcal{T}_h}\oint_{\partial e_{out}\setminus\Gamma} \vec{b}\cdot\,\omega[\![\psi_h]\!]\,ds$ for the Adjoint state.

Thus we have.

$$B(\omega,\psi_h) \equiv \sum_{e\in\mathcal{T}_h}\int_{\Omega_h}\omega(-\nabla\psi_h\cdot\vec{b} + c\psi_h)\,d\Omega_h - \sum_{e\in\mathcal{T}_h}\int_{\partial e_{out}\setminus\Gamma}\vec{b}\cdot\omega[\![\psi_h]\!]\,ds$$

$$-\sum_{\sigma\in B_h}\int_{\Gamma_{in}}\vec{b}\cdot\vec{n}\,\omega\,\psi_h\,ds + \sum_{\sigma\in B_h}\int_{\Gamma_{out}}\vec{b}\cdot\vec{n}\,\omega(\psi_h - j_\Gamma)\,ds\ \sum_{e\in\mathcal{T}_h}\int\omega\, j_\Omega\,d\Omega_h.$$

The scheme is Adjoint consistent if $B(\omega,\psi) \equiv 0$.

Let $\psi_h \to \psi$

$$B(\omega,\psi) = \sum_{e\in\mathcal{T}_h}\int_{\Omega_h}\omega(-\nabla\psi\cdot\vec{b} + c\psi - j_\Omega)\,d\Omega - \sum_{e\in\mathcal{T}_h}\int_{\partial e_{out}\setminus\Gamma}\vec{b}\cdot\omega[\![\psi]\!]\,ds$$

No ψ specified at inflow.

$$-\sum_{\sigma\in B_h}\int_{\Gamma_{in}}\vec{b}\cdot\vec{n}\,\hat{\omega}^{0}\,\psi\,ds + \sum_{\sigma\in B_h}\int_{\Gamma_{out}}\vec{b}\cdot\vec{n}\,\omega\,\psi - j_\Gamma\,\omega\,ds = 0$$

$-\nabla\psi\cdot\vec{b} + c\psi + j_\Omega = 0$ by definition of continuous Adjoint

$[\![\psi]\!] = 0$ by definition of the jump $\Rightarrow \vec{b}\cdot\omega[\![\psi]\!] = 0$. inter is ok.

$\vec{b}\cdot\vec{n}\,\psi - j_\Gamma = 0$ by Adjoint B.C.

Thus the DG scheme is Adjoint Consistent.

Ex 2. Poisson Equation using SIP.

$\nabla^2 u = -f \qquad u = a$ on $\Gamma^D \quad \nabla u \cdot \vec{n} = a_n$ on Γ^N

i). Continuous Adjoint,

$$\langle Lu, \psi \rangle = \int_\Omega \nabla^2 u \, \psi \, d\Omega = -\int_\Omega \nabla u \cdot \nabla \psi \, d\Omega + \oint \nabla u \cdot \vec{n} \, \psi \, ds$$

$$= \int_\Omega u \nabla^2 \psi \, d\Omega + \oint \nabla u \cdot \vec{n} \, \psi \, ds - \oint u \nabla \psi \cdot \vec{n} \, ds$$

Recall the identity reads

$$\langle Lu, \psi \rangle + \langle Bu, C^* \psi \rangle_\Gamma = \langle u, L^* \psi \rangle + \langle Cu, B^* \psi \rangle_\Gamma$$

Re-arranging the equality gives

$$\int_\Omega \nabla^2 u \, \psi \, d\Omega + \oint u \nabla \psi \cdot \vec{n} \, ds = \int_\Omega u \nabla^2 \psi \, d\Omega + \oint \nabla u \cdot \vec{n} \, \psi \, ds.$$

Thus $L^* \equiv \nabla^2$ gives

$$\boxed{L^* \psi \equiv \nabla^2 \psi = j_\Omega}$$

The Boundaries are a bit more complicate. Let's analyze them all

$-\oint u \nabla \psi \cdot \vec{n} \, ds; \quad -\oint_{\Gamma^D} u \nabla \psi \, \vec{n} \, ds - \oint_{\Gamma^N} u \nabla \psi \cdot \vec{n} \, ds$

Thus on Γ^D $C^* \psi = -\nabla \psi \cdot \vec{n}$, $Bu = a$, Γ^N: $Cu = u$, $B^* \psi = -\nabla \psi \, \vec{n} \, ds$

$\oint \nabla u \cdot n \, \psi \, ds. \quad \oint_{\Gamma^D} \nabla u \cdot \vec{n} \, \psi \, ds + \oint_{\Gamma^N} \nabla u \cdot \vec{n} \, \psi \, ds \Rightarrow$

on Γ^D $Cu = \nabla u \cdot \vec{n}$, $B^* \psi = \psi$ Γ^N: $C^* \psi = \psi$, $Bu = \nabla u \cdot \vec{n}$

Thus the B.C. Are.

Γ^N $\nabla u \cdot \vec{n} = a_n$, $\quad -\nabla \psi \cdot \vec{n} = j_N$

Γ^D $\quad u = a, \quad \psi = j_D$

the continuous Adjoint,
$-\nabla^2 \psi = j_\Omega, \quad \psi = j_D$ on Γ^D $\quad \nabla(\psi) \cdot \vec{n} = j_N$ on Γ^N,

a) Consistency of the Primal Problem:

$$B(u_h, v_h) \equiv -\sum_{e\in \mathcal{T}_h}\int_{\Omega_h} \nabla u_h \cdot \nabla v_h \, d\Omega_h + \sum_{i\in \mathcal{I}_h}\int_{\Sigma_i} \{\nabla u_h\}[\![v_h]\!] + \{\nabla v_h\}[\![u_h]\!] \, ds$$

$$-\int \mu [\![u_h]\!][\![v_h]\!] ds + \sum_{b\in \mathcal{B}_h}\int_{\Gamma_b^D} \nabla u_h^+ v_h^+ + \nabla v_h^+ (u-a) - \mu (u-a) v_h^+ ds$$

$$+\sum_{b\in \mathcal{B}_h}\int_{\Gamma^N} v_h^+ a_N \, ds.$$

This is in a face-based form. It can be converted to an element based form for the interior face terms.

$$B(u_h, v_h) \equiv -\sum_{e\in \mathcal{T}_h}\int \nabla u_h \cdot \nabla v_h \, d\Omega + \sum_{e\in \mathcal{T}_h}\oint_{\partial e \setminus \Gamma^N} \{\nabla u_h\}\vec{n}^+ v_h^+ + \tfrac{1}{2}\nabla v_h^+ \cdot \vec{n}^+ [\![u_h]\!] ds$$

$$-\sum_{e\in \mathcal{T}_h}\int_{\partial e \setminus \Gamma} \mu [\![u_h]\!] v_h^+ \cdot \vec{n} \, ds + \int_{\Gamma^D} \nabla u_h^+ \cdot \vec{n}\, v_h^+ \; \nabla v_h^+ (u-a)\cdot \vec{n} + \mu [u-a] v_h^+ ds$$

$$+\int_{\Gamma^N} a_N v_h^+ \, ds = \sum_{e\in \mathcal{T}_h}\int_{\Omega_h} v_h f \, d\Omega$$

$$a_N = \nabla u \cdot \vec{n} |_{\Gamma^N}$$

Integrating By parts

$$B(u_h, v_h) \equiv \sum_{e\in \mathcal{T}_h}\int \nabla^2 u_h \cdot v_h - f v_h \, d\Omega + \sum_{e\in \mathcal{T}_h}\oint_{\partial e \setminus \Gamma} \{\nabla u_h\}\vec{n}^+ v_h^+ + \tfrac{1}{2}\nabla v_h^+ \cdot [\![u_h]\!] - \nabla u_h^+ \cdot \vec{n}\, v_h^+ ds$$

$$-\sum_{e\in \mathcal{T}_h}\int_{\partial e \setminus \Gamma} \mu [\![u_h]\!] v_h^+ \vec{n} \, ds + \sum_{b\in \mathcal{B}_h}\int_{\Gamma^D} \nabla u_h^+ \vec{n}\, v_h^+ - \nabla u_h^+ \cdot \vec{n}\, v_h^+ + \nabla v_h^+ (u-a)\vec{n}$$

$$+\mu (u-a) v_h^+ ds \, ds + \int_{\Gamma^N} a_N v_h^+ - \nabla u_h^+ v_h^+ \cdot \vec{n} \, ds = 0$$

Using the identity

$$\nabla u_h^+ \cdot \vec{n}^+ v_h^+ = \{\nabla u_h\}\cdot \vec{n}^+ v_h^+ + \tfrac{1}{2}[\![\nabla u_h]\!] v_h^+$$

gives

$$\sum_{e\in \mathcal{T}_h}\oint_{\partial e \setminus \Gamma} \cancel{\{\nabla u_h\}\cdot \vec{n}\, v_h^+} + \tfrac{1}{2}\nabla v_h^+ \cdot [\![u_h]\!] - \cancel{\{\nabla u_h\}\cdot \vec{n}\, v_h^+} - \tfrac{1}{2}[\![\nabla u_h]\!] v_h^+ ds$$

$$= \sum_{e\in \mathcal{T}_h}\oint_{\partial e \setminus \Gamma} \tfrac{1}{2}\nabla v_h^+ [\![u_h]\!] - \tfrac{1}{2}[\![\nabla u_h]\!] v_h^+ ds$$

Grouping the $[\![\nabla u_h]\!]$ term with the penalty term.

$$B(u_h, v_h) \equiv \sum_{e\in\mathcal{E}_h} \int v_h(\nabla^2 u_h + f)\,d\Omega + \sum_{e\in\mathcal{E}_h}\int_{\partial e\setminus\Gamma} \tfrac{1}{2}\nabla v_h^+ [\![u_h]\!]\,ds - \sum_{e\in\mathcal{E}_h}\int_{\partial e\setminus\Gamma} \mu[\![u_h]\!]\cdot\vec{n}^+ v_h^+ + \tfrac{1}{2}[\![\nabla u_h]\!]\cdot\vec{n}^+ v_h^+\,ds$$

$$+\sum_{\sigma\in\mathcal{B}_h}\int_{\Gamma^D} \nabla v_h^+\cdot\vec{n}\,(u_h - a) + \mu(u_h - a)v_h^+\,ds + \sum_{\sigma\in\mathcal{B}_h}\int_{\Gamma^N} a_N v_h^+ - \nabla u_h\cdot\vec{n}\, v_h^+\,ds = 0$$

For clarity we'll check each term seperately

1) Volume: $\nabla^2 u_h + f =$

Let $u_h \to u \Rightarrow \nabla^2 u + f = 0.$ ✓

2). Interior Surface.

$\tfrac{1}{2}\nabla v_h^+ [\![u_h]\!] - \mu[\![u_h]\!]\cdot\vec{n}\, v_h^+ - \tfrac{1}{2}[\![\nabla u_h]\!]\,\vec{n}\, v_h^+$

Let $u_h \to u. \Rightarrow \tfrac{1}{2}\nabla v_h^+ [\![u]\!] - \mu[\![u]\!]\cdot\vec{n}^+ v_h^+ - \tfrac{1}{2}[\![\nabla u]\!]\cdot\vec{n}\, v_h^+ = 0$ ✓ by $[\![u]\!]=0$, $[\![\nabla u]\!]=0.$

3). Γ^D:

$\nabla v_h^+\cdot\vec{n}\,(u_h - a) + \mu(u_h - a)v_h^+$

Let $u_h \to u \Rightarrow u|_{\Gamma^D} = a \Rightarrow \nabla v_h^+\cdot\vec{n}\,(a-a) + \mu(a-a) = 0.$ ✓

4). Γ^N:

$a_N v_h^+ - \nabla u_h\cdot\vec{n}\, v_h^+\,ds$

Let $u_h \to u \Rightarrow \nabla u\cdot\vec{n}\, v_h^+\big|_{\Gamma^N} = a_N v_h^+ \Rightarrow (a_N - a_N)v_h^+ = 0$ ✓.

Thus the Scheme is consistent.

b). Adjoint Consistency: (9)

Inserting w_h for u_h and ψ_h for v_h in $B(u_h, v_h)$ i.e. the weak form

First recall

$$\nabla^2 \psi = -j_\Omega \quad \vec{x} \in \Omega, \quad \psi = j_D \quad \vec{x} \in \Gamma_D, \quad \nabla\psi \cdot \vec{n} = j_N \quad \vec{x} \in \Gamma^N$$

$$J(u) = \int_\Omega u j_\Omega \, d\Omega + \int_{\Gamma_D} \nabla u \cdot \vec{n} j_D \, ds + \int_{\Gamma^N} u j_N \, ds$$

The discrete Adjoint is given by

$$B(w_h, \psi_h) = -\sum_{e \in \mathcal{T}_h} \int_{\Omega_h} \nabla w_h \cdot \nabla \psi_h \, d\Omega_h + \sum_{e \in \mathcal{I}_h} \int_{\Gamma_i} \{\nabla w_h\} [\![\psi_h]\!] + \{\nabla \psi_h\} [\![w_h]\!] - \mu [\![w_h]\!] [\![\psi_h]\!] \, ds$$

$$+ \sum_{b \in B_h} \int_{\Gamma_D} \nabla w_h \cdot \vec{n} \psi_h + \nabla \psi_h^+ \cdot \vec{n} w_h - \mu w_h \psi_h \, ds + \sum_{b \in B_h} \int_{\Gamma_N} w_h^+ \nabla \psi_h^+ \cdot \vec{n} \, ds$$

The above can be written in an element Based form as

$$B(w_h, \psi_h) = -\sum_{e \in \mathcal{T}_h} \int \nabla w_h \cdot \nabla \psi_h \, d\Omega_h + \sum_{e \in \mathcal{T}_h} \int_{\partial e \setminus \Gamma} \tfrac{1}{2} \nabla w_h^+ [\![\psi_h]\!] + \{\nabla \psi_h\} w_h^+ \vec{n} - \mu w_h [\![\psi_h]\!] \cdot \vec{n}$$

$$+ \sum_{b \in B_h} \int_{\Gamma_D} \nabla w_h^+ \cdot \vec{n} \psi_h^+ + \nabla \psi_h^+ \cdot \vec{n} w - \mu w_h^+ \psi_h^+ \, ds + \sum_{b \in B} \int_{\Gamma_N} w_h^+ \nabla \psi_h^+ \cdot \vec{n} \, ds$$

Integrate the volume term by parts once gives

$$B(w_h, \psi_h) = \sum_{e \in \mathcal{T}_h} w (\nabla^2 \psi_h) \, d\Omega + \sum_{e \in \mathcal{T}_h} \int_{\partial e \setminus \Gamma} \tfrac{1}{2} \nabla w_h^+ [\![\psi_h]\!] + \underbrace{\{\nabla \psi_h\} w_h^+ \vec{n}}_{①} - \mu w_h [\![\psi_h]\!] \cdot \vec{n}$$

$$\underbrace{- w_h^+ \nabla \psi_h^+ \cdot \vec{n}}_{①} \, ds + \sum_{b \in B_h} \int_{\Gamma_D} \nabla w_h^+ \cdot \vec{n} \psi_h^+ + \cancel{\nabla \psi_h^+ \cdot \vec{n} w_h^+} - \mu w_h^+ \psi_h^+ - \cancel{w_h^+ \psi_h^+ \cdot \vec{n}} \, ds$$

$$+ \sum_{b \in B_h} \int_{\Gamma_N} w_h^+ \nabla \psi_h^+ \cdot \vec{n} \, ds$$

Using

$$w_h^+ \nabla \psi_h^+ \cdot \vec{n} = \{\nabla \psi_h\} \cdot \vec{n} \, w_h^+ + \tfrac{1}{2} [\![\nabla \psi_h]\!] w_h$$

$$B(w_h,\psi_h)=\sum_{e\in T_h}\int w_h(\nabla^2\psi_h)\,d\Omega+\sum_{e\in T_h}\int_{\partial e\setminus\Gamma}\tfrac{1}{2}\nabla w_h^+[\![\psi_h]\!]+\cancel{\tfrac{1}{2}\{\nabla\psi_h\}w_h^+\cdot\vec n}-\mu w_h^+[\![\psi_h]\!]\cdot\vec n$$ (10)

$$-\cancel{\{\nabla\psi_h\}w_h^+\cdot\vec n-\tfrac{1}{2}[\![\nabla\psi_h]\!]w_h^+}\,ds+\sum_{6\in B_h}\int_{\Gamma_D}\nabla w_h^+\cdot\vec n\,\psi_h-\mu w_h^+\psi_h^+\,ds+\sum_{6\in B_h}\int_{\Gamma_N}w_h^+\nabla\psi_h^+\cdot\vec n\,ds$$

Again proceed term by term. Let $\psi_h\to\psi$,

$$B(w_h,\psi)-J(w_h)=\sum_{e\in T_h}\int_{\Omega_e}w_h\nabla^2\psi\,d\Omega+\sum_{e\in T_h}\int_{\partial e\setminus\Gamma}\tfrac{1}{2}\nabla w_h\cancel{[\![\psi]\!]}^{0}-\mu w_h^+\cancel{[\![\psi]\!]}^{0}\cdot\vec n-\tfrac{1}{2}[\![\nabla\psi]\!]w_h$$

$$+\sum_{6\in B_h}\int_{\Gamma_D}\nabla w_h^+\cdot\vec n\,\psi-\mu w_h^+\psi\,ds+\sum_{6\in B_h}\int_{\Gamma_N}w_h^+\nabla\psi\cdot\vec n\,ds-\sum_{e\in T_h}\int_{\Omega_e}-w_h j_\Omega\,d\Omega$$

$$-\sum_{6\in B_h}\int_{\Gamma_D}\nabla w_h^+\cdot\vec n\,j_D\,ds-\sum_{6\in B_h}\int_{\Gamma_N}w_h^+ g_N\,ds$$

Term by term

1. Volume $\sum_{e\in T_h}\int_{\Omega_e}\nabla^2\psi+j_\Omega\,d\Omega=0$ by def. of continuous solution ✓

2. Interior surfaces
$\sum_{e\in T_h}\int_{\partial e\setminus\Gamma}0\,ds=0$ ✓

3. Neumann B.C
$\sum_{6\in B_h}\int_{\Gamma_N}w(\nabla\psi\cdot\vec n-j_N)\,ds$, on Γ^N $\nabla\psi\,\vec n=j_N$ by def.
Hence $=0$ ✓

4. Dirichlet B.C.
$\sum_{6\in B_h}\int_{\Gamma_D}\nabla w_h\cdot\vec n(\psi-j_D)-\mu w\psi\,ds=\sum_{6\in B_h}\int_{\Gamma_D}\mu w\psi\,ds$

This is a problem however if we modify the $J_\Gamma\to\tilde J_\Gamma=J_\Gamma-\int\mu(v-a)j_D\,ds$ the

$\tilde J_{[w]\Gamma}\equiv\int\nabla w\cdot\vec n\,j_D-\mu w j_D\,ds$ hence the target J_Γ is modified on the Boundary such that the linearization is as above. With $\psi\to j_D$ on Γ^D by Def.

$$B(w_h,\psi_h)-\tilde J(w_h)=\underset{\text{vol}}{0}+\underset{\Gamma_i}{0}+\underset{\Gamma^N}{0}+\int\nabla w_h\cdot\vec n(\underbrace{\psi-j_D}_{0})-\mu w(\underbrace{\psi-j_D}_{0})\,ds=0\ ✓$$

We now get the heart of the matter gradient based source terms like those encountered in turbulence models.

