= DG - HYBRID ELEMENTS =

INGREDIENTS.

1) New DATA structure

1.1. Allow various types of elements

1.2. Allow different "p" order for fields.

1.3. Continuous + random access.

1.4. Optimizer for MATMUL or VEC-VEC operations.

1.5. Allow different "p" for geometry.

2.

2) New/additional HBasis.

2.1 Use L2-basis. (p=0,1..) hierarchical.

2.2 Use H1-basis. (p=1,2..) hierarchical (old)

3) Solvers.

- 9NEJ → only [D]
- LEJ/LGS → [D]+[O]
- GS using only [D] → cell based solver:
  - each cell is computed, solved → move to next cell

4) CUDA GPU

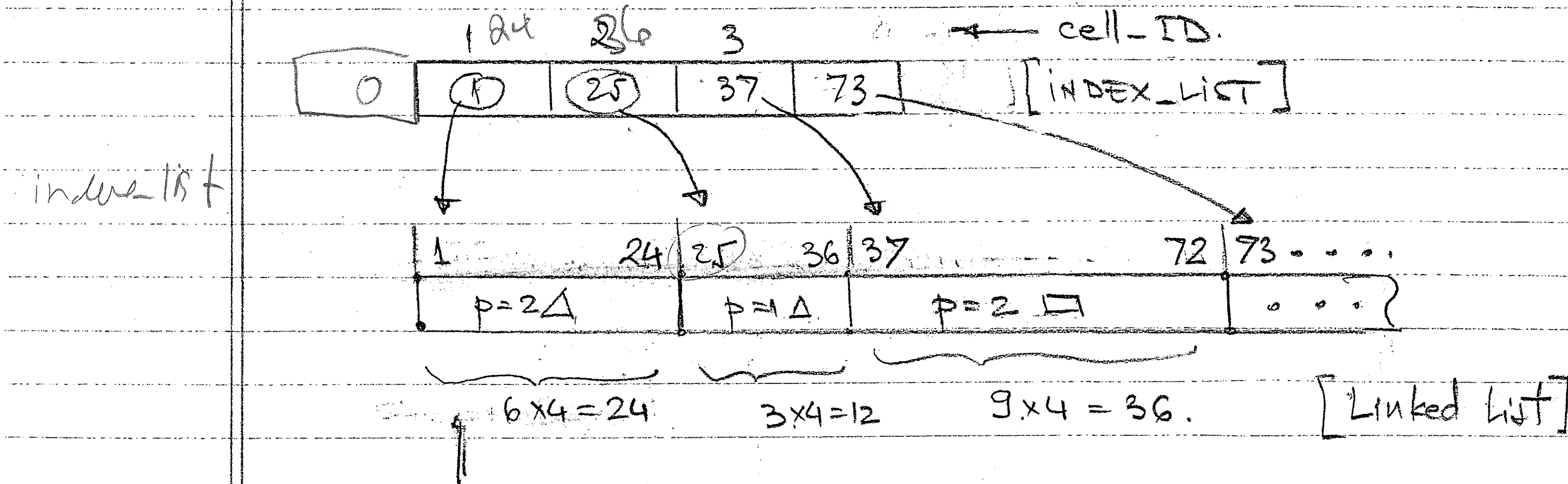
- study if feasible in 3D

5) GCL

1) NEW DATA STRUCTURE

Linked-list with random acces.

a) Residual (R_p), state vector (U_p), Update (Δu)



each block size = $\underline{NDOF}$ = NMODE × NFLDS.

- [INDEX LIST] is SAME for R_p, U_p, ΔU
- [LINKED LIST] → different for R_p, U_p, ΔU.

- Each block $\boxed{NDOF = NMODE \times NFLDS}$ — count

  used as. • [NMODE, NFLDS]

  or • [NFLDS, NMODE] 2D arrays.

  The choise is dictated by the way the Residual Jacobian is represented. → see next.

- We also need either

  - store polynomial order
  - store # of modes.

  } in each cell (or both)

p=2	p=1	p=2	p=3	. . . .

$$NMODE = \frac{(p+1)(p+2)}{2} \quad ; \quad \text{2D TRI} \quad (\text{type}=2)$$

$$NMODE = (p^a+1)(p^b+1) \quad ; \quad \text{2D QUAD} \quad (\text{type}=3)$$

⇒ We have 2 cases.:

(i) If store "p"/cell we also need. to store the cell type.

p^1	p^2	p^3	p^4 .	- - -
type[1]	type[2]	type[3]	type[4] -	- .
1	2	3	4	5

← cell_iD.

prefer this more costly

∴ To get NMODE just use forumula!

(ii) Just store NMODE/cell

NMODE	NMODE	NMODE	. . . .
1	2	3	4 - - -

← cellID.

which one is best?

(i) is best because:

- we will need the cell_type latter for edges.
- one can change the basis fn → use a different forumula./basis_type @ each cell_type.

Not sure here!! - - -

b) Residual Jacobian: [D], [O_L], [O_R]

In the current version:

$[D] = [NDOF, NDOF, NCELL]$ as 3DIM array.

$[O]_L$ & $[O_R]$ same representation since for visc. flows, we need all MODES within the cell

However, it has the assumption that.

- L, R cells have same order → same NDOF
- L, R cells have same topology → square matrices $[NDOF, NDOF]$

On hybrid cell / variable "p" these assumptions are no longer true. This implies that:

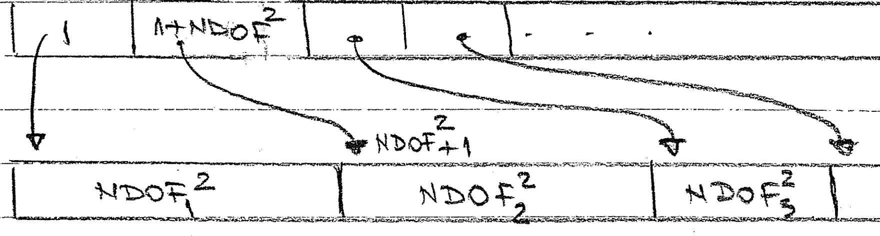
- L cell : $[NDOF_L]^2$
- R cell : $[NDOF_R]^2$ $\Longrightarrow$ $[D]_L \neq$ size $[D]_R$

- $[O]_L = [NDOF_L \times NDOF_R]$
- $[O]_R = [NDOF_R \times NDOF_L]$ } NOT SQUARE !

⟹ Need a linke list like before:

INDEX LIST (contains iDOF)

Lowest Leve Routines
A(NDOF, NDOF)



~~Cool~~

Routine (A(iptr))

LINKED LIST contains the [D] entries.

For [D] is easy to convert from 1D → 2D array. since the matrix is SQUARE and.

~~NDOF = NMODE × NFLDS.~~
NFDS = const.
NMODE = f (p, cell_type) → use formula.

~~BETTER IDEA~~ !! Some info is redundant in both Residual R(u) and Jacobian ∂R/∂u lists.

- if we store a list of [p-order] and [cell_type] then we can construct the INDEX_LIST on the fly.
- ~~if we store [NMODE] list and [cell_type] list again we can construct INDEX_LIST for both. R(u) & ∂R/∂u on the fly.~~

Ex: store [p-order] & [cell_type].

- for each element (icell)
  - NMODE = f(p, cell_type).
  - NFLDS = const
  - NDOF = NMODE × NFLDS.
  - we can reference the start & end entry in R(u), u, Δu list as: istart & iend,

! see list figure.!

iend = 0. (init)

icell	NDOF	istart	iend
1	24	1	24
2	12	25	36
3	36.	37	72
4.		73	

Loop [istart = iend + 1
iend = istart + NDOF − 1

recurrence formula → NOT ok!

No don't use this store IS, IE.

Similarly, we can get a LINKE_LIST for $[X_p]$.
In the case of coordinates $NFLD = NDIM$

$\Rightarrow$ $NDOF = NMODE \times NDIM$. where $NMODE \neq NMODE$ for fields.

$\Rightarrow$ Need a list to store. geo_Order / cell

p-geo	p-geo	p-geo.	. . .

Then
- $NMODE = f(\text{p-geo}, \text{cell\_type})$.
- $NDOF_{geo} = NMODE_{geo} \times NDIMS$.
- construct INDEX_LIST
- allocate. $[X_p]$ 1D vector.

Edge based $[O]_L$ & $[O]_R$.

As we mentioned the general case, for every EDGE,

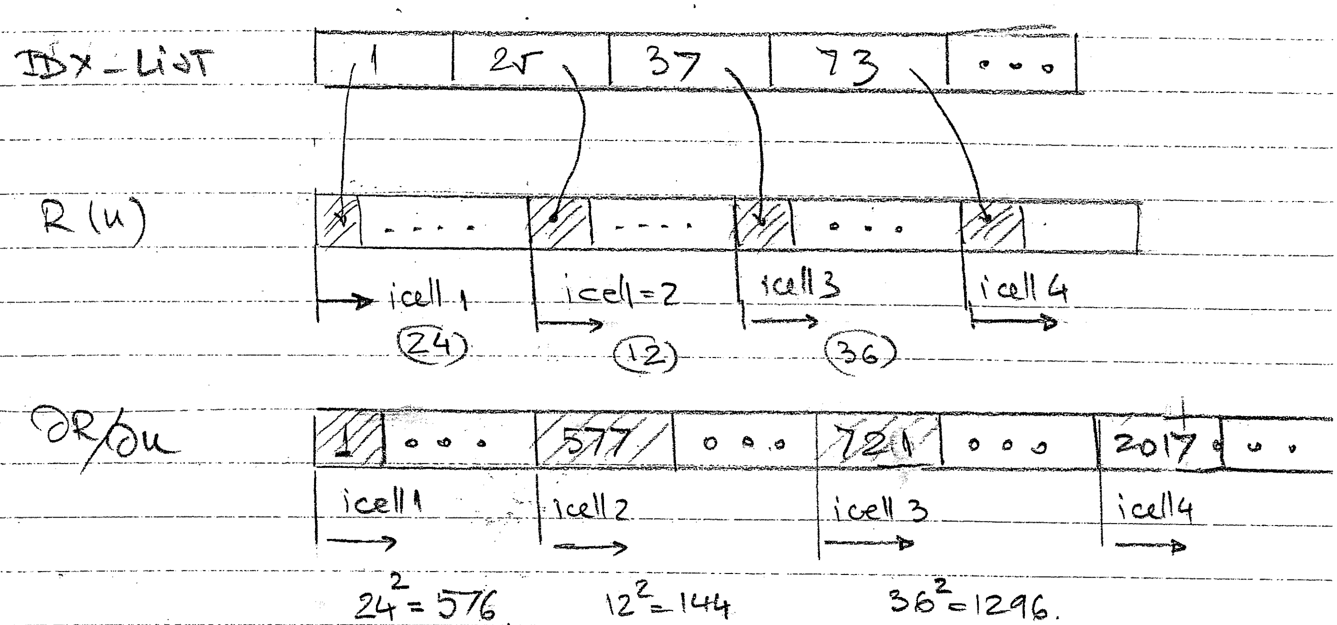
- $[O]_L = [NDOF_L \times NDOF_R]$
- $[O]_R = [NDOF_R \times NDOF_L]$

$\Rightarrow$ each require. an individual INDEX_LIST.

⟹ NOT a good idea since the list entries (istart & iend) cannot be accessed randomly!

Use this algorithm to generate the INDEX_LIST for $R(u)$, u, Δu.

- How about use this INDE_LIST to create an index_list for [D].



= Again RECURENCE formula NOT OK!:

```
iend = 0
loop [ istart = iend + 1
     [ iend = istart - 1 + NDOI
```

CONCLUSION:

- need 1 INDEX_LIST for $R(u)$, u, Δu, $S(u)$ ....
- need 1 INDEX_LIST for [D].
- need 1 index_list for $[O]_L$.
- need 1 index_LIST for $[O]_R$.
- need 1 index_LIST for $[X_p]$ (nodal coord.)

I think this is complete

Other variables

- Metrics : $\partial x/\partial \xi$, $|\partial x/\partial \xi|$, n_x, n_y ~~etc.~~

  Jac.

2 choices.

a) compute on the fly. for each step, each cell/edge for each Q-pts.

b) store them in 2 lists :

- list 1 : straight elements $\rightarrow$ const values.
- list 2 : curved elements $\rightarrow$ values at Qpts.

For now, let's use (a) since we don't know the 3D implementation for curved surfaces.

- MASS MATRIX :

$$M = \int_K \phi_i \phi_j \, d\Omega_K = [NMODE \times NMODE] \times NQPTS.$$

- We need a better idea !! for curved elements.
  - compute on the fly

OPERATIONS IN "ξ" transformed space.

- All quantities must be calculated & stored:
  - $\phi_i \rightarrow$ quadrature points. (cell)
  - $d\phi_i \rightarrow$ — " — (cell)
  - $\phi_i \rightarrow$ — " — (edge)
  - $d\phi_i \rightarrow$ — " — (edge)

We also have to consider. p-multigrid. → need to pre-compute them @ qpts for each level.

$\Rightarrow \quad \phi_i = [NNODE, NQPS, NLEVS] \times$ cell_type.

Need a better data structure! → idea? - linked list works here, but we don't waste much memory on these

OPERATIONS.

- Interpolation : $\sum \phi_i \hat{u}_i = [\hat{u}][\phi]^T = [\phi][\hat{u}]^T$

- $\frac{\partial R}{\partial u} \cdot \Delta u \rightarrow$ MATMUL or just DOT_PRODUCT
  Need to construct the operator using

Interpolation:

- $[u]$ = 1D array (linked list) one can recast $[\phi]$ vector so a simple DOT_PRODUCT can be used.

- recast $[u]$ as $[NFLDS \times NMODE]_{2D}$ and use.

  $MATMUL(u, \phi)_{icell}$

$\partial R / \partial u \cdot \Delta u$.

- recast/reshape $\partial R/\partial u$ as $[NDOF, NDOF]_{2D}$ and use

  $MATMUL(\partial R/\partial u, u)_{icell}$

- multiple DOT_PRODUCTS. (lines • colums)

  => need to extract "rows" from $\partial R/\partial u$ which is 1D vector $[NDOF]^2$.

Therefore, the latter, dictates the choice of ordering data within each block of $R(u)$, $\partial R/\partial u$. The "block" is the continuous entries in $R, \partial R/\partial u$ vectors corresponding to a given cell.

Next, let's detail →

$100 \times 100 = 10{,}000$

Discontinous Galerkin Methods: A Basic Intro, w/out Flux FVM theory.

Some mention of function spaces

we have 3 important ones H^1, $\vec{H}(curl)$, $\vec{H}(div)$.

Let Ω be a domain in the d-dimensional real space denoted as $\mathbb{R}^d$

The scalar Hilbert Space is called H^1

$$H^1 = \{ v \in L^2(\Omega) ; \frac{\partial v}{\partial x_i} \in L^2(\Omega), 1 \le i \le d \}$$ where

v is a scalar valued fn.

We also have 2 vector Hilbert spaces.

$$\vec{H}(curl) = \{ \vec{v} \in [L^2(\Omega)]^d ; \ curl\, \vec{v} \in [L^2(\Omega)]^d \}$$

$$\vec{H}(div) = \{ \vec{v} \in [L^2(\Omega)]^d ; \ div(\vec{v}) \in L^2(\Omega) \}$$

Recall that a Hilbert Space is a space in which every sequence of vectors (Functions for us) is convergent to a vector (Function) that is also in the space. This simply ensures that our answer stays in the space of our basis f'ns.

Our 3 spaces above are such spaces for scalar and vector valued f'ns v, $\vec{v}$.

Unisolvency: The finite element which is comprised of

$\mathbb{K}(K, P, \Sigma)$

K - domain - like triangle

P - polynomial space on K

$\dim(P) = N_P$

Σ - set of linear forms

$L_i : P \to \mathbb{R}$, $i = 1, 2, \dots N_P$

basically L is evaluation points for polynomials in P so that we get values in $\mathbb{R}$. ②

The finite element is said to be unisolvent if and only if there is a unique basis $B = \{\theta_1, \dots, \theta_{N_p}\}$ contained in $(\subset)$ P satisfying $L_i(\theta_j) = \delta_{ij}$ – kronecker δ – called δ property

<u>Proof:</u> we seek a fn θ_j using basis $\{g_1 \dots g_{N_p}\}$ $\subset P$.

$$\theta_j = \sum_{k=1}^{N_p} a_{kj} g_k \Rightarrow$$

δ - property is $L_i(\theta_j) = \delta_{ij}$

$$L_i(\theta_j) = L_i\left(\sum_{k=1}^{N_p} a_{kj} g_k\right) = \sum_{k=1}^{N_p} a_{kj} L_i(g_k) = \delta_{ij}$$

this gives a Matrix of equation

$$[L][A] = [I]$$

$$A = \begin{array}{c|c} & \xrightarrow{\;j\;} \\ k \downarrow & \begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & \ddots & \\ & & \ddots \end{bmatrix} \end{array} \qquad L = \begin{bmatrix} L_1(g_1) & L_1(g_2) & \cdots \\ L_2(g_1) & & \\ \vdots & & \ddots \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$ Identity. The Matrices A, L are seen by perhaps writing out a simple example for $N_p = 2$ or 3.

If we assume that the coluns of L are linearly dependant they there is at least 1 set of coefficents ③

$\alpha_1 \ldots \alpha_{N_p}$ such that

$$\sum_{k=1}^{N_p} \alpha_K L_i(g_K) = L_i\left(\sum_{k=1}^{N_p} \alpha_K g_K\right) = 0 \quad \text{for } i = 1, 2, \ldots, N_p$$

$\sum \alpha_K g_K$ is non-trivial. there should be

$L_i(\alpha_K g_K) = 1$ for $i=k$, the above says there is not. Thus the above is not unisolvent.

The the colums must be linearly independant providing a unique set of coefficients for each

L_i

θ_j is uniquely identify by

$$A = L^{-1} I.$$

Is θ_j linearly independant

$$\sum_{j=1}^{N_p} \beta_j \theta_j = 0 \Rightarrow \beta_i = 0$$

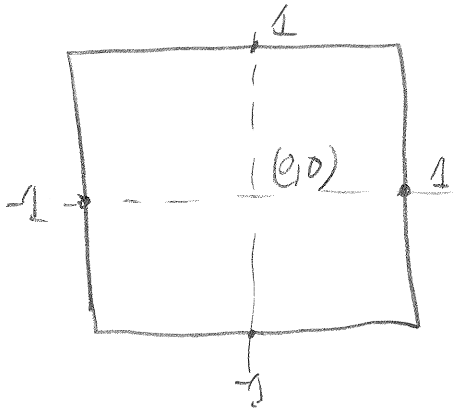
since $L_i(\theta_j) = \delta_{ij}$ then

$$L_i\left(\sum_{j=1}^{N_p} \beta_j \theta_j\right) = 0 \text{ for all } i \text{ if } \beta_i = 0 \text{ bc}$$

$$0 = L_i\left(\sum_{j=1}^{N_p} \beta_j \theta_j\right) = \sum_{j=1}^{N_p} \beta_j L_i(\theta_j) = \sum_{j=1}^{N_p} \beta_j \delta_{ij} = 0$$

$\beta_j \delta_{ij} = 0$ for all $i \neq j$ but if $i=j$ then $\delta_{ij} = 1$ thus $\beta_i = 0$ for that i. This holds for any i since $i = 1, 2, \ldots N_p$ that it holds for all i, which means all β's are 0 givins θ is linearly independent.

Example of checking unisolvency:



$K = (-1, 1)$ in x, $(-1, 1)$ in y

$P = \text{span}\{1, x_1, x_2, x_1 \cdot x_2\}$

$L = \{[-1,0], [1,0], [0,-1], [0,1]\}$

just points $\quad L : P \to \mathbb{R}^2$

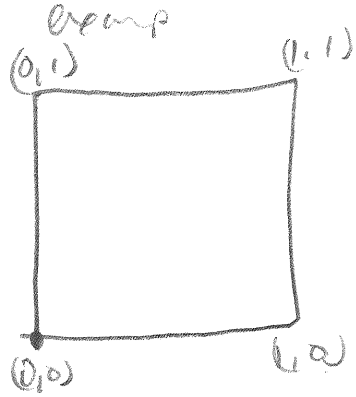
$L_1(g) = g(-1, 0)$ etc.

$$\begin{array}{c} i=1 \\ i=2 \\ \\ \\ \end{array} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

if L is singular then it's not unisolvent.

$\text{Det}(L) = 0$, check with Matlab.

The Elements we consider a unisolvent, take for example



gives

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$\text{Det}(L) = 1$

it's invertible

and $A = L^{-1}[I]$

so it's unisolvent.

Basis Functions:

Before we do anything we need to figure out a heirechal way to implement of Lobatto shape functions

Their basic definition is

$\ell_0(x) = \frac{1-x}{2}$, $l_1(x) = \frac{x+1}{2}$ these are 1-D vertex functions

the 1-D "Bubble" fns (these become edge fns in 2D)

are given as.

$$\ell_K(x) = \frac{1}{\|L_{K-1}\|_2} \int_{-1}^{x} L_{K-1}(\xi)\,d\xi, \quad 2 \le K \quad \text{where}$$

L_K - Legendre polynomials. which are just Jacobi polynomials of $P^{\alpha,\beta}$ where $\alpha, \beta = 0$.

Some basic Identities and definitions:

Recurrence Relation for Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\right]$$

If $n=0$ we set

$$P_0^{(\alpha,\beta)} = \frac{(-1)^0}{2^0 0!}\left[\cancel{(1-x)^{-\alpha}}\cancel{(1+x)^{-\beta}}\right]\left[\cancel{(1-x)^{\alpha}}\cancel{(1+x)^{\beta}}\right] = 1$$

$$P_1 = \frac{-1}{2 \cdot 1}\cancel{(1-x)^{-\alpha}}\cancel{(1+x)^{-\beta}}\left[\frac{(1-x)^{\cancel{\alpha+1}}(\alpha+1)(1+x)^{\cancel{\beta+1}}}{\cancel{1-x}} + \frac{(1-x)^{\cancel{\alpha+1}}(1+x)^{\cancel{\beta+1}}(\beta+1)}{\cancel{(1+x)}}\right]$$

$$= -\frac{1}{2}\left[+(\alpha+1)(1+x) - (1-x)(\beta+1)\right]$$

$$P_1 = \frac{1}{2}\left[\alpha + \alpha x + \cancel{1} + x - \cancel{\beta} - \cancel{1} + \beta x + x\right] = \frac{1}{2}\left[\alpha - \beta + (\alpha+\beta+2)x\right]$$

from here we can use the Recurrence Relation given in Sherwin and Karnidakis.

$$a_n^1 P_{n+1}^{\alpha,\beta}(x) = (a_n^2 + a_n^3 x) P_n^{\alpha,\beta}(x) - a_n^4 P_{n-1}^{\alpha,\beta}(x)$$

$$a_n^1 = 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)$$
$$a_n^2 = (2n+\alpha+\beta+1)(\alpha^2-\beta^2)$$
$$a_n^3 = (2n+\alpha+\beta)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)$$
$$a_n^4 = 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)$$

Integration:

$$2n\int_{-1}^{x}(1-y)^{\alpha}(1+y)^{\beta}P_n^{\alpha,\beta}(y)\,dy = -(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{n-1}^{\alpha+1,\beta+1}(x)$$

How this applies to our chosen Lobatto basis

if we want to construct the higher order Lobatto M

$K \geq 2$

$$\ell_K(x) = \frac{1}{\|L_{K-1}\|_2}\int_{-1}^{x} L_{K-1}(\xi)\,d\xi, \quad 2 \leq K$$

using the integration rule

$$2(K-1)\int_{-1}^{x} L_{K-1}(\xi)\,d\xi = -(1-x)^{1}(1+x)^{1}P_{K-2}^{(1,1)}$$

$\Rightarrow$ for orthonormal

$$\ell_K = \frac{1}{\sqrt{\frac{2}{(2K-1)}}}\,\frac{-1}{2(K-1)}(1-x)(1+x)P_{K-2}^{1,1}(x)$$

$$= \frac{1}{\sqrt{\frac{2}{2K-1}}}\,\frac{-1}{(K-1)}\,\frac{(1-x)(1+x)}{2}P_{K-2}^{(1,1)}(x)$$

Thus giving use a simple relation for any Lobatto fn.

$$\ell_K = \frac{-2\sigma}{K-1} \cdot \ell_1 \cdot \ell_0 \, P^{(1,1)}_{K-2}(x).$$

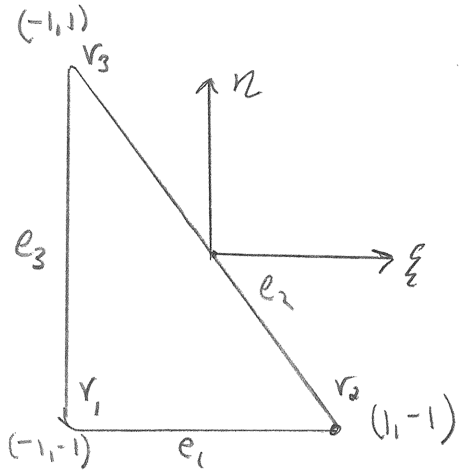
kernel is written as a product $\ell_1 \cdot \ell_2$.

$$\ell_K = \ell_1 \cdot \ell_2 \cdot \varphi_{K-2} \quad \text{when} \quad \varphi_{K-2} = \frac{-2\sigma}{(K-1)} \cdot P^{(1,1)}_{K-2}(x)$$

with $\sigma = \dfrac{1}{\sqrt{\dfrac{2}{2K-1}}}$

Remark: The Lobatto basis is Not orthonormal, however it is orientation invariant and Heirachical, and relatively simple.

2D Basis fns:



Our domain of choice is the reference element. The element is in the space $P(-1,1)$ for both ξ, η. Thus it spans the space defined for Jacobi polynomials (and all the special cases).

Notation: K - order of element.

Basis f'ns: We need 3 fns that are uniquely 1 at a vertex and zero at the rest. We define the fns

$$L_1 = -\frac{(\xi+\eta)}{2} \quad \text{- has propert } L_1(-1,-1) = 1$$

$$L_2 = \frac{\xi+1}{2} \quad - L_2(1,-1) = 1$$

$$L_3 = \frac{\eta+1}{2} \quad - L_3(-1,1) = 1$$

We can easily see that this is the 2D analogue of the vertex lobatto fns.

We now define edge flns which we want to be Lobatto (1-D) fns along the prescribed edge.

$$\left.\begin{aligned} \phi^{e_1} &= L_1 \cdot L_2 \, \psi_{K-2}(L_2 - L_1) \\ \phi^{e_2} &= L_2 \cdot L_3 \, \psi_{K-2}(L_3 - L_2) \\ \phi^{e_3} &= L_3 \cdot L_1 \, \psi_{K-2}(L_1 - L_3) \end{aligned}\right\}$$

the $L_j L_i$ product gives zero on all edges except the target and the kernel ψ gives a lobatto fln on the edge.

for $K \geq 2$

Lastly we need to represent the solution on the element interior, for this we use bubble fns.

$$\phi^b_{n_1,n_2,t} = L_1 \cdot L_2 \cdot L_3 \; \psi_{n_1-1}(L_2-L_1) \cdot \psi_{n_2-1}(L_1-L_3) \text{ for}$$

$$1 \le n_1, n_2 \text{ and } n_1+n_2 \le p^b-1 \quad \text{and} \quad 3 \le p^b$$

basically what the above says is that n_1, n_2 span all possible combinations such that they are each ≥ 1 by $n_1+n_2 \le p^b-1$.

for example take $p^b=3$ the $n_1+n_2 \le 2$ with the other constraint we see $n_1=n_2=1$ is the only possible values of n_1, n_2 thus for $p=3$ we get $\underline{\underline{1}}$ bubble fn.

<u>example 2</u> take $p^b=4$

3rd order bubble already done.

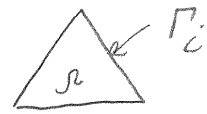
$n_1+n_2 \le 3$ gives $n_1=1, n_2=1$

$n_1=1, n_2=2$ - gives 4th order #1

$n_1=2, n_2=1$ - gives 4th order #2

Thus $p=4$ has 3 bubble fns total and 2 in addition to the one already specified for the 3rd order.

Formulation for 2D - Linear Wave Equation:



$$\frac{\partial u}{\partial t} + a \cdot \frac{\partial u}{\partial x} + b \cdot \frac{\partial u}{\partial y} = 0 \qquad (1)$$

First we apply the Method of weighted residuals

1). Multiply by test fn ϕ_i

$$\phi_i \cdot \frac{\partial u}{\partial t} + a \cdot \phi_i \frac{\partial u}{\partial x} + b \cdot \phi_i \frac{\partial u}{\partial y} = 0$$

2) Integrate over the domain.

$$\iint_\Omega \underset{①}{\phi_i \frac{\partial u}{\partial t}} + \underset{②}{a \cdot \phi_i \frac{\partial u}{\partial x}} + \underset{③}{b \cdot \phi_i \cdot \frac{\partial u}{\partial y}} \, dx dy = 0 \qquad ②$$

Integrate the 2nd and 3rd term by parts.

② and ③ can be written as

$$\iint_\Omega \phi_i \nabla \cdot (au\hat{i} + bu\hat{j}) \, dx dy$$

using Integrate by parts this become

$$\int_\Gamma \phi_i (au\hat{i} + bu\hat{j}) \cdot \vec{n} \, d\Gamma - \iint_\Omega (au\hat{i} + bu\hat{j}) \cdot \nabla(\phi_i) \, dx dy \qquad ③$$

substitute ③ into ② gives

$$\iint_\Omega \phi_i \frac{\partial u}{\partial t} - (au\hat{i} + bu\hat{j}) \cdot \nabla(\phi_i) \, dx dy + \int_\Gamma \phi_i (au\hat{i} + bu\hat{j}) \cdot \vec{n} \, d\Gamma = 0 \qquad ④$$

If we expand out our dot products we have

$$\iint_\Omega \phi_i \frac{\partial u}{\partial t} - au \cdot \frac{\partial \phi_i}{\partial x} - bu \frac{\partial \phi_i}{\partial y} \, dx dy + \int_\Gamma \phi_i (au n_x + bu n_y) \, d\Gamma = 0 \qquad ⑤$$

If we now introduce the following approximation for u

$$u \approx u_h = \sum_{j=1}^{NP} \alpha_j(t) \, \phi_j(x, y) \qquad ⑥$$

substitution of ⑥ into ⑤ gives

$$\iint \phi_i \frac{\partial\left(\sum_{j=1}^{N_p} \alpha_j(t)\,\phi_j(x,y)\right)}{\partial t} - a \sum_{j=1}^{N_p} \alpha_j(t)\,\phi_j(x,y)\,\frac{\partial \phi_i}{\partial x} - b \sum_{j=1}^{N_p} \alpha_j(t)\,\phi_j(x,y)\,\frac{\partial \phi_i}{\partial y}$$

$$dx\,dy = -\sum_{k=1}^{N_{face}} \phi_i \left[(aU)^* n_x + (bU)^* n_y\right]\Big|_{\Gamma_k}$$ where $(aU)^*$ is the $i=1;N_p$ numerical flux.

