Technical Report
The Deduction System of Matching Logic

Formal Systems Laboratory[1]

[1]University of Illinois, Urbana-Champaign, USA

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Abstract

Abstract goes here.

1 Syntax

Formulas of matching logic, called *patterns*, are written in a formal language, denoted as $\mathcal{L}$, whose grammar is listed in (1). The language $\mathcal{L}$ is many-sorted. A signature of $\mathcal{L}$ contains not only a finite set Σ of *symbols*, but also a finite nonempyty set S of *sorts*. Each symbol $\sigma \in \Sigma$ is, of course, sorted, with a fixed nonempty arity. We write $\sigma \in \Sigma_{s_1,\ldots,s_n,s}$ to emphasize that σ takes n arguments (with argument sorts $s1, \ldots, s_n$) and returns a pattern in sort s, but we hope in most cases sorting is clear from context.

The grammar for $\mathcal{L}$, as defined below, is almost identical to first-order logic, except that in $\mathcal{L}$ there is no difference between relational (predicate) and functional symbols, and we accept first-order terms as patterns in matching logic.

$$\begin{aligned} P ::= &\ x \\ &\mid P_1 \wedge P_2 \\ &\mid \neg P \\ &\mid \forall x.P \\ &\mid \sigma(P_1, \ldots, P_n), \end{aligned} \tag{1}$$

where the universal quantifier ($\forall x$) behaves the same as in first-order logic, with alpha-renaming always assumed.

For simplicity, we did not mention sorting in the grammar definition, and assume it should be clear to all readers. For example, in $P_1 \wedge P_2$, both patterns P_1 and P_2 should have the same sort, and that sort is the sort of $P_1 \wedge P_2$. The sort of $\forall x.P$ is the sort of P, while the sort of variable x does not matter. To see why it is the case, consider the pattern $\exists x.list(x, 1 \cdot 3 \cdot 5)$, which is the set of all memory configurations that has a list $(1, 3, 5)$ in it.

Propositional connectives are always assumed, including disjunction ($\vee$), implication ($\rightarrow$), and equivalence ($\leftrightarrow$). Existential quantifier ($\exists x$) is defined by universal quantifier ($\forall x$) in the normal way. The bottom pattern ($\bot_s$) and the top pattern ($\top_s$) in sort s are given by $x \wedge \neg x$ and $\neg \bot_s$, respectively, where x is a variable in sort s. It does not matter which variable we pick.

1.1 Extended Syntax

The formal language $\mathcal{L}$ is often extended with *definedness* symbols. For s_1, s_2 are two sorts, the definedness symbol $\lceil \_ \rceil_{s_1}^{s_2} \in \Sigma_{s_1,s_2}$ is a unary symbol with one argument sort s_1 and the result sort s_2. For a pattern P who has sort s_1, the pattern $\lceil \_ \rceil_{s_1}^{s_2}(P)$ is often written as $\lceil P \rceil_{s_1}^{s_2}$, or simply $\lceil P \rceil$.

Definedness symbols carry specific intended semantics. For each definedness symbol $\lceil \_ \rceil_{s_1}^{s_2}$, we add the pattern $\lceil x \rceil_{s_1}^{s_2}$ as an axiom to the deductive system, where x is a variable who has sort s_1. It does not matter which variable we pick.

With definedness symbols, we extends the formal language $\mathcal{L}$ with

$$
\begin{aligned}
&\lfloor P \rfloor_{s_1}^{s_2} := \neg \lceil \neg P \rceil_{s_1}^{s_2} \\
&P_1 =_{s_1}^{s_2} P_2 := \lfloor P_1 \leftrightarrow P_2 \rfloor_{s_1}^{s_2} \\
&P_1 \neq_{s_1}^{s_2} P_2 := \neg (P_1 =_{s_1}^{s_2} P_2) \\
&P_1 \subseteq_{s_1}^{s_2} P_2 := \lfloor P_1 \rightarrow P_2 \rfloor_{s_1}^{s_2} \\
&x \in_{s_1}^{s_2} P := x \subseteq_{s_1}^{s_2} P.
\end{aligned}
$$

