The Derivation of TCA

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1. **How to get the formulation of TCA (Transfer Component Analysis)?**

   Solution:

   We give two kinks of expressions: the first one is to show JDA paper, while the second one is for TCA paper.

$$
\begin{aligned}
&\left\| \frac{1}{n_s} \sum_{i=1}^{n_s} \mathbf{A}^{\mathrm{T}} \mathbf{x}_i - \frac{1}{n_t} \sum_{j=n_s+1}^{n_s+n_t} \mathbf{A}^{\mathrm{T}} \mathbf{x}_j \right\|^2 \\
&= \left\| \frac{1}{n_s} \mathbf{A}^{\mathrm{T}} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{n_s} \end{bmatrix}_{1\times n_s} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n_s\times 1} - \frac{1}{n_t} \mathbf{A}^{\mathrm{T}} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{n_t} \end{bmatrix}_{1\times n_t} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n_t\times 1} \right\|^2 \\
&= \operatorname{tr}\left(\frac{1}{n_s^2} \mathbf{A}^{\mathrm{T}} \mathbf{X}_s \mathbf{1} (\mathbf{A}^{\mathrm{T}} \mathbf{X}_s \mathbf{1})^{\mathrm{T}} + \frac{1}{n_t^2} \mathbf{A}^{\mathrm{T}} \mathbf{X}_t \mathbf{1} (\mathbf{A}^{\mathrm{T}} \mathbf{X}_t \mathbf{1})^{\mathrm{T}} - \frac{1}{n_s n_t} \mathbf{A}^{\mathrm{T}} \mathbf{X}_s \mathbf{1} (\mathbf{A}^{\mathrm{T}} \mathbf{X}_t \mathbf{1})^{\mathrm{T}} - \frac{1}{n_s n_t} \mathbf{A}^{\mathrm{T}} \mathbf{X}_t \mathbf{1} (\mathbf{A}^{\mathrm{T}} \mathbf{X}_s \mathbf{1})^{\mathrm{T}} \right) \\
&= \operatorname{tr}\left(\frac{1}{n_s^2} \mathbf{A}^{\mathrm{T}} \mathbf{X}_s \mathbf{1}\mathbf{1}^{\mathrm{T}} \mathbf{X}_s^{\mathrm{T}} \mathbf{A} + \frac{1}{n_t^2} \mathbf{A}^{\mathrm{T}} \mathbf{X}_t \mathbf{1}\mathbf{1}^{\mathrm{T}} \mathbf{X}_t^{\mathrm{T}} \mathbf{A} - \frac{1}{n_s n_t} \mathbf{A}^{\mathrm{T}} \mathbf{X}_s \mathbf{1}\mathbf{1}^{\mathrm{T}} \mathbf{X}_t^{\mathrm{T}} - \frac{1}{n_s n_t} \mathbf{A}^{\mathrm{T}} \mathbf{X}_t \mathbf{1}\mathbf{1}^{\mathrm{T}} \mathbf{X}_s^{\mathrm{T}} \right) \\
&= \operatorname{tr}\left[\mathbf{A}^{\mathrm{T}} \left(\frac{1}{n_s^2} \mathbf{1}\mathbf{1}^{\mathrm{T}} \mathbf{X}_s^{\mathrm{T}} \mathbf{X}_s + \frac{1}{n_t^2} \mathbf{1}\mathbf{1}^{\mathrm{T}} \mathbf{X}_t^{\mathrm{T}} \mathbf{X}_t - \frac{1}{n_s n_t} \mathbf{1}\mathbf{1}^{\mathrm{T}} \mathbf{X}_s^{\mathrm{T}} \mathbf{X}_t - \frac{1}{n_s n_t} \mathbf{1}\mathbf{1}^{\mathrm{T}} \mathbf{X}_t^{\mathrm{T}} \mathbf{X}_s \right) \mathbf{A} \right] \\
&= \operatorname{tr}\left(\mathbf{A}^{\mathrm{T}} \begin{bmatrix} \mathbf{X}_s & \mathbf{X}_t \end{bmatrix} \begin{bmatrix} \frac{1}{n_s^2} \mathbf{1}\mathbf{1}^{\mathrm{T}} & \frac{-1}{n_s n_t} \mathbf{1}\mathbf{1}^{\mathrm{T}} \\ \frac{-1}{n_s n_t} \mathbf{1}\mathbf{1}^{\mathrm{T}} & \frac{1}{n_t^2} \mathbf{1}\mathbf{1}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{X}_s \\ \mathbf{X}_t \end{bmatrix} \mathbf{A} \right) \\
&= \operatorname{tr}\left(\mathbf{A}^{\mathrm{T}} \mathbf{X} \mathbf{M} \mathbf{X}^{\mathrm{T}} \mathbf{A} \right)
\end{aligned}
$$