If we write this in a Matrix Form for the element Ω_e

$$[M]^{\Omega_e} \frac{\partial\{\alpha(t)\}}{\partial t} - [S]^{\Omega_e}\{\vec{\alpha}(t)\} = -\sum_{k=1}^{N_{face}} \{\phi\}\left[(aU)^* n_x + (bU)^* n_y\right]\Big|_{\Gamma_k}$$

the $(aU)^*$, $(bU)^*$ will take into account the ambiguity across the faces in the value of U, since it will have 2 values, (just like finite volume schemes).

$$[M]^{\Omega_e} = \iint \phi_i\,\phi_j\,dx\,dy \qquad [S]^{\Omega_e} = \iint \phi_j \left(a\frac{\partial \phi_i}{\partial x} + b\frac{\partial \phi_i}{\partial y}\right) dx\,dy$$

$|J|\,d\xi\,d\eta$

Thus we have formed our location equations for the element. There are a few remaining issues.

1). Using the standard element.

2). Numerical flux.

We get.

$$S_{ij} = \int_{-1}^{1}\int_{-1}^{-\xi_1} \phi_j \left[a\left(J_{11}^{-1} \cdot \frac{\partial \phi_i}{\partial \xi} + J_{12}^{-1} \frac{\partial \phi_i}{\partial \eta}\right) + b\left(J_{21}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{22}^{-1} \frac{\partial \phi_i}{\partial \xi}\right)\right] |J| \, d\eta \, d\xi$$

where $J_{ij}^{-1} = J_{ij}^{-1}(\xi, \eta)$.

$$S_{ij} = \int_{-1}^{1}\int_{-1}^{\xi} \phi_j \left[\frac{\partial \phi_i}{\partial \xi}\left(a \cdot J_{11}^{-1} + b J_{21}^{-1}\right) + \frac{\partial \phi_i}{\partial \eta}\left(a \cdot J_{12}^{-1} + b J_{22}^{-1}\right)\right] |J| \, d\eta \, d\xi$$

Thus we now have, a formula ready for quadrature.

Further we see that for the Stiffness Matrix we need $\frac{\partial \phi_i}{\partial \xi}$.

In general the following is true

$\phi_i = \phi_i(L_1(\xi,\eta), L_2(\xi,\eta), L_3(\xi,\eta))$ were 1 and 2 represent the 2 appropriate vertex fns.

$$\frac{\partial \phi_i}{\partial \xi} = \frac{\partial \phi_i}{\partial L_1} \cdot \frac{\partial L_1}{\partial \xi} + \frac{\partial \phi_i}{\partial L_2} \cdot \frac{\partial L_2}{\partial \xi} + \frac{\partial \phi_i}{\partial L_3} \cdot \frac{\partial L_3}{\partial \xi}$$

$$\frac{\partial \phi_i}{\partial \eta} = \frac{\partial \phi_i}{\partial L_1} \cdot \frac{\partial L_1}{\partial \eta} + \frac{\partial \phi_i}{\partial L_2} \cdot \frac{\partial L_2}{\partial \eta} + \frac{\partial \phi_i}{\partial L_3} \cdot \frac{\partial L_3}{\partial \eta}$$

For edge fns

$$\phi^e = L_i L_j \cdot \psi_{n-2}(L_j - L_i) \Rightarrow \frac{\partial \phi^e}{\partial \xi} = \left(L_j \psi_{n-2}(L_j - L_i) - L_i L_j \frac{d\psi}{dL_i}\right) \cdot \frac{\partial L_i}{\partial \xi}$$

$$+ \left(L_i \psi_{n-2}(L_j - L_i) + L_i L_j \frac{d\psi_{n-2}}{dL_j}\right) \frac{\partial L_j}{\partial \xi}$$

$$\phi^6 = L_1 \cdot L_2 \cdot L_3 \; \varphi_{n_1-1}(L_2 - L_1)\, \varphi_{n_2-1}(L_1 - L_3)$$

$$\frac{\partial \phi^6}{\partial L_1} = L_2 \cdot L_3 \, \varphi_{n_1-1}(L_2-L_1) \cdot \varphi_{n_2-1}(L_1-L_3) - L_1 L_2 L_3 \frac{d\varphi_{n-1}}{d(L_2-L_1)} \cdot \varphi_{n_2-1}(L_1-L_3) +$$

$$L_1 L_2 L_3 \, \varphi_{n_1-1}(L_2-L_1) \frac{d\varphi_{n_2-1}}{d(L_1-L_3)}$$

$$\frac{\partial \phi^6}{\partial L_2} = L_1 L_3 \, \varphi_{n_1-1}(L_2-L_1)\, \varphi_{n_2-1}(L_1-L_3) + L_1 L_2 L_3 \frac{d\varphi_{n-1}}{d(L_2-L_1)} \varphi_{n_2-1}(L_1-L_3)$$

$$\frac{\partial \phi^6}{\partial L_3} = L_1 L_2 \, \varphi_{n_1-1}(L_2-L_1)\, \varphi_{n_2-1}(L_1-L_3) - L_1 L_2 L_3 \, \varphi_{n_1-1} \frac{d\varphi_{n_2-1}}{d(L_1-L_3)}$$

$$\frac{\partial \phi^6}{\partial \xi} = \frac{\partial \phi^6}{\partial L_1} \cdot \frac{\partial L_1}{\partial \xi} + \frac{\partial \phi^6}{\partial L_2} \cdot \frac{\partial L_2}{\partial \xi} + \frac{\partial \phi^6}{\partial L_3} \frac{\partial L_3}{\partial \xi}$$

$$\frac{\partial \phi^6}{\partial \eta} = \frac{\partial \phi^6}{\partial L_1} \cdot \frac{\partial L_1}{\partial \eta} + \frac{\partial \phi^6}{\partial L_2} \frac{\partial L_2}{\partial \eta} + \frac{\partial \phi^6}{\partial L_3} \cdot \frac{\partial L_3}{\partial \eta}$$

Proving stability of D.G. discretizations

Goal: For a given discretization show $\sum_{e \in T_h} \int_{\Omega_e} \frac{d}{dt} u_h^2 \, d\Omega \le 0$

this is a necessary condition.

Consider

$\frac{\partial u}{\partial t} + \frac{\partial (a u)}{\partial x} = 0$ a simple Linear wave equation

discretize using D.G. gives

$$\sum_{e \in T_h} \int_{\Omega_e} v_h \frac{d u_h^+}{dt} - a u_h^+ \frac{\partial v_h}{\partial x} \, d\Omega_e + \sum_{e \in T_h} \oint_{\partial \Omega_e} v_h f^*(u_h^+, u_h^-) \, ds + \sum_{b \in b_h} \oint a u_h^b(u_h^+) v_h \, ds = 0$$

Where +/- denote inside/outside the element e. Furthermore note that we have assumed $u_h \in V_h$ and $v_h \in V_h$ as per the formal FEM statement.

Now make the substitution $v_h = u_h$ which is still $\in V_h$ and is ok since v_h can be arbitrary at this point.

$$\sum_{e \in T_h} \int \underbrace{u_h \frac{d}{dt} u_h}_{1} - \underbrace{a u_h \frac{\partial u_h}{\partial x} \, d\Omega_e}_{2} + \underbrace{\sum_{e \in T_h} \oint u_h f^*(u_h^+, u_h^-) \, ds}_{3} + \underbrace{\sum_{b \in b_h} \oint a u_h^b(u_h^+) v_h \, ds.}_{4}$$

Term by term

1: This is just $\frac{1}{2}\frac{d}{dt}\|u_h\|^2$, because $\frac{1}{2}\frac{d}{dt}\|u_h\|^2 = \sum_{e \in T_h} \frac{1}{2}\frac{d}{dt}\|u_h\|^2_{\Omega_e} =$

$$\sum_{e \in T_h} \int_{\Omega_e} \frac{1}{2}\frac{d}{dt}(u_h)^2 \, d\Omega_e = \sum_{e \in T_h} \int_{\Omega_e} u_h \frac{d u_h}{dt} \, d\Omega_e \checkmark$$

2 This is similar to 1 $\quad a u_h \frac{\partial u_h}{\partial x} = \frac{1}{2} a \frac{d u_h^2}{dx} \Rightarrow \sum_{e \in T_h} \int_{\Omega_e} \frac{1}{2} a \frac{d}{dx}(u_h^2) \, d\Omega_e$

Integration by parts gives $\sum_{e \in T_h} - \int_{\Omega_e} \cancelto{0}{\frac{\partial (a)}{\partial x}} a \frac{1}{2} u_h^2 + \oint_{\partial \Omega_e} \frac{1}{2} a u_h^2 \, ds$

$$= \oint_{\partial \Omega_e} \frac{1}{2} a u_h^2 \, ds$$

3, 4: More difficult. They require knowledge of B.C.

For simplicity consider a Periodic Domain, the ④ = 0 because all fluxes are internal. So far we have

$$\frac{d}{dt}\|u_h\|^2 + \sum_{e\in\mathcal{T}_h}\int_{\partial\Omega_e} u_h f^*(u_h^+,u_h^-) - \tfrac{1}{2}a u_h^2\, ds = 0$$

or

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 = \sum_{e\in\mathcal{T}_h}\int_{\partial\Omega_e} \tfrac{1}{2}a u_h^2 - u_h f^*(u_h^+,u_h^-)\, ds$$

Now we need to know about f^*, Let's use upwind / central flux $f^* = \frac{1}{2}(au_h^+ + au_h^-) + \frac{1}{2}(1-\alpha)|a|(u_h^+ - u_h^-)$ if $\alpha = 1$, central $\alpha = 0$, upwind, or L-F.

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 = \sum_{e\in\mathcal{T}_h}\int_{\partial\Omega_e} \tfrac{1}{2}a u_h^2 - u_h^+\left[\tfrac{1}{2}au_h^+ + au_h^- + \tfrac{1}{2}(1-\alpha)|a|(u_h^+-u_h^-)\right] ds$$

There are two ways to handle this.

Method A: Element wise. Consider a sample stencil and Let's prove stability for element i only; ie. sum o $i-1, i, i+1$

$i-1$, contribution

$$\tfrac{1}{2}a\cancel{(U^i_{i-\frac12})^2} - \cancel{U^i_{i-\frac12}}\left[\cancel{\tfrac{1}{2}aU^i_{i-\frac12}} + \tfrac{1}{2}a\cancel{U^{i-1}_{i-\frac12}} + \frac{1-\alpha}{2}|a|(U^i_{i-\frac12} - U^{i-1}_{i-\frac12})\right] +$$

$$\tfrac{1}{2}a\cancel{(U^i_{i+\frac12})^2} - U^i_{i+\frac12}\left[\tfrac{1}{2}a\cancel{U^i_{i+\frac12}} + \tfrac{1}{2}a\cancel{U^{i+1}_{i+\frac12}} + \frac{1-\alpha}{2}|a|(U^i_{i+\frac12} - U^{i+1}_{i+\frac12})\right]$$

$i-1$: $i-\frac12$ fluxes only

$$-\tfrac{1}{2}a\cancel{(U^{i-1}_{i-\frac12})^2} + U^{i-1}_{i-\frac12}\left[\tfrac{1}{2}aU^i_{i-\frac12} + \tfrac{1}{2}a\cancel{U^{i-1}_{i-\frac12}} + \frac{1-\alpha}{2}|a|(U^i_{i-\frac12} - U^{i-1}_{i-\frac12})\right]$$

$i+1$: $i+\frac12$ fluxes only

$$-\tfrac{1}{2}a\cancel{(U^{i+1}_{i+\frac12})^2} + U^{i+1}_{i+\frac12}\left[\tfrac{1}{2}a\cancel{U^i_{i+\frac12}} + \tfrac{1}{2}\cancel{a}U^{i+1}_{i+\frac12} + \frac{1-\alpha}{2}|a|(U^i_{i+\frac12} - U^{i+1}_{i+\frac12})\right]$$

3

The remaing terms are written as

$$-\frac{1-\alpha}{2}|a|\left(U^{i}_{i-\frac12}-U^{i-1}_{i-\frac12}\right)U^{i}_{i-\frac12}+\frac{1-\alpha}{2}|a|\left(U^{i}_{i-\frac12}-U^{i-1}_{i-\frac12}\right)U^{i-1}_{i-\frac12}$$

$$-\frac{1-\alpha}{2}|a|\left(U^{i}_{i+\frac12}-U^{i+1}_{i+\frac12}\right)U^{i}_{i+\frac12}+\frac{1-\alpha}{2}|a|\left(U^{i}_{i+\frac12}-U^{i+1}_{i+\frac12}\right)U^{i+1}_{i+\frac12}=$$

$$=\frac{1-\alpha}{2}|a|\left[-\left(U^{i}_{i-\frac12}-U^{i-1}_{i-\frac12}\right)^2-\left(U^{i}_{i+\frac12}-U^{i+1}_{i+\frac12}\right)^2\right]$$

if $\alpha=0$ the this is a negative #. Also notice that if we sumed this term over all cells we would have the same thing and it is stable.

An easy way to write this is in interface form.

$$\sum_{e\in T_h}\int_{\partial\Omega_e}\tfrac12 a(u_h^+)^2+u_h^+f^*(u_h^+,u_h^-)\,ds=\sum_{i\in I_h}\int \tfrac12 a(u_h^+)^2+u_h^+f^*(u_h^+,u_h^-)\,ds$$

for an interface $i\in I_h$ there are two contribution each cell + and - written as

$$\left(\cancel{\tfrac12(au_h^+)^2}+\cancel{\tfrac12 a u_h^{+2}}-\tfrac12 a u_h^+u_h^- - \tfrac{1-\alpha}{2}|a|u_h^+[\![u_h]\!]\right)n^+ +$$

$$\left(\cancel{\tfrac12 a u_h^{-2}}-\cancel{\tfrac12 a u_h^{-2}}-\tfrac12 a u_h^+u_h^- - \tfrac{1-\alpha}{2}|a|u_h^-[\![u_h]\!]\right)n^-$$

$n^-=-n^+$ gives

$$-\cancel{\tfrac12 a u_h^+u_h^-}+\tfrac12 a u_h^+u_h^- \quad \tfrac{1-\alpha}{2}|a|[\![u_h]\!](u_h^- - u_h^+)|n|$$

$$=-\left(\frac{1-\alpha}{2}|a|[\![u_h]\!]^2\right)$$

Thus

$$\frac{d}{dt}\|u_h\|^2=-\left(\frac{1-\alpha}{2}|a|\right)\sum_{i\in I_h}[\![u_h]\!]^2 \Rightarrow \text{Stable}$$

This is how proof is constructed.

The wave equation cast as a non-linear Problem:

Consider a more general scalar conservation Law

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot F(u) = 0 \quad (1)$$ where u is a scalar and $\vec{\nabla}$ is 2D.

for the wave equation we can write the flux as $a \cdot u\hat{x} + b u\hat{y}$. We solve the above by using DG.

$$\int_{\Omega_e} \phi_i \frac{\partial u}{\partial t} d\Omega_e + \int_{\Omega_e} \phi_i \vec{\nabla} \cdot F(u) d\Omega_e = 0 \quad (2)$$ using integration by parts.

$$\int_{\Omega_e} \phi_i \vec{\nabla} \cdot F(u) d\Omega_e = \int_{\Gamma_e} \phi_i \vec{F}(u) \cdot \vec{n} \, d\Gamma_e - \int_{\Omega_e} \vec{\nabla}(\phi_i) \cdot F(u) \, d\Omega_e \quad (3)$$

substituting (3) into (2) gives.

$$\int_{\Omega_e} \phi_i \frac{\partial u}{\partial t} d\Omega_e - \int_{\Omega_e} \vec{\nabla}(\phi_i) \cdot F(u) d\Omega_e = -\int_{\Gamma_e} \phi_i \vec{F}(u) \cdot \vec{n} \, d\Gamma_e \quad (4)$$

In the 1st term you can substitute $u \approx \sum_{j=1}^{N_{node}} \hat{u}_j \phi_j$ giving

$$\underbrace{\int_{\Omega_e} \phi_i \frac{\partial \sum_{j=1}^{N_{node}} \hat{u}(t) \phi_j}{\partial t} d\Omega_e}_{(1)} - \underbrace{\int_{\Omega_e} \vec{\nabla}(\phi_i) \cdot \vec{F}(u) \, d\Omega_e}_{(2)} = -\underbrace{\int_{\Gamma_e} \phi_i \cdot \vec{F}(u) \cdot \vec{n} \, d\Gamma_e}_{(3)}$$

For $F(u)$ we can't pull out a vector of $\{\hat{u}\}$'s thus we'll use an iterative approach and treat this term as a a volume Residual term $R_v(u)$. Thus each of our 3 terms needs an evaluation of an integral.

(1) - Mass Matrix $M_{ij} = \int_{\Omega_e} \phi_i \phi_j \, d\Omega_e$

(2) - $R_v(u)$ - Volume Residual

(3) - $R_s(u)$ - Surface Residual.

Evaluation of Mass Matrix

$$M_{ij} = \int_{\Omega_e} \phi_i \phi_j \, d\Omega_e$$

Using the specified (previously) rules for quadrature on a triangle

$$M_{ij} = \int_{\Omega_e} \phi_i \phi_j \, d\Omega_e = \int_{-1}^{1}\int_{-1}^{-\xi} \phi_i(\xi,\eta)\cdot\phi_j(\xi,\eta)\cdot|J|\, d\eta\, d\xi$$ - this was shown previously.

Using gauss-Legendre quadrature.

$$M_{ij} \approx \sum_{k=1}^{N_{gp}} \phi_i(\xi_k,\eta_k)\cdot\phi_j(\xi_k,\eta_k)\cdot|J(\xi_k,\eta_k)|\cdot w_k$$

Evaluation of Volume Residual

$$R_v = \int_{\Omega_e} \nabla\phi_i \cdot \vec{F}(U)\, d\Omega_e = \int_{-1}^{1}\int_{-1}^{-\xi} \vec{\nabla}(\phi_i(\xi,\eta)) \cdot \vec{F}(U(\xi,\eta))\, d\Omega_e$$

since F is assume non-linear we will use

$$U \approx \sum_{j=1}^{N_{nod}} \hat{U}^{OLD}_j \phi_j(\xi,\eta)$$ - thus we do quadrature on this

$$R_{v_i} = \int_{-1}^{1}\int_{-1}^{-\xi} \vec{\nabla}(\phi_i(\xi,\eta))\cdot\vec{F}(U)\,|J|\,d\eta\,d\xi \approx \sum_{k=1}^{N_{gp}} \vec{F}\left(\sum_{j=1}^{N_{nod}} \hat{U}^{OLD}_j \phi_j(\xi_k,\eta_k)\right)\cdot\vec{\nabla}\phi_i(\xi_k,\eta_k)$$

$|J(\xi$ w_k

Thus R_{v_i} - is nnodes x nfields for each element.

We still need to deal with $\vec{\nabla}(\phi_i(\xi,\eta))$

$$\vec{\nabla} = \begin{Bmatrix} \dfrac{\partial \phi_i(\xi,\eta)}{\partial x} \\ \dfrac{\partial \phi_i(\xi,\eta)}{\partial y} \end{Bmatrix} = \begin{matrix} \dfrac{\partial\phi_i}{\partial\xi}\cdot\dfrac{\partial\xi}{\partial x} + \dfrac{\partial\phi_i}{\partial\eta}\cdot\dfrac{\partial\eta}{\partial x} \\ \dfrac{\partial\phi_i}{\partial\xi}\cdot\dfrac{\partial\xi}{\partial y} + \dfrac{\partial\phi_i}{\partial\eta}\cdot\dfrac{\partial\eta}{\partial y} \end{matrix} = \begin{bmatrix} \dfrac{\partial\xi}{\partial x} & \dfrac{\partial\eta}{\partial x} \\ \dfrac{\partial\xi}{\partial y} & \dfrac{\partial\eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial\phi_i}{\partial\xi} \\ \dfrac{\partial\phi_i}{\partial\eta} \end{Bmatrix} =$$

$$[J]^{-1}\begin{Bmatrix} \dfrac{\partial\phi_i}{\partial\xi} \\ \dfrac{\partial\phi_i}{\partial\eta} \end{Bmatrix} = [J]^{-1}\, \vec{\nabla}_{\xi,\eta}(\phi_i)$$

$$Rv = \sum_{K=1}^{N_{gp}} \vec{F}\left(\sum_{j=1}^{N_{mode}} \hat{U}^{OLD} \phi_j(\xi_k, \eta_k)\right) \cdot [J]^{-1} \vec{\nabla}_{\xi,\eta}(\{\phi\})$$ gives

nmodes by nfields. or for each mode

$$Rv_i = \sum_{K=1}^{N_{gp}} \vec{F}\left(\sum_{j=1}^{N_{mode}} \hat{U}^{OLD} \phi_j(\xi_k, \eta_k)\right) \cdot [J]^{-1} \vec{\nabla}_{\xi,\eta}(\phi_i)\, w_K$$

For the wave equation we can write out

$$\vec{F} = \begin{Bmatrix} a\tilde{U} \\ b\tilde{U} \end{Bmatrix} \qquad \tilde{U} = \sum_{j=1}^{N_{mode}} \hat{U}_j \phi_j$$

$$Rv_i = \sum_{K=1}^{N_{gp}} \begin{Bmatrix} a\tilde{U}(\xi_k, \eta_k) \\ b\tilde{U}(\xi_k, \eta_k) \end{Bmatrix} \cdot [J]^{-1} \begin{Bmatrix} \frac{\partial \phi_i}{\partial \xi} \\ \frac{\partial \phi_j}{\partial \eta} \end{Bmatrix} |J(\xi_k, \eta_k)|\, W_K$$

$$Rv_i = \sum_{K=1}^{N_{gp}} \begin{Bmatrix} a\tilde{U}(\xi_k, \eta_k) \\ b\tilde{U}(\xi_k, \eta_k) \end{Bmatrix} \cdot \begin{Bmatrix} J_{11}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{12}^{-1} \frac{\partial \phi_i}{\partial \eta} \\ J_{21}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{22}^{-1} \frac{\partial \phi_i}{\partial \eta} \end{Bmatrix} |J|\, w_k$$

$$Ru_i = \sum_{K=1}^{N_{gp}} \left(a\tilde{U}(\xi_k, \eta_k)\left[J_{11}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{12}^{-1} \frac{\partial \phi_i}{\partial \eta}\right] + b\tilde{U}\left[J_{21}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{22}^{-1} \frac{\partial \phi_i}{\partial \eta}\right]\right) \cdot w_K |J(\xi_k, \eta_k)|$$

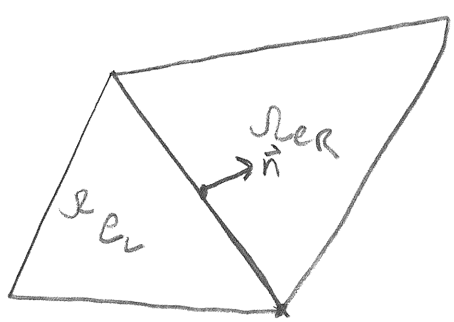
- This is 1 row of Rv_{ij}.

Surface term evaluation: The Numerical Flux.

So far we have delt with the terms $\int_{\Omega_e}$ - which are called volume terms because they go over the element. We have 1 term left $\oint_{\Gamma_e} \phi_i \cdot \vec{F}(U)\cdot \vec{n}\, d\Gamma_e$ - which is an integral over the elements Boundary. Hence it is called the Surface term and it is also part of the residual.

Numerical flux: Notice that this term is very similar to the finite volume expression $\oint_{\Gamma_e} \vec{F}(U)\cdot\vec{n}\, d\Gamma_e$. Which should inspire use to have the same concerns as in F.V.M.

Consider the following 2 elements Ω_{e_L}, Ω_{e_R} that share a 2D face (edge).



The question arise what is $\vec{F}(U)$ because U is defined on the edge from both Ω_{e_L}, Ω_{e_R} seperately.

Answer: Replace $\vec{F}(U)$ with $\vec{F}^*(U_L, U_R)$ just like in finite volume methods.

There is however a key difference, namely that FVM deals only with cell averages where D.G. has modal coefficients and thus our strategy with be to evaluate the trace of ϕ at the quadrature points on that edge.

$U = \sum_{j=1}^{N_{mod}} U_j \phi_j$ at the quadrature points

which is really what the integral says.

$$\oint_{\Gamma_e} \phi_i \vec{F}(U) \cdot \vec{n} \, d\Gamma_e = \oint_{\Gamma_e} \phi_i \cdot \vec{F}^*(U_L, U_R) \cdot \vec{n} \, d\Gamma_e \approx \sum_{F=1}^{N_{face}} \sum_{k=1}^{N_{qp}} \phi_i\Big|_{\Gamma_i}(\xi_k, \eta_k) \cdot \vec{F}(U_L\Big|_{\Gamma_i}(\xi_k,\eta_k), U_R(\xi_k,\eta_k)) \cdot \vec{n} \, w_k |J|$$

– thus if we do this on a face basis we can deal with a single face.

Single Face:

$$\oint_{\Gamma_{e_{f_i}}} \phi_i \vec{F}^*(U_L, U_R) \cdot \vec{n} \, d\Gamma_e \approx \sum_{k=1}^{N_{qp}} \phi_i\Big|_{\Gamma_i}(\xi_k, \eta_k) \cdot \vec{F}^*(U_L(\xi_k,\eta_k), U_R(\xi_k,\eta_k)) \cdot \vec{n} \, w_k |J|$$

We see that for modes ϕ_i where $\phi_i\Big|_{\Gamma_i} = 0$ for all points on Γ_i then we see that we don't need to even deal with mode i because it's zero. Further recall that the edge f'ns are simply lobatto f'ns on the edge to which they belong, then we can do the following for the shape f'n ϕ_i

If $\phi_i \neq 0 \;\; \forall \, \xi_k, \eta_k\Big|_{\Gamma_i}$ then replace

$\phi_i = l_i$ where l_i is the 1D lobatto f'n.

We replace ξ_k, η_k with ζ_k – which is a 1D coordinate.

$$\oint_{\Gamma_K} \phi_i \vec{F}^*(U_L, U_R)\cdot\vec{n}\, d\Gamma_K \approx \sum_{k=1}^{N_{edgeqp}} \phi_i \cdot \vec{F}^*(U_L, U_R)\cdot\vec{n}\cdot w_K |J|$$

where

$$U_L = \sum_{j=1}^{N_{edge\_modes}} \hat{U}_K \ell_K$$

Where K is an index used to map local edge mode j to global mode K

$K = \text{edgemap}(j)$

Recall that we doing a trace along an edge thus "$|J|$" takes new meaning in terms of a 1-D coordinate along said edge namely $|J| = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$

Ex: Edge Map

Consider a $p=2$ soln which has 6 modes total.

If we want to evaluate the flux on local face 1 of some element e then

edgemap(1) = 1
edgemap(2) = 2
edgemap(3) = 4

$\Big\}$ these are the 3 non-zero modes on this edge.

for face 2

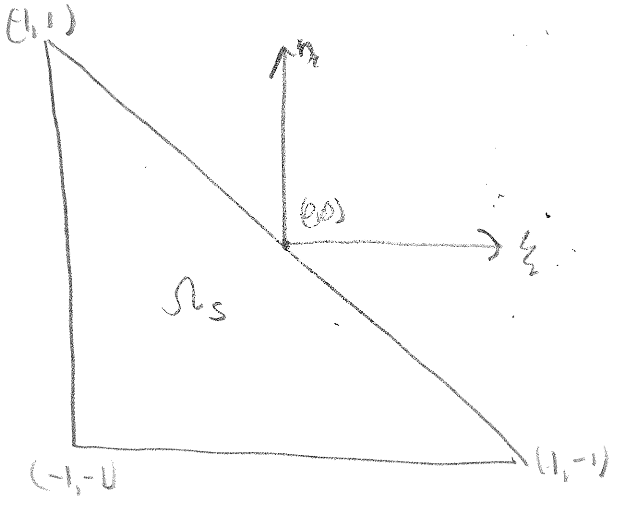
edgemap(1) = 2
edgemap(2) = 3
edgemap(3) = 5

Note: All bubble flns will never be present they are zero at the faces.

Conclusion: Surface terms are tricky but b/c of our basis we can 1D gauss lobatto flns to get U_L, U_R and ϕ_i. Mathematically this is equivalent to computing the trace of a function on face i and further corresponds to computing that trace with a change of variables and relocation of the origin.

Using the standard element:

Rather then do integration on each element's domain it would be more convient to do it on the standard domain Ω^s



and map it to the physical domain.

For now we will always use a linear map.

$$x = L_1 \beta_1 + L_2 \beta_2 + L_3 \beta_3$$
$$y = L_1 \gamma_1 + L_2 \gamma_2 + L_3 \gamma_3$$

so for 3 vertices we have

$$\begin{bmatrix} L_1(-1,-1), & L_2(-1,-1) & L_3(-1,-1) \\ L_1(1,-1) & L_2(1,-1) & L_3(1,-1) \\ L_1(-1,1) & L_2(1,-1) & L_3(1,-1) \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Same for the y - coordinate solving for γ_i.

Evaluating the above matrix we find that the matrix $[L] = [I]$ as expected thus our coefficients essentially the are the vertex values.

$$\beta_i = x_i \quad i=1 \ldots 3$$
$$\gamma_i = y_i \quad i=1 \ldots 3$$

Now that we have established a mapping ~~for~~ $x(\xi,\eta)$; $y(\xi,\eta)$ we will need to be able to evaluate $[M]^{\Omega e}$, $[S]^{\Omega e}$.