Remark 1. *To prevent writing tangled subscripts and superscripts that indicate sorts of variables and patterns all the time, we omit them as much as possible, unless there is a chance of confusing things. A statement with sorting subscripts and superscripts omitted is treated as (possibly many) statements with the omitting sorting subscripts and superscripts completed in all possible well-formed ways.*

2 Deductive system

A deductive system is a recursive set of patterns as *axioms* and a finite set of *inference rules*. The deductive system of matching logic that we introduce in this section has been proved *sound* and *complete*.

2.1 The Deductive System

Axioms are given by the following axiom schemata where P, Q, R are arbitrary patterns and x, y are logic variables.

- (K1) $P \rightarrow (Q \rightarrow P)$
- (K2) $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$
- (K3) $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$

- (K4) $\forall x.P \to P[y/x]$
- (K5) $\forall x.(P \to Q) \to (P \to \forall x.Q)$ if x does not occur free in P
- (K6) $P_1 = P_2 \to (Q[P_1/x] \to Q[P_2/x])$
- (M1) $x \in y = (x = y)$
- (M2) $x \in P \wedge Q = (x \in P) \wedge (x \in Q)$
- (M3) $x \in \neg P = \neg(x \in P)$
- (M4) $x \in \forall y.P = \forall y.x \in P$ where x is distinct from y
- (M5) $x \in \sigma(\dots, P_i, \dots) = \exists y.y \in P_i \wedge x \in \sigma(\dots, y, \dots)$ where y occurs free in the left hand side of the equation.

Remark 2. *Substitution is denoted as $Q[P/x]$. Alpha-renaming is always assumed in order to avoid free variables capturing.*

Inference rules include

- (Modus Ponens) From P and $P \to Q$, deduce Q.
- (Universal Generalization) From P, deduce $\forall x.P$.
- (Membership Introduction) From P, deduce $\forall x.(x \in P)$, where x does not occur free in P.
- (Membership Elimination) From $\forall x.(x \in P)$, deduce P, where x does not occur free in P.

Theorem 3. *The proof system is sound and complete.*

Proof. No proof. □

2.2 Metatheorems of the deductive system

Proposition 4 (Tautology)**.** *For any propositional tautology $\mathcal{A}(p_1, \dots, p_n)$ where $p_1, \dots, p_n$ are all propositional variables in $\mathcal{A}$, and for any patterns $P_1, \dots, P_n$,*

$$\vdash \mathcal{A}(P_1, \dots, P_n).$$

Proof. No proof. □

Corollary 5. $\vdash \top$.

Proof. By definition, $\top = \neg\bot = \neg(x \wedge \neg x)$, where x is a matching logic variable who has the same sort with $\top$. Let proposition $\mathcal{A} = \neg(p \wedge \neg p)$ with p is a propositional variable. Then $\mathcal{A}$ is a propositional tautology. By Proposition 4, $\top = \mathcal{A}[x/p]$ is derivable in the proof system, i.e., $\vdash \top$. □

Equalities plays an important role in matching logic. Axiom (K6) is very powerful even though it looks quite simple. It basically says that whenever one establishes that $P = Q$, then the two patterns are interchangeable everywhere in any patterns, as concluded in the next lemma.

Lemma 6. *If* $\vdash P_1 = P_2$ *and* $\vdash Q[P_1/x]$*, then* $\vdash Q[P_2/x]$.

Proof.

$$\dfrac{\dfrac{\dfrac{\cdot}{P_1 = P_2} \quad \dfrac{\cdot}{P_1 = P_2 \to (Q[P_1/x] \to Q[P_2/x])}\ \text{(K6)}}{\vdash Q[P_1/x] \to Q[P_2/x]}\ \text{(MP)} \qquad \dfrac{\cdot}{\vdash Q[P_1/x]}}{\vdash Q[P_2/x]}\ \text{(MP)}$$

□

The next proposition is useful when one wants to establish an equality pattern.