Gradient based source terms:

Consider a modified form of Laplaces equation.

$$-\nabla^2 U = S(U, \nabla U), \quad \vec{x} \in \Omega \qquad U = a \ \vec{x} \ \Gamma^D \qquad \nabla \cdot \vec{U} = a_N \ \vec{x} \in \Gamma^N.$$

We will proceed with the naive discretization 1st and show that it is in fact dual-inconsistent. We will then show what can be done to fix this.

Preliminary: The continuous Adjoint

Note that the source term is non-linear, and thus we'll need to linearize it. Giving

$$-\nabla^2 z = \frac{\partial S}{\partial U}[U, \nabla U] z \cdot \frac{\partial S}{\partial \nabla U}[U, \nabla U] \cdot \nabla z$$

when $\frac{\partial S}{\partial U}[\cdot,\cdot]$ is a frechet derivative evaluated at state $(\cdot,\cdot)$.

Where z is the variable we seek & $U, \nabla U$ are points about which the linearization is performed.

The continuous Adjoint becomes

$$\int -\nabla^2 z \, \psi \, d\Omega = \int_\Omega \nabla z \nabla \psi \, d\Omega - \oint_{\partial\Omega} \nabla z \psi \cdot \vec{n} \, ds = -\int_\Omega \nabla^2 \psi z \, d\Omega - \oint \nabla z \psi \cdot \vec{n} \, ds + \oint z \nabla \psi \cdot \vec{n}$$

$$= \int_\Omega \psi \frac{\partial S}{\partial U}[U, \nabla U] z \, d\Omega + \int_\Omega \psi \frac{\partial S}{\partial \nabla U}[U, \nabla U] \cdot \nabla z$$

Re-arranging.

$$-\int_\Omega \nabla^2 z \, \psi \, d\Omega + \oint \nabla z \psi \cdot \vec{n} \, ds = -\int_\Omega \nabla^2 \psi z \, d\Omega + \oint z \nabla \psi \cdot \hat{n} \, ds$$

Now normally the product on the RHS would have been $\int f \psi \, d\Omega$ which must be put all in terms of U to get the volume integrals to tie together. This was easy b/c $\int_\Omega f \psi \, d\Omega = \int_\Omega U j_\Omega \, d\Omega$ thus we immediately that the RHS of the Adjoint equation is j_Ω. ~~the~~

~~However~~ Now this not so trivial. The LHS is as in the previous pages and so we shall not dwell on it. as are the B.C.

The Primary concern now is the volume integral of the Source term this must be cast as something operating on z so that we can recover the Source terms of the Adjoint equation.

We have

$$\int \psi \frac{\partial S}{\partial u}[u,\nabla u]\, z + \psi \frac{\partial S}{\partial \nabla u}[u,\nabla u]\cdot \nabla z \, d\Omega$$

The first term poses no problem as it is the linearized equivalent to f, and that term is easily taken care of,

The second term is the issue.

$$\int_\Omega \psi \frac{\partial S}{\partial u}[u,\nabla u]\, z + \psi\frac{\partial S}{\partial \nabla u}[u,\nabla u]\cdot\nabla z \, d\Omega$$

$$= \int_\Omega \left(\psi \frac{\partial S}{\partial u}[u,\nabla u] - \nabla\cdot\left(\frac{\partial S}{\partial \nabla u}[u,\nabla u]\,\psi\right)\right) z \, d\Omega + \oint_{\partial\Omega} \psi \frac{\partial S}{\partial \nabla u}[u,\nabla u]\cdot(z\vec{n})\, ds$$

We see immediately that the second term is the convection of the Adjoint state by the $\frac{\partial S}{\partial \nabla u}$ term. (Source gradient jacobian).

Thus the final Adjoint equation (Not about the Boundary term just yet) is.

Frechet Derivative of Sampling function.

$$\boxed{-\nabla^2\psi - \frac{\partial S}{\partial u}[u,\nabla u]\,\psi + \nabla\cdot\left(\frac{\partial S}{\partial \nabla u}[u,\nabla u]\,\psi\right) = \frac{\partial j_\Omega}{\partial u}[u]}$$

Now consider a weak form discretization of the primal problem

$$B(u_h, v_h) - \int_\Omega v_h\, S(u,\nabla u)\, d\Omega = 0.$$

Where $B(u_h, v_h)$ is a dual consistent weak form of the Laplacian just such as the SIP from the previous sections then we have, for the Adjoint Problem.

$$B(w,\psi_h) - \int_\Omega \psi_h\left(\frac{\partial S}{\partial u}[u,\nabla u]\, w + \frac{\partial S}{\partial \nabla u}[u,\nabla u]\cdot\nabla w\right) = 0.$$

Leaving $B(\omega, \psi_h)$ be as we know it's dual consistent $-\nabla^2\psi$ (13)

We only work about now $-\int_\Omega \psi_h \left(\frac{\partial S}{\partial U}[U,\nabla U]\omega + \frac{\partial S}{\partial \nabla U}[U,\nabla U]\cdot\nabla\omega\right)$ compares to the exact in the limit $\psi_h \to \psi$.

Let's begin.

I.B.P on src term.

$$-\int_\Omega \psi_h \frac{\partial S}{\partial U}[U,\nabla U]\omega - \nabla\cdot\left(\frac{\partial S}{\partial \nabla U}[U,\nabla U]\cdot\psi\right)\omega \, d\Omega - \sum_{e\in\mathcal{E}_h}\int \left\{\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi_h\right\}\omega\cdot\vec{n}\, ds$$

This can be written is the form

$$-\sum_{e\in\mathcal{E}_h}\int_{\Omega_h} \psi_h \frac{\partial S}{\partial U}[U,\nabla U]\omega - \nabla\cdot\left(\frac{\partial S}{\partial \nabla U}[U,\nabla U]\cdot\psi_h\right)\omega\, d\Omega_h - \sum_{e\in\mathcal{E}_h}\int_{\Gamma_i}\left\{\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi_h\right\}[\![\omega]\!] + \{\omega\}\left[\!\left[\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi_h\right]\!\right]ds$$

$$-\sum_{e\in B_h}\int_{\Gamma_B}\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi_h\,\omega\cdot\vec{n}\,ds$$

Now let $\psi_h \to \psi$.

$$-\sum_{e\in\mathcal{E}_h}\int_{\Omega_h}\omega\left(\psi\frac{\partial S}{\partial U}[U,\nabla U] - \nabla\cdot\left(\frac{\partial S}{\partial \nabla U}[U,\nabla U]\cdot\psi\right)\right)d\Omega - \sum_{e\in\mathcal{E}_h}\int \frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi\cdot[\![\omega]\!] + 0\,ds$$

$$-\sum_{e\in B_h}\int_{\Gamma_B}\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi\,\omega\cdot\vec{n}\,ds.$$

Thus we have two extra terms

$$\left.\begin{aligned}&-\sum_{e\in\mathcal{E}_h}\int\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi[\![\omega]\!]\,ds\\ &-\sum_{e\in B_h}\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi\,\omega\cdot\vec{n}\,ds\end{aligned}\right\}$$ Two extra terms. Not dual consistent.

This can be remedied by adding terms to original discretization.

1) Fixing the dual inconsistency:

Let's add two terms to the $B(u_h, v_h)$ term.

$$B(u_h,v_h) - \sum_{e\in\mathcal{T}_h}\int_{\Omega_h} v_h\, S(u_h,\nabla u_h)\,d\Omega_h + \sum_{e\in\mathcal{T}_h}\int_{\Sigma_i} \vec{\beta}_i\left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]v_h\right)\cdot[\![u_h]\!]\,ds$$

$$+\sum_{\sigma\in\mathcal{B}_h}\int \vec{\beta}_b\left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]v_h\right)\cdot(\vec{u}_h\cdot\vec{n})\,ds$$

with The fluxes $\vec{\beta}_i\left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]v_h\right)$ satisfy $\vec{\beta}_i\left(\frac{\partial S}{\partial \nabla u}[u,\nabla u]v_h\right) = \frac{\partial S}{\partial \nabla u}[u,\nabla u]v_h$

Proposition: The above form of the primal problem is dual consistent.

The discrete Adjoint problem is.

$$B(w,\psi_h) - \sum_{e\in\mathcal{T}_h}\int_{\Omega_h} w\frac{\partial S}{\partial u}[u_h,\nabla u_h]\psi_h + \frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\nabla w\, d\Omega + \sum_{e\in\mathcal{T}_h}\int_{\Sigma_i}\vec{\beta}_i\left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\right)[\![w]\!]\,ds$$

$$+\sum_{\sigma\in\mathcal{B}_h}\int \vec{\beta}_b\left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\right)\cdot\vec{n}\, w\, ds$$

Now integrating the volume term by parts

$$B(w,\psi_h) - \sum_{e\in\mathcal{T}_h}\int_{\Omega_h} w\left[\frac{\partial S}{\partial u}[u_h,\nabla u_h]\psi_h - \nabla\cdot\left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\cdot\psi_h\right)\right]d\Omega_h$$

$$+\sum_{e\in\mathcal{T}_h}\int_{\Sigma_i}\vec{\beta}_i\left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\right)[\![w]\!] - \left\{\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\right\}[\![w]\!] - \{w\}\left[\!\left[\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\right]\!\right]ds$$

$$+\sum_{\sigma\in\mathcal{B}_h}\int_{\Gamma_b}\vec{\beta}_b\left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\right)\vec{n}\,w - \left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\right)\cdot\vec{n}\,w\,ds$$

Now Let $u_h\to u$, $\psi_h\to\psi$ then, we have

$$B(w,\psi) - \sum_{e\in\mathcal{T}_h}\int w\left(\frac{\partial S}{\partial u}[u,\nabla u]\psi - \nabla\cdot\left(\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi\right)\right)d\Omega_h$$

$$+\sum_{e\in\mathcal{T}_h}\int_{\Sigma_i}\cancel{\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi[\![w]\!]} - \cancel{\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi\cdot[\![w]\!]} - \{w\}\overset{\text{no}}{\cancel{\left[\!\left[\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi\right]\!\right]}}\,ds$$

$$+\sum_{\sigma\in\mathcal{B}_h}\int_{\Gamma_b}\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi\cdot\vec{n}\,w - \cancel{\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi\cdot\vec{n}\,w}\,ds$$

The continuous Adjoint for this equation is

$$B(\omega,\psi) - \frac{\partial S}{\partial U}[U,\nabla U]\psi + \nabla\cdot\left(\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi\right) d\Omega_h = 0$$

Thus the proposed scheme is dual consistent. All we really did was introduce primal fluxes to cancel the extra Adjoint terms. The fluxes will Also be consistent in the primal solution by The $[\![U]\!]=0$ term.

The interface fluxes $\vec{\beta}_i\left(\frac{\partial S}{\partial \nabla U}[U_h,\nabla U_h]V_h\right)$ are really a surface discretization of the $\nabla\cdot\left(\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi\right)$ term in continuous Adjoint equation. The question is what type of term is this? The only requirement is $\vec{\beta}_i\left(\frac{\partial S}{\partial \nabla U}[U,\nabla U]V_h\right)\Big|_{U_1} = \frac{\partial S}{\nabla U}[U,\nabla U]V_h$ the Boundary flux should be the same $\vec{\beta}_b\left(\frac{\partial S}{\partial \nabla U}[U,\nabla U]V_h\right) = \frac{\partial S}{\partial \nabla U}[U,\nabla U]V_h$ on the Boundary

This really just a statement of consistency.

To figure out what to do with these we'll need to analyze the characteristics of the $\nabla\cdot\left(\frac{\partial S}{\partial \nabla U}[U,\nabla U]\psi\right)$ flux.

<u>An Alternative Method</u>:

Consider using a mixed formulation of the gradient in the source term consistent with the SIP mixed formulation. Then the primal discretization is,

$$R(U_h,V_h) = B(U_h,V_h) - \int_\Omega V_h\, S(U,\nabla U + \vec{R}_h)$$

The Linearized Residual is when $\vec{z} = \nabla U_h + \vec{R}_h$ → the mixed gradient

$$\frac{\partial R}{\partial U_h}(U_h,V_h) = B(U_h,V_h) - \int_\Omega V_h \frac{\partial S}{\partial U}[U,\nabla U + \vec{R}_h]U_h + V_h \frac{\partial S}{\partial \vec{z}}[U,\vec{z}]\cdot\vec{z}\, d\Omega_h.$$

Then the adjoint residual is

$$\frac{\partial R}{\partial U_h}(\omega,\psi_h) = B(\omega,\psi_h) - \int_\Omega \psi_h \frac{\partial S}{\partial U}[U,\vec{z}]\omega + \psi_h \frac{\partial S}{\partial \vec{z}}[U,\vec{z}]\left(\nabla\omega + \vec{R}_h(\omega)\right) ds$$

$$\frac{\partial R}{\partial u}[u,\nabla u]\psi_h = B(\omega,\psi_h) - \int_{\Omega_h} \psi_h \frac{\partial S}{\partial u}[u_h, \vec{z}] - \nabla\cdot\left(\frac{\partial S}{\partial \nabla u}[u,\nabla u]\cdot\psi_h\right)\omega \, d\Omega$$

$$-\int_{\Omega_h} \psi_h \frac{\partial S}{\partial \nabla u}[u,\nabla u]\cdot \vec{R}_h(\omega)\, d\Omega \mp \int_{\Sigma_h} \left\{\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi_h\right\}[\![\omega]\!] + [\![\frac{\partial S}{\partial \nabla u}[u,\nabla u]]\!]\{\omega\}$$

$$-\int_\Gamma \frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi_h\cdot\vec{n}\,\omega\, ds$$

Recall from the Mixed Method notes that.

$$\int \vec{g}_h\cdot\vec{R}_h\, d\Omega = -\int_{\Sigma_h}\{g\}[\![u]\!]ds - \int_{\Gamma_D} \vec{g}_h\cdot\vec{n}\,(u_h - u_D)\, ds$$

using ω for u_h and assuming the $\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi_h \in [V_h^p]^d$ i.e. this operator can be put into the finite element space

We have. Let $\vec{g}_h = \frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h$

$$B(\omega,\psi_h) - \int_\Omega \omega\psi_h \frac{\partial S}{\partial u}[u_h,\nabla u_h] - \nabla\cdot\left(\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\cdot\psi_h\right)\omega\, d\Omega$$

$$+\int_{\Sigma_h}\left\{\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\right\}[\![\omega]\!]\, ds + \int_\Gamma \frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\cdot\omega\cdot\vec{n}\, ds$$

$$-\int_{\Sigma_h}\left\{\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]\psi_h\right\}[\![\omega]\!]ds + [\![\frac{\partial S}{\partial \nabla u}[u_h,\nabla u_h]]\!]\{\omega\}\, ds$$

$$-\int_\Gamma \frac{\partial S}{\partial \nabla u}[u_h,\nabla u]\psi_h\cdot\vec{n}\,\omega\, ds.$$

Let $u_h \to u$, $\psi_h \to \psi$ gives.

$$B(\omega,\psi_h) - \int_\Omega \omega\left[\psi\frac{\partial S}{\partial u}[u,\nabla u] - \nabla\cdot\left(\frac{\partial S}{\partial \nabla u}[u,\nabla u]\cdot\psi\right)\right]d\Omega + \overset{①}{\int_{\Sigma_h}\left\{\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi\right\}[\![\omega]\!]ds}$$

$$+\overset{②}{\int_\Gamma \frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi\,\omega\cdot\vec{n}\,ds} - \overset{①}{\int_{\Sigma_h}\left\{\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi\right\}[\![\omega]\!]ds} + [\![\frac{\partial S}{\partial \nabla u}[u,\nabla u]]\!]\xrightarrow{0 \text{ by def.}} \psi$$

$$-\overset{②}{\int_\Gamma \frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi\cdot\vec{n}\,\omega\, ds}$$

Thus we have

$$B(w,\psi) - \int_{\Omega} w\left[\psi \frac{\partial S}{\partial u}[u,\nabla u] - \nabla\cdot\left(\frac{\partial S}{\partial \nabla u}[u,\nabla u]\,\psi\right)\right] d\Omega \ \text{to}$$

Thus the scheme is dual consistent under the assumption $\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi \in [V_h^p]^d$. If this is statified for all h and p i.e. all orders and Meshes then the scheme is dual consistent. However This is not gaurenteed unless $h \to 0$ b/c as $h \to 0$ the $[V_h^p]^d$ approaches an infinite dimensional space and thus when $\frac{\partial S}{\partial \nabla u}[u,\nabla u]\psi \notin [V_h^p]^d$ the scheme is said to asymptotically dual consistent.