<u>Mass Matrix: $[M]^{\Omega e}$</u>

$$M_{ij} = \iint \phi_i \phi_j \, dx\,dy \rightarrow \int_{-1}^{1}\int_{-1}^{1} \phi_i(\xi,\eta)\phi_j(\xi,\eta)\,|J|\,d\eta\,d\xi$$

This is easy, requiring only the $|J|$ ~~of~~.

$$S_{ij} = \iint \phi_j \left[a\frac{\partial \phi_i}{\partial x} + b\frac{\partial \phi_i}{\partial y}\right] dx\,dy$$

? What are $\frac{\partial \phi_i}{\partial x}$, $\frac{\partial \phi_i}{\partial y}$ in terms of ξ,η.

Let's construct the Jacobian.

$$\frac{\partial \phi_i}{\partial \xi} = \frac{\partial \phi_i}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial \phi_i}{\partial y}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \phi_i}{\partial \eta} = \frac{\partial \phi_i}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial \phi_i}{\partial y}\cdot\frac{\partial y}{\partial \eta} \rightarrow \begin{Bmatrix} \frac{\partial \phi_i}{\partial \xi} \\ \frac{\partial \phi_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_i}{\partial x} \\ \frac{\partial \phi_i}{\partial y} \end{Bmatrix}$$

Thus all we really need to do is invert this 2x2 Jacobian.

$$\begin{Bmatrix} \frac{\partial \phi_i}{\partial x} \\ \frac{\partial \phi_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} J \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial \phi_i}{\partial \xi} \\ \frac{\partial \phi_i}{\partial \eta} \end{Bmatrix}.$$

- Using this relation expanded in our S_{ij}

Evaluation of the Integrals. Numerical Quadrature.

We see that to get the values for the Mass and Stiffness matrices we have to evaluate integrals over the interval $[-1,1]$, $[-1,1]^2$ inclusive. Thus for a variable p-order code we need to do this numerically. This procedure is known as quadrature. There are many different ways of doing this. We will follow the rules in Solin's book.

Solin recommends the use of Gauss Quadrature.

We could use Gauss-Labotto Quadrature as an alternative.

As our example we will use the M_{ij} expression.

$$M_{ij} = \int_{-1}^{1}\int_{-1}^{-\xi_1} \phi_i(\xi,\eta)\,\phi_j(\xi,\eta)\,|J(\xi,\eta)|\,d\xi\,d\eta \approx$$

$$\sum_{K=1}^{N_{gp}} \phi_i(\xi_K,\eta_K)\,\phi_j(\xi_K,\eta_K)\,|J(\xi_K,\eta_K)|\,w_K$$ - This is our formula.

Thus each entry in the Matrix requires the sum over all the quadrature points.

The points and weights ξ_K, η_K, w_K are tabulated in Solin's Book up to $p=7$ and up to $p=21$ using the provided Mathematica Notebooks. We have tabulated the points in the web.

<u>Remark:</u> For a rational polynomial of order p the Gauss Quadrature formula ~~of~~ order p will evaluate the Integral exactly.

Ex. $\int_{-1}^{1} x\,dx$ - We will use 1D quadrature to show this. Further if one uses order higher the p the integral is also exact.

$$\int_{-1}^{1} x\,dx = \left.\frac{x^2}{2}\right|_{-1}^{1} = \frac{1}{2} - \left(\frac{(-1)^2}{2}\right) = 0$$

$$\int_{-1}^{1} x\,dx \approx \sum_{k=1}^{2} w_1 x_1 = 1.0(-.57735) + 1.0(.57735)$$

$$= 0$$

for higher order

$$\int_{-1}^{1} x\,dx \approx w_1 \cdot (0) + (.775457).55555 + .55555(-.775459)$$

$$= 0.$$ Thus the integration is in fact exact for these types of polynomials.

Projection of the initial conditions to the Modal Space:

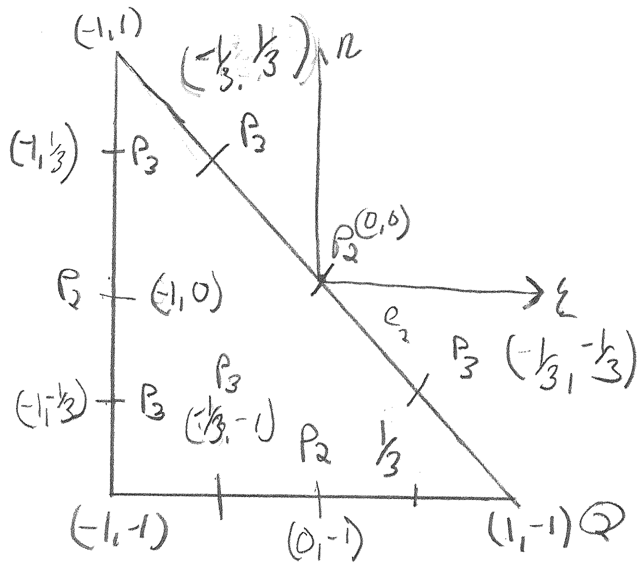
At the out set of our solution process we need to project the values at the physical locations to the modal space. In plain english we need to get the modal coefficients corresponding to the initial conditions.

Recall the form of our solution.

$$U = \sum_{j=1}^{nmode} \hat{U}_j \phi_j$$ - we must solve for the

we first need to pick values at X_i, eta

$$U(X(\xi, \eta)) = \sum_{j=1}^{nmode} \hat{U}_j \phi_j(\xi, \eta)$$ - thus we need to pick ξ, η points to evaulete my locations from using our mapping.



for $p=1$ we just use the verticies

for $p=2$ we use points labeled $P2$ + vertex

for $p=3$ we use points labeled P_3

points on e_2 are found via the vector equation

$$\vec{X} = \vec{X}_2 + \lambda \sqrt{8}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$ where λ is a fraction, $\sqrt{8}$ is $\|e_2\|$

Interior points are every edge combination that keeps you off the edges and in the triangle where combinations come from edge points not on the vertex.

Interior points

- As an take $p=3$.

$-\frac{1}{3}, -\frac{1}{3}$ is in the triangle, $\frac{1}{3}, \frac{1}{3}$ is on the edge doesn't count.

take $p=4$

$(-.5, -.5)$, $(-.5, 0)$ $(0, -.5)$ - All others are

Using these restrictions and the reference element I have devised an algorithm to get these points.

Using our reference triangle we use 2 vectors equidistant x, y arrays from $-1, 1$

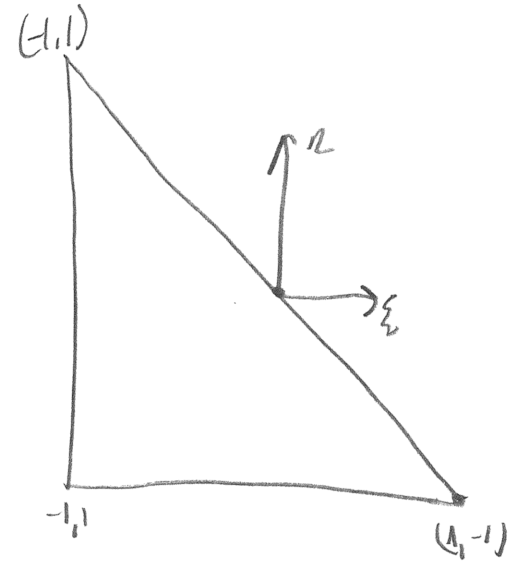
$x = [-1 : \frac{2}{p} : 1]$, $y = x$

combining all combinations of x, y we would get a rectangle

so we do the following

```
Npt = 0
for i = 1: p+1
    for j = 1: p+1
        func = - x(j)
        if (y(i) ≤ func)
            Npts = Npts + 1
            Xp(Npts) = X(j)
            Yp(Npts) = y(i)
        end if
    end for
end for.
```

The only real trick is the "func" term if we draw our ref. triangle



We see that the hypotenuse is the line $y_{hyp} = -x$ thus if the y(i) we're only is $\leq y_{hyp} = -x$ then we're inside the triangle thus we'll gets all points in the triangle and on the edges.

<u>Projection operation:</u>

Now that we have ξ, η points to evaluate the basis fn at we are ready to do the projection.

$$u(x) = \sum_{j=1}^{N_{mode}} \hat{u}_j \phi_j$$

if $x = x(\xi, \eta)$ — through our mapping fn. then we need to generate N_{mode} equations using the ξ, η points specified by our algorithm above.

Then for each point ξ_i, η_i we have

$$u(x(\xi_i, \eta_i)) = \sum_{j=1}^{N_{mod}} \hat{u}_j \phi_j(\xi_i, \eta_i)$$ — giving linear system

$$\{U(x(\{\xi\},\{\eta\}))\} = [V]\{\hat{U}\}$$

where $V_{ij} = \phi_j(\xi_i, \eta_i)$ - which is the jth mode evaluated at the ith point.

Here V essentially plays the role of a Vandermonde Matrix.

Remark: Equidistant nodes are not the optimal choice. If conditioning problems are encountered then we should move to the nodes specified in Hesthaven and Warburton's Book.

Mass Matrix Inversion:

Rather then directly form $[M]^{-1}$ we will do $[M]=[L][U]$ as a preprocessing step and store the $[L]$ $[U]$, in the compact form of course giving for the explicit scheme.

$$[M]\frac{d\{\hat{U}\}}{dt}=\{\vec{R}\} \quad = \quad [L][U]\left(\frac{\{\hat{U}\}^{n+1}-\{\hat{U}\}^{n}}{\Delta t}\right)A_k=\{\vec{R}\}$$

$$[L][U]\{\hat{U}\}^{n+1}=\frac{\Delta t}{A_k}\{R\}+[L][U]\{\hat{U}\}^{n}$$

$$[L]\{\hat{U}\}_1^{n+1} = RHS$$

$$[U]\{\hat{U}\}^{n+1}=\{\hat{U}\}_1^{n+1}$$

this is just like

$A=LU$

$LU\{x\}=\{b\}$

$U\{x\}=\{y\}$ - solve 2nd

$L\{y\}=\{b\}$ - solve 1st

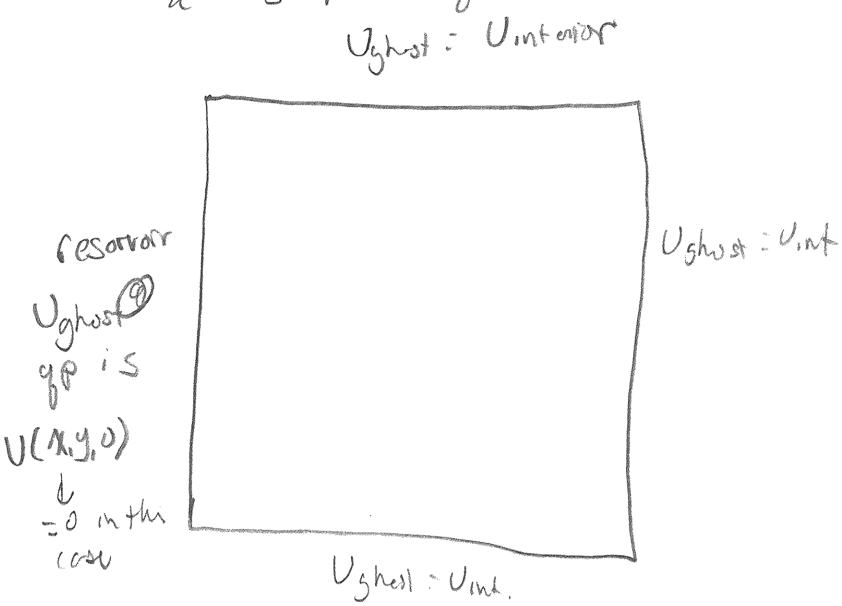
Boundary conditions:

To enforce boundary conditions we would like to use ghost cells. The question is how to derive expressions for the U variable in these ghost cells.

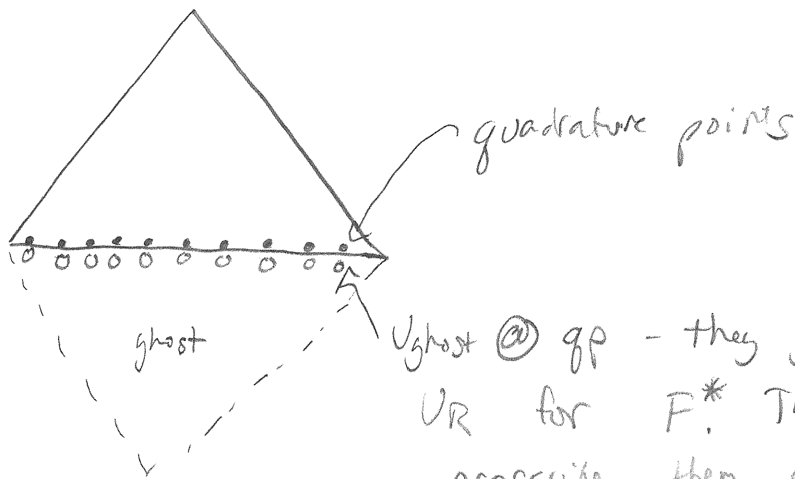
The answer is that we don't actually derive expressions for modes or any kind of cell centered value. Instead we derive an expression for the U variable at each quadrature point. Thus we don't actually need an array holding ghost cell modes.

Wave Equation B.C.:

An appropriate B.C. for the 2D wave equation is to ~~AA~~ feed the domain from a reservoir and continue the wave on all other exterior boundaries. Thus for a simple square.



To visualize what quantities U_{ghost} represents take a cell.



U_{ghost} @ qp - they just provide the U_R for F^*. That's why we prescribe them directly and not as modes.

DG Formulation for the Euler Equations:

We start with the Euler Equations in conservative form

$$\frac{\partial \vec{q}}{\partial t} + \frac{\partial \vec{F}(\vec{q})}{\partial x} + \frac{\partial (\vec{G}(\vec{q}))}{\partial y} = 0 \quad \text{where} \quad (1)$$

$$\vec{q} = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{Bmatrix}, \quad \vec{F}(\vec{q}) = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (\rho E + p) u \end{Bmatrix}, \quad \vec{G}(\vec{q}) = \begin{Bmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ (\rho E + p) v \end{Bmatrix}$$

Of course these are non-linear but as we have previously cast the wave equation in a way that can handle this, it's not a problem.

Apply Method of weighted residuals with test f'n $v = \phi_i$

$$\underbrace{\iint_{\Omega_e} \phi_i \frac{\partial \vec{q}}{\partial t}\, d\Omega_e}_{(1)} + \iint_{\Omega_e} \underbrace{\phi_i \frac{\partial \vec{F}(\vec{q})}{\partial x}}_{(2)} + \underbrace{\phi_i \frac{\partial G(\vec{q})}{\partial y}}_{(3)}\, d\Omega_e = 0 \quad (2)$$

Using Div. Theorem of (2) and (3) we get

4x2 matrix - b/c 4 equations

$$\iint_{\Omega_e} \phi_i \frac{\partial \vec{F}}{\partial x} + \phi_i \frac{\partial G}{\partial y}\, d\Omega_e = \iint_{\Omega_e} \phi_i \vec{\nabla} \cdot (\vec{F}, \vec{G})\, d\Omega_e =$$

$$\oint \phi_i (\vec{F}, \vec{G}) \cdot \vec{n}\, ds - \iint_{\Omega_e} \nabla(\phi_i) \cdot (\vec{F}, \vec{G}) \quad \text{if we write} \quad (3)$$

$(\vec{F}(\vec{q}), \vec{G}(\vec{q})) = \tilde{F}$ then we see $\tilde{F}$ is

$$\tilde{F} = \begin{Bmatrix} \rho u n_x + \rho v n_y \\ (\rho u^2 + p) n_x + \rho u v n_y \\ \rho u v n_x + (\rho v^2 + p) n_y \\ (\rho E + p) u n_x + (\rho E + p) v n_y \end{Bmatrix} - \text{Just like F.V.M} \quad (4)$$

using ③ and ④ in ② gives

$$\iint_{\Omega_e} \phi_i \frac{\partial(\hat{q})}{\partial t}\, d\Omega_e - \iint_{\Omega_e} \vec{\nabla}(\phi_i)\cdot(\vec{F},\vec{G})\, d\Omega_e + \oint \phi_i \tilde{F}\, ds = 0 \quad ⑤$$

a slight re-write of the 2nd term gives

colm vec'2 4x2 matrix

$$\underbrace{\iint_{\Omega_e} \phi_i \frac{\partial(\vec{q})}{\partial t}\, d\Omega_e}_{①} - \underbrace{\iint_{\Omega_e} [\vec{F},\vec{G}] \begin{Bmatrix} \frac{\partial \phi_i}{\partial x} \\ \frac{\partial \phi_i}{\partial y} \end{Bmatrix} d\Omega_e}_{②} + \underbrace{\oint \phi_i \tilde{F}\, ds}_{③} = 0$$

Thus we now have nnodes·nfields (4 in this case) equations for each element. We have put #'s next to each term above. We'll treat each one as it's own little piece. The terms are identified as follows.

① - Mass Matrix - this has some nice properties for storage.

② - Volume Residual - Applatly named due to an integral over the volume

③ - Surface Residual - Like F.V.M integral but slightly more complicated in actual evaluation

After introduction of $\rho = \sum_{j=1}^{N_{nodes}} \hat{\rho}_j \phi_j$, $\rho u = \sum_{j=1}^{N_{nodes}} \hat{\rho u}_j \phi_j$, $\rho v = \sum_{j=1}^{N_{nodes}} \hat{\rho v}_j \phi_j$, $\rho E = \sum_{j=1}^{N_{nodes}} \hat{\rho E}_j \phi_j$

Mass Matrix:

The evaluation of the mass matrix can be done quite easily. However, I believe that it must be looked at correctly. I believe the correct view point is to view each component of $\vec{q}$, $\vec{F},\vec{G}$ as a seperate equation. Thus yielding a block diagonal Mass Matrix that is the same on each block. Thus lowering storage requirements.

Our term is

$$\iint_{\Omega_e} \phi_i \frac{\partial \vec{q}}{\partial t} \, d\Omega_e \quad \text{for } i = 1, \text{nmodes}$$

We have 2 options,

1) Hold i constant and go over the whole $\vec{q}$ vector.
2). Hold the $\vec{q}$ vector component constant and go over all i.

We can show that 2) is simply a re-ording of 1)
this re-ording makes for far more efficient storage.

As a Model Consider the we have only 2 unknowns $\rho, \rho u$.
further we use a $p=1$ disretization giving 3 modes. Thus we
write out the equations as follows.

Let $(\ )'$ denote $\frac{d(\ )}{dt}$ and $\overline{(\ )} = \iint_{\Omega_e} (\ ) \, d\Omega_e$

$\boxed{1}$ $\overline{\phi_1 \phi_1} \hat{\rho}_1' + \overline{\phi_1 \phi_2} \hat{\rho}_2 + \overline{\phi_1 \phi_3} \hat{\rho}_3 + R_{V\rho}(1) - R_{S\rho}(1) = 0$

$\boxed{2}$ $\overline{\phi_1 \phi_1} \hat{\rho u}_1' + \overline{\phi_1 \phi_2} \hat{\rho u}_2 \ \overline{\phi_1 \phi_3} \hat{\rho u}_3 + R_{V\rho u}(1) - R_{S\rho u}(1) = 0$

$\boxed{3}$ $\overline{\phi_2 \phi_1} \hat{\rho}_1' + \overline{\phi_2 \phi_2} \hat{\rho}_2' + \overline{\phi_2 \phi_3} \hat{\rho}_3' + R_{V\rho}(2) - R_{S\rho}(2) = 0$

$\boxed{4}$ $\overline{\phi_2 \phi_1} \hat{\rho u}_1' + \overline{\phi_2 \phi_2} \hat{\rho u}_2' + \overline{\phi_2 \phi_3} \hat{\rho u}_3' + R_{V\rho u}(2) - R_{S\rho u}(2) = 0$

$\boxed{5}$ $\overline{\phi_3 \phi_1} \hat{\rho}_1' + \overline{\phi_3 \phi_2} \hat{\rho}_2' + \overline{\phi_3 \phi_3} \hat{\rho}_3' + R_\rho(3) - R_{S\rho}(3) = 0$

$\boxed{6}$ $\overline{\phi_3 \phi_1} \hat{\rho u}_1' + \overline{\phi_3 \phi_2} \hat{\rho u}_2' + \overline{\phi_3 \phi_3} \hat{\rho u}_3' + R_{\rho u}(3) - R_{S\rho u}(3) = 0$

We can write this in matrix form for $\hat{\rho}_j', \hat{\rho u}_j'$

$$
\begin{bmatrix}
\overline{\phi_1\phi_1} & \overline{\phi_1\phi_2} & \overline{\phi_1\phi_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{\phi_1\phi_1} & \overline{\phi_{12}} & \overline{\phi_{13}} \\
\overline{\phi_2\phi_1} & \overline{\phi_2\phi_2} & \overline{\phi_2\phi_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{\phi_2\phi_1} & \overline{\phi_2\phi_2} & \overline{\phi_2\phi_3} \\
\overline{\phi_3\phi_1} & \overline{\phi_3\phi_2} & \overline{\phi_3\phi_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{\phi_3\phi_1} & \overline{\phi_3\phi_2} & \overline{\phi_3\phi_3}
\end{bmatrix}
\begin{Bmatrix}
\hat{\rho}_1' \\ \hat{\rho}_2' \\ \hat{\rho}_3' \\ \hat{\rho u}_1' \\ \hat{\rho u}_2' \\ \hat{\rho u}_3'
\end{Bmatrix}
=
\begin{Bmatrix}
-R_1 = R_\rho(1) \\ -R_2 = R_{\rho u}(1) \\ -R_3 = R_\rho(2) \\ -R_4 = R_{\rho u}(2) \\ -R_5 = R_\rho(3) \\ -R_6 = R_{\rho u}(3)
\end{Bmatrix}
$$

Careful examination shows no coupling between the $\hat{\rho}_j'$, $\hat{\rho u}_j'$ ~~the~~ further it we switch the following rows. $2 \leftrightarrow 3$, then $3 \leftrightarrow 5$, then $4 \leftrightarrow 5$, give

$$
\begin{bmatrix}
\overline{\phi_1\phi_1} & \overline{\phi_1\phi_2} & \overline{\phi_1\phi_3} & 0 & 0 & 0 \\
\overline{\phi_2\phi_1} & \overline{\phi_2\phi_2} & \overline{\phi_2\phi_3} & 0 & 0 & 0 \\
\overline{\phi_3\phi_1} & \overline{\phi_3\phi_2} & \overline{\phi_3\phi_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{\phi_1\phi_1} & \overline{\phi_1\phi_2} & \overline{\phi_1\phi_2} \\
0 & 0 & 0 & \overline{\phi_2\phi_1} & \overline{\phi_2\phi_2} & \overline{\phi_2\phi_3} \\
0 & 0 & 0 & \overline{\phi_3\phi_1} & \overline{\phi_3\phi_2} & \overline{\phi_3\phi_3}
\end{bmatrix}
\begin{Bmatrix}
\hat{\rho}_1' \\ \hat{\rho}_2' \\ \hat{\rho}_3' \\ \hat{\rho u}_1' \\ \hat{\rho u}_2' \\ \hat{\rho u}_3'
\end{Bmatrix}
=
\begin{Bmatrix}
-R_\rho(1) \\ -R_\rho(2) \\ -R_\rho(3) \\ -R_{\rho u}(1) \\ -R_{\rho u}(2) \\ -R_{\rho u}(3)
\end{Bmatrix}
$$

It is easily seen that this is the matrix implied by selecting option 2). Thus we use option 2 b/c now we need on store M_{ij} as nmode x nmode and then use it on each set of unknowns.

We use the standard Gauss Quadrature mentioned for the wave equation.

$$M_{ij} = \iint_{\Omega_e} \phi_i \phi_j \, d\Omega_e = \int_{-1}^{1}\int_{-1}^{1} \phi_i(\xi,\eta)\,\phi_j(\xi,\eta)\,|J|\,d\eta\,d\xi \approx \sum_{k=1}^{N_{GP}} \phi_i(\xi_k,\eta_k)\cdot \phi_j(\xi_k,\eta_k)\,w(k)\,|J(\xi_k,\eta_k)|$$

Algorithm Remark:

If the mapping is linear $|J|$ is not a fn of ξ_k, η_k thus we can pull it out. Thus we only store a prototype $[M]$ and for each element in the solver we'll multiply by $|J|$ for that element's nodes. For Boundary elements we'll need to make a special $[M]$ on the fly. Storing this thing is very memory intensive.

Volume Residual (R_v):

given by

$$\iint_{\Omega_e} [\vec{F}, \vec{G}] \begin{Bmatrix} \frac{\partial \phi_i}{\partial x} \\ \frac{\partial \phi_i}{\partial y} \end{Bmatrix} d\Omega_e =$$

Again we use our normal standard element with the mapping based on the basis fns.

$$= \int_{-1}^{1}\int_{-1}^{1} [\vec{F}(\vec{q}), \vec{G}(\vec{q})]\,[J]^{-1} \begin{Bmatrix} \frac{\partial \phi_i}{\partial \xi} \\ \frac{\partial \phi_i}{\partial \eta} \end{Bmatrix} |J|\,d\eta\,d\xi \approx$$

$$\sum_{k=1}^{N_{GP}} \left[\vec{F}(\vec{q}(\xi_k,\eta_k)), \vec{G}(\vec{q}(\xi_k,\eta_k))\right] \begin{Bmatrix} J_{11}^{-1}(\xi_k,\eta_k)\,\frac{\partial \phi_i}{\partial \xi}\Big|_{\xi_k,\eta_k} + J_{12}^{-1}(\xi_k,\eta_k)\cdot\frac{\partial \phi_i}{\partial \eta}\Big|_{\xi_k,\eta_k} \\ J_{21}^{-1}(\xi_k,\eta_k)\,\frac{\partial \phi_i}{\partial \xi}\Big|_{\xi_k,\eta_k} + J_{22}^{-1}(\xi_k,\eta_k)\cdot\frac{\partial \phi_i}{\partial \eta}\Big|_{\xi_k,\eta_k} \end{Bmatrix} |J|\,w(k)$$

We'll formally write out all 4 residual fields for the Mode i.

$$R^i_{V\rho} = \sum_{K=1}^{N_{QP}} \rho u(\xi_k, \eta_k) \cdot \left(J_{11}^{-1}(\xi_k,\eta_k) \cdot \frac{\partial \phi_i}{\partial \xi} + J_{12}^{-1}(\xi_k,\eta_k) \frac{\partial \phi_i}{\partial \eta} \right) + \rho v(\xi_k, \eta_k) \left(J_{21}^{-1}(\xi_k,\eta_k) \frac{\partial \phi_i}{\partial \xi} + J_{22}^{-1} \frac{\partial \phi_i}{\partial \eta} \right)$$

$w(k) |J|$

$$R_{V\rho u} = \sum_{K=1}^{N_{QP}} \left(\rho u^2(\xi_k,\eta_k) + p(\xi_k,\eta_k) \right) \cdot \left(J_{11}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{12}^{-1} \frac{\partial \phi_i}{\partial \eta} \right) + \rho u v(\xi_k,\eta_k) \left(J_{21}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{22}^{-1} \frac{\partial \phi_i}{\partial \eta} \right)$$

$w(k) |J|$

$$R_{V\rho v} = \sum_{K=1}^{N_{QP}} \left(\rho u v(\xi_k,\eta_k) \cdot \left(J_{11}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{12}^{-1} \frac{\partial \phi_i}{\partial \eta} \right) + (\rho v^2 + p)\big|_{\xi_k \eta_k} \cdot \left(J_{21}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{22}^{-1} \frac{\partial \phi_i}{\partial \eta} \right) \right) w(k) |J|$$

$$R_{V\rho E} = \sum_{K=1}^{N_{QP}} (\rho E + p) u\big|_{\xi_k \eta_k} \cdot (\quad) + (\rho E + p) v\big|_{\xi_k, \eta_k} (\quad)$$

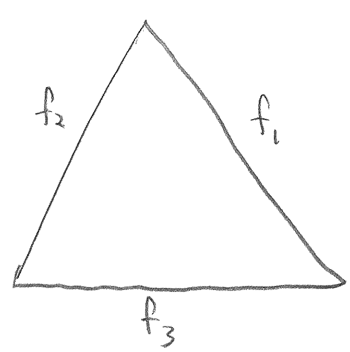
Surface Residual (R_S):

$\oint \phi_i \cdot \tilde{F}^* ds$ - where $\tilde{F}^* = \vec{F}^* \cdot \hat{n}$ - $\vec{F}^*$ is the numerical flux of which we have serveral options. See FVM Notes.

Given that we have a good $\tilde{F}^*$ then we can proceed with evaluating this integral.

Recall our topology of choice is triangle (or any finite # of sides shape e.g. quad, hexagon). Also recall

$$\tilde{F}^* = \tilde{F}^*(\vec{q}_L, \vec{q}_R)$$

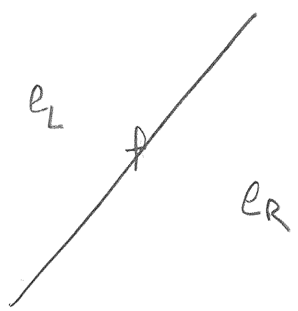


Thus we first break our integral into a Σ over the faces

$$\oint \phi_i \cdot \tilde{F}^* ds = \sum_{f=1}^{N_{face}} \int \phi_i \cdot \tilde{F}^* ds.$$ Thus we are still doing a line integral.

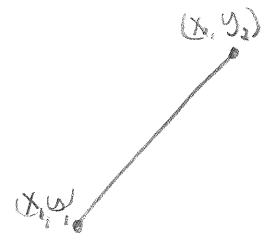
As with the wave equation it becomes immediately that we should loop over the faces. Thus we can do this line integral 1 face at a time

Single Face:



We want to evaluate $\int \phi_i \cdot \tilde{F}^* ds$. Since we are on a face we will evaluate the trace $(\phi_i)\big|_S$ and parametrize S with the appropriate mapping (linear if possible).

The mapping should take -1 —— 1 to



If we parametrize with ξ then we have $x(\xi) = x$, $y = y(\xi)$.

thus $ds = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2} \cdot d\xi$ giving

$$\int \phi_i \cdot \tilde{F}^* ds = \int_{-1}^{1} \text{trace}(\phi_i) \cdot \tilde{F}^*(g_L, g_R) \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2} \cdot d\xi$$

Note: For modes like bubble modes there is ~~no~~ the trace $(\phi_{bubble}) = 0$ for all edges (faces).