Proposition 7. $\vdash P \leftrightarrow Q$ *iff* $\vdash P = Q$.

Proof. That the right hand side implies the left is easy, so we only prove that the left implies the right. By definition, $(P = Q) = \neg\lceil\neg(P \leftrightarrow Q)\rceil$ □

Proposition 8 ($\vee$-Introduction)**.** $\vdash P$ *implies* $\vdash P \vee Q$.

Proof. Use Proposition 4 and Modus Ponens. Note that in general, $\vdash P \vee Q$ does not imply $\vdash P$ or $\vdash Q$. □

Proposition 9 (Functional Substitution)**.** $\vdash \exists y.(Q = y) \to (P[Q/x] \to \exists x.P(x))$.

Proposition 10 ($\wedge$-Introduction and Elimination)**.** $\vdash P$ *and* $\vdash Q$ *iff* $\vdash P \wedge Q$.

Proof. Use Proposition 4 and Modus Ponens. □

Proposition 11 (Equality Introduction)**.** $\vdash P = P$.

Proof. Use Membership Introduction and Proposition 4. □

Corollary 12. $\vdash P$ *implies* $\vdash P = \top$.

Proposition 13. $\vdash x \in \lceil y \rceil$.

Proof.

$$\begin{aligned} &\vdash x \in \lceil y \rceil \\ \text{if } &\vdash \forall x.(x \in \lceil y \rceil) && (K5,\ K6, \text{ and Modus Ponens}) \\ \text{iff } &\vdash \lceil y \rceil. \end{aligned}$$

□

Proposition 14. $\vdash P \to \lceil P \rceil$.

Proof.

$$
\begin{aligned}
&\vdash P \to \lceil P \rceil\\
\text{iff } &\vdash \forall x.(x \in P \to \lceil P \rceil)\\
\text{if } &\vdash x \in P \to \lceil P \rceil\\
\text{iff } &\vdash x \in P \to x \in \lceil P \rceil\\
\text{iff } &\vdash x \in P \to \exists y.(y \in P \wedge x \in \lceil y \rceil)\\
\text{iff } &\vdash x \in P \to \neg \forall y.(y \notin P \vee x \notin \lceil y \rceil)\\
\text{iff } &\vdash \forall y.(y \notin P \vee x \notin \lceil y \rceil) \to x \notin P\\
\text{if } &\vdash x \notin P \vee x \notin \lceil x \rceil \to x \notin P\\
\text{iff } &\vdash x \in P \to x \in P \wedge x \in \lceil x \rceil\\
\text{iff } &\vdash x \in P \to x \in \lceil x \rceil\\
\text{if } &\vdash x \in \lceil x \rceil
\end{aligned}
$$

Remark Similarly we can show $\vdash \lfloor P \rfloor \to P$. □

Proposition 15. $\vdash \forall x.(x \in P) = \lfloor P \rfloor$, *where x occurs free in P.*

