Some Questions:

$\vec{n}^T A_v \vec{n}$ is Looks like what?

Let's begin by looking @ the normal dot product with the flux f_c

$$\vec{n} \cdot \vec{f}_c(\vec{g}) = \begin{Bmatrix} \rho u n_x + \rho v n_y \\ (\rho u^2 + p) n_x + \rho v u n_y \\ \rho u v n_x + (\rho v^2 + p) n_y \\ (\rho E + up) n_x + (\rho v E + pv) n_y \end{Bmatrix} \Rightarrow \vec{n} \cdot \vec{f}_c(\vec{g}) \text{ is } \begin{bmatrix} \{E\}, \{F\} \end{bmatrix} \cdot \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$$

? How becomes what does

$\vec{n}^T A_v \nabla \vec{g}$ looks like?

$$\lfloor n_x, n_y \rfloor \underbrace{\begin{bmatrix} [G_{11}] & [G_{12}] \\ [G_{21}] & [G_{22}] \end{bmatrix}}_{\text{Block } 2\times 2} \underbrace{\begin{bmatrix} \lfloor \frac{\partial \vec{g}}{\partial x} \rfloor \\ \lfloor \frac{\partial \vec{g}}{\partial y} \rfloor^T \end{bmatrix}}_{2\times 1} = \lfloor n_x, n_y \rfloor \begin{Bmatrix} [G_{11}]\{\frac{\partial \vec{g}}{\partial x}\} + [G_{12}]\{\frac{\partial \vec{g}}{\partial y}\} \\ [G_{21}]\{\frac{\partial \vec{g}}{\partial x}\} + [G_{22}]\{\frac{\partial \vec{g}}{\partial y}\} \end{Bmatrix}$$

$= n_x E_v + n_y F_v$ - as expected. Thus we'll do asymmetric term example:

$$\lfloor \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \rfloor \begin{bmatrix} [G_{11}] & [G_{12}] \\ [G_{21}] & [G_{22}] \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} =$$

$$\lfloor \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \rfloor \begin{Bmatrix} [G_{11}] n_x + [G_{12}] n_y \\ [G_{21}] n_x + [G_{22}] n_y \end{Bmatrix} = \begin{matrix} \overset{\frac{\partial E_v}{\partial g_x}}{[G_{11}]\{\frac{\partial \phi}{\partial x}\} n_x} + \overset{\frac{\partial E_v}{\partial g_y}}{[G_{12}]\{\frac{\partial \phi}{\partial x}\} n_y} \\ [G_{21}]\{\frac{\partial \phi}{\partial y}\} n_x + [G_{22}]\{\frac{\partial \phi}{\partial y}\} n_y \end{matrix}$$

This not the same as my original derivation or Kosoro's version. I think the it should be written as $\nabla \phi^T [A_v]^T \vec{n}$ then it will come out.

Thus if the symetry term is in fact $\nabla \phi^T \cdot A_v \vec{n}$ then the equation to be solved for the auxilary variables is given by

$$\int_{\Omega_e} \phi^{\pm} \delta_n^{\pm} \, d\Omega = -\frac{1}{2} \int_{\sigma_f} \phi^{\pm} (\vec{n}^T A_v^T \vec{n})^{\pm} \cdot (q^{\mp} - \overset{\pm}{q}) \, ds$$

where σ_f is the sum over all faces on the element face based.

essentially the rhs is

$\delta_n = \sum_j \delta n_j \, \phi_j$ then we do the volume integral

$$M_{ij}^{\pm} = \int_{\Omega_e} \phi_i^{\pm} \, \phi_j^{\pm} \, d\Omega$$

The surface integral will gives a RHS.

$$-\frac{1}{2} \int_{\sigma_f} \phi_i^{\pm} (\vec{n}^T A_v^T \vec{n}) \cdot (\vec{q}^{\mp} - \overset{\pm}{\vec{q}}) \, ds$$

So for the element +

$$M_{ij} = \int_{\Omega_e} \phi_i^{+} \phi_j^{+} \, d\Omega$$

Remark:

1) The $\delta n = \delta_1 n_x + \delta_2 n_y$ per field we do it normally where the MIT Paper does NOt.

2). The δn is face based L, R. Thus in the loop over the faces wee would need to Do this on L, R.

Viscous fluxes for N.S. Equations. (Surface).

We have previously derived terms for Euler and Volume viscous flux. This leaves us with the following integral.

$$-\int_{\partial\Omega_K} \overset{①}{[\![\phi_i]\!]^T\{[A_v]\cdot\vec{\nabla}\vec{q}\}}\, ds - \int_{\partial\Omega_K} \overset{②}{[\![\vec{q}]\!]\{[A_v]^T\vec{\nabla}\phi_i\}}\, ds$$

The above is designed such that it's transpose is the same thus consider transposing it.

① Becomes ② Becomes

$$-\int_{\partial\Omega_K} \overset{①^*}{[\![\vec{q}]\!]\{[A_v]^T\vec{\nabla}\phi_i\}}\, ds - \int \overset{②^*}{[\![\phi_i]\!]^T\{[A_v]\vec{\nabla}\vec{q}\}}\, ds$$

Thus ① = ②*, ② = ①* giving a Symetric viscous operator.

This has implementation implications because our symmetry term has to be correct.

The integral 1 is easily written as

$$-\int_{\partial\Omega_K} \phi_i^+\{E_v n_x + F_v n_y\}\, ds = ①$$

A_v

The integral in 2 is a bit trickier. Thus this gives the following

$$[A_v]^T = \begin{bmatrix} G_{11}^T & G_{21}^T \\ G_{12}^T & G_{22}^T \end{bmatrix},$$

$$-\int_{\partial\Omega_K} (q^+ - q^-)\left\{\left(G_{11}^T\frac{\partial\phi}{\partial x} + G_{21}^T\frac{\partial\phi}{\partial y},\ G_{12}^T\frac{\partial\phi}{\partial x},\ G_{22}^T\frac{\partial\phi}{\partial y}\right)\cdot(n_x, n_y)\right\} ds$$

Then the penalty piece is written out as.

$$\int \mu \{ G_{11} + G_{22} \} \phi_i^+ (q^+ - q^-) \, ds$$

thus giving the entire surface integral as.

$$-\int_{\partial\Omega} \phi_i^+ \{ E_v n_x + F_v n_y \} - (\vec{q}^+ - \vec{q}^-) \left\{ \left[G_{11}^T \frac{\partial \phi_i}{\partial x} + G_{21}^T \frac{\partial \phi_i}{\partial y} \right] n_x + \left[G_{12}^T \frac{\partial \phi_i}{\partial x} + G_{22}^T \frac{\partial \phi_i}{\partial y} \right] n_y \right\}$$
$$+ \mu \{ G_{11} + G_{22} \} (\vec{q}^+ - \vec{q}^-) \cdot \phi_i^+ \, ds$$

with

$G_{11} = \frac{\partial E_v}{\partial q_x}$

$G_{12} = \frac{\partial E_v}{\partial q_y}$

$G_{21} = \frac{\partial F_v}{\partial q_x}$

$G_{22} = \frac{\partial F_v}{\partial q_y}$

Remark in Symmetry term $\frac{\partial \phi_i}{\partial x}, \frac{\partial \phi_i}{\partial y}$ are scalar multiplying Matrices thus terms like

$\left[G_{11}^T \frac{\partial \phi_i}{\partial x} + G_{21}^T \frac{\partial \phi_i}{\partial y} \right]$ are 4x4 Matrices and they operate on 4x1 $(q^L - q^R)$ to give 4x1 vector for Residual.

Linearization of Viscous Residuals:

$$R_v = \int_{\Omega_k} \frac{\partial \phi_i}{\partial x} \vec{E}_v + \frac{\partial \phi_i}{\partial y} \vec{F}_v \, d\Omega_k$$

$$\frac{\partial R_v}{\partial \hat{q}} = \int_{\Omega_k} \frac{\partial \phi_i}{\partial x} \frac{\partial \vec{E}_v}{\partial \hat{q}} + \frac{\partial \phi_i}{\partial y} \frac{\partial \vec{F}_v}{\partial \hat{q}} \, d\Omega_k$$

We are left to derive 2 terms: $\frac{\partial \vec{E}_v}{\partial \hat{q}}, \frac{\partial \vec{F}_v}{\partial \hat{q}}$

Recall $\vec{E}_v = \vec{E}_v(\vec{q}, \vec{\nabla} \vec{q})$, $\vec{F}_v = \vec{F}_v(\vec{q}, \vec{\nabla} \vec{q})$

1). $$\frac{\partial \vec{E}_v}{\partial \hat{q}} = \frac{\partial \vec{E}_v}{\partial (q_x)} \cdot \overset{\nearrow \frac{\partial \phi_j}{\partial x}}{\frac{\partial (q_x)}{\partial \hat{q}}} + \frac{\partial E_v}{\partial (q_y)} \overset{\nearrow \frac{\partial \phi_j}{\partial y}}{\frac{\partial (q_y)}{\partial \hat{q}}} + \frac{\partial E_v}{\partial \vec{q}} \cdot \overset{\nearrow \phi_j}{\frac{\partial \vec{q}}{\partial \hat{q}}}$$

$$\frac{\partial E_v}{\partial \hat{q}} = [G_{11}]\left[\frac{\partial \phi_j}{\partial x}\right] + [G_{12}]\left[\frac{\partial \phi_j}{\partial y}\right] + \left[\frac{\partial E_v}{\partial \vec{q}}\right][\phi_j]$$

$\frac{\partial E_v}{\partial \vec{q}}$ has yet to be derived.

2) $$\frac{\partial F_v}{\partial \hat{q}} = \frac{\partial \vec{F}_v}{\partial (q_x)} \cdot \overset{\nearrow \frac{\partial \phi_j}{\partial x}}{\frac{\partial (q_x)}{\partial \hat{q}}} + \frac{\partial \vec{F}_v}{\partial (q_y)} \cdot \overset{\nearrow \frac{\partial \phi_j}{\partial y}}{\frac{\partial (q_y)}{\partial \hat{q}}} + \frac{\partial F_v}{\partial \vec{q}} \overset{\nearrow \phi_j}{\frac{\partial \vec{q}}{\partial \hat{q}}}$$

$$= [G_{21}]\left[\frac{\partial \phi_j}{\partial x}\right] + [G_{22}]\left[\frac{\partial \phi_j}{\partial y}\right] + \left[\frac{\partial F_v}{\partial \vec{q}}\right][\phi_j]$$

$\frac{\partial F_v}{\partial \vec{q}}$ - has yet to derived.

Derivation of $\frac{\partial E_v}{\partial \vec{q}}, \frac{\partial F_v}{\partial \vec{q}}$

Consider

$$E_v = \begin{bmatrix} 0 \\ \mu \frac{M_\infty}{Re_\infty}\left\{ \frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y} \right\} \\ \mu \frac{M_\infty}{Re_\infty}\left\{ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right\} \\ \mu \frac{M_\infty}{Re_\infty}\left[\frac{\gamma}{Pr}\frac{\partial e}{\partial x} + u\left\{ \frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y} \right\} + v\left\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\} \right] \end{bmatrix}$$

Recall $\mu = \mu(\vec{q})$ thus

$$\frac{\partial E_v}{\partial q} = \begin{bmatrix} 0 \\ \frac{\partial \mu}{\partial q}\frac{M_\infty}{Re_\infty}\left\{ \frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y} \right\} + \mu\frac{M_\infty}{Re_\infty}\left\{ \frac{4}{3}\frac{\partial}{\partial q}\left(\frac{\partial u}{\partial x}\right) - \frac{2}{3}\frac{\partial}{\partial q}\left(\frac{\partial v}{\partial y}\right) \right\} \\ \frac{\partial \mu}{\partial q}\frac{M_\infty}{Re_\infty}\left\{ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right\} + \mu\frac{M_\infty}{Re_\infty}\left\{ \frac{\partial}{\partial q}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial y}\right) \right\} \\ \frac{\partial \mu}{\partial q}\frac{M_\infty}{Re_\infty}\left[\frac{\gamma}{Pr}\cdot\frac{\partial e}{\partial x} + u\left\{ \frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y} \right\} + v\left\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\} \right] + \\ \mu\frac{M_\infty}{Re_\infty}\left[\frac{\gamma}{Pr}\cdot\frac{\partial}{\partial q}\left(\frac{\partial e}{\partial x}\right) + \frac{\partial u}{\partial q}\left\{ \frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y} \right\} + u\left\{ \frac{4}{3}\frac{\partial}{\partial q}\left(\frac{\partial u}{\partial x}\right) - \frac{2}{3}\frac{\partial}{\partial q}\left(\frac{\partial v}{\partial y}\right) \right\} + \right. \\ \left. \frac{\partial v}{\partial q}\left\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\} + v\left\{ \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial q}\left(\frac{\partial v}{\partial x}\right) \right\} \right] \end{bmatrix}$$

Thus we Need

$$\frac{\partial}{\partial q}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial q}\cdot\frac{\partial \vec{q}}{\partial x}\right) = \frac{\partial^2 u}{\partial \vec{q}^2}\cdot\frac{\partial \vec{q}}{\partial x}, \quad \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial \vec{q}^2}\cdot\frac{\partial \vec{q}}{\partial y}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial q}\left(\frac{\partial v}{\partial q}\frac{\partial \vec{q}}{\partial x}\right) = \frac{\partial^2 V}{\partial \vec{q}^2}\cdot\frac{\partial \vec{q}}{\partial x}, \quad \frac{\partial}{\partial q}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial \vec{q}^2}\cdot\frac{\partial \vec{q}}{\partial y}$$

$$\frac{\partial}{\partial \vec{q}}\left(\frac{\partial e}{\partial x}\right) = \frac{\partial^2 e}{\partial \vec{q}^2}\cdot\frac{\partial \vec{q}}{\partial x}, \quad \frac{\partial}{\partial q}\left(\frac{\partial e}{\partial y}\right) = \frac{\partial^2 e}{\partial \vec{q}^2}\cdot\frac{\partial \vec{q}}{\partial y}$$

This further leads to needs for 2nd order derivatives but these are Matrices operating on vectors thus we will write them as such. To avoid doing Matrix Multiplication in the code. By hand we will derive the Matrices in term of Vectors like $\frac{du}{dq}$ etc.

$$\frac{\partial^2 U}{\partial \hat{g}^2} = \begin{bmatrix} -\left(\left\lfloor \frac{\partial u}{\partial \hat{g}} \frac{1}{\rho} - \frac{u}{\rho^2}\frac{\partial \rho}{\partial \hat{g}} \right\rfloor\right) \\ \left\lfloor -\frac{1}{\rho^2}\frac{d\rho}{d\hat{g}} \right\rfloor \\ \lfloor 0 \rfloor \\ \lfloor 0 \rfloor \end{bmatrix}$$

$\frac{1}{\rho}\ \frac{-u}{\rho}\ -\frac{u}{\rho^2} : +\frac{\partial u}{\rho^2}, -\frac{1}{\rho^2}, 0, 0$

$$\Rightarrow \frac{\partial}{\partial \hat{g}}\left(\frac{\partial u}{\partial \hat{g}}\frac{\partial \hat{g}}{\partial x}\right) = \begin{Bmatrix} -\left\lfloor \frac{\partial u}{\partial \hat{g}}\frac{1}{\rho} - \frac{u}{\rho^2}\frac{\partial \rho}{\partial \hat{g}} \right\rfloor \cdot \left\{\frac{\partial \hat{g}}{\partial x}\right\} \\ \left\lfloor -\frac{1}{\rho}\frac{d\rho}{d\hat{g}} \right\rfloor \cdot \left\{\frac{\partial \hat{g}}{\partial x}\right\} \\ 0 \\ 0 \end{Bmatrix}$$

Same for $\frac{\partial}{\partial \hat{g}}\left(\frac{\partial u}{\partial \hat{g}}\cdot\frac{\partial \hat{g}}{\partial y}\right)$ except Replace $\frac{\partial \hat{g}}{\partial x}$, w $\frac{\partial \hat{g}}{\partial y}$

$$\frac{\partial^2 V}{\partial \hat{g}^2} = \begin{bmatrix} -\left(\left\lfloor \frac{\partial v}{\partial \hat{g}}\frac{1}{\rho} - \frac{v}{\rho^2}\frac{\partial \rho}{\partial \hat{g}} \right\rfloor\right) \\ \lfloor 0 \rfloor \\ \left\lfloor -\frac{1}{\rho^2}\frac{\partial \rho}{\partial \hat{g}} \right\rfloor \\ \lfloor 0 \rfloor \end{bmatrix}$$

$\left\lfloor +\frac{v}{\rho^2} + \frac{v}{\rho^2}, 0, -\frac{1}{\rho^2}, 0 \right\rfloor$

$$\Rightarrow \frac{\partial}{\partial \hat{g}}\left(\frac{\partial v}{\partial \hat{g}}\cdot\frac{\partial \hat{g}}{\partial x}\right) = \begin{Bmatrix} -\left\lfloor \frac{\partial v}{\partial \hat{g}}\cdot\frac{1}{\rho} - \frac{v}{\rho^2}\frac{\partial \rho}{\partial \hat{g}} \right\rfloor \cdot \left\{\frac{\partial \hat{g}}{\partial x}\right\} \\ 0 \\ \left\lfloor -\frac{1}{\rho}\frac{\partial \rho}{\partial \hat{g}} \right\rfloor \cdot \left\{\frac{\partial \hat{g}}{\partial x}\right\} \\ 0 \end{Bmatrix}$$

$$\frac{\partial^2 e}{\partial \hat{g}^2} = \begin{bmatrix} \frac{1}{\rho}\left[2u\left\lfloor\frac{\partial u}{\partial \hat{g}}\right\rfloor + 2v\left\lfloor\frac{\partial v}{\partial \hat{g}}\right\rfloor - \left\lfloor\frac{\partial E}{\partial \hat{g}}\right\rfloor\right] - \frac{1}{\rho^2}\left(u^2+v^2-E\right)\frac{\partial \rho}{\partial \hat{g}} \\ -\left(\left\lfloor\frac{\partial u}{\partial \hat{g}}\right\rfloor\frac{1}{\rho} - \frac{u}{\rho^2}\frac{\partial \rho}{\partial \hat{g}}\right) \\ -\left(\left\lfloor\frac{\partial v}{\partial \hat{g}}\right\rfloor\frac{1}{\rho} - \frac{v}{\rho^2}\frac{\partial \rho}{\partial \hat{g}}\right) \\ -\frac{1}{\rho^2}\left\lfloor\frac{\partial \rho}{\partial \hat{g}}\right\rfloor \end{bmatrix}$$

$-\frac{u^2}{\rho^2} - \frac{2u^2}{\rho^2}$

gives

$$\frac{\partial}{\partial \xi}\left(\frac{\partial e}{\partial \xi}\cdot\frac{\partial e}{\partial x}\right) = \begin{cases} \left[\frac{1}{\rho}\left(\partial u\left[\frac{\partial u}{\partial \xi}\right] + \partial v\left[\frac{\partial v}{\partial \xi}\right] - \left[\frac{\partial E}{\partial \xi}\right]\right) - \frac{1}{\rho^2}\left[u^2+v^2-E\right]\left[\frac{\partial \rho}{\partial \xi}\right]\right]\cdot\left\{\frac{\partial \xi}{\partial x}\right\} \\ -\left[\frac{1}{\rho}\frac{\partial u}{\partial \xi} - \frac{u}{\rho^2}\frac{d\rho}{d\xi}\right]\cdot\left\{\frac{\partial \xi}{\partial x}\right\} \\ -\left[\frac{1}{\rho}\frac{\partial v}{\partial \xi} - \frac{v}{\rho^2}\frac{d\rho}{d\xi}\right]\cdot\left\{\frac{\partial \xi}{\partial x}\right\} \\ -\frac{1}{\rho^2}\left[\frac{d\rho}{d\xi}\right]\cdot\left\{\frac{\partial \xi}{\partial x}\right\} \end{cases}$$

$$F_v = \begin{cases} 0 \\ \mu\frac{M_\infty}{Re_\infty}\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\} \\ \mu\frac{M_\infty}{Re_\infty}\left\{\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right\} \\ \mu\frac{M_\infty}{Re_\infty}\left[\frac{\gamma}{Pr}\cdot\frac{\partial e}{\partial y} + u\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + v\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right)\right] \end{cases}$$

$$\frac{\partial}{\partial \xi}(\vec{F}_v) = \begin{bmatrix} 0 \\ \frac{\partial \mu}{\partial \xi}\frac{M_\infty}{Re_\infty}\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\} + \mu\frac{M_\infty}{Re_\infty}\left\{\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial x}\right)\right\} \\ \frac{\partial \mu}{\partial \xi}\frac{M_\infty}{Re_\infty}\left\{\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right\} + \mu\frac{M_\infty}{Re_\infty}\left\{\frac{4}{3}\frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial y}\right) - \frac{2}{3}\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)\right\} \\ \frac{\partial \mu}{\partial \xi}\frac{M_\infty}{Re_\infty}\left[\frac{\gamma}{Pr}\cdot\frac{\partial e}{\partial y} + u\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + v\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right)\right] + \\ \mu\frac{M_\infty}{Re_\infty}\left[\frac{\gamma}{Pr}\cdot\frac{\partial}{\partial \xi}\left(\frac{\partial e}{\partial y}\right) + \frac{\partial u}{\partial \xi}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + u\left(\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial x}\right)\right) + \frac{\partial v}{\partial \xi}\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right) + v\left(\frac{4}{3}\frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial y}\right) - \frac{2}{3}\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)\right)\right] \end{bmatrix}$$

Remark:

1) The above uses the same stuff as $\frac{\partial E_v}{\partial \xi}$ thus we need do NO further derivation we have all the building blocks ~~that~~ we need.

2) In the code there will be **NO** **MATMUL** instead we will build $\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)$ and similar 1 component at a time since so many terms are common or zero.

Thus if we write this out fully

$$\frac{\partial R_v}{\partial \hat{q}} = -\int \frac{\partial \phi_i}{\partial x}\left[G_{11}\left[\frac{\partial \phi_j}{\partial x}\right] + G_{12}\left[\frac{\partial \phi_j}{\partial y}\right] + \frac{\partial E_v}{\partial q}\left[\phi_j\right]\right] + \frac{\partial \phi_i}{\partial y}\left[G_{21}\left[\frac{\partial \phi_j}{\partial x}\right] + G_{22}\left[\frac{\partial \phi_j}{\partial y}\right] + \frac{\partial F_v}{\partial q}\phi_j\right]$$

Surface term:

We have 4 terms to work out here $\frac{\partial R_s^L}{\partial \hat{q}_L}$, $\frac{\partial R_s^R}{\partial \hat{q}_R}$, $\frac{\partial R_s^L}{\partial \hat{q}_R}$, $\frac{\partial R_s^R}{\partial \hat{q}_L}$.

Recall R_s:

$$R_s^L = \int -\phi_i^L \tfrac{1}{2}\left[(E_v^L + E_v^R)n_x + (F_v^L + F_v^R)n_y\right] - \tfrac{1}{2}\left(\left[G_{11}^{T\,L}\frac{\partial \phi_i^L}{\partial x} + G_{21}^{T\,L}\frac{\partial \phi_i^L}{\partial y}\right]\cdot n_x + \left[G_{12}^{T\,L}\frac{\partial \phi_i^L}{\partial x} + G_{22}^{T\,L}\frac{\partial \phi_i^L}{\partial y}\right]\cdot n_y\right)(q^L - q^R) + \mu\tfrac{1}{2}(G_{11}^L + G_{22}^L + G_{11}^R + G_{22}^R)(q^L - q^R)\, ds$$

$$R_s^R = \int \phi_i^R \tfrac{1}{2}\left[(E_v^L + E_v^R)\cdot n_x + (F_v^L + F_v^R)\cdot n_y\right] - \tfrac{1}{2}\left(\left[G_{11}^{T\,R}\frac{\partial \phi_i^R}{\partial x} + G_{21}^{T\,R}\frac{\partial \phi_i^R}{\partial y}\right]\cdot n_x + \left[G_{12}^{T\,R}\frac{\partial \phi_i^R}{\partial x} + G_{22}^{T\,R}\frac{\partial \phi_i^R}{\partial y}\right]\cdot n_y\right)(q^L - q^R) - \mu\tfrac{1}{2}(G_{11}^L + G_{22}^L + G_{11}^R + G_{22}^R)(q^L - q^R)\, ds$$

First we consider R_s^L, we have derived these for Vol. term.

$$\frac{\partial R_s^L}{\partial \hat{q}^L} = \int -\phi_i^L \tfrac{1}{2}\left[\frac{\partial E_v}{\partial \hat{q}}\cdot n_x + \frac{\partial F_v}{\partial \hat{q}}\cdot n_y\right] - \tfrac{1}{2}\left(\left[\frac{\partial G_{11}^T}{\partial \hat{q}}\frac{\partial \phi_i}{\partial x} + \frac{\partial G_{21}^T}{\partial \hat{q}}\frac{\partial \phi_i}{\partial y}\right]\cdot n_x + \left[\frac{\partial G_{12}^T}{\partial \hat{q}}\frac{\partial \phi_i}{\partial x} + G_{22}^T\frac{\partial \phi_i}{\partial y}\right]\cdot n_y\right)(q^L - q^R)$$

$$- \tfrac{1}{2}\left(\left[G_{11}^T\frac{\partial \phi_i}{\partial x} + G_{21}^T\frac{\partial \phi_i}{\partial y}\right]\cdot n_x + \left[G_{12}^T\frac{\partial \phi_i}{\partial x} + G_{22}^T\frac{\partial \phi_i}{\partial y}\right]\cdot n_y\right)\cdot \underset{4\times 4}{[I]}[\phi_j]$$

$$+ \mu\tfrac{1}{2}\left(\frac{\partial G_{11}}{\partial \hat{q}} + \frac{\partial G_{22}}{\partial \hat{q}}\right)(q^L - q^R) + \mu\tfrac{1}{2}(G_{11}^L + G_{22}^L + G_{11}^R + G_{22}^R)[I][\phi_j]$$

Thus we are left only to ponder what do terms like $\frac{\partial G_{11}}{\partial \hat{g}}$ — look like.