Recall that edge fns are simply 1D bubble fns thus we can do each side completely in 1D including evaluating q_L, q_R.

$$\int_{-1}^{1} \text{trace}(\phi_i) \cdot \tilde{\hat{F}}^*(q_L, q_R) \cdot \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}\, d\xi = \sum_{k=1}^{N_{gp}} \text{trace}(\phi_i(\xi_k)) \cdot \tilde{\hat{F}}^*(q_L(\xi_k), q_R(\xi_k))$$

$$\sqrt{\left(\frac{dx}{d\xi}\right)^2\Big|_{\xi_k} + \left(\frac{dy}{d\xi}\right)^2} \cdot w(k).$$

where $\vec{q}_L\big|_{\xi_k} = \sum_{j=1}^{N_{mode}} \hat{q}_{j e_L} \cdot \text{trace}(\phi_j)\Big|_{\xi_k}$ · Note Not all Modes contribute again we are on a specific edge.
similar for $\vec{q}_R$.

$$\vec{R}_S^i = \sum_{k=1}^{N_{gp}} \text{trace}(\phi_i(\xi_k)) \cdot \tilde{\hat{F}}^*(q_L(\xi_k), q_R(\xi_k)) \cdot \sqrt{\left(\frac{dx}{d\xi}\right)^2\Big|_{\xi_k} + \left(\frac{dy}{d\xi}\right)^2}\; w(k)$$

As an example Mapping consider using the 1D linear Lobatto fns ℓ_0, ℓ_1 thus $\ell_0 = \frac{(1-\xi)}{2}$ $\ell_1 = \frac{(1+\xi)}{2}$

$$x = \ell_0 \cdot x_{n_1} + \ell_1 x_{n_2} \qquad \frac{dx}{d\xi} = \frac{d\ell_0}{d\xi} x_{n_1} + \frac{d\ell_1}{d\xi} \cdot x_{n_2}$$

$$y = \ell_0 \cdot y_{n_1} + \ell_1 y_{n_2} \qquad \frac{dy}{d\xi} = \frac{d\ell_0}{d\xi} y_{n_1} + \frac{d\ell_1}{d\xi} y_{n_2}$$

But really anything will work, you can set curved sides using higher order 1D Lobatto fns.

We now have the following discretized equation

$$[M] \frac{\partial \vec{\hat{q}}}{\partial t} - \vec{R}_V + R_S = 0.$$ where we have expressions for R_V, R_S.

? What do we do with time ?

1) Explicit $[M]\left(\frac{\hat{q}^{n+1}-\hat{q}^{n}}{\Delta t}\right) - \vec{R}_v^{\,n} + \vec{R}_s^{\,n} = 0$

2) Implicit $[M]\left(\frac{\hat{q}^{n+1}-\hat{q}^{n}}{\Delta t}\right) - \vec{R}_v^{\,n+1} + \vec{R}_s^{\,n+1} = 0$

for S.S. $-\vec{R}_v^{\,n+1} + \vec{R}_s^{\,n+1} = 0$

For right now we'll use 4th order explicit Rung-Kutta. But one can see that once the residuals are formed there is virtually no-difference between this and F.V.M in the sense of getting an implicit or explicit scheme. The key difference is that we have to solve not just nfields x nnodes x nelem equations

nfields x nelem.

RK4:

$$[M]\,\hat{q}^{(2)} = .25\Delta t\left(\vec{R}_v^{(n)} - \vec{R}_s^{(n)}\right) + [M]\,\hat{q}^{(n)}$$

$$[M]\,\hat{q}^{(3)} = \tfrac{1}{3}\Delta t\left(\vec{R}_v^{(2)} + \vec{R}_s^{(2)}\right) + [M]\,\hat{q}^{(n)}$$

$$[M]\,\hat{q}^{(4)} = \tfrac{1}{2}\Delta t\left(\vec{R}_v^{(3)} + \vec{R}_s^{(3)}\right) + [M]\,\hat{q}^{(n)}$$

$$[M]\,\hat{q}^{(n+1)} = \Delta t\left(\vec{R}_v^{(4)} + \vec{R}_s^{(3)}\right) + [M]\,\hat{q}^{(n)}$$

The implicit schemes are much harder to write down and we'll defer them to their own section.

Diffusion- Advection Problem :

bounded by Γ

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = \nabla^2 u$$

using DG with weight ϕ_i

$$\iint_\Omega \phi_i \frac{\partial u}{\partial t} + \underbrace{\phi_i a \frac{\partial u}{\partial x} + \phi_i b\frac{\partial u}{\partial y}}_{2} - \underbrace{\phi_i\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\}}_{③} d\Omega = 0$$

Integration by parts of 2 gives

$$\iint_\Omega \phi_i a \frac{\partial u}{\partial x} = \oint_\Gamma \phi_i \cdot (au\hat{i} + bu\hat{j})\cdot\vec{n}\, d\Gamma - \iint_\Omega \left(\frac{\partial \phi_i}{\partial x}\hat{i} + \frac{\partial \phi_i}{\partial y}\hat{j}\right)\cdot(au\hat{i} + bu\hat{j})\, d\Omega$$

Integration by parts of ③ gives

$$\int_\Omega \phi_i \nabla^2 u\, d\Omega = \int_\Omega \phi_i \vec{\nabla}\cdot\nabla(u) = \oint_\Gamma \phi_i \vec{\nabla}(u)\cdot\vec{n}\, d\Gamma - \iint \vec{\nabla}(\phi_i)\cdot\nabla u\, d\Omega$$

Thus giving the following weak form (I think) check by parts formula.

$$\iint_\Omega \phi_i \frac{\partial u}{\partial t} - \frac{\partial \phi_i}{\partial x}\cdot au - \frac{\partial \phi_i}{\partial y}\cdot bu + \nabla(\phi_i)\cdot\nabla(u)\, d\Omega = -\oint_\Gamma \phi_i (au\hat{i} + bu\hat{j})\cdot\vec{n}\, d\Gamma$$

$$+ \underbrace{\oint_\Gamma \phi_i \vec{\nabla}(u)\cdot\vec{n}\, d\Gamma}_{\text{Tricky part}}$$

Interior Penalty Methods for Diffusion:

$-\nabla^2 u = f$ $\qquad$ v = test fn

$$-\int_{\Omega_e} \phi \nabla^2 u = \int_{\Omega_e} \phi f \, d\Omega \quad (1)$$

$$\int_{\Omega_e} \nabla u \cdot \nabla \phi \, d\Omega - \int_{\partial\Omega_e} \nabla u \cdot n \, \phi \, d\partial\Omega = \int \phi f \, d\Omega_e$$ - Replace the $\int_{\partial\Omega}$ with Numerical Flux (2)

$\partial\Omega_e \rightarrow$ Boundary / Numerical Flux

$$\int_{\Omega_e} \nabla u \cdot \nabla \phi \, d\Omega_e - \int \frac{(\nabla u^+ \cdot n^+ + \nabla u^- \cdot n^-)}{2} \phi \, d\partial\Omega_e = \int_{\Omega_e} f \phi \, d\Omega_e \quad (3)$$

The above is not elliptic. (3)

an operator is elliptic if $a(u,u) > 0$

we want to add a term to make it elliptic

given

$$\int_{\Omega_e} \nabla u \cdot \nabla \phi \, d\Omega - \int_{\partial\Omega_e} \frac{(\nabla u^+ \cdot n^+ + \nabla u^- \cdot n^+)}{2} \phi \, d\partial\Omega + \int_{\partial\Omega_e} \mu (u^+ - u^-) \phi \, d\Omega$$

$$= \int_{\Omega} f \phi \, d\Omega \quad (4)$$

If this is valid on a piece of the Domain Ω then we write it for the whole Domain Ω_{int}.

$$\int_{\Omega_{int}} \nabla u \cdot \nabla \phi - \int_{\Gamma_I} \frac{\nabla u^+ + \nabla u^-}{2} (\phi^+ n^+ + \phi^- n^-) + \int_{\Gamma_I} \mu (u^+ n^+ + u^- n^-)(\phi^+ n^+ + \phi^- n^-) = \int_{\Omega} f \cdot \phi \, d\Omega$$

$\Gamma_I \uparrow$ Interior edges

$$\int_\Omega \nabla u \cdot \nabla \phi - \int_{\Gamma_I} \{\nabla u\}^{\text{avg}} \cdot [\phi]^{\text{Jum}}\, ds + \int \mu [u][\phi]\, ds = \int f \phi$$

If μ is sufficiently Large then we can get $a(u,u)$. We also want to retain the symmetry of $a(u,u)$ so we add a term

$$\int_{\partial\Omega_e} \frac{(\nabla\phi^+ \cdot n^+ + \nabla\phi^- \cdot n^-)}{2}(u^+ - u^-)\, d\Omega$$ which globall is written

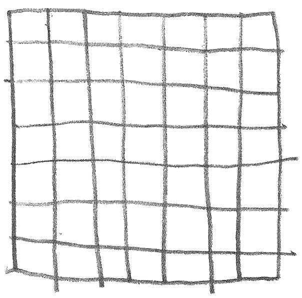
$$\int_{\Gamma_I} \{\nabla\phi\} \cdot [u]\, d\Gamma_I$$

How to output Results:

Most plotting programs do not have higher order methods for output. thus we must find a way to trick them.

1). Treat element as a single zone and do the following.

a) Define an evenly spaced grid on a quad

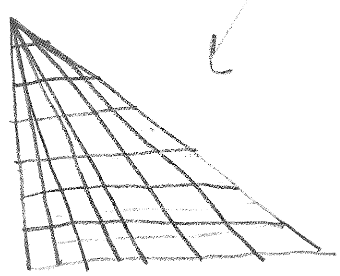


b) Map this into a triangle as

$$x = \frac{(\xi+1)(\eta+1)}{2}$$

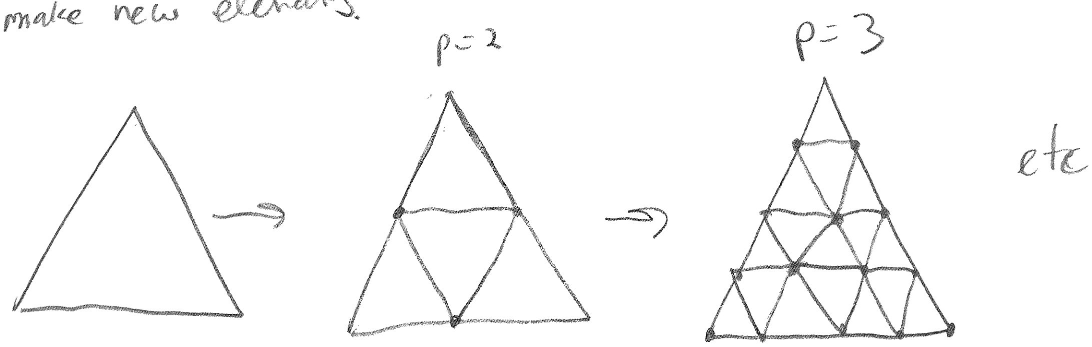
$$y = \eta$$

gives



Remark: As a guideline use $p+1$ points on 3,2 sides of the quad and make a cartesian grid. Each zone is ordered.

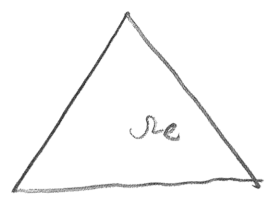
2). take Karthik's idea and do midpoint subdivision to make new elements.



Timestep Estimation:

For explicit schemes (and damped implicit schemes) we need to have well form way of computing Δt.

Essentially we have U, V on Ωe



If for given face we define $\bar{U} = (U_L + U_R)\frac{1}{2}$, $\bar{V} = (V_L + U_R)\frac{1}{2}$, $\bar{a} = (a_L + a_R)\frac{1}{2}$, then one way to define the CFL # is

$$CFL = \frac{\left(\left|\oint (\bar{U} n_x + \bar{V} n_y)\, ds\right| + \oint a\, ds\right) \cdot \Delta t}{A}$$

giving

$$\Delta t = \frac{CFL \cdot A}{\left(\left|\oint (\bar{U} n_x + \bar{V} n_y)\, ds\right| + \oint a\, ds\right)}$$

For finite volumes this is rather trivial, but for DG we are essentially evaluating an additional surface term, the faces

we do this in a loop over

$$\Delta t = \frac{CFL \cdot A}{\sum_{f=1}^{N_{face}} \left| \sum_{k=1}^{N_g} (\bar{U}(\xi_k) n_x + \bar{V}(\xi_k) n_y) \omega(k) \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2} \right| + \sum_{k=1}^{N_g} \bar{a}(\xi_k)\, \omega(k) \cdot \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{ds}{d\xi}\right)^2}}$$

Isentropic Vortex Test Case:

To test a DG scheme one must ensure that the Higher order method is in fact functioning. The isentropic vortex is a vortex that satisfies the isentropic thermodynamic relationship. It is prescribed as a perturbation from the free stream. It should be noted that these relations specifically δT are sensitive to the non-dimensionalization scheme. The ones prescribed below work for my scheme where $\bar{P}_\infty = \frac{1}{\gamma}$, $T_\infty = 1$. Specifically we remove a $\frac{1}{\gamma}$ from δT.

Isentropic Vortex:

$$f = \frac{\beta}{2\pi} e^{\phi(1-r^2)}$$

$$r^2 = (x - x_0)^2 + (y - y_0)^2$$

$$\delta \bar{u} = -(y - y_0) f$$

$$\delta \bar{v} = (x - x_0) f$$

$$\delta \bar{T} = -\frac{(\gamma-1)\beta^2}{8\pi^2} e^{2\phi(1-r^2)}$$

giving the

$$\bar{U} = U_\infty + \delta u = M_\infty \cos(\alpha) + \delta \bar{u}$$

$$\bar{V} = V_\infty + \delta v = M_\infty \sin(\alpha) + \delta \bar{v}$$

$$\bar{T} = \bar{T}_\infty + \delta \bar{T} = 1 - \frac{(\gamma-1)\beta^2}{8\pi^2} e^{2\phi(1-r^2)}$$

using thermo

$$\bar{\rho} = (\bar{T})^{1/(\gamma-1)} = \left[1 - \frac{(\gamma-1)\beta^2}{8\pi^2} e^{2\phi(1-r^2)}\right]^{\frac{1}{\gamma-1}}$$

$$q(1) = \bar{\rho}$$

$$q(2) = \bar{\rho}\bar{U}$$

$$q(3) = \bar{\rho}\bar{V}$$

$$\bar{P} = \frac{1}{\gamma}(\bar{\rho})^{\gamma}$$

$$q(4) = \frac{\bar{P}}{\gamma-1} + \frac{1}{2}\bar{\rho}(\bar{U}^2 + \bar{V}^2)$$

This directly prescribe the q variables for projection onto the Modal coefficients

Implicit Methods:

Up to now only explicit schemes have been considered. Below we will out line the formulation for steady state and time-accurate solution methods. Recall the semi-discrete form of the Euler Equations.

$$[M]\frac{d\hat{q}}{dt} - \vec{R}_v(\hat{q}) + \vec{R}_s(\hat{q}) = 0 \qquad (1)$$

1). For steady state drop $\frac{d\hat{q}}{dt}$ gives

$$-\vec{R}_v(\hat{q}) + \vec{R}_s(\hat{q}) = 0$$

denote $-\vec{R}_v(\hat{q}) + \vec{R}_s(\hat{q}) = \vec{R}_f(\hat{q})$ for flow Residual then solution Method.

We can use this to form a Newton's Method

$$\vec{R}_f(\hat{q}) = 0 \quad \rightarrow \quad \left[\frac{\partial R_f}{\partial \hat{q}^K}\right]\Bigg|_{\hat{q}^K} \delta\hat{q}^{K+1} = -\vec{R}_f(\hat{q}_f^K) \qquad (2)$$

So far we have all the pieces except for the Jacobian, $\left[\frac{\partial R_f}{\partial \hat{q}^K}\right]$. From the expression for $\vec{R}_f$ it is easily seen that we can break this computation up into

$$\left[\frac{\partial \vec{R}_f}{\partial \hat{q}^K}\right] = \left[\underbrace{\frac{\partial \vec{R}_v}{\partial \hat{q}^K}}_{\text{Denote as Volume Jac - Diagonal only}} + \overbrace{\frac{\partial \vec{R}_s}{\partial \hat{q}^K}}^{\text{Denote as Surface Jac - Diagonal + OD.}}\right] \qquad (3)$$

Volume Jac:

$\frac{\partial \vec{R}_v}{\partial \hat{\vec{q}}}$ - This can be most easily evaluated using the follow chain rule.

$$\left[\frac{\partial R_v}{\partial \hat{\vec{q}}}\right] = \frac{\partial R_v}{\partial \vec{q}} \cdot \frac{\partial \vec{q}}{\partial \hat{\vec{q}}} \qquad (4)$$

The 1st term is simply the 1st order F.V. Jacobian of the Native flux. Thus we can construct this in a 2 part proceedure. We have previously in F.V.M Notes derived for the Jacobia written out it's very long.

$\frac{\partial R_v}{\partial \vec{q}}$ see FVM Notes

$$\frac{\partial R_v}{\partial \vec{q}} \cdot \frac{\partial \vec{q}}{\partial \hat{\vec{q}}} \nearrow 4 \times N_{modes} \cdot 4$$

$$\vec{q}_1 = \rho_0 = \sum_{j=1}^{N_{mode}} \hat{\rho}_j \phi_j \qquad \frac{\partial \rho}{\partial \hat{\vec{q}}} = \left[\phi_1 \quad \phi_2 \quad \phi_3 \quad \cdots \quad 0 \quad \cdots \quad 0\right]$$

$$\frac{\partial \rho u}{\partial \hat{\vec{q}}} = \sum_{j=1}^{N_{mode}} (\hat{\rho u})_j \phi_j = \left[\underbrace{0 \ldots 0}_{N_{mode}\text{ wid}} \quad \phi_1 \quad \phi_2 \quad \phi_3 \cdots \underbrace{0 \ldots \ldots 0}_{2 \times N_{mode}}\right] \qquad (6)$$

$$\frac{\partial \rho v}{\partial \hat{\vec{q}}} = \sum_{j=1}^{N_{mode}} (\hat{\rho v})_j \phi_j = \left[\underbrace{0 \ldots 0 \ldots 0}_{N_{mode} \times 2} \quad \phi_1 \quad \phi_2 \quad \phi_3 \ldots 0 \ldots 0\right] \qquad (7)$$

$$\frac{\partial \rho E}{\partial \hat{\vec{q}}} = \sum_{j=1}^{N_{mode}} (\hat{\rho E})_j \phi_j = \left[\underbrace{0 \ldots 0}_{3 \times N_{mode}\text{ wide}} \quad \phi_1 \quad \phi_2 \quad \phi_3 \quad \cdots\right] \qquad (8)$$

$$\frac{\partial R_v}{\partial \hat{\vec{q}}} = \begin{bmatrix} \frac{\partial R^{(1,1)}}{\partial \vec{q}} \phi_1 & \frac{\partial R^{(1,1)}}{\partial \vec{q}} \phi_2 & \frac{\partial R^{(1,1)}}{\partial \vec{q}} \phi_3 & \frac{\partial R^{(1,1)}}{\partial \vec{q}} \phi_4 \cdots & \frac{\partial R^{(1,2)}}{\partial \vec{q}} \phi_1 & \frac{\partial R^{(1,2)}}{\partial \vec{q}} \phi_2 & \cdots \\ \\ \\ \end{bmatrix} \begin{Bmatrix} \delta \hat{\rho}_1 \\ \delta \hat{\rho}_2 \\ \delta \hat{\rho}_3 \\ \vdots \\ \delta \hat{\rho u}_1 \\ \delta \hat{\rho u}_2 \end{Bmatrix}$$

Recall that

$$R_v = \sum_{k=1}^{N_{gp}} \left[\vec{F}(\vec{q}(\xi_k, \eta_k)), \vec{G}(\xi_k, \eta_k)\right] \begin{Bmatrix} J_{11}^{-1}(\xi_k, \eta_k) \frac{\partial \phi_i}{\partial \xi_k} + J_{12}^{-1}(\xi_k, \eta_k) \frac{\partial \phi_i}{\partial \eta_k} \\ J_{21}^{-1}(\xi_k, \eta_k) \frac{\partial \phi_i}{\partial \xi}\Big|_{\xi_k, \eta_k} + J_{22}^{-1} \frac{\partial \phi_i}{\partial \eta}\Big|_{\xi_k, \eta_k} \end{Bmatrix} |J| w(k)$$

To make writing this a bit easier we'll define some terms

$$B_1 = J_{11}^{-1} \frac{\partial \phi_i}{\partial \xi_k} + J_{21}^{-1} \frac{\partial \phi_i}{\partial \eta}, \quad B_2 = J_{21}^{-1} \frac{\partial \phi_i}{\partial \xi} + J_{22}^{-1} \frac{\partial \phi_i}{\partial \eta}$$

$$R_v = \sum_{k=1}^{N_{gp}} \left(\vec{F}(q(\xi_k, \eta_k)) B_1(\xi_k, \eta_k) + \vec{G}(q(\xi_k, \eta_k)) B_2(\xi_k, \eta_k)\right) |J| w(k)$$

$$\frac{\partial R_v}{\partial \hat{q}} = \sum_{k=1}^{N_{gp}} \left(\frac{\partial \vec{F}}{\partial \vec{q}} \cdot \frac{\partial \vec{q}}{\partial \hat{q}} \cdot B_1(\xi_k, \eta_k) + \frac{\partial G}{\partial \vec{q}} \cdot \frac{\partial \vec{q}}{\partial \hat{q}} B_2(\xi_k, \eta_k)\right) |J| w(k)$$

where $\frac{\partial \vec{q}}{\partial \hat{\vec{q}}}$ is a given in 5, 6, 7, 8

Let's do the P^1 example: 3 modes

$$\hat{\vec{q}} = [\hat{\rho}_1 \; \hat{\rho}_2 \; \hat{\rho}_3 \; \hat{\rho u}_1 \; \hat{\rho u}_2 \; \hat{\rho u}_3 \; \hat{\rho v}_1 \; \hat{\rho v}_2 \; \hat{\rho v}_3 \; \hat{\rho E}_1 \; \hat{\rho E}_2 \; \hat{\rho E}_3$$

$$\frac{\partial \vec{q}}{\partial \hat{q}} = \begin{bmatrix} \frac{\partial \rho}{\partial \hat{\rho}_1} & \frac{\partial \rho}{\partial \hat{\rho}_2} & \frac{\partial \rho}{\partial \hat{\rho}_3} & \frac{\partial \rho}{\partial \hat{\rho u}_1} & \frac{\partial \rho}{\partial \hat{\rho u}_2} & \frac{\partial \rho}{\partial \hat{\rho u}_3} & \cdots \\ \frac{\partial \rho u}{\partial \hat{\rho}_1} & \frac{\partial \rho u}{\partial \hat{\rho}_2} & \frac{\partial \rho u}{\partial \hat{\rho}_3} & \frac{\partial \rho u}{\partial \hat{\rho u}_1} & & & \\ \vdots & \vdots & \vdots & \vdots & & & \end{bmatrix}$$

gives

$$\frac{\partial \vec{q}}{\partial \hat{q}} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{bmatrix}$$

We will need $\frac{\partial F}{\partial \vec{q}}$, $\frac{\partial G}{\partial \vec{q}}$ seperately

$$\frac{\partial F}{\partial \vec{q}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(\gamma-3)u^2+(\gamma-1)v^2}{2} & -(\gamma-3)u & -(\gamma-1)v & \gamma-1 \\ -uv & v & u & 0 \\ -u\left[\frac{a^2}{\gamma-1}+(\frac{\gamma}{2}-1)(u^2+v^2)\right] & \frac{a^2}{(\gamma-1)}+(\frac{3}{2}-\gamma)u^2+\frac{1}{2}v^2 & -uv(\gamma-1) & \gamma u \end{bmatrix}$$

$$\frac{\partial G}{\partial \vec{q}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ \frac{(\gamma-3)v^2+(\gamma-1)u^2}{2} & -u(\gamma-1) & -v(\gamma-3) & \gamma-1 \\ \frac{-va^2}{(\gamma-1)}+(\frac{\gamma}{2}-1)(u^2v+v^3) & -uv(\gamma-1) & \frac{a^2}{(\gamma-1)}+(\frac{3}{2}-\gamma)u^2+\frac{1}{2}v^2 & v\gamma \end{bmatrix}$$

Surface Jacobian

Recall $\vec{R}_s = \sum_{K=1}^{N_g} \text{trace}(\phi_i) \cdot \tilde{F}^*(q_L(\hat{q}_L), q_R(\hat{q}_R)) \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}\; w(K)$

Thus we will endup with both contributions to the Left and right cells. We have

$$\frac{\partial \vec{R}_s}{\partial \hat{q}_L}, \quad \frac{\partial \vec{R}_s}{\partial \hat{q}_R}$$

$$\frac{\partial \vec{R}_s}{\partial \hat{q}_L} = \sum_{K=1}^{N_g} \text{trace}(\phi_i) \cdot \left[\frac{\partial \tilde{F}^*}{\partial q_L}\right]\left[\frac{\partial q_L}{\partial \hat{q}_L}\right] \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}\; w(K)$$

$$\frac{\partial \vec{R}_s}{\partial \hat{q}_R} = \sum_{K=1}^{N_g} \text{trace}(\phi_i) \left[\frac{\partial \tilde{F}^*}{\partial q_R}\right]\left[\frac{\partial q_R}{\partial \hat{q}_R}\right] \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}\; w(K)$$

We already have $\frac{\partial \tilde{F}^*}{\partial q_L}, \frac{\partial \tilde{F}^*}{\partial q_R}$ from the previous F.V.M Notes, they depend on the flux f'ns in question.

Spectral Multi-grid (P- multigrid):

If we have a non-linear problem we can cast it (steady of time Accurate) as

$R_p(\vec{q}) = 0$. We the apply newton's method

$$\left[\frac{\partial R_p}{\partial \vec{q}}^K\right] \delta \vec{q}^{K+1} = -R^K$$

This becomes our linear problem to solve

Since we know that we have a matrix ~~that~~ whose size depends on the order p of the computation. we will treat the varying levels of p as MG "levels" though they do not represent grids.

Level $p = p_{max}$

$[A]\, \delta \vec{q}^{K+1} = -R^K$ - smooth get $\delta \tilde{q}^{K+1}$

$\delta \vec{q}^{K+1} = \delta \tilde{q} + \Delta \delta q$ gives

$[A](\delta \tilde{q} + \Delta \delta q) = -R$

written as operator

$L_h(\delta \tilde{q}_h + \Delta \delta q_h) = -R$

we want to acheive a correction $\Delta \delta q$ thus our equation is

$$L_h \Delta \delta q = -R - L_h(\delta \tilde{q}) = -r \quad \text{where}$$

$$r_h = L_h(\delta \tilde{q}_h) - (-R_h)$$

Restrict to coarse level.

$$L_H(\Delta \delta q_H) = -I_h^H(r_h)$$ where I_h^H is the restriction

Now smooth

$$L_H(\Delta\delta q_H) = -I_h^H(r_h)$$

Then
$\delta q^{K+1} = \delta\tilde{q} + \Delta\delta q$ where $\Delta\delta q = I_H^h \Delta\delta q_H$. I_H^h is prolongation operator.

Definition of Resctriction:

$I_h^H(r_h)$ - set all modes for current level $p = 0$
Basically pretend they are not in the Matrix or residual.

Prolongation:

for this operator $I_H^h(\Delta\delta q_H)$ we take $\delta\tilde{q}(1:Nmode_H) +$

$\Delta\delta q_H(1:Nmode_H)$

Essentially we ~~add~~ correct all Modes ~~from~~ in the level coarser $\delta\tilde{q}$ that were corrected in

Ex. $p=4 \to p=3$
Say we are on $p=4$ which would be our finest level we want to correct $\delta\tilde{q}$ then our correction would be only the $p=3$ subset of modes that are in our $p=4$ computation

Basically
$\delta\tilde{q}(7:10,:,e) = \Delta\delta q(7:10,:,e) + \tilde{\delta q}(7:10,:,e)$
- thus only $p3$ modes get changed

Spectral or p Multigrid:

C-GC type algorithm $\quad \hat{q} = \begin{Bmatrix} \hat{\rho}_1 \\ \vdots \\ \hat{\rho}_{E N_{mode}} \end{Bmatrix}$ - Vector of Modal Coefficients

Begin with $\left[\frac{\partial R_p}{\partial \hat{q}}\right]^n \delta \hat{q}_p^{n+1} = S_p - R_p(\hat{q}_p^n)$

Define $[J_p^n] = \left[\frac{\partial R_p}{\partial \hat{q}}\right]$, $\quad \hat{w}_p^K = \delta \hat{q}_p^K$, $\quad f_p^K = S_p^K - R_p(\hat{q}_p^n)$

1). Smooth

$$[J_p^n]\, \hat{w}_p^K = f_p^K$$

define $r_p^K = f_p^K - [J_p^n]\, \hat{\omega}_p^K$

2). Define RHS for coarse grid by

$$f_{p-1}^K = I_p^{p-1} r_p^K$$

Solve with I.C. $\Delta w_{p-1}^k = 0$

$$[J_{p-1}^K]\, \Delta \hat{\omega}_{p-1}^K = f_{p-1}^K$$

$\rightarrow$ where $[J_{p-1}^K]$ is subset of $[J_p^K]$

3) Prolong $\hat{\omega}_{p-1}^k$ Back to corre w_p^k

$$w_p^{K+1} = \hat{\omega}_p^K + I_{p-1}^p \Delta \hat{\omega}_{p-1}^K$$ gives b/c $w^k_p = \delta \hat{q}$

$$\delta \tilde{q}^{K+1} = \delta \hat{q}^K + I_{p-1}^p \Delta \hat{\omega}_{p-1}^K$$

This is Basic 2 grid stategy that can be applied recursively to go all the way to $p=1$. After which the restriction to $p=0$ is given as

$$q_h = \frac{1}{3} \sum_{i=1}^{3} \hat{q}^i_{p=1}, \qquad R_h(q_h) = \sum_{i=1}^{3} R_p(\hat{q}^i_{p=1})$$

The spectral restriction operator for restriction up to $p=1$ is to simply ommit the modes for the "fine" level.

e.g. If the fine level $= 4$ then we use only modes $1 \to 10$ for each field in the residual when we solve the coarse grid problem.