Proof. By Proposition 7 and 10, it suffices to show

$$\vdash \forall x.(x \in P) \to \lfloor P \rfloor \tag{2}$$

and

$$\vdash \lfloor P \rfloor \to \forall x.(x \in P). \tag{3}$$

To show (2),

$$
\begin{aligned}
&\vdash \forall x.(x \in P) \to \lfloor P \rfloor\\
\text{iff } &\vdash \forall x.\lceil x \wedge P \rceil \to \neg \lceil \neg P \rceil\\
\text{iff } &\vdash \lceil \neg P \rceil \to \exists x.\neg \lceil x \wedge P \rceil\\
\text{iff } &\vdash \forall y.(y \in (\lceil \neg P \rceil \to \exists x.\neg \lceil x \wedge P \rceil))\\
\text{if } &\vdash y \in (\lceil \neg P \rceil \to \exists x.\neg \lceil x \wedge P \rceil\\
\text{iff } &\vdash \exists z_1.(z_1 \notin P \wedge y \in \lceil z_1 \rceil) \to\\
&\qquad \exists x.\neg(\exists z_2.(z_2 = x \wedge z_2 \in P \wedge y \in \lceil z_2 \rceil))\\
\text{iff } &\vdash \exists z_1.(z_1 \notin P \wedge \top) \to \qquad\qquad \text{(Proposition 13, 6, and Corollary 12)}\\
&\qquad \exists x.\neg(\exists z_2.(z_2 = x \wedge z_2 \in P \wedge \top))\\
\text{iff } &\vdash \exists z_1.(z_1 \notin P) \to \exists x.\neg(\exists z_2.(z_2 = x \wedge z_2 \in P))\\
\text{iff } &\vdash \forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \to \forall z_1.(z_1 \in P)\\
\text{if } &\vdash \forall z_1.(\forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \to (z_1 \in P))\\
\text{if } &\vdash \forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \to (z_1 \in P).
\end{aligned}
$$

Since $\vdash \forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \to \exists z_2.(z_2 = z_1 \wedge z_2 \in P)$, it suffices to show

$$\begin{aligned}
&\vdash \exists z_2.(z_2 = z_1 \wedge z_2 \in P) \to (z_1 \in P)\\
\text{iff } &\vdash z_1 \notin P \to \forall z_2.(z_2 \neq z_1 \vee z_2 \notin P)\\
\text{if } &\vdash \forall z_2.(z_1 \notin P \to z_2 \neq z_1 \vee z_2 \notin P)\\
\text{if } &\vdash z_1 \notin P \to z_2 \neq z_1 \vee z_2 \notin P\\
\text{if } &\vdash z_2 = z_1 \wedge z_2 \in P \to z_1 \in P.
\end{aligned}$$

And we proved (2).

Similarly, to show (3),

$$\begin{aligned}
&\vdash \lfloor P \rfloor \to \forall x.(x \in P)\\
\text{iff } &\vdash \exists x.\neg\lceil x \wedge P \rceil \to \lceil \neg P \rceil\\
\text{iff } &\vdash \forall y.(y \in \exists x.\neg\lceil x \wedge P \rceil \to \lceil \neg P \rceil)\\
\text{if } &\vdash y \in \exists x.\neg\lceil x \wedge P \rceil \to \lceil \neg P \rceil\\
\text{iff } &\vdash \exists x.\neg\exists z_2.(z_2 = x \wedge z_2 \in P) \to \exists z_1.(z_1 \notin P)\\
\text{iff } &\vdash \forall z_1.(z_1 \in P) \to \exists z_2.(z_2 = x \wedge z_2 \in P)\\
\text{iff } &\vdash x \in P \to \exists z_2.(z_2 = x \wedge z_2 \in P).
\end{aligned}$$

We proved (3).

Remark If x occurs free in P, the result does not hold. For example, let P be $\mathit{upto}(x)$ where $\mathit{upto}(\cdot)$ is interpreted to $\mathit{upto}(n) = \{0, 1, \ldots, n\}$ on $\mathbb{N}$. □

Remark From Membership Introduction and Elimination inference rules and Proposition 15, $\vdash P$ iff $\vdash \lfloor P \rfloor$.

Proposition 16 (Classification Reasoning)**.** *For any P and Q, from $\vdash P \to Q$ and $\vdash \neg P \to Q$ deduce $\vdash Q$.*

Proof. From $\vdash \neg P \to Q$ deduce $\vdash \neg Q \to P$. Notice that $\vdash P \to Q$, so we have $\vdash \neg Q \to Q$, i.e., $\vdash \neg\neg Q \vee Q$ which concludes the proof. □

Corollary 17. *For any P_1, P_2, and Q are patterns with $\vdash P_1 \vee P_2$, from $\vdash P_1 \to Q$ and $\vdash P_2 \to Q$, deduce $\vdash Q$.*

Definition 18 (Predicate Pattern)**.** *A pattern P is called a predicate pattern or a predicate if $\vdash (P = \top) \vee (P = \bot)$.*

Remark Predicate patterns are closed under all logic connectives.