Remark: In their Numerical expirements Oliver and Darmofal have not noticed any difference between the dual consistent and asymptotically dual consistent approach.

Non Linear PDE's:

Here the definition of the continuous Adjoint is a bit different

Consider a general Nonlinear problem

$$Nu = 0 \quad \vec{x} \in \Omega \qquad \text{and} \qquad Bu = 0 \quad \vec{x} \in \Gamma$$

with the Non-linear functional

$$J(u) = \int_{\Omega} j_{\Omega}(u)\,dx + \int_{\Gamma} j_{\Gamma}(Cu)\,ds,$$

The functional can be linearized as

$$\frac{\partial J}{\partial u}[u](w) = \int_{\Omega} \frac{\partial j_{\Omega}}{\partial u}[u]\, w\, d\Omega + \int_{\Gamma} \frac{\partial j_{\Gamma}}{\partial u}[Cu]\frac{\partial C}{\partial u}[u] w\, ds$$

The $[\cdot]$ is state about which the Nonlinear functions are linearized. Here J_{Ω}, j_{Γ} are differentiable Non-linear functions And Cu, Bu are ~~Aon~~ potentially non-linear operators.

Remark: Here w is just some other variable $\in V$ b/c $u \in V$

The Adjoint identity for the Non-linear problem is given by. (18)

$$\left\langle \frac{\partial N}{\partial U}[U]\psi, \psi \right\rangle + \left\langle \frac{\partial B}{\partial U}[U]\omega, \left(\frac{\partial C}{\partial U}[U]\right)^{*}\psi \right\rangle_{\Gamma} = \left\langle \omega, \left(\frac{\partial N}{\partial U}[U]\right)^{*}\psi \right\rangle + \left\langle \frac{\partial C}{\partial U}[U]\omega, \left(\frac{\partial B}{\partial U}[U]\right)^{*}\psi \right\rangle$$

This results in the following Adjoint problem for the Non-linear operator and it's boundary condition.

$$\left(\frac{\partial N}{\partial U}[U]\right)^{*}\psi = \frac{\partial J_{\Omega}}{\partial U}[U] \quad \vec{x} \in \Omega. \qquad \left(\frac{\partial B}{\partial U}[U]\right)^{*}\psi = \frac{\partial J_{\Gamma}}{\partial U}[U] \quad \vec{x} \in \Gamma$$

These are the continuous Adjoint variables and operators for $NU=0$ and the defined functional.

The discrete Adjoint must satisfy

$$\frac{\partial N}{\partial U}[U](\omega, \psi) = \frac{\partial J}{\partial U}[U]\omega \quad \forall \omega \in V.$$

It should be noted that in optimization ~~this~~ our Adjoint identity ensures that. By definition of continuous Adjoint problem

$$\frac{\partial J}{\partial [U]}(\omega) = \left\langle \omega, \frac{\partial J_{\Omega}}{\partial U}[U] \right\rangle + \left\langle \frac{\partial C}{\partial U}[U]\omega, \frac{\partial J_{\Gamma}}{\partial U}[U] \right\rangle_{\Gamma} = \left\langle \omega, \left(\frac{\partial N}{\partial U}[U]\right)^{*}\psi \right\rangle +$$

$$\left\langle \frac{\partial C}{\partial U}[U]\omega, \left(\frac{\partial B}{\partial U}[U]\right)^{*}\psi \right\rangle_{\Gamma} = \left\langle \frac{\partial N}{\partial U}[U]\omega, \psi \right\rangle + \left\langle \frac{\partial B}{\partial U}[U]\omega, \left(\frac{\partial C}{\partial U}[U]\right)^{*}\psi \right\rangle_{\Gamma}$$

This is just a duality statement for the ~~objective $\frac{\partial J}{\partial U}[U]\omega$~~, linearized objective $\frac{\partial J}{\partial U}[U](\omega)$, (linearization of $J(U)$).

The Euler Equations

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \vec{\nabla} \cdot \vec{F}^{c}(\underset{\sim}{u})$$

where $\vec{(\ )}$ denotes vector in space. and $\underset{\sim}{(\ )}$ vector in fields

$$J(U) \equiv \int_{\Gamma} j(C(U))\,ds \Rightarrow \frac{\partial J}{\partial U}[U] = \int_{\Gamma} \frac{\partial j}{\partial J}[CU]\frac{\partial C}{\partial U}[U]\omega\,ds$$

$$J(U) = \int_{\Gamma_{wall}} p(U) \cdot n \cdot \vec{z}_{\Gamma_{wall}}\,ds \rightarrow \text{surface pressure.}$$

$$CU = p(U), \quad j = \vec{n} \cdot \vec{z}_{\Gamma_{wall}}$$

$\vec{z}$ can be lift or drag. vector i.e.

$$z_{lift} = (-\sin(\alpha), \cos(\alpha))^{T}$$

$$\frac{\partial J}{\partial U}[U] = \int_{\Gamma_{wall}} \frac{\partial p}{\partial U}[U] \cdot \vec{n} \cdot z_{\Gamma_{wall}}\,ds$$

Further Let $[\sim]$ denote a matrix in fields i.e. nfld x nfld.
and $[\rightarrow]$ denote a matrix in ~~dim~~ space i.e. d x d.

Step 1: Continuous Adjoint equation for steady euler equations

$$\nabla \cdot (\vec{F}^c(U)) = 0. \qquad F^c(U) = (F(U), G(U)), \quad \frac{\partial \vec{F}^c}{\partial U}[U]$$

Linearize about U.

$$\nabla \cdot \left(\frac{\partial \vec{F}^c}{\partial U} w \right) = 0$$

Multiply by ψ - test function and integrate by Parts

$$\int_\Omega \psi \, \nabla \cdot \left(\frac{\partial \vec{F}^c}{\partial U}[U] w \right) d\Omega = -\int_\Omega \nabla \psi \cdot \frac{\partial \vec{F}_c}{\partial U}[U] \, w \, d\Omega + \int_\Gamma \psi \frac{\partial F^c}{\partial U} w \cdot \vec{n} \, ds$$

$$= -\int w \left(\frac{\partial F^c}{\partial U}[U] \right)^T \cdot \nabla \psi \, d\Omega + \int_\Gamma w \left(n \cdot \left[\frac{\partial F^c}{\partial U} \right]^T \right) \psi \, ds = \frac{\partial J}{\partial U}[U](w) \quad \forall w \in V$$

where $(\ )^T \Rightarrow$ a transpose of the fields only. Not the space

i.e. in 2D $w \left(\frac{\partial F^c}{\partial U} \right)^T \cdot \nabla \psi = \frac{\partial F}{\partial U}^T \frac{\partial \psi}{\partial x} + \frac{\partial G}{\partial U}^T \frac{\partial \psi}{\partial y}$

The continuous Adjoint equation is

$$-\left(\frac{\partial F^c}{\partial U}[U] \right)^T \nabla \psi = 0 \ ; \ \vec{x} \in \Omega \qquad \vec{n} \cdot \frac{\partial F^c}{\partial U}[U] \psi = \frac{\partial j}{\partial U}[U] \quad \vec{x} \in \Gamma$$

on Γ_w $\quad n \cdot F^c(U) = (0, P n_x, P n_y, 0)^T$

Using the definition of the functional.

$$\text{The } \partial \cdot n \cdot \left[\frac{\partial F^c}{\partial U} \right] = \frac{\partial P}{\partial U}[U](0, n_x, n_y, 0) \cdot \psi = \frac{\partial P}{\partial U}[U] \, n \cdot \vec{z} \text{ on } \Gamma_w.$$

~~Thus~~ the ~~Boundary~~ condition ~~on the Adjoint~~ are,

$$\cancel{\left(\frac{\partial B}{\partial U}^*[U] \right) \vec{\psi}} = \vec{n} \cdot \vec{z}$$

Further recall $CU = P[U] \Rightarrow \frac{\partial C}{\partial U}[U] = \frac{\partial P}{\partial U}[U]$

Also Recall the piece of the Adjoint identity that Defines $\left(\frac{\partial B}{\partial U}[U] \right)^*$ is $\langle \frac{\partial C}{\partial U}[U] w, \left(\frac{\partial B}{\partial U}[U] \right)^* \psi \rangle_\Gamma$

$\therefore \frac{\partial P}{\partial U}[U] = \frac{\partial C}{\partial U}[U]$ the

$$\int_\Gamma w \left(n \cdot \left[\frac{\partial F^c}{\partial U} \right]^T \right) \psi \, ds = \int w \frac{\partial P}{\partial U} (0, n_x, n_y, 0) \cdot \psi \, ds$$

Thus $\left(\frac{\partial B}{\partial U}\right)^* \psi = (0, n_x, n_y, 0) \cdot \psi = \psi_2 n_x + \psi_3 n_y = ?$

We ll by defention of the functional the RHS is $\frac{\partial j}{\partial U} = \vec{n} \cdot \vec{z}_{wall}$

So

$\left(\frac{\partial B}{\partial U}\right)^* \psi = \psi_2 n_x + \psi_3 n_y = \vec{n} \cdot \vec{z}_{wall}$, where $\vec{z}_{wall}$ is the sampling fn for the objective.

In summary the Adjoint equation is

$$-\left[\frac{\partial F^c}{\partial U}\right]^T \cdot \nabla \psi = 0 \quad \vec{x} \in \Omega \qquad \text{with} \quad n_i \psi_i = \vec{n} \cdot \vec{z} \quad i = 1, \ldots, d.$$

where d is the # of Dimensions

Discrete problem:

DG discretization of the euler equations is

$$N(U_h, v_h) = -\sum_{e \in T_h} \int_{\Omega_h} \vec{F}^c(U_h) : \nabla v_h \, d\Omega + \sum_{e \in T_h} \int_{\partial e \setminus \Gamma} H(U^+, U^-, \vec{n}^+) v_h^+ \, ds$$

$$+ \int_\Gamma \bar{H}(U^+, \psi(U^+), \vec{n}^+) v_h^+ \, ds = 0 \qquad \forall v \in V_h^p,$$

Here $H(U^+, U^-, \vec{n}^+)$ is a consistent Numerical flux and $\bar{H}(U^+, U^-, \vec{n}^+)$ is the same but on a Boundary

$\bar{H}$ may be different from H.

1). Consistency of ~~ADG~~ Primal DG scheme.

$$N(U, v_h) = -\sum_{e \in T_h} \int_{\Omega_h} \vec{F}^c(U) \cdot \nabla v_h \, d\Omega + \sum_{e \in T_h} \int_{\partial e \setminus \Gamma} H(U, U, \vec{n}^+) v_h^+ \, ds$$

$$+ \int_\Gamma \bar{H}(U, U, \vec{n}^+) v_h \, ds = 0$$

I.B.P.

$$N(U, v_h) = \sum_{e \in T_h} \int_{\Omega_h} \nabla \cdot (\vec{F}^c(U)) v_h \, d\Omega + \sum_{e \in T_h} \int_{\partial e \setminus \Gamma} \left(-\vec{F}^c_c \cdot \vec{n}^+ + H(U, U, \vec{n}^+)\right) v_h^+ \, ds$$

$$+ \int_\Gamma \left(\bar{H}(U, U, \vec{n}^+) - \vec{F}^c(U) \cdot \vec{n}\right) v_h^+ \, ds$$

a). There are 3 terms.

Volume: $\nabla \cdot (\vec{F}^c(u))$

: By definition this is 0 ✓

b) Surface term.

$-\vec{F}^c(u) \cdot \vec{n} + H(u, u, \vec{n}^+) \to$ by $H(u,u,\vec{n}^+) = \vec{F}^c \cdot \vec{n}$

gives

$-\vec{F}^c(u) \cdot \vec{n} + \vec{F}^c(u) \cdot \vec{n} = 0$ ✓

c). Boundary

$-\vec{F}^c(u) \cdot \vec{n} + \tilde{H}(u, u, \vec{n}^+) \to$ by $\tilde{H}(u,u,\vec{n}^+) = \vec{F}^c \cdot \vec{n}$

$-\vec{F}^c(u) \vec{n} + \vec{F}^c(u) \cdot \vec{n} = 0$ ✓

Thus the scheme is consistent.

2). Dual Consistency.

From the Non-linear operator the ~~Adj~~ discrete Adjoint equation is

$$\frac{\partial N}{\partial u}[u_h](w; \psi_h) = -\sum_{e \in T_h} \int_{\Omega} \left(\frac{\partial \vec{F}^c}{\partial u}[u_h] w \right)^T \cdot \nabla \psi_h \, d\Omega + \sum_{e \in T_h} \int_{\partial \Omega} \left(\frac{\partial H}{\partial u^+}[u_h^+, u_h^-, \vec{n}^+] w^+ + \frac{\partial H}{\partial u^-}[u_h^+, u_h^-, \vec{n}^+] w^- \right)^T \psi_h^+ \, ds$$

$$+ \int_{\Gamma} \left(\frac{\partial \tilde{H}}{\partial u^+}[u_h^+, u_h^-, \vec{n}^+] + \frac{\partial \tilde{H}}{\partial u_\Gamma}[u_h^+, u_\Gamma(u_h^+), \vec{n}] \frac{\partial u_\Gamma}{\partial u^+} \right)^T w^+ \psi_h^+ \, ds$$

If we translate the interior flux terms to ~~th~~ an interface form

by $\int_{\Gamma_I} \frac{\partial H}{\partial u^-}[u_h^+, u_h^-, \vec{n}] w^- \psi_h^+ = -\int_{\Gamma_I} \frac{\partial H}{\partial u^+}[u_h^+, u_h^-, \vec{n}] w^+ \psi_h^- \, ds$ when

+ is now on the other side. then,

$$\frac{\partial N}{\partial u}[u_h](w, \psi_h) = -\sum_{e \in T_h} \int_{\Omega} \left(\frac{\partial \vec{F}^c}{\partial u}[u_h] w \right)^T \nabla \psi \, d\Omega + \sum_{i \in I_h} \int_{\Sigma_i} \left(\frac{\partial H}{\partial u^+}[u_h^+, u_h^-, \vec{n}] \right)^T w^+ [\![\psi_h]\!] \cdot \vec{n} \, ds$$

$$+ \sum_{b \in B_h} \int_{\Gamma_b} \left(\frac{\partial \tilde{H}}{\partial u^+}[u_h^+, u_h^-, \vec{n}] + \frac{\partial \tilde{H}}{\partial u_\Gamma}[u_h^+, u_h^-, \vec{n}] \frac{\partial u_\Gamma}{\partial u^+} \right)^T w^+ \psi_h^+ \, ds$$

There are 3 term

1). Volume

$$\left(\frac{\partial \vec{F}_c}{\partial U}[U_h]\right)^T \nabla \psi_h$$

Let $U_h \to U$, $\psi_h \to \psi$ then

$$\left(\frac{\partial \vec{F}_c}{\partial U}[U]\right)^T \nabla \psi = 0 \quad \text{by Def of Continuous Adj}$$

2) Interface

$$-\left(\frac{\partial \hat{H}}{\partial U^+}[U_h^+, U_h^-, \vec{n}]\right)^T [\![\psi_h]\!] \cdot \vec{n}$$

Let $U_h \to U$, $\psi_h \to \psi$ gives

$$\left(\frac{\partial \hat{H}}{\partial U^+}[U, U, \vec{n}]\right)^T [\![\psi]\!] \cdot \vec{n} = 0 \quad \text{by } [\![\psi]\!] = 0.$$

3) Boundary interfaces

$$\frac{\partial j_\Gamma}{\partial U} - \left(\frac{\partial \hat{H}}{\partial U^+}[U_h, U_h, \vec{n}] + \frac{\partial \hat{H}}{\partial U_\Gamma}[U_h, U_\Gamma, \vec{n}]\frac{\partial U_\Gamma}{\partial U^+}\right)\psi_h$$

Let $U_h \to U$, $\psi_h \to \psi$ gives

$$\frac{\partial j_\Gamma}{\partial U} - \left(\frac{\partial \hat{H}}{\partial U^+}[U, U, \vec{n}] + \frac{\partial \hat{H}}{\partial U}[U, U, \vec{n}]\frac{\partial U_\Gamma}{\partial U}\right)\psi$$

For this to cancel $\left(\frac{\partial \hat{H}}{\partial U^+}[U, U, \vec{n}] + \frac{\partial \hat{H}}{\partial U}[U, U, \vec{n}]\frac{\partial U_\Gamma}{\partial U}\right)$ must be $= \vec{n} \cdot \vec{z}$ to satisfy the Adjoint B.C.