Let's write this out as an example

take $p_{fine} = 2$ $\quad p_{coarse} = 1$

Notation R_k^f denotes fine $f=$ field, $k=$ mode so

R_2^1 is residual of ρ for mode weight ϕ_2

thus

$$R_{p=2} = \left\{ \begin{array}{c} R_1^1 \\ R_2^1 \\ R_3^1 \\ R_4^1 \\ R_5^1 \\ R_6^1 \\ R_1^2 \\ R_2^2 \\ R_3^2 \\ R_4^2 \\ R_5^2 \\ R_6^2 \\ R_1^3 \\ R_2^3 \\ R_3^3 \\ R_4^3 \\ R_5^3 \\ R_6^3 \\ R_1^4 \\ R_2^4 \\ R_3^4 \\ R_4^4 \\ R_5^4 \\ R_6^4 \end{array} \right\} \to R_{p=1} = \left\{ \begin{array}{c} R_1^1 \\ R_2^1 \\ R_3^1 \\ R_1^2 \\ R_2^2 \\ R_3^2 \\ R_1^3 \\ R_2^3 \\ R_3^3 \\ R_1^4 \\ R_2^4 \\ R_3^4 \end{array} \right\}$$

? What is the operator I_p^{p-1} that gives us this.

for simplicity let's assume we have a 2 field ~~[illegible]~~ problem

then the first 3 modes simply carry over, which $\Rightarrow$ an identity operator is used, we can cast this as a rectangular matrix times the fine level residual.

for 2 fields it looks like

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0 \end{bmatrix} \begin{Bmatrix} R^1_1\\ R^1_2\\ R^1_3\\ R^1_4\\ R^1_5\\ R^1_6\\ R^2_1\\ R^2_2\\ R^2_3\\ R^2_4\\ R^2_5\\ R^2_6 \end{Bmatrix} =$$

$$\begin{Bmatrix} R^1_1\\ R^1_2\\ R^1_3\\ R^2_1\\ R^2_2\\ R^2_3 \end{Bmatrix}.$$

From the h-multi-grid we know that the prolongation should be transpose of restriction thus for our example we write out the prolongation.

$$I_{p=1}^{p=2} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \begin{Bmatrix} \hat{q}^1_1\\ \hat{q}^1_2\\ \hat{q}^1_3\\ \hat{q}^2_1\\ \hat{q}^2_2\\ \hat{q}^2_3 \end{Bmatrix} = \begin{Bmatrix} \hat{q}^1_1\\ \hat{q}^1_2\\ \hat{q}^1_3\\ 0\\ 0\\ 0\\ \hat{q}^2_1\\ \hat{q}^2_2\\ \hat{q}^2_3\\ 0\\ 0\\ 0 \end{Bmatrix}$$

Thus the prolongation is simply to use the $p-1$ correction in the update $w_p^{k+1} = w_p^k + I_{p-1}^{p} w_{p-1}^k$ → thus add only to the $p-1$ subset of the $p=2$ level.

The choice is not as instructive ~~the~~ with $p=1$, $p=2$, thus we will do another ~~e~~ example with $p=2$, $p=3$ with 2 fields again. We write the restriction.

R shift

$$I_p^{p-1} = \left[\begin{array}{ccccccc|ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \cdots\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\ 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\ 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\ 0 \\
0 & 0 & 0 & 1 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0\ \cdots & & & & & & & \\
0 & \cdots & & & & & 0 & 1 & 0 & \cdots & & & & 0 \\
0 & \cdots & & & & & 0 & 0 & 1 & 0 & \cdots & & & 0 \\
0 & \cdots & & & & & 0 & 0 & 0 & 1 & 0 & \cdots & & 0 \\
0 & \cdots & & & & & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0\ \cdots & 0 \\
0 & \cdots & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 1\ 0 \cdots & 0
\end{array}\right]
\begin{Bmatrix} R_1^1 \\ R_2^1 \\ R_3^1 \\ R_4^1 \\ R_5^1 \\ R_6^1 \\ R_7^1 \\ R_8^1 \\ R_9^1 \\ R_{10}^1 \\ R_1^2 \\ R_2^2 \\ R_3^2 \\ R_4^2 \\ R_5^2 \\ R_6^2 \\ R_7^2 \\ R_8^2 \\ R_9^2 \\ R_{10}^2 \end{Bmatrix}$$

$$= \left\{ \begin{array}{l} \left.\begin{array}{l} R_1^1 \\ R_2^1 \\ R_3^1 \\ R_4^1 \\ R_5^1 \\ R_6^1 \end{array}\right\} R^1 \\ \left.\begin{array}{l} R_1^2 \\ R_2^2 \\ R_3^2 \\ R_4^2 \\ R_5^2 \\ R_6^2 \end{array}\right\} R^2 \end{array}\right.$$

Taking the transpose of this operator

$$I_{p-1}^{p} = \left[\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & & & & & & \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & \\
\vdots & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & & \vdots & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & \vdots & 0 & 0 & 1 & 0 \\
 & & & & & & & & \vdots & 0 & 0 & 1 \\
0 & & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{Bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 \\ q_4^1 \\ q_5^1 \\ q_6^1 \\ q_1^2 \\ q_2^2 \\ q_3^2 \\ q_4^2 \\ q_5^2 \\ q_6^2 \end{Bmatrix}
=
\begin{Bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 \\ q_4^1 \\ q_5^1 \\ q_6^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ q_1^2 \\ q_2^2 \\ q_3^2 \\ q_4^2 \\ q_5^2 \\ q_6^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

This tells us that we add all lower rows of $p-1$ to p when we prolong

Due to the form of our prolongation operator there is a very nice low memory way to implement the method.

We seek corrections to $\delta q^{p-\text{Prime}}$ where w_p^k is smoothed δq^k.

$$\delta q_p^{k+1} = W_p^k + I_{p-1}^{p} \Delta w_{p-1}^{k+1}$$

If we use M.G. recursively $\Delta w_{p-1}^{k+1} = \Delta w_{p-1}^{k} + I_{p-2}^{p-1} \Delta w_{p-2}^{k+1}$

thus giving

$$\delta q_p^{k+1} = w_p^k + I_{p-1}^{p}\left(\Delta w_{p-1}^{k} + I_{p-2}^{p-1} \Delta w_{p-2}^{k+1} \right)$$

$$\Delta w_{p-2}^{k+1} = \Delta w_{p-2}^{k} + I_{p-3}^{p-2} \Delta w_{p-3}^{k+1} \quad \text{give}$$

$$\delta q_p^{k+1} = w_p^k + I_{p-1}^{p}\left(\Delta w_{p-1}^{k} + I_{p-2}^{p-1}\left(\Delta w_{p-2}^{k} + I_{p-3}^{p-2} \Delta w_{p-3}^{k+1} \right)\right)$$

Due to the simple nature of our prolongation operation this means the follow

$$\delta q_p^{k+1} = w_p^k + I_{p-1}^{p} \Delta w_{p-1}^{k} + I_{p-1}^{p} I_{p-2}^{p-1} \Delta w_{p-2}^{k} + I_{p-1}^{p} I_{p-2}^{p-1} I_{p-3}^{p-2} \Delta w_{p-3}^{k+1}$$

Basically as we smooth we can the correction back to the initial fine grid to correct it, thus negating storage

Aliasing an instability: Hesthaven Wharburton: Do this to understand Filter.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(a(x)u) = 0$$ if we apply an interpolation operator to our nodal DG I_N flux we have

$$f_h^K(x,t) = I_N(a(x)\,U_h^K(x,t)) = \sum_{i=1}^{N_p} a(x_i)\cdot U_h^k(x_i,t)\, l_i(x)$$

- the "error" here is that $a(x)$ is sampled only at x_i nodal points this is not technically correct because they should be projecting the solution to each point using $f_h^K(x) = P_N(a(x)U_h^k(x))$ thus at each x $f_h^K = \sum_{i=1}^{N_p} a(x)\cdot \underbrace{U^K(x_i)}_{\text{Nodal coefficient}}\cdot l_i(x)$ - true projection based interpolation. They want to get around this as it is expensive.

Note: We don't have this problem as for each point in physical space we desire our $f(u)$ the $f(u) \to f(u_h)$ we do $u_h = \sum_{j=1}^{N_m} \hat{u}_j \phi_j(x)$

Consider $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(a(x)u) = 0$$

Restrict our Attention to $I_N(a(x)u)$ as flux essentially are aliased Interpolated flux

$$\frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x}(I_N(a u_h)) = 0$$

re-write

$$\frac{\partial u_h}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}[I_N(a u_h)] + \frac{1}{2} I_N\left(a\frac{\partial u_h}{\partial x}\right) +$$

$$+\frac{1}{2} I_N \frac{\partial}{\partial x} a u_h - \frac{1}{2} I_N\left(a \frac{\partial u_h}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial x}(I_N(a u_h)) - \frac{1}{2} I_N \frac{\partial}{\partial x}(a u_h)$$

Simplified ∞

$$\frac{\partial u_h}{\partial t} + N_1 + N_2 + N_3 = 0$$

$$N_1 = \frac{1}{2}\left[\frac{\partial}{\partial x}\left[I_N(a u_h)\right] + I_N\left(a \frac{\partial u_h}{\partial x}\right)\right]$$

$$N_2 = \frac{1}{2}\left[I_N \frac{\partial}{\partial x} a u_h - I_N\left(a \frac{\partial u_h}{\partial x}\right)\right]$$

$$N_3 = \frac{1}{2}\left[\frac{\partial}{\partial x}\left[I_N(a u_h)\right] - I_N \frac{\partial (a u_h)}{\partial x}\right]$$

Apply local DG scheme to this domain

$$\int_{-1}^{1} l_i(x)\left[\frac{\partial u_h}{\partial t} + N_1 + N_2 + N_3\right] dx = \frac{1}{2}\oint_{-1}^{1} \hat{n}\cdot[[I_N(a u_h)]]\, l_i(x)\, dx$$

gives

coeff, Inner prod

$$\sum_{j=1}^{N_p} M_{ij}\frac{\partial u_j}{\partial t} + (N_1, l_i)_\Omega + (N_2, l_i) + (N_3, l_i) =$$

$$\frac{1}{2}\oint_{-1}^{1} \hat{n}[[I_N(a u_h)]]\, l_i(x)\, dx$$

to create an energy term multiply by $u_h(x_i)$ from left.

$$u_h(x_i)\sum_{j=1}^{N_p} M_{ij}\frac{\partial u_j}{\partial t} + u_h(x_i)\cdot(N_1, l_i) + u_h(x_i)(N_2, l_i) +$$

$$u_h(x_i)(N_3, l_i) = u_h(x_i)\cdot\frac{1}{2}\oint_{-1}^{1}\hat{n}[[I_N(a u_h)]]\, l_i\, dx$$

$\sum_{i=1}^{N_p}$ gives - recall $u_h(x_i) = u_i$ - Nodal coefficient

$$\sum_{i=1}^{N_p}\left[u_i \sum_{j=1}^{N_p}\int_{-1}^{1} l_i(x)\, l_j(x)\frac{\partial u_j}{\partial t} + u_i(N_1, l_i) + u_i(N_2, l_i) + u_i(N_3, l_i) = \frac{1}{2}u_i\oint_{-1}^{1}[[I_N(a u_h)]]\, l_i\, dx \right.$$

The Mass term can be written as

$$\int_{-1}^{1} \sum_{i=1}^{N_p} u_i l_i(x) \sum_{j=1}^{N_p} u_j l_j(x)\, dx$$ b/c u_i, u_j are consts as far as integration is concerned gives

$$\int_{-1}^{1} u_h \cdot u_h \, dx = \frac{1}{2} \| u_h \|_{\Omega}^2$$ - on this domain ("element"). thus the Mass term is

2-Norm square

$$\frac{1}{2} \frac{d}{dt} \| u_h \|_{\Omega}^{2} = \sum_{i=1}^{N_p} u_i M_{ij} \frac{\partial u_j}{\partial t}$$

Further consider terms like.

$$\sum_{i=1}^{N_p} u_i \cdot (N_1, l_i) = (N_1, u_h)$$ b/c $(N_1, u_h) = \int_{-1}^{1} N_1 \sum_{i=1}^{N_p} u_i l_i(x)\, dx$

N_1 is independent of the $\sum_{i=1}^{N_p}$ gives for

our equation

$$\frac{d}{dt} \| u_h \|_{\Omega}^2 = -(N_1, u_h) - (N_2, u_h) - (N_3, u_h) + \frac{1}{2} \oint_{-1}^{1} [[I_N(a u)]] \cancel{\sum_{i}^{N_p}} \ldots$$

$$\sum_{i=1}^{N_p} u_i l_i(x)\, dx \rightarrow$$

$$\frac{d}{dt} \| u_h \|_{\Omega}^2 = -(N_1, u_h) - (N_2, u_h) - (N_3, u_h) + \frac{1}{2} \oint_{-1}^{1} [[I_N(a u)]]\, u_h\, dx$$

We are looking for terms that are bounded by something that depends on N - order of approx.

We want to consider this term by term:

A: $-a(N_1, U_h) + \oint_{-1}^{1} \hat{n}\cdot[[I_N(aU_h)]]\, U_h\, dx =$

Recall that through integration by parts

$$\int_{-1}^{1} U_h \frac{\partial}{\partial x}(I_N(aU_h))\,dx = -\int_{-1}^{1} \frac{\partial U_h}{\partial x} I_N(aU_h)\,dx + \oint_{-1}^{1} \hat{n}[[I_N(aU_h)]]\,U_h\,dx$$

gives

$$= -a(N_1, U_h) + \int_{-1}^{1} U_h \frac{\partial}{\partial x}(I_N(aU_h))\,dx + \int_{-1}^{1} \frac{\partial U_h}{\partial x} I_N(aU_h)\,dx \rightarrow$$

$$\cancel{-2\tfrac{1}{2}\int_{-1}^{1} \frac{\partial}{\partial x}[I_N(aU_h)]U_h\,dx} - 2\tfrac{1}{2}\int_{-1}^{1} I_N\left(a\frac{\partial U_h}{\partial x}\right) U_h\,dx + \cancel{\int_{-1}^{1} U_h \frac{\partial}{\partial x}(I_N(aU_h))\,dx} +$$

$$\int_{-1}^{1} \frac{\partial U_h}{\partial x} I_N(aU_h)\,dx \rightarrow$$

$$= -\left(I_N\left(a\frac{\partial U_h}{\partial x}\right), U_h\right) + \left(\frac{\partial U_h}{\partial x}, I_N(aU_h)\right)$$

Interpolation points are not affected by $\frac{\partial}{\partial x}$ or multiplication

$U_h I_N\left(a\frac{\partial U_h}{\partial x}\right) = I_N(U_h a U_h)$ - b/c a = const.

Further the L_2 inner product is symmetric gives $\rightarrow$

$$= -\int_{-1}^{1} U_h I_N\left(a\frac{\partial U_h}{\partial x}\right) dx + \int_{-1}^{1} \frac{\partial U_h}{\partial x} I_N(aU_h)\,dx$$

$$= -\int_{-1}^{1} I_N\left(U_h \cdot a\frac{\partial U_h}{\partial x}\right) dx + \int_{-1}^{1} I_N\left(\frac{\partial U_h}{\partial x} aU_h\right) dx = 0$$

This is seen by writing it out

from polynomial approx. theory

$$\|v - I_N v\|_\Omega^2 \leq C \frac{h^{\sigma-1}}{N^{p-1}} |v|_{\Omega,p} \quad \text{gives}$$

$$\|I_N \frac{du}{dx} - \frac{d}{dx} I_N v\|^2 \leq \|\frac{du}{dx} - I_N \frac{du}{dx}\|^2 + \|\frac{du}{dx} - \frac{d}{dx} I_N v\|^2$$

$$\leq C \frac{h^{\sigma-1}}{N^{p-1}} |v|_{\Omega,p} \qquad \leq C \frac{h^{\sigma-1}}{N^{p-1}} |v|_{\Omega,p}$$

$$\leq C \frac{h^{\sigma-1}}{N^{p-1}}$$

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 \leq C_1 \|u_h\|_\Omega + C_2(h, a) N^{1-p} |u|_{\Omega,p}$$

All this is caused by

$$\|I_N \frac{\partial a u_h}{\partial} - \frac{\partial}{\partial x} I_N(a u_h)\|^2 \neq 0 \quad \text{b/c}$$

$$I_N \frac{\partial u}{\partial x} \neq \frac{\partial}{\partial x}(I_N a v)$$ Basically the interpolation of derivative $\neq$ derivative of interpolant

$$B: \; -a(N_2, U_h) = -\int_{-1}^{1} I_N \frac{\partial(aU_h)}{\partial x} \cdot U_h \, dx + \int_{-1}^{1} I_N\left(a \frac{\partial U_h}{\partial x}\right) \cdot U_h \, dx$$

$$a \frac{\partial U_h}{\partial x} = \frac{\partial(aU_h)}{\partial x} - U_h \frac{\partial a}{\partial x}$$

$$= -\int_{-1}^{1} I_N\left(\cancel{\frac{\partial(aU_h)}{\partial x}}\right) \cdot U_h \, dx + \int_{-1}^{1} I_N\left(\cancel{\frac{\partial(aU_h)}{\partial x}} - U_h \frac{\partial a}{\partial x}\right) U_h \, dx$$

$$= -\left(I_N\left(\frac{\partial a}{\partial x} U_h\right), U_h\right) \leq \max_x |a_x| \, \|U_h\|_\Omega^2$$

I know this but can't remember the inequality

This is cont. out the source.

$$C: \; -a(N_3, U_h) = -\int_{-1}^{1} \frac{\partial}{\partial x}\left[I_N(aU_h)\right] U_h \, dx + \int_{-1}^{1} I_N\left(\frac{\partial}{\partial x}(aU_h)\right) U_h \, dx$$

By triangle ineq.

$$= \left(I_N\left(\frac{\partial}{\partial x} aU_h\right) - \frac{\partial}{\partial x}\left(I_N(aU_h)\right), U_h \right) \leq$$

$$\left\| I_N\left(\frac{\partial}{\partial x}(aU_h)\right) - \frac{\partial}{\partial x} I_N(aU_h) \right\|_\Omega^2 + \|U_h\|_\Omega^2$$

this is bounded

Using section 4,3 of Book

$$\left\| I_N\left(\frac{\partial}{\partial x}(aU_h)\right) - \frac{\partial}{\partial x} I_N(aU_h) \right\|^2 \quad \text{is of for}$$

$$\left\|^2 I_N \frac{dv}{dx} - \frac{d}{dx}(I_N v)\right\|^2 \leq \left\| \frac{dv}{dx} - I_N \frac{dv}{dx} \right\|^2 +$$

$$\left\| \frac{dv}{dx} - \frac{d}{dx}(I_N v) \right\|_\Omega^2$$

<u>The $P=0$ Problem:</u> Up to now only $P>=1$ has been considered. however in the application of the P-multigrid Method the $P=0$ Level plays a critical role. Due to the fact that our basis has no constant term in it's heirarchy we will need to form $p=0$ operators to use in P-Multigrid.

1). For CGC methods only corrections are computed for each level thus we only need to generate a $p=0$ Jacobian. This requires a $p=0$ $\vec{q}$ which is basiscally a single mode for each field on each element. To compute the Jacobian we will write out our Steady state Residual as follows

$$\iint_{\Omega_e} \phi_i \frac{\partial \vec{q}}{\partial t} + \phi_i \frac{\partial \vec{F}}{\partial x} + \phi_i \frac{\partial \vec{G}}{\partial y} \, d\Omega_e = 0$$ Application of I.B.P.

$$\iint_{\Omega_e} \phi_i \frac{\partial \hat{q}}{\partial t} - \iint_{\Omega_e} \frac{\partial \phi_i}{\partial x} \vec{F} + \frac{\partial \phi_i}{\partial y} \cdot \vec{G} \, d\Omega_e + \int_{\partial \Omega_e} \phi_i \, (\vec{F}, \vec{G}) \cdot \vec{n} \, dS = 0$$

For $p=0$ we set $\phi_i = 1$ and $q \approx \tilde{q} = \sum_{i=1}^{1} \hat{q}_i \phi_i \Rightarrow$ [illegible]

$$\iint \frac{\partial \hat{q}}{\partial t} \, d\Omega_e - 0 + \int_{\partial \Omega_e} (\vec{F}, \vec{G}) \cdot \vec{n} \, dS = 0$$

Dropping $\frac{\partial \hat{q}}{\partial t}$ - Steady State

$$R_p = \int_{\partial \Omega_e} (\vec{F}, \vec{G}) \cdot \vec{n} \, dS = 0$$ - this our FVM scheme with the difference that we will do $\int_{\partial \Omega_e} (\ ) \, dS$ in a more DG sense

We expect that this will give exactly an FVM method thus for a face

$$\int_{\partial\Omega_e} (\tilde{F})^* \, ds \approx (\hat{F})^* \cdot \vec{n} \quad \text{- where } \vec{n} \text{ is dimensional.}$$

? If we apply a quadrature rule to this, will it work out the same

$(F)^* = (F)^*(\hat{q}) = (F)^*\left(\sum_{i=1}^{1} \hat{q} \cdot 1\right)$ our integral is

$$\int_{\partial\Omega_e} \overset{\wedge}{\phi_1} (F^*(\hat{q}))^* \cdot \vec{n} \, ds = \int_{-1}^{1} (F(\hat{q}))^* \cdot \hat{n} \overbrace{\sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}}^{\text{end.}} \, d\xi \sim$$

If we 1 quadrature point then

$$\sim \int_{-1}^{1} (F(\hat{q}(0.0))^* \cdot \hat{n} \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}\Bigg|_{\xi=0.0} \cdot 2.0$$

If we have a straight side $\sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2} = \frac{|ds|}{\text{ref side length}}$

$= \frac{|ds|}{2}$ this gives

$F(\hat{q}(0.0))^* \cdot \hat{n} \cdot |ds|$ which since $\hat{q}$ is a constant value over the whole cell we have

$= F(\hat{q})^* \cdot \vec{n}$ - our FVM expression.

Thus over all interior faces we do our FVM exactly to get R_p thus $\frac{\partial R_p}{\partial \hat{q}} = \frac{\partial R_p}{\partial q}$ and is again the same as an F.V.M.

For our curved Boundary faces. we have

$$R_{\rho_f} = \int F(\hat{q})^{*} \cdot \hat{n} \sqrt{\left(\frac{dx}{d\zeta}\right)^2 + \left(\frac{dy}{d\zeta}\right)^2}\Bigg|_{\xi=0} \; d\zeta$$

— since the boundary is curved we'll need to use the quadrature formula to evaluate

$$\sqrt{\left(\frac{dx}{d\zeta}\right)^2 + \left(\frac{dy}{d\zeta}\right)^2}\Bigg|_{\xi=0}$$

Poisson using DG with interior Penalty Method:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f$$

Apply weighting function ϕ_i and Integrate over the cell.

$$\int_{\Omega_K} \phi_i \frac{\partial^2 u}{\partial x^2} + \phi_i \frac{\partial^2 u}{\partial y^2} \, d\Omega_K = -\int_{\Omega_K} \phi_i f \, d\Omega_K \qquad (1)$$

Apply integration by parts.

$$\int_{\Omega_K} \phi_i \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) d\Omega_K = \int_{\partial\Omega_K} \phi_i \nabla u \cdot \vec{n} \, d\partial\Omega_K - \int_{\Omega_K} \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi_i}{\partial y} \cdot \frac{\partial u}{\partial y} \, d\Omega_K$$

$$= \int_{\partial\Omega_K} \phi_i \nabla u \cdot \vec{n} \, d\partial\Omega_K - \int_{\Omega_K} \nabla u \cdot \nabla \phi_i \, d\Omega_K$$ Substitution into (1)

$$-\int_{\Omega_K} \nabla u \cdot \nabla \phi_i \, d\Omega_K + \int_{\partial\Omega_K} \phi_i \vec{\nabla} u \cdot \vec{n} \, d\partial\Omega_K = -\int_{\Omega_K} \phi_i f \, d\Omega_K$$

$$\Downarrow$$

$$\int_{\Omega_K} \nabla u \cdot \nabla \phi_i \, d\Omega_K - \int_{\partial\Omega_K} \phi_i \vec{\nabla} u \cdot \vec{n} \, d\partial\Omega_K = \int_{\Omega_K} \phi_i f \, d\Omega_K$$

We need to re-ellipticize the surface term by adding $+\int_{\partial\Omega_K} \mu (u^+ - u^-) \phi_i \, d\partial\Omega_K$.

$$\int_{\Omega_K} \nabla u \cdot \nabla \phi_i \, d\Omega_K - \int_{\partial\Omega_K} \phi_i \cdot \vec{\nabla} u \cdot \vec{n} \, d\partial\Omega_K + \int_{\partial\Omega_K} \mu (u^+ - u^-) \phi_i \, d\partial\Omega_K = \int_{\Omega_K} \phi_i f \, d\Omega_K$$

As with Finite volume viscous face terms we use the average of ∇U^+ and ∇U^-

Let us denote the cell we are in as Left "l" and the neighbor on said face as right "r" giving a surface term as

$$\int_{\partial\Omega_K} \nabla U \cdot \vec{n} \, \phi \, d\partial\Omega_K = \sum_{f=1}^{N_f} \pm \int \frac{1}{2}(\nabla U_L + \nabla U_R) \cdot \vec{n} \cdot \phi_{i L/R} \, ds \qquad \pm \begin{cases} + & \text{if } e = e_L \\ - & \text{if } e = e_R \end{cases}$$

the penalty term given by

$$\int_{\partial\Omega_K} \mu(U_L - U^R) \phi_{i L/R} \, d\Omega_K = \sum_{f=1}^{N_f} \pm \frac{1}{2}(\nabla U_L + U_R) \cdot \vec{n} \, \phi_{iL} \, ds$$

giving for a cell whose always on the left

$$\int_{\Omega_K} \nabla U \cdot \nabla \phi_i \, d\Omega_K - \sum_{f=1}^{N_f} + \int \frac{1}{2}(\nabla U_L + \nabla U_R) \cdot \vec{n} \cdot \phi_{iL} \, ds +$$

$$\sum_{f=1}^{N_f} \int \mu(U_L - U_R) \cdot \phi_L \, dS$$

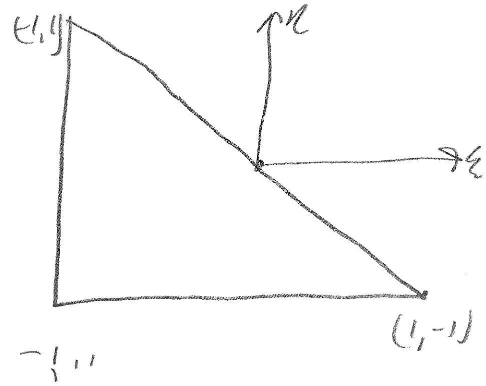
For a Right cell:

$$\int_{\Omega_K} \nabla U \cdot \nabla \phi_i \, d\Omega_K - \sum_{f=1}^{N_f} - \int \frac{1}{2}(\nabla U_L + \nabla U_R) \cdot \vec{n} \cdot \phi_{iR} \, ds +$$

$$\sum_{f=1}^{N_f} \int \mu(U_R - U_L) \cdot \phi_{iR} \, dS$$

Consider the usual standard element

$$K_\epsilon = \{ \xi, \eta \in \mathbb{R}^2 ; -1 < \xi, \eta ; \eta + \xi < 0 \}$$



Take $$u \simeq u_h = \sum_{j=1}^{N_{node}} \hat{u}_j(t) \phi_j(\xi, \eta)$$

substitute u_h for u gives (We will leave out discrete forces for now)

$$\int_{\Omega_K} \nabla u_h \cdot \nabla \phi_i \, d\Omega_K - \int_{\partial\Omega_K} \frac{1}{2}(\nabla u_{h_L} + \nabla u_{h_R}) \cdot \phi_i \vec{n} \, d\partial\Omega_K + \int_{\partial\Omega_K} \phi_i \mu (u_{h_L} - u_{h_R}) \, d\partial\Omega_K$$

$$= \int_{\Omega_K} \phi_i f \, d\Omega_K$$

Making the substitution for u_h gives

$$\int_{\Omega_K} \sum_{j=1}^{N_{node}} \nabla \phi_j \cdot \phi_i \, d\Omega_K \cdot \hat{u}_j - \int_{\partial\Omega_K} \frac{1}{2}\left(\sum_{j=1}^{} \hat{u}_{j_L}(t) \nabla \phi_j + \sum_{j=1}^{N_{node}} \hat{u}_{jR} \nabla \phi_j \right) \vec{n} \cdot \phi_i \, d\partial\Omega_K$$

$+$

To treat this explicitly we'll add a time term to LHS $\frac{\partial u_h}{\partial t}$ gives

$$\int_{\Omega_K} \phi_i \sum_{j=1}^{N_{node}} \phi_j \frac{\partial \hat{u}_j(t)}{\partial t} \, d\Omega_K + \int_{\Omega_K} \sum_{j=1}^{N_{node}} \nabla \phi_j \cdot \hat{u}_j \cdot \phi_i \, d\Omega_K - \int_{\partial\Omega_K} \frac{1}{2}\left(\sum_{j=1}^{N_{node}} \hat{u}_{jL}(t) \nabla \phi_{jL} + \right.$$

$$\left. \sum_{j=1}^{N_{node}} \hat{u}_{jR}(t) \nabla \phi_{jR} \right) \cdot \vec{n} \, \phi_i \, d\partial\Omega_K + \int_{\partial\Omega_K} \phi_i \mu \left(\sum_{j=1}^{N_{node}} \hat{u}_{jL} \phi_{jL} - \sum_{j=1}^{N_{node}} \hat{u}_{jR} \phi_{jR} \right) d\partial\Omega_K$$

$$= \int f \phi_i \, d\Omega_K$$

However explicitly this will take forever to converge. Thus we'll go for a matrix route:

Volume term

$$\int \sum_{j=1}^{N_{nod}} \nabla \phi_j \cdot \hat{\theta}_j \nabla \phi_i \, d\Omega_k$$ - will form to a Matrix

$$\int \nabla \phi_j \cdot \nabla \phi_i \, d\Omega_k \{\theta_j\}$$

$$\nabla \phi_j = \frac{\partial \phi_j}{\partial x}\hat{i} + \frac{\partial \phi_j}{\partial y}\hat{j} \Rightarrow \nabla \phi_j \cdot \nabla \phi_i = \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \cdot \frac{\partial \phi_j}{\partial y}$$

$$\frac{\partial \phi_i}{\partial x} = \frac{\partial \phi_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \phi_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \phi_i}{\partial y} = \frac{\partial \phi_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \phi_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\int \nabla \phi_j \cdot \nabla \phi_i \, d\Omega_k = \int_{-1}^{1}\int_{-1}^{-\xi} \left(\frac{\partial \phi_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \phi_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}\right) \cdot \left(\frac{\partial \phi_j}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \phi_j}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}\right) +$$

$$\left(\frac{\partial \phi_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \phi_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right) \cdot \left(\frac{\partial \phi_j}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \phi_j}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right) |J_{\xi,\eta}| \, d\eta \, d\xi$$

It has been shown previously that

$$\frac{\partial \phi_i}{\partial x} = [J] \begin{Bmatrix} \frac{\partial \phi_i}{\partial \xi} \\ \frac{\partial \phi_i}{\partial \eta} \end{Bmatrix} \quad J^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \Rightarrow$$

$$\int \nabla \phi_j \cdot \nabla \phi_i \, d\Omega_k = \int_{-1}^{1}\int_{-1}^{-\xi} \left(\frac{\partial \phi_i}{\partial \xi} \cdot J_{11}^{-1} + \frac{\partial \phi_i}{\partial \eta} \cdot J_{12}^{-1}\right) \cdot \left(\frac{\partial \phi_j}{\partial \xi} \cdot J_{11}^{-1} + \frac{\partial \phi_j}{\partial \eta} \cdot J_{12}^{-1}\right) +$$

$$\left(\frac{\partial \phi_i}{\partial \xi} \cdot J_{21}^{-1} + \frac{\partial \phi_i}{\partial \eta} \cdot J_{22}^{-1}\right) \cdot \left(\frac{\partial \phi_j}{\partial \xi} \cdot J_{21}^{-1} + \frac{\partial \phi_j}{\partial \eta} \cdot J_{22}^{-1}\right) |J_{\xi,\eta}| \, d\eta \, d\xi$$

This integral will be done with our Normal volume Quadrature:

$$\int_{-1}^{1}\int_{-1}^{-\xi} (\ )\,d\eta\,d\xi \approx \sum_{K=1}^{N_G}\Bigg[\left(\frac{\partial\phi_i}{\partial\xi}\cdot J_{11}^{-1}+\frac{\partial\phi_i}{\partial\eta}\cdot J_{12}^{-1}\right)\cdot\left(\frac{\partial\phi_j}{\partial\xi}\cdot J_{11}^{-1}+\frac{\partial\phi_j}{\partial\eta}\cdot J_{12}^{-1}\right)\Big|_{\xi_K,\eta_K}\{\hat{U}_j\}+$$

$$\{\hat{U}_j\}\left(\frac{\partial\phi_i}{\partial\xi}\cdot J_{21}^{-1}+\frac{\partial\phi_i}{\partial\eta}\cdot J_{22}^{-1}\right)\cdot\left(\frac{\partial\phi_j}{\partial\xi}\cdot J_{21}^{-1}+\frac{\partial\phi_j}{\partial\eta}\cdot J_{22}^{-1}\right)\Big|_{\xi_K,\eta_K}\Bigg]\,|J_{\xi_K,\eta_K}|\,W(K)$$

This gives a Matrix $[A_V]\{\hat{U}\}$ - which is block diagonal in nmode x nmode Blocks for each element. An explicit expression for μ of Interior Penalty Method

$$A_{V_{ij}}=\int_{-1}^{1}\int_{-1}^{-\xi}\Bigg[\left(\frac{\partial\phi_i}{\partial\xi}\cdot J_{11}^{-1}+\frac{\partial\phi_i}{\partial\eta}J_{12}^{-1}\right)\cdot\left(\frac{\partial\phi_j}{\partial\xi}J_{11}^{-1}+\frac{\partial\phi_j}{\partial\eta}\cdot J_{12}^{-1}\right)+$$

$$\left(\frac{\partial\phi_i}{\partial\xi}\cdot J_{21}^{-1}+\frac{\partial\phi_i}{\partial\eta}\cdot J_{22}^{-1}\right)\left(\frac{\partial\phi_j}{\partial\xi}\cdot J_{21}^{-1}+\frac{\partial\phi_j}{\partial\eta}\cdot J_{22}^{-1}\right)\Bigg]\,|J_{\xi,\eta_K}|\,d\eta\,d\xi \approx$$

can be approximated by quadrature.