Remark For any P, $\lceil P \rceil$ is a predicate pattern.

Proposition 19. $\vdash (\lceil P \rceil = \bot) = (P = \bot)$ *and* $\vdash (\lfloor P \rfloor = \top) = (P = \top)$.

Proof. It is easy to prove one derivation from the other, so we only prove the first one. By Proposition 7, it suffices to prove

$$\vdash (\lceil P \rceil = \bot) \to (P = \bot) \tag{4}$$

and

$$\vdash (P = \bot) \to (\lceil P \rceil = \bot) \tag{5}$$

The proof of (5) is trivial and we left it as an exercise. We now prove (4) through the following backward reasoning.

$$\begin{aligned}
& \vdash (\lceil P \rceil = \bot) \to (P = \bot) \\
\text{iff} \quad & \vdash \forall y.(y \in ((\lceil P \rceil = \bot) \to (P = \bot))) \\
\text{if} \quad & \vdash y \in ((\lceil P \rceil = \bot) \to (P = \bot)) \\
\text{iff} \quad & \vdash (y \in (\lceil P \rceil = \bot) \to (y \in (P = \bot)).
\end{aligned} \tag{6}$$

While for any pattern Q,

$$\begin{aligned}
& \vdash y \in (Q = \bot) \\
\text{iff} \quad & \vdash y \in \neg\lceil \neg(Q \leftrightarrow \bot) \rceil \\
\text{iff} \quad & \vdash y \in \neg\lceil Q \rceil \\
\text{iff} \quad & \vdash \neg\exists z.(z \in Q \wedge y \in \lceil z \rceil) \\
\text{iff} \quad & \vdash \neg\exists z.(z \in Q)
\end{aligned}$$

So we continue to prove (6) by showing

$$\begin{aligned}
& \vdash (y \in (\lceil P \rceil = \bot)) \to (y \in (P = \bot)) \\
\text{iff} \quad & \vdash \neg\exists z.(z \in \lceil P \rceil) \to \neg\exists z.(z \in P) \\
\text{iff} \quad & \vdash \exists z.(z \in P) \to \exists z.(z \in \lceil P \rceil) \\
\text{iff} \quad & \vdash \exists z.(z \in P) \to \exists z.(\exists z_1.(z_1 \in P \wedge z \in \lceil z_1 \rceil)) \\
\text{iff} \quad & \vdash \exists z.(z \in P) \to \exists z.\exists z_1.(z_1 \in P) \\
\text{iff} \quad & \vdash \exists z_1.(z_1 \in P) \to \exists z.\exists z_1.(z_1 \in P).
\end{aligned}$$

And we finish the proof by noticing the fact that for any pattern Q and variable x,

$$\vdash Q \to \exists x.Q.$$

□

Proposition 20. *For any predicate P, $\vdash (P \neq \top) = (P = \bot)$ and $\vdash (P \neq \bot) = (P = \top)$.*

Proof. We only prove the first derivation, by showing both

$$\vdash (P \neq \top) \to (P = \bot) \tag{7}$$

and

$$\vdash (P = \bot) \to (P \neq \top). \tag{8}$$

Proving (8) is trivial. We now prove (7), which is also trivial by transforming disjunction to implication. □

Proposition 21. *For any pattern Q and any predicate pattern P, $\vdash P \vee Q$ iff $\vdash P \vee \lfloor Q \rfloor$.*

Proof. $(\Leftarrow)$ is obtained immediately by the remark of Proposition 14. We now prove $(\Rightarrow)$.

Because $\vdash Q = \top \vee Q \neq \top$, it suffices to show

$$\vdash Q = \top \to (P \vee \lfloor Q \rfloor = \top) \tag{9}$$

and

$$\vdash Q \neq \top \to (P \vee \lfloor Q \rfloor = \top) \tag{10}$$

by Corollary 17, and the fact that $\vdash P \vee \lfloor Q \rfloor = \top$ and $\vdash \top$ imply $\vdash P \vee \lfloor Q \rfloor$.