   Important property:

   1. $||\mathbf{A}||^2 = \operatorname{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}})$, which is used in the second equation.

   2. $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$, which is used in the fourth equation.

   If used in kernel trick, then

$$
\begin{aligned}
&\left\| \frac{1}{n_s} \sum_{i=1}^{n_s} \phi(\mathbf{x}_i) - \frac{1}{n_t} \sum_{j=1}^{n_t} \phi(\mathbf{x}_j) \right\|^2 \\
&= \mathrm{tr}\left(\begin{bmatrix} \phi(\mathbf{x}_s) & \phi(\mathbf{x}_t) \end{bmatrix} \begin{bmatrix} \frac{1}{n_s^2}\mathbf{1}\mathbf{1}^{\mathrm{T}} & \frac{-1}{n_s n_t}\mathbf{1}\mathbf{1}^{\mathrm{T}} \\ \frac{-1}{n_s n_t}\mathbf{1}\mathbf{1}^{\mathrm{T}} & \frac{1}{n_t^2}\mathbf{1}\mathbf{1}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \phi(\mathbf{x}_s)^{\mathrm{T}} \\ \phi(\mathbf{x}_t)^{\mathrm{T}} \end{bmatrix} \right) \\
&= \mathrm{tr}\left(\begin{bmatrix} \phi(\mathbf{x}_s)^{\mathrm{T}} \\ \phi(\mathbf{x}_t)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \phi(\mathbf{x}_s) & \phi(\mathbf{x}_t) \end{bmatrix} \begin{bmatrix} \frac{1}{n_s^2}\mathbf{1}\mathbf{1}^{\mathrm{T}} & \frac{-1}{n_s n_t}\mathbf{1}\mathbf{1}^{\mathrm{T}} \\ \frac{-1}{n_s n_t}\mathbf{1}\mathbf{1}^{\mathrm{T}} & \frac{1}{n_t^2}\mathbf{1}\mathbf{1}^{\mathrm{T}} \end{bmatrix} \right) \\
&= \mathrm{tr}\left(\begin{bmatrix} < \phi(\mathbf{x}_s), \phi(\mathbf{x}_s) > & < \phi(\mathbf{x}_s), \phi(\mathbf{x}_t) > \\ < \phi(\mathbf{x}_t), \phi(\mathbf{x}_s) > & < \phi(\mathbf{x}_t), \phi(\mathbf{x}_t) > \end{bmatrix} \mathbf{M} \right) \\
&= \mathrm{tr}\left(\begin{bmatrix} K_{s,s} & K_{s,t} \\ K_{t,s} & K_{t,t} \end{bmatrix} \mathbf{M} \right)
\end{aligned}
$$

where

$$
(M)_{ij} = \begin{cases} \frac{1}{n_s n_s}, & \mathbf{x}_i, \mathbf{x}_j \in \mathcal{D}_s \\ \frac{1}{n_t n_t}, & \mathbf{x}_i, \mathbf{x}_j \in \mathcal{D}_t \\ \frac{-1}{n_s n_t}, & \text{otherwise} \end{cases}
$$