Following Hartmann the function $\frac{\partial j}{\partial U}$ is modified by

$$\tilde{j}(U) = j(i(U)) \quad \text{with} \quad i(U) = U_\Gamma$$

Further we will require $\frac{\partial \hat{H}}{\partial U^+} = 0$ so that $\frac{\partial \hat{H}}{\partial U^-}[U, U, \vec{n}]\frac{\partial U_\Gamma}{\partial U}$ can define a consistent B.C. for the Adjoint equation

If we Let

$$U_P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-n_x^2 & -n_x n_y & 0 \\ 0 & -n_x n_y & 1-n_y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \rho^+ \\ \rho u_1^+ \\ \rho u_2^+ \\ \rho E^+ \end{Bmatrix}.$$

Which is just setting $U_P \cdot \vec{n} = 0$ using U_h^+ and $\vec{n}$.

and the Let $\tilde{H}(U_h^+, U_P(U_h^+), \vec{n}) \equiv \vec{n} \cdot \vec{F}^c(U_P(U_h^+))$

the

$$\frac{\partial \tilde{H}}{\partial U} \cdot \frac{\partial U_P}{\partial U^+} = \frac{\partial \vec{F}^c}{\partial U}[U] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-n_x^2 & -n_x n_y & 0 \\ 0 & +n_x n_y & 1-n_y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Boundary condition therefore reduces to $\left(\vec{n} \cdot \frac{\partial \vec{F}^c}{\partial U}[U]\right)^T = \frac{\partial P}{\partial U}(0, n_x, n_y, 0)$,

~~Thus~~ The Boundary residual reduces to

$$J_\Gamma - \left(\vec{n} \cdot \left(\frac{\partial \vec{F}}{\partial U}[U]\right)\right)^T \psi,$$

The continuous ~~B~~ Adjoint B.C. $J_\Gamma = \frac{\partial P}{\partial U}[U](0, n_x, n_y, 0) \cdot \psi$

so we have

$$\cancel{J_\Gamma} \quad \frac{\partial P}{\partial U}[U](0, n_x, n_y, 0) \cdot \psi - \frac{\partial P}{\partial U}(0, n_x, n_y, 0)\,\psi = 0$$

So this ~~Adjoi~~ Boundary condition of using $\vec{n} \cdot \vec{F}^c(U_P)$ is Adjoint Consistent.

Remark: If $\tilde{H}(U_h^+, U_h^-, \vec{n}) = H(U_h^+, U_h^-, \vec{n})$ then in general $\frac{\partial \tilde{H}}{\partial U^-} \neq 0$ thus these types of B.C. may not be Adjoint consistent by $\vec{F}^c(U) \cdot \vec{n}$ is Adjoint consistent.

Thus (5) and (6) can be written as

$$(5) \to -\int_{\Omega_h} \nabla v_h \cdot \vec{z}_h \, d\Omega + \int_{\Sigma_h} \widehat{(\vec{z}_h)} \cdot [\![v_h]\!] \, ds + \int_{\Gamma_h^D} \widehat{(\vec{z}_h)} \cdot \vec{n} \, v_h \, ds + \int_{\Gamma_h^N} (\vec{z} \cdot \vec{n})^b \, v_h \, ds \quad (7)$$

$$(6) \to \int_{\Omega_h} \vec{z}_h \cdot \vec{g}_h \, d\Omega = -\int_{\Omega_h} u_h \nabla \cdot \vec{g}_h \, d\Omega + \int_{\Sigma_h} \widehat{(u_h)} [\![\vec{g}_h]\!] \, ds + \int_{\Gamma_h^D} u_b \, \vec{g}_h \cdot \vec{n} \, ds + \int_{\Gamma_h^N} u_h \vec{g}_h \cdot \vec{n} \quad (8)$$

It should be understood that (7) and (8) hold for arbitrary test function $v_h \in V_h$ and $\vec{g}_h \in \vec{G}_h$. From Here what defines the various DG schemes is the choice of numerical fluxes $\widehat{u_h}$ and $\widehat{(\vec{z}_h)}$.

Here in 3 schemes will be derived

1). Bassi-Rebay 1 (BR1)
2). Bassi-Rebay 2 (BR2)
3) Symmetric Interior Penalty (SIP)

1). BR1 Scheme

We begin by selecting $\widehat{(u_h)} = \{u_h\}$ and manipulating equation.

(8) First,

$$\int_{\Omega_h} \vec{z}_h \cdot \vec{g}_h \, d\Omega = -\int_{\Omega_h} u_h \nabla \cdot \vec{g}_h \, d\Omega + \int_{\Sigma_h} \{u_h\} [\![\vec{g}_h]\!] \, ds + \int_{\Gamma_h^D} u_b \, \vec{g}_h \cdot \vec{n} \, ds + \int_{\Gamma_h^N} u_h \vec{g}_h \cdot \vec{n} \, ds$$

We will need some useful identities, (the Mathematicians would write these as Lemmas but, I'm not that snobby it's much simpler than that.)

<u>Identity 1:</u> $(\alpha_h \vec{\beta}_h \cdot \vec{n})^- + (\alpha_h \vec{\beta}_h \cdot \vec{n})^+ = \{\alpha_h\} [\![\vec{\beta}_h]\!] + \{\vec{\beta}_h\} [\![\alpha_h]\!]$

Proof: Let $\{\ \}$ and $[\![\ ]\!]$ be defined as previously. then the above is just

$$\frac{\alpha_h^- + \alpha_h^+}{2} \left[(\vec{\beta}_h \cdot \vec{n})^- + (\vec{\beta}_h \cdot \vec{n})^+ \right] + \frac{\vec{\beta}_h^- + \vec{\beta}_h^+}{2} \left[(\alpha_h \vec{n})^- + (\alpha_h \vec{n})^+ \right]$$

Let $\vec{n}^+ = -\vec{n}^- = -\vec{n}$ with $\vec{n}^- = \vec{n}$

$$\frac{\alpha_h^- + \alpha_h^+}{2}\left[\vec{\beta}_h^- \vec{n} - \vec{\beta}_h^+ \vec{n}\right] + \frac{\vec{\beta}_h^- + \vec{\beta}_h^+}{2}\left[\alpha_h^- \vec{n} - \alpha_h^+ \vec{n}\right] =$$

$$= \frac{\alpha_h^- \beta_h^- \vec{n}}{2} - \cancel{\frac{\alpha_h^- \beta_h^+ \vec{n}}{2}} + \cancel{\frac{\alpha_h^+ \beta_h^- \cdot \vec{n}}{2}} - \frac{\alpha_h^+ \beta_h^+ \cdot \vec{n}}{2} + \frac{\vec{\beta}_h^- \cdot \alpha_h^- \vec{n}}{2} - \cancel{\frac{\vec{\beta}_h^- \cdot \alpha_h^+ \vec{n}}{2}} + \cancel{\frac{\alpha_h^- \vec{\beta}_h^+ \vec{n}}{2}}$$

$$- \frac{\alpha_h^+ \cdot \vec{\beta}_h^+ \vec{n}}{2}$$

$$= \alpha_h^- \vec{\beta}_h^- \vec{n} - \alpha_h^+ \vec{\beta}_h^+ \vec{n} = \boxed{(\alpha_h \vec{\beta}_h \vec{n})^- + (\alpha_h \vec{\beta}_h \cdot \vec{n})^+}$$ the result

proven.

Identity 2: $\int_{\Omega_h} \alpha_h(\nabla \cdot \vec{\beta}_h) + \vec{\beta}_h \cdot \nabla \alpha_h \, d\Omega =$
$\int_{\Sigma_h} [\{\alpha_h\} [\![\vec{\beta}_h]\!] + \{\vec{\beta}_h\} \cdot [\![\alpha_h]\!] \, ds + \int_{\Gamma_h} \alpha_h \vec{\beta}_h \cdot \vec{n} \, ds$

Proof:

Begin by considering

$$\int_{\Omega_h} \alpha_h(\nabla \cdot \vec{\beta}_h) + \vec{\beta}_h \cdot \nabla \alpha_h \, d\Omega = \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} \alpha_h(\nabla \cdot \vec{\beta}_h) + \vec{\beta}_h \cdot \nabla \alpha_h \, d\Omega$$

By inspection this is obviously.

$$= \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} \nabla \cdot (\alpha_h \vec{\beta}_h) \, d\Omega$$

using integration by parts give

$$= \sum_{e \in \mathcal{T}_h} \oint_{\partial \Omega_e} \alpha_h \vec{\beta}_h \cdot \vec{n} \, ds$$

which using the edge based equivalent is written as

$$= \sum_{i \in \mathcal{I}_h} \int_{\Sigma_i} [(\alpha_h \vec{\beta}_h \cdot \vec{n})^- + (\alpha_h \vec{\beta}_h \cdot \vec{n})^+] \, ds + \sum_{b \in \mathcal{B}_h} \int_{\Gamma_b^D} \alpha_h \vec{\beta}_h \cdot \vec{n} \, ds$$

using identity 1 this becomes

$$= \sum_{i \in \mathcal{I}_h} \int_{\Sigma_i} [\{\alpha_h\} [\![\vec{\beta}_h]\!] + \{\vec{\beta}_h\} [\![\alpha_h]\!] \, ds + \sum_{b \in \mathcal{B}_h} \int_{\Gamma_b^D} \alpha_h \vec{\beta}_h \cdot \vec{n} \, ds$$

Now using our $\sum_{i \in \mathcal{I}_h}(\ )$ $\sum_{b \in \mathcal{B}_h}(\ )$ ~~need~~ notation changes all

$$\boxed{\int_{\Omega_h} \alpha_h (\nabla\cdot\beta_h) + \beta_h\cdot\nabla\alpha_h = \int_{\Sigma_h} \left[\{\alpha_h\}[\![\beta_h]\!] + \{\beta_h\}[\![\alpha_h]\!]\right] ds + \int_{\Gamma_h} \alpha_h\, \vec{\beta}_h\cdot\vec{n}\, ds} \quad (7)$$

Which is the desired result.

Identities 1 and (1) can be used to simplify. (6)

by

$$-\int_{\Omega_h} u_h \nabla\cdot\vec{g}_h\, d\Omega = \int_{\Omega_h} \vec{g}_h\cdot\nabla u_h\, d\Omega - \int_{\Sigma_h} \left[\{u_h\}[\![\vec{g}_h]\!] + \{\vec{g}_h\}[\![u_h]\!]\right] ds$$

$$\left. - \int_{\Gamma_h^N} u_h \vec{g}_h\cdot\vec{n}\, ds - \int_{\Gamma_h^D} u_h \vec{g}_h\cdot\vec{n}\, ds \right\}$$

Note: here we split $\int_{\Gamma_h}$ into the Dirchlet $\int_{\Gamma^D}$ and Neuman $\int_{\Gamma^N}$ parts

Using this in (6) gives.

$$\int_{\Omega_h} \vec{z}_h\cdot\vec{g}_h\, d\Omega = \int_{\Omega_h} \vec{g}_h\cdot\nabla u_h\, d\Omega - \int_{\Sigma_h} \left[\{u_h\}\cancel{[\![\vec{g}_h]\!]} + \{\vec{g}_h\}[\![u_h]\!]\right] ds - \int_{\Gamma_h^N} \cancel{u_h \vec{g}_h\cdot\vec{n}}\, ds$$

$$+ \int_{\Sigma_h} \{u\}\cancel{[\![g]\!]}\, ds + \int_{\Gamma_h^N} \cancel{u_h \vec{g}_h\cdot\vec{n}}\, ds + \int_{\Gamma_h^D} u_b \vec{g}_h\cdot\vec{n}\, ds$$

Results in

$$\int_{\Omega_h} \vec{z}_h\cdot\vec{g}_h\, d\Omega - \int_{\Omega_h} \vec{g}_h\cdot\nabla u_h\, d\Omega = -\int_{\Sigma_h} \{\vec{g}\}[\![u]\!]\, ds + \int_{\Gamma_h^D} u_b \vec{g}_h\cdot\vec{n}\, ds$$

$$- \int_{\Gamma_h^D} \vec{g}_h\cdot\vec{n}\, u_h\, ds$$

Re-arranging gives

$$\int_{\Omega_h} \vec{z}_h\cdot\vec{g}_h\, d\Omega - \int_{\Omega_h} \vec{g}_h\cdot\nabla u_h\, d\Omega + \int_{\Sigma_h} \{\vec{g}\}[\![u_h]\!]\, ds + \int_{\Gamma_h^D} \vec{g}_h\cdot\vec{n}(u - u^b)\, ds = 0$$

We could follow Bassi-Rebay and define an extended jump operator but this is unnecessary and can cause confusion.

What really defines the BR1 scheme is the discretization of the ~~Laplace~~ Poisson equation.

$$-\int_{\Omega_h} \nabla v_h \cdot \vec{z}_h \, d\Omega + \int_{\Sigma_h} (\hat{\vec{z}}_h) \cdot [\![v_h]\!] ds + \int_{\Gamma_h^D} (\hat{\vec{z}}_h) \cdot \vec{n} \, v_h \, ds + \int_{\Gamma_h^N} (\vec{z} \cdot \vec{n})^b v_h \, ds = -\int_{\Omega_h} v_h f \, d\Omega$$

The BR1 scheme is defined by the choice of $(\hat{\vec{z}}_h) = \{\vec{z}_h\}$.
That's it.

The Auxiliary variable equation can be manipulated

$$\int_{\Omega_h} \vec{g}_h \cdot (\vec{z}_h - \nabla u_h) \, d\Omega = -\int_{\Sigma_h} \{\vec{g}_h\} [\![u_h]\!] \, ds - \int_{\Gamma_h^D} \vec{g}_h \cdot \vec{n} \, (u_h - u^b) \, ds$$

Define the Lifting operator $\vec{R}_h = (\vec{z}_h - \nabla u_h)$

$$\int_{\Omega} \vec{g}_h \cdot \vec{R}_h \, d\Omega = -\int_{\Sigma_h} \{\vec{g}_h\} [\![u_h]\!] ds - \int_{\Gamma_h^D} \vec{g}_h \cdot \vec{n} (u - u^b) \, ds$$

<u>Symmetric Interior Penalty</u> ;

Let $\hat{\vec{z}}_h = \{\nabla u_h\} + \mu [\![u]\!]$
and $\hat{u}_h = \{u_h\}$ → This the same as B.R. 2 thus the lifting operator is defined exactly the same as B.R. 2

Thus the Primal equation is

$$-\int_{\Omega_h} \nabla v_h \cdot z_h \, d\Omega + \oint \{\nabla u_h\} [\![v_h]\!] + \mu [\![u_h]\!] \cdot [\![v]\!] \, ds = -\int_{\Omega_h} v_h f \, d\Omega$$

Auxiliary variable equation is

$$\int_{\Omega_h} \vec{g}_h \cdot \vec{z}_h = \int_{\Omega_h} \nabla u_h \cdot \vec{g}_h \, d\Omega - \int_{\Sigma_h} \{\vec{g}\} [\![u_h]\!] \, ds - \int_{\Gamma_h^D} \vec{g}_h \cdot \vec{n} \, (u - u^b) \, ds$$

Using $\vec{g} = \nabla v_h$ in the second equation and substituting the result into 1st equation gives the Primal form as

$$-\int_{\Omega_h} \nabla v_h \cdot \nabla u_h \, d\Omega + \int_{\Sigma_h} \{\nabla v_h\} [\![u_h]\!] + \{\nabla u_h\} [\![v_h]\!] ds - \int_{\Sigma_h} \mu [\![u_h]\!] [\![v_h]\!] ds$$
$$+ \int_{\Gamma_h^D} \nabla u_h \cdot \vec{n} \, v_h + \nabla v_h \cdot \vec{n} (u - u^b) \, ds - \int_{\Gamma_h^D} \mu (u - u^b) v_h \, ds.$$

The SIP method is therefore a mixed method with choice (9)

$$\hat{u}_h = \{u_h\} \quad \hat{z}_h = \{\nabla u_h\} - \mu[\![u_h]\!]\,ds.$$

The resulting equation is given as

$$-\int_{\Omega_h} \nabla u_h \cdot \nabla v_h \, d\Omega + \int_{\Sigma_h} \{\nabla u_h\}\cdot[\![v_h]\!] + \{\nabla v_h\}[\![u_h]\!]\,ds - \int_{\Sigma_h} \mu[\![u_h]\!][\![v_h]\!]\,ds$$
$$\quad (1) \qquad\qquad (2) \qquad\qquad (3) \qquad\qquad (4)$$

$$+ \int_{\Gamma_h^D} \nabla u_h\cdot\vec{n}\, v_h + \nabla v_h\cdot\vec{n}(u-u^0)\,ds - \int_{\Gamma_h^D} \mu(u-u^0)v_h\,ds$$

(1): Volume term

(2): Part 1 of $\hat{z}_h$ flux

(3): Symmetry term given from definition of the Lifting operator.

(4): Penalty Term

The rest are Boundary versions of (2) - (4)

Remark! For Poisson problems the above is sufficient to solve. In fact a generalized variant of this (Almost trivial) is the MS discretization.
However for turbulence model equations the source terms depend on gradients. These gradients cannot be computed from the ∇u_h only term in fact we shall derive what they should be for

$\nabla^2 u = S(u, \nabla u)$ in the Next Section.

Source terms: with Gradients

As a simple example consider the follow modified poisson problem,

$$\nabla^2 u = -S(u, \nabla u).$$

subject to $u = u^b$ on $\Gamma^D \in \partial\Omega$.
$\left(\frac{\partial u}{\partial n}\right)^b = z^b$ on $\Gamma^N \in \partial\Omega$ } Dirichlet and Neumann cf on

Application of the mixed method gives

$$\nabla \cdot \vec{z} = -S(u, \vec{z}).$$
$$\nabla u = \vec{z}$$

The variation statement is as follows.

For $u \in \mathbb{R}$ and $\vec{z} \in \mathbb{R}^d$ seek $u, \vec{z}$ such that

$$\int_\Omega v\, \nabla \cdot \vec{z}\, d\Omega = -\int_\Omega v\, S(u, \vec{z})\, d\Omega \qquad \forall\ v \in \mathbb{R}$$

and

$$\int_\Omega \vec{g} \cdot \vec{z}\, d\Omega = \int_\Omega \vec{g} \cdot \nabla u\, d\Omega \qquad \forall\ \vec{g} \in \mathbb{R}^d$$

~~These solutions as before are sought in the following discontinuous Finite element spaces.~~

$$\cancel{V_h := \{ v \in L^2(\Omega_h) : v|_e \in P^k(e)\ \forall e \in \mathcal{T}_h \}}$$
$$\cancel{\vec{G}_h := \{ \vec{g} \in [L^2(\Omega_h)]^d : \vec{g}|_e \in [P^k(e)]^d\ \forall e \in \mathcal{T}_h \}}$$

] The english versions are on page 1.