Well we know that each time we take a vector derivative we increase the Rank of the tensor.

Thus terms like

$$\frac{\partial G_{11}}{\partial \hat{g}} = \frac{\partial G_{11}}{\partial \vec{g}} \cdot \frac{\partial \vec{g}}{\partial \hat{g}} \quad \text{are Rank 3 tensors.}$$

Now it looks as though the dimensions won't match but that's Not ~~and~~ true because we really ~~want~~ want things like

$$\frac{\partial (G_{11} \cdot (\vec{g}_L - \vec{g}_R))}{\partial \hat{g}} = \left[G_{11}[I] + \frac{\partial G_{11}}{\partial \hat{g}} \cdot (\vec{g}_L - \vec{g}_R) \right] \frac{\partial \vec{g}}{\partial \hat{g}} \quad \text{thus}$$

we can see that $\frac{\partial G_{11}}{\partial \hat{g}} \cdot (\vec{g}_L - \vec{g}_R)$ will become Rank 2 again thus we're ok.

We have 2 terms involving G's thus we need to build the Rank 3 tensors for each G. We will Denote $\frac{\partial G}{\partial \vec{g}}$ as $G'(\vec{g})$, Thus we can begin to work out what G' looks like for each G_{ij} term

Thus we write out the volume Linearization.

$$\frac{\partial R_v}{\partial \hat{q}} = -\int \frac{\partial \phi_i}{\partial x}\left[G_{11}\left[\frac{\partial \phi_j}{\partial x}\right] + G_{12}\left[\frac{\partial \phi_j}{\partial y}\right] + \frac{\partial E_v}{\partial \hat{q}}[\phi_j]\right] + \frac{\partial \phi_i}{\partial y}\left[G_{21}\left[\frac{\partial \phi_j}{\partial x}\right] + G_{22}\left[\frac{\partial \phi_j}{\partial y}\right] + \frac{\partial F_v}{\partial \hat{q}}\right]$$

Thus we can Make Some definitions that will be helpful for Surface term

$$\frac{\partial E_v}{\partial \hat{q}} = G_{11}\left[\frac{\partial \phi_j}{\partial x}\right] + G_{12}\left[\frac{\partial \phi_j}{\partial y}\right] + \frac{\partial E_v}{\partial \hat{q}}[\phi_j]$$

$$\frac{\partial F_v}{\partial \hat{q}} = G_{21}\left[\frac{\partial \phi_j}{\partial x}\right] + G_{22}\left[\frac{\partial \phi_j}{\partial y}\right] + \frac{\partial F_v}{\partial \hat{q}}[\phi_j]$$

Surface terms.

We will Work through these 1 at a time

$$R_s^L = \int -\phi_i^L \tfrac{1}{2}\left[(E_v^L + E_v^R)\cdot n_x + (F_v^L + F_v^R) n_y\right] - \text{symflux} + \text{pflux}\,\phi_i^L$$

$$\frac{\partial R_s^L}{\partial \hat{q}_L} = \int -\phi_i^L \tfrac{1}{2}\left[\left(\frac{\partial E_v^L}{\partial \hat{q}_L}\right) n_x + \left(\frac{\partial F_v^L}{\partial \hat{q}_L}\right) n_y\right] - \frac{\partial}{\partial \hat{q}_L}(\text{symflux}) + \frac{\partial}{\partial \hat{q}_L}(\text{pflux})\cdot \phi_i^L$$

$$\frac{\partial R_s^L}{\partial \hat{q}_R} = \int -\phi_i^L \tfrac{1}{2}\left[\left(\frac{\partial E_v^R}{\partial \hat{q}_R}\right) n_x + \left(\frac{\partial F_v^R}{\partial \hat{q}_R}\right) n_y\right] - \frac{\partial}{\partial \hat{q}_R}(\text{symflux}) + \frac{\partial}{\partial \hat{q}_R}(\text{pflux})\cdot \phi_i^L$$

$$R_5^R = \int \phi_i^R \tfrac{1}{2}\left[(E_v^L + E_v^R) n_x + (F_v^L + F_v^R) \cdot n_y\right] - \text{symvec}$$

$$- \phi_i^R \,\text{pflux} \;\; ds$$

$$\frac{\partial R_5^R}{\partial \hat{q}_R} = \int \phi_i^R \tfrac{1}{2}\left[\frac{\partial E_v^R}{\partial \hat{q}_R} n_x + \frac{\partial F_v^R}{\partial \hat{q}_R} \cdot n_y\right] - \frac{\partial}{\partial \hat{q}_R}(\text{symvec})$$

$$- \phi_i^R \frac{\partial(\text{pflux})}{\partial \hat{q}_R} \; ds$$

$$\frac{\partial R_5^R}{\partial \hat{q}_R} = \int \phi_i^R \tfrac{1}{2}\left[\frac{\partial E_v^L}{\partial \hat{q}_L} n_x + \frac{\partial F_v^L}{\partial \hat{q}_L}\right] - \frac{\partial}{\partial \hat{q}_L}(\text{symvec}) -$$

$$\phi_i^R \frac{\partial(\text{pflux})}{\partial \hat{q}_L} \; ds.$$

Thus we need to get 1st $\frac{\partial(\text{pflux})}{\partial \hat{q}_L}$

<u>Linearization of pflux</u>

$$\text{pflux} = \text{pen.}\, \frac{\mu_\infty}{Re_\infty} \begin{bmatrix} 0 \\ \tilde{A}_2 \Delta q_1 + \tilde{B} \Delta q_2 \\ \tilde{A}_3 \Delta q_1 + \tilde{C}_3 \Delta q_3 \\ \hat{A}_4 \Delta q_1 \;\; \tilde{B}_4 \Delta q_2 + \tilde{C}_4 \Delta q_3 + \tilde{D}_4 \Delta q_4 \end{bmatrix}$$

where $\tilde{(\,)}_i = (\tilde{\,})_i (q_L, q_R)$ thus

we take

$$\frac{\partial\,\text{pflux}}{\partial \vec{q}} \cdot \frac{\partial \vec{q}}{\partial \hat{q}} = \frac{\partial(\text{pflux})}{\partial \vec{q}} \cdot [\phi_j]$$

$$\frac{\partial\, pflux}{\partial \vec{q}_L} = pen.\frac{M_{\infty}}{Re_{\infty}} \left[\begin{array}{l} 0 \\ \frac{\partial \hat{A}_2}{\partial \vec{q}_L} \cdot \Delta q_1 + \hat{A}_2 \frac{\partial(\Delta q_1)}{\partial \vec{q}_L} + \frac{\partial \hat{B}_2}{\partial \vec{q}_L}\Delta q_2 + \hat{B}_2 \frac{\partial(\Delta q_2)}{\partial \vec{q}_L} \\ \frac{\partial \hat{A}_3}{\partial \vec{q}_L} \cdot \Delta q_1 + \hat{A}_3 \frac{\partial \Delta q_1}{\partial \vec{q}_L} + \frac{\partial \tilde{C}_3}{\partial \vec{q}_L} \cdot \Delta q_3 + \tilde{C}_3 \frac{\partial(\Delta q_3)}{\partial \vec{q}_L} \\ \frac{\partial \hat{A}_4}{\partial \vec{q}_L} \cdot \Delta q_1 + \hat{A}_4 \frac{\partial(\Delta q_1)}{\partial \vec{q}} + \frac{\partial \tilde{B}_4}{\partial \vec{q}_L} \Delta q_2 + \tilde{B}_4 \frac{\partial(\Delta q_2)}{\partial \vec{q}_L} + \end{array} \right.$$

$$\frac{\partial \tilde{C}_4}{\partial \vec{q}_L} \Delta q_3 + \tilde{C}_4 \frac{\partial(\Delta q_3)}{\partial \vec{q}_L} + \frac{\partial \tilde{D}_4}{\partial \vec{q}_L} \Delta q_4 + \tilde{D}_4 \frac{\partial(\Delta q_4)}{\partial \vec{q}_L}$$

The right will be the same except $\frac{\partial}{\partial q_R}$

Thus we are now ready to write out explicitly the 4 jacobians:

$$\frac{\partial R_q^L}{\partial \hat{q}^L} = \int -\phi_i^L \tfrac{1}{2}\Big[\Big([G_{11}]^L\Big[\frac{\partial \phi_j}{\partial x}\Big] + [G_{12}]^L\Big[\frac{\partial \phi_j^L}{\partial y}\Big] + \Big[\frac{\partial F_v}{\partial q}\Big][\phi_j^L]\Big)n_x +$$

$$\Big([G_{21}]^L\Big[\frac{\partial \phi_j}{\partial x}\Big] + [G_{22}]\Big[\frac{\partial \phi_j^L}{\partial y}\Big] - \Big[\frac{\partial F_v}{\partial q}\Big][\phi_j^L]\Big)n_y\Big] +$$

$$-\phi_i^L \frac{\partial\, pflux}{\partial \hat{q}^L}[\phi_j^L]$$

$$\phi_i^L \, R_{visc} \, [\phi_j^L]$$

$$\frac{\partial R_S^L}{\partial \hat{q}_R} = \int -\phi_i^L \tfrac{1}{2}\Big[\Big([G_{11}]^R\Big[\frac{\partial \phi_j}{\partial x}\Big]^R + [G_{21}]^R\Big[\frac{\partial \phi_j}{\partial y}\Big]^R + \Big[\frac{\partial F_v}{\partial q}\Big]^R[\phi_j^R]\Big)n_x +$$

$$\Big([G_{21}]^R\Big[\frac{\partial \phi_j}{\partial x}\Big]^R + [G_{22}]^R\Big[\frac{\partial \phi_j}{\partial y}\Big]^R + \Big[\frac{\partial F_v}{\partial q}\Big]^R[\phi_j^R]\Big)n_y\Big]$$

$$+\phi_i^L \frac{\partial\, pflux}{\partial \hat{q}^R}[\phi_j^R]$$

The right side is exactly the same except it's on the right with the sign reversed.

The P0 Residual (Viscous)

For the viscous $p=0$ residual we do the following.

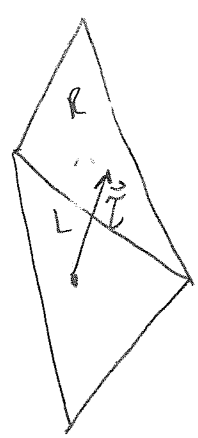
For a face

$$E_v^* = \begin{cases} 0 \\ \left(\frac{\mu_L+\mu_R}{2}\right)\frac{M_\infty}{Re_\infty}\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right) \\ \left(\frac{\mu_L+\mu_R}{2}\right)\frac{M_\infty}{Re_\infty}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \left(\frac{\mu_L+\mu_R}{2}\right)\frac{M_\infty}{Re_\infty}\left(\frac{\gamma}{Pr}\cdot\frac{\partial e}{\partial x} + \left(\frac{U_L+U_R}{2}\right)\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right) + \left(\frac{V_L+V_R}{2}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right) \end{cases}$$

$$F_v^* = \begin{cases} 0 \\ \frac{\mu_L+\mu_R}{2}\frac{M_\infty}{Re_\infty}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \left(\frac{\mu_L+\mu_R}{2}\right)\frac{M_\infty}{Re_\infty}\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right) \\ \left(\frac{\mu_L+\mu_R}{2}\right)\frac{M_\infty}{Re_\infty}\left(\frac{\gamma}{Pr}\cdot\frac{\partial e}{\partial y} + \left(\frac{U_L+U_R}{2}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \left(\frac{V_L+V_R}{2}\right)\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right)\right) \end{cases}$$

Thus the surface Normal flux $\tilde{F}^* = E_v^* n_x + F_v^* n_y$

How to compute $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ etc.



$$\frac{\partial(\,)}{\partial x} = \left[\frac{(\,)_R - (\,)_L}{|\vec{\tau}|^2}\right]\tau_x$$

$$\frac{\partial(\,)}{\partial y} = \left[\frac{(\,)_R - (\,)_L}{|\vec{\tau}|^2}\right]\cdot\tau_y$$

The P=0 viscous residual using Penalty formulation

As an alternative to the previously discretized Method for computing the p=0 viscous residual we will consider the following form given by K. Shahbazi, which looks like it is Based on the BR2 implementation by the MIT people.

$\tilde{F}_V$ - Normal viscous flux at the face is given by

$$\tilde{F}_V = \int_{\Gamma} \nu_0 \{G_{ij}\} [\![\vec{q}_h]\!] \cdot [\![\phi_i]\!] \, ds$$

thus we need only an expression for ν_0 as $\{G_{ij}\} \cdot [\![\vec{q}_h]\!]$ has been derived previously.

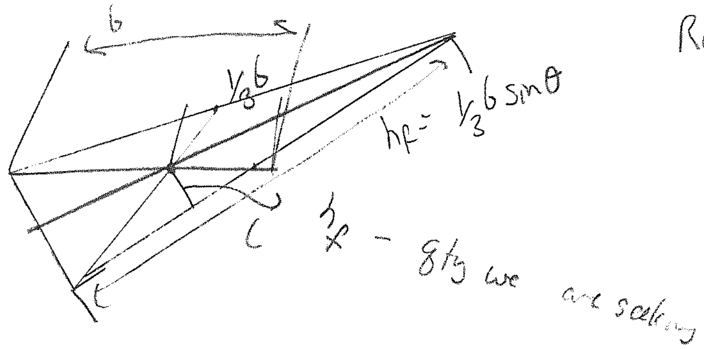
According to Shahbazi ν_0 is the following

$$\nu_0 = \begin{cases} \text{sum of } \perp \text{ distance from cell centers to the face } f \text{ on interior faces. take inverse of this } (\perp) \\ \perp \text{ distance from cell center to face } f \text{ on Boundary faces} \end{cases}$$

? What is the $\perp$ distance from the cell center to the face.

- Consider a triangle as shown

$h = b\sin\theta$



Recall $A = \frac{1}{2} c \cdot h$

$$b\sin\theta = \frac{2A}{c} \Rightarrow$$

$$\boxed{h_f = \frac{1}{3}\left(\frac{2A}{c}\right)}$$

Thus the penalty parameter is

for interior $\left[\frac{1}{3}\left(\frac{2(A_{e_L} + A_{e_R})}{|\vec{n}_f|}\right)\right]^{-1}$

for Boundary $\left[\frac{1}{3}\left(\frac{2(A_e)}{|\vec{n}_{f_{bc}}|}\right)\right]^{-1}$

GMRES - Preconditioned Newton-Krylov Solver:

Follows K. Shahbazi. We will now construct our Newton Solver to follow a Krylov subspace method.

Notation x - vector $[\ ]$ - Matrix, scalars will be underlined.

Review of Basic Krylov Subspaces:

1) Consider a simple Iteration with Preconditioner M.

set $x_{k+1} = x_k + [M]^{-1}(b - [A]x_k)$

If $M = D$ then this is Jacobi

Define

$e_k \equiv [A]^{-1}b - x_k \qquad x_k = x_{k-1} + [M^{-1}](b - [A]x_{k-1})$

If $[M]^{-1} \sim [A]^{-1}$ then $x_k = x_{k-1} + [M^{-1}]b - [M]^{-1}[A]x_{k-1}$

$$e_k = [A^{-1}]b - x_{k-1} - [M]^{-1}b + [M]^{-1}[A]x_{k-1}$$

If we factor this, we have

$$= -(I - [M]^{-1}[A])x_{k-1} + \underbrace{([A]^{-1} - [M]^{-1})}_{\text{Factor out } [A]^{-1} \text{ to the right}} b$$

$$= -(I - [M]^{-1}[A])x_{k-1} + ([I] - [M]^{-1}[A])[A]^{-1}b$$

$$= (I - [M]^{-1}[A])\underbrace{([A]^{-1}b - x_{k-1})}_{\equiv e_{k-1}}$$

$\Rightarrow$

$$e_k = ([I] - [M]^{-1}[A])e_{k-1} = (I - [M]^{-1}[A])^2 e_{k-2} \ldots = (I - [M]^{-1}[A]$$

$$e_k = ([I] - [M]^{-1}[A])^k e_0$$

Taking the norm of Both side

$$\|e_k\| = \|([I]-[M]^{-1}[A])^k e_0\| \leq \|([I]-[M]^{-1}[A])\|^k \cdot \|e_0\|$$

This $\Rightarrow$ that if $\lim_{k\to\infty} \|[I]-[M]^{-1}[A]\| \to 0$ then the iteration with converge. (There is a more rigorous proof that for any e_0 it will converge.

<u>Orthomin(1)</u>:

an improvement on simple iteration is to put dynamically computed parameters in the iteration.

$$x_{k+1} = x_k + \alpha_k(b-[A]x_k)$$

where $r_{k+1} = b-[A]x_{k+1}$
$= b-[A]x_k - \alpha_k[A]r_k = r_k - \alpha_k[A]$

$$\alpha_k = \frac{\langle r_k, [A]r_k\rangle}{\langle [A]r_k, [A]r_k\rangle}$$

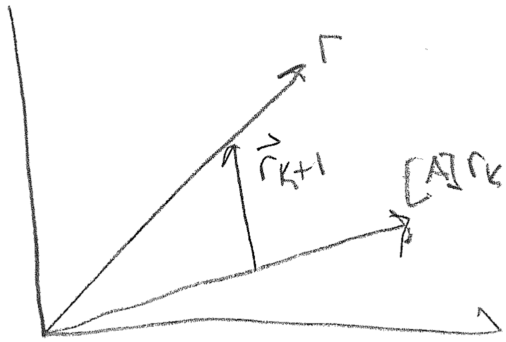
Recall that vector projection a onto b is given as $(\vec{a}\cdot\vec{b})\hat{b}$ $\hat{b}$ - is unit vector in direction of b. Thus look closely at $\alpha_k = \dfrac{\langle r_k, [A]r_k\rangle}{\|[A]r_k\|^2} \sim$ "$(\vec{a}\cdot\vec{b})\cdot\hat{b}$"

Thus it is the projection onto $[A]r_k$, then

r_{k+1} is the orthogonal to $[A]r_k$ because the orthogonal component of $\vec{a}$ to $\vec{b}$ is our ex. is given as $\vec{a}_\perp = \vec{a} - (\vec{a}\cdot\vec{b})\hat{b}$ which say

$r_{k+1} = r_k - \alpha_k[A]r_k$ is just the orthogonal piece of r_k to $[A]r_k$.

pictorally for 2D- $\vec{r}_{K}$

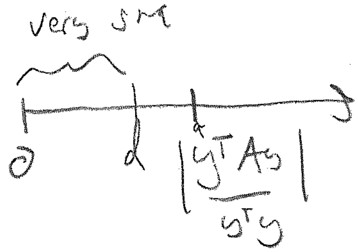


Theorem: The ℓ_2 norm of the residual given by orthomin (1) decreases strictly monotonically for every initial r_0 vector ~~provided~~ if and only if $0 \notin F(A)$ where $F(A) = \frac{y^TAy}{y^Ty}$ for any vector y. ($F(A)$ is kind of like energy of A normalized by y^Ty)

1) We show that ℓ_2-norm of r_{K+1} decreases @ each step.

- Because $0 \notin F(A)$ then it is a closed set i.e. for any vector y $\frac{y^TAy}{y^Ty} \neq 0$ thus there is always a # d this distance from the origin to $F(A)$ such that $\left|\frac{y^TAy}{y^TAy}\right| \geq d$ for all vectors $y \neq 0$

Pictorally



We now begin our proof. Start by taking

$r_{K+1} = r_K - \alpha_K [A] r_K$ - shown previously

take $\langle r_{K+1}, r_{K+1} \rangle = \|r_{K+1}\|^2 =$

$$\langle r_K - \alpha_K [A] r_K, r_K - \alpha_K [A] r_K \rangle = \|r_K\|^2 - 2\alpha_K \langle [A] r_K, r_K \rangle$$
$$+ \alpha_K^2 \langle [A] r_K, [A] r_K \rangle$$

Recall the formula for $\alpha_K = \dfrac{\langle r_K, [A] r_K \rangle}{\langle [A] r_K, [A] r_K \rangle}$

Expanding and simplifying gives

$$\|r_{K+1}\|^2 = \|r_K\|^2 - 2 \frac{\langle r_K, Ar_K \rangle^2}{\|Ar_K\|^2} + \frac{\langle r_K, Ar_K \rangle^2}{\langle Ar_K, Ar_K \rangle^2} \langle Ar_K, Ar_K \rangle$$

$$= \|r_K\|^2 - \frac{\langle r_K, Ar_K \rangle^2}{\|Ar_K\|^2}$$

without loss of generality we can set $\langle r_K, Ar_K \rangle^2 = |\langle r_K, Ar_K \rangle|^2$

$$= \|r_K\|^2 - \frac{|\langle r_K, Ar_K \rangle|^2}{\|Ar_K\|^2}$$

this can be re-written as

recall $\|r_K\|^2 = r^T r$

$$= \|r_K\|^2 \left(1 - \left| \frac{r_K^T A^T r_K}{r_K^T r_K} \right|^2 \cdot \frac{(r^T r)}{\|Ar_K\|^2} \right)$$

Recalling our in-equality $\left| \frac{r_K^T A^T r_K}{r_K^T r_K} \right|^2 \geq d \Rightarrow$

further the schwartz inequality says $\|Ar_K\|^2 \leq \|A\|^2 \|r_K\|^2$

gives

$$\|r_{K+1}\|^2 \leq \|r_K\|^2 \left(1 - d^2 \frac{\cancel{\|r_K\|^2}}{\|A\|^2 \cancel{\|r_K\|^2}} \right) \Rightarrow$$

$$\|r_{K+1}\|^2 \leq \|r_K\|^2 \left(1 - \frac{d^2}{\|A\|} \right)$$

Remark: $\left(1 - \frac{d^2}{\|A\|}\right) < 1$ thus it is convergent and by that much at least.

thus orthomin (1) converges to $A^{-1}b$.

We Are now 1 step closer to the Krylov methods.

Orthomin (2):

Consider iterations as in orthomin(1) by

$X_{K+1} = X_K + \alpha_K P_K$ — in orthomin (1) $P_K = Ar_K$ which gave $r_{K+1} \perp Ar_K$

Remark: In orthomin (1) α_K was chosen to minimize $\|r_{K+1}\|$, how?

$$\|r_{K+1}\|^2 = \langle r_K - \alpha_K Ar_K, r_K - \alpha_K Ar_K \rangle =$$

$$= \|r_K\|^2 - 2\alpha_K (Ar_K, r_K) + \alpha_K^2 \|Ar_K\|^2 =$$

$$\frac{\partial}{\partial \alpha_K} \|r_{K+1}\|^2 = -2\langle Ar_K, r_K \rangle + 2\alpha_K \|Ar_K\|^2$$ — set $= 0$

to find α_K that minimizes this residual.

