# Towards an Efficient and Economic Deductive System of Matching Logic

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    We aim for a Hilbert style deductive system which has a relatively large number of axioms but only a few inference rules.

## 1 Grammar and extended grammar

The formal language $\mathcal{L}$ we use to write matching logic patterns is defined as follows.

$$
\begin{aligned}
P ::=\ & x \\
& |\ P_1 \rightarrow P_2 \\
& |\ \neg P \\
& |\ \forall x.P \\
& |\ \sigma(P_1, \ldots, P_n) \\
& |\ P_1 = P_2 \\
(*\,*\,*\ &\texttt{extended}\ *\,*\,*) \\
& |\ P_1 \vee P_2 \\
& |\ P_1 \wedge P_2 \\
& |\ P_1 \leftrightarrow P_2 \\
& |\ \exists x.P \\
& |\ P_1 \neq P_2 \\
& |\ \top \\
& |\ \bot \\
& |\ \lceil P \rceil \\
& |\ \lfloor P \rfloor \\
& |\ P_1 \subseteq P_2 \\
& |\ x \in P
\end{aligned}
$$

with the extended grammar defined as

$$P_1 \lor P_2 := \neg P_2 \to P_1$$
$$P_1 \land P_2 := \neg(\neg P_1 \lor \neg P_2)$$
$$P_1 \leftrightarrow P_2 := (P_1 \to P_2) \land (P_2 \to P_1)$$
$$\exists x.P := \neg \forall x.\neg P$$
$$P_1 \neq P_2 := \neg(P_1 = P_2)$$
$$\top := x_1 = x_1$$
$$\bot := x_1 \neq x_1$$
$$\lceil P \rceil := P \neq \bot$$
$$\lfloor P \rfloor := P = \top$$
$$P_1 \subseteq P_2 := \lfloor P_1 \to P_2 \rfloor$$
$$x \in P := x \subseteq P$$

## 2  Hilbert proof system

Axioms in $\mathcal{L}$ are given by the following nine axiom schemata where $P, Q, R$ are arbitrary patterns and $x, y$ are variables.

- (K1) $P \to (Q \to P)$

- (K2) $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

- (K3) $(\neg P \to \neg Q) \to (Q \to P)$

- (K4) $\forall x.(P \to Q) \to (P \to \forall x.Q)$ if $x$ does not occur free in $P$

- (K5) $\forall x.P \to P$ if $x$ does not occur free in $P$

- (K6) $\forall x.P(x) \to P(y)$

- (K7) $P = P$

- (K8) $P_1 = P_2 \to (Q[P_1/x] \to Q[P_2/x])$

- (K9) $\exists y.Q = y \to (\forall x.P(x) \to P[Q/x])$ if $Q$ is free for $x$ in $P$

Inference rules include

- (Modus Ponens) From $P$ and $P \to Q$, deduce $Q$.

- (Universal Generalization) From $P$, deduce $\forall x.P$.

**Proposition 1** (Deduction Theorem). *If $\Gamma \cup \{P\} \vdash Q$ and the proof does not use $\forall x$-Generalization where $x$ is free in $P$, then $\Gamma \vdash P \to Q$.*

**Proposition 2** (Tautology). *For any tautology $\mathcal{A}(p_1, \ldots, p_n)$ where $p_1, \ldots, p_n$ are propositional variables,*
$$\vdash \mathcal{A}(P_1, \ldots, P_n).$$

2

**Proposition 3** (More Theorems in $\mathcal{L}$)**.**

$$\vdash \exists x.x$$
$$\vdash \lceil x \rceil$$
$$\vdash \exists y.x = y$$
$$\vdash P_1 = P_2 \rightarrow Q[P_1/x] = Q[P_2/x]$$

# 3 Inference rules

**Axioms**

$$\frac{\cdot}{\Gamma \vdash A}$$

where $A$ is an axiom.

**Inclusion**

$$\frac{\cdot}{\Gamma \vdash P}$$

where $P \in \Gamma$.

**Modus Ponens**

$$\frac{\Gamma \vdash Q \rightarrow P \quad \Gamma \vdash Q}{\Gamma \vdash P}$$

**Closed-Form Deduction Theorem**

$$\frac{\Gamma \cup \{P\} \vdash Q}{\Gamma \vdash P \rightarrow Q}$$

where $P$ is closed.

**Universal Generalization**

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P} \ (\forall x)$$

**Existential Instantiation**

$$\frac{\Gamma \vdash P[Q/x] \quad \Gamma \vdash \exists y.Q = y}{\Gamma \vdash \exists x.P} \ (\exists x)$$

**Conjunction Splitting**

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$$

# Examples of proof.sty

`\infer` draws beautiful proof figures easily:

(1)
$$\cfrac{\cfrac{B11 \,\&\, B12 \,\&\, B13 \quad B21 \,\&\, B22 \,\&\, B23}{B} \quad C}{A}$$

(2)
$$\cfrac{\cfrac{B11 \,\&\, B12 \,\&\, B13 \quad B21 \,\&\, B22 \,\&\, B23}{B} \quad C}{A1 \,\&\, A2 \,\&\, A3 \,\&\, A4 \,\&\, A5 \,\&\, A6}$$

(3)
$$\cfrac{C \quad \cfrac{B11 \,\&\, B12 \,\&\, B13 \quad B21 \,\&\, B22 \,\&\, B23}{B}}{A1 \,\&\, A2 \,\&\, A3 \,\&\, A4 \,\&\, A5 \,\&\, A6}$$

You can use also some variations:

(4)
$$\cfrac{\cfrac{B11 \,\&\, B12 \,\&\, B13 \quad \vdots \quad B21 \,\&\, B22 \,\&\, B23}{B} \quad C}{A} \;(1)$$

(5)
$$\cfrac{\cfrac{B11 \,\&\, B12 \,\&\, B13 \quad B21 \,\&\, B22 \,\&\, B23}{\sum \atop B} \quad {(2) \atop C}}{A1 \,\&\, A2 \,\&\, A3 \,\&\, A4 \,\&\, A5 \,\&\, A6} \quad \vdots \;(1)$$

(6)
$$\cfrac{A \,\&\, B \,\&\, C}{A}$$

Here are more practical examples:

(7)
$$\cfrac{A \quad B}{A \,\&\, B}\;(\&I) \qquad \cfrac{A \,\&\, B}{A}\;(\&E_l) \qquad \cfrac{A \,\&\, B}{B}\;(\&E_r)$$

$$\cfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B}\;(\to I) \qquad \cfrac{A \to B \quad A}{B}\;(\to E)$$

Some techniques: Use `\vcenter` for an equation of proofs.

(8)
$$\pi = \cfrac{A \quad \cfrac{B \quad C}{D}}{E}$$

Use `\kern` to adjust the form of a proof.

(9)

$$A \quad \overset{\displaystyle B \quad C}{\underset{\displaystyle E}{\rule{0pt}{0pt}D}}$$