~~Replacing all the continuous functions by their discrete finite element equivalents give,~~

Integration By parts gives

$$-\int_\Omega \vec{z} \cdot \nabla v\, d\Omega + \oint_{\partial\Omega} v\, \vec{z}^* \cdot \vec{n}\, ds = -\int_\Omega v\, S(u, \vec{z})\, d\Omega$$

$$\int_\Omega \vec{g} \cdot \vec{z}\, d\Omega = -\int_\Omega u\, \nabla \cdot \vec{g}\, d\Omega + \oint_{\partial\Omega} u^* \vec{g} \cdot \vec{n}\, d\Omega.$$

We will discretize this using discontinuous finite elements. And thus introduce the same 2 discontinuous finite elements spaces as

$$V_h := \{ v \in L^2(\Omega_h) : v|_e \in P^k(\Omega_h) \ \forall e \in \mathcal{T}_h \}$$
$$\vec{G}_h := \{ \vec{g} \in [L^2(\Omega_h)]^d : \vec{g}|_e \in [P^k(\Omega_h)]^d \ \forall e \in \mathcal{T}_h \}$$

where as usual $\mathcal{T}_h$ is the discretization of space into canonical shapes in quads, tri's etc.

We seek solutions as find $u_h \in V_h$, $\vec{z} \in \vec{G}_h$ such that,

$$\sum_{e\in\mathcal{T}_h} \int_{\Omega_h} \vec{z}_h \cdot \nabla v_h \, d\Omega_h + \sum_{e\in\mathcal{T}_h} \oint_{\partial\Omega_h} v_h (\widehat{\vec{z}_h}) \cdot \vec{n} \, ds = \sum_{e\in\mathcal{T}_h} \int_{\Omega_h} v_h \, S(u_h, \vec{z}_h) \, d\Omega_h$$

$$\sum_{e\in\mathcal{T}_h} \int_{\Omega_h} \vec{g}_h \cdot \vec{z}_h \, d\Omega_h = -\sum_{e\in\mathcal{T}_h} \int_{\Omega_h} u_h \nabla \cdot \vec{g}_h \, d\Omega + \sum_{e\in\mathcal{T}_h} \oint_{\partial\Omega} (\hat{u}_h) \vec{g}_h \cdot \vec{n} \, ds$$

$\forall \ v_h \in V_h$ and $\vec{g}_h \in \vec{G}_h$.

As before we exploit the $\sum_{e\in\mathcal{T}_h}$ on the interface terms and write them as sums of the faces of $\mathcal{T}_h$

$$\sum_{e\in\mathcal{T}_h} \oint_{\partial\Omega} v_h (\hat{\vec{z}}_h) \cdot \vec{n} \, ds = \sum_{i\in\mathcal{I}_h} \int_{\Sigma_i} (\widehat{\vec{z}_h}) \left[(v_h \vec{n})^- + (v_h \vec{n})^+ \right] ds + \sum_{b\in\mathcal{B}_h} \int (\hat{\vec{z}}_h) \cdot v_h \vec{n} \, ds$$

$$\sum_{e\in\mathcal{T}_h} \oint_{\partial\Omega} \hat{u}_h \, \vec{g}_h \cdot \vec{n} \, ds = \sum_{i\in\mathcal{I}_h} \int_{\Sigma_i} \hat{u}_h \left[(\vec{g}\cdot\vec{n})^- + (\vec{g}\cdot\vec{n})^+ \right] ds + \sum_{b\in\mathcal{B}_h} \int \hat{u}_h \, \vec{g}_h \cdot \vec{n} \, ds$$

Using these and the jump and average notations.

$$\sum_{e\in\mathcal{T}_h} \int_{\Omega_h} \vec{z}_h \cdot \nabla v_h \, d\Omega_h + \sum_{i\in\mathcal{I}_h} \int_{\Sigma_i} (\widehat{\vec{z}_h}) \cdot [\![v_h]\!] \, ds + \sum_{b\in\mathcal{B}_h} \int (\hat{\vec{z}}_h) \cdot v_h \vec{n} \, ds$$

$$\sum_{e\in\mathcal{T}_h} \int_{\Omega_h} \vec{g}_h \cdot \vec{z}_h \, d\Omega_h = -\sum_{e\in\mathcal{T}_h} \int_{\Omega_h} u_h \nabla \cdot \vec{g}_h \, d\Omega_h + \sum_{i\in\mathcal{I}_h} \int_{\Sigma_i} \hat{u}_h [\![\vec{g}_h]\!] \, ds + \sum_{b\in\mathcal{B}_h} \int \hat{u}_h \, \vec{g}_h \cdot \vec{n} \, ds$$

The Boundary terms can be Broken up into the Neumann and Dirichlet parts. Also consider the short hand (12)

$$\sum_{e \in \Omega_h} \int_{\Omega_h} (\,) d\Omega \equiv \int_{\Omega_h}$$

$$\sum_{e \in \Sigma_h} \int_{\Sigma_i} (\,) ds \equiv \int_{\Sigma_h}$$

$$\left.\begin{array}{l} \displaystyle\sum_{b \in B_h^D} \int_{\Sigma_i^D} (\,) ds = \int_{\Gamma_h^D} \\ \displaystyle\sum_{b \in B_h^N} \int_{\Sigma_i^N} (\,) ds = \int_{\Gamma_h^N} \end{array}\right\} \begin{array}{l} \text{Dirichlet Boundary integral} \\ \text{Neumann Boundary integral} \end{array}$$

Using those results in.

$$-\int_{\Omega_h} \nabla v_h \cdot \vec{z}_h \, d\Omega_h + \int_{\Sigma_h} \widehat{(\vec{z}_h)} \cdot [\![v_h]\!] ds + \int_{\Gamma_h^D} (\hat{z}_h \cdot \vec{n}) v_h \, ds + \int_{\Gamma_h^N} (\vec{z} \cdot \vec{n})^b v_h \, ds = \int_{\Omega_h} s(u_h, \vec{z}_h)$$

$$-\int_{\Omega_h} \vec{g}_h \cdot \vec{z}_h \, d\Omega_h = -\int_{\Omega_h} u_h \nabla \cdot \vec{g}_h \, d\Omega_h + \int_{\Sigma_h} (\hat{u}_h) \cdot [\![\vec{g}_h]\!] ds + \int_{\Gamma_h^D} u^b \cdot \vec{g}_h \cdot \vec{n} \, ds + \int_{\Gamma_h^N} (\hat{u}_h) \cdot \vec{g}_h \cdot \vec{n} \, ds$$

Those are the final weighted residual equations. Now we must pick $\hat{u}_h$ $\widehat{(\vec{z}_h)}$

<u>The SIP Method</u>

Let $\widehat{(\vec{z}_h)} = \{\nabla u_h\} - \mu [\![u_h]\!]$

$(\hat{u}_h) = \{u_h\}$

The poisson equation is

$$\boxed{\begin{aligned} &-\int_{\Omega_h} \nabla v_h \cdot \vec{z}_h \, d\Omega_h + \int_{\Sigma_h} \{\nabla v\} \cdot [\![v_h]\!] ds - \int_{\Sigma_h} \mu [\![u_h]\!] [\![v_h]\!] ds \\ &+ \int_{\Gamma_h^D} \nabla u_h \cdot \vec{n} \, v_h \, ds - \int_{\Gamma_h^D} \mu (u - u^b) v_h \, ds + \int_{\Gamma_h^N} (\vec{z} \cdot \vec{n})^b v_h \, ds = -\int_{\Omega_h} v_h (u_h, \vec{z}_h) d\Omega_h \end{aligned}}$$

SIP poisson with $\vec{z}_h$

The 2^{nd} equation for the Auxilary variable $\vec{z}h$ is exactly the same as the original poisson problem. Thus we have. (13)

$$\int_{\Omega h} \vec{g}h \cdot \vec{z}h \, d\Omega = \int_{\Omega h} \nabla Uh \cdot \vec{g}h \, d\Omega - \int_{\Sigma h} \{\vec{g}h\} [\![Uh]\!] \, ds - \int_{\Gamma_h^D} \vec{g}_h \cdot \vec{n} (Uh - U^g) \, ds$$

Again replacing $\vec{g}h$ by the choice $\vec{g}h = \nabla vh$ given

$$\int_{\Omega h} \nabla vh \cdot \vec{z}h \, d\Omega = \int_{\Omega h} \nabla Uh \cdot \nabla vh \, d\Omega - \int_{\Sigma h} \{\nabla vh\} \cdot [\![Uh]\!] \, ds - \int_{\Gamma_h^D} \nabla vh \cdot \vec{n} (U_h - U^g) ds$$

Thus using this in the poisson equation.

$$\boxed{\begin{aligned} -\int_{\Omega h} \nabla vh \cdot \nabla Uh \, d\Omega h + \int_{\Sigma h} \{\nabla Uh\} [\![vh]\!] + \{\nabla vh\} [\![Uh]\!] \, ds - \int_{\Sigma h} \mu [\![Uh]\!] [\![vh]\!] ds \\ + \int_{\Gamma_h^D} \nabla Uh \cdot \vec{n} \, vh + \nabla vh \cdot \vec{n} (U_h - U^g) \, ds - \int_{\Gamma_h^D} \mu vh (Uh - U^g) ds + \int_{\Gamma_h^N} (\vec{z} \cdot \vec{n})^g vh \, ds \\ = -\int_{\Omega h} vh \, S(Uh, \vec{z}h) \, d\Omega h \end{aligned}}$$

For a standard poisson equation this would be enough but the source term still has $\vec{z}h$ in it.
Since the linearity of $S(Uh, \nabla Uh)$ is not gaurenteed or specified. We'll need to resort to actually compute $\vec{z}h$ just for this term.

Computing $\vec{z}h$ for the source term;

In order to completely eliminate $\vec{z}h$ from the LHS of the poisson equation $\vec{z}h$ was taken a ∇vh however this does not work for actually computing $\vec{z}h$. In fact when Bassi-Rebay compute there lifting operator for BR2 they use $\vec{g}h = \nabla vh$ for Eliminating the lifting operator from the volume term out when they compute the local lift on the faces $\vec{g}h = (vh \vec{n})$

Though not rigorously correct we will do an analogous procedure for directly computing a volume lifting operator for the gradient based source term.

The auxiliary equation can be written as

$$\int_{\Omega_h} \vec{g}_h \cdot (\vec{z}_h - \nabla u_h)\, d\Omega = -\int_{\Sigma_h} \{\vec{g}_h\} [\![u_0]\!] ds - \int_{\Gamma_h^D} \vec{g}_h \cdot \vec{n} (u_h - u^b) ds$$

Define $\vec{z}_h - \nabla u_h = \vec{R}_h$ - the lifting operator:

then $\vec{z}_h = \nabla u_h + \vec{R}_h$

Thus we'll need to compute $\vec{R}_h$ from

$$\int_{\Omega_h} \vec{g}_h \cdot (\vec{R}_h)\, d\Omega = -\int_{\Sigma_h} \{\vec{g}\} [\![U]\!] ds - \int_{\Gamma_h^D} \vec{g}_h \cdot \vec{n} (u_h - u^b) ds$$

We will simply split $\vec{g}_h$ into a single test function for each ~~dire~~ spatial direction $\vec{g}_k = \delta_{kj} v_h \ \forall j \in [1, d]$.

So for 2D. $\vec{g}_1 = (v_h, 0)$, $\vec{g}_2 = (0, v_h)$

Thus for each k-direction $\vec{R}_h$ component the equation is

$$\boxed{\begin{gathered} \int_{\Omega} v_h^{L/R} \cdot \vec{R}_{hk}^{L/R} = -\int_{\Sigma_h} \frac{v_h^{L/R}}{2} \cdot (u_h^- - u_h^+) n_k\, ds - \int v_h^+ \cdot n_k (u_h - u^b) ds \\ k = 1, 2 \text{ for 2D} \quad \text{Very simple} \end{gathered}}$$

once we have $\vec{R}_{hk}$ as a modal expansion.

$\vec{R}_h = \sum_{j=} \hat{R}_{hj} \phi_j$ then we're ready

The final equation is

$$-\int_{\Omega_h} \nabla u_h \cdot \nabla v_h\, d\Omega + \int_{\Sigma_h} \{\nabla u_h\} [\![v_h]\!] + \{\nabla v_h\} [\![u_h]\!] ds - \int_{\Sigma_h} \mu [\![u_h]\!] [\![v_h]\!] ds$$

$$+ \int_{\Gamma_h^D} \nabla u_h \cdot \vec{n}\, v_h + \nabla v_h \cdot \vec{n} (u_h - u^b) ds - \int_{\Gamma_h^D} \mu v_h (u_h - u^b) ds$$

$$+ \int_{\Gamma_h^N} v_h (\vec{z} \cdot \vec{n})^b ds = -\int_{\Omega_h} v_h \cdot S(u_h, \nabla u_h + \vec{R}_h) \quad \vec{R}_h \text{ computed from}$$

Implementation of Mixed Gradient Computations

From the derivation the equation for the Auxiliary variable is

$$\int_{\Omega_h} v_h \cdot \vec{R}_h \, d\Omega = -\int_{\Sigma_h} \frac{v_h}{2} \cdot (u_h^- - u_h^+)\vec{n}\, ds - \int_{\Gamma_D} v_h (u_h^- - u^b)\vec{n}\, ds$$

Recall

$$\int_{\Omega_h} = \sum_{e \in \mathcal{T}_h} \int_{\Omega_e}$$

$$\int_{\Sigma_h} = \sum_{i \in \mathcal{I}_h} \int_{\Sigma_i}, \qquad \int_{\Gamma_D} = \sum_{b \in \mathcal{B}_h} \int_{\Gamma_b^D}$$

To implement these equations replace v_h by ϕ_i gives

$$\int_{\Omega_h} \phi_i \cdot \vec{R}_h \, d\Omega = -\int_{\Sigma_h} \frac{\phi_i}{2} \cdot (u_h^- - u_h^+)\vec{n}\, ds - \int_{\Gamma_D} \phi_i (u_h^- - u_h^+)\vec{n}\, ds$$

Looking at a single element e in the mesh we have

$$\int_{\Omega_e} \phi_i \cdot \vec{R}_{h_e} \, d\Omega = -\oint_{\partial\Omega_e} \frac{\phi_i}{2} (u_h^- - u_h^+)\vec{n}\, ds ,$$ where if part of $\partial\Omega_e$ is a Dirichlet boundary face the $u_h^+ \to u^b$

We could do this as a non-linear formulation but it is a linear equation so we can in essence define the jacobian and directly use that to produce the lifting operator for the element.

$$\int_{\Omega_e} \phi_i \cdot \sum_j \phi_j \vec{R}_{h_j} \, d\Omega = -\oint_{\partial\Omega_e} \frac{\phi_i^-}{2} \Big(\sum_j \phi_j^- \hat{u}_j^- - \sum_j \phi_j^+ \hat{u}_j^+ \Big) \vec{n}\, ds$$

For each direction.

$$\sum_j \int_{\Omega_e} \phi_i \, \phi_j \, ds \, \hat{R}_{h x_j} = -\left[\sum_j \oint_{\partial\Omega_e} \frac{\phi_i^-}{2} \phi_j^- n_x \, ds \, \hat{u}_j^- - \sum_j \oint_{\partial\Omega_e} \frac{\phi_i^-}{2} \phi_j^+ n_x \, ds \, \hat{u}_j^+ \right]$$

$$\sum_j \int_{\Omega_e} \phi_i \, \phi_j \, ds \, \hat{R}_{h y_j} = -\left[\sum_j \oint_{\partial\Omega_e} \frac{\phi_i^-}{2} \phi_j^- n_y \, ds \, \hat{u}_j^- - \sum_j \oint \frac{\phi_i^-}{2} \phi_j^+ n_y \, ds \, \hat{u}_j^+ \right]$$

Of course the $\int_{\Omega_e} \phi_i \phi_j \, d\Omega$ is the Mass Matrix for each field.

Let $[M] = \int_{\Omega_e} \phi_i \phi_j \, d\Omega$ then

$$\{\hat{R}_{hx}\} = -[M]^{-1}\left(\oint_{\partial\Omega_e} \frac{\phi_i^-}{2} \phi_j^- n_x \, ds \{\hat{u}_j\}^- - \oint \frac{\phi_i^-}{2} \phi_j^+ n_x \, ds \{\hat{u}_j\}^+\right)$$

$$\{\hat{R}_{hy}\} = -[M]^{-1}\left(\oint_{\partial\Omega_e} \frac{\phi_i^-}{2} \phi_j^- n_y \, ds \{\hat{u}_j\}^- - \oint \frac{\phi_i^-}{2} \phi_j^+ n_y \, ds \{\hat{u}_j\}^+\right)$$

Thus we'll stor

$[M]^{-1}\left(\oint_{\partial\Omega_e} \frac{\phi_i}{2} \phi_j \, n_{(\,)} \, ds\right)$ like terms which are the jacobians of $\vec{R}_h$ and need to be stored any way for implicit soluc

Thus we can build the lifting operator for the volume term out of it's jacobians with out extra quadrature.

This is Not true for Boundary faces where we will have to retain the full non-linear formulation.

Adjoint Based Error estimation for hp- adaptation.

Consider a functional of interest $L(u)$. Let us assume we have two "Meshes" $\mathcal{T}_H, \mathcal{T}_h$. where $\mathcal{T}_H$ is a coarse mesh and $\mathcal{T}_h$ is the fine Mesh. We want to estimate the functional value on $\mathcal{T}_h$ but without solving the fine mesh problem. (Here the fine mesh could be the same grid with p+1 order where $\mathcal{T}_H$ has p order).