$0 = -\cancel{2}\langle Ar_k, r_k \rangle + \cancel{2}\alpha_k \|Ar_k\|^2$ gives $\alpha_k = \dfrac{\langle Ar_k, r_k \rangle}{\langle Ar_k, Ar_k \rangle}$

which we have shown gave us

$r_{k+1} \perp Ar_k.$

End Remark:

Thus if we consider the more generic iteration.

$x_{k+1} = x_k + \alpha_k p_k.$

Instead of trying to minimize $\| r_{k+1} \|_2$ - we instead follow the orthogonality path of thinking but instead of $r_{k+1} \perp Ap_k$ we force $r_{k+1} \perp Ap_{k-1}$ so that it is orthogonal to the previous p-vector (i.e. takes history into account).

Thus define a new $\tilde{p}_k$ vector such that

$\tilde{p}_k = r_k - \dfrac{\langle Ar_k, Ap_{k-1} \rangle}{\langle Ap_{k-1}, Ap_{k-1} \rangle} \cdot p_{k-1}.$ thus $r_{k+1} \perp \text{span}\{Ar_k, Ap_{k-1}\}$

r_k - projection of Ar_k onto Ap_{k-1}. Does this give us some orthogonality.

yes: $\langle r_{k+1}, A\tilde{p}_k \rangle = 0$

show:

$\langle r_k - \alpha_k Ap_k, Ar_k - \beta_k Ap_{k-1} \rangle = 0$

Lot of Algebra but it does come out

Further:

$$\langle r_{k+1}, Ap_{k-1} \rangle = \langle r_k, Ap_{k-1} \rangle - \alpha_k \langle A\tilde{p}_k, Ap_{k-1} \rangle = 0$$

Thus the algorithm is given by

x_0, $r_0 = b - Ax_0$, set $p_0 = r_0$

for $k = 1, 2.$

Comput Ap_{k-1}

$$x_k = x_{k-1} + \alpha_{k-1} p_{k-1}, \quad \alpha_{k-1} = \frac{\langle r_{k-1}, Ap_{k-1} \rangle}{\langle Ap_{k-1}, Ap_{k-1} \rangle}$$

compute $r_k = r_{k-1} - \alpha_{k-1} Ap_{k-1}$,

set $p_k = r_k - \beta_{k-1} p_{k-1}$, $\beta_k = \dfrac{\langle Ar_k, Ap_{k-1} \rangle}{\langle Ap_{k-1}, Ap_{k-1} \rangle}$

Repeat.

The above can be played with until C.G for symmetric systems is derived. However, we are not interested in that. We want something for non-symmetric Matrices.

GMRES:

We want a method that minimizes the ℓ_2-norm of $\|r_{k+1}\|$ over the space $r_0 + \text{span}\{Ap_0, Ap_1, \dots Ap_k\} =$
$r_0 + \text{span}\{Ar_0, A^2 r_0, \dots, A^k r_0\}$.

The GMRES method computes an orthonormal basis for the Krylov space given by span $\{r_0, Ar_0, \ldots, A^k r_0\}$

Let us work out GMRES by considering the general example.

Consider solving $[A]\{x\} = \{b\}$.

Let K_n denote the Krylov subspace spanned by

$\langle b, Ab, \ldots, A^{n-1}b \rangle$

Concept: At step n we shall approximate the exact soln $\{x\}_*$ by vector $x_n \in K_n$ that minimizes the norm of residual $\{r_n\} = \{b\} - [A]\{x_n\}$

This is a least squares problem.

- Definition of a ~~least~~ least squares problem: solve $[A]\{x\} = \{b\}$ - rectangular such that nrows > ncols. We "solve" this by minimizing the 2-norm of $\{b\} - [A]\{x\}$.

Setting up the least squares problem ~~via~~ for GMRES.

Let K_n be the nrows $\times n$ Krylov matrix.

$$AK_n = \left[Ab \,\middle|\, A^2 b \,\middle|\, \cdots \,\middle|\, A^n b \,\middle|\, \right]$$

the column space is AK_n.

Our goal is to find a vector $c \in \mathbb{C}^n$ - (real $\mathbb{R}^n$ for us) such that

$\| [A][K_n]\{c\} - \{b\} \|_2 =$ minimum, this could be done via QR factorization with $[A][K_n] = [Q][R]$

once you have $\{c\}$ set $\{X_n\} = [K_n]\{c\}$ and you have your answer.

There is however a better way, use Arnoldi procedure to construct a sequence of Krylov ~~vectors~~ matricies Q_n whose columns ~~span~~ $q_1, q_2, \ldots$ span successive Krylov ~~spaces~~ subspace K_n. Then we write $X_n = Q_n y$ and our problem is

$$\| A Q_n y - b \|_2 = \text{minimum.}$$

Using Arnoldi we can write $[A][Q_n] = [Q_{n+1}][\tilde{H}_n]$ were $\tilde{H}$ is $(n+1) \times n$ upper-left section of the Hessenberg matrix. The Arnoldi procedure is designed to give the Hesenberg reduction $[A] = [Q][H][Q]^T$.

The problem (goal) can be re-writen as

$$\| [Q_{n+1}][\tilde{H}_n]\{y\} - \{b\} \|_2 = \text{minimum.}$$

Now we do some tricky stuff. Recall that orthogonal Matricies like Q_n have the property $Q_n^T = Q_n^{-1}$. thus without changing the norm we can left multiply by Q_n^T

giving.

$$\|\tilde{H}_n y - Q_{n+1}^T b\|_2 = \min.$$

by the construction of the Krylov Matrices

$$Q_{n+1}^T b = \|b\|_2 \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} n$$

Arnoldi's procedure ~~holds~~ shows that

$$K_n = \langle \vec{b}, A\vec{b}, \ldots, A^{n-1}\vec{b} \rangle = \langle \vec{q}_1, \vec{q}_2, \ldots \vec{q}_n \rangle$$

thus Q_{n+1}^T has products that look $\{b\}^T\{b\}$ which

$$= \|\{b\}\|_2$$

Thus our final problem becomes the least squares minimization of

$$\|\tilde{H}\{y\} - Q_{n+1}^T \|b\|_2 \{\begin{smallmatrix} 1 \\ \vdots \\ 0 \end{smallmatrix}\}\|_2 = \min.$$

Thus our algorithm has 3 main parts

1). Find $\tilde{H}$, Q_{n+1}^T

2). Solve least squares

3). $x^{n+1} = Q_n\{y\}$

Step 1) This is the Arnoldi procedure. as outlined in the Book by Trefethen

Arnoldi Procedure:

We need a way to find the Hessenberg Matrix $\tilde{H}_n$ and the vector $\{q_1\} \dots \{q_n\}$. This the arnoldi procedure.

Given q

for $n = 1, 2, 3 \dots$ — Note: for iteration n of GMRES we perform step n of Arnoldi.

$V = [A]\{q_n\}$

for $j = 1, n$

$h_{jn} = \langle \vec{V}, \vec{q}_j \rangle = \tilde{H}_n(j, n)$

$\vec{V} = \vec{V} - h_{jn} \vec{q}_j$

end

$h_{n+1,n} = \|\vec{V}\|$

$q_{n+1} = \frac{V}{\|V\|}$

end

This leaves us with a least squares problem to solve namely

$\|\tilde{H}_n \underline{y} - \|b\| e_1\| = \min.$

$$\tilde{H}_n = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & & \vdots \\ & \ddots & \ddots & \\ & & h_{n,n-1} & h_{n,n} \\ & & & h_{n+1,n} \end{bmatrix}$$

thus it's 1 row too Big.

Which gives us an over determined system written as

$\tilde{H}_n y = \|b\| e_1$ Which must be solved.

$\underline{x} = \underline{x_0} + [Q_n]\{y\}$

Least Squares Problem: let $||b||_2 = \beta$

$[\tilde{H}_n]\{y\} = \beta\, \vec{z}(k+1)$ for the k^{th} iteration of GMRES

Note $\vec{z}_{k=1} = \vec{z}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}$ But as we go through and expand our basis it fills in.

The best way to solve this problem is by take the QR factorization of $[\tilde{H}_n]$ and solving $[R]_{k\times k}\, y = \beta\, \bar{z}_{k\times 1}$ which is simple because we don't care about the Q part of $[\tilde{H}_n]$ we just need to get R part which we store in $[\tilde{H}_n]$ in the code.

Computation of R. If we had a general Matrix then computing QR using Householder would be a good idea. However $\tilde{H}_n$ is upper Hessenberg further if we fine R as we go through the Krylov space (we can save a lot of time.

Consider the Matrix for $n=1$

$[\tilde{H}_n] = \begin{bmatrix} h_{11} \\ h_{21} \end{bmatrix}$ it needs to $\rightarrow 1\times 1$ thus we should use a givens rotation targeted at the 2,1 spot to make it zero.

Given we know how to compute the angle θ or more aptly $\sin(\theta)$ $\cos(\theta)$ we can do the following.

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{Bmatrix} h_{11} \\ h_{21} \end{Bmatrix} : \rightarrow \begin{bmatrix} ||\vec{h}|| \\ 0 \end{bmatrix}$$

this is the givens rotation. The method to compute θ came from R. Fl/gnl.

$||\vec{h}|| = c\cdot h_{11} + s h_{21}$

$\Rightarrow$

$0 = -s h_{11} + c h_{21}$

Going a bit further we has essentially tangled spot 2ol thus this was easy our general Multi-dimensional rotation matrix is identity with $ii = c$, $ji = s$, $ij = -s$, $jj = c$

$$\begin{bmatrix} c & s & \\ -s & c & \\ & & I \end{bmatrix}$$

Now consider when $n = 2$.

$$[\tilde{H}]_2 = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ & h_{32} \end{bmatrix}$$

we seek to make $h_{21} = 0$, $h_{22} = 0$,

Thus as before we set $\hat{h}_{11}$ - (e.g. new value of h_{11}) as $\hat{h}_{11} = c \cdot h_{11}$ $s \cdot h_{21}$ $\hat{h}_{21} \to 0$.

? what happen in 2nd column that's easy

$$\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ 0 & h_{32} \end{bmatrix} = \begin{bmatrix} \hat{h}_{11} & (c \cdot h_{12} + s h_{22}) \\ 0 & (-s h_{12} + c h_{22}) \\ 0 & h_{32} \end{bmatrix} = \begin{matrix} \hat{h}_{12} \\ \hat{h}_{22} \end{matrix}$$

But now we need to apply rotation for h_{32}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix} \begin{bmatrix} \hat{h}_{11} & \hat{h}_{12} \\ 0 & \hat{h}_{22} \\ 0 & h_{32} \end{bmatrix} \to \begin{bmatrix} \hat{h}_{11} & \hat{h}_{12} \\ 0 & (\hat{h}_{22} c + \hat{h}_{32} s) \\ 0 & 0 \end{bmatrix}$$

If we did a few more it would be a bundly clear it it isn't already appent that if we store all columns $n-1$ then we can just apply $n-1$ givens rotations to column and when that is done make a new angle for $h_{n+1,n}$ and h_{nn} apply the last rotation. and that is what's done in the code.

Recall that we must also suitable deal with givens rotations applied to the RHS to set the system consistent.

consider $n = 1$.

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \beta \\ 0 \end{bmatrix} = \begin{Bmatrix} c\beta \\ -s\beta \end{Bmatrix} = \begin{Bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{Bmatrix}$$ now with. $n = 2$

$$\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \beta \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} c\beta \\ -s\beta \\ 0 \end{Bmatrix}$$

Now apply the transform for h_{32}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix} \begin{Bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \hat{\beta}_1 \\ c\hat{\beta}_2 \\ -s\hat{\beta}_2 \end{Bmatrix}.$$

Thus we see the same pattern as with $\tilde{H}$ we can do it as we expand the Basis

As $\hat{\beta}_1$ has not changed each time we add the rotation for $h_{n,n}$ $h_{n+1,n}$ as we rotate $z(n)$, $z(n+1)$ as

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} z(n) \\ z(n+1) \end{Bmatrix}$$

How this is done is very evident in the code.

<u>Re-projection:</u>

once we have formed the $Ry = z$ we solve for y. We now need to project this back to x

$x = x_0$ $Q_n y$ $\quad Q_n = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$

Implementation issues:

Rather than solve for y at each Krylov step we estimate the linear residual as $\frac{\zeta(K+1)}{\|b\|_2}$. If this is small enough then stop increasing the Krylov space and get y and x otherwise increase the space and continue.

Further we cannot afford to store very large Krylov subspaces thus we store only m vectors after that we update X and restart the GMRES again. The code makes this quite clear.

Force Output:

To compute the forces we need to do integrals over each Boundary faces. The pressure based forces will be computed as in the finite volume code. However, we need to be able to compute the viscous based forces.

Recall:

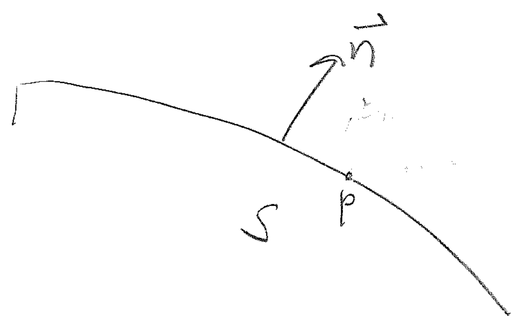
$$\tau_{xx} = \mu\left(\frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y}\right), \quad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$$

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau_{yy} = \mu\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial u}{\partial x}\right)$$

$$F_{vx} = \tau_{xx} \cdot A_{face} \cdot n_x + \tau_{yx} \cdot A_{face} \cdot n_y$$

$$F_{vy} = \tau_{yy} \cdot A_{face} \cdot n_y + \tau_{xy} \cdot A_{face} \cdot n_x$$

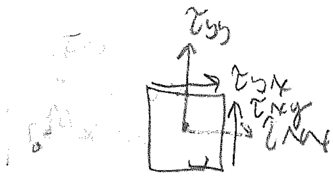
Really the above formulas are finding the component of the surface Area in the x, y dir.



$$ds_x = S \cdot n_x$$

$$ds_y = S \cdot n_y$$

Thus if I have a stress a a point p.



thus the force in the x-dir

$$\tau_{xx}\, ds_x + \tau_{yx} \cdot ds_y.$$

Thats why the normals go with the components it's not to put the stress in the right dir. but to get the proper component of the Area in that dir.

This is <u>1</u> way to physically reason this formula.

Multi-grid Theory

Smoothing factors (Derivation)

Model Problem 1. $-\nabla^2 u = f$

If we regard $L = -\nabla^2$, then a discrete operator L_h can be regarded as the finite Difference approximation of L.

If we want to write L_h as an "iteration" operator it would look like

$$\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} = L_h$$

Which is the equivalent of (Note this is Not a Matrix) It is written in "stencil Notation".

$$L_h = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2}$$

The above operator written stencil Notation indicates that if the center point is i,j then the spots occupied are the indicies the top one $i, j+1$, etc. and the values there are the coefficients

For an iterative scheme consider L_h split as

$$L_h^+ \bar{w}_h + L_h^- w_h = f_h,$$

where w_h - is old approx to u_h
$\bar{w}_h$ is the new approx to u_h.

$\bar{w}_h = S_h w_h$ - where S_h is the smoothing operator.

The Local Fourier Analysis is based on grid functions of the form.

$$\phi(\vec{\theta}, x) = e^{i\vec{\theta}\vec{x}_h}, \quad \text{where } \vec{\theta}, \vec{x} \text{ are vectors of size } D$$

$$= e^{i\theta_1 \frac{x_1}{h_1}} e^{i\theta_2 \frac{x_2}{h_2}}$$

D=2, for 2D, D=3, for 3D etc.

Further

$$L_h \phi(\vec{\theta}, \vec{x}) = \tilde{L}_h(\vec{\theta})\phi(\vec{\theta}, \vec{x}) \quad \text{with}$$

$$\tilde{L}_h = \sum_{K} S_K e^{i\vec{\theta}\cdot\vec{K}} \quad \text{where } \vec{K} = (K_1, K_2)$$

Let's illustrate how to derive $\tilde{L}_h(\vec{\theta})$ using the Laplacian

Ex 1. $\tilde{L}_h(\vec{\theta})$ for discrete Laplacian. $-\nabla^2 u$

a) Recall $\phi(\vec{\theta}, \vec{x}) = e^{i\theta_1 \frac{x_1}{h_1}} e^{i\theta_2 \frac{x_2}{h_2}}$. if $h_1 = h_2$, then $e^{i\theta_1 \frac{x_1}{h}} e^{i\theta_2 \frac{x_2}{h}}$

b) Recall that the laplacian (discrete) is given by

$$\frac{1}{h^2}\Big(4U(x_i, y_i) - U(x_i+h, y_i) - U(x_i-h, y_i) - U(x_i, y_i+h) - U(x_i, y_i-h)\Big)$$

Begin by $L_h \phi =$

$$= \frac{1}{h^2}\Big[4\phi_{i,j} - \phi(x_i+h, y_i) - \phi(x_i-h, y_i) - \phi(x_i, y_i+h) - \phi(x_i, y_i-h)\Big]$$

$$\phi(x_i+h, y_i) = e^{i\theta_1 \left(\frac{x+h}{h}\right)} e^{i\theta_2 \frac{y}{h}} = e^{i\theta_1}\left[e^{i\theta_1 \frac{x}{h}} e^{i\theta_2 \frac{y}{h}}\right] = e^{i\theta}\phi(\vec{x}, \vec{\theta})$$

the rest of the terms follow very similarly to give.

$$Lh = \frac{1}{h^2}(4\phi - e^{i\theta_1}\phi + e^{-i\theta_1}\phi - e^{i\theta_2}\varphi - e^{-i\theta_2}\phi) \Rightarrow$$

Gy

$$Lh = \tilde{L}h(\vec{\theta})\,\phi(\vec{\theta}, \vec{x})$$

$$\tilde{L}h(\vec{\theta}) = \frac{1}{h^2}(4 - e^{i\theta_1} - e^{-i\theta_1} - e^{i\theta_2} - e^{-i\theta_2}),$$ the imaginary part is the frequency, Damping factor is real part.

End Ex. 1.

If we apply an iterative smooth with a splitting such that $Lh = Lh^+ + Lh^-$ then it we write the smoothing operator a $\tilde{S}h\,\phi(\vec{\theta}, \vec{x}) = \tilde{S}h(\vec{\theta})\,\phi(\vec{\theta}, \vec{x})$

$$\tilde{S}h = -\frac{\hat{L}h^-(\theta)}{\hat{L}h^+(\theta)}.$$

We now have the tools needed to compute a smoothing factor.

Ex. Analyzing G/S (Regular Not Red-Black).

$$Lh\,Uh = \frac{-U^m_{i+1,j} + 4U^{m+1}_{i,j} - U^{m+1}_{i-1,j}}{h^2} + \frac{U^m_{i,j+1},\ U^{m+1}_{i,j-1}}{h^2}$$

where $m+1$ denotes an updated value.

$$Lh^+ = \frac{1}{h^2}\begin{bmatrix} & 0 & \\ -1 & 4 & 0 \\ & -1 & \end{bmatrix}, \quad Lh^- = \begin{bmatrix} & -1 & \\ 0 & 0 & -1 \\ & 0 & \end{bmatrix}$$

gives.

$$\hat{L}h^+ = \frac{1}{h^2}(4 - e^{-i\theta_1} - e^{-i\theta_2})$$

$$\hat{L}h^- = -\frac{1}{h^2}(e^{i\theta_1} + e^{i\theta_2})$$

$$\hat{S}_h = \frac{-\hat{L}_h^-(\theta)}{\hat{L}_h^+(\theta)} = \frac{e^{i\theta_1} + e^{i\theta_2}}{(4 - e^{-i\theta_1} + e^{-i\theta_2})}$$

$T_h = \left(-\frac{\pi}{4} \ldots -\frac{\pi}{2}\right)$ and $\frac{\pi}{2} \ldots \pi$

$\mu_{loc} = \max |\hat{S}_h|$ such that $\vec{\theta} \in T_h$

μ_{loc} is really the maximum of the absolute value or magnitude since it is complex.

$$\mu_{loc} = \max \left| \frac{e^{i\theta_1} + e^{i\theta_2}}{4 - e^{-i\theta_1} - e^{-i\theta_2}} \right|, = .5 \text{ for } \theta_1 = \pi/2,\ \theta_2 = \arccos(4/5).$$

Let us now conider ω-Jac. given as,

$$U_h^{m+1} = U_h^m + \omega(Z_h^{m+1} - U_h^m)$$

Note: in practice coded as $Z_h^{m+1} - U_h = -D^{-1}\vec{r}$

$U_h^{m+1} = U_h^m - \omega D^{-1}(\vec{r})$

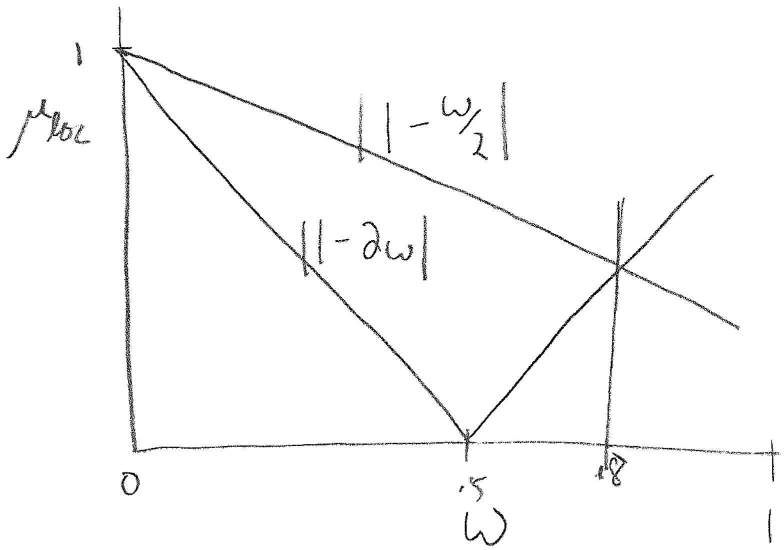
$\vec{r} = A\vec{x} - \vec{b}$

$$\frac{4}{h^2\omega} U_h^{m+1} = \frac{4U_h^m}{h^2\omega} \cdot \frac{U_{i+1,j}^m}{h^2} - \frac{U_{i-1,j}}{h^2} - \frac{U_{i,j+1}}{h^2} - \frac{U_{i,j-1}}{h^2} + 4U_{ij}^m$$

$$L_h^+ = \frac{1}{h^2}\begin{bmatrix} & 0 & \\ 0 & 4/\omega & 0 \\ & 0 & \end{bmatrix}$$

$$L_h^- = \begin{bmatrix} & -1 & \\ -1 & 4(1-\frac{4}{\omega}) & -1 \\ & -1 & \end{bmatrix}$$

If these are plotted. they look like



Thus the point where they meet is the absolute lowest possible μ_{loc} thus $\omega = .8$ is the optimal value for this problem as we use it as this for CFD.

for $\omega = .8$

$\mu_{loc}(.8) = \frac{3}{5} = .6$ which is almost as good as G/S Lex $= .5$.

However R/B G.S. has $\mu_{loc} = .25$ which is the best.