Surface terms.

$$-\int_{\partial\Omega_K}\frac{1}{2}\left(\vec{\nabla}U_{h_L}+\nabla U_{h_R}\right)\cdot\phi_i\cdot\vec{n}\,d\partial\Omega_K+\int_{\partial\Omega_K}\phi_i\cdot\mu_K(U_{h_L}-U_{h_R})\,d\partial\Omega_K=$$

$$-\int_{\partial\Omega_K}\frac{1}{2}\left(\vec{\nabla}\Big(\sum_{j=1}^{N}\hat{U}_{j_L}\phi_j\Big)+\vec{\nabla}\Big(\sum_{j=1}^{N}\hat{U}_{j_R}\phi_{j_R}\Big)\right)\phi_i\cdot\vec{n}\,d\partial\Omega_K+\int\phi_i\,\mu_K\Big(\sum_{j=1}^{N}\hat{U}_j\phi_{j_L}-\sum_{j=1}^{N}\hat{U}_j\phi_{j_R}\Big)\,d\partial\Omega_K$$

gives a general Matrix format

$$A_S^{Diag}=-\int_{\partial\Omega_K}+\frac{1}{2}\vec{\nabla}\phi_j\cdot\phi_i\cdot\vec{n}\pm\phi_i\mu_K\phi_j\,d\partial\Omega_K\{\hat{U}_L\}\text{ or }-\{\hat{U}_R\}$$

Each element L/R gets one of these added to the A diagonal at the appropriate Nodes (i.e. ones that are valid). The element also get's 2 of these on the off diagonal e.g.

$$A_S^{OD}=\int_{\partial\Omega_K}\frac{1}{2}D\vec{\nabla}\phi_j\cdot\phi_i\cdot\vec{n}+-\phi_i\mu_K\phi_j$$ - for Left.

For the face terms we have to compute the trace of $\vec{\nabla}\phi_i$ which will follow from our evaulation of trace(ϕ_i).

$$\text{trace}(\nabla\phi_i\cdot\hat{n}) = \frac{\partial\phi_i}{\partial s}\cdot\frac{ds}{dx}\hat{n}_x + \frac{\partial\phi_i}{\partial s}\cdot\frac{ds}{dy}\hat{n}_y$$

$$\frac{\partial\phi_i}{\partial x} = \frac{\partial\phi_i}{\partial s}\cdot\frac{\partial s}{\partial x} \qquad \frac{ds}{dx} = \frac{1}{dx/ds} \qquad \frac{ds}{dy} = \frac{1}{dy/ds}$$

$$x(s) = \left(\frac{1-s}{2}\right)\cdot x_1 + \left(\frac{1+s}{2}\right)\cdot x_2 \qquad \frac{dx}{ds} = -\frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}(x_2 - x_1)$$

$$\frac{dy}{ds} = \frac{1}{2}(y_2 - y_1)$$

$$\vec{\nabla}\phi_j\cdot\vec{n} = \frac{\partial\phi_j}{\partial s}\cdot\frac{ds}{dx}n_x + \frac{\partial\phi_j}{\partial s}\frac{ds}{dy}\cdot n_y = \frac{\partial\phi_j}{\partial s}\left[\frac{1}{2}(x_2-x_1)\cdot n_x + \frac{1}{2}(y_2-y_1)\cdot n_y\right]$$

For the boundary the Penalty will Enforce the B.C.

For a Left Element

$$\int$$

We will Now write Down explicit expressions for L/R Adiagg and Aoffdiag from $\int_{\partial\Omega_K} \frac{1}{2}(\vec{\nabla} U_L^h + \vec{\nabla} U_R^h) \cdot \vec{n} \phi_i \, d\partial\Omega_K + \int \mu \phi_i (U_L^h - U_R^h) \, d\partial\Omega_K$

Left:

Adiagg: $\displaystyle\int_{\partial\Omega_K} \frac{1}{2} \nabla U_L^h \cdot \vec{n} \cdot \phi_{i_L} + \mu \phi_i U_L^h \, d\partial\Omega_K = \int_{\partial\Omega_K} \frac{1}{2}\left(\frac{\partial\phi_j^L}{\partial s} \cdot \frac{\partial s}{\partial x} n_x + \frac{\partial\phi_j^L}{\partial s} \cdot \frac{\partial s}{\partial y} \cdot n_y\right) \cdot \phi_i^L +$

$\mu \phi_i^L \phi_j^L \, d\partial\Omega_K$

A^{ODLR}: $\displaystyle\int_{\partial\Omega_K} \frac{1}{2} \nabla U_R^h \cdot \vec{n} \, \phi_{i_L} + \mu \phi_i^L \phi_j^R = \int \frac{1}{2}\left(\frac{\partial\phi_j^R}{\partial s} \cdot \frac{\partial s}{\partial x} \cdot n_x + \frac{\partial\phi_j^R}{\partial s} \cdot \frac{\partial s}{\partial y} \cdot n_y\right) \phi_i^L +$

$\mu \phi_i^L \cdot \phi_j^R \, d\partial\Omega_K$

Right:

Adiagg: $\displaystyle\int_{\partial\Omega_K} -\frac{1}{2} \vec{\nabla} U_R^h \cdot \vec{n} \, \phi_i^R + \mu \phi_i^R \phi_j^R \, d\partial\Omega_K = -\int_{\partial\Omega_K} \frac{1}{2}\left(\frac{\partial\phi_j^R}{\partial s} \cdot \frac{\partial s}{\partial x} \cdot n_x + \frac{\partial\phi_j^R}{\partial s} \cdot \frac{\partial s}{\partial y} \cdot n_y\right) \cdot \phi_i^R - \mu \phi_i^R \phi_j^R \, d\partial\Omega_K$

A^{ODRL}: $\displaystyle -\int_{\partial\Omega_K} -\frac{1}{2} \vec{\nabla} U_L^h \cdot \vec{n} \, \phi_i^R - \mu \phi_i^R \cdot \phi_j^L \, d\partial\Omega_K = -\int_{\partial\Omega_K} \frac{1}{2}\left(\frac{\partial\phi_j^L}{\partial s} \cdot \frac{\partial s}{\partial x} \cdot n_x + \frac{\partial\phi_j^R}{\partial s} \cdot \frac{\partial s}{\partial y} \cdot n_y\right) \cdot \phi_i^R +$

$\mu \phi_i^R \cdot \phi_j^L \, d\partial\Omega_K$

For Boundary Elements:

$$\int_{\partial\Omega_K} \frac{1}{2}(\vec{\nabla} U_L^h + \vec{\nabla} U_R^h) \cdot \vec{n} \, \phi_i + \mu \phi_i (U_L^h - U_R^h) \, d\partial\Omega_K$$

$\vec{\nabla} U_R^h = 0 \qquad U_R^h = U_b =$ Known

Adiag: $\displaystyle\int_{\partial\Omega_K} \frac{1}{2} \nabla U_L^h \cdot \vec{n} \, \phi_i + \mu \phi_i U_L \, d\partial\Omega_K$ ← evaluated just like "Left" above.

The RHS. "Source" term.

$$\int_{\partial\Omega_K} -\mu \phi_i^L u_b \, d\Omega_K \qquad \int \mu \phi_i (u_i - u_R)$$

Poisson As N-L Prob:

Rather then the previous approach we will form the poisson as a Non-linear problem nomely

$$R_p(u) = -F \quad \text{where } -F = -\int \phi_i f \, d\Omega_K.$$

Thus following the original formula w/ penalty term we have

$$R_p = -\int_{\Omega_K} \vec{\nabla} \cdot u^h \cdot \vec{\nabla} \phi_i \, d\Omega_K + \int_{\partial\Omega_K} \tfrac{1}{2} \phi_i (\vec{\nabla} U_L^h + \vec{\nabla} U_R^h) \cdot \vec{n} \, d\partial\Omega_K +$$

$$\int_{\partial\Omega_K} \mu_K \phi_i (U_L^h - U_R^h) \, d\partial\Omega_K = -\int_{\Omega_K} \phi_i f \, d\Omega_K$$

Thus we have 3 terms

Vol Resid:

$$\int_{\Omega_K} \vec{\nabla} \cdot u^h \cdot \vec{\nabla} \phi_i \, d\Omega_K = \int_{\Omega_K} \left(\frac{\partial u^h}{\partial x} \cdot \frac{\partial \phi_i}{\partial x} + \frac{\partial u^h}{\partial y} \cdot \frac{\partial \phi_i}{\partial y} \right) d\Omega_K =$$

$$\int_{\Omega_K} \overbrace{\left(\frac{\partial u^h}{\partial \xi} \cdot J_{11}^{-1} + \frac{\partial u^h}{\partial \eta} \cdot J_{12}^{-1} \right) \cdot \left(\frac{\partial \phi_i}{\partial \xi} \cdot J_{11}^{-1} + \frac{\partial \phi_i}{\partial \eta} \cdot J_{12}^{-1} \right)}^{\frac{\partial u^h}{\partial x} \cdot \frac{\partial \phi_i}{\partial x}} +$$

$$\underbrace{\left(\frac{\partial u^h}{\partial \xi} J_{21}^{-1} + \frac{\partial u^h}{\partial \eta} \cdot J_{22}^{-1} \right) \cdot \left(\frac{\partial \phi_i}{\partial \xi} \cdot J_{21}^{-1} + \frac{\partial \phi_i}{\partial \eta} \cdot J_{22}^{-1} \right)}_{\frac{\partial u^h}{\partial y} \cdot \frac{\partial \phi_i}{\partial y}} |\operatorname{Det} J| \cdot d\eta \, d\xi$$

Surf Resid:

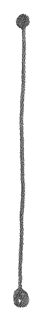
$$\int_{\partial\Omega_K} \tfrac{1}{2} (\vec{\nabla} U_L^h + \vec{\nabla} U_R^h) \cdot \vec{n} \, d\partial\Omega_K + \int_{\partial\Omega_K} \mu \phi_i (U_L^h - U_R^h) \, d\Omega_K$$

$$\int_{\partial\Omega_K} \frac{1}{2}\left(\frac{\partial u_L^h}{\partial x} + \frac{\partial u_R^h}{\partial x}\right) n_x + \frac{1}{2}\left(\frac{\partial u_L^h}{\partial y} + \frac{\partial u_R^h}{\partial y}\right)\cdot n_y \, d\partial\Omega_K$$

normally

$$\frac{\partial u^h}{\partial x} = \frac{\partial u^h}{\partial s}\cdot\frac{\partial s}{\partial x}$$ - which would be the trace$(\vec{\nabla} u^h)$

Unfortunately consider a vertical line



$$\frac{\partial s}{\partial x} = \frac{1}{\frac{\partial x}{\partial s}} \quad \text{but} \quad \frac{\partial x}{\partial s} = 0$$

Thus we have a problem.

Thus we use the cell's $\vec{\nabla}\phi$ to do this and
pick the ξ,η points such that we accomplish a trace
operation. For example if we are on side $= 1$ then
$\xi = s$, $\eta = -1$ thus this gives us the trace (Integrand) by using
s to get ξ,η. thus our integral becomes

$$\int_{\partial\Omega_K} \text{trace}\left\{ \frac{1}{2}\left[\left(\overbrace{\frac{\partial u_L^h}{\partial \xi} J_{11}^{-1} + \frac{\partial u_L^h}{\partial \eta}\cdot J_{12}^{-1}}^{\left(\frac{\partial u_L}{\partial x}\right)}\right) + \left(\frac{\partial u_R^h}{\partial \xi} J_{11}^{-1} + \frac{\partial u_R^h}{\partial \eta}\cdot J_{12}^{-1}\right)\right] n_x + \frac{1}{2}\left[\left(\overbrace{\frac{\partial u_L^h}{\partial \xi} J_{21}^{-1} + \frac{\partial u_L^h}{\partial \eta}\cdot J_{22}^{-1}}^{\left(\frac{\partial u_L}{\partial y}\right)}\right) + \left(\frac{\partial u_R^h}{\partial \xi} J_{21}^{-1} + \frac{\partial u_R^h}{\partial \eta} J_{22}^{-1}\right)\right]\right\} \cdot \phi_i \Big|_{face} \, \mathrm{Det}J\Big|_{face} \, ds$$

for all stuff in trace we define ξ,η as

$\xi(s), \eta(s)$.

Further we can Add a Symmetrizing term to the face flux.

$$\int_\Gamma [\![U]\!]\{\nabla\phi_i\}\,ds = \int_\Gamma (U^+n^+ - U^-n^+)\cdot\frac{(\nabla\phi_i^+ + \nabla\phi_i^-)}{2}\,ds$$

Since on cell + the ϕ_i^- does not exist then for a given Left face we can write.

$$\int_\Gamma (U^+ - U^-)\frac{\vec\nabla\phi_i}{2}\cdot\vec n\,ds.$$ Thus the entire face integral

$$\int_\Gamma \underbrace{\frac{(\vec\nabla U_L^h + \vec\nabla U_R^h)}{2}}_{\text{visc flux}}\cdot\vec n\,\phi_i + \underbrace{(U^L - U^R)\frac{\vec\nabla\phi_i}{2}\cdot\vec n}_{\text{Sym.}} + \underbrace{\mu\,\phi_i(U^L - U^R)}_{\text{penalty}}\,ds$$

$$-\int_\Gamma (U^+ - U^-)\frac{\nabla\phi_i}{2}\cdot\vec n\,ds$$

Recall on Boundary face the it's

$$(U^L - U^R)\cdot\vec n\,\frac{\vec\nabla\phi_i^{\text{inte.}}}{2}$$

Linearization of the poisson Equation!

Recall that the Residual was Broken into Vol. and Surf. terms.

Volume:

$$R_v = -\int_{\Omega_K} \vec{\nabla} u^h \cdot \vec{\nabla}\phi_i \, d\Omega_K = \sum_{k=1}^{N_g} \left(\frac{\partial u^h}{\partial x} \cdot \frac{\partial \phi_i}{\partial x} + \frac{\partial u^h}{\partial y} \cdot \frac{\partial \phi_i}{\partial y} \right) \cdot \text{Det} J \cdot d\Omega_K$$

Thus $R_v = R_v(\nabla u^h)$ thus

$$\frac{\partial R_v}{\partial \hat{u}} = \frac{\partial R_v}{\partial(\nabla u^h)} \cdot \frac{\partial \nabla u^h}{\partial \hat{u}} = \frac{\partial R_v}{\partial\left(\frac{\partial u^h}{\partial x}\right)} \cdot \frac{\partial u^h_x}{\partial \hat{u}} + \frac{\partial R_v}{\partial\left(\frac{\partial u^h}{\partial y}\right)} \frac{\partial u^h_y}{\partial \hat{u}}$$

$$1 \cdot \frac{\partial u^h}{\partial \hat{u}} + 1 \cdot \frac{\partial u^h_y}{\partial \hat{u}}$$

$$\frac{\partial u^h_x}{\partial \hat{u}} = \sum_j \hat{u}_j \cdot \frac{\partial \phi_j}{\partial x}$$ thus $$\frac{\partial u^h_x}{\partial \hat{u}_1} = \frac{\partial \phi_1}{\partial x}$$ thus our Jacobian is

Thus

$$\frac{\partial R_v}{\partial(\nabla u^h_x)} = \frac{\partial \phi_i}{\partial x}$$

$$\frac{\partial R_v}{\partial(u^h_y)} = \frac{\partial \phi_i}{\partial y} \Rightarrow$$

$$\frac{\partial R_v}{\partial \hat{u}} = \sum_{k=1}^{N_{gp}} \left(\frac{\partial \phi_i}{\partial x} \cdot \frac{\partial u^h_x}{\partial \hat{u}} + \frac{\partial \phi_i}{\partial y} \cdot \frac{\partial u^h_y}{\partial \hat{u}} \right) \text{Det} J \, w_g(k)$$

$$\frac{\partial R_v}{\partial \hat{u}_{ij}} = -\sum_{k=1}^{N_g} \left(\frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \cdot \frac{\partial \phi_j}{\partial y} \right) \text{Det} \, w_g(k)$$

Surface:

For a left face

$$R_{S_L} = \int_\Gamma \underbrace{(\nabla U_h^L + \nabla U_h^R)}_{2} \cdot \vec{n} \, \phi_i^L + (U^L - U^R) \underbrace{\vec{\nabla} \phi_i^L}_{2} \cdot \vec{n} - \mu \phi_i^L (U^L - U^R) \, ds$$

$$R_{S_R} = \int_\Gamma -\frac{(\nabla U_h^L + \nabla U_h^R)}{2} \cdot \vec{n} \cdot \phi_i^R + (U^L - U^R) \underbrace{\vec{\nabla} \phi_i^R}_{2} \cdot \vec{n} + \mu \phi_i^R (U^L - U^R) \, ds$$

$$\frac{\partial R_S}{\partial \hat{U}} = \frac{\partial R_S}{\partial \hat{U}}\left(\frac{\partial U_h}{\partial x}, \frac{\partial U_h}{\partial y}, U\right) = \frac{\partial R_S}{\partial U_{hx}} \cdot \frac{\partial U_{hx}}{\partial \hat{U}} + \frac{\partial R_S}{\partial U_{hy}} \cdot \underbrace{\frac{\partial U_{hy}}{\partial \hat{U}}}_{\frac{\partial \phi_j}{\partial y}} + \frac{\partial R_S}{\partial U} \cdot \frac{\partial U}{\partial \hat{U}}$$

Break down for either right or Left.

$$R_S^{L/R} = \int_\Gamma \pm \frac{1}{2}\left((U_{hx}^L + U_{hx}^R) \cdot n_x + (U_{hy}^L + U_{hy}^R) \cdot n_y \right) \cdot \phi_i^{L/R} + \frac{1}{2}(U^L - U^R)\left[\frac{\partial \phi_i^{L/R}}{\partial x} n_x + \frac{\partial \phi_i^{L/R}}{\partial y} \cdot n_y\right] \mp \mu \phi_i^{L/R} (U^L - U^R) \, ds$$

$$\left.\begin{aligned} \frac{\partial R_S^{L/R}}{\partial U_{hx}} &= \int_\Gamma \pm \frac{1}{2} \cdot n_x \cdot \phi_i^{L/R} \\ \frac{\partial R_S}{\partial U_{hy}} &= \pm \frac{1}{2} n_y \phi_i^{L/R} \end{aligned}\right\}$$ Diagonal. – i over face nodes

i – over element nodes

$$\frac{\partial R_S}{\partial U} = +\left(\frac{\partial \phi_i^{L/R}}{\partial x} \cdot n_x + \frac{\partial \phi_i^{L/R}}{\partial y} n_y\right) \mp \mu \phi_i^{L/R} (\pm 1)$$ – for Left or right.

Rember that $\frac{\partial U^h}{\partial x}, \frac{\partial U^h}{\partial y}$ goes over all the nodes element

$$\frac{\partial U_{hx}}{\partial \hat{U}_j} = \cancel{\phi_j} \frac{\partial \phi_j}{\partial x}$$ for $j = 1 \to N_{enode}$.

6·6 = 36

3·6 +

Thus

$$\frac{\partial R_{S_{ij}}}{\partial \hat{U}} = \pm \frac{1}{2}\left(n_x \cdot \phi_i^{L/R} \frac{\partial \phi_j}{\partial x} + n_y \phi_i^{L/R} \frac{\partial \phi_j}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial \phi_i^{L/R}}{\partial x} n_x + \frac{\partial \phi_i^{L/R}}{\partial y} n_y\right)\left(\pm \phi_j^{L/R}\right)$$

$$\mp \mu \phi_i^{L/R} (\pm \phi_j^{L/R})$$

We now explicitly write down the 4 linearizations of the surface Residual.

$$R_S^L(\hat{U}_L, \hat{U}_R) = \int \phi_i^L \frac{1}{2}\left[\left(\frac{\partial U^L}{\partial x} n_x + \frac{\partial U^L}{\partial y} n_y\right) + \left(\frac{\partial U^R}{\partial x} \cdot n_x + \frac{\partial U^R}{\partial y} n_y\right)\right] + \frac{1}{2}(U^L - U^R)\left(\frac{\partial \phi_i^L}{\partial x} n_x + \frac{\partial \phi_i^L}{\partial y} n_y\right) - \mu \phi_i^L (U^L - U^R)\, ds$$

$$R_S^R(\hat{U}_R, \hat{U}_L) = \int -\phi_i^R \frac{1}{2}\left[\left(\frac{\partial U^L}{\partial x} n_x + \frac{\partial U^L}{\partial y} n_y\right) + \left(\frac{\partial U^R}{\partial x} \cdot n_x + \frac{\partial U^R}{\partial y} n_y\right)\right] + \frac{1}{2}(U^L - U^R)\left(\frac{\partial \phi_i^R}{\partial x} n_x + \frac{\partial \phi_i^R}{\partial y} \cdot n_y\right) + \mu \phi_i^R (U^L - U^R)$$

Linearizations

$$\frac{\partial R_S^L}{\partial \hat{U}_{L_j}} = \int \phi_i^L \frac{1}{2}\left[\left(\frac{\partial \phi_j^L}{\partial x} \cdot n_x + \frac{\partial \phi_j^L}{\partial y} \cdot n_y\right)\right] + \frac{1}{2} \phi_j^L \left(\frac{\partial \phi_i^L}{\partial x} \cdot n_x + \frac{\partial \phi_i^L}{\partial y} \cdot n_y\right) - \mu \phi_i^L \cdot \phi_j^L \, ds$$

$$\frac{\partial R_S^L}{\partial \hat{U}_{R_j}} = \int \phi_i \frac{1}{2}\left[\left(\frac{\partial \phi_j^R}{\partial x} \cdot n_x + \frac{\partial \phi_j^R}{\partial y} n_y\right)\right] - \frac{1}{2} \phi_j^R \left(\frac{\partial \phi_i^L}{\partial x} \cdot n_x + \frac{\partial \phi_i^L}{\partial y} \cdot n_y\right) + \mu \phi_i^L \cdot \phi_j^R \, ds$$

$$\frac{\partial R_S^R}{\partial \hat{U}_{R_j}} = \int -\phi_i^R \frac{1}{2}\left[\left(\frac{\partial \phi_j^R}{\partial x} \cdot n_x + \frac{\partial \phi_j^R}{\partial y} \cdot n_y\right)\right] - \frac{1}{2} \phi_j^R \left(\frac{\partial \phi_i^R}{\partial x} \cdot n_x + \frac{\partial \phi_i^R}{\partial y}\right) - \mu \phi_i^R \phi_j^R \, ds$$

$$\frac{\partial R_S^R}{\partial \hat{U}_j^L} = \int -\phi_i^R \frac{1}{2}\left[\left(\frac{\partial \phi_j^L}{\partial x} \cdot n_x + \frac{\partial \phi_j^L}{\partial y} \cdot n_y\right)\right] + \frac{1}{2} \phi_j^L \left(\frac{\partial \phi_i^R}{\partial x} \cdot n_x + \frac{\partial \phi_i^R}{\partial y} n_y\right) + \mu \phi_i^R (\phi_j^L)$$

Navier Stokes Non-dimensionalization:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p - \tau_{xx}) + \frac{\partial}{\partial y}(\rho u v - \tau_{yx}) = 0$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho v u - \tau_{xy}) + \frac{\partial}{\partial y}(\rho v^2 + p - \tau_{yy}) = 0$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}\left[\rho u E + up + q_x - u\tau_{xx} - v\tau_{xy}\right] + \frac{\partial}{\partial y}\left[\rho v E + vp + q_y - u\tau_{xy} - v\tau_{yy}\right] = 0$$

$$q_x = -\lambda \frac{\partial T}{\partial x}, \quad q_y = -\lambda \frac{\partial T}{\partial y} \qquad \lambda = C_p \frac{\mu}{Pr} \qquad T = \frac{e}{C_v}$$

$$e = E - \tfrac{1}{2}(u^2+v^2). \qquad \tau_{ij} = 2\mu S_{ij}^{*} = 2\mu\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]$$

$$= \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]$$

Issues,

1) Stress term, won't be too Bad.
2). Heat conduction how to rep T.
   - essentially we'll do this as $T = \frac{e}{c_v}$ $\quad e = E - \frac{1}{2}(u^2+v^2)$

   thus the flux is easy and the linearization will need to be done in 2 parts. $\frac{\partial T}{\partial e} \cdot \frac{\partial e}{\partial \vec{q}} \cdot \frac{\partial \vec{q}}{\partial \vec{q}}$.

ND: $\bar{\vec{u}} = \frac{\vec{u}}{a_\infty}$, $\bar{\rho} = \frac{\rho}{\rho_\infty}$, $\bar{p} = \frac{p}{\rho_\infty a_\infty^2}$, $\bar{x} = \frac{x}{L}$ $\quad \bar{e} = \frac{e}{\cancel{\rho_\infty} a_\infty^2}$

$$\bar{T} = \frac{T}{T_\infty} = \frac{\bar{p}\cdot\cancel{\rho_\infty a_\infty^2}}{\bar{\rho}\cdot\cancel{\rho_\infty} R} \cdot \frac{\cancel{R \rho_\infty}}{p_\infty \cdot \cancel{\rho_\infty a_\infty^2}} = \frac{\bar{p}}{\bar{\rho}}\frac{1}{\bar{p}_\infty} \qquad \bar{p}_\infty = \frac{1}{\gamma} \qquad \bar{T} = \frac{\gamma \bar{p}}{\bar{\rho}}$$

$$\tau = \frac{t\, a_\infty}{L} \Rightarrow t = \frac{L\tau}{a_\infty} \qquad \bar{\mu} = \frac{\mu}{\mu_\infty}$$

We now do 1 equation at a time.