Let $\tilde{U}_H$ denote a solution on $\mathcal{T}_H$ and $\tilde{U}_h$ a solution on $\mathcal{T}_h$.

Then the functional can be expanded in a taylor series about the coarse soln interpolated/projected to the fine mesh $\tilde{U}_H^h = I_H^h \tilde{U}_H$.

$$L(\tilde{U}_h) = L(\tilde{U}_H^h) + \left[\frac{\partial L}{\partial \tilde{U}_h}\right]_{\tilde{U}_H^h} (\tilde{U}_h - \tilde{U}_H^h) + \dots \quad (1)$$

$\left[\frac{\partial L}{\partial \tilde{U}_h}\right]_{\tilde{U}_H^h}$ denotes the objectives sensitivities to the fine mesh solution evaluated at the $\tilde{U}_H^h$ state.

The fine mesh must also obey the residual.

$R(\tilde{U}_h) = 0$ we don't have $\tilde{U}_h$ then we can do a taylor expansion for this.

$$R(\tilde{U}_h) = R(\tilde{U}_H^h) + \left[\frac{\partial R}{\partial \tilde{U}_h}\right]_{\tilde{U}_H^h} (\tilde{U}_h - \tilde{U}_H^h) + \dots \quad (2)$$

gives

$$0 = R(\tilde{U}_H^h) + \left[\frac{\partial R}{\partial \tilde{U}_h}\right]_{\tilde{U}_H^h} (\tilde{U}_h - \tilde{U}_H^h) + \dots \quad (3)$$

We can write an expression for $\tilde{U}_h - \tilde{U}_H^h$ as

$$(\tilde{U}_h - \tilde{U}_H^h) \simeq - \left[\frac{\partial R}{\partial \tilde{U}_h}\right]^{-1}_{\tilde{U}_H^h} R(\tilde{U}_H^h) \quad (4)$$

Using this in the functionals taylor series gives

$$L(\tilde{U}_h) = L(\tilde{U}_H) + \left[\frac{\partial L}{\partial \tilde{U}_h}\right]_{\tilde{U}_H^h} \left(- \left[\frac{\partial R}{\partial \tilde{U}}\right]^{-1}_{\tilde{U}_H^h} R(\tilde{U}_H^h)\right) \quad (5)$$

$$L_h(\tilde{U}_h) = L(\tilde{U}_H^h) - \left[\frac{\partial L}{\partial \tilde{U}_h}\right]_{\tilde{U}_H^h} \left[\frac{\partial R}{\partial \tilde{U}_h}\right]^{-1}_{\tilde{U}_H^h} R(\tilde{U}_H^h) \quad (6)$$ (2)

We can define a new operator

$$\Lambda_h^T = \left[\frac{\partial L}{\partial \tilde{U}_h}\right]_{\tilde{U}_H^h} \left[\frac{\partial R}{\partial \tilde{U}_h}\right]^{-1} \Rightarrow \left[\frac{\partial R}{\partial \tilde{U}_h}\right]^T \Lambda_h = \left[\frac{\partial L}{\partial \tilde{U}_h}\right]^T$$ Adjoint (7) problem.

$$L(\tilde{U}_h) = L(\tilde{U}_H^h) - \Lambda_h^T R(\tilde{U}_H^h). \quad (8)$$

The error is estimated in each cell on the coarse mesh by $\left|\sum_{\lambda=1}^{N\lambda} -\Lambda_h^T\big|_n R(\tilde{U}_H^h)\big|_n\right|$ were n is the # of daughter cells in the fine mesh.

Problem! The Adjoint problem is still on the fine level. This is too expensive. ~~Solution~~

Solution: Solve the coarse level Adjoint problem and interpolate/project this to the fine level.

This amounts to solving a modified (7) as

$$\left[\frac{\partial R}{\partial \tilde{U}_H}\right]^T \Lambda_H = \left[\frac{\partial L}{\partial \tilde{U}_H}\right]^T \quad (9)$$

and the defining $\Lambda_H^h = I_H^h \Lambda_H$

Now replace Λ_h^T by Λ_H^{Th} in (8) giving

$$L(\tilde{U}_h) = L(\tilde{U}_H^h) - \Lambda_H^{Th} R(\tilde{U}_H^h) \quad (10)$$

This is the final error equation. Again the error in each coarse cell is estimated as

$$\sum_{n=1}^{N\lambda} -\Lambda_H^{Th} R(\tilde{U}_H^h)$$

Let the error in functional be written as

$$\mathcal{E} = L(\tilde{U}_h) - L(\tilde{U}_H^h) = -\Lambda_H^{hT} R(\tilde{U}_H^h)$$

Then the fine level functional error is made up of inner products over the estimated fine level Adjoint / Residual.

The error from each coarse mesh cell can be estimate by looking at ~~However~~ how much error each coarse grid cell contributes to the fine grid functional error as.

$$\mathcal{E}_{K \in T_H} = -\sum_{n=1}^{N_d} \Lambda_H^{hT} R(\tilde{U}_H^h)\big|_n \qquad n - \text{daughter on fine level of } K.$$

Thus refinement criteria will be done such that some % of the error is refined probably between 80-90%. This corresponds to equi-distributing the error over the grid and then "global" refinement.

<u>Projection Operators</u>:

1) Patch wise Least Squares projection.

The adjoint and solution variables will be projected to the fine mesh by a patchwise least squares procedure on the fine mesh.

Let K denote an element in T_h i.e. $K \in T_h$. Let P_K denote the patch $\in T_h$ made up of the element K and his neighbors. The difference over the patch between the coarse solution and interpolated fine solution is given as

$$d = \sum_{\ell \in P_K} \left\| \sum_{j=1}^{M^*} (\hat{\Lambda}^h_{H_{jK}} \phi_j)\big|_\ell - \sum_{j=1}^{M} (\hat{\Lambda}_{H_{jK}} \phi_j)_\ell \right\|_2^2$$

where M is # of nodes on mesh T_H in each cell and M^* is the # of nodes on T_h on each cell.

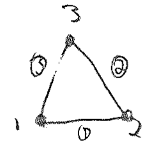
We wish to minimize $\mathcal{L}$ w.r.t. each coefficient of the expansion of Λ^h_π in cell K.

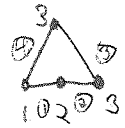
$$\frac{\partial \mathcal{L}}{\partial (\Lambda^h_{\pi_{jK}})} = 0.$$

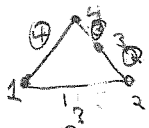
Non-Conforming Mesh Refinement

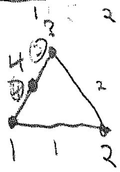
1). Non-conforming types

a) Triangle.

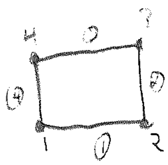
Type 0:  conforming

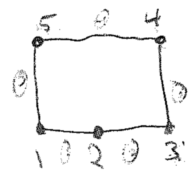
Type 1:  Node 2 hangs; faces 1 and 2 have same local id #

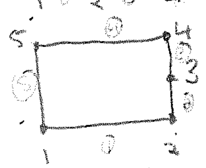
Type 2:  Node 3 hangs; faces 2 an 3 have same local id #

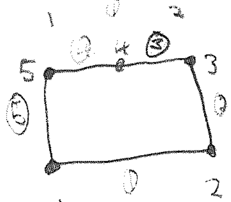
Type 3:  Node 4 hangs faces 3 and 4 have same local id #

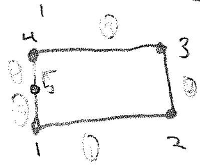
b). Quad

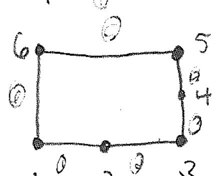
Type 0:  Conforming.

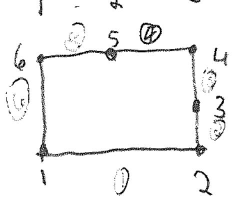
Type 1:  Node 2 hangs.: faces 1 and 2 have same local id #

Type 2:  Node 3 hangs; faces 2 and 3 have the same local id #

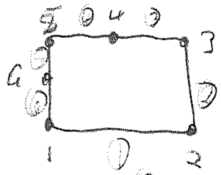
Type 3:  Node 4 hangs, faces 3 and 4 have the same local id #

Type 4:  Node 5 hang: faces 4 and 5 have the same local id #

Type 5:  Nodes 2 and 4 hang: faces 1 and 2, 3 and 4 have same local id #

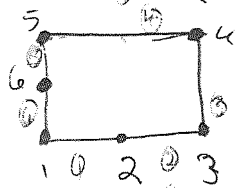
Type 6:  Nodes 3 and 5 hang: faces 2 and 3, 4 and 5 have the same local id #

Type 7:



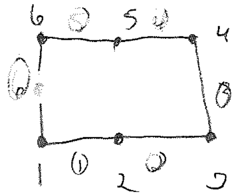
Nodes 4 and 6 hang: faces 3 and 4, 5 and 6 have the same local id #

Type 8:



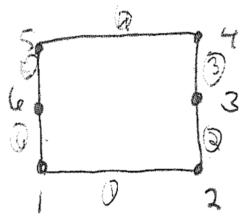
Nodes 6 and 2 hang: faces 5 and 6, 1 and 2 have the same local id #

Type 9:



Nodes 2 and 5 hang: faces 1 and 2, 4 and 5 have the same local id #

Type 10:



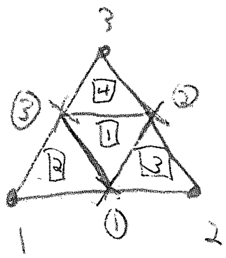
Nodes 3 and hang: faces 2 and 3, 5 and 6 have the same local id #

Refinement connections

1) Triangle

Consider a triangle with "o" representing an "old" Node and "X" representing a "New" node.

☐# is new cell #.



The new cells are defined as follows

New cell 1: ① ② ③ ⇒ OLD Node 1, New Node 1, New Node 3. ! The english will be omitted for Brevity just know a # is an old Node and ⓘ is a new node i.e. new nodes have a circle around them.

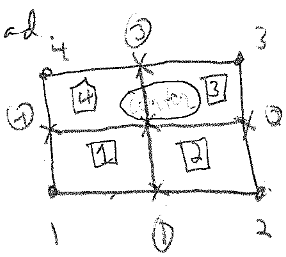
$\boxed{1}$: ① ② ③

$\boxed{2}$: 1 ① ③ → Check hanging Node on face1., and face 5

$\boxed{3}$: 2 ② ① → Check hanging Node on face2., and face 3

$\boxed{4}$: 3 ③ ② → Check hanging Node on face 3 and face 4

2). Quad



$\boxed{1}$: 1 ① (center) ④

$\boxed{2}$: 2 ② (center) ①

$\boxed{3}$: 3 ③ (center) ②

$\boxed{4}$: 4 ④ (center) ③

Non - conforming types that affect.

face 1: 1, 5, 8, 9, first face 4: 4, 7, 8, 10 } Last

face 4: 1, 5, 8, 9 ①

face 1: 2, 6, 10/5. face 4: 2, 6, 10/5 ③

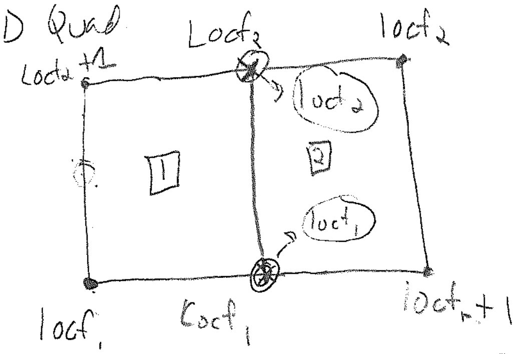
face 1: 3, 7/ 8, 9 ② ④

focal face

→ face 1 4/ 7, 8, 10 ④ ⑤ face 4: 3, 7 / 6, 9 ④ ⑤

Connection type 1: Anisotropic Line refinement.

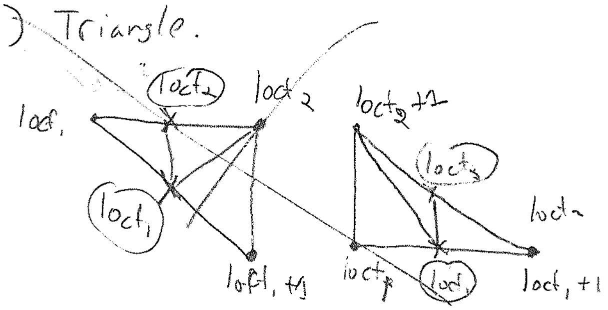
1) Quad



1: locf1 loct1 loct2 locf2 +1 check locf1 - 1 e2node new.

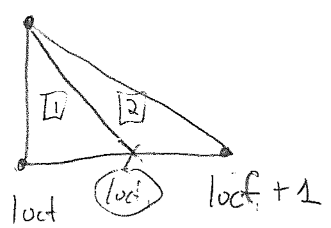
2 locf2 loct2 loct1 locf1+1 check locf1 +1 e2node new

2) Triangle.



? Should I do this

2.) Triangle End. of line

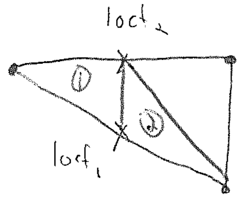


face

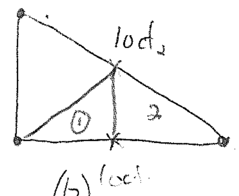
1 loct loct loct1 + 2 check n locf -1

2 locf+2 loct1 locf +1 check face locf +1

3). Triangle interior to Line.



(a).



(b)

①: loct$_1$ loct$_1$ loct$_2$

②: loct$_1$+1 loct$_2$ loct$_1$

Now define integer l_2 as

l_2 = loct$_2$ + 1
if (loct$_2$ == 3)
 l_2 = 1
else
 l_2 = loct$_2$ + 1
endif.

Now if l_2 == loct$_1$ then we have senario (a).

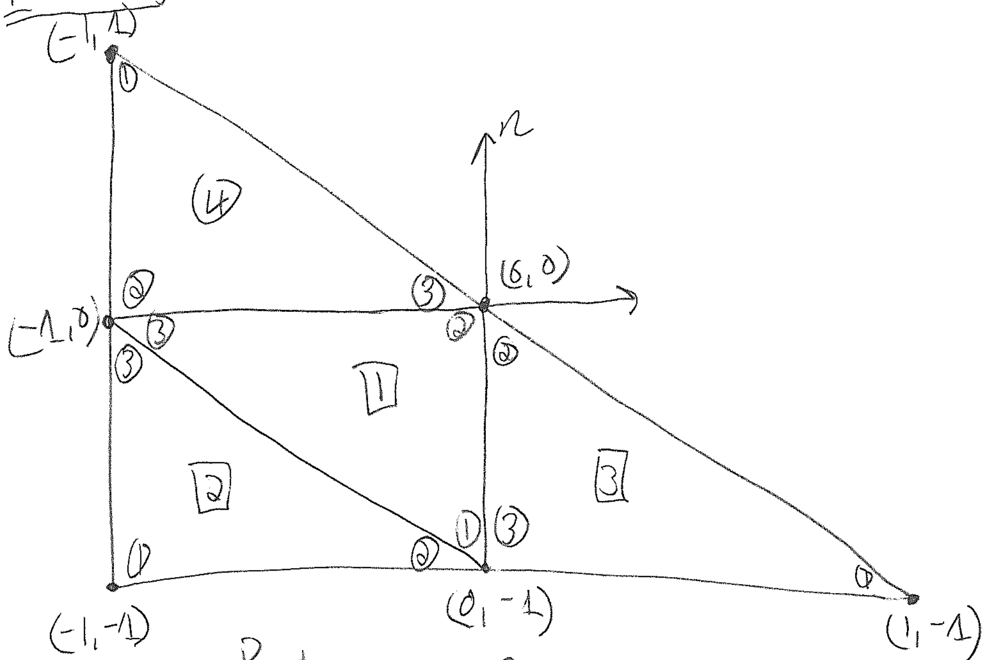
connect as.
③: opposite Node of loct$_1$, loct$_2$, loct$_1$+1, check last face (loct$_1$+1) for extra Node

else we have senario (b).

connect as
③ opposite Node of loct$_1$, check loct$_2$+1 for extra Node, loct$_1$ loct$_2$

Mapping Child standard element points to Parent for geometry and or solution projection

1 Triangle.



Child 1:

$$\begin{Bmatrix}\xi' \\ \eta'\end{Bmatrix} = \begin{Bmatrix}0 \\ -1\end{Bmatrix}, \begin{Bmatrix}0 \\ 0\end{Bmatrix}, \begin{Bmatrix}-1 \\ 0\end{Bmatrix}$$

Point 1, 2, 3

Child 2:

$$\begin{Bmatrix}\xi' \\ \eta'\end{Bmatrix} = \begin{Bmatrix}-1 \\ -1\end{Bmatrix}, \begin{Bmatrix}0 \\ -1\end{Bmatrix}, \begin{Bmatrix}-1 \\ 0\end{Bmatrix}$$

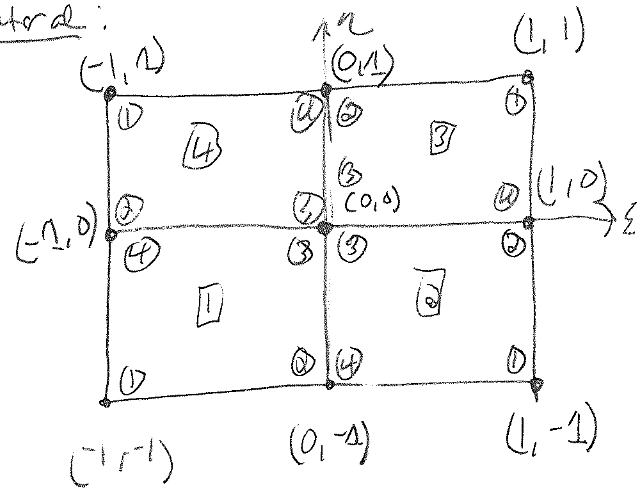
Child 3:

$$\begin{Bmatrix}\xi' \\ \eta'\end{Bmatrix} = \begin{Bmatrix}1 \\ -1\end{Bmatrix}, \begin{Bmatrix}0 \\ 0\end{Bmatrix}, \begin{Bmatrix}0 \\ -1\end{Bmatrix}$$

Child 4:

$$\begin{Bmatrix}\xi' \\ \eta'\end{Bmatrix} = \begin{Bmatrix}-1 \\ 1\end{Bmatrix}, \begin{Bmatrix}-1 \\ 0\end{Bmatrix}, \begin{Bmatrix}0 \\ 0\end{Bmatrix}$$

2) Quadrilateral:



Point 1, Point 2, Point 3, Point 4

Child 1

$$\begin{Bmatrix} \xi' \\ \eta' \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$$

child 2:

$$\begin{Bmatrix} \xi' \\ \eta' \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

child 3:

$$\begin{Bmatrix} \xi' \\ \eta' \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

child 4:

$$\begin{Bmatrix} \xi' \\ \eta' \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}, \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

New formulas for Viscous fluxes. (Laminar / RANS).