Thus we are motivated to examine a colored G/S type iteration for both parallel and good smoothing.

$$\tilde{S}h(\vec{\theta}) = -\frac{\tilde{L}h^-(\vec{\theta})}{Lh^+(\vec{\theta})}$$

$$Lh^+(\vec{\theta}) = \frac{4}{\omega}$$

$$Lh^-(\vec{\theta}) = -e^{-i\theta_1} - e^{i\theta_1} - e^{-i\theta_2} - e^{i\theta_2} + 4(1-\tfrac{4}{\omega})$$

$$\tilde{S}h(\vec{\theta}) = \frac{+e^{-i\theta} + e^{i\theta_1} + e^{-i\theta_2} + e^{i\theta_2} - 4(1-\frac{4}{\omega})}{\frac{4}{\omega}}$$

$$= \frac{\cos(\theta_1) - i\cancel{\sin(\theta_1)} + \cos(\theta_1) + i\cancel{\sin(\theta_1)} + \cos(\theta_2) - i\cancel{\sin(\theta_2)} + \cos(\theta_2) + i\cancel{\sin(\theta_2)} - 4(1-\frac{4}{\omega})}{\frac{4}{\omega}}$$

$$= \frac{\omega}{4} 2\cos\theta_1 + \frac{\omega}{4} 2\cos(\theta_2) + 1 - \omega$$

$$\hat{S}h = 1 - \frac{\omega}{2}(2 - \cos(\theta_1) - \cos(\theta_2))$$

The points of extrema are easy to find possibilities

$\theta_1 = \pi/2, \quad \theta_2 = \pi/2$ gives $1-\omega$

$\theta_1 = \pi/2, \quad \theta_2 = \pi$ gives $1 - \frac{\omega}{2}(2-1) = 1 - \frac{\omega}{2}$

$\theta_1 = \pi, \quad \theta_2 = \frac{\pi}{2}$ same as

$\theta_1 = \pi, \quad \theta_2 = \pi$ gives $1 - \frac{\omega}{2}(2-(-1)-(-1)) = 1 - 2\omega$

thus

$$\mu_{loc} = \max(|1-\tfrac{\omega}{2}|, |1-2\omega|).$$

An Inplace Approach to Do Matmul for [A] = [L][U] stored in

LI - Matrix

Consider $[L][U]\{x\} = \{z\}$

Let's do a 4x4 example. We also work out

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ & & U_{33} & U_{34} \\ & & & U_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \left\{ \sum_{j=i}^{N} U_{ij} x_j \right\} \quad i = 1, 2, 3, 4 \quad = \{y\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix} \{y\} = \left\{ y_i + \sum_{j=1}^{i-1} L_{ij} y_j \right\}$$

Now if we store $LU \to [A]$ then

$[L][U]\{x\} \Rightarrow [L]\{y\}$ with $y = [U]x$

$$\{y\} = \left\{ \sum_{j=i}^{N} A_{ij} x_j \right\}.$$

$$\{z\} = \left\{ y_i + \sum_{j=1}^{i-1} A_{ij} y_j \right\}$$

Preconditioning:

Consider $[A]x = b$ we would like to Right Precondition this to to solve the Modified System. The following is known as Flexible GMRES

$$[A][M^{-1}]\hat{x} = b \quad \text{with} \quad [M]x = \hat{x}$$

Thus we need to redefine the Residual as

$\vec{r} = b - [A][M^{-1}]\hat{x}$, $= b - [A]x$ thus the residual is unchanged.

Applications to GMRES:

The above applys to GMRES.

1) We need to do the Residual as above

2) We have the product $[A]\{q_n\} \rightarrow [A][M^{-1}]q_n$

thus $[A][M]^{-1}q_n = v$, define $z_n = [M]^{-1}q_n$

thus $v = [A]z_n$ with z_n the solution of

$$[M]z_n = \vec{q}_n$$

3) After we set $\hat{x} = \hat{x}_0 + [Q_n]\{y\}$ but we need x thus $x = [M]^{-1}\hat{x} \Rightarrow$ that

$$x = x_0 + [M]^{-1}[Q_n]\{y\}$$ thus

Solve.

$$[M]x = \underbrace{[M]x_0 + [Q_n]\{y\}}_{\text{vektor,}}$$ or

cols are z_n

getting $x = x_0 \quad [Z_n]\{y\}$

Thus for this we have 2 sets of vectors to Store. B.C. we still need to q's to do the normalization the z's are computed only when we make the Krylov space Bigger.

A Generic and Efficient interpretation of Iterative Methods

We wish to solve $[A]x = b$

There are 2 ways to view this

1). View as $[A]\{x\}^{k+1} = \{b\}$ where $\{x^{k+1}\} = \{x\}^k + \{\delta x\}^{k+1}$

gives

$[M]\{\delta x\}^{k+1} = \{b\} - [A]\{x\}^k$ - then take $[M]$ on the LHS to be some Approximation of $\hat{A}$ as $[M] \approx [D]$ is a jacobi iteration.

- This method is more expensive because it use $[A]\{x\}^k$ rather we should try another way.

2). View as $[A]\{x\} = \{b\}$ - factor $[A]$ as $[A] = [M] + [N]$ where $[M]$ multiplies x^{k+1} and N multiplies x^k. Using the point of view in 1 and inserting $[M] + [N] = [A]$ we get the following..

$$[M]\{x^{k+1} - x^k\} = \{b\} - [A]\{x\} \quad \text{gives}$$

$$M\{x\}^{k+1} = \{x\}^k + M^{-1}(\{b\} - [A]\{x^k\}) \quad \text{gives}$$

$$\{x\}^{k+1} = \cancel{\{x\}^k} + M^{-1}(\{b\} - [N]\{x^k\}) - \cancel{\{x\}^k}$$

yet another way to write this is to add an under relaxation factor.

$$[M](\{x\}^{k+1} - \{x\}^k) = w(\{b\} - [A]\{x\}^k)$$

$$\{x\}^{k+1} = \{x\}^k + [M]^{-1}w(\{b\} - ([M] + [N])\{x\}^k) =$$

$$\{x\}^{k+1} = (1-w)\{x\}^k + [M^{-1}]w(\{b\} - [N]\{x\}^k)$$

To get Different Methods we select [M] to be different things.

Examples

1). [M] = diagonal (or Block Diagonal) - we get Jacobi. if $\omega \neq 1$ we get under relaxed Jacobi

2). If [M] = lower triangular part of [A] we get G.S. if $\omega \neq 1$ we get SOR.

3). If [M] = tri-diag of a line in the Mesh we get a line solve.

Naturally each choice of [M] yields its corresponding choice of [N].

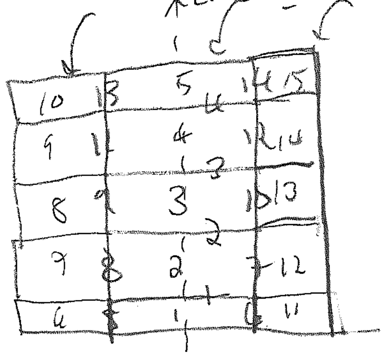
The choices of [N] for 1, 2 are easy.

<u>How this applies to line-implicit solver.</u>

In a line implicit solver we take a line of cells in the Mesh and make the solution implicit along those lines. This results in a Block tridiagonal system. Thus [M] becomes this Block tri-diagonal Matrix. [N] is the off diagonal pieces that are <u>Not</u> in the line.

For example consider the following physical example.

These faces are not in the line.



Let's say we are on line 1.

for element 1 in line 1 we would have row 1 of

$$M = \begin{bmatrix} [[J D]_1, [JOD]_{RL_2} \\ \end{bmatrix}$$

for this same element the row of [N] =

$[0, \ldots, [JOD]_{RL_5}, \ldots, [JOD]_{RL_6}]$ - these are the faces on in the line,

It really is this simple. Just look at all the faces on the element and use only the off diagonals for the faces on that element that are not in the line.

<u>Implementation of the Thomas Algorithm for a line Solver,</u>

Our goal is to use the line solver and a tool for smoothing in M.G. and with GMRES. Both of these approaches require the computation of $[A]\delta U$ or $[A]x$. Thus we need to do as much Thomas preprocessing as possible but maintain an easy way to get the linear residual with the lines.

From a text book Thomas for Blocks can be written as

$C_1' = B_1^{-1} C_1$ for $[A, B, C] = R$

$R_1'' = B_1^{-1} R_1$

For $j = 2, \ldots, m-1$

$B_j' = B_j - A_j C_{j-1}'$

$R_j' = R_j - A_j R_{j-1}''$

$C_j' = B_j'^{-1} C_j$

$R_j'' = B_j'^{-1} R_j'$

end

$B_m' = B_m - A_m C_{m-1}'$

$R_m' = R_m - A_m R_{m-1}''$

$R_m'' = B_m'^{-1} R_m'$

For $j = m-1, \ldots 1$

$R_j''' = R_j'' - C_j' R_{j+1}'''$

end.

The most efficient method from Thomas Point of view is the Jacobian Diagonal with $LU(B_j')$ to over write the appropriate off diagonal with and over the C_j'.

However this results in a difficulty computing the residual.

Thus we can compromise on the processing and factorization stages.

We note that the Thomas in Block form is a Block LU Factorization.

As a setup procedure we will form $LU(B_j')$ and store this

Thus the setup loop is,

$$B_1' = B_1$$

For $j = 2, \ldots, m-1$

$$B_j' = B_j - A_j \underbrace{(B_{j-1}'^{-1} C_{j-1})}_{C'_{j-1}}$$

$LU(B_j')$

end.

$$B_m' = B_m - A_m \underbrace{(B_{m-1}'^{-1} C_{m-1})}_{C'_{m-1}}$$

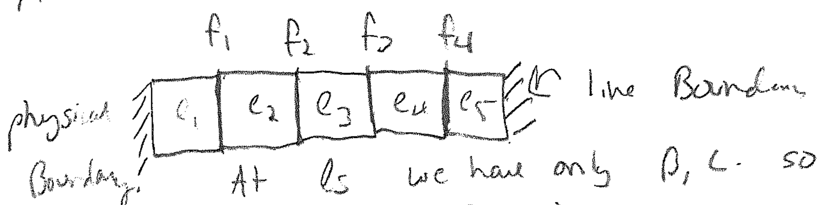
The residual calculation will involve going over the lines and computing.

for $j = 2, m-1$

$$B_j = B_j' + A_j(B_{j-1}'^{-1} C_{j-1})$$

end.

Accessing the elements and faces in the grid.



Linked list indicate.
ls – start
le – end.

At l_s we have only B, C. so as we

aface = ls_face(ls)

As we move up the line ~~we~~ for preprocessing we want out Cface to log is 1 so

for $j = 2, m-1$

cface = ls_face(j-1)

aface = ls_face(j-1)

end.

The B index is ls_elem(j).

The solution loop is

$R_1'' = B_1^{-1} R_1$

For $j = 2, m-1$

$R_j' = R_j - A_j R''_{j-1}$

$R_j'' = B_j'^{-1} R_j'$

end

$R_m' = R_m - A_m R''_{m-1}$

$R_m''' = B_m'^{-1} R_m'$

For $j = m-1, \ldots, 1$

$C_j' = B_j'^{-1} C_j$

$R_j''' = R_j'' - C_j' R_{j+1}'''$

e

Block Thomas

Consider Triadiagonal Matrix stored $[A, B, C]x = f$

Pre-processing Matrix:

$B_1' = B_1$

for $j = 2, m$.

$B_j' = B_j - A_j(B_{j-1}'^{-1} C_{j-1}) \rightarrow B_j' = B_j - A_j(\Gamma_{j-1})$

end.

Solution procedure:

$y_1 = B_1'^{-1} f_1$

for $j = 2, m-1$

$y_j = B_j'^{-1}(f_j - A_j y_{j-1})$

end.

$y_m = B_m'^{-1}(f_m - A_m \cdot y_{j-1})$

$x_m = y_m$

for $j = m-1, \dots, 1$

$x_j = y_j - (B_j'^{-1} C_j)(x_{j+1}) \rightarrow x_j = y_j - z_j$

where

$[B']_j z_j = C_j x_{j+1}$

Multi-Step ODE Integration.

Solving ODE's via multi-step methods

Solving $y' = f(x,y)$ $\quad y' \equiv \frac{dy}{dx}$

Adams Bashfort.

- In constrast to single step methods A-B methods need a starting procedure to give $y_1, \ldots, y_{K-1}$ which are approx. solns at point $x_0 + h, \ldots, x_0 + (K-1)h$ and then a multi-step procedure for computing $y(x_0 + Kh)$. This is then applied recursively to ~~of~~ all unknowns using K steps.

Explicit Adams Method.

denote $x_i = x_0 + ih$.

We want numerical approximations at y_{n+1} and we know $y_n, y_{n-1}, \ldots, y_{n-K+1}$

Equation

$y' = f(x,y)$ $\quad y(x_0) = y_0$ - known initial condition.

Integration of this from x_n to x_{n+1}

$$\int_{x_n}^{x_{n+1}} y'(t)\,dt = \int_{x_n}^{x_{n+1}} f(t, y(t))\,dt \rightarrow$$

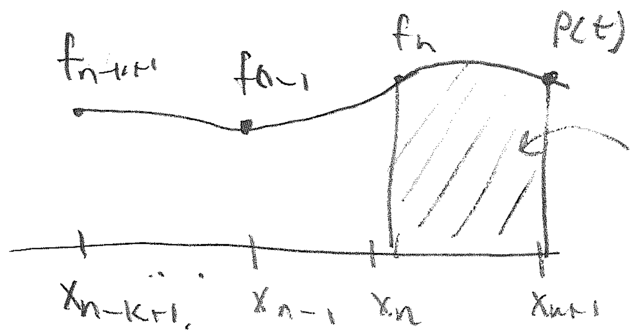
$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(t, y(t))\,dt$$

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(t, y(t))\,dt$$

Since we know $y_{n-K+1}, \ldots, y_n$ then $f(x_i, y_i)$ is known for $i = n-K+1, \ldots, n$

The function $f(t, y(t))$ can then be replaced by the interpolation polynomial $p(t)$ that passes through

$$\{ (x_i, f_i) \mid i = n-k+1, n \}$$



For the interpolation polynomial $P(t)$ we use Lagrange polynomial.

Normally we would Express $P(t)$ as

$$P(t) = \sum_{k=1}^{n} \phi_k L_{k,n}(x) \quad \text{where} \quad L_{k,n} = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

However it is more convient here to do the interpolation in divided Difference form namely

$$P_n(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)(t - t_1) + \dots$$

or in Backwards form

$$P_n(t) = b_0 + (x - x_n)b_1 + (x - x_n)(x - x_{n-1})b_2 +$$

$$(x - x_n)(x - x_{n-1})(x - x_{n-2})b_3 + \dots (x - x_n) \dots (x - x_1)b_n. \quad (1)$$

The form in (1) is going to be how we derive interpolation for Adams Bashford.

To see the connection between divided difference form and Lagrange form consider

$$P_K(x) = P_K(x) + g_K(x) \quad \text{for } n = 1, 2, \ldots, n \quad (1)$$

$$P_n = P_{n-1}(x) + g_n(x)$$

where $g_i(x) = a_K \prod_{j=0}^{K-1} (X - X_j)$ (2)

Using 1 and 2 recursively gives

$$P_n (((a_0 + g_1) + g_2) + g_3) + \ldots \quad \text{gives}$$

$$P_n = a_0 + a_1 (x - x_0) + a_2 (X - X_0)(X - X_1) + \ldots (X - X_{n-1}) \cdots a_n$$

Recall that a given unique interpolation polynomial of order n is given by

$$P_n = \sum_{j=0}^{n} f(x_j) \prod_{\substack{K=0 \\ K \neq j}}^{n} \frac{x - x_K}{x_j - x_K}$$

We can permute the multiplications and additions due to associativity + commutativity,

$$P_n = \sum_{j=0}^{n} \left[\frac{f(x_j)}{\prod_{\substack{K=0 \\ K \neq j}}^{n} (x_j - x_K)} \left(\prod_{\substack{K=0 \\ K \neq j}}^{n} (X - X_K) \right) \right]$$

thus comparing with (2) gives

$$a_n = \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{\substack{K=0 \\ K \neq j}}^{n} (x_j - x_K)}$$

Using the interpolation polynomial P and from lagrange polynomials we know $P_n(t_i) = f(x_i)$. Examing the polynomial we see

$$b_0 = f(x_n)$$

$$f_{n-1} = f(x_n) + (x_{n-1} - x_n) b_1 \Rightarrow \quad b_1 = \frac{f_{n-1} - f_n}{x_{n-1} - x_n} = \frac{f_n - f_{n-1}}{x_n - x_{n-1}} = f[x_n, x_{n-1}]$$

$$f_{n-2} = f(x_n) + (x_{n-2} - x_n) f[x_n, x_{n-1}] + b_2 (x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$b_2 = \frac{f_{n-2} - f x_n - (x_{n-2} - x_n) f[x_n, x_{n-1}]}{(x_{n-2} - x_n)(x_{n-2} - x_{n-1})}$$

adding $\dfrac{f_{n-1} - f_{n-1}}{(x_{n-2} - x_n)(x_{n-2} - x_{n-1})} = 0$ gives

$$= \frac{f_{n-2} - f_{n-1}}{x_{n-2} - x_{n-1} (x_{n-2} - x_n)} + \frac{f[x_n, x_{n-1}](x_{n-1} - x_n)}{(x_{n-2} - x_n)(x_{n-2} - x_{n-1})} - \frac{f[x_n, x_{n-1}](x_{n-2} - x_n)}{(x_{n-2} - x_n)(x_{n-2} - x_{n-1})}$$

$$= \frac{f[x_{n-2}, x_{n-1}]}{x_{n-2} - x_n} + \frac{f[x_n, x_{n-1}](\cancel{x_{n-1} - x_{n-2}})}{(\cancel{x_{n-2} - x_{n-1}})(x_{n-2} - x_n)}$$

$$= \frac{f[x_{n-1}, x_{n-1}] - f[x_{n-1}, x_{n-2}]}{(x_n - x_{n-2})} = f[x_n, x_{n-1}, x_{n-2}] = b_2$$

Thus

$$P_n(x) = f[x_0] + f[x_n, x_{n-1}](x - x_n) + f[x_n, x_{n-1}, x_{n-2}](x-x_n)(x-x_{n-1})$$

$$+ \cdots$$

Thus the interpolation polynomial is known completely ~~by~~ the with the coefficients given by Newton divided differences

If we consider even step sizes h the

$$x = x_n + sh \qquad s \in [0,1]$$

replacing this in our ~~polynom~~ interpolation polynomial,

$$\text{B.C. } x_n - (x_n + sh) = -sh$$

$$P_n(x) = P_n(s) = f[x_n] + f[x_n + sh, x_{n-1}](sh) + f[x_n, x_{n-1}, x_{n-2}](x_n + sh - x_n)$$

$$(x_n + sh - x_{n-1}) + \cdots = f[x_n] - [f(x_{n-1}) - f(x_n)]s +$$

$$\frac{(f(x_{n-2}) - f(x_{n-1}) - f(x_{n-1}) - f(x_n))}{2\cancel{h^2}} \cancel{h}(s+1)s\cancel{h} + \cdots$$

$$= f(x_n) + [f(x_{n-1}) - f(x_n)]s + \frac{s(s+1)}{2}$$

If we denote $f(x_{n-1}) - f(x_n) \equiv \nabla f$ and

$$f(x_{n-2}) - f(x_{n-1}) - f(x_{n-1}) - f(x_n) \equiv \nabla^2 f \text{ then}$$

we obtain the compact form of P_n in terms of Binomial coefficients s of order ~~k~~ j

$$P_n(x) = p(x_n + sh) = \sum_{j=0}^{k-1} (-1)^j \binom{-s}{j} \nabla^j f_n \quad \text{~~thus~~ with}$$

$$\binom{-s}{j} = (-1)^j \frac{s(s+1)(s+2) + \cdots (s + (j-1))}{j!}$$

Note: $-s$ terms come from terms like $x_{n-1} - x_n = -h$

s terms come from $(x_{n-1} - x_n)(x_{n-2} - x_n) = (-h)(-2h) = 2h^2$

Examp~~e~~ for $\cancel{J=2}$ $K=1$

$p_n(x) = f_n - s\nabla^1 f_n$

Using the Expression in

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(t, y(t))\,dt \quad \text{gives}$$

$$y(x_{n+1}) = y(x_n) + h\sum_{j=0}^{K-1}\int_0^1 (-1)^j\binom{-s}{j}\,ds\,\nabla^j f_n = y(x_n) + h\sum_{j=0}^{K-1}(-1)^j\int_0^1\binom{-s}{j}\,ds$$

$$= y(x_n) + h\sum_{j=0}^{K-1}\gamma_j\nabla^j f_n \quad \text{with } \gamma_j = (-1)^j\int_0^1\binom{-s}{j}ds$$

Let's do this for $K=1$

$$\gamma_0 = h\int_0^1 (1)\,ds = h(1-0) = h \quad \text{gives}$$

$$y(x_{n+1}) = y(x_n) + h f_n$$

for $K=2$ $j=0, 1$

$$\gamma_0 = h$$

$$\gamma_1 = -\int_0^1 s\,ds = \frac{s^2}{2} = -\frac{1}{2}$$

$$y(x_{n+1}) = y(x_n) + h\left(f_n + -\frac{1}{2}(f_{n-1} - f_n)\right) = y(x_n) + h\left(\frac{3}{2}f_n - \frac{1}{2}f_{n-1}\right)$$

for $K=3$ $j_{max}=2$

$$\binom{-s}{j} = (-1)^j\left[\frac{s(s+1)+\dots(s+(j-1))}{j!}\right]$$

Need $\gamma_2 = \cancel{(?)}$ for $j=2$ $\binom{-s}{2} = (1)\left[\frac{s(s+1)}{2}\right]$

$$\gamma_2 = -1\int_0^1 \frac{s^2+s}{2}\,ds = \frac{s^3}{6} + \frac{s^2}{4}\Big|_0^1 = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$$

Thus as we find coefficients for lower order we keep ~~finding them for~~ them for higher order

So for $k=3$ $\quad \gamma_0 = 1,\ \gamma_1 = 1/2,\ \gamma_2 = \frac{5}{12}$ gives

$$y(x_{n+1}) = y(x_n) + h\left(f_n - \tfrac{1}{2}(f_{n-1} - f_n) + \tfrac{5}{12}\left(f(x_{n-2}) - f(x_{n-1}) - f(x_{n-1}) + f(x_n)\right)\right)$$

$$= y(x_n) + h\left(\frac{18}{12} f_n + \frac{5}{12} f_n - \frac{6}{12} f_{n-1} - \frac{10}{12} f_{n-1} + \frac{5}{12} f_{n-2}\right)$$

$$= y(x_n) + h\left(\frac{23}{12} f_n - \frac{16 f_{n-1}}{12} + \frac{5}{12} f_{n-2}\right).$$

Those all match the Book and they all come from my Math.

This is enough Adams Method we are really interested in BDF Formulas.

BDF Methods.

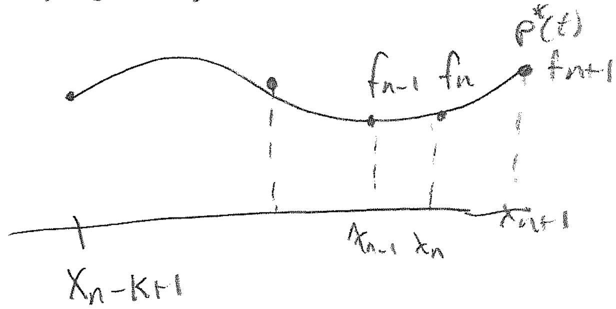
These Methods are not Based on Integration (Quadrature) instead they numerically differentiate a function.

Assume approximations $y_{n-k+1}, \dots y_n$ are known. to derive expression for y_{n+1} we use the interpolating polynomial $q(x)$

$$q(x) = q(x_n + sh) = \sum_{j=0}^{k} (-1)^j \binom{-s+1}{j} \nabla^j y_{n+1}$$

One then asks where does this polynomial come from.

It is the same polynomial we have been using in Adams methods except it includes y_{n+1}, x_{n+1} in it's domain



To derive this polynomial proceed as before (2 terms is enough to verify Book expression)

$$q(x) = a_0 + a_1(x - x_{n+1}) + a_2(x - x_{n+1})(x - x_n) + \ldots$$

$$q(x_{n+1}) = f_{n+1} \Rightarrow a_0 = f_{n+1}$$

$$q(x_n) = f_n = f_{n+1} + a_1(x_n - x_{n+1}) \Rightarrow a_1 = \frac{f_{n+1} - f_n}{x_{n+1} - x_n} = f[x_{n+1}, x_n]$$

$$f_{n-1} = f_{n+1} + f[x_{n+1}, x_n](x_{n-1} - x_{n+1}) + a_2(x_{n-1} - x_{n+1})(x_{n-1} - x_n)$$

$$\frac{f_{n-1} - f_{n+1}}{(x_{n-1} - x_{n+1})(x_{n-1} - x_n)} - \frac{f[x_{n+1}, x_n](x_{n-1} - x_{n+1})}{(x_{n-1} - x_{n+1})(x_{n-1} - x_n)} = a_2$$

add $+\dfrac{f_n - f_n}{(x_{n-1} - x_{n+1})(x_{n-1} - x_n)}$ gives

$$\frac{(f_n - f_{n+1}) + (f_{n-1} - f_n)}{(x_{n-1} - x_{n+1})(x_{n-1} - x_n)} - \frac{f[x_{n+1}, x_n](x_{n-1} - x_{n+1})}{(x_{n-1} - x_{n+1})(x_{n-1} - x_n)} = a_2$$

$\downarrow$

$$+\frac{f[x_n, x_{n-1}]}{(x_{n-1} - x_{n+1})} - \frac{f[x_{n+1}, x_n](x_{\cancel{n+1}} - x_n)}{(x_{n-1} - x_{n+1})(x_{n-1} - x_n)} - \frac{f[x_{n+1}, x_n](x_{n-1} - x_{\cancel{n+1}})}{(x_{n-1} - x_{n+1})(x_{n-1} - x_n)} = a_2$$

$$a_2 = \frac{\left(f[x_n, x_{n-1}] - f[x_{n+1}, x_n]\right)}{x_{n-1} - x_{n+1}} = \frac{f[x_{n+1}, x_n] - f[x_n, x_{n-1}]}{x_{n+1} - x_n}$$

$$= f[x_{n+1}, x_n, x_{n-1}]$$

For $g(x+sh)$ and equal spacing $x_{n+1}-x_n=h$ we get the following

$$g(x_n+sh)=f_{n+1}+\frac{f_{n+1}-f_n}{\cancel{h}}\overset{(-1+s)\cancel{h}}{(x_n+sh-x_{n+1})}+\frac{(f_{n+1}-f_n-f_n+f_{n-1})}{2h^{\cancel{2}}}$$

$$(x_n+sh-x_{n+1})(x_n+sh-x_n)\cancel{h^2}$$

$$\downarrow$$

$$\frac{(-1+s)(s)}{2}=\frac{(s^2-s)}{2}$$

Thus again we have binomial coefficients except now they are give by

$$(-1)^k\frac{(s-1)(s-1+1)\dots(s-1+(k-1))}{k!}$$

check: $k=1$

$(-1)^1(-1)^1(s-1)=(s-1)$ ✓

$(-1)^2(-1)^2\frac{(s-1)(s-1+1)}{2}=\frac{s^2-s}{2}$ ✓

Thus we can represent our polynomial

$$g(x+sh)=\sum_{j=0}^{k}(-1)^j\binom{-s+1}{j}\nabla^j y_{n+1}$$ - Note here we interp the solution y not $y'=f$.