Cty: $\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$

$$\frac{a_\infty \cancel{\rho_\infty}}{\cancel{L}}\frac{\partial \bar\rho}{\partial \bar t} + \frac{\cancel{a_\infty \rho_\infty}}{\cancel{L}}\frac{\partial(\bar\rho\bar u)}{\partial \bar x} + \frac{\cancel{\rho_\infty a_\infty}}{\cancel{L}}\frac{\partial(\bar\rho\bar v)}{\partial \bar y} = 0 \Rightarrow$$

$$\boxed{\text{N.D Cty: } \frac{\partial \bar\rho}{\partial \bar t} + \frac{\partial(\bar\rho\bar u)}{\partial \bar x} + \frac{\partial(\bar\rho\bar v)}{\partial \bar y}}$$

X-Mtm: $\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}\left[\rho u^2 + p - 2\mu\frac{\partial u}{\partial x} + \frac{2}{3}\mu\frac{\partial u}{\partial x} + \frac{2}{3}\mu\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial y}\left[\rho u v - \mu\frac{\partial u}{\partial y} - \mu\frac{\partial v}{\partial x}\right] = 0$

$$\frac{\cancel{\rho_\infty a_\infty^2}}{\cancel{L}}\frac{\partial \bar\rho\bar u}{\partial \bar t} + \frac{1}{\cancel{L}}\frac{\partial}{\partial \bar x}\left[\bar\rho \cancel{\rho_\infty a_\infty^2}\bar u^2 + \bar p \cancel{\rho_\infty a_\infty^2} - 2\bar\mu\frac{\mu_\infty a_\infty}{\rho_\infty a_\infty L}\frac{\partial \bar u}{\partial \bar x} + \frac{2}{3}\bar\mu\frac{\mu_\infty a_\infty}{L}\frac{\partial \bar u}{\partial \bar x} + \frac{2}{3}\bar\mu\frac{\mu_\infty a_\infty}{L}\frac{\partial \bar v}{\partial \bar y}\right]$$
$$+\frac{1}{L}\frac{\partial}{\partial \bar y}\left[\rho_\infty\bar\rho a_\infty^2\bar u\bar v - \bar\mu\frac{\mu_\infty a_\infty}{L}\frac{\partial \bar u}{\partial \bar y} - \bar\mu\frac{\mu_\infty a_\infty}{L}\frac{\partial \bar v}{\partial \bar x}\right]$$

$$\frac{\partial(\bar\rho\bar u)}{\partial \bar t} + \frac{\partial}{\partial \bar x}\left[\bar\rho\bar u^2 + \bar p - 2\bar\mu\frac{\mu_\infty}{\rho_\infty a_\infty L}\frac{\partial \bar u}{\partial \bar x} + \frac{2}{3}\bar\mu\frac{\mu_\infty}{\rho_\infty a_\infty L}\frac{\partial \bar u}{\partial \bar x} + \frac{2}{3}\bar\mu\frac{\mu_\infty}{\rho_\infty a_\infty L}\frac{\partial \bar v}{\partial \bar y}\right]$$
$$+\frac{1}{L}\frac{\partial}{\partial \bar y}\left[\bar\rho\bar u\bar v - \bar\mu\frac{\mu_\infty}{\rho_\infty a_\infty L}\frac{\partial \bar u}{\partial \bar y} - \bar\mu\frac{\mu_\infty}{\rho_\infty a_\infty L}\frac{\partial \bar v}{\partial \bar x}\right]$$

Define $Re_\infty = \frac{\rho_\infty U_\infty L}{\mu_\infty} = \frac{\rho_\infty M_\infty a_\infty L}{\mu_\infty}$; $\frac{U_\infty}{a_\infty} = M \quad \frac{U_\infty}{M_\infty}$

$\Rightarrow \frac{\rho_\infty a_\infty L}{\mu_\infty} = \frac{\rho_\infty U_\infty L}{M_\infty \mu_\infty} = \frac{Re_\infty}{M_\infty} \rightarrow$ gives

$$\frac{\partial \bar\rho\bar u}{\partial \bar t} + \frac{\partial}{\partial \bar x}\left[\bar\rho\bar u^2 + \bar p - 2\bar\mu\frac{M_\infty}{Re_\infty}\frac{\partial \bar u}{\partial \bar x} + \frac{2}{3}\bar\mu\frac{M_\infty}{Re_\infty}\frac{\partial \bar u}{\partial \bar x} + \frac{2}{3}\bar\mu\frac{M_\infty}{Re_\infty}\frac{\partial \bar v}{\partial \bar y}\right] +$$
$$\frac{\partial}{\partial \bar y}\left[\bar\rho\bar u\bar v - \bar\mu\frac{M_\infty}{Re_\infty}\frac{\partial \bar u}{\partial \bar y} - \bar\mu\frac{M_\infty}{Re_\infty}\frac{\partial \bar v}{\partial \bar x}\right] = 0$$

N.D. X-mtm.

$$\boxed{\frac{\partial \bar\rho\bar u}{\partial \bar t} + \frac{\partial}{\partial \bar x}\left[\bar\rho\bar u^2 + \bar p - \frac{M_\infty\bar\mu}{Re_\infty}\left(\frac{4}{3}\frac{\partial \bar u}{\partial \bar x} - \frac{2}{3}\frac{\partial \bar v}{\partial \bar y}\right)\right] + \frac{\partial}{\partial \bar y}\left[\bar\rho\bar u\bar v - \bar\mu\frac{M_\infty}{Re_\infty}\left(\frac{\partial \bar u}{\partial \bar y} + \frac{\partial \bar v}{\partial \bar x}\right)\right] = 0}$$

y-mtm:

$$\frac{\rho_\infty a_\infty^2}{L}\frac{\partial(\bar\rho\bar v)}{\partial\tau} + \frac{1}{L}\frac{\partial}{\partial\bar x}\left[\bar\rho\bar v\bar u\,\rho_\infty a_\infty^2 - \frac{\bar\mu\mu_\infty a_\infty}{\rho_\infty L a_\infty}\frac{\partial\bar v}{\partial\bar x} - \frac{\bar\mu\mu_\infty a_\infty}{\rho_\infty a_\infty L}\frac{\partial\bar u}{\partial\bar y}\right] +$$

$$\frac{1}{L}\frac{\partial}{\partial\bar y}\left[\bar\rho\bar v^2\rho_\infty a_\infty^2 + \rho_\infty a_\infty^2\bar p - \frac{2\bar\mu\mu_\infty a_\infty}{\rho_\infty a_\infty L}\frac{\partial\bar v}{\partial\bar y} + \frac{2}{3}\frac{\bar\mu\mu_\infty a_\infty}{\rho_\infty a_\infty L}\frac{\partial\bar u}{\partial\bar x} + \frac{2}{3}\frac{\bar\mu\mu_\infty a_\infty}{\rho_\infty a_\infty L}\frac{\partial\bar v}{\partial\bar y}\right] = 0$$

N.D. y-mtm

$$\boxed{\frac{\partial(\bar\rho\bar v)}{\partial\tau} + \frac{\partial}{\partial\bar x}\left[\bar\rho\bar v\bar u - \bar\mu\frac{M_\infty}{Re_\infty}\left(\frac{\partial\bar v}{\partial\bar x} + \frac{\partial\bar u}{\partial\bar y}\right)\right] + \frac{\partial}{\partial\bar y}\left[\bar\rho\bar v^2 + \bar p + \bar\mu\frac{M_\infty}{Re_\infty}\left(-\frac{4}{3}\frac{\partial\bar v}{\partial\bar y} + \frac{2}{3}\frac{\partial\bar u}{\partial\bar x}\right)\right] = 0}$$

Energy:

$$\frac{a_\infty^3\rho_\infty}{L}\frac{\partial\bar\rho\bar E}{\partial\tau} + \frac{1}{L}\frac{\partial}{\partial\bar x}\Bigg[\rho_\infty a_\infty a_\infty^2\bar\rho\bar u\bar E + \rho_\infty a_\infty^2 a_\infty\bar p\bar u - \frac{\lambda}{L}\frac{\partial\bar T\,T_\infty}{\partial\bar x} - \bar u a_\infty\frac{\mu_\infty a_\infty}{L}\left[\frac{4}{3}\bar\mu\frac{\partial\bar u}{\partial\bar x} - \frac{2}{3}\bar\mu\frac{\partial\bar v}{\partial\bar y}\right]$$

$$- a_\infty\bar v\frac{\mu_\infty a_\infty}{L}\left[\bar\mu\frac{\partial\bar u}{\partial\bar y} + \bar\mu\frac{\partial\bar v}{\partial\bar x}\right]\Bigg] + \frac{1}{L}\frac{\partial}{\partial\bar y}\Bigg[\rho_\infty\bar\rho a_\infty^3\bar v\bar E + \rho_\infty a_\infty^3\bar p\bar v -$$

$$\frac{\lambda T_\infty}{L}\frac{\partial\bar T}{\partial\bar y} - \frac{a_\infty\bar u\mu_\infty a_\infty}{L}\left[\bar\mu\frac{\partial\bar v}{\partial\bar x} + \frac{\partial\bar u}{\partial\bar y}\right] - \frac{a_\infty\mu_\infty a_\infty\bar v}{L}\left[\frac{4}{3}\bar\mu\frac{\partial\bar v}{\partial\bar y} - \frac{2}{3}\bar\mu\frac{\partial\bar u}{\partial\bar x}\right]\Bigg]$$

$$= 0$$

Multiply by $\frac{L}{\rho_\infty a_\infty^3}$ gives

$$\frac{\partial\bar\rho\bar E}{\partial\tau} + \frac{\partial}{\partial\bar x}\Bigg[\bar\rho\bar u\bar E + \bar p\bar u - \frac{\lambda a_\infty^2}{C_v L\rho_\infty a_\infty^3}\frac{\partial\bar e}{\partial\bar x} - \frac{\mu_\infty}{\rho_\infty a_\infty L}\bar u\left[\frac{4}{3}\bar\mu\frac{\partial\bar u}{\partial\bar x} - \frac{2}{3}\bar\mu\frac{\partial\bar v}{\partial\bar y}\right] -$$

$$\frac{\mu_\infty}{\rho_\infty a_\infty L}\bar v\left[\bar\mu\frac{\partial\bar u}{\partial\bar y} + \bar\mu\frac{\partial\bar v}{\partial\bar x}\right]\Bigg] + \frac{\partial}{\partial\bar y}\Bigg[\bar\rho\bar v\bar E + \bar p\bar v - \frac{\lambda a_\infty^2}{C_v L\rho_\infty a_\infty^3}\frac{\partial\bar e}{\partial\bar y} - \frac{\mu_\infty}{\rho_\infty a_\infty L}$$

$$\bar u\left[\bar\mu\frac{\partial\bar v}{\partial\bar x} + \bar\mu\frac{\partial\bar u}{\partial\bar y}\right] - \frac{\mu_\infty}{\rho_\infty a_\infty L}\bar v\left[\frac{4}{3}\bar\mu\frac{\partial\bar v}{\partial\bar y} - \frac{2}{3}\bar\mu\frac{\partial\bar u}{\partial\bar x}\right]\Bigg]$$

It has been previously shown

$$\frac{\mu_\infty}{\rho_\infty a_\infty L} = \frac{\mu_\infty M_\infty}{\rho_\infty U_\infty L} = \frac{M_\infty}{Re_\infty}$$

using this gives

$$\frac{\partial\bar\rho\bar E}{\partial\tau} + \frac{\partial}{\partial\bar x}\left[\bar\rho\bar u\bar E + \bar p\bar u - \frac{\lambda}{C_v L\rho_\infty a_\infty}\frac{\partial\bar e}{\partial\bar x} - \frac{M_\infty}{Re_\infty}\bar u\bar\mu\left[\frac{4}{3}\frac{\partial\bar u}{\partial\bar x} - \frac{2}{3}\frac{\partial\bar v}{\partial\bar y}\right] - \frac{M_\infty}{Re_\infty}\bar v\bar\mu\left[\frac{\partial\bar u}{\partial\bar y} + \frac{\partial\bar v}{\partial\bar x}\right]\right]$$

$$+ \frac{\partial}{\partial\bar y}\left[\bar\rho\bar v\bar E + \bar p\bar v - \frac{\lambda a_\infty^2}{C_v L\rho_\infty a_\infty^3}\frac{\partial\bar e}{\partial\bar y} - \frac{M_\infty}{Re_\infty}\bar u\bar\mu\left[\frac{\partial\bar v}{\partial\bar x} + \frac{\partial\bar u}{\partial\bar y}\right] - \frac{M_\infty}{Re_\infty}\bar v\bar\mu\left[\frac{4}{3}\frac{\partial\bar v}{\partial\bar y} - \frac{2}{3}\frac{\partial\bar u}{\partial\bar x}\right]\right]$$

We have to play with heat conduction term

$$\frac{\lambda a_\infty^2}{C_v \rho_\infty L a_\infty^3} \sim \text{The constant for heat conduction term.}$$

$$\cancel{T_\infty = \frac{\bar{P}_\infty \rho_\infty a_\infty^2}{\rho_\infty R}} \quad \text{give}$$

$$\frac{\lambda \bar{P}_\infty a_\infty^2}{R \rho_\infty L a_\infty^3} = \frac{\lambda\left(\frac{1}{\gamma}\right)}{R \rho_\infty L a_\infty}$$

$$\lambda = \frac{\bar{\mu}\mu_\infty C_p}{Pr}$$

$$\frac{C_p}{R} = \frac{\gamma}{\gamma - 1}$$

$$\frac{\mu_\infty C_p\left(\frac{1}{\gamma}\right)}{Pr\, \rho_\infty a_\infty L C_v} =$$

$$\frac{\bar{\mu}\mu_\infty \gamma\left(\frac{1}{\gamma}\right)}{Pr\, \rho_\infty a_\infty L} = \frac{\gamma M_\infty \bar{\mu}}{Pr\, Re_\infty}$$ thus this is the coefficient for heat conduction terms.

This was done by setting $T_\infty = e_\infty = \frac{a_\infty^2}{C_v}$

Thus giving the final form,

$$\boxed{\begin{aligned}\frac{\partial \bar{\rho}\bar{E}}{\partial \tau} &+ \frac{\partial}{\partial \bar{x}}\left[\bar{u}(\bar{\rho}\bar{E}+\bar{p}) - \frac{\gamma M_\infty \bar{\mu}}{Pr Re_\infty}\frac{\partial \bar{e}}{\partial \bar{x}} - \frac{M_\infty}{Re_\infty}\bar{u}\bar{\mu}\left\{\frac{4}{3}\frac{\partial \bar{u}}{\partial \bar{x}} - \frac{2}{3}\frac{\partial \bar{v}}{\partial \bar{y}}\right\} - \frac{M_\infty}{Re_\infty}\bar{v}\bar{\mu}\left[\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right]\right] + \\ &\frac{\partial}{\partial \bar{y}}\left[\bar{v}(\bar{\rho}\bar{E}+\bar{p}) - \frac{\gamma M_\infty \bar{\mu}}{Pr Re_\infty}\frac{\partial \bar{e}}{\partial \bar{y}} - \frac{M_\infty}{Re_\infty}\bar{u}\bar{\mu}\left\{\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}}\right\} - \frac{M_\infty}{Re_\infty}\bar{v}\bar{\mu}\left\{\frac{4}{3}\frac{\partial \bar{v}}{\partial \bar{y}} - \frac{2}{3}\frac{\partial \bar{u}}{\partial \bar{x}}\right\}\right]\end{aligned}}$$

Thus we define $\vec{q}$, E, E_v, F, F_v. So as to give

$$\frac{\partial \vec{q}}{\partial \tau} + \frac{\partial (E - E_v)}{\partial x} + \frac{\partial (F - F_v)}{\partial y} = 0$$

$$E = \left\{\begin{array}{l} \bar{\rho}\bar{u} \\ \bar{\rho}\bar{u}^2 + \bar{p} - \bar{\mu}\frac{M_\infty}{Re_\infty}\left\{\frac{4}{3}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{2}{3}\frac{\partial \bar{v}}{\partial \bar{y}}\right\} \\ \bar{\rho}\bar{u}\bar{v} - \bar{\mu}\frac{M_\infty}{Re_\infty}\left\{\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}}\right\} \\ \bar{u}[\bar{\rho}\bar{E} + \bar{p}] - \gamma\frac{M_\infty}{Pr Re_\infty}\frac{\partial \bar{e}}{\partial \bar{x}} - \frac{M_\infty}{Re_\infty}\bar{u}\bar{\mu}\left\{\frac{4}{3}\frac{\partial \bar{u}}{\partial \bar{x}} - \frac{2}{3}\frac{\partial \bar{v}}{\partial \bar{y}}\right\} - \frac{M_\infty}{Re_\infty}\bar{v}\bar{\mu}\left\{\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right\} \end{array}\right\}$$

$$E_v = \begin{Bmatrix} 0 \\ \frac{\bar{\mu} M_\infty}{Re_\infty}\left\{\frac{4}{3}\frac{\partial \bar{u}}{\partial x} - \frac{2}{3}\frac{\partial \bar{v}}{\partial y}\right\} \\ \frac{\bar{\mu} M_\infty}{Re_\infty}\left\{\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y}\right\} \\ \frac{M_\infty}{Re_\infty}\left[\frac{\gamma \bar{\mu}}{Pr}\cdot\frac{\partial \bar{e}}{\partial x} + \bar{u}\bar{\mu}\left\{\frac{4}{3}\frac{\partial \bar{u}}{\partial x} - \frac{2}{3}\frac{\partial \bar{v}}{\partial y}\right\} + \bar{v}\bar{\mu}\left\{\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right\}\right] \end{Bmatrix}$$

$$F = \begin{Bmatrix} \bar{\rho}\bar{v} \\ \bar{\rho}\bar{v}\bar{u} \\ \bar{\rho}\bar{v}^2 + \bar{p} \\ \bar{v}\left[\bar{\rho}\bar{E} + \bar{p}\right] \end{Bmatrix}$$

$$F_v = \begin{Bmatrix} 0 \\ \bar{\mu}\frac{M_\infty}{Re_\infty}\left\{\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right\} \\ \bar{\mu}\frac{M_\infty}{Re_\infty}\left\{\frac{4}{3}\frac{\partial \bar{v}}{\partial y} - \frac{2}{3}\frac{\partial \bar{u}}{\partial x}\right\} \\ \frac{M_\infty}{Re_\infty}\left[\frac{\gamma}{Pr}\cdot\frac{\partial \bar{e}}{\partial y} + \bar{u}\bar{\mu}\left\{\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right\} + \bar{v}\bar{\mu}\left\{\frac{4}{3}\frac{\partial \bar{v}}{\partial y} - \frac{2}{3}\frac{\partial \bar{u}}{\partial x}\right\}\right] \end{Bmatrix}$$

Always recall it's $E - E_v$, $F - F_v$

Before Proceeding to the Navier-Stokes terms. consider:

$$-\vec{\nabla}\cdot(A\vec{\nabla}u) = f$$

$$\int_{\Omega_K} \phi_i \vec{\nabla}\cdot(A\vec{\nabla}u)\, d\Omega_K = -\int_{\Omega_K} \phi_i f \, d\Omega_K$$

$\Downarrow$

I.B.P

$$-\int_{\Omega_K} \vec{\nabla}\phi_i \cdot A\vec{\nabla}u \, d\Omega_K + \int_{\partial\Omega_K} \phi_i A\vec{\nabla}u\cdot\vec{n}\, ds$$ - rewrite for generic face as

$$\int_{\partial\Omega_K} [[\phi_i]]\cdot\{A\vec{\nabla}u\} + [[A u]]\{A\nabla\phi_i\} - \mu\,\phi_i [[Au]]\cdot\vec{n}\, ds.$$

So for a left face gives

$$\iint_{\partial\Omega_K} \phi_i \{\underbrace{A\vec{\nabla}u}_{E_v,\,F_v}\}\cdot\vec{n} + \left[(u)^L - u^R\right] \frac{A\vec{\nabla}\phi_i}{2}\cdot\vec{n} - \mu\,\phi_i\left[(Au)^L - (Au)^R\right] ds$$

$$\{A\nabla\phi_i\} = \frac{1}{2}\left(A^+\nabla\phi_i^+ + A^-\nabla\phi_i^-\right)$$

Computation of viscous flux terms

Once inside the subroutine we are going to need to compute the components of E_v, $F_v(w)$ or some combination of $(E_v, F_v)\cdot\vec{n}$..(surf)

1) Volume:

$$-E_v: \left\{ \begin{array}{l} 0 \\ \bar{\mu}\dfrac{M_\infty}{Re_\infty}\left\{ \dfrac{4}{3}\dfrac{\partial \bar{u}}{\partial x} - \dfrac{2}{3}\dfrac{\partial \bar{v}}{\partial y} \right\} \\ \bar{\mu}\dfrac{M_\infty}{Re_\infty}\left\{ \dfrac{\partial \bar{v}}{\partial x} + \dfrac{\partial \bar{u}}{\partial y} \right\} \\ \dfrac{M_\infty \bar{\mu}}{Re_\infty}\left[\dfrac{\gamma}{Pr}\cdot\dfrac{\partial \bar{e}}{\partial x} + \bar{u}\bar{\mu}\left\{ \dfrac{4}{3}\dfrac{\partial \bar{u}}{\partial x} - \dfrac{2}{3}\dfrac{\partial \bar{v}}{\partial y} \right\} + \bar{v}\bar{\mu}\left\{ \dfrac{\partial \bar{u}}{\partial y} + \dfrac{\partial \bar{v}}{\partial x} \right\} \right] \end{array} \right\}$$

Requires

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x}, \frac{\partial \bar{u}}{\partial y}, \frac{\partial e}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y}, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial q}\cdot\frac{\partial q}{\partial x}; \quad \frac{\partial e}{\partial q}\cdot\frac{\partial q}{\partial x}$$

$$F_v: \left\{ \begin{array}{l} 0 \\ \mu\dfrac{M_\infty}{Re_\infty}\left\{ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \right\} \\ \mu\dfrac{M_\infty}{Re_\infty}\left\{ \dfrac{4}{3}\dfrac{\partial \bar{v}}{\partial y} - \dfrac{2}{3}\dfrac{\partial \bar{u}}{\partial x} \right\} \\ \dfrac{M_\infty \mu}{Re_\infty}\left[\dfrac{\gamma}{Pr}\cdot\dfrac{\partial \bar{e}}{\partial y} + \bar{u}\mu\left\{ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \right\} + \bar{v}\mu\left\{ \dfrac{4}{3}\dfrac{\partial v}{\partial y} - \dfrac{2}{3}\dfrac{\partial u}{\partial x} \right\} \right] \end{array} \right\}$$

only additional requirement is

$$\frac{\partial e}{\partial y} = \frac{\partial e}{\partial q}\cdot\frac{\partial q}{\partial y}$$

Done in all Jac routines

Thus our 1st task is to compute $\dfrac{\partial w}{\partial q}, \dfrac{\partial u}{\partial q}, \dfrac{\partial e}{\partial q}$

D.G. Discretization for N.S.: Dropping Bugs

$$\frac{\partial \vec{q}}{\partial t} + \frac{\partial (E-E_v)}{\partial x} + \frac{\partial (F-F_v)}{\partial y} = 0$$

Using weighted residuals.

$$\int_{\Omega_K} \phi_i \frac{\partial \vec{q}}{\partial t} + \overset{②}{\phi_i \frac{\partial (E-E_v)}{\partial x}} + \overset{③}{\phi_i \frac{\partial (F-F_v)}{\partial y}} \, d\Omega_K = 0$$

Using I.B.P. on 2 and 3.

$$\int_{\Omega_K} \phi_i \vec{\nabla}\cdot(E-E_v, F-F_v)\, d\Omega_K = -\int \vec{\nabla}\phi_i \cdot (E-E_v, F-F_v)\, d\Omega_K + \int_{\partial\Omega_K} \phi_i (E-E_v, F-F_v)\cdot \vec{n}\, d$$

We now split this into an Euler part or viscous part.

$$-\int_{\Omega_K} \vec{\nabla}\phi_i \cdot (E, F)\, d\Omega_K + \int_{\Omega_K} \vec{\nabla}\phi_i \cdot (E_v, F_v)\, d\Omega_K + \int_{\partial\Omega_K} \phi_i (E^*, F^*)\cdot \vec{n}\, ds - \int_{\partial\Omega_K} \phi_i (E_v^*, F_v^*)\cdot \vec{n}\, d\Omega_K$$

where the ~~stc~~ $(\,)^*$ is the numerical flux.

We have previously shown the Euler fluxes.

Viscous terms:

Volume:

$$+\int_{\Omega_K} \vec{\nabla}\phi_i (E_v, F_v)\, d\Omega_K = \int_{\Omega_K} \frac{\partial \phi_i}{\partial x} E_v + \frac{\partial \phi_i}{\partial y} F_v =$$

$$\int_{-1}^{1}\int_{-1}^{-\xi} \left(\frac{\partial \phi_i}{\partial \xi} J_{11}^{-1} + \frac{\partial \phi_i}{\partial \eta} J_{12}^{-1}\right) E_v + \left(\frac{\partial \phi_i}{\partial \xi} J_{21}^{-1} + \frac{\partial \phi_i}{\partial \eta} J_{22}^{-1}\right) F_v \, d\eta\, d\xi$$

Surface:

$$-\int_{\partial\Omega_K} \phi_i\,(E_v^*, F_v^*)\cdot\vec{n}\, d\,\partial\Omega_K = -\int_{\partial\Omega_K} \phi_i\left(\tfrac{1}{2}(E_v^L + E_v^R)\cdot n_x + \tfrac{1}{2}(F_v^L + F_v^R)\cdot n_y\right) + (q_L - q_R)\tfrac{1}{2}\nabla\phi_i\cdot\vec{n}[A]$$

$$-\mu\,\phi_i([A]^L q_L - [A]^R q_R)\, ds$$

Because we have ϕ_i, $\nabla\phi_i$ we split this into a ~~few~~ 2 parts because 1 loop is over faces nodes (stuff with ϕ_i) and the other is over element nodes (stuff w $\nabla\phi_i$).

ϕ_i: Face based nodes contribution given as

$$-\left\{\int_{\partial\Omega_K} \phi_i\left[\tfrac{1}{2}(E_v^L + E_v^R)\cdot n_x + \tfrac{1}{2}(F_v^L + F_v^R)n_y\right] - \mu\phi_i([A]^L q_L - [A]^R q_R)\, ds\right.$$

$$= \int_{\partial\Omega_K} -\phi_i\left[\tfrac{1}{2}(E_v^L + E_v^R)n_x + \tfrac{1}{2}(F_v^L + F_v^R)n_y\right] + \mu\phi_i(q_L - q_R)\, ds$$

$\nabla\phi_i$: Element based.

$$-\int_{\partial\Omega_K} (q_L - q_R)\tfrac{1}{2}[A]\left(\frac{\partial\phi_i}{\partial x}\cdot n_x + \frac{\partial\phi_i}{\partial y}\cdot n_y\right) d\,\partial\Omega$$

On Boundary face we have

ϕ_i: face based

$$\int_{\partial\Omega_K} -\phi_i\left(E_v^L n_x + F_v^L n_y\right) + \mu\phi_i(q_L A - q_R A)\, ds$$

$$\int_{\partial\Omega_K} -\tfrac{1}{2}(q_L - q_R)\left(\frac{\partial\phi_i}{\partial x} n_x + \frac{\partial\phi_i}{\partial y}\cdot n_y\right)[A]^L$$

Thus we have our challenge to find $[A] = \frac{\partial E_v}{\partial \nabla \vec{q}}$

$[A]$ is not really a matrix but rather a combination such that

$E_v = [A_E] \vec{\nabla}(\vec{q})$ thus - $[A_E]$ is 4×8 thus A can be writte

$$[A] = \begin{bmatrix} \frac{\partial E_v}{\partial\left(\frac{\partial \vec{q}}{\partial x}\right)} & \frac{\partial E_v}{\partial\left(\frac{\partial \vec{q}}{\partial y}\right)} \\ \frac{\partial F_v}{\partial\left(\frac{\partial \vec{q}}{\partial x}\right)} & \frac{\partial F_v}{\partial\left(\frac{\partial \vec{q}}{\partial y}\right)} \end{bmatrix}$$

thus the vector

$$\begin{Bmatrix} E_v \\ F_v \end{Bmatrix} = [A] \begin{Bmatrix} \frac{\partial \vec{q}}{\partial x} \\ \frac{\partial \vec{q}}{\partial y} \end{Bmatrix}$$

Thus we have 4 block matricies to form that we will form and store seperately. But remba that we can always write it as 1 big system. We will the "Big" system to determ how [A] operates on other things.

We will now explicitly derive expressions for each of the above 4 matrixies.