The proof of (9) is straightforward as follows.

$$\begin{array}{ll}
 & \vdash Q = \top \to (P \vee \lfloor Q \rfloor = \top) \\
\text{if} & \vdash Q = \top \to (P \vee \lfloor \top \rfloor = \top) \\
\text{if} & \vdash Q = \top \to (\top = \top) \\
\text{if} & \vdash \top.
\end{array}$$

The proof of (10) needs more effort:

$$\begin{array}{ll}
 & \vdash Q \neq \top \to (P \vee \lfloor Q \rfloor = \top) \\
\text{iff} & \vdash (Q = \top) \vee (P \vee \lfloor Q \rfloor = \top) \\
\text{iff} & \vdash (\lfloor Q \rfloor = \top) \vee (P \vee \lfloor Q \rfloor = \top) \\
\text{iff} & \vdash \lfloor Q \rfloor \neq \top \to (P \vee \lfloor Q \rfloor = \top) \\
\text{iff} & \vdash \lfloor Q \rfloor = \bot \to (P \vee \lfloor Q \rfloor = \top) \\
\text{if} & \vdash \lfloor Q \rfloor = \bot \to (P \vee \bot = \top) \\
\text{iff} & \vdash \lfloor Q \rfloor = \bot \to (P = \top) \\
\text{if} & \vdash Q = \top \vee P = \top.
\end{array}$$

Notice that P is a predicate pattern, so it suffices to show

$$\vdash P = \top \to (Q = \top \vee P = \top),$$

whose validity is obvious, and

$$\vdash P = \bot \to (Q = \top \vee P = \top),$$

which is proved by showing

$$\vdash P = \bot \to Q = \top. \tag{11}$$

Because $\vdash P \vee Q$, it suffices to show

$$\begin{array}{ll}
 & \vdash P = \bot \to (P \vee Q) \to (Q = \top) \\
\text{if} & \vdash P = \bot \to (\bot \vee Q) \to (Q = \top) \\
\text{iff} & \vdash P = \bot \to Q \to (Q = \top) \\
\text{if} & \vdash Q \to (Q = \top) \\
\text{iff} & \vdash (Q \neq \top) \to \neg Q \\
\text{iff} & \vdash (\lfloor Q \rfloor = \bot) \to \neg Q.
\end{array}$$

Notice we have $\vdash Q \to \lfloor Q \rfloor$, which means $\vdash \neg\lfloor Q \rfloor \to \neg Q$, so it suffices to show

$$
\begin{array}{ll}
& \vdash (\lfloor Q \rfloor = \bot) \to \neg\lfloor Q \rfloor \\
\text{iff} & \vdash (\lfloor Q \rfloor = \bot) \to \neg\bot \\
\text{iff} & \vdash (\lfloor Q \rfloor = \bot) \to \top \\
\text{iff} & \vdash \top.
\end{array}
$$

And this concludes the proof. □

Proposition 22 (Deduction Theorem)**.** *If $\Gamma \cup \{P\} \vdash Q$ and the derivation does not use $\forall x$-Generalization where x is free in P, then $\Gamma \vdash \lfloor P \rfloor \to Q$.*

Proof. The proof is by induction on n, the length of the derivation of Q from $\Gamma \cup \{P\}$.

Base step: $n = 1$, and Q is an axiom, or P, or a member of Γ. If Q is an axiom or a member of Γ, then $\Gamma \vdash Q$ and as a result, $\Gamma \vdash \lfloor P \rfloor \to Q$. If Q is P, then $\Gamma \vdash \lfloor P \rfloor \to Q$ by Proposition 14.

Induction step: Let $n > 1$. Suppose that if P' can be deduced from $\Gamma \cup \{P\}$ without using $\forall x$-Generalization where x is free in P, in a derivation containing fewer than n steps, then $\Gamma \vdash \lfloor P \rfloor \to P'$.