For BDF formula we have a constraint the $\left.\frac{dg}{dx}\right|_{x_{n+1-r}}=f(x_{n+1-r},y_{n+1-r})$ for $r=1$ we get explicit formulae for $r=0$ we get implicit formulae.

$$\frac{dg}{dx}=\frac{dg}{ds}\cdot\frac{ds}{dx}=\left.\frac{dg}{ds}\cdot\frac{1}{h}\right|_{n+1}=f_{n+1}\Rightarrow h f_{n+1}=\sum_{j=0}^{k}(-1)^j\left.\frac{d}{ds}\binom{-s+1}{j}\right|_{s=1}\nabla^j y_{n+1}$$

Note x_{n+1} is $x(s=1)$

$$=\sum \delta_j^*\nabla^j y_{n+1}=f_{n+1}\cdot h$$ this ~~gives~~

We need δ_j^*

$$\delta_j^* = (-1)^j \frac{d}{ds}\binom{-s+1}{j}$$

Recall $(-1)^j \binom{-s+1}{j} = \frac{1}{j!}(s-1)s(s+1)\cdots(s+j-2)$

for $j=1$ $\quad \delta_1^* = 1$

for $j=2$ $\quad \delta_2^* = \frac{d}{ds}\frac{1}{j!}(s^2-s)\Big|_1 = \frac{1}{j}(2s-1)\Big|_1 = \frac{1}{j}$

$\delta_j^* = \frac{1}{j} \quad j \geq 1 \qquad \delta_0^* = 0$

Thus our formula is

$$\sum_{j=0}^{k} \delta_j^* \nabla^j y_{n+1} = h f_{n+1}$$

for $k=0$ we get $0 = h f_{n+1}$ – garbage useless

for $k=1$ $\quad y_{n+1} - y_n = h f_{n+1}$; BDF1

for $k=2$ $\quad \frac{(y_{n+1} - y_n) - (y_n - y_{n-1})}{2} + y_{n+1} - y_n = h f_{n+1}$

$$\frac{y_{n+1}}{2} - y_n + \frac{1}{2}y_{n-1} + y_{n+1} - y_n = h f_{n+1}$$

gives $\frac{3y_{n+1}}{2} - 2y_n + \frac{y_{n-1}}{2} = h f_{n+1}$; BDF2

for $k=3$

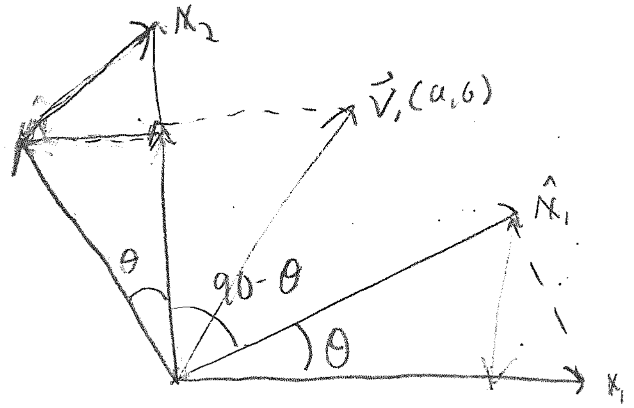
$$y_{n+1} - y_n + \frac{(y_{n+1} - y_n) - (y_n - y_{n-1})}{2} + \frac{(y_{n+1} - 2y_n + y_{n-1} - y_n + 2y_{n-1} - y_{n-2})}{3} = \frac{11}{6}y_{n+1} - 3y_n + \frac{3}{2}y_{n-1} - \frac{1}{3}y_{n-2}$$

; BDF 3.

Note: That compared to Adams we get K order accuracy (I have note yet proven this out I known from expierence $K=2$: BDF2 is 2nd order) not $K+1$ order. But we need to store solutions not residuals. Also we have yet to study the stability of BDF's v.s. Adams, However BDF's are well known for their amenability to Stiff equations (which we have) Therefore we will begin time-accurate computations with BDF-2, then move to IRK type schemes.

Coordinate Transforms:

Consider going from 1 coordinate system to another
i.e. from x_1, x_2, $\hat{x}_1, \hat{x}_2$ with angle θ between the



given $\vec{V} \in (x_1, x_2)$ what are it's components in $\hat{x}_1, \hat{x}_2$ $\quad \vec{V} = \begin{Bmatrix} a \\ b \end{Bmatrix}$

$\hat{x}_1$: The components of $\vec{V}$ in $\hat{x}_1$

$a\cos\theta + b\sin\theta$

$\hat{x}_2$: The components of $\vec{V}$ in $\hat{x}_2$

$-a\sin\theta + b\cos\theta$

gives
$[T]\begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} c \\ d \end{Bmatrix}$ where $[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\vec{V}_{\in \hat{x}_1 \hat{x}_2} = [T]\vec{V} \Rightarrow \quad \vec{V}_{\in x_1, x_2} = [T]^{-1}\{\vec{V}_{\in \hat{x}_1, \hat{x}_2}\}$

is this true, Let's check

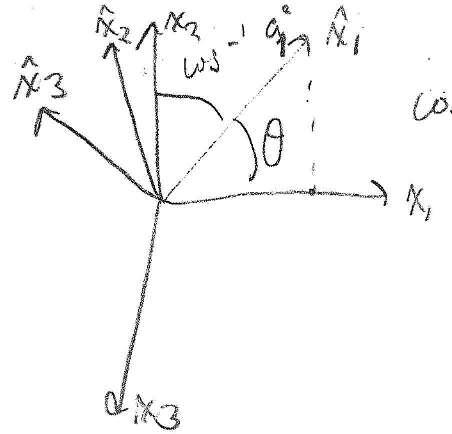
gives $\vec{v}_{\in \hat{x}_1, \hat{x}_2} = \begin{Bmatrix} c \\ d \end{Bmatrix}$

x_i: Components in x
$c\cdot\cos\theta - d\sin\theta$
$c\sin\theta + d\cos\theta$ $\Rightarrow [T]^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$[T]^{-1}[T]^{-1} = \begin{bmatrix} \cos & \sin \\ -\sin & \cos \end{bmatrix}\begin{bmatrix} \cos & -\sin \\ \sin & \cos \end{bmatrix} = \begin{bmatrix} 1 & \overset{0}{\cancel{-\cos\sin + \cos\sin}} \\ \underset{0}{\cancel{-\sin\cos + \cos\sin}} & 1 \end{bmatrix}$ ✓

More formally:

We can define the Rotation Matrix in terms of direction cosines.



$\cos\theta = \hat{x}_1 \cos$

$a_1^2 = \cos(\theta_{\hat{x}_1}^{\hat{x}_2})$

thus $\cos^{-1}(a_1^2) = \theta_{x_1}^{\hat{x}_2}$ - angle between $\hat{x}_1$ and $\hat{x}_2$.

Thus we have a direction cosine for each ~~vector~~ unit vect $\hat{i}$ relative to the original coordinate system

i.e. $a_1^3 = \dfrac{\hat{x}_3 \cdot x_3}{|\hat{x}_3|} = \cos(\theta_{\hat{x}_1}^{x_3})$ this is the direction cosine For 3D there are 3 of these

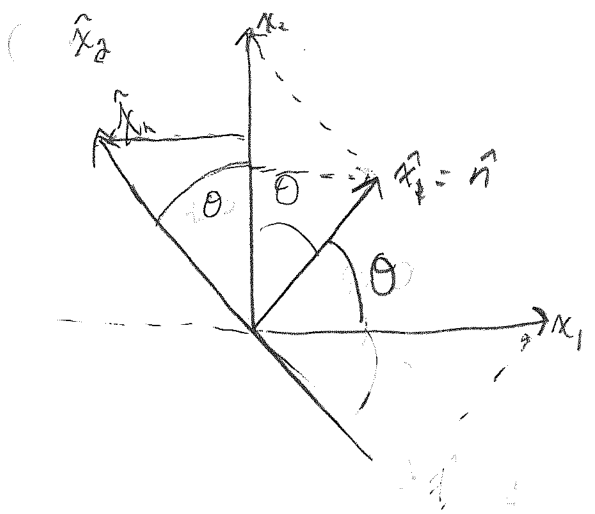
$a_i^{\hat{j}}$

$$\begin{array}{cccc} & \hat{i} & \hat{j} & \hat{k} \\ i & a_1^1 & a_2^1 & a_3^1 \\ & & a_2^2 & a_3^2 \\ j & a_1^2 & & \\ k & a_1^3 & a_2^3 & a_3^3 \end{array}$$

This can be represented as a matrix.

$$\begin{bmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{bmatrix} = [A]$$

A specific interest are relations to a normal coordinate system



$\hat{x}_1 \cos\theta - x \sin\theta$

$x_1 \sin\theta + x_2 \cos\theta$

$\hat{x}_1 = 1$, $\hat{x}_2 = 1$ i.e. $\|\vec{n}\|, \|\vec{\tau}\| = 1$

thus

$\hat{x}_1 \cos\theta = n_x \Rightarrow n_x = \cos\theta$

$\hat{x}_2 \sin\theta = n_y \qquad n_y = \sin\theta$

$-\hat{x}_2 \sin\theta = \tau_x = -n_y$ we already new this.

$\hat{x}_2 \cos\theta = \tau_y = n_x$

Thus for transformations to and from a normal coordinate systems we have

$$[T]^{-1} = \begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix}, \quad T = \begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix}$$

Let's use a 2D version of our rotation matrix $[A]$ and see if we can work out what we discovered above.

$$a_1^1 = \cos(\theta_{\hat{x}_1}^{x_1}) = \frac{(1,0)\cdot(n_x, n_y)}{\|\vec{n}\| \|(1,0)\|} = n_x$$

$$a_1^2 = \cos(\theta_{\hat{x}_1}^{x_2}) = (0,1)\cdot(n_x, n_y) = n_y$$

$$a_2^1 = \cos(\theta_{\hat{x}_2}^{x_1}) = (1,0)(-n_y, n_x) = -n_y$$

$$a_2^2 = \cos(\theta_{\hat{x}_2}^{x_2}) = (0,1)(-n_y, n_x) = n_y$$

$\Rightarrow [A] = [T]^{-1}$ i.e. transform from $\hat{x} \to x$. a_{col}^{row}

$$\begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} \begin{Bmatrix} \bar{v}_x \\ \bar{v}_y \end{Bmatrix} = \begin{Bmatrix} v_x \\ v_y \end{Bmatrix} = \begin{matrix} a_1^1 \bar{v}_x + a_2^1 \bar{v}_y \\ a_1^2 \bar{v}_x + a_2^2 \bar{v}_y \end{matrix} \Rightarrow v_j = a_i^j \bar{v}_i$$

So to transform unit vector $e_j = a_i^j \bar{e}_i$ or for the transpose $\bar{e}_i = a_i^j e_j$

Using these to manipulate tensors:

Let us first define a tensor component

$$\tau_{ij} = \vec{e}_i \underset{\sim}{\tau} \vec{e}_j \quad \text{ie} \quad [1,0,0] \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = [1,0,0] \begin{Bmatrix} \tau_{11} \\ \tau_{21} \\ \tau_{31} \end{Bmatrix} = \tau_{11}$$

So we need a way to go from

$\vec{e} \to \bar{\vec{e}}$

$$[A]^T = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_{33} \end{bmatrix} \quad \text{which takes use from } e_j \to \bar{e}_j$$

$$\bar{v} = [A]^T\{v\} = \begin{matrix} a_1^1 v_1 + a_1^2 v_2 + a_1^3 v_3 \\ a_2^1 v_1 + a_2^2 v_2 + a_2^3 v_3 \\ a_3^1 v_1 + a_3^2 v_2 + a_{33} v_3 \end{matrix}$$

$\bar{v}_i = a_i^j v_j$ here v_j, v_i etc are scalar components

To express T_{ij} by rotation from $\bar{T}_{ij}$

$e_i = a_j^i \bar{e}_j$ gives

$$T_{kl} = a_i^k \bar{e}_i \underset{\sim}{T} \cdot a_j^l \bar{e}_j = a_i^k a_j^l \bar{T}_{ij}$$

$$T_{11} = a_i^1 a_j^1 \bar{T}_{ij}$$

Thus for 2D

$$\bar{T} = \begin{bmatrix} \bar{T}_{11} & \bar{T}_{12} \\ \bar{T}_{21} & \bar{T}_{22} \end{bmatrix}$$

$$T_{11} = a_1^1 \cdot (a_1^1 \bar{T}_{11} + a_2^1 \bar{T}_{12}) + a_2^1 (\bar{T}_{21} a_1^1 + \bar{T}_{22} \cdot a_2^1)$$

$$\begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} \begin{bmatrix} \bar{T}_{11} & \bar{T}_{12} \\ \bar{T}_{21} & \bar{T}_{22} \end{bmatrix} \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{bmatrix} = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} \bar{T}_{11} a_1^1 + \bar{T}_{12} a_2^1 & ; (&) \\ \bar{T}_{21} a_1^1 + \bar{T}_{22} a_2^1 & ; (&) \end{bmatrix}$$

give $T_{11} = a_1^1 (a_1^1 \bar{T}_{11} + a_2^1 \bar{T}_{12}) + a_2^1 (\bar{T}_{21} a_1^1 + \bar{T}_{22} a_2^1)$

Thus we can say the transformation from $\bar{x} \to x$ of tensor $\bar{T}$ is given by the following convient Matrix Notation.

$$[T] = [A][\bar{T}][A]^T \text{ and } \bar{T} = [A]^{-1}[T][A]^{-T}$$ howevere $[A]$ is orthogonal thus $[A]^{-1} = [A]^T$ and $[A]^{-T} = [A]$ so

$\bar{T} = [A]^T[T][A]$ - thus we simply figure out $[A]$ as

$$\begin{bmatrix} \cos(\theta_{\hat{x}_1}^{x_1}) & \cos(\theta_{\hat{x}_2}^{x_1}) & \cos(\theta_{\hat{x}_3}^{x_1}) \\ \cos(\theta_{\hat{x}_1}^{x_2}) & \cos(\theta_{\hat{x}_2}^{x_2}) & \cos(\theta_{\hat{x}_3}^{x_2}) \\ \cos(\theta_{\hat{x}_1}^{x_3}) & \cos(\theta_{\hat{x}_2}^{x_3}) & \cos(\theta_{\hat{x}_3}^{x_3}) \end{bmatrix}$$

Note: These are best computes as do products of old $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and New $\vec{\hat{x}}_1, \vec{\hat{x}}_2, \vec{\hat{x}}_3$.

Ex: Rotation from Cartesian to Normal axis

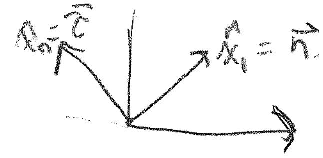
Since one thing we are interested in is the rotation from cartesian coordinates to normal axes we will use this as a nice 2D Example.

Consider the viscous stress tensor.

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{bmatrix}$$

If we want the component of this in the $\vec{\tau}$ direction we can 1). Rotate the tensor, 2) Find the traction in cartesian coordinates and then dot product with $\vec{\tau}$.

Method 1:



a) Find $[A]$

$$a_1^1 = \cos(\theta_{\hat{x}_1}^{x_1}) = \frac{(1,0)(n_x,n_y)}{1} = n_x$$

$$a_1^2 = \cos(\theta_{\hat{x}_1}^{x_2}) = (0,1)(n_x,n_y) = n_y$$

$$a_2^1 = \cos(\theta_{\hat{x}_2}^{x_1}) = (1,0)(-n_y,n_x) = -n_y$$

$$a_2^2 = \cos(\theta_{\hat{x}_2}^{x_2}) = (0,1)(-n_y,n_x) = n_x$$

$$\begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} = [A]$$

$$\begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix}\begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix}\begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} = \begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix}\begin{bmatrix} \tau_{xx}n_x + \tau_{xy}n_y & -\tau_{xx}n_y + \tau_{xy}n_x \\ \tau_{yx}n_x + \tau_{yy}n_y & -\tau_{yx}n_y + \tau_{yy}n_x \end{bmatrix}$$

$$= \begin{bmatrix} n_x^2\tau_{xx} + n_xn_y\tau_{xy} + n_xn_y\tau_{yx} + \tau_{yy}n_y^2 & -n_xn_y\tau_{xx} + n_x^2\tau_{xy} - \tau_{yx}n_y^2 + \tau_{yy}n_yn_x \\ -n_yn_x\tau_{xx} - \tau_{xy}n_y^2 + \tau_{yx}n_x^2 + \tau_{yy}n_xn_y & \tau_{xx}n_y^2 - \tau_{xy}n_xn_y - \tau_{xy}n_xn_y + \tau_{yy}n_x^2 \end{bmatrix}$$

Method 2):

$$[-n_y, n_x]\begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix}\begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = [-n_y, n_x]\begin{Bmatrix} \tau_{xx}n_x + \tau_{xy}n_y \\ \tau_{yx}n_x + \tau_{yy}n_y \end{Bmatrix} = -\tau_{xx}n_yn_x - \tau_{xy}n_y^2 + \ldots \tau_{yx}n_x^2 + \tau_{yy}n_xn_y$$

Note: The above and the off. diagonals from 1 are exactly the same thus to get a shear stress tangent to a wall with normal $\hat{n}$ we can use either Method (1) or 2), 2 is more computationally efficient

If $\tau_{yx} = \tau_{xy}$ then

$$\tau_w = (n_yn_x)(-\tau_{xx}) + \tau_{xy}(n_x^2 - n_y^2) + \tau_{yy}n_xn_y$$

Computing Coefficient of Friction:

Using the tensor Math. we can compute the Stress component tangential to the surface.

$$[A] = \begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix}$$

$$\bar{\tau}_{\text{normal axis}} = [A]^T [\tau][A] =$$

$$\begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix} \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{bmatrix} \begin{bmatrix} n_x & n_y \\ n_y & n_x \end{bmatrix} =$$

$$\left[\begin{array}{c|c} n_x^2 \tau_{xx} + 2 n_x n_y \tau_{xy} + \tau_{yy} n_y^2 & -n_x n_y \tau_{xx} + \tau_{xy}(n_x^2 - n_y^2) + \tau_{yy} n_x n_y \\ \hline -n_y n_x \tau_{xx} + \tau_{xy}(n_x^2 - n_y^2) + \tau_{yy} n_x n_y & \tau_{xx} n_y^2 - \tau_{xy} n_x n_y - \tau_{xy} n_x n_y + \tau_{yy} n_x^2 \end{array}\right]$$

$$\tau_w \sim \frac{a_\infty}{L_{ref}} \frac{\partial \bar{u}}{\partial \bar{y}}$$

$$C_f \sim \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} \sim \frac{\cancel{a_\infty} \mu_\infty \bar{\mu} \frac{\partial \bar{u}}{\partial \bar{y}}}{L_{ref} \frac{1}{2} \rho_\infty a_\infty^2 M_\infty^2} \sim$$

$$\sim \frac{\bar{\mu} \frac{\partial \bar{u}}{\partial \bar{y}}}{} \frac{\mu_\infty}{L_{ref} \rho_\infty a_\infty M_\infty M_\infty}$$

$$\sim \frac{\bar{\mu} \frac{\partial \bar{u}}{\partial \bar{y}}}{\frac{1}{2} Re_\infty M_\infty}$$

Computing viscous lift forces.

1) Compute traction vector in cartesian coordinates

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \{S\}$$ traction vector.

$$S_x = (\tau_{xx} n_x + \tau_{xy} n_y)$$

$$S_y = (\tau_{xy} n_x + \tau_{yy} n_y)$$

Then we have to integrate these tractions over the surface

$$F_{x_v} = \oint S_x \, ds$$

$$F_{y_v} = \oint S_y \, ds$$

$$C_{x_v} = \frac{F_{x_v}}{\frac{1}{2} \rho_\infty U_\infty^2 C_{ref}} = \frac{\mu_\infty \bar{\mu} \left(\frac{4}{3} \frac{\partial \bar{u}}{\partial \bar{x}} - \frac{2}{3} \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) \frac{\cancel{a_\infty}}{L_{ref}}}{\frac{1}{2} \rho_\infty U_\infty \cancel{a_\infty} M_\infty L_{ref} C_{ref}}$$

$$= \frac{\bar{\mu}(\ )}{\frac{1}{2}} \frac{\mu_\infty}{\rho_\infty U_\infty L_{ref}} \frac{1}{M_\infty C_{ref}} = \frac{\bar{\mu}(\ )}{\frac{1}{2} Re M_\infty C_{ref}}$$

Unsteady Solution using BDF2:

The formula has been derived previous and for

$$[M]\frac{\partial \vec{q}}{\partial t} + R(\vec{q}) = 0$$

gives

Note: Super script n denotes time level. Sub-

$$\frac{[M]}{\Delta t}\left(\frac{3}{2}\vec{q}^{\,n+1} - 2\vec{q}^{\,n} + \frac{\vec{q}^{\,n-1}}{2}\right) + R(\vec{q}^{\,n+1}) = 0$$

This makes up the time accurate residual, defined as $R_t(\vec{q}^{\,n+1}, \vec{q}^{\,n}, \vec{q}^{\,n-1}) = 0$

Thus our new non-linear problem is given as

$$R_t(\vec{q}^{\,n+1}, \vec{q}^{\,n}, \vec{q}^{\,n-1}) = 0$$

Application of Newton's Method

$$\left[\frac{\partial R_t}{\partial \vec{q}^{\,n+1}_{K}}\right] \delta\vec{q}^{\,n+1}_{K+1} = -R(\vec{q}^{\,n+1}_{K}, \vec{q}^{\,n}, \vec{q}^{\,n-1})$$

where subscript K denotes our Newton iteration index

$$\frac{\partial R_t}{\partial \vec{q}^{\,n+1}_{K}} = \left(\frac{3}{2}\frac{[M]}{\Delta t} + \left[\frac{\partial \vec{R}}{\partial \vec{q}^{\,n+1}_{K}}\right] - 0 + 0\right)$$

The time accurate residual can further be broken up as

$$R(\vec{q}^{\,n+1}_{K}) + \frac{3}{2}\frac{[M]}{\Delta t}(\vec{q}^{\,n+1}) + S(\vec{q}^{\,n}, \vec{q}^{\,n-1}) = R_t$$

where $S(\vec{q}^{\,n}, \vec{q}^{\,n-1})$ is a source term given by

$$\vec{S}(\vec{q}^{\,n}, \vec{q}^{\,n-1}) = -2\frac{[M]}{\Delta t}\vec{q}^{\,n} + \frac{[M]}{2\Delta t}\vec{q}^{\,n-1}$$

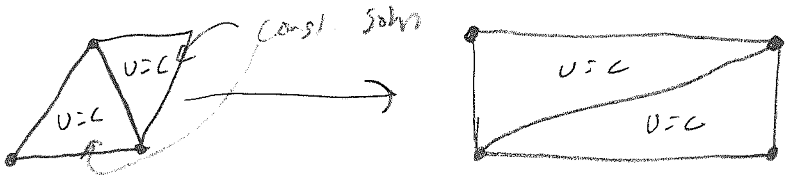
Moving Mesh's - Ensuring Conservation of qts over a time step

The GCL:

Start with the Euler Equations in Moving Coordinate

$$\frac{d(AU)}{dt} + \int_{B(t)} (F(U) - \dot{x}\, U)\cdot\vec{n}\, dB \quad ①$$

Say we go from where c is a uniform Soln.



