Supplementary Note 9: Link Quality, Link Criticality Score, and Network Reliability

In the following we show the connection between link quality, the corresponding criticality score, and the proposed percolation-based demand-serving reliability of networks, as given by Eq. 4 in main text. The criticality score of a link measures the proportion of the total flow demand that traverses the link as the section with the lowest performance on the optimal path between all O-D pair of nodes. Let ψ be a sequence of links corresponding to a directed path on the network. For any path ψ we define the quality of the path (q_ψ) as the minimum quality among all links on the path; i.e. $q_\psi = \min_{e_{ij}\in\psi} q_{ij}$.

We deem a network as reliable, where despite presence of local disruptions (reflected as lack of quality), the network provides alternative paths with high quality links for the demanded flows between known origins and destinations. Therefore, for an origin node O and a reachable destination node D on the network ($O,D \in V$), from a non-empty set of all directed paths connecting them (denoted as Ψ_{OD}), we take the maximum quality path as the primary (optimal) path. The optimal path maximizes the minimum quality of the links on the path. The set of all optimal paths between a pair of origin-destination nodes is defined as:

$$\sigma_{OD} = \underset{\psi\in\Psi_{OD}}{\operatorname{argmax}}\, q_\psi . \tag{S5}$$

During the percolation process on the network, paths in σ_{OD} are the last paths to be removed before node D becomes unreachable from node O; i.e. when the flow demand f_{OD} becomes affected. The problem of finding σ_{OD} is known as the maximum capacity paths problem (*26*) with link weights being their quality attribute. With optimal paths between all O-D pairs of nodes found, the criticality score s_{ij} for the link e_{ij} can be formulated formally as:

$$s_{ij} = \sum_{O,D\in V}\sum_{\psi\in\sigma_{OD}} \frac{f_{OD}.\lambda(e_{ij},\psi)}{\left(\mathbf{1}_n^{\mathrm{T}} F\,\mathbf{1}_n\right).\left|\sigma_{OD}\right| .\left|\underset{e_{kl}\in\psi}{\operatorname{argmin}}\, q_{kl}\right|}, \tag{S6}$$

where $\lambda(e_{ij}, \psi)$ is equal to 1 if e_{ij} is (or one of) the minimum-quality link(s) on the path ψ and 0 otherwise. By iterating over each link e_{kl} on the path ψ, $|\operatorname{argmin}_{e_{kl}\in\psi} q_{kl}|$ counts the number of links which have equally the minimum link-quality on that path.

In the following we show the connection between link quality, the corresponding criticality score, and the proposed percolation-based demand-serving reliability of networks. Using Eq. S6, we can write:

$$\sum_{e_{ij}\in E} s_{ij}.q_{ij} = \frac{1}{\left(\mathbf{1}_n^{\mathrm{T}} F\,\mathbf{1}_n\right)} \sum_{O,D\in V} \frac{f_{OD}}{\left|\sigma_{OD}\right|} \sum_{\psi\in\sigma_{OD}} \sum_{e_{ij}\in E} \frac{q_{ij}.\lambda(e_{ij},\psi)}{\left|\underset{e_{kl}\in\psi}{\operatorname{argmin}}\, q_{kl}\right|} \tag{S7}$$

as for each path ψ, when iterating over all links in the network, $\lambda(e_{ij}, \psi)$ becomes 1 for only links from the set of minimum-quality links on ψ,

$$= \frac{1}{\left(\mathbf{1}_n^{\mathrm{T}} F\,\mathbf{1}_n\right)} \sum_{O,D\in V} \frac{f_{OD}}{\left|\sigma_{OD}\right|} \sum_{\psi\in\sigma_{OD}} \sum_{e_{ij}\in \operatorname{argmin}_{e_{kl}\in\psi} q_{kl}} \frac{q_{ij}}{\left|\underset{e_{kl}\in\psi}{\operatorname{argmin}}\, q_{kl}\right|} \tag{S8}$$

$$= \frac{1}{\left(\mathbf{1}_n^{\mathrm{T}} F \mathbf{1}_n\right)} \sum_{O,D \in V} \frac{f_{OD}}{|\sigma_{OD}|} \sum_{\psi \in \sigma_{OD}} \sum_{e_{ij} \in \operatorname{argmin}_{e_{kl} \in \psi} q_{kl}} \frac{\min_{e_{kl} \in \psi} q_{kl}}{\left|\operatorname{argmin}_{e_{kl} \in \psi} q_{kl}\right|} \tag{S9}$$

and as the number of links minimum link-quality on ψ is $|\operatorname{argmin}_{e_{kl} \in \psi} q_{kl}|$,