An observation I made recently has lead to the need to derive new flux formulas for the symmetry and penalty terms.

1). Viscous fluxes. $\vec{E}_v$, $\vec{F}_v$

$$\vec{E}_v = \begin{cases} 0 \\ \frac{M_\infty}{Re_\infty}(\mu+\mu_T)\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right) \\ \frac{M_\infty}{Re_\infty}(\mu+\mu_T)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \frac{M_\infty}{Re_\infty}\left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\gamma\frac{\partial e}{\partial x} + \frac{M_\infty}{Re_\infty}(\mu+\mu_T)\left[u\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right) + v\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \end{cases}$$

Let $\frac{M_\infty}{Re_\infty}(\mu+\mu_T) \equiv \alpha$ and $\frac{M_\infty}{Re_\infty}\left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right) \equiv \beta$

Then

$$\vec{E}_v = \begin{cases} 0 \\ \alpha\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right) \\ \alpha\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \beta\gamma\frac{\partial e}{\partial x} + \alpha\left[u\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right) + v\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \end{cases}$$

$$\vec{F}_v = \begin{cases} 0 \\ \frac{M_\infty}{Re_\infty}(\mu+\mu_T)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{M_\infty}{Re_\infty}(\mu+\mu_T)\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right) \\ \frac{M_\infty}{Re_\infty}\left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\gamma\frac{\partial e}{\partial y} + \frac{M_\infty}{Re_\infty}(\mu+\mu_T)\left[u\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + v\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right)\right] \end{cases}$$

$$\vec{F}_v = \begin{cases} 0 \\ \alpha\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \alpha\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right) \\ \beta\gamma\frac{\partial e}{\partial y} + \alpha\left[u\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + v\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right)\right] \end{cases}$$

These can be written as

$$\vec{E}_v = \begin{cases} 0 \\ \alpha\left(\frac{4}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\} - \frac{2}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\cdot\left\{\frac{\partial \vec{q}}{\partial y}\right\}\right) \\ \alpha\left(\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\} + \left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\}\right) \\ \beta\gamma\left\lfloor\frac{\partial e}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\} + \alpha\left[u\left(\frac{4}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\cdot\frac{\partial \vec{q}}{\partial x} - \frac{2}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\}\right) + v\left(\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\} + \left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor \ldots\right.\right. \end{cases}$$

$$F_v = \begin{Bmatrix} \overset{0}{\alpha}\left(\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor \left\{\frac{\partial \vec q}{\partial y}\right\} + \left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor \frac{\partial \vec q}{\partial x}\right) \\ \alpha\left(\tfrac{4}{3}\left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor \frac{\partial \vec q}{\partial y} - \tfrac{2}{3}\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor \frac{\partial \vec q}{\partial x}\right) \\ \beta\gamma\left\lfloor \frac{\partial e}{\partial \vec q}\right\rfloor \cdot \left\{\frac{\partial \vec q}{\partial y}\right\} + \alpha\left[u\left(\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor \left\{\frac{\partial \vec q}{\partial y}\right\} + \left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor \left\{\frac{\partial \vec q}{\partial x}\right\}\right) + v\left(\tfrac{4}{3}\left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor \left\{\frac{\partial \vec q}{\partial y}\right\} - \tfrac{2}{3}\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor \frac{\partial \vec q}{\partial x}\right)\right] \end{Bmatrix}$$

This gives

$$G_{11} = \frac{\partial E_v}{\partial \vec q_x} = \begin{bmatrix} \lfloor 0 \rfloor \\ \alpha \tfrac{4}{3}\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor \\ \alpha \left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor \\ \beta\gamma\left\lfloor \frac{\partial e}{\partial \vec q}\right\rfloor + \alpha\left[u\left(\tfrac{4}{3}\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor\right) + v\left(\left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor\right)\right] \end{bmatrix}$$

$$G_{12} = \frac{\partial E_v}{\partial \vec q_y} = \begin{bmatrix} \lfloor 0 \rfloor \\ -\alpha \tfrac{2}{3}\left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor \\ \alpha \left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor \\ \alpha\left[u\left(-\tfrac{2}{3}\left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor\right) + v\left(\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor\right)\right] \end{bmatrix}$$

$$G_{21} = \frac{\partial F_v}{\partial \vec q_x} = \begin{bmatrix} \lfloor 0 \rfloor \\ \alpha \left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor \\ -\alpha \tfrac{2}{3}\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor \\ \alpha\left[u\left(\left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor\right) + v\left(-\tfrac{2}{3}\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor\right)\right] \end{bmatrix}$$

$$G_{22} = \frac{\partial F_v}{\partial \vec q_y} = \begin{bmatrix} \lfloor 0 \rfloor \\ \alpha \left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor \\ \alpha \tfrac{4}{3}\left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor \\ \beta\gamma\left\lfloor \frac{\partial e}{\partial \vec q}\right\rfloor + \alpha\left[u\left(\left\lfloor \frac{\partial u}{\partial \vec q}\right\rfloor\right) + v\left(\tfrac{4}{3}\left\lfloor \frac{\partial v}{\partial \vec q}\right\rfloor\right)\right] \end{bmatrix}$$

2) Symmetry flux.

As per previous Notes this is written as

$$Sflux^{L/R} = \frac{1}{2}\left[\begin{bmatrix} G_{11}^{L/R} & G_{21}^{L/R} \\ G_{12}^{L/R} & G_{22}^{L/R} \end{bmatrix}\begin{Bmatrix} \frac{\partial\phi}{\partial x}^{L/R} \\ \frac{\partial\phi}{\partial y}^{L/R} \end{Bmatrix}\vec{n}\right]\{\Delta\vec{q}\} =$$

$$\frac{1}{2}\left(\left[G_{11}^{L/R}\frac{\partial\phi}{\partial x} + G_{21}^{L/R}\frac{\partial\phi}{\partial y}^{L/R}\right]n_x + \left[G_{12}^{L/R}\frac{\partial\phi}{\partial x}^{L/R} + G_{22}^{L/R}\frac{\partial\phi}{\partial y}\right]n_y\right)\{\Delta\vec{q}\}$$

The above is ok for Boundaries. Also just get rid of the $\frac{1}{2}$ part. It is logically correct But No efficient.

Instead Break it as

$$sflux = \frac{1}{2}{}_L\left(\left[G_{11}^{L/R}n_x + G_{12}^{L/R}n_y\right]\frac{\partial\phi}{\partial x}^{L/R} + \left[G_{21}n_x + G_{22}n_y\right]\frac{\partial\phi}{\partial y}^{L/R}\right)\{\Delta q\}$$

even better is

$$sflux^{L/R} = \frac{1}{2}\left(\left[G_{11}^{L/R}n_x + G_{12}^{L/R}n_y\right]\{\Delta q\}\frac{\partial\phi}{\partial x}^{L/R} + \left[G_{21}^{L/R}n_x + G_{22}^{L/R}n_y\right]\{\Delta q\}\frac{\partial\phi}{\partial y}^{L/R}\right)$$

For efficiency and so that we only compute non-linear terms 1 time per quadrature point. This should be written as

$$sf_x^{L/R} = \left[G_{11}^{L/R}n_x + G_{12}^{L/R}n_y\right]\{\Delta q\} = Ev(\{\Delta q\}\vec{n}_x)$$ i.e. replace $\nabla\vec{q}$ with $\{\Delta\vec{q}\}\vec{n}$ in the Ev term

$$sf_y^{L/R} = \left[G_{21}^{L/R}n_x + G_{22}^{L/R}n_y\right]\{\Delta q\} = Fv(\{\Delta q\}\vec{n})$$ i.e. replace $\nabla\vec{q}$ with $\{\Delta\vec{q}\}\cdot\vec{n}$ in Fv term

$$sflux^{L/R} = \frac{1}{2}\left(sf_x^{L/R}\frac{\partial\phi}{\partial x}^{L/R} + sf_y^{L/R}\frac{\partial\phi}{\partial y}^{L/R}\right)$$

So at gp compute $sf_x^{L/R}$, $sf_y^{L/R}$ i.e. 4 vectors then just do the multiplication and add with each test fn ϕ_i.

With this very helpful simplification the sf_x's and sf_y

$$E_v = \begin{Bmatrix} 0 \\ \alpha\left(\frac{4}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\} - \frac{2}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\}\right) \\ \alpha\left(\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\} + \left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\}\right) \\ \beta\gamma\left\lfloor\frac{\partial e}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\} + \alpha\left[u\left(\frac{4}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\} - \frac{2}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\}\right) + v\left(\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\} + \left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\}\right)\right] \end{Bmatrix}$$

$$F_v = \begin{Bmatrix} 0 \\ \alpha\left(\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\} + \left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\}\right) \\ \alpha\left(\frac{4}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\} - \frac{2}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\}\right) \\ \beta\gamma\left\lfloor\frac{\partial e}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\} + \alpha\left[u\left(\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\} + \left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\}\right) + v\left(\frac{4}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial y}\right\} - \frac{2}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\frac{\partial \vec{q}}{\partial x}\right\}\right)\right] \end{Bmatrix}$$

The sf_x, sf_y are easy to derive replace $\frac{\partial \vec{q}}{\partial x}$ with $\Delta \vec{q}\, n_x$ and $\frac{\partial \vec{q}}{\partial y}$ with $\Delta \vec{q}\, n_y$ in the above fluxes

$$sf_x = \begin{Bmatrix} 0 \\ \alpha\left(\frac{4}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_x\right\} - \frac{2}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_y\right\}\right) \\ \alpha\left(\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_x\right\} + \left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_y\right\}\right) \\ \beta\gamma\left\lfloor\frac{\partial e}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_x\right\} + \alpha\left[u\left(\frac{4}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_x\right\} - \frac{2}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_y\right\}\right) + v\left(\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_y\right\} + \left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_x\right\}\right)\right] \end{Bmatrix}$$

$$sf_y = \begin{Bmatrix} 0 \\ \alpha\left(\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_y\right\} + \left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_x\right\}\right) \\ \alpha\left(\frac{4}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_y\right\} - \frac{2}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_x\right\}\right) \\ \beta\gamma\left\lfloor\frac{\partial e}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_y\right\} + \alpha\left[u\left(\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_y\right\} + \left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_x\right\}\right) + v\left(\frac{4}{3}\left\lfloor\frac{\partial v}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_y\right\} - \frac{2}{3}\left\lfloor\frac{\partial u}{\partial \vec{q}}\right\rfloor\left\{\Delta \vec{q}\, n_x\right\}\right)\right] \end{Bmatrix}$$

In pratice will define variables $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial e}{\partial x}, \frac{\partial e}{\partial y}$ in the code.

Likewise for the symmetry flux we can do $dunx$, $duny$, $dvnx$, $dvny$, $denx$, $deny$ as analogous to their derivative counterparts

Using these definitions we get.

$$du = \left\lfloor \frac{\partial u}{\partial \vec{q}} \right\rfloor \{\Delta \vec{q}\} = -\frac{u}{\rho}\Delta\rho + \frac{\Delta\rho u}{\rho}$$

$$dv = \left\lfloor \frac{\partial v}{\partial \vec{q}} \right\rfloor \{\Delta \vec{q}\} = -\frac{v}{\rho}\Delta\rho + \frac{\Delta\rho v}{\rho}$$

$$de = \left\lfloor \frac{\partial e}{\partial \vec{q}} \right\rfloor \{\Delta \vec{q}\} = \left(-\frac{E}{\rho} + \frac{U^2}{\rho}\right)\Delta\rho - \frac{u}{\rho}\Delta\rho u - \frac{v}{\rho}\Delta\rho v + \frac{\Delta\rho E}{\rho}$$

Obviously $dun_x = (du)\cdot(n_x)$, and so on.

gives

$$Sf_x = \left\{ \begin{array}{l} 0 \\ \alpha\left(\frac{4}{3}dun_x - \frac{2}{3}dvn_y\right) \\ \alpha\left(dun_y + dvn_x\right) \\ \beta\gamma\, den_x + \alpha\left[U\left(\frac{4}{3}dun_x - \frac{2}{3}dvn_y\right) + V\left(dun_y + dvn_x\right)\right] \end{array} \right\}$$

$$Sf_y = \left\{ \begin{array}{l} 0 \\ \alpha\left(dun_y + dvn_x\right) \\ \alpha\left(\frac{4}{3}dvn_y - \frac{2}{3}dun_x\right) \\ \beta\gamma\, den_y + \alpha\left[U\left(dun_y + dvn_x\right) + V\left(\frac{4}{3}dvn_y - \frac{2}{3}dun_x\right)\right] \end{array} \right\}$$

This is so much easier and clearer than the way it was originally explained.

3) Penalty flux

Following R. Hartmann JCP '08 we will write a far more intuitive and efficient Penalty flux.

He writes for and edge that the penalty flux is given by

$$C_{ip}\frac{p^2}{h_e}\{\{[G(U_h)]\}\}[[U_h]]$$ where $C_{ip}\frac{p^2}{h_e}$ is our penalty term called ν thus we have

$$\nu\{\{[G(U_h)]\}\}[[U_h]]$$

Now the surface integral reads.

$$\int_{\Gamma_e} \nu\{\{[GU_h]\}\}[[U_h]] : [[\phi_a]]\, ds$$ where $[[\alpha]] = \alpha^L \vec{n} - \alpha^R \vec{n}$

$$\{\{\alpha\}\} = \frac{\alpha^L + \alpha^R}{2}$$

Again in comparison to the way we used to write this down. we can greatly simplify by expanding this term. For an interior edge.

$$\frac{1}{2}\nu\left(\begin{bmatrix} G_{11}^L+G_{11}^R & G_{12}^L+G_{12}^R \\ G_{21}^L+G_{21}^R & G_{22}^L+G_{22}^R \end{bmatrix}\begin{Bmatrix}\Delta q\, n_x \\ \Delta q\, n_y\end{Bmatrix}\right)\cdot\begin{Bmatrix}\phi_i n_x \\ \phi_i n_y\end{Bmatrix}$$

This can be expanded.

$$\frac{1}{2}\nu\left[(G_{11}^L+G_{11}^R)\Delta q n_x + (G_{12}^L+G_{12}^R)\Delta q n_y\right]\phi_i n_x + \frac{1}{2}\nu\left[(G_{21}^L+G_{21}^R)\Delta q n_x + (G_{22}^L+G_{22}^R)\Delta q n_y\right]$$

$\phi_i n_y$

It is immediatly apparent that this is just.

$$\left[\frac{1}{2}\left(E_v(q_i^L,\Delta q\vec{n})+E_v(q^R,\Delta q\vec{n})\right)n_x+\frac{1}{2}\left(F_v(q_i^L,\Delta q\vec{n})+F_v(q_i^R,\Delta q\vec{n})\right)n_y\right]\cdot\phi_i$$

Exactly analogous to the viscous flux by with $\vec{\nabla} q$ replaced by $\Delta q\vec{n}$ which is the same L and R.

For computational efficiency this will be fully expanded.

Recall our du, dv, de terms. from the Symmetry flux, we will define exactly the same here but

$$du^L = \left\lfloor \frac{\partial u^L}{\partial q_L}\right\rfloor\{\Delta q\}, \quad du^R = \left\lfloor\frac{\partial u^R}{\partial q^R}\right\rfloor\{\Delta q\} \quad \ldots \text{ and so on.}$$

In fact for the implementation of the Symmetry flux we'll need $du^L, du^R, dv^L, dv^R, de^R, de^R$ to get

Sf_x^L
Sf_y^L
Sf_x^R
Sf_y^R

because we are going to compute the whole flux in 1 routine to minimize non-linear operations.