Insert in to ①

$$\frac{d A\phi}{dt} - \int_{B(t)} \dot{x}\,\phi\cdot\vec{n}\, dB = 0$$

gives

$$\frac{dA}{dt} - \int_{B(t)} \dot{x}\cdot\vec{n}\, dB = 0$$ - This the GCL equation.

For BDF1. V_c

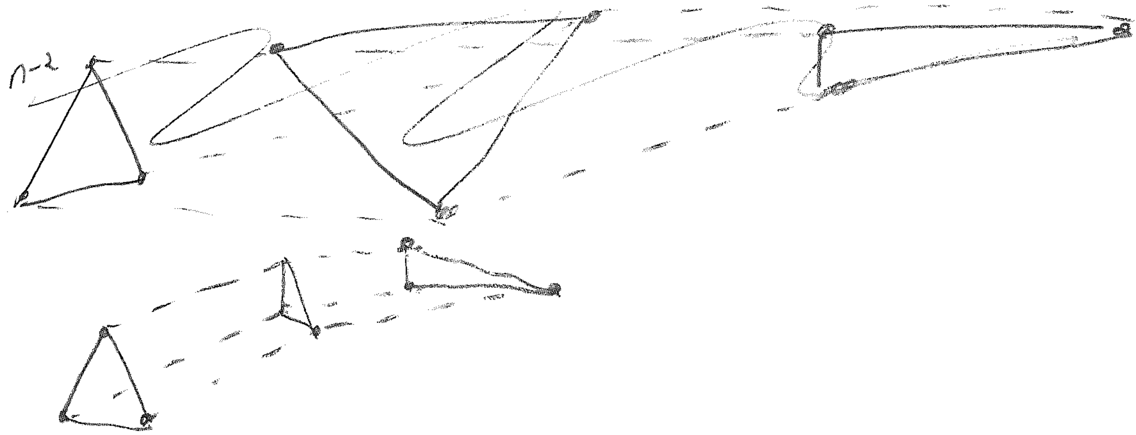
$$\frac{A^n - A^{n-1}}{\Delta t} = \int \dot{x}\cdot\vec{n}\, dB$$ - solve for V_c

$$\frac{d(AU)}{dt} + \int_{B(t)} [F(\vec{U})\cdot\vec{n} - V_c\vec{U}]\, dB = 0$$ where V_c is Soln of

To do this correctly GCL must be solved with same temporal accuracy as the flow.

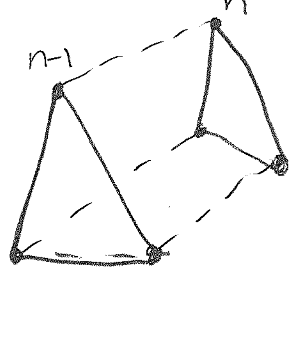
For BDF2

$$\frac{\frac{3}{2}A^n - 2A^{n-1} + \frac{1}{2}A^{n-2}}{\Delta t} - \sum_{i=1}^{3} V_{e_i}\cdot B_i = 0 \qquad \int_{B(t)} \dot{x}\cdot\hat{n}\, dB$$



$$= \frac{\gamma_1}{\Delta t}\left(A^n - A^{n-1}\right) + \frac{\gamma_2}{\Delta t}\left(A^{n-1} - A^{n-2}\right) -$$

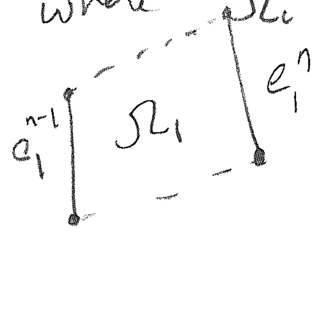
$$= \frac{\gamma_1 A^n}{\Delta t} + \frac{(\gamma_2-\gamma_1)A^{n-1}}{\Delta t} - \frac{\gamma_2}{\Delta t}A^{n-2} -$$



$$(A^n - A^{n-1}) = \Delta t \cdot \sum_{i=1}^{3} \Omega_i^{n}$$

$$(A^{n-1} - A^{n-2}) = \Delta t \cdot \sum_{i=1}^{3} \Omega_i^{n-1}$$

where Ω_i is area swept by edge i



$$= \frac{\gamma_1}{\Delta t} \sum_{i=1}^{3} \Omega_i^{n/n-1} + \frac{\gamma_2}{\Delta t} \sum_{i=1}^{3} \Omega_i^{n-1/n-2}$$

$$= \gamma_1 \sum_{i=1}^{3} \frac{\Omega_i^{n/n-1}}{\Delta t} + \gamma_2 \sum_{i=1}^{3} \frac{\Omega_i^{n-1/n-2}}{\Delta t}$$

Electro Magnetics   Ref: E.J. Rothwell, M.J. Cloud, Electromagnetics, CRC Press, Boca Raton, FL, 2009.   ①

1). Introduction

- Fundamental difference between the field point of view and mechanical point of view.
  - ~~The~~ Mechanical says you can describe the system by the particle charges, position + momenta.
  - Field instead says the system is described by field variables $\vec{E}, \vec{D}, \vec{B}, \vec{H}$ - the field point of view ~~is~~ is the E-M phenomena affect all space.
  - Unlike charged particle mechanics the field point of view like fluid mechanics is not represented by a finite # of state variables
    - e.g. for a mechanical system of N particles you have 6N state vars.
  - Field is described by functions of continuous variables.

- Field Theory description
  - Need 2 types of fields
    1). Mediating field generated by ~~the~~ source
    2). Field describing the source itself.
      - In free-space $\vec{E}, \vec{B}$ are mediating field while source is current
  - Fields are unobservable
    - Need link to mechanics to observe
    Lorentz force provides the link and allows us to observe the field.

- Constitutative Relations
  - describe affect of supporting medium on field.

2) Sources of E-M field.

- We are interested in Macro-scopic E-M thus we want the effects of sources given by continuous sources (es charge + current densities)

- we know on a micro-scopic level
  1). Charge is fundamental property of Matter.
  2). charge is quantized (exists in discrete quantum amounts)
  3). quantum is assoc to smallest sub-atomic particle.
  4) Charge is invariant with speed (Mass is not)
  5). Charge is the source of E-M field, field is entity that carry energy + momentum away from charge via propogating waves.

- For problems on our scale of interest, we can do an ~~a~~ averaging space as long as enough particles are in the laboratory size we can regard them as a continuum.

- Averaging.
  Need convient weighting function.
  Consider using gaussian.

$$f(\vec{r}) = (\pi a^2)^{-3/2} e^{-r^2/a^2}$$

$$\int f(\vec{r}) dV = 1.$$

a - ~~extent~~ averaging radius

for Micro-scopic quantity $\vec{F}(\vec{r}, t)$

$$\langle F(\vec{r},t)\rangle = \int F(\vec{r}-\vec{r}', t) f(\vec{r}') dV' \text{ - } r' \text{ is origin.}$$

Where $\vec{r}'$ - is some origin, and V' is Volume surrounding ~~said~~ said origin.

Macroscopic Volume charge density.

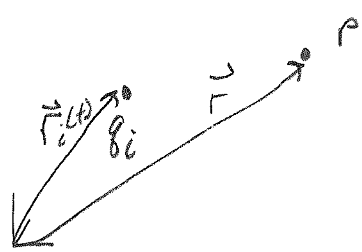
- We do not distinguish between "free" charge that is unattached to molecular structure, and the charge found near the surface of a conductor.
- Ignore dipole nature of polarizable materials.

- For "free-space" E-M assume charge has 3 DOF (volume charge) 2 DOF (surface " ") 1 DOF (Line " ")

- Typically microscopic variations have length scale of 10^{-10} m.
  - for solids, liquids, dense gases. $a = 10^{-8}$ m is good contain $\sim 10^6$ particles $\Rightarrow$ continuous.

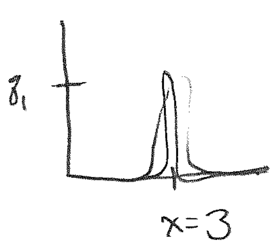
The volume density of charge can be written as

$$\rho^{\circ}(\vec{r},t) = \sum_i q_i \, \delta(\vec{r} - \vec{r}_i(t))$$ – each of these is charge at point i

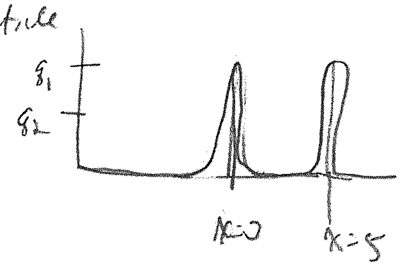


In 1D. for q_i $i = 1$, $r_i = 3$.

$i=1$, $r_1 = 3$
$i=2$, $r_2 = 5$.



for 2 particle



This is a summed micro-scopic quantity. So we apply at assume distribution and average

$$\rho(\vec{r},t) = \langle \rho^{0}(\vec{r},t) \rangle = \int \sum_i q_i \delta(\vec{r}-\vec{r}_i) \cdot f(\vec{r}_i)\, dV_i$$

$$\rho(\vec{r},t) = \sum_i q_i f(\vec{r}-\vec{r}_i(t))$$

Total charge in volume.

$$Q(t) = \int_V \rho(\vec{r},t)\, dv = \int \sum_i q_i f(\vec{r}-\vec{r}_i(t))\, dV =$$

$$\sum_{\substack{i \\ r_i(t) \in V}} q_i$$

~~Macro~~ Macroscopic Current Density

- charge in motion is called current.

$$\vec{v}_i(t) = \frac{d\vec{r}_i}{dt}$$

$$J^{0}(\vec{r},t) = \sum_i q_i \vec{v}_i(t) \delta(\vec{r}-\vec{r}_i(t))$$

Again apply ~~average~~ spatial averaging at time t. - this eliminate uncorrelated motion on scale of averaging radius

$$J(\vec{r},t) = \langle J^{0}(\vec{r},t) \rangle = \int \sum_i q_i \vec{v}_i(t) \delta(\vec{r}-\vec{r}_i(t)) f(r_i(t))\, dV_i(t)$$

$$= \sum_i q_i \vec{v}_i(t) f(\vec{r}-r_i(t))$$

- Assume sufficiently large a gives

$$J(\vec{r},t) = \rho(\vec{r},t)\, v(\vec{r},t)$$

Current flux $I(t)$

$$I(t) = \int_S J(\vec{r},t)\cdot\hat{n}\,dS = \sum_i q_i \left(\frac{d}{dt}\vec{r}_i(t)\cdot\hat{n}\right)\int_S f(\vec{r}-\vec{r}_i(t))\,dS$$

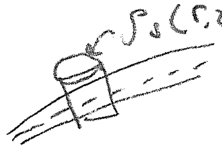
deforming surface

- Many types of current.
  - each described by $\vec{J} = \rho\vec{v}$. Isolated charged particles in space give convection currents, Neg-charged particles in + background are called conduction currents. Electrolytic current, results from the flow of positive or negative ions through a fluid.

- Impressed vs Secondary sources
  - this depends on action setting charge in motion.
    - Impressed (or primary) are independent of the field they source
    - Secondary result from interactions between the source fields and the medium in which the ~~field~~ field exists.

Surface + Line Sources:

- Surface charge is a contrious volume charge distributed in a thin layer. If we assign the charge density in a cylinder on the surface as



$\rho_s(\vec{r},t)$

S - surface $\rho_S(\vec{r},t)$

Then we write the volume charge as

$$\rho(\vec{r},w,t) = \rho_s(r,t)\,f(w,\Delta) -$$

where Δ - thickness and w is the distance $\perp$ to S inside cylinder.

The density function $f(x, \Delta)$

$$\int_{-\infty}^{\infty} f(x, \Delta)\, dx = 1 \qquad \lim_{\Delta \to 0} f(x, \Delta) = \delta(x)$$

~~The~~ take for example $f(x, \Delta) = \dfrac{e^{-x^2/\Delta^2}}{\Delta\sqrt{\pi}}$

with this the total charge in a cylinder of normal to the surface is

$$dQ(t) = \int_{-\infty}^{\infty} \rho_s(r, t)\, dS\, f(w, \Delta)\, dw = \rho_s(r, t)\, dS$$

$$Q(t) = \int_S \rho_s(\vec{r}, t)\, dS$$

The Line charge is similar see text.

Surface Current Density: Defined the same as charge.

$$\vec{J}(\vec{r}, w, t) = \vec{J}_s(\vec{r}, t) f(w, \Delta)$$

$$dI(t) = \int_{-\infty}^{\infty} [\vec{J}_s(\vec{r}, t) \cdot \hat{n}_\ell]\, d\ell\, f(w, \Delta) = J_s(r, t) \cdot \hat{n}_\ell(r)\, d\ell$$

This is the total current flowing through a strip $d\ell$ of width $d\ell$ $\perp$ to S at $\vec{r}$.

Charge Conservation:

Net charge in any closed system remains the same with time.

- Essentially we are deriving a cty equation for charge.

Using Reynolds transport

$$\frac{DQ}{Dt} = \frac{D}{Dt}\int_{V(t)} \rho\, dV = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \oint \rho \vec{v} \cdot dS = 0$$

$$= \int_V \frac{\partial \rho}{\partial t} dV + \oint \vec{J}(\vec{r},t)\, dS = 0$$ - Using divergence theorem

$$= \int_V \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}(\vec{r},t)\, dv = 0$$ gives

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}(\vec{r},t) = 0$$

Ex. Using cty

given $\rho = \rho_0 r e^{-\beta t}$

$$\frac{\partial}{\partial t}\int_V \rho\, dv = 4\pi \int_V \rho_0 r e^{-\beta t} r^2 dr = -\oint J_r \hat{r}\, dS$$

Using calc text: $dS = r \sin\phi \cdot d\theta\, r\, d\phi$

$$-\int_0^{2\pi}\int_0^{\pi} J_r a^2 \sin\phi\, d\phi\, d\theta = J_r a^2 \left(\overset{(-1 - 1)(2\pi)}{\cos(\pi) - \cos(0)} \right) = -4\pi a^2 J_r(a)$$

$$4\pi \int_0^a -\rho_0 \beta r^3 e^{-\beta t} dr = \cancel{-4\pi}\, \beta \rho_0 \frac{a^{\cancel{4}\,2}}{\cancel{4}} e^{-\beta t} = \cancel{-4\pi} J_r \cancel{a^2}$$

$$J_r = \beta \rho_0 \frac{a^2}{4} e^{-\beta t}$$ $a = r$ give

$$\vec{J} = \hat{r}\, \beta \rho_0 \frac{r^2}{4} e^{-\beta t}$$

$$\nabla\cdot J = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) \cdot \frac{1}{\cancel{r^2}}\left(\cancel{4}\frac{r^{3}}{\cancel{4}}\beta\rho_0 r e^{-\beta t}\right) = \beta\rho_0 r e^{-\beta t}$$

$$\vec{v} = \frac{\vec{J}}{\rho} = \hat{r}\,\frac{\frac{\beta\cancel{\rho_0}\, r^{\cancel{2}}\cancel{e^{-\beta t}}}{4}}{\cancel{\rho_0}\,\cancel{r}\cancel{e^{-\beta t}}} = \hat{r}\,\frac{\beta r}{4} = \vec{v}$$

$$\frac{\partial\rho}{\partial t} = -\beta\rho_0 r e^{-\beta t} + \beta\rho_0 r e^{-\beta t} = 0 \quad \checkmark$$ this verifies conservation

<u>Magnetic charges:</u>

~~Created~~ Magnetic fields are created only by time varying magnetic fields

- Elemental Source is magnetic dipole,
- So far we have found no magnetic charges
- Even though these guys don't exist we will use them for convience,

Maxwell's Equation (Minkowski):

(1)

$\vec{\nabla} \times \vec{E}(\vec{x},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{x},t)$ - Faraday's law

$\vec{\nabla} \times \vec{H}(\vec{x},t) = \vec{J}(\vec{x},t) + \frac{\partial}{\partial t}(\vec{D}(\vec{x},t))$ - Ampere's Law

$\nabla \cdot \vec{B}(\vec{x},t) = 0$ - Gauss's Law

$\nabla \cdot \vec{D}(\vec{x},t) = \rho(\vec{x},t)$ - Gauss's Magnetic Law

$\vec{E}$ - Electric field -strength
$\vec{B}$ - Magnetic flux density
$\vec{J}$ - current density
ρ - charge density.
$\vec{H}$ - Magnetic ~~field~~
$\vec{D}$ - Electric Excitation

Also use cty equation i.e. charge conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Inter dependance:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \cdot \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

Recall that $\nabla \cdot (\nabla \times \vec{E}) = 0$ Divergence of curl = 0

give

$\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B}) = 0$ - I.e. rate of change of divergence in time is 0

Thus $\vec{\nabla} \cdot \vec{B} = C_B(\vec{x})$ - i.e. the divergence of $\vec{B}$ is only a function of space

We choose $C_B(\vec{x}) = 0$ gives

$$\vec{\nabla} \cdot \vec{B} = 0$$

Similar work can be done on Ampere law to get.

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial t}(\vec{D})$ gives

$$0 = -\frac{\partial}{\partial t}\rho + \vec{\nabla} \cdot \frac{\partial}{\partial t}(\vec{D}) = \frac{\partial}{\partial t}(\rho - \vec{\nabla} \cdot \vec{D}) = 0$$

I.e. $\rho - \vec{\nabla}\cdot D = C_D$ take $C_D = 0$ gives

$\vec{\nabla}\cdot\vec{D} = \rho$

If we can define a time prior to which C_B and C_D are both = to zero then the setting of $C_B, C_D = 0$ is perfectly value because they cannot change in time, We will regard this setting as initial conditions for Ampere's Law and Faraday's Law,

Constitutive Relations:

To close the system a set of Constitutive Relations must be applied.

- Relations split; 1) for elio EM field Qty's the others for mechanical interaction between sources + fields

- Involve - parameters + operators.

- If the relations involve time derivative of integrals the medium is temporally dispersive

Many types of Materials - see pg. 28 for description

General form,

$$\left.\begin{aligned}\vec{D} &= \vec{D}[\vec{E},\vec{B}]\\ \vec{H} &= \vec{H}[\vec{E},\vec{B}]\end{aligned}\right\}$$ i.e, these are related to $\vec{E},\vec{B}$ in some Mathematical way.

For now we will restrict our selves to linear iso-tropic Media.

i.e. $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$

For conducting Media $\vec{J} = \sigma \vec{E}$

For a Vacuum we have ε_0, - peremittivity of free space

μ_0 - peremeability of free space.

Thus for linear isotropic Media we will define our

ε, μ as $\varepsilon = \varepsilon_r \varepsilon_0$ $\mu = \mu_r \mu_0$ - i.e. relative permittivity and peremeability.

Dimensionless.

Recall. $c = \dfrac{1}{(\mu_0 \varepsilon_0)^{1/2}}$

$\mu_0 = 4\pi \times 10^{-7}$ H/m

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

For Now this is all we need for Maxwell's Equations constitutive Relations.

Solving Linear Maxwell's Equation:

1) Maxwell's Equations

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{B})$$
$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial}{\partial t}(\vec{D})$$

with consistency conditions

$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\cdot\vec{D} = \rho$$

Charge conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

for Linear isotropic materials we can write the following constitutive Relations.

$$\vec{D} = \varepsilon\vec{E}, \quad \vec{B} = \mu\vec{H} \quad \text{with Ohms law} \quad \vec{J} = \sigma\vec{E}$$

$$\left.\begin{aligned} \nabla\times\vec{E} &= \frac{\partial}{\partial t}(\vec{B}) \\ \vec{\nabla}\times\left(\frac{1}{\mu}\vec{B}\right) &= \vec{J} + \frac{\partial}{\partial t}(\varepsilon\vec{E}) \end{aligned}\right\} \text{ or } (1)$$

$$\left.\begin{aligned} \vec{\nabla}\times\vec{E} &= -\frac{\partial}{\partial t}(\vec{B}) \\ \vec{\nabla}\times\vec{B} &= \mu\vec{J} + \varepsilon\mu\frac{\partial}{\partial t}(\vec{E}) \end{aligned}\right\} \text{ Second form } (2)$$

Alternatively we can Rewrite this for $\vec{E}, \vec{H}$ ala, Hesthaven + Warburton.

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\mu\vec{H})$$
$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial}{\partial t}(\varepsilon\vec{E})$$

Using Ohms Law.

$$\left.\begin{aligned} \vec{\nabla}\times\vec{E} &= -\frac{\partial}{\partial t}(\mu\vec{H}) \\ \vec{\nabla}\times\vec{H} &= \sigma\vec{E} + \frac{\partial}{\partial t}(\varepsilon\vec{E}) \end{aligned}\right\} (3)$$

- We prefer this form as we solve for magnetic field $\vec{H}$.

We will work with (3) ~~only~~

Introducing our relative permittivity and permeability give

$$\vec{\nabla} \times \vec{E} = -\mu_r \mu_0 \frac{\partial}{\partial t}(\vec{H})$$
$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon_r \varepsilon_0 \frac{\partial}{\partial t}(\vec{E})$$

Non-Dimensionalization, $\vec{E}^* = \frac{\vec{E}}{E_0}$, $\vec{H}^* = \frac{\vec{H}}{H_0}$, $\vec{J}^* = \frac{\vec{J}}{J_0}$

$$x^* = \frac{x}{L_{ref}} \qquad \tau = \frac{t}{L_{ref}/C_0}$$

As usual we won't be able to specify E_0, H, J_0 explicitly without introducing extra N/D groups, rather we will seek to get $E_0(H_0, \mu_0, \varepsilon_0)$

$$\frac{E_0}{L_{ref}} \vec{\nabla}^* \vec{E}^* = -\mu_r \frac{\mu_0 C_0 H_0}{L_{ref}} \frac{\partial}{\partial \tau}(\vec{H}^*) \Rightarrow E_0 = \mu_0 C_0 H_0 = \frac{\mu_0}{\sqrt{\mu_0 \varepsilon_0}} H_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} H_0$$

define $\sqrt{\frac{\mu_0}{\varepsilon_0}} = Z_0$ give $E_0 = Z_0 H_0$ gives

$$\vec{E}^* = \frac{\vec{E}}{Z_0 H_0}$$ thus our N/D equations

$$\vec{\nabla}^* \times \vec{E}^* = -\mu_r \frac{\partial}{\partial \tau}(\vec{H}^*)$$

$$\frac{H_0}{L_{ref}} \vec{\nabla} \times \vec{H} = \frac{J_0 \vec{J}^*}{L_{ref}} + \varepsilon_r \varepsilon_0 \frac{Z_0 H_0 C_0}{\cancel{L_{ref}}} \frac{\partial}{\partial \tau}(\vec{E}^*)$$

$$\vec{\nabla}^* \times \vec{H}^* = \frac{L_{ref} J_0}{H_0} \vec{J}^* + \varepsilon_r \cancel{\varepsilon_0} \sqrt{\frac{\mu_0}{\cancel{\varepsilon_0}}} \cdot \frac{1}{\sqrt{\mu_0} \mu_0} \frac{\cancel{H_0}}{\cancel{H_0}} \frac{\partial}{\partial \tau}(\vec{E}^*)$$

$$J_0 = \frac{H_0}{L_{ref}} \quad \text{give} \quad \vec{J}^* = \boxed{\frac{\vec{J} L_{ref}}{H_0}}$$

$$\vec{\nabla}^* \times \vec{H}^* = \vec{J}^* + \varepsilon_r \frac{\partial}{\partial \tau}(\vec{E}^*)$$

Thus Dropping the $(\,)^*$ Notation Maxwell's equations Become.

$$\vec{\nabla} \times \vec{E} = -\mu_r \frac{\partial}{\partial \tau}(\vec{H})$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon_r \frac{\partial}{\partial \tau}(\vec{E})$$

Ohms law $\vec{J}^* = \sigma \vec{E}^*$, $\vec{J} = \sigma \vec{E}$

Dropping $(\,)^*$ $(\,)$

Now we should Reconsider writing these equations in some kind of Diversence form.