$$= \frac{1}{\left(\mathbf{1}_n^{\mathrm{T}} F \mathbf{1}_n\right)} \sum_{O,D \in V} \frac{f_{OD}}{|\sigma_{OD}|} \sum_{\psi \in \sigma_{OD}} \min_{e_{kl} \in \psi} q_{kl} \tag{S10}$$

and as the minimum link qualities on paths in σ_{OD} are equal,

$$= \frac{1}{\left(\mathbf{1}_n^{\mathrm{T}} F \mathbf{1}_n\right)} \sum_{O,D \in V} f_{OD} . q_{OD}^{*}, \tag{S11}$$

where q_{OD}^{*} is the quality of any path in σ_{OD}.

By definition of the reachability matrix (see Methods in the main text, especially the text regarding Eq. 5) as a function of percolation threshold on the network:

$$r_{OD}^{\rho} = \begin{cases} 1 & \rho \le q_{OD}^{*} \\ 0 & \rho > q_{OD}^{*} \end{cases}, \tag{S12}$$

where r_{OD}^{ρ} is the (O, D) entry of the reachability matrix R_{ρ} associated with origin node O and destination node D on the network G_{ρ}. Note that the integral of r_{OD}^{ρ} with respect to ρ between the limits $\rho = 0$ and $\rho = 1$ is equal to q_{OD}^{*}. Now, from Eq. S11 and Eq. S12 we can write:

$$\sum_{e_{ij} \in E} s_{ij} . q_{ij} = \frac{1}{\left(\mathbf{1}_n^{\mathrm{T}} F \mathbf{1}_n\right)} \sum_{O,D \in V} f_{OD} . \int_0^1 r_{OD}^{\rho} d\rho, \tag{S13}$$

where the right-hand-side can be simplified with matrix operations to obtain Eq. 2 from the main article which is the definition of the reliability measure α, so we can conclude:

$$\sum_{e_{ij} \in E} s_{ij} . q_{ij} = \alpha . \tag{S14}$$

Implication of link criticality score. To further investigate the implications of the link criticality score, we determine the scenarios in which increasing a link quality does not affect the link criticality scores on the network. As long as increasing the quality of a link does not change the criticality scores on the network, the exact impact on network reliability can be calculated according to Eq. S14.

Let us assume for two different links $e_{ij} \neq e_{kl}$ on the network the probability of having equal qualities is zero, i.e. $P(q_{ij} = q_{kl}) = 0$. This in theory is correct assuming that link qualities come from a continuous distribution, and is approximately correct in practice with link qualities calculated as decimals with high enough precision. Based on this assumption, there will be only a single optimal path between any (O, D) pair of nodes on the network, i.e. $|\sigma_{OD}| = 1$ for $O,D \in V$, as the minimum qualities on two paths cannot be equal. Also, the minimum quality link on any path on the network will be unique; i.e. $\forall \psi : \left|\operatorname{argmin}_{e_{kl} \in \psi} q_{kl}\right| = 1$. Let us denote the unique optimal path between an origin node O and a destination node D as ψ_{OD} ($\sigma_{OD} = \{\psi_{OD}\}$) with the path quality of $q_{\psi_{OD}} = q_{OD}^{*}$. Then, the criticality score of a link can be rewritten as:

$$s_{ij} = \sum_{\substack{O,D \in V \\ q_{ij} = q^*_{OD}}} \frac{f_{OD}}{\left(\mathbf{1}_n^{\mathrm{T}} F \mathbf{1}_n\right)}, \tag{S15}$$

while it can be easily seen that Eq. S14 (Eq. 4 from the main text) still holds true.

As Eq. S15 shows, criticality score of a link is the proportion of total flow demand for which the link is the minimum quality link on the optimal path between all pairs of O-D nodes on the network. Here, for a chosen link e_{ij} we investigate the impact of increasing its quality q_{ij}, on pairwise disjoint subsets of all (O, D) node pairs on the network with non-zero flow demand ($f_{OD} > 0$) between them. Let us denote the set of all ordered pairs of nodes with non-zero flow demand, as V^p. Then, with respect to any link e_{ij}, V^p can be partitioned into maximum three subsets:

i) $V_1^p(e_{ij})$, containing (O, D) pairs for which e_{ij} is on at least one optimal path but has quality higher than the minimum link quality on that path,

ii) $V_2^p(e_{ij})$, containing (O, D) pairs for which e_{ij} appears only on suboptimal paths but has the minimum quality among all links on at least one of those paths,

iii) $V_3^p(e_{ij})$, including only pairs for which e_{ij} is on the optimal path and is the minimum quality link,

For which we can write:

$$V^p = V_1^p(e_{ij}) \cup V_2^p(e_{ij}) \cup V_3^p(e_{ij}), \quad \forall e_{ij} \in E. \tag{S16}$$

For any link e_{kl} equal or different from e_{ij}, criticality score can be calculated as:

$$s_{kl} = \sum_{c=1,2,3} \sum_{\substack{(O,D) \in V_c^p(e_{ij}) \\ q_{kl} = q^*_{OD}}} \frac{f_{OD}}{\left(\mathbf{1}_n^{\mathrm{T}} F \mathbf{1}_n\right)}, \tag{S17}$$

where c is the index associated with each of the three classes of node pairs on the network.