Consider:

$$E_v = \begin{Bmatrix} 0 \\ \frac{\bar{\mu} M_\infty}{Re_\infty}\left\{\frac{4}{3}\frac{\partial(u)}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial y}\right\} \\ \bar{\mu}\frac{M_\infty}{Re_\infty}\left\{\frac{\partial v}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial x} + \frac{\partial u}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial y}\right\} \\ \frac{M_\infty}{Re_\infty}\left[\frac{\gamma}{Pr}\bar{\mu}\frac{\partial e}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial x} + \bar{u}\bar{\mu}\left\{\frac{4}{3}\cdot\frac{\partial u}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial y}\right\} + v\bar{\mu}\left\{\frac{\partial u}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial y} + \frac{\partial v}{\partial \vec{q}}\cdot\frac{\partial \vec{q}}{\partial x}\right\}\right] \end{Bmatrix}$$

Thus it becomes very easy to take $\frac{\partial E_v}{\partial\left(\frac{\partial \vec{q}}{\partial x}\right)}$ given as follows

$\lfloor \quad \rfloor$ - row vector

$$\frac{\partial E_v}{\partial\left(\frac{\partial \vec{\xi}}{\partial x}\right)} = \begin{bmatrix} \lfloor 0 \rfloor \\ \mu\frac{M_\infty}{Re_\infty}\left\lfloor \frac{4}{3}\frac{\partial u}{\partial \vec{\xi}}\right\rfloor \\ \mu \frac{M_\infty}{Re_\infty}\left\lfloor \frac{\partial v}{\partial \vec{\xi}}\right\rfloor \\ \frac{M_\infty \mu}{Re_\infty}\left\lfloor \frac{\gamma}{Pr}\mu\frac{\partial e}{\partial \vec{\xi}} + u\mu\left\{\frac{4}{3} \frac{\partial u}{\partial\vec{\xi}}\right\} + v\mu\left\{\frac{\partial v}{\partial \vec{\xi}}\right\}\right\rfloor\end{bmatrix} \equiv G_{11}$$

Similarly

$$\frac{\partial E_v}{\partial\left(\frac{\partial \vec{\xi}}{\partial y}\right)} = \begin{bmatrix} \lfloor 0 \rfloor \\ \mu\frac{M_\infty}{Re_\infty}\left\lfloor -\frac{2}{3}\frac{\partial v}{\partial \vec{\xi}}\right\rfloor \\ \mu \frac{M_\infty}{Re_\infty}\left\lfloor \frac{\partial u}{\partial \vec{\xi}}\right\rfloor \\ \frac{M_\infty \mu}{Re_\infty}\left\lfloor u\mu\left\{-\frac{2}{3} \frac{\partial v}{\partial\vec{\xi}}\right\} + v\mu\left\{\frac{\partial u}{\partial \vec{\xi}}\right\}\right\rfloor\end{bmatrix} \equiv G_{12}$$

Consider F_v written as

$$F_v = \begin{Bmatrix} 0 \\ \mu\frac{M_\infty}{Re_\infty}\left\{\frac{\partial u}{\partial \vec{\xi}}\cdot\frac{\partial \vec{\xi}}{\partial y} + \frac{\partial v}{\partial \vec{\xi}}\frac{\partial \vec{\xi}}{\partial x}\right\} \\ \mu\frac{M_\infty}{Re_\infty}\left\{\frac{4}{3}\frac{\partial v}{\partial \vec{\xi}}\cdot\frac{\partial \vec{\xi}}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial \vec{\xi}}\cdot\frac{\partial \vec{\xi}}{\partial x}\right\} \\ \frac{M_\infty \mu}{Re_\infty}\left[\frac{\gamma}{Pr}\cdot\frac{\partial e}{\partial \vec{\xi}}\cdot\frac{\partial \vec{\xi}}{\partial y} + u\mu\left\{\frac{\partial u}{\partial \vec{\xi}}\cdot\frac{\partial \vec{\xi}}{\partial y} + \frac{\partial v}{\partial \vec{\xi}}\cdot\frac{\partial \vec{\xi}}{\partial x}\right\} + v\mu\left\{\frac{4}{3}\frac{\partial v}{\partial \vec{\xi}}\cdot\frac{\partial \vec{\xi}}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial \vec{\xi}}\cdot\frac{\partial \vec{\xi}}{\partial x}\right\}\right. \end{Bmatrix}$$

Thus

$$\frac{\partial F_v}{\partial\left(\frac{\partial \vec{\xi}}{\partial x}\right)} = \begin{bmatrix} \lfloor 0 \rfloor \\ \mu\frac{M_\infty}{Re_\infty}\left\lfloor \frac{\partial v}{\partial \vec{\xi}}\right\rfloor \\ \mu \frac{M_\infty}{Re_\infty}\left\lfloor -\frac{2}{3}\frac{\partial u}{\partial \vec{\xi}}\right\rfloor \\ \frac{M_\infty \mu}{Re_\infty}\left\lfloor u\mu\left\{\frac{\partial v}{\partial\vec{\xi}}\right\} + v\mu\left\{-\frac{2}{3}\frac{\partial u}{\partial \vec{\xi}}\right\}\right\rfloor\end{bmatrix} \equiv G_{21}$$

$$\frac{\partial F_v}{\partial\left(\frac{\partial \vec{\xi}}{\partial y}\right)} = \begin{bmatrix} \lfloor 0 \rfloor \\ \mu\frac{M_\infty}{Re_\infty}\left\lfloor \frac{\partial u}{\partial \vec{\xi}}\right\rfloor \\ \mu \frac{M_\infty}{Re_\infty}\left\lfloor \frac{4}{3}\frac{\partial v}{\partial \vec{\xi}}\right\rfloor \\ \frac{\mu M_\infty}{Re_\infty}\left\lfloor \frac{\gamma}{Pr}\mu\frac{\partial e}{\partial \vec{\xi}} + u\mu\left\{\frac{\partial u}{\partial\vec{\xi}}\right\} + v\mu\left\{\frac{4}{3}\frac{\partial v}{\partial \vec{\xi}}\right\}\right\rfloor\end{bmatrix} \equiv G_{22}$$

Thus we now have everything we need to create [A]. With [A] in hand we are ready to see how this will operate on the terms it's involved in.

Consider 1st the symmetrizing term.

$$(q^L - q_R)\,[A]\,\frac{\vec{\nabla}\phi_i \cdot \vec{n}}{2} = \frac{1}{2}(q^L - q_R)\,[A]\cdot\vec{\nabla}\phi_i\cdot\vec{n}$$

This for the Matrix vector product is written as

$$\begin{array}{cc} G_{11} & G_{12} \\ \end{array}$$
$$\begin{bmatrix} \frac{\partial E_v}{\partial(\partial q/\partial x)} & \frac{\partial E_v}{\partial(\partial q/\partial y)} \\ \frac{\partial F_v}{\partial(\partial q/\partial x)} & \frac{\partial F_v}{\partial(\partial q/\partial y)} \end{bmatrix} \begin{Bmatrix} \left\{\frac{\partial \phi_i}{\partial x}\right\} \\ \vdots \\ \left\{\frac{\partial \phi_i}{\partial y}\right\} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial E_v}{\partial(\partial q/\partial x)}\left\{\frac{\partial\phi_i}{\partial x}\right\} + \frac{\partial E_v}{\partial(\partial q/\partial y)}\cdot\left\{\frac{\partial\phi_i}{\partial y}\right\} \\ \frac{\partial F_v}{\partial(\partial q/\partial x)}\left\{\frac{\partial\phi_i}{\partial x}\right\} + \frac{\partial F_v}{\partial(\partial q/\partial y)}\left\{\frac{\partial\phi_i}{\partial y}\right\} \end{Bmatrix}$$

then dot product with $\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$ gives

$$(q^L - q_R)\underbrace{\left[\left(\frac{\partial E_v}{\partial(q_x)}\left\{\frac{\partial\phi_i}{\partial x}\right\} + \frac{\partial E_v}{\partial(q_y)}\cdot\frac{\partial\phi_i}{\partial y}\right)n_x + \left(\frac{\partial F_v}{\partial q_x}\left\{\frac{\partial\phi_i}{\partial x}\right\} + \frac{\partial F_v}{\partial(q_y)}\left\{\frac{\partial\phi_i}{\partial y}\right\}\right)n_y\right]}_{[A]\cdot\vec{\nabla}\phi_i\cdot\vec{n}}$$

At this stage it is my opinion that [A] goes with the penalty term. the $\vec{\nabla}$ operator thus it does not belong in

Thus penalty looks like

$$\mu\phi_i(q_L - q_R)\left[\frac{1}{2}\left(\frac{\partial E_v}{\partial(q_x)}^L + \frac{\partial F_v}{\partial(q_y)}^L + \frac{\partial E_v}{\partial(q_x)}^R + \frac{\partial F_v}{\partial(q_y)}^R\right)\right]^T$$

Essentially since [A] is 8x8 we take Avg of the block trace of A in the penalty term.

If we denote $[G] = \frac{1}{2}\left[\left(\frac{\partial E_v^L}{\partial q_x} + \frac{\partial F_v^L}{\partial q_y}\right) + \left(\frac{\partial E_v^R}{\partial q_x} + \frac{\partial F_v^R}{\partial q_y}\right)\right]$ then

$$\text{pflux} = \mu\,\phi_i\,[G]^T\{\Delta q\}$$

Penalty Flux (Details):

$$\mu \phi_i \{ G_{ii} \} (\vec{q}_L - \vec{q}_R)$$

$$\{ G_{ii} \} = \frac{1}{2} (G^L_{11} + G^L_{22} + G^R_{11} + G^R_{22})$$

We want to write

$\{ [G_{ii}] \}^T (\vec{q}_L - \vec{q}_R)$ - we must Transpose it

Using Maple we have derived the operation as follows we with define $pflux = \underset{\text{penalty coefficient}}{pen} \cdot \{G_{ii}\}(\vec{q}_L - \vec{q}_R)$

$$pflux = pen \cdot \frac{M_\infty}{Re_\infty} \cdot \left[\begin{array}{l} 0 + 0 \\ -\frac{1}{2}\left(\frac{7}{3}\frac{U_L \mu_L}{\rho_L} + \frac{7}{3}\frac{U_R \mu_R}{\rho_R} \right) \Delta\rho + \frac{1}{2}\left(\frac{7\mu_L}{3\rho_L} + \frac{7\mu_R}{3\rho_R} \right) \Delta\rho U \\ -\frac{1}{2}\left(\frac{7}{3}\frac{V_L \mu_L}{\rho_L} + \frac{7}{3}\frac{V_R \mu_R}{\rho_R} \right) \Delta\rho + \frac{1}{2}\left(\frac{7}{3}\frac{\mu_L}{\rho_L} + \frac{7}{3}\frac{\mu_R}{\rho_R} \right) \Delta\rho V \\ \frac{1}{2}\left\{ \mu_L \left[\frac{\partial\gamma}{Pr\rho_L}(U_L^2 + V_L^2 - E_L) - \frac{7}{3}\left(\frac{U_L^2}{\rho_L} + \frac{V_L^2}{\rho_L} \right) \right] + \right. \end{array} \right.$$

$$\left. \mu_R \left[\frac{\partial\gamma}{Pr\rho_R}(U_R^2 + V_R^2 - E_R) - \frac{7}{3}\left(\frac{U_R^2}{\rho_R} + \frac{V_R^2}{\rho_R} \right) \right] \right\} \Delta\rho +$$

$$\frac{1}{2}\left\{ \mu_L \left[\left(-\frac{\partial\gamma U_L}{Pr\rho_L} + \frac{7}{3}\frac{U_L}{\rho_L} \right] \right) + \mu_R \left[-\frac{\partial\gamma U_R}{Pr\rho_R} + \frac{7}{3}\frac{U_R}{\rho_R} \right] \right\} \Delta\rho U$$

$$+\frac{1}{2}\left\{ \mu_L \left[-\frac{\partial\gamma V_L}{Pr\rho_L} + \frac{7}{3}\frac{V_L}{\rho_L} \right] + \mu_R \left[-\frac{\partial\gamma V_R}{Pr\rho_R} + \frac{7}{3}\frac{V_R}{\rho_R} \right] \right\} \Delta\rho V$$

$$+\frac{1}{2}\left\{ \frac{\partial\mu_L\gamma}{Pr\cdot\rho_L} + \frac{\partial\mu_R\gamma}{Pr\cdot\rho_R} \right\}$$

We can simplify the Energy term abit

$$\text{pflux}(4) = \text{pen}\cdot\frac{M\alpha}{Re\alpha}\left[\frac{1}{2}\left\{\frac{\mu_L}{\rho_L}\left[\frac{\partial\cdot\gamma}{P_r}(U_L^2+V_L^2-E_R)-\frac{7}{3}\left(U_L^2+V_L^2\right)\right]+\right.\right.$$
$$\left.\left.\frac{\mu_R}{\rho_R}\left[\frac{\partial\gamma}{P_r}(U_R^2+V_R^2-E_R)-\frac{7}{3}(U_R^2+V_R^2)\right]\right\}\Delta f \dots\right.$$

We see that this operation of getting the pflux is simply $\frac{1}{2}\left[(G_{11}^L+G_{22}^L)\Delta\vec{g}+(G_{11}^R+G_{22}^R)\Delta\vec{g}\right]\cdot\text{pen}$ thus in the code we will form it as pflux_left, and pflux_right. which are the same except one has Left vectors. and one has right vectors. Thus we write the generic ~~code~~ formulas below.

$$(G_{11}+G_{22})\Delta\vec{g} = \begin{bmatrix} 0 \\ -\frac{7}{3}\frac{\mu_L}{\rho_L}U\Delta\rho+\frac{7}{3}\frac{\mu_L}{\rho_L}\Delta\rho U \\ -\frac{7}{3}\frac{\mu}{\rho_L}V\Delta\rho+\frac{7}{3}\frac{\mu}{\rho}\Delta\rho V \\ \frac{\mu}{\rho}\left[\frac{\partial\gamma}{P_r}\cdot\frac{(U_L^2+V^2-E)}{\rho}-\frac{7}{3}\frac{U^2}{\rho}-\frac{7}{3}V^2\right]\Delta\rho+ \\ \frac{\mu}{\rho}\left[-\frac{\partial\gamma}{P_r}U+\frac{7}{3}U\right]\Delta\rho U+ \\ \frac{\mu}{\rho}\left[-\frac{\partial\gamma V}{P_r}+\frac{7}{3}V\right]\Delta\rho V+ \\ \frac{\mu}{\rho}\frac{\partial\gamma}{P_r}\Delta\rho E \end{bmatrix}\cdot\frac{M\alpha}{Re\alpha}$$

for each Left and Right.

We will code this directly as ~~an~~ seperate routine for boundary terms and interior terms.

We can thus define the pflux

pflux = pen $\frac{1}{2}\left[G_{11}^{L} + G_{22}^{L} + G_{11}^{R} + G_{22}^{R} \right]$ as

$$\text{pen}\cdot\frac{M_{\infty}}{Re_{\infty}}\begin{bmatrix} 0 \\ \tilde{A}_2\cdot\Delta q_1 + \tilde{B}_2\cdot\Delta q_2 \\ \tilde{A}_3\cdot\Delta q_1 + \tilde{C}_3\,\Delta q_3 \\ \tilde{A}_4\,\Delta q_1 + \tilde{B}_4\,\Delta q_2 + \tilde{C}_4\,\Delta q_3 + \tilde{D}_4\,\Delta q_4 \end{bmatrix}$$

Where $\tilde{A}_i = \tilde{A}_i(q_L, q_R)$, $\tilde{B}_i = \tilde{B}_i(q_L, q_R)$ .... etc. for $\tilde{C}$, $\tilde{D}$

Thus we have

$$\tilde{A}_1 = 0$$
$$\tilde{A}_2 = -\frac{7}{6}\left(\frac{\mu_L u_L}{\rho_L} + \frac{\mu_R\cdot u_R}{\rho_R}\right)$$
$$\tilde{A}_3 = -\frac{7}{6}\left(\frac{\mu_L v_L}{\rho_L} + \frac{\mu_R v_R}{\rho_R}\right)$$
$$\tilde{A}_4 = \frac{1}{2}\frac{\mu_L}{\rho_L}\left[\frac{2\gamma(u_L^2+v_L^2-E_L)}{Pr\,\rho_L} - \frac{7}{3}\frac{u_L^2}{\rho_L} - \frac{7}{3}\frac{v_L^2}{\rho_L}\right] +$$
$$\frac{1}{2}\frac{\mu_R}{\rho_R}\left[\frac{2\gamma(u_R^2+v_R^2-E_R)}{Pr\cdot\rho_R} - \frac{7}{3}u_R^2 - \frac{7}{3}v_R^2\right]$$

$$\tilde{B}_1 = 0$$
$$\tilde{B}_2 = \frac{7}{6}\left[\frac{\mu_L}{\rho_L} + \frac{\mu_R}{\rho_R}\right]$$
$$\tilde{B}_3 = 0$$
$$\tilde{B}_4 = \frac{1}{2}\mu_L\left(-\frac{2\gamma u_L}{Pr\,\rho_L} + \frac{7}{3}\frac{u_L}{\rho_L}\right) + \frac{1}{2}\mu_R\left(-\frac{2\gamma u_R}{Pr\cdot\rho_R} + \frac{7}{3}\frac{u_R}{\rho_R}\right) =$$
$$\tilde{B}_4 = \frac{1}{2}\frac{\mu_L u_L}{\rho_L}\left(\frac{7}{3} - \frac{2\gamma}{Pr}\right) + \frac{1}{2}\frac{\mu_R u_R}{\rho_R}\left(\frac{7}{3} - \frac{2\gamma}{Pr}\right)$$
$$\tilde{C}_2 = 0$$
$$\tilde{C}_3 = \frac{7}{6}\left(\frac{\mu_L}{\rho_L} + \frac{\mu_R}{\rho_R}\right)$$

$$\tilde{C}_4 = \frac{1}{2}\mu_L\left[-\frac{\partial\gamma v_L}{P_r \cdot \rho_L} + \frac{7}{3}\frac{v_L}{\rho_L}\right] + \frac{1}{2}\mu_R\left[-\frac{\partial\gamma v_R}{P_r \cdot \rho_R} + \frac{7}{3}\frac{v_R}{\rho_R}\right]$$

$$\tilde{C}_4 = \frac{1}{2}\mu_L\frac{v_L}{\rho_L}\left[\frac{7}{3} - \frac{\partial\gamma}{P_r}\right] + \frac{1}{2}\mu_R\frac{v_R}{\rho_R}\left[\frac{7}{3} - \frac{\partial\gamma}{P_r}\right]$$

$$\tilde{D}_1 = 0$$

$$\tilde{D}_2 = 0$$

$$\tilde{D}_3 = 0$$

$$\tilde{D}_4 = \frac{\mu_L\gamma}{P_r \cdot \rho_L} + \frac{\mu_R\gamma}{P_r \cdot \rho_R} = \frac{\gamma}{P_r}\left(\frac{\mu_L}{\rho_L} + \frac{\mu_R}{\rho_R}\right)$$

For Boundary Faces

our penalty flux reads

$\text{pen}\cdot\left(G_{11}^L + G_{22}^R\right)\Delta\vec{q}$ thus we need to only 1 side of this

Thus we can write this as

$$\text{pen}\,\frac{\mu_{\infty}}{Re_{\infty}}\begin{bmatrix} 0 \\ \tilde{A}_2^B \Delta q_1 + \tilde{B}_2^B \Delta q_2 \\ \tilde{A}_3^B \Delta q_1 + \tilde{C}_3^B \Delta q_3 \\ \tilde{A}_4^B \Delta q_1 + \tilde{B}_4^B \Delta q_2 + \tilde{C}_4^B \Delta q_3 + \tilde{D}_4^B \Delta q_4 \end{bmatrix}$$

$$\tilde{A}_2^B = -\frac{7}{3}\frac{\mu_L U_L}{\rho_L}$$

$$\tilde{A}_3^B = -\frac{7}{3}\frac{\mu_L V_L}{\rho_L}$$

$$\tilde{A}_4^B = \mu_L\left[\frac{\partial\gamma}{P_r \cdot \rho_L}\left(U_L^2 + V_L^2 - E_L\right) - \frac{7}{3}\frac{U_L^2}{\rho_L} - \frac{7}{3}\frac{V_L^2}{\rho_L}\right] = \frac{\mu_L}{\rho_L}\left(\frac{\partial\gamma}{P_r}\left(U_L^2 + V_L^2 - E_L\right) - \frac{7}{3}U_L^2 - V_L^2\right)$$

$$\tilde{B}_2^B = \frac{7}{3}\frac{\mu_L}{\rho_L}$$

$$\tilde{B}_4^B = \mu_L\left[-\frac{\partial\gamma U_L}{P_r \cdot \rho_L} + \frac{7}{3}\frac{U_L}{\rho_L}\right]$$

$$\tilde{C}_3^B = \frac{7}{3}\frac{\mu_L}{\rho_L}$$

$$\tilde{C}_4^B = \mu_L\left[-\frac{2\gamma V_L}{Pr\cdot\rho_L} + \frac{7}{3}\frac{V_L}{\rho_L}\right]$$

$$\tilde{D}_4^B = \frac{2\gamma\mu_L}{Pr\cdot\rho_L}$$

The symmetry term:

We will treat the symmetry term as we treat the penalty term.

Mathematically the symmetry term (According to Koshoro's ...)

$$\left[\begin{bmatrix} G_{11}^{L} & G_{21}^{L} \\ G_{12}^{L} & G_{22}^{L} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi^L}{\partial x} \\ \frac{\partial \phi^L}{\partial y} \end{Bmatrix} \cdot \vec{n} \right] \{\Delta \vec{q}\} = \frac{1}{2}\left[\left[[G_{11}^L]\frac{\partial \phi^L}{\partial x} + [G_{21}^L]\frac{\partial \phi^L}{\partial y}\right] n_x + \left[[G_{12}^L]\frac{\partial \phi^L}{\partial x} + [G_{22}^L]\frac{\partial \phi^L}{\partial y}\right] n_y\right]\{\Delta \vec{q}\}$$

$$\{\vec{q}_L - \vec{q}_R\}$$

The term:

$$\left[\left[[G_{11}^L]\frac{\partial \phi_i^L}{\partial x} + [G_{21}^L]\frac{\partial \phi_i^L}{\partial y}\right] n_x + \left[[G_{12}^L]\frac{\partial \phi_i^L}{\partial x} + [G_{22}^L]\frac{\partial \phi_i^L}{\partial y}\right] \cdot n_y\right]$$

is a 4x4 matrix, which operates on the $\{\Delta \vec{q}\}$ vector.

Thus as with the penalty term we will derive a flux vector for the symmetry term.

$$\text{Symflux} = \left[\left[[G_{11}^L]\frac{\partial \phi_i^L}{\partial x} + [G_{21}^L]\frac{\partial \phi_i^L}{\partial y}\right] n_x + \left[[G_{12}^L]\frac{\partial \phi_i^L}{\partial x} + [G_{22}^L]\frac{\partial \phi_i^L}{\partial y}\right] n_y\right]\{\Delta \vec{q}\}$$

We will now write out the vector as we did with penalty.

Remark: the Symflux is written as on the left

$$\frac{1}{2}\left(\left[[G_{11}^L]\frac{\partial \phi_i^L}{\partial x} + [G_{21}^L]\frac{\partial \phi_i^L}{\partial y}\right] n_x + \left[G_{12}^L\frac{\partial \phi_i^L}{\partial x} + G_{22}^L\frac{\partial \phi_i^L}{\partial y}\right] \cdot n_y + [G_{11}^R]\cdot 0 + [G_{21}^R]\cdot 0 + \ldots\right.$$

Thus for the right we just do the same on the Right side.

$$\text{Sym\_flux} = \frac{M_\infty}{Re_\infty}\begin{bmatrix} 0 \\ A_2\Delta q_1 + B_2\Delta q_2 + C_2\Delta q_3 \\ A_3\Delta q_1 + B_3\Delta q_2 + C_3\Delta q_3 \\ A_4\Delta q_1 + B_4\Delta q_2 + C_4\Delta q_3 + D_4\Delta q_4 \end{bmatrix}$$

Where

$$A_2 = \frac{\mu_L}{\rho_L}\left[-n_x\left(\frac{4}{3}\frac{U_L}{\rho_L}\frac{\partial\phi_i}{\partial x} + \frac{V_L}{\rho_L}\cdot\frac{\partial\phi_i}{\partial y}\right) + n_y\left(\frac{2}{3}\frac{V_L}{\rho_L}\frac{\partial\phi_i}{\partial x} - \frac{U_L}{\rho_L}\frac{\partial\phi_i}{\partial y}\right)\right]$$

$$B_2 = \frac{\mu_L}{\rho_L}\left[\frac{4}{3}\frac{\partial\phi_i}{\partial x}\cdot n_x + \frac{\partial\phi_i}{\partial y}\cdot n_y\right]$$

$$C_2 = \frac{\mu_L}{\rho_L}\left[\frac{n_x}{\rho_L}\cdot\frac{\partial\phi_i}{\partial y} - \frac{2}{3}\frac{n_y}{\rho_L}\cdot\frac{\partial\phi_i}{\partial x}\right]$$

$$A_3 = \frac{\mu_L}{\rho_L}\left[n_x\left(\frac{2}{3}\frac{U_L}{\rho_L}\cdot\frac{\partial\phi_i}{\partial y} - \frac{V_L}{\rho_L}\frac{\partial\phi_i}{\partial x}\right) + n_y\left(-\frac{U_L}{\rho_L}\cdot\frac{\partial\phi_i}{\partial x} - \frac{4}{3}\frac{V_L}{\rho_L}\frac{\partial\phi_i}{\partial y}\right)\right]$$

$$B_3 = \frac{\mu_L}{\rho_L}\left[-\frac{2}{3}n_x\cdot\frac{\partial\phi_i}{\partial y} + n_y\frac{\partial\phi_i}{\partial x}\right]$$

$$C_3 = \frac{\mu_L}{\rho_L}\left[n_x\frac{\partial\phi_i}{\partial x} + \frac{4}{3}n_y\frac{\partial\phi_i}{\partial y}\right]$$

$$A_4 = \frac{\mu_L}{\rho_L}\left[n_x\left\{\frac{\partial\phi_i}{\partial x}\left(\frac{\gamma}{Pr}(U_L^2+V_L^2-E_L) - \frac{4}{3}U_L^2 - V_L^2\right) - \frac{U_LV_L}{3}\frac{\partial\phi_i}{\partial y}\right\} + n_y\left\{-\frac{U_LV_L}{3}\frac{\partial\phi_i}{\partial x} + \frac{\partial\phi_i}{\partial y}\left(\frac{\gamma}{Pr}(U_L^2+V_L^2-E_L) - U_L^2 - \frac{4}{3}V_L^2\right)\right\}\right]$$

$$B_4 = \frac{\mu_L}{\rho_L}\left[n_x \left\{ \frac{\partial \phi_i^L}{\partial x}\left(-\frac{\gamma}{P_r} u_L + \frac{4}{3} u_L\right) - \frac{2}{3} v_L \frac{\partial \phi_i^L}{\partial y} \right\} + n_y \left\{ \frac{\partial \phi_i}{\partial x} v_L + \frac{\partial \phi_i}{\partial y}\left(-\frac{\gamma}{P_r} u_L + u_L\right) \right\}\right]$$

$$C_4 = \frac{\mu_L}{\rho_L}\left[n_x \left\{ \frac{\partial \phi_i^L}{\partial x}\left(-\frac{\gamma}{P_r} v_L + v_L\right) + u_L \frac{\partial \phi_i^L}{\partial y} \right\} + n_y \left\{ -\frac{2}{3} u_L \frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial y}\left(-\frac{\gamma}{P_r} v_L + \frac{4}{3} v_L\right) \right\}\right]$$

$$D_4 = \frac{\mu_L}{\rho_L}\left(n_x \frac{\gamma}{P_r} \frac{\partial \phi_i^L}{\partial x} + \frac{\gamma}{P_r} n_y \frac{\partial \phi_i}{\partial y} \right)$$

Thus from here we can linearize as with the penalty flux.

BR2 for viscous flows: Method of Bassi-Rebay

$$\frac{\partial \vec{q}}{\partial t} + \nabla \cdot f_c(\vec{q}) + \nabla \cdot f_v(\vec{q}, \nabla \vec{q}) = 0$$

Applying Method of weighted residual $\quad f = f_c + f_v$

$$\sum_e \left[\int_{\Omega_e} \phi \frac{\partial \vec{q}}{\partial t} d\Omega + \oint_{\partial \Omega_e} \phi n \cdot f(\vec{q}, \nabla \vec{q}) - \int_{\Omega_e} \nabla \phi \cdot f(\vec{q}, \nabla \vec{q}) \right] d\Omega = 0 \quad \text{- whole system}$$

Let Γ denote union of interior faces, and Σ denote the union of Boundary faces the

$$\int_\Omega \phi \frac{\partial \vec{q}}{\partial t} d\Omega - \int_\Omega \nabla \phi \cdot f(\vec{q}, \nabla \vec{q}) d\Omega + \int_\Gamma \phi^- n^- \cdot f(\vec{q}^-, \nabla \vec{q}^-) +$$

$$\phi^+ n^+ \cdot f(\vec{q}^+, \vec{\nabla} \vec{q}^+) ds + \int_\Sigma \phi n^- \cdot f(\vec{q}^*, \vec{\nabla} \vec{q}^*) ds$$

Note: Here $- =$ interior, $+ =$ exterior e.s. $- = L$, $+ = R$.

We can re-write this as

$$\int_\Omega \phi \frac{\partial \vec{q}}{\partial t} d\Omega - \int_\Omega \nabla \phi \cdot f(\vec{q}, \nabla \vec{q}) d\Omega + \int_\Gamma \phi^L f(\vec{q}^L, \nabla \vec{q}^L) - \phi^R f(\vec{q}^R, \nabla \vec{q}^R) ds$$

$$+ \int_\Sigma \phi^L n \cdot f(\vec{q}^*, \nabla \vec{q}^*) ds$$

The arguments $\vec{q}^*, \nabla \vec{q}^*$ are the boundary values.

Surface terms.

$$\int_\Gamma \phi^- n^- f(\vec{q}^-, \nabla\vec{q}^-) + \phi^+ n^+ f(\vec{q}^+, \nabla\vec{q}^+)\, ds =$$

$$\int_\Gamma (\phi^- h^- + \phi^+ h^+)\, ds + \frac{1}{2}\int_\Gamma \underbrace{[(\nabla\phi^T A_v \cdot n)^+ - (\nabla\phi^T A_v n)^-](\vec{q}^+ - \vec{q}^-)\, ds}_{\text{Sym vector}}$$

$$+\frac{1}{2}\int_\Gamma \overbrace{(\phi^+ - \phi^-)[(n^T A_v \nabla\vec{q})^+ - (n^T A_v \nabla\vec{q})^-]\, d\sigma}^{\text{viscous flux term}} +$$

$$\frac{1}{2}\int_\Gamma (\phi^+ - \phi^-)(\delta_n^+ - \delta_n^-)\, d\sigma + \int_\Sigma \phi \cdot n \cdot f_c(\vec{q}^b)\, d\sigma -$$

$$\int_\Sigma (\nabla\phi^T \cdot A_v n)^- (\vec{q}^b - \vec{q}^-)\, ds + \int_\Sigma \phi (n^T A_v \nabla\vec{q})^b\, d\sigma +$$

$$\phi\, \delta_n^b\, d\sigma = 0 \qquad \forall \phi$$

What we need is a method to get $\delta^\pm$, δ^b

$\delta_n^\pm$ is defined by the following Equation.

$$\int_{\Omega_e} \phi^\pm \delta_n^\pm\, d\Omega = -\frac{1}{2}\int_{\partial\Gamma} \phi^\pm (\vec{n}^T A_v \vec{n})^\pm (\vec{q}^\mp - \vec{q}^\pm)\, d\sigma$$

Similar for the Boundary face.

$$\int_{\Omega_e^-} \phi^- \delta_n^b\, d\Omega = \int_{\partial b} \phi^- (\vec{n}^T A_v \vec{n})^b (\vec{q}^b - \vec{q}^-)\, ds$$

The varicibel $\delta_n(\vec{x}) = \sum_j \hat{\delta}_j \phi_j(\vec{x})$

$$q_n = \sum_j \hat{q}_{nj} \phi_j(\vec{x})$$