It can be shown that $\vec{\nabla} \times \vec{E} = -\vec{\nabla} \cdot (\vec{e}_i \times \vec{E})_{i=1,2,3}$

This gives.

$$\varepsilon_r \frac{\partial}{\partial \tau}(\vec{E}) - \vec{\nabla} \times \vec{H} = \vec{J} = \varepsilon_r \frac{\partial}{\partial \tau}(\vec{E}) + \vec{\nabla} \cdot (e_i \times \vec{H})_{i=1,2,3} = 0.$$

$$\mu_r \frac{\partial \vec{H}}{\partial \tau} + \vec{\nabla} \times \vec{E} = 0 = \mu_r \frac{\partial \vec{H}}{\partial \tau} - \vec{\nabla} \cdot (e_i \times \vec{E})_{i=1,2,3}$$

This can be verified as follows. ~~for~~

- we will do it only for $\varepsilon_r \frac{\partial}{\partial \tau}(\vec{E}) - \vec{\nabla} \times \vec{H} = \vec{J}$

$$\varepsilon_r \frac{\partial}{\partial \tau}(E_x) - \overbrace{\frac{\partial}{\partial y} H_z + \frac{\partial}{\partial z} H_y}^{-\nabla \times \vec{H}(1)} = J_x$$

$$\varepsilon_r \frac{\partial}{\partial \tau}(E_y) + \frac{\partial}{\partial x} H_z - \frac{\partial}{\partial z} H_x = J_y$$

$$\varepsilon_r \frac{\partial}{\partial \tau}(E_z) - \frac{\partial}{\partial x} H_y + \frac{\partial}{\partial y} H_x = J_z$$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ H_x & H_y & H_z \end{vmatrix} = 0\hat{i} - H_z\hat{j} + H_y\hat{k} \Rightarrow \vec{\nabla} \cdot (e_1 \times H) = -\vec{\nabla} \times \vec{H}_1$$

Thus we can we write Maxwell's as

$$\frac{\partial}{\partial \tau}(\vec{q}) + \nabla \cdot F(\vec{q}) = S$$

$$q = \begin{Bmatrix} \mu_r \vec{H} \\ \varepsilon_r \vec{E} \end{Bmatrix} \quad F_i = \begin{Bmatrix} -(e_i \times \vec{E})_{i=1,2,3} \\ (e_i \times \vec{H}) \end{Bmatrix}$$

Using this form of Maxwell's Equations

we have,

$$\frac{\partial \vec{q}}{\partial t} + \nabla \cdot F(\vec{q}) = \vec{S}$$

For 2D problems we have, 2 form TM, TE polar

TM, H_x, H_y, E_z all else zero

TE, E_x, E_y, H_z all else zero.

TM:

$$\mu_r \frac{\partial H^x}{\partial t} + \frac{\partial E_z}{\partial y} = 0$$

$$\mu_r \frac{\partial H^y}{\partial t} - \frac{\partial E_z}{\partial x} = 0$$

$$\varepsilon_r \frac{\partial E^z}{\partial t} - \frac{\partial H^y}{\partial x} + \frac{\partial H_x}{\partial y} = J_z \sim \sigma E_z$$ using Ohm's Law

$$\vec{q} = \begin{Bmatrix} H^x \\ H^y \\ E^z \end{Bmatrix}, \quad F_{TM} = \begin{Bmatrix} (0, E_z) \\ (-E_z, 0) \\ (-H^y, H_x) \end{Bmatrix} \qquad S_{TM} = \begin{Bmatrix} 0 \\ 0 \\ J_z \end{Bmatrix}$$

TE:

$$\mu_r \frac{\partial H_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

$$\varepsilon_r \frac{\partial E_x}{\partial t} - \frac{\partial H_z}{\partial y} = J_x$$

$$\varepsilon_r \frac{\partial E_y}{\partial t} + \frac{\partial H_z}{\partial x} = J_y$$

$$F_{TE} \begin{Bmatrix} (E_y, -E_x) \\ (0, -H_z) \\ (H_z, 0) \end{Bmatrix}$$

$$S_{TE} = \begin{Bmatrix} 0 \\ J_x \\ J_y \end{Bmatrix}$$

DG Formulation for TM: Maxwell's:

$$\frac{\partial}{\partial t}(\vec{q}) + \nabla \cdot \vec{F}_{TM} = \vec{S}$$

Begin by $\langle \phi_i, (\cdot) \rangle = 0$

$$\int_\Omega \phi_i \frac{\partial}{\partial t}(\vec{q}) d + \int_\Omega \vec{\nabla} \cdot \vec{F}_{TM} \phi_i \, d\Omega - \int_\Omega \vec{S} \cdot \phi_i \, d\Omega = 0$$

For one element i τ_h we have

$$\int_{\Omega_e} \phi_i \frac{\partial}{\partial t}(\vec{q}) \, d\Omega_e + \int_{\Omega_e} \vec{\nabla} \cdot F_{TM} \phi_i \, d\Omega_e = \int_{\Omega_e} \vec{S} \cdot \phi_i \, d\Omega_e$$

Do integration by parts on second term

$$\underbrace{\int_{\Omega_e} \phi_i \frac{\partial}{\partial t}(\vec{q}) \, d\Omega_e}_{\text{Time term}} - \underbrace{\int_{\Omega_e} \vec{\nabla} \phi_i \cdot \vec{F}_{TM} \, d\Omega_e}_{\text{Volume Resid}} + \underbrace{\oint_{\partial\Omega_e} \phi_i \vec{F}_{TM} \cdot \hat{n} \, ds}_{\text{Surface Resid}} = \underbrace{\int_{\Omega_e} \vec{S} \cdot \phi_i \, d\Omega}_{\text{Source term}}$$

Introduce $\vec{q} = \sum_{j=1}^{N} \{\hat{q}_j\} \phi_j = \tilde{q}$ $\quad \vec{q} = [\Phi]\{\hat{q}\}$ $\quad [\Phi] = \begin{bmatrix} \phi & 0 & 0 \cdots \phi_i & 0 \\ 0 & \phi & & \\ 0 & 0 & & \\ 0 & & & \\ \vdots & & \vdots & \vdots \end{bmatrix}$

$\{\hat{q}\} = \{ \text{[illegible]} \}$

$$\underbrace{\int \phi_i [\Phi] \, d\Omega_e}_{[M]} \frac{\partial}{\partial t}\{\hat{q}_j\} - \underbrace{\int_{\Omega_e} \vec{\nabla}\phi_i \cdot \vec{F}_{TM}(\textstyle\sum_j \{\hat{q}_j\}\phi_j)}_{VR} +$$

$$\underbrace{\oint \vec{F}_{TM}(\textstyle\sum_j \{\hat{q}_j\}\phi_j) \cdot \hat{n} \, dd}_{SR} = \underbrace{\int \vec{S} \cdot \phi_i \, d\Omega_e}_{ST}$$

<u>Mass Matrix:</u> This ha been done previously and we have for the Euler and Navier-Stokes equations given a computation + storage and implementation scheme.

<u>Volume Resid:</u>

$$\int_{\Omega_e} \vec{\nabla}\cdot\phi_i \cdot \vec{F}_{TM}(\tilde{q})\, d\Omega_e.$$

~~We~~ For each ϕ_i we should have a vector of size 3.

Thus for a ϕ_i we get

$$\int_{\Omega_{standard}/tri/quad} \left(\frac{\partial\phi_i}{\partial x}, \frac{\partial\phi_i}{\partial y}\right) \cdot \begin{Bmatrix} (0, E_z) \\ (-E_z, 0) \\ (-H_y, H_x) \end{Bmatrix} \text{Det}J\, d\Omega_{standard.}$$ ~~ξ_{xy}~~

Using Quadrature.

$$R_v\{:\} = \sum_{K=1}^{N_{gp}} \left.\begin{Bmatrix} \frac{\partial\phi_i}{\partial y}\cdot E_z \\ -\frac{\partial\phi_i}{\partial x} E_z \\ -\frac{\partial\phi_i}{\partial x} H_y + \frac{\partial\phi_i}{\partial y} H_x \end{Bmatrix}\right|_{\xi,\eta=\xi_K,\eta_K} w(K)\, \text{Det}J(\xi_K, \eta_K)$$

Further we will bring in the μ_r, ε_r to the flux as

$$R_v\{:\} = \sum_{K=1}^{N_{gp}} \left.\begin{Bmatrix} \frac{\partial\phi_i}{\partial y}\cdot \frac{E_z}{\mu_r} \\ -\frac{\partial\phi_i}{\partial x} \frac{E_z}{\mu_r} \\ -\frac{\partial\phi_i}{\partial x} \frac{H_y}{\varepsilon_r} + \frac{\partial\phi_i}{\partial y} \frac{H_x}{\varepsilon_r} \end{Bmatrix}\right|_{\xi,\eta=\xi_K,\eta_K} w(K)\, \text{Det}_J(\xi_K, \eta_K)$$

Surface Residual.

$\oint \phi_i \vec{F}_{TM} \cdot \hat{n}\, ds$:

~~We~~ Since we have discontinous basis functions there is the usual ambiguity. Thus for a face we have

$$\int \phi_i F^*_{TM}\, ds \approx \sum_{K=1}^{NGP} \phi_i|_{S_K} \vec{F}^*(\vec{q}_L, \vec{q}_R)|_{S_K} w(K) \, \mathrm{Det}\, J|_{face}$$

The only ? is what do we do for a numerical flux

$\vec{F}^*$ = numerical flux. for maxwell's equation.

To derive a numerical flux function the first thing that we need is the eigen values. recall that we have the normal flux vectors → / as

$$\vec{E}_{TM} \cdot \hat{n} = \left\{ \begin{array}{c} \dfrac{E_z \cdot \hat{n}_y}{\mu_r} \\ \dfrac{-E_z \hat{n}_x}{\mu_r} \\ -\dfrac{H_y \hat{n}_x}{\varepsilon_r} + \dfrac{H_x \cdot \hat{n}_y}{\varepsilon_r} \end{array} \right\}$$

which can be written as

$$= \begin{bmatrix} 0 & 0 & \hat{n}_y/\mu_r \\ 0 & 0 & -\dfrac{\hat{n}_x}{\mu_r} \\ \dfrac{\hat{n}_y}{\varepsilon_r} & -\dfrac{\hat{n}_x}{\varepsilon_r} & 0 \end{bmatrix} \begin{Bmatrix} H_x \\ H_y \\ E_z \end{Bmatrix} \equiv [\hat{A}]\{q\}$$

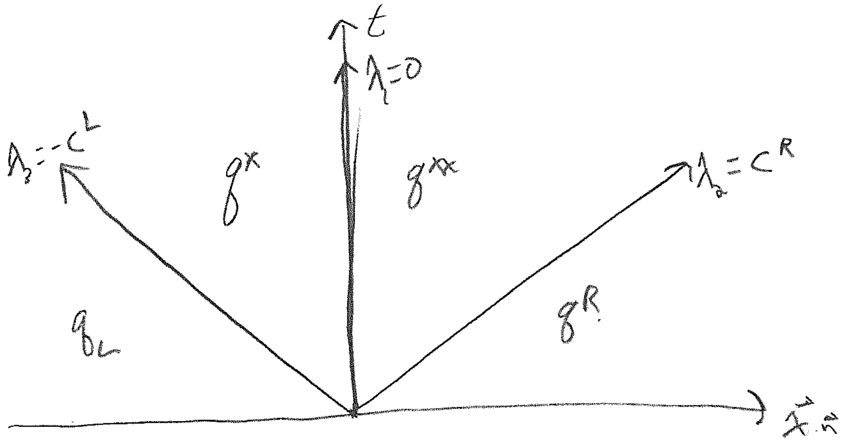
The eigen values of $[\hat{A}]$ are

$$|\lambda I - \hat{A}| = \begin{vmatrix} \lambda & 0 & -\dfrac{\hat{n}_y}{\mu_r} \\ 0 & \lambda & \dfrac{\hat{n}_x}{\mu_r} \\ -\dfrac{\hat{n}_y}{\varepsilon_r} & \dfrac{\hat{n}_x}{\varepsilon_r} & \lambda \end{vmatrix} = \lambda\left(\lambda^2 - \frac{\hat{n}_x^2}{\varepsilon_r \mu_r}\right) - \frac{\hat{n}_y}{\mu_r}\left(\frac{\hat{n}_y}{\varepsilon_r}\lambda\right) = 0$$

$$\lambda\left(\lambda^2 - \frac{1}{\varepsilon_r \mu_r}\right) = 0 \qquad \lambda_1 = 0,\ \lambda_2 = c,\ \lambda_3 = -c$$

$$\lambda^2 = \frac{1}{\varepsilon_r \mu_r} = c^2$$

The wave Diagram



We can apply the Rankine-Hugoniot Jump Conditions

$\lambda=0$: $\quad -0(q^* - q^{**}) + (F(q^*) - F(q^{**})) = 0$

$\lambda=c$: $\quad -c^R(q^{**} - q_R) + [F(q^{**}) - F(q^R)] = 0$

$\lambda=c$: $\quad -[-c^L(q^L - q^*)] + [F(q^L) - F(q^*)] = 0$

$\quad = -c^L(q^* - q^L) - [F(q^*) - F(q^L)] = 0$

$\quad = c^L(q^* - q^L) + [F(q^*) - F(q^L)] = 0$

We note that across the contact ~~discon~~ wave $F(q^*) = F(q^{**})$ gives a system of equations for $F(q^*)$ and q^*, q^{**}

The 1st equation:

$F(q^*) - F(q^{**}) = 0 \Rightarrow F(q^*) = F(q^{**})$ since a $F(q) = \hat{A} q$ the $\hat{A} q^* = \hat{A} q^{**} \Rightarrow q^* = q^{**}$ for linear problems

Using Maple

$$F(q^*) = F(q)^* = \frac{c_R c_L q_L + c_R F_L - c_R c_L q_R + F_R c_L}{(c_L + c_R)}$$

which can be re-written as

$$\frac{c_R F_L + c_L F_R + c_R c_L (q_L - q_R)}{(c_L + c_R)}$$

Special Case $C_L = C_R$ (continuous ~~tt~~ const medium)

$$\frac{\partial(F_L+F_R)}{\partial \ell} + \frac{\ell^{?}(q_L - q_R)}{\partial \ell} = \frac{1}{2}(F_L + F_R) + C(q_L - q_R)$$ which

is the L-F type flux which is exact for linear problem

$$F^*_{TM} = \frac{1}{2}\left(\begin{Bmatrix} \frac{E_z^L \hat{n}_y}{\mu_r} \\ \frac{-E_z^L \hat{n}_x}{\mu_r} \\ \frac{-H_y^L \hat{n}_x}{\varepsilon_r} + \frac{H_x^L \cdot \hat{n}_y}{\varepsilon_r} \end{Bmatrix} + \begin{Bmatrix} \frac{E_z^R \hat{n}_y}{\mu_r} \\ \frac{-E_z^R \hat{n}_x}{\mu_r} \\ \frac{-H_y^R \hat{n}_x}{\varepsilon_r} + \frac{H_x^L \hat{n}_y}{\varepsilon_r} \end{Bmatrix} \right) +$$

$$\frac{1}{2}C \begin{Bmatrix} H_x^L - H_x^R \\ H_y^L - H_y^R \\ E_z^L - E_z^R \end{Bmatrix}$$

Note: B.C. will Be enforced weakly as usual.

Linearization:

$$\frac{\partial F^*_{TM}}{\partial q_L} = \begin{bmatrix} \frac{1}{2}C & 0 & \frac{1}{2}\frac{\hat{n}_y}{\mu_r} \\ 0 & \frac{1}{2}C & -\frac{1}{2}\frac{\hat{n}_x}{\mu_r} \\ \frac{1}{2}\frac{\hat{n}_y}{\varepsilon_r} & -\frac{1}{2}\frac{\hat{n}_x}{\varepsilon_r} & \frac{1}{2}C \end{bmatrix}$$

$$\frac{\partial F^*_{TM}}{\partial q_R} = \begin{bmatrix} -\frac{1}{2}C & 0 & \frac{1}{2}\frac{\hat{n}_y}{\mu_r} \\ 0 & -\frac{1}{2}C & -\frac{1}{2}\frac{\hat{n}_x}{\mu_r} \\ \frac{1}{2}\frac{\hat{n}_y}{\varepsilon_r} & -\frac{1}{2}\frac{\hat{n}_x}{\varepsilon_r} & -\frac{1}{2}C \end{bmatrix}$$

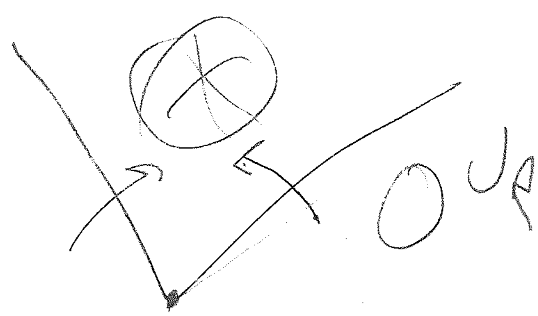
$\frac{\partial \rho}{\partial t} + \rho_0$

$\frac{\partial \{U\}}{\partial t} \quad [\lambda]\{\omega\} \quad [\lambda]^{-1}\{U\} = \{\omega\}$

$\{\omega\} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \quad \begin{bmatrix} \rho_0 & \rho_0 \\ -a & a \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \end{Bmatrix} = \begin{matrix} \rho_0\omega_1 + \rho_0\omega_2 = U \\ -a\omega_1 + a\omega_2 \end{matrix}$

$\begin{bmatrix} \frac{1}{2\rho_0} & -\frac{1}{2a} \\ \frac{1}{2\rho_0} & \frac{1}{2a} \end{bmatrix} \begin{Bmatrix} \rho_L \\ U_L \end{Bmatrix} = \begin{matrix} \omega_1 = \frac{\rho_L}{2\rho_0} - \frac{U_L}{2a} \\ \omega_2 = \frac{\rho_L}{2\rho_0} + \frac{U_L}{2a} \end{matrix}$

$\frac{\rho_0 \rho}{2\rho_0} - \frac{U}{2a}\rho_0 + \frac{\rho_0\rho}{2\rho_0} + \frac{U\rho_0}{2a} = \rho$



$\nabla \cdot \mu \vec{H} =$

Scattered field TM Maxwell's Equations

$\vec{E} = \vec{E}^i + \vec{E}^s$, $\vec{H} = \vec{H}^i + \vec{H}^s$

regarding E^i as particular solution to maxwells equations

$$\mu_r^i \frac{\partial}{\partial t} H_x^i + \frac{\partial E_z^i}{\partial y} = 0$$

$$\mu_r^i \frac{\partial}{\partial t} H_y^i - \frac{\partial}{\partial x} E_z^i = 0$$

$$\varepsilon_r^i \frac{\partial}{\partial t} E_z^i - \frac{\partial}{\partial x} H_y^i + \frac{\partial}{\partial y} H_x^i = 0$$

The total formulation is given as

$$\mu_r \frac{\partial H_x^i}{\partial t} + \mu_r \frac{\partial H_x^s}{\partial t} + \frac{\partial}{\partial y} E_z^i + \frac{\partial}{\partial y} E_z^s = 0$$

using $\frac{\partial}{\partial y} E_z^i = -\mu_r^i \frac{\partial}{\partial t} H_x^i$ gives

$$\boxed{\mu_r \frac{\partial H_x^s}{\partial t} + (\mu_r - \mu_r^i) \frac{\partial}{\partial t} H_x^i + \frac{\partial}{\partial y} E_z^s = 0}$$

$$\mu_r \frac{\partial H_y^s}{\partial t} + \mu_r \frac{\partial}{\partial t} H_y^i - \frac{\partial}{\partial x} E_z^i - \frac{\partial}{\partial x} E_z^s = 0$$

using $\frac{\partial}{\partial x} E_z^i = \mu_r^i \frac{\partial}{\partial t} H_y^i$ gives

$$\boxed{\mu_r \frac{\partial H_y^s}{\partial t} + (\mu_r - \mu_r^i) \frac{\partial H^i}{\partial t} - \frac{\partial}{\partial x} E_z^s}$$

$$\varepsilon_r \frac{\partial E_z^s}{\partial t} + \varepsilon_r \frac{\partial E_z^i}{\partial t} - \frac{\partial}{\partial x} H_y^i - \frac{\partial}{\partial x} H_y^s + \frac{\partial}{\partial y} H_x^i + \frac{\partial}{\partial y} H_x^s = 0$$

using $-\varepsilon_r^i \frac{\partial E_z^i}{\partial t} = -\frac{\partial}{\partial x} H_y^i + \frac{\partial}{\partial y} H_x^i$ gives

$$\boxed{\varepsilon_r \frac{\partial E_z^s}{\partial t} + (\varepsilon_r - \varepsilon_r^i) \frac{\partial E_z}{\partial t} - \frac{\partial}{\partial x} H_y^s + \frac{\partial}{\partial y} H_x^s = 0}$$

Scattered B.C. on solid metallic surface.

Recall to set $E_z = 0$ on a metallic Boundary we have

$$E_z^R = -E_z^L$$ using scattered field formulation

we have

$$E_z^{iR} + E_z^{sR} = -E_z^{iL} - E_z^{sL} =$$

we seek E_z^{sR}, $E_z^{iR} = E_z^{iL}$. this is known gives

$$\boxed{E_z^{sR} = -2E_z^{i} - E_z^{sL}}$$ this is our new B.C.

for scattering. Note we will project E_z^i to the modes at every time step.

Also Note that this the is the only Mode by which scattering enters the equations for a vaccum medium illuminating a vacuum medium.

Linearizations:

Volume Resid.:

$$F = \begin{Bmatrix} 0 \\ -E_z/\mu_r \\ -H_y/\varepsilon_r \end{Bmatrix}, \quad G = \begin{Bmatrix} E_z/\mu_r \\ 0 \\ H_x/\varepsilon_r \end{Bmatrix}$$

$$\frac{\partial E}{\partial q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/\mu_r \\ 0 & -1/\varepsilon_r & 0 \end{bmatrix}$$

$$\frac{\partial G}{\partial q} = \begin{bmatrix} 0 & 0 & 1/\mu_r \\ 0 & 0 & 0 \\ 1/\varepsilon_r & 0 & 0 \end{bmatrix}$$

Plane wave solutions

$$\phi = \phi_0 \exp^{-i \vec{K}(\omega)\cdot(x\hat{i} + y\hat{j})}$$

$\vec{K}$ is the wave vector i.e.

$$K_x = \omega\cos(\theta)\cancel{\hat{i}} + \cancel{\omega\sin(\theta)\hat{j}}$$
$$K_y = \omega\sin\theta$$

$$Re(\phi) = Re\left(\phi_0\left[\cos(\vec{K}\cdot\vec{r}) - i\sin(\vec{K}\cdot\vec{r})\right]\right) =$$

$$= \phi_0 \cos(\omega\cos\theta\cdot x + \omega\sin\theta\cdot y)$$

We need to figure out how to realize this as a radar type wave.

Boundary Conditions:

For any Boundary in Maxwell's equations we require continuity in the tangential ~~~~ components of the fields

$\hat{n} \times (E_1 - E_2) = 0$, $\hat{n} \times (H_1 - H_2) = 0$. Also the normal fluxes must be equal to the surface charge or current densities i.e.

$\hat{n} \cdot (B_1 - B_2) = \vec{J}_s$, $\hat{n} \cdot (D_1 - D_2) = \rho_s$.

In the case of a perfectly conducting Boundary i.e. a "solid" surface of Metal. It has surface charges But the fields cannot penetrate the Boundary.

For general Materials we have

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0, \quad \hat{n} \times (H_1 - H_2) = 0$$

$$\hat{n} \cdot (B_1 - B_2) = 0, \quad \hat{n} \cdot (D_1 - D_2) = \rho_s$$

This is because they support No surface charges or currents

This expresses

1). Continuity of tangential field components
2). Continuity of Normal fluxes and
3). discontinuity of Normal fields

For a perfectly conducting surface in 2D TM equation

$\vec{n} \times E = 0$

gives

$\mu_r \hat{n} \cdot \vec{H} = 0$

$$\vec{n} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ n_x & n_y & 0 \\ 0 & 0 & E_z \end{vmatrix} = 0\hat{i} - (n_x)E_z\hat{j} + 0\hat{k} = 0 \Rightarrow E_z = 0 \text{ on Boundary.}$$

i.e. E_z is dirichlet B.C. set ghost $-E_z$.

For $\vec{n} \cdot \vec{H} = 0$ we simply have the mirror condition.