Case 1: Q is an axiom, or P, or a member of Γ. Precisely as in the Base step, we show that $\vdash \lfloor P \rfloor \to Q$.

Case 2: Q follows from two previous patterns in the derivation by an application of Modus Ponens. These two patterns must have the forms Q_1 and $Q_1 \to Q$, and each one can certainly be deduced from $\Gamma \cup \{P\}$ by a derivation with fewer than n steps, by just omitting the subsequent members from the original derivation from $\Gamma \cup \{P\} \vdash Q$. So we have $\Gamma \cup \{P\} \vdash Q_1$ and $\Gamma \cup \{P\} \vdash Q_1 \to Q$, and, applying the hypothesis of induction, $\Gamma \vdash \lfloor P \rfloor \to Q_1$ and $\Gamma \vdash \lfloor P \rfloor \to (Q_1 \to Q)$. It follows immediately that $\Gamma \vdash \lfloor P \rfloor \to Q$.

Case 3: Q follows from a previous pattern in the derivation by an application of $\forall x_i$-Generalization where x_i does not occur free in P. So Q is $\forall x_i.Q_1$, say, and Q_1 appears previously in the derivation. Thus $\Gamma \cup \{P\} \vdash Q_1$, and the derivation has fewer than n steps, so $\Gamma \vdash \lfloor P \rfloor \to Q_1$, since there is no application of Universal Generalization involving a free variable of P. Also x_i cannot occur free in P, as it is involved in an application of Universal Generalization in the deduction of Q from $\Gamma \cup \{P\}$. So we have a derivation of $\Gamma \vdash \lfloor P \rfloor \to Q$ as follows.

$$
\begin{array}{ll}
& \Gamma \vdash \lfloor P \rfloor \to Q \\
\text{iff} & \Gamma \vdash \lfloor P \rfloor \to \forall x_i.Q_1 \\
\text{if} & \Gamma \vdash \forall x_i.(\lfloor P \rfloor \to Q_1) \\
\text{if} & \Gamma \vdash \lfloor P \rfloor \to Q_1.
\end{array}
$$

So $\Gamma \vdash \lfloor P \rfloor \to Q$ as required.

Case 4: Q follows from a previous pattern in the derivation by an application of Membership Introduction. So Q is $\forall x_i.(x_i \in Q_1)$ with x_i is free in Q_1, say, and Q_1 appears previously in the derivation. Thus $\Gamma \cup \{P\} \vdash Q_1$, and the derivation has fewer than n steps, so $\Gamma \vdash \lfloor P \rfloor \to Q_1$, since there is no application of Universal Generalization involving a free variable of P. So we have a derivation of $\Gamma \vdash \lfloor P \rfloor \to Q$ as follows.

$$\begin{aligned}
& \Gamma \vdash \lfloor P \rfloor \to Q \\
\text{iff} \quad & \Gamma \vdash \lfloor P \rfloor \to \forall x_i.(x_i \in Q_1) \\
\text{iff} \quad & \Gamma \vdash \lfloor P \rfloor \to \lfloor Q_1 \rfloor,
\end{aligned}$$

which follows by the hypothesis of induction $\Gamma \vdash \lfloor P \rfloor \to Q_1$ and the fact that $\Gamma \vdash Q_1 \to \lfloor Q_1 \rfloor$ (by the Remark in Proposition 14).

Case 5: Q follows from a previous pattern in the derivation by an application of Membership Elimination. The previous pattern must have the form $\forall x_i.(x_i \in Q)$, and can be deduced from $\Gamma \cup \{P\}$ by a derivation with fewer than n steps, by just omitting the subsequent members from the original derivation from $\Gamma \cup \{P\} \vdash Q$. So we have $\Gamma \cup \{P\} \vdash \forall x_i.(x_i \in Q)$, and, applying the hypothesis of induction, $\Gamma \vdash \lfloor P \rfloor \to \forall x_i.(x_i \in Q)$. So we have a derivation of $\Gamma \vdash \lfloor P \rfloor \to Q$ as follows.