Now, we investigate the effect of ameliorating the road condition on a link to increase its performance indicated by link quality attribute, on demand-serving reliability of the network. For a network with initial demand-serving reliability of α, the impact of amelioration on a link e_{ij} can be studied by increasing its original quality value q_{ij} to q'_{ij} ($q'_{ij} > q_{ij}$) and re-calculating the reliability of the network which we denote with α'. For all (O,D) pairs in $V_1^p(e_{ij})$, ameliorating the condition on e_{ij} and thus increasing q_{ij} does not change the minimum quality link on connecting paths, hence the component associated with c = 1 in all link criticality scores remains unchanged. For each origin-destination pair of nodes in $V_2^p(e_{ij})$, increase in quality of the link e_{ij} over a threshold can change the optimal path iff there is at least one path including e_{ij} for which all other links have quality above q^*_{OD}. Even in that case, increasing the quality with $q'_{ij} < q^*_{OD}$ does not alter any optimal path between the pairs of nodes in $V_2^p(e_{ij})$, so the component associated with c = 2 in criticality score of all links remains unchanged. For the link e_{ij} with $s_{ij} = 0$, the third class of (O,D) pairs is definitely empty (i.e. $V_3^p(e_{ij}) = \{\}$), thus we can conclude that:

$$(s_{ij} = 0) \wedge \left(q'_{ij} < \min\left(\{1\} \cup \left\{q^*_{OD} | (O,D) \in \hat{V}_2^p(e_{ij})\right\}\right)\right) \Rightarrow \alpha' = \alpha, \tag{S18}$$

where $\hat{V}_2^p$ is a subset of $V_2^p(e_{ij})$ for which increasing q_{ij} above some threshold, changes a suboptimal path including e_{ij} into the optimal path, and can be defined formally as:

$$\left\{(O,D)\in V_2^p(e_{ij}) \,|\, \exists\, \psi \in \Psi_{OD} : (e_{ij} \in \psi) \wedge (\forall e_{kl} \in \psi : q_{kl} > q_{OD}^{*})\right\}. \tag{S19}$$

Finally, for a link e_{ij} with $s_{ij} > 0$ there is at least one pair of nodes in $V_3^p(e_{ij})$ and if the link quality is increased, the third component of the all link criticality scores in Eq. S17 will remain fixed as long as the link maintains to be the lowest quality link on the optimal path. Let us denote the second lowest quality on the optimal connecting the node O to the node D as q_{OD}^{**}, then as long $q'_{ij} < q_{OD}^{**}$ for all node pairs in $V_3^p(e_{ij})$, the component associated with c = 3 in all link criticality scores remains unchanged. But since s_{ij} is positive, increasing q_{ij} improves the reliability of the network. With the help of Eq. S14 we can write:

$$\begin{aligned}(s_{ij} > 0) \wedge q'_{ij} < \min\left(\left\{q_{OD}^{*} | (O,D) \in \hat{V}_2^p(e_{ij})\right\} \cup \left\{q_{OD}^{**} | (O,D) \in V_3^p(e_{ij})\right\}\right) \\ \Rightarrow \alpha' - \alpha = s_{ij}.\left(q'_{ij} - q_{ij}\right).\end{aligned} \tag{S20}$$

To summarize all the above, consider increasing the quality q_{ij} of the link e_{ij} to q'_{ij} ($q'_{ij} > q_{ij}$). From Eq. S18 and Eq. S20 it is seen that for any link with criticality score of zero, there exists a non-empty range of values for q'_{ij} above the link's original quality q_{ij} that network reliability will certainly remain unchanged. However, if the link criticality score is non-zero, there exists a non-empty range of values for q'_{ij} above link's initial quality q_{ij}, for which no link criticality score changes in the network. Thus according to Eq. S14 increasing q_{ij} to q'_{ij} within the abovementioned range, changes the reliability of the network to $s_{ij}.(q'_{ij} - q_{ij})$, that is directly proportional to the criticality score calculated for the link.