Boundary Mapping

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1 Introduction

Computing the residual contribution from domain boundaries requires the evalutation of surface integrals. Computing these integrals requires quadrature surface entities of the mesh (lines in 2-D and faces in 3-D). Employing a surface quadrature rule can be somewhat confusing since the quarature rule specifies points and weights on a surface which is one dimension smaller than the mesh entities. For example if one is solving a PDE in 2-D then surface quadrature is specified by a set of points $\xi_{qp} = \left\{(s_{qp})\right\}$, where s is a surface coordinate. If one is solving a PDE in 3-D then surface quadrature is specified by a set of points $\vec{\xi}_{qp} = (u_{qp}, v_{qp})$, where u, v is a surface coordinate pair.

2 Evaluating Polynomials on Boundaries

The surface quadrature points are specified in a different lower dimension set of coordinates than the element as a whole. However, the basis functions are defined in the standard element and therefore require element and not face coordinates. Essentially we require a map $X(s) : \mathbb{R}^1 \mapsto \mathbb{R}^d$ for mapping a 1-D entity to a d-D entity. We also require a map $X(u, v) : \mathbb{R}^2 \mapsto \mathbb{R}^d$, for mapping 2-D to d-D.

2.1 Example mapping from 1-D to 2-D

Consider an edge on a triangle (depited in Figure 1).

Now it's relatively trival to see what the coordinates are for any particular edge of the triangle. However, given a coordinate $s \in [-1, 1]$ in 1-D on an arbitrary line segment one can obtain any edge of the triangle. All that one has to do is define $\xi(s)$ and $\eta(s)$ and since we are mapping a straight line segment to a straight triangular edge we can use linear functions to define these mappings. In fact you should have guessed by now which functions you should use, which are $\phi_1 = (1-s)/2$ and $\phi_2 = (1+s)/2$. The maps are defined as

$$\xi(s) = \frac{1-s}{2}\xi_1 + \frac{1+s}{2}\xi_2 \eta(s) = \frac{1-s}{2}\eta_1 + \frac{1+s}{2}\xi_2 \tag{1}$$

The following table shows the coefficients and final expression for mapping each edge of the triangle in Figure 1.

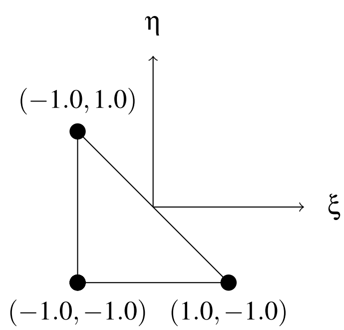

Figure 1: 2-D Triangle element.

Table 1: Energy norm surface contribution of common external aerodynamics boundary conditions.

Triangle Edge Number	(ξ_1, η_1)	(ξ_2, η_2)	$\xi(s)$	$\eta(s)$
1	$(-1, -1)$	$(1, -1)$	$\xi(s) = s$	$\eta(s) = -1$
2	$(1, -1)$	$(-1, 1)$	$\xi(s) = s$	$\eta(s) = -s$
3	$(-1, 1)$	$(-1, -1)$	$\xi(s) = -1$	$\eta(s) = s$

This same techniques can be applied for faces and elements in 3-D and on any element where a linear basis can be defined. The general proceedure is to formulate a nodal basis on the edge/face and then expand the (ξ, η, ζ) coordinates in a linear nodal basis on the edge/face.