Thus for perfectly conducting surface (i.e. BC type = 2)
we have, for the ghost cell values.

$$E_{z_g} = -E_z$$

$$\left.\begin{aligned} H_{x_g} &= H_x - 2 n_x (H_x n_x + H_y n_y) \\ H_{y_g} &= H_y - 2 n_y (H_x n_x + H_y n_y) \end{aligned}\right\}$$ Same as velocity in CFD.

Far field.

Silver Mueller B.C.: $\vec{n} \times \vec{E} Z_0 + \vec{n} \times (\vec{n} \times \vec{H}) = 0$

$$\begin{vmatrix} i & j & k \\ n_x & n_y & 0 \\ H_x & H_y & 0 \end{vmatrix} = (n_x H_y - H_x n_y)\hat{k}$$

$$\begin{vmatrix} i & j & k \\ n_x & n_y & 0 \\ 0 & 0 & T \end{vmatrix} = n_x T \hat{i} - n_y T \hat{j} + 0 \hat{k}$$

$$\begin{vmatrix} i & \cancel{k} & k \\ n_x & n_y & 0 \\ 0 & 0 & E_z \end{vmatrix} = \hat{i}(n_y E_z) - \hat{j}(E_z \cdot n_x)$$

given 2 eqns.

$$Z_0 E_z \cdot n_y + n_x^2 H_y - H_x n_x n_y = 0$$

$$-Z_0 E_z n_x - n_y n_x H_y + H_x n_y^2 = 0$$

$$\left.\begin{matrix} n_x H_y = 0 \\ n_x . E_z = 0 \end{matrix}\right\} \text{x- face}$$

$$\left.\begin{matrix} n_y E_z = 0 \\ H_x n_y = 0 \end{matrix}\right\} \text{y- face}$$

The above is implemented as 2 separate B.C. for xfaces, y - faces

Non-Linear p-Multigrid:

For clarity we will define $R_p(U_p^n)$ - as non-linear residual and r_p to be the multi-grid residual.

In general the non-linear equations have the form

$R_p(U_p^n) = S_p$, and the residual $r_p = S_p - R(U_p^n)$

Note: in the case $p = P_{max}$ $S_p = 0$, $r_p = -R(U_p^n)$

The coarse level source term is given as

$S_{p-1} = I_p^{p-1} r_p$, $U_{p-1}^n = \hat{I}^{p-1} U_p^n$

2-level Procedure:

1) Do some smoothing on fine level and get $R(U_p^n)$ form correction equation if we had $\bar{U}_p^{n+1}$ then $R(U_p^{n+1}) = S_p$ - U_p^{n+1} satisfies this

Thus,

$R(U_p^{n+1}) - R(U_p^n) = S_p - R(U_p^n)$ - correction equation

ie This measures how far we need to go to correct current value of U_p^n to make it satisfy the $R(U_p^{n+1}) = S_p$.

Restrict this to the coarse level.

$R(U_{p-1}^{n+1}) - R(I_p^{p-1} U_p^n) = I_p^{p-1}(\underbrace{S_p - R(U_p^n)}_{r_p})$

~~$U_{p-1}^n = I_p^{p-1} U_p^n$~~ thus our coarse ~~source~~ equation to solve is

$R(U_{p-1}^{n+1}) = I_p^{p-1} r_p + R(U_{p-1}^n) = D_{p-1}$

Iterate on U_{p-1}^{n+1} this gives a correction

$U_p^{n+1} = U_p^n + I_{p-1}^p(\underbrace{U_{p-1}^{n+1} - I_p^{p-1} U_p^n}_{\text{error between interp and computed.}})$

To really get Multigrid we would apply this again to $R_{p-1}(U^n_{p-1}) = D_{p-1}$

generate some iterations for U^n_{p-1}

generate residual as.

$$r_{p-1} = D_{p-1} - R_{p-1}(U^n_{p-1})$$

$$R_{p-2}(U^n_{p-2}) - R_{p-2}(I^{p-2}_{p-1} U^n_{p-1}) = I^{p-2}_{p-1} r_{p-1}$$

$$R_{p-2}(U^n_{p-2}) = I^{p-2}_{p-1} r_{p-1} + R_{p-2}(I^{p-2}_{p-1} U^n_{p-1})$$

This generates the correction

$$\hat{U}^n_{p-1} = U^n_{p-1} + I^{p-1}_{p-2}(U^n_{p-2} - I^{p-2}_{p-1} U^n_{p-1})$$

<u>Implementation Note:</u>

Due the form of our prolongation i.e. just add to those nodes. We can easily see the following.

$$U^{n+1}_p = U^n_p + I^p_{p-1}(\hat{U}^n_{p-1} - I^{p-1}_p U^n_p) =$$

$$U^n_p + I^p_{p-1}\left[U^n_{p-1} + I^{p-1}_{p-2}(U^n_{p-2} - I^{p-2}_{p-1} U^n_{p-1}) - I^{p-1}_p U^n_p\right]$$

This equation shows that as we generate a correction on a level and it's correction it can be directly prolongated to fine level as seen by splitting in the following way.

$$U^{n+1}_p = U^n_p + \underbrace{I^p_{p-1}\left[U^n_{p-1} - I^{p-1}_p U^n_p\right]}_{p-1\text{ - correction}} + \underbrace{I^p_{p-1} I^{p-1}_{p-2}(U^n_{p-2} - I^{p-2}_{p-1} U^n_{p-1})}_{p-2\text{ correction to level } p}$$

Thus these can be prolonged "on the fly".

$$U_p^{n+1} = U_p^n + \Delta U_p^n + I_{p-1}^{p}\left(\cancel{I_p^{p-1} U_p^n} + \Delta U_{p-1}^n - \cancel{I_p^{p-1} U_p^n}\right) + I_{p-1}^{p} I_{p-2}^{p-1}\left(\Delta U_{p-2}^n\right)$$

$$\cancel{\frac{\partial R_{p-2}}{\partial I_{p-1}^{p-2} U_{p-2}^n}} \qquad \frac{\partial R_{p-1}}{\partial I_p^{p-1} U_p^n} \Delta U_{p-1}^n = I_{p}^{p-1} r_p + R_{p-1}\left(I_p^{p-1} U_p^n\right)$$

TVM Filter:

TVM: Total Variation Minimization

q_f: Filter solution
q_0: Unfilter Solution

$$q_f = \min_{q_f} \int_{\Omega_K} |\nabla q_f| \, d\Omega + \frac{\lambda}{2} \| q_f - q_0 \|_{L_2, \Omega_K}$$

Where
$$|\nabla q_f| = \sqrt{\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2}$$
$$\| q_f - q_0 \|_{L_2, \Omega_K} = \int_{\Omega_K} (q_f - q_0)^2 \, d\Omega_K$$

Note: Each of the above operations is done per field.

L_2-Norm term is a control factor for balancing limiting procedure vs original Solution. I.E. The filtered solution should be as close as possible to the original sol.

1st minimize the above functional giving Euler-Lagrange equation.

$$\nabla \cdot \left(\frac{\nabla q_f}{|\nabla q_f| + \alpha} \right) - \lambda (q_f - q_0) = 0 \text{ with B.C. } \frac{\partial q_f}{\partial n} = 0 \text{ on } \partial\Omega_K \; \forall \Omega_K$$

Now discretize with weighted Residuals

$$\int_{\Omega_K} \phi_j \nabla \cdot \left(\frac{\nabla q_f}{|\nabla q_f| + \alpha} \right) d\Omega_K - \int_{\Omega_K} \phi_j \lambda (q_f - q_0) \, d\Omega_K$$

$\downarrow$

I.B.P

$$M \frac{\partial u}{\partial t} = -\int \frac{\nabla \phi_j \cdot \nabla q_f}{|\nabla q_f| + \alpha} d\Omega_K + \oint \phi_j \frac{\nabla q_f \cdot \hat{n}}{|\nabla q_f| + \alpha} ds - \int \lambda \phi_j (q_f - q_0) \, d\Omega_K$$

Note: The surface term is kept there to allow for possible implementation

Tune-able constants:

$\alpha = 20{,}000\, e^{-|\nabla q_0|} + 10^{-6}$

$$\lambda = \frac{\beta}{\int_{\Omega_K} (q_f - \bar{q}_f)^2 d\Omega_K + \beta} \quad \beta = 10^{-10}$$

when $\beta < 1 \; ? \; 0$

where $\bar{q}_f = \frac{1}{|\Omega_K|} \int_{\Omega_K} q_f \, d\Omega_K \rightarrow$ Avg. Soln.

yields stiff system as (using $q_f = \sum_{j=1}^{N} \hat{q}_{f_j} \phi_j$)

$$M \frac{d\tilde{q}_f}{dt} + R_f(\hat{q}_f) = 0$$

To solve consider Rosenbrock

$$(1 + \mu \Delta t J_0) K_1 = -\Delta t \, R_f(\hat{q}_f)$$

$$(1 + \mu \Delta t J_0) K_2 = -\Delta t \, R_f(\tilde{q}_f + \tfrac{2}{3} K_1) + \tfrac{4}{3} \mu \Delta t J_0 K_1$$

$$\tilde{q}_f^{n+1} = \tilde{q}_f^{n} + \tfrac{1}{4} K_1 + \tfrac{3}{4} K_2$$

$$J_0 = \left[\frac{\partial R_f}{\partial \tilde{q}_f} \right]_{\tilde{q}_f^n}$$

TVM Filter Surface term:

In the integration by parts stuff we have

$$M\frac{\partial \hat{g}_f}{\partial t} = -\int \frac{\nabla\phi_j \cdot \nabla g_f}{|\nabla g_f| + \alpha} \, d\Omega_k + \oint \phi_j \frac{\nabla g_f \, \vec{n}}{|\nabla g_f| + \alpha} \, d\partial\Omega_k - \int \lambda \phi_j (g_f - g_0) \, d\Omega_k$$

$$\Downarrow$$

$$\sqrt{\nabla g_f \cdot \nabla g_f + \alpha^2}$$

We are seeking

$$\oint \phi_j \frac{\nabla g_f \, \vec{n}}{|\nabla g_f| + \alpha} \, d\partial\Omega_k \simeq \oint [\![\phi_j]\!] \left\{ \frac{\nabla g_f}{|\nabla g_f| + \alpha} \right\} \cdot \vec{n} \, d\ + $$

$$[\![g_f]\!] \left\{ \begin{bmatrix} G_{f11} & G_{f12} \\ G_{f21} & G_{f22} \end{bmatrix} \nabla\phi_j \right\} - \{G_{fii}\} \gamma^{IP} (g_f^L - g_R^R) [\![\phi_j]\!] \, d\partial\Omega_k$$

On a perfield Basis ie for 1 field w.

$$\simeq \oint [\![\phi_j]\!] \left\{ \frac{\nabla w_f \cdot \vec{n}}{|\nabla w_f| + \alpha} \right\} + [\![g_f]\!] \left\{ \begin{bmatrix} G_{f11}^w & G_{f12}^w \\ G_{21}^w & G_{22}^w \end{bmatrix} \nabla\phi_j \right\} \cdot \vec{n} - \{G_{fii}^w\} \gamma (g_f^L - g_R^R) [\![\phi_j]\!] \, d\partial\Omega_k$$

$$G_{f11}^w = \frac{1}{(|\nabla w_f| + \alpha)} - \frac{\left(\frac{\partial w_f}{\partial x}\right)^2}{(|\nabla w_f| + \alpha)^3}, \quad G_{f12}^w = -\frac{\frac{\partial w_f}{\partial x}\frac{\partial w_f}{\partial y}}{(|\nabla w_f| + \alpha)^3}, \quad G_{21} = G_{12}, \quad G_{22} = \frac{1}{(|\nabla w_f| + \alpha)} - \frac{\left(\frac{\partial w_f}{\partial y}\right)^2}{(|\nabla w_f| + \alpha)^3}$$

$$\simeq \oint [\![\phi_j]\!] \left\{ \frac{\nabla w_f \cdot \vec{n}}{|\nabla w_f| + \alpha} \right\} + [\![w_f]\!] \left\{ \left(G_{f11}\frac{\partial\phi_j}{\partial x} + G_{12}\frac{\partial\phi_j}{\partial y}\right)\hat{n}_x + \left(G_{f21}\frac{\partial\phi_j}{\partial x} + G_{f22}\frac{\partial\phi_j}{\partial y}\right)\hat{n}_y \right\}$$

$$- \{G_{f11} + G_{f22}\} \gamma (g_f^L - g_f^R) [\![\phi_j]\!] \, d\partial\Omega_k$$

We will proceed by writing this term out piece by piece for a general field.

Term 1: The regular flux.

For the Left

$$\phi_j^L \cdot \frac{1}{2}\left[\left(\frac{\frac{\partial w_f^L}{\partial x}}{(|\nabla w_f|+\alpha)}\right) + \left(\frac{\frac{\partial w_f^R}{\partial x}}{(|\nabla \hat{w}_f|+\alpha)}\right)\right]\hat{n}_x + \frac{1}{2}\left[\left(\frac{\frac{\partial w_f^L}{\partial y}}{(|\nabla w_f|+\alpha)}\right) + \left(\frac{\frac{\partial w_f^R}{\partial y}}{(|\nabla w|+\alpha)}\right)\right]\cdot \hat{n}_y$$

negative for right.

Term 2:

For both Left and Right

0 for Left

$$(w_f^L - w_f^R)\ \frac{1}{2}\left[\left(G_{f11}^w \frac{\partial \phi_j}{\partial x} + G_{f12}^w \frac{\partial \phi_j}{\partial y}\right)^L + \left(G_{f11}^w \frac{\partial \phi_j}{\partial x} + G_{f12}^w \frac{\partial \phi_j}{\partial y}\right)^R\right]\cdot \hat{n}_x +$$
$$\frac{1}{2}\left[\left(G_{f21}^w \frac{\partial \phi_j}{\partial x} + G_{f22}^w \frac{\partial \phi_j}{\partial y}\right)^R + \left(G_{f21}^w \frac{\partial \phi_j}{\partial x} + G_{f12} \frac{\partial \phi_j}{\partial y}\right)^R\right]\cdot \hat{n}_y$$

Term 3:

For Left.

$$\frac{1}{2}\left[\left(G_{f11}^w + G_{f22}^w\right)^L + \left(G_{f11}^w + G_{f22}^w\right)^R\right]\cdot \hat{\nabla}\phi_j\,(w_f^L - w_f^R)$$

TVM Filter Linearization:

If we define R_f as

$$R_f = -\int_{\Omega_K} \underbrace{\frac{\nabla \phi_i \cdot \nabla \vec{q}_f}{|\nabla \vec{q}_f| + \alpha}}_{A} - \int_{\Omega_K} \lambda \phi_i (\vec{q}_f - \vec{q}_0)\, d\Omega_K$$

- Note: No surface term here.

Then we define J as.

$$\left[\frac{\partial R_f}{\partial \hat{q}}\right] = -\int_{\Omega_K} \left(\frac{\partial R_f}{\partial (q_x)} \cdot \frac{\partial q_x}{\partial \hat{q}} + \frac{\partial R_f}{\partial (q_y)} \cdot \frac{\partial q_y}{\partial \hat{q}}\right) d\Omega_K - \int_{\Omega_K} \frac{\partial R_f}{\partial q_f} \cdot \frac{\partial q_f}{\partial \hat{q}_f}\, d\Omega_K$$

ϕ_j

This makes it a simple Matter of finding $\frac{\partial R_f}{\partial q_x}$ / $\frac{\partial R_f}{\partial q_y}$

In this case no field's R_f depends on any other field

Thus $\frac{\partial R_f}{\partial q_x}$ is diagonal

$$\begin{bmatrix} \frac{\partial R_{f_1}}{\partial \rho_x} & 0 & 0 & 0 \\ 0 & \frac{\partial R_{f_2}}{\partial \rho u_x} & 0 & 0 \\ 0 & 0 & \frac{\partial R_{f_3}}{\partial \rho v_x} & 0 \\ 0 & 0 & 0 & \frac{\partial R_{f_4}}{\partial \rho E_x} \end{bmatrix}$$

Thus we can write the $\frac{\partial R_f}{\partial q_x}$ for a field and it is the same for each.

Consider a generic field w then for any w.

$$\frac{\partial R_{f_w}}{\partial w_{x_f}} = \frac{\partial}{\partial w_{x_f}}\left[-\frac{\nabla \phi_i \cdot \nabla w_f}{|\nabla w_f| + \alpha}\right]$$

$$\frac{\partial R_{f_w}}{\partial (w_{y_f})} = \frac{\partial}{\partial (w_{y_f})}\left[-\frac{\nabla \phi_i \cdot \nabla w_f}{|\nabla w_f| + \alpha}\right]$$

$$\frac{\partial R_f}{\partial w_f} = \frac{\partial}{\partial w_f}\left[-\lambda (w_f)\, \phi_i\, (w_f - w_0)\right]$$

Thus we seed that parts (A) and (B) actually are the $\frac{\partial R_f}{\partial \nabla w_f}$, $\frac{\partial R_f}{\partial w_f}$ terms them selves. because as with viscous fluxes we Regard R_f as $R_f(\nabla w, w)$ where the gradient is a seperate variable.

Details:

First we will work out $\frac{\partial R_f}{\partial (w_{xf})}$ and $\frac{\partial R_f}{\partial (w_{yf})}$

$$\frac{\partial R_f}{\partial (w_{xf})} = \frac{\partial}{\partial (w_{xf})}\left[\frac{\frac{\partial \phi_i}{\partial x}\frac{\partial w_f}{\partial x} + \frac{\partial \phi_i}{\partial y}\cdot\frac{\partial w_f}{\partial y}}{\sqrt{\left(\frac{\partial w_f}{\partial x}\right)^2 + \left(\frac{\partial w_f}{\partial y}\right)^2} + \alpha}\right]$$

define $\sqrt{\left(\frac{\partial w_f}{\partial x}\right)^2 + \left(\frac{\partial w_f}{\partial y}\right)^2} = |\nabla w_f|$

$$= \frac{\frac{\partial \phi_i}{\partial x}\left(|\nabla w_f| + \alpha\right) - \left(\frac{\partial \phi_i}{\partial x}\frac{\partial w_f}{\partial x}\right)\cdot\left[\frac{1}{|\nabla w_f|}\frac{\partial w_f}{\partial x}\right] - \frac{\partial \phi_i}{\partial y}\cdot\frac{\partial w_f}{\partial y}\left[\frac{1}{|\nabla w_f|}\frac{\partial w_f}{\partial x}\right]}{\left[|\nabla w_f| + \alpha\right]^2}$$

$$\frac{\partial R_f}{\partial (w_y)} = \frac{\partial}{\partial (w_{yf})}\left[\frac{\frac{\partial \phi_i}{\partial x}\cdot\frac{\partial w_f}{\partial x} + \frac{\partial \phi_i}{\partial y}\cdot\frac{\partial w_f}{\partial y}}{\sqrt{\left(\frac{\partial w_f}{\partial x}\right)^2 + \left(\frac{\partial w_f}{\partial y}\right)^2} + \alpha}\right] =$$

$$= \frac{-\frac{\partial \phi_i}{\partial x}\cdot\frac{\partial w_f}{\partial x}\left(\frac{1}{|\nabla w_f|}\cdot\frac{\partial w_f}{\partial y}\right) + \frac{\partial \phi_i}{\partial y}\cdot\left(|\nabla w_f| + \alpha\right) - \frac{\partial \phi_i}{\partial y}\cdot\frac{\partial w_f}{\partial y}\left(\frac{1}{|\nabla w_f|}\frac{\partial w_f}{\partial y}\right)}{\left[|\nabla w_f| + \alpha\right]^2}$$

We will take $\frac{\partial R_f}{\partial \hat{\omega}_f}$ directly for practical reasons. this way these Notes will match ~~exactly~~ very closely to the code.

$$\frac{\partial R_f}{\partial \hat{\omega}_f} = \frac{\partial}{\partial \hat{\omega}_f}\left[\lambda(\hat{\omega}_f)\phi_i(\omega_f - \omega_0)\right]$$

$\phi_j \quad \forall j$

$$= \phi_i\left[\frac{\partial \lambda}{\partial \hat{\omega}_f}(\omega_f - \omega_0) + \lambda(\omega_f)\frac{\partial \omega_f}{\partial \hat{\omega}_f}\right]$$

$\phi_j \qquad \frac{\partial \omega_f}{\partial \hat{\omega}_f}$

$$\frac{\partial \lambda}{\partial \hat{\omega}_f} = \frac{-1}{\left[\int_{\Omega_K}(\omega_f - \bar{\omega}_f)^2\, d\Omega_K\right]^2}\cdot\left[\int_{\Omega_K} 2(\omega_f - \bar{\omega}_f)\cdot\left\{\frac{\partial \omega}{\partial \hat{\omega}_f} - \frac{1}{|\Omega_K|}\int_{\Omega_K}\phi_j\, d\Omega_K\right\}\right]$$

Thus we will need to build and store $\frac{d\lambda}{d\hat{\omega}_f}$

Shock Detection.

1). Person and Peraire Element-wise Resolution indicator.

$$S_K = \frac{\langle q - \check{q}, q - \check{q} \rangle_{L_2, \Omega_K}}{\langle q, q \rangle_{L_2, \Omega_K}}$$

where
$$q = \sum_{j=1}^{M} \tilde{q}_j \phi_j$$
$$\check{q} = \sum_{j=1}^{\tilde{M}} \tilde{q}_j \phi_j \quad \text{where } \tilde{M} = \# \text{ of Modes } p-1.$$

2). Jump indicator

$$S_K = \frac{1}{|\partial\Omega_K|} \int_{\partial\Omega_K} \left| \frac{q^+ - q^-}{\frac{1}{2}(q^+ + q^-)} \right| ds.$$

Remark: This can be done for as many fields as you like but I will expirement with using density which <u>should</u> not be zero

Where Does the Euler - Lagrange Eqn Come from ?

This is quite simple. It is a variational calculus principle very closely related to energy methods in Structural Dynamics and Mechanics.

The statement

$$S = \int_{\Omega_{1c}} |\nabla g_t| \, d\Omega_{1c} + \frac{\lambda}{2} \| g_t - g_0 \|$$

is a functional that we would like the minimum of. Thus we take the variation and set the total variation $= 0$ to find the minimum (or stationary point).

Let's derive this proceedure by a simple example and then we'll see how the E/L eqn is derived and then it can be applied to our functional.

Consider a general functional $L = \int f(t, g, \dot{g}) \, dt$ where t is a single independent variable and $\dot{g} = \frac{dg}{dt}$.

$$\delta L = \delta \int f(t, g, \dot{g}) \, dt = \int \frac{\partial f}{\partial t} \cancelto{0}{\delta t} + \frac{\partial f}{\partial g} \delta g + \frac{\partial f}{\partial \dot{g}} \delta\left(\frac{dg}{dt}\right) dt \quad ①$$

Integration by parts of the 2nd term. Linear operators can be reversed (assoc.)

$U = \frac{\partial f}{\partial \dot{g}}$, $dv = d(\delta g) \rightarrow$

$dU = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{g}}\right) dt$, $v = \delta g$ gives No variation at end points (part of var calc) see S.D. Notes D. H.

$$UV \Big|_{t_1}^{t_2} - \int V \, dU \rightarrow \cancelto{0}{\delta g \frac{\partial f}{\partial \dot{g}} \Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \delta g \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{g}}\right) dt \quad ②$$

Using ② in ①.

$$\delta L = \int \left[\frac{\partial f}{\partial g} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{g}}\right) \right] \delta g \, dt.$$

Thus if we want $\delta L = 0$ then f must satisfy

$$\frac{\partial f}{\partial g} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{g}}\right) = 0 \quad \text{-- E/L.}$$

Application the our function $\dot{g} \rightarrow \nabla g_f$ $t \rightarrow \Omega_K$.

$$\frac{\partial f}{\partial \dot{g}} \rightarrow \frac{\partial\left(\sqrt{\nabla g \cdot \nabla g}\right)}{\partial \nabla g} \rightarrow \frac{1/2}{\sqrt{\nabla g \cdot \nabla g}} \cdot (\nabla g + \nabla g) = \frac{\nabla g}{|\nabla g|}$$

$$\frac{\partial f}{\partial g} = \frac{\lambda}{\cancel{2}}\, \cancel{2}(g_f - g_0) \cdot (1)$$

Thus we have

$$\nabla \cdot \left(\frac{\nabla g}{|\nabla g|}\right) - \lambda(g_f - g_0) = 0$$

We add the α term for computer implementation.

$$\nabla \cdot \left(\frac{\nabla g}{|\nabla g| + \alpha}\right) - \lambda(g_f - g_0) = 0$$

That is really simple.