The penalty flux is given component wise as.

$$pflux(1) = 0$$

$$pflux(2) = \frac{1}{2}\nu\Big[\Big(\alpha^L\big(\tfrac{4}{3}du^L n_x - \tfrac{2}{3}dv^L n_y\big) + \alpha^R\big(\tfrac{4}{3}du^R n_x - \tfrac{2}{3}dv^R n_y\big)\Big)n_x +$$
$$\Big(\alpha^L\big(du^L n_y + dv^L n_x\big) + \alpha^R\big(du^R n_y + dv^R n_x\big)\Big)n_y\Big]$$

$$pflux(3) = \frac{1}{2}\nu\Big[\Big(\alpha^L\big(du^L n_y + dv^L n_x\big) + \alpha^R\big(du^R n_y + dv^R n_x\big)\Big)n_x +$$
$$\Big(\alpha^L\big(\tfrac{4}{3}dv^L n_y - \tfrac{2}{3}du^L n_x\big) + \alpha^R\big(\tfrac{4}{3}dv^R n_y - \tfrac{2}{3}du^R n_x\big)\Big)n_y\Big]$$

$$pflux(4) = \frac{1}{2}\nu\Big[\Big\{\beta^L\gamma\, de^L n_x + \alpha^L\big[u^L\big(\tfrac{4}{3}du^L n_x - \tfrac{2}{3}dv^L n_y\big) + v^L\big(du^L n_y + dv^L n_x\big)\big] +$$
$$\beta^R\gamma\, de^R n_y + \alpha^R\big[u^R\big(\tfrac{4}{3}du^R n_x - \tfrac{2}{3}dv^R n_y\big) + v^R\big(du^R n_y + dv^R n_x\big)\big]\Big\}n_x +$$
$$\Big\{\beta^L\gamma\, de^L n_y + \alpha^L\big[u^L\big(du^L n_y + dv^L n_x\big) + v^L\big(\tfrac{4}{3}dv^L n_y - \tfrac{2}{3}du^L n_x\big)$$
$$\beta^R\gamma\, de^R n_y + \alpha^R\big(u^R\big(du^R n_y + dv^R n_x\big) + v^R\big(\tfrac{4}{3}dv^R n_y - \tfrac{2}{3}du^R n_x\big)\Big\}n_y\Big]$$

This is really just

$\frac{\nu}{2}(Sf_x^L + Sf_x^R)n_x + \frac{\nu}{2}(Sf_y^L + Sf_y^R)n_y = pflux$, implement like this

SA Symmetry flux Implementation: New version should be more efficient.

For the Symmetry flux of the S.A. Model the G_{ij} matrices are quite Simple

$$G_{11} = \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}(\mu+\rho\tilde{\nu})\left\lfloor\frac{\partial\tilde{\nu}}{\partial\vec{q}}\right\rfloor$$

$$G_{12} = 0$$

$$G_{21} = 0$$

$$G_{22} = \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}(\mu+\rho\tilde{\nu})\left\lfloor\frac{\partial\tilde{\nu}}{\partial\vec{q}}\right\rfloor$$

Gives the symmetry flux as.

$$sflux = \frac{1}{2}\frac{M_\infty}{Re_\infty}\frac{1}{\sigma}(\mu+\rho\tilde{\nu})\left\lfloor\frac{\partial\tilde{\nu}}{\partial\vec{q}}\right\rfloor\left(\frac{\partial\phi_i}{\partial x}n_x + \frac{\partial\phi_i}{\partial y}n_y\right)\{\Delta\vec{q}\} = \frac{1}{2}\left(\frac{M_\infty}{Re_\infty}\frac{1}{\sigma}(\mu+\rho\tilde{\nu})n_x\{\Delta q\}\frac{\partial\phi_i}{\partial x} + \text{" "}n_y\{\Delta q\}\frac{\partial\phi_i}{\partial y}\right)$$

Thus in an exactly analogous way to how we did this for the N/S fluxes we define.

$$Sflux = \frac{1}{2}\left(Sf_x\frac{\partial\phi_i}{\partial x} + Sf_y\frac{\partial\phi_i}{\partial y}\right) \quad \text{gives}$$

$$Sf_x = \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}(\mu+\rho\tilde{\nu})\left\lfloor\frac{\partial\tilde{\nu}}{\partial\vec{q}}\right\rfloor n_x\{\Delta q\}$$

$$Sf_y = \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}(\mu+\rho\tilde{\nu})\left\lfloor\frac{\partial\tilde{\nu}}{\partial\vec{q}}\right\rfloor n_y\{\Delta q\}$$

with $\frac{\partial\tilde{\nu}}{\partial\vec{q}} = \left\lfloor -\frac{\tilde{\nu}}{\rho}, 0, 0, 0, \frac{1}{\rho}\right\rfloor$ gives.

$$\boxed{\begin{aligned} Sf_x &= \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}(\mu+\rho\tilde{\nu})n_x\left(-\frac{\tilde{\nu}}{\rho}\Delta\rho + \frac{\Delta\rho\tilde{\nu}}{\rho}\right) \\ Sf_y &= \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}(\mu+\rho\tilde{\nu})n_y\left(-\frac{\tilde{\nu}}{\rho}\Delta\rho + \frac{\Delta\rho\tilde{\nu}}{\rho}\right)\end{aligned}}$$

SA Penalty flux just like the RANS terms is

$$\left(\frac{1}{2}\gamma(G_{11}^L + G_{11}^R)\{\Delta q n_x\}n_x + \frac{\gamma}{2}(G_{22}^L + G_{22}^R)\{\Delta q n_y\}n_y\right)\phi_i$$

Thus the flux part is just.

$$\tfrac{1}{2}(G_{11}^L + G_{11}^R)(\Delta q n_x)\, n_x + \tfrac{1}{2}(G_{22}^L + G_{22}^R)(\Delta q n_y)\, n_y$$

with $G_{11}^L = \dfrac{M_\infty}{Re_\infty}\dfrac{1}{\sigma}\left(\mu^L + (\rho\tilde{\nu})^L\right)\left[\dfrac{\partial\tilde{\nu}^L}{\partial q^L}\right]$

$$G_{11}^R = \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}\left(\mu^R + (\rho\tilde{\nu})^R\right)\left[\frac{\partial\tilde{\nu}}{\partial q^R}\right]$$

$$G_{22}^L = \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}\left(\mu^L + (\rho\tilde{\nu})^L\right)\left[\frac{\partial\tilde{\nu}^L}{\partial q^L}\right]$$

$$G_{22}^R = \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}\left(\mu^R + (\rho\tilde{\nu})^R\right)\left[\frac{\partial\tilde{\nu}^R}{\partial q^R}\right]$$

$$\frac{\partial\tilde{\nu}}{\partial q} = \left[-\frac{\tilde{\nu}}{\rho}, 0, 0, 0, \frac{1}{\rho}\right]$$

$$\text{pflux} = \tfrac{1}{2}\nu\left(\frac{M_\infty}{Re_\infty}\frac{1}{\sigma}\left(\mu^L + \rho\tilde{\nu}\right)^L n_x\left(-\frac{\tilde{\nu}^L}{\rho^L}\Delta\rho + \frac{\Delta\rho\tilde{\nu}}{\rho^L}\right) + \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}\left(\mu^R + (\rho\tilde{\nu})^R\right) n_x\left(-\frac{\tilde{\nu}^R}{\rho^R}\Delta\rho + \frac{\Delta\rho\tilde{\nu}}{\rho^R}\right)\right)$$
$$\cdot n_x + \tfrac{1}{2}\nu\left(\frac{M_\infty}{Re}\frac{1}{\sigma}\left(\mu^L + \rho\tilde{\nu}^L\right) n_y\left(-\frac{\tilde{\nu}^L}{\rho^L}\Delta\rho + \frac{\Delta\rho\tilde{\nu}}{\rho^L}\right) + \frac{M_\infty}{Re_\infty}\frac{1}{\sigma}\left(\mu^R + (\rho\tilde{\nu})^R\right) n_y\left(-\frac{\tilde{\nu}^R}{\rho^R}\Delta\rho + \frac{\Delta\rho\tilde{\nu}}{\rho^R}\right)\right)$$
$$\cdot n_y$$

Which as before is just.

$$\text{pflux} = \frac{\nu}{2}\left(Sf_x^L + Sf_x^R\right) n_x + \frac{\nu}{2}\left(Sf_y^L + Sf_y^R\right) n_y$$

IRK Schemes for CFD:

For CFD FIRK is not really feasible, the linear system is too large. However ESDIRK can provide benefit. Thus these notes will be confined to that subject.

General R-K Form.

$$\frac{dy}{dt} = f(y,t)$$

$$y^{n+1} = y^n + \Delta t \sum_{j=1}^{s} b_i \; f(t^n + c_i \Delta t, y_i)$$

$$y^{i} = y^n + \Delta t \sum_{j=1}^{s} a_{ij} \; f(t^n + c_i \Delta t \, y_j) \quad i = 1, 2, 3, \ldots s$$

Form of CFD.

$$[M] \frac{\partial \hat{q}}{\partial t} + R(\hat{q}) = 0 \qquad \text{analogous to} \qquad \frac{dy}{dt} - f(y) = 0$$

$$\hat{q}^i_i = \hat{q}^n - [M]^{-1} \Delta t \sum_{j=1}^{s} a_{ij} R(\hat{q}^j) \rightarrow [M]\frac{(\hat{q}^i - \hat{q}^n)}{\Delta t} + \sum_{j=1}^{s} a_{ij} R(\hat{q}^j)$$

$$\hat{q}^{n+1} = \hat{q}^n - [M]^{-1} \Delta t \sum_{j=1}^{s} b_j R(\hat{q}^j) \rightarrow \frac{[M](\hat{q}^{n+1} - \hat{q}^n)}{\Delta t} + \sum_{j=1}^{s} b_j R(\hat{q}^j)$$

There a 2 specific schemes that I will use. I'll write the 4 stage 3rd order one here, the 4th order is similar ~~???~~ just a bit larger.

1). ESDIRK 3 ; 4-stage 3rd order.

Stage 1:
$$\hat{q}^1 = q^n$$

Stage 2:
$$[M]\frac{(\hat{q}^2 - \hat{q}^n)}{\Delta t} + a_{21} R(\hat{q}^1) + a_{22} R(\hat{q}^2) = 0$$

Stage 3:
$$[M]\frac{(\hat{q}^3 - \hat{q}^n)}{\Delta t} + a_{31} R(\hat{q}^1) + a_{32} R(\hat{q}^2) + a_{33} R(\hat{q}^3) = 0$$

Stage 4:
$$[M]\frac{(\hat{q}^4 - \hat{q}^n)}{\Delta t} + a_{41} R(\hat{q}^1) + a_{42} R(\hat{q}^2) + a_{43} R(\hat{q}^3) + a_{44} R(\hat{q}^4) = 0$$

$$\hat{q}^{n+1} = \hat{q}^n - \Delta t \, [M]^{-1} \left[b_1 R(\hat{q}^1) + b_2 R(\hat{q}^2) + b_3 R(\hat{q}^3) + b_4 R(\hat{q}^4) \right]$$

This is implemented by storing the residuals in q^n as

$$qn(:,1) = q^n \quad , \quad qn(:, 2:s) = R(\hat{q}^{1:s-1})$$

We always want to solve the implicit problem as

$$\left[\alpha \frac{[M]}{\Delta t} \hat{q}^k - \beta \frac{\partial R}{\partial \hat{q}} \right] \Delta \hat{q}^k = - R_t$$

where. $R_t = \beta R(\hat{q}^k) + S$

Thus we'll need to form the S, α, β for each stage.

Stage 2: Implicit problem form.

$$R_t = [M] \frac{(\hat{q}^2 - \hat{q}^n)}{\Delta t} + a_{21} R(\hat{q}^1) + a_{22} R(\hat{q}^2)$$

$$S_2 = a_{21} R(\hat{q}^1) - \frac{[M]}{\Delta t} \hat{q}^n$$

$$\frac{[M]}{\Delta t} \hat{q}^2 + a_{22} R(\hat{q}^2) + S_2 \qquad \alpha = 1, \quad \beta = a_{22}$$

Stage 3:

$$R_t = [M] \frac{(\hat{q}^3 - \hat{q}^n)}{\Delta t} + a_{31} R(\hat{q}^1) + a_{32} R(\hat{q}^2) + a_{33} R(\hat{q}^3)$$

$$S_3 = a_{31} R(\hat{q}^1) + a_{32} R(\hat{q}^2) - \frac{[M]}{\Delta t} (\hat{q}^n)$$

$$\frac{[M]}{\Delta t} \hat{q}^3 + a_{33} R(\hat{q}^3) + S_3 \qquad \alpha = 1, \quad \beta = a_{33}$$

Stage 4:

$$R_t = [M] \frac{(\hat{q}^4 - \hat{q}^n)}{\Delta t} + a_{41} R(\hat{q}^1) + a_{42} R(\hat{q}^2) + a_{43} R(\hat{q}^3) + a_{44} R(\hat{q}^4)$$

$$S_4 = a_{41} R(\hat{q}^1) + a_{42} R(\hat{q}^2) + a_{43} R(\hat{q}^3) - \frac{[M]}{\Delta t} \hat{q}^n$$

$$\frac{[M]}{\Delta t} \hat{q}^4 + a_{44} R(\hat{q}^4) \qquad \alpha = 1, \quad \beta = a_{44}$$

$$[M] q^{n+1} = [M] q^n - \Delta t \left[b_1 R(\hat{q}^1) + b_2 R(\hat{q}^2) + b_3 R(\hat{q}^3) + b_4 R(\hat{q}^4) \right)$$

A Appendix

c_i	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$
0	0	0	0	0
$\frac{1767732205903}{2027836641118}$	$\frac{1767732205903}{4055673282236}$	$\frac{1767732205903}{4055673282236}$	0	0
0.6	$\frac{2746238789719}{10658868560708}$	$\frac{-640167445237}{6845629431997}$	$\frac{1767732205903}{4055673282236}$	0
1	$\frac{1471266399579}{7840856788654}$	$\frac{-4482444167858}{7529755066697}$	$\frac{11266239266428}{11593286722821}$	$\frac{1767732205903}{4055673282236}$
b_i	$\frac{1471266399579}{7840856788654}$	$\frac{-4482444167858}{7529755066697}$	$\frac{11266239266428}{11593286722821}$	$\frac{1767732205903}{4055673282236}$

Table A.1
Butcher Tableau for four-stage third-order accurate diagonally implicit Runge-Kutta scheme (DIRK3)

c_i	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$	$a_{i,5}$	$a_{i,6}$
0	0	0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0
$\frac{83}{250}$	$\frac{8611}{62500}$	$\frac{-1743}{31250}$	$\frac{1}{4}$	0	0	0
$\frac{31}{50}$	$\frac{5012029}{34652500}$	$\frac{-654441}{2922500}$	$\frac{174375}{388108}$	$\frac{1}{4}$	0	0
$\frac{17}{20}$	$\frac{15267082809}{155376265600}$	$\frac{-71443401}{120774400}$	$\frac{730878875}{902184768}$	$\frac{2285395}{8070912}$	$\frac{1}{4}$	0
1	$\frac{82889}{524892}$	0	$\frac{15625}{83664}$	$\frac{69875}{102672}$	$\frac{-2260}{8211}$	$\frac{1}{4}$
b_i	$\frac{82889}{524892}$	0	$\frac{15625}{83664}$	$\frac{69875}{102672}$	$\frac{-2260}{8211}$	$\frac{1}{4}$

Table A.2
Butcher Tableau for six-stage fourth-order accurate diagonally implicit Runge-Kutta scheme (DIRK4)

Finite Volume N/C Diffusion term for SA.

The SA diffusion term is normally written as

$$\frac{1}{\sigma}\left[\nabla[(\mu+\rho\tilde{\nu})\nabla\tilde{\nu}] + \rho C_{b2}(\nabla\tilde{\nu})^2\right]$$

Employing the identity $(\nabla\tilde{\nu})^2 = \nabla(\tilde{\nu}\nabla\tilde{\nu}) - \tilde{\nu}\nabla^2\tilde{\nu}$ gives

$$\frac{1}{\sigma}\left[\nabla[(\mu+\rho\tilde{\nu})\nabla\tilde{\nu}] + C_{b2}\nabla(\rho\tilde{\nu}\nabla\tilde{\nu}) - \rho C_{b2}\tilde{\nu}\nabla^2\tilde{\nu}\right]$$

Simplification gives

$$\frac{1}{\sigma}\left[\nabla[(\mu+\rho\tilde{\nu}+C_{b2}\rho\tilde{\nu})\nabla\tilde{\nu}] - \rho\tilde{\nu}C_{b2}\nabla^2\tilde{\nu}\right]$$

To discretize hold $\rho\tilde{\nu}C_{b2} = \text{const}$ in $\rho\tilde{\nu}C_{b2}\nabla^2\tilde{\nu}$

$$\frac{1}{\sigma}\int_{\Omega_K}\nabla\left[\underline{\frac{(\mu+\rho\tilde{\nu}+C_{b2}\rho\tilde{\nu})}{\sigma}}\nabla\tilde{\nu}\right] - \underline{\frac{\rho\tilde{\nu}C_{b2}}{\sigma}}\nabla^2\tilde{\nu}\,]\,d\Omega \quad \text{gives}$$

$$\int_{\partial\Omega_K}\underline{\frac{(\mu+\rho\tilde{\nu}(1+C_{b2})}{\sigma}}_{face}\nabla\tilde{\nu}\cdot\vec{n} - (\rho\tilde{\nu})^K C_{b2}\nabla\tilde{\nu}\cdot\vec{n}\,)\,ds$$

Thus in edge/Face for

$$\text{Left} = \int_{\Gamma_i}\frac{M_\infty}{Re}\left\{\underline{\frac{\mu+\rho\tilde{\nu}(1+C_{b2})}{\sigma}}\right\}\nabla\tilde{\nu}\cdot\vec{n} - \frac{M_\infty}{Re}(\rho\tilde{\nu})^L\frac{C_{b2}}{\sigma}\nabla\tilde{\nu}\cdot\vec{n}\,ds$$

$$\text{Righ} = \int_{\Gamma_i}\frac{M_\infty}{Re}\left\{\underline{\frac{\mu+\rho\tilde{\nu}(1+C_{b2})}{\sigma}}\right\}\nabla\tilde{\nu}\cdot\vec{n} - \frac{M_\infty}{Re}(\rho\tilde{\nu})^R\frac{C_{b2}}{\sigma}\nabla\tilde{\nu}\cdot\vec{n}\,ds$$

$\{\ \} = \frac{1}{2}(\ )^L + (\ )^R$ - the average.