$$\begin{aligned}
& \Gamma \vdash \lfloor P \rfloor \to Q \\
\text{iff} \quad & \Gamma \vdash \neg\lfloor P \rfloor \vee Q \\
\text{iff} \quad & \Gamma \vdash \neg\lfloor P \rfloor \vee \lfloor Q \rfloor && \text{(Proposition 21)} \\
\text{iff} \quad & \Gamma \vdash \neg\lfloor P \rfloor \vee \forall x_i.(x_i \in Q) \\
\text{iff} \quad & \Gamma \vdash \lfloor P \rfloor \to \forall x_i.(x_i \in Q),
\end{aligned}$$

which is the hypothesis of induction. And this concludes our inductive proof. □

Corollary 23 (Closed-form Deduction Theorem)**.** *If P is closed, $\Gamma \cup \{P\} \vdash Q$ implies $\Gamma \vdash \lfloor P \rfloor \to Q$.*

Theorem 24 (Frame Rule)**.** *Let $\sigma \in \Sigma$ be a symbol in the signature. From $P_1 \to P_2$, deduce $\sigma(P_1) \to \sigma(P_2)$. In its most general form, $P_1 \to P_2$ deduces $\sigma(Q_1, \ldots, P_1, \ldots, Q_n) \to \sigma(Q_1, \ldots, P_2, \ldots, Q_n)$.*

Proof. we write $\sigma(Q_1, \ldots, P_i, \ldots, Q_n)$ as $\sigma(P_i, \vec{Q})$ for short, for any $i \in \{1, 2\}$.

$$\begin{aligned}
& \vdash \sigma(P_1, \vec{Q}) \to \sigma(P_2, \vec{Q}) \\
\text{iff} \quad & \vdash y \in (\sigma(P_1, \vec{Q}) \to \sigma(P_2, \vec{Q})) \\
\text{iff} \quad & \vdash (y \in \sigma(P_1, \vec{Q})) \to (y \in \sigma(P_2, \vec{Q})) \\
\text{iff} \quad & \vdash \exists z_1.\exists \vec{z}.(z_1 \in P_1 \wedge \vec{z} \in \vec{Q} \wedge y \in \sigma(z_1, \vec{z})) \\
& \to \exists z_2.\exists \vec{z}.(z_2 \in P_2 \wedge \vec{z} \in \vec{Q} \wedge y \in \sigma(z_2, \vec{z})) \\
\text{iff} \quad & \vdash \exists z_1.\exists \vec{z}.(z_1 \in P_1 \wedge \vec{z} \in \vec{Q} \wedge y \in \sigma(z_1, \vec{z}) \\
& \qquad \to z_1 \in P_2 \wedge \vec{z} \in \vec{Q} \wedge y \in \sigma(z_1, \vec{z})) \\
\text{iff} \quad & \vdash \exists z_1.\exists \vec{z}.(z_1 \in P_1 \to z_1 \in P_2) \\
\text{if} \quad & \vdash \exists z_1.(z_1 \in P_1 \to z_1 \in P_2) \\
\text{if} \quad & \vdash P_1 \to P_2.
\end{aligned}$$

$\square$

Corollary 25 (Frame Rule as Implication)**.** $\vdash \lfloor P \to Q \rfloor \to (\sigma(P) \to \sigma(Q))$

3 Inference rules

Axioms

$$\frac{\cdot}{\Gamma \vdash A}$$

where A is an axiom.

Inclusion

$$\frac{\cdot}{\Gamma \vdash P}$$

where $P \in \Gamma$.

Modus Ponens

$$\frac{\Gamma \vdash Q \to P \quad \Gamma \vdash Q}{\Gamma \vdash P}$$

Closed-Form Deduction Theorem

$$\frac{\Gamma \cup \{P\} \vdash Q}{\Gamma \vdash P \to Q}$$

where P is closed.

Universal Generalization

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P}\ (\forall x)$$

Conjunction Splitting

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$$