Proof System of Matching Logic

1 SYNTAX AND SEMANTICS OF MATCHING LOGIC

Definition 1.1 (Signatures and Patterns). A matching logic signature is a triple $\bar{\Sigma} = (\text{VAR}, S, \Sigma)$ that consists of an S-indexed family of countably infinite variable sets $\text{VAR} = \{\text{VAR}_s\}_{s \in S}$, a countable (finite or infinite) sort set S, and an $(S^* \times S)$-indexed countable (finite or infinite) symbol set $\Sigma = \{\Sigma_{s_1 \ldots s_n, s}\}_{s_1, \ldots, s_n, s \in S}$. We write $\Sigma_{s_1 \ldots s_n, s}$ as $\Sigma_{\lambda, s}$ if $n = 0$. The set of all $\bar{\Sigma}$-patterns of sort s, denoted as $\text{PATTERN}_s(\bar{\Sigma})$, is defined by the following grammar

$$
\begin{array}{llll}
\varphi_s & ::= & x \in \text{VAR}_s & \text{// Variable} \\
& | & \sigma(\varphi_{s_1}, \ldots, \varphi_{s_n}) \text{ with } \sigma \in \Sigma_{s_1 \ldots s_n, s} & \text{// Structure} \\
& | & \neg \varphi_s & \text{// Complement} \\
& | & \varphi_s \wedge \varphi_s & \text{// Intersection} \\
& | & \exists x . \varphi_s \text{ with } x \in \text{VAR}_{s'} \text{ (of any sort } s' \in S) & \text{// Binding}
\end{array}
$$

Let $\text{PATTERN}(\bar{\Sigma}) = \{\text{PATTERN}_s(\bar{\Sigma})\}_{s \in S}$ be the S-indexed family set of all patterns.

When VAR is clear from the context, we often omit it and just write $\bar{\Sigma} = (S, \Sigma)$. If S is also clear, we refer to a signature by just Σ. When the signature $\bar{\Sigma}$ is clear we write PATTERN_s to denote the set of all $\bar{\Sigma}$-patterns of sort s. Throughout the paper, we use S to denote a fixed sort set and $s, s_1, s_2, \ldots$ to denote individual sorts.

By abuse use of language, we blur the distinction between a family of sets and their union set. In particular, we write $\text{VAR} = \bigcup_{s \in S} \text{VAR}_s$ and $\text{PATTERN} = \bigcup_{s \in S} \text{PATTERN}_s$ to denote the set of all variables and patterns respectively. To ease our notation, $\varphi \in \text{PATTERN}$ means φ is a pattern, and $\varphi_s \in \text{PATTERN}$ or $\varphi \in \text{PATTERN}_s$ means it has sort s. Similarly, $\sigma \in \Sigma$ means σ is a symbol. If $\sigma \in \Sigma_{\lambda, s}$, we say σ is a constant symbol of sort s, and we write σ instead of $\sigma()$. We write $x{:}s$ to emphasize that x is a variable of sort s.

Derived constructs in propositional and FOL reasoning are defined in their usual way:

$$
\begin{array}{rclcrcl}
\top_s & \equiv & \exists x{:}s . x{:}s & & \bot_s & \equiv & \neg \top_s \\
\varphi_1 \vee \varphi_2 & \equiv & \neg(\neg \varphi_1 \wedge \neg \varphi_2) & & \varphi_1 \rightarrow \varphi_2 & \equiv & \neg \varphi_1 \vee \varphi_2 \\
\varphi_1 \leftrightarrow \varphi_2 & \equiv & (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1) & & \forall x . \varphi & \equiv & \neg(\exists x . \neg \varphi)
\end{array}
$$

We feel free to drop their sorts when there is no confusion. The precedence of these constructs are the same as in FOL. Negation ($\neg$) binds the tightest, followed by conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$), and double implication ($\leftrightarrow$). Binders ($\exists$ and $\forall$) bind to the rightmost. Implication is right associative, so $\varphi_1 \rightarrow \varphi_2 \rightarrow \varphi_3$ means $\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3)$, but we are generous in using parenthesis to prevent potential parsing ambiguity from humans.

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As in FOL, we can define the notions of free variables, variable-capture-free substitution, and alpha-renaming.

Definition 1.2 (Free variables, substitution, and alpha-renaming). Let $\varphi \in$ PATTERN. The set of free variables in φ, denoted as $FV(\varphi)$, is defined as follows:

$$FV(x) = \{x\} \quad FV(\neg\varphi) = FV(\varphi) \quad FV(\varphi_1 \wedge \varphi_2) = FV(\varphi_1) \cup FV(\varphi_2)$$
$$FV(\sigma(\varphi_1, \ldots, \varphi_n)) = \bigcup_{1 \le i \le n} FV(\varphi_i) \quad FV(\exists x.\varphi) = FV(\varphi) \setminus \{x\}$$

Variable-capture-free substitution is defined in the usual way:

$$x[\psi/x] = \psi \quad y[\psi/x] = y \text{ if } y \text{ is distinct from } x \quad (\neg\varphi)[\psi/x] = \neg(\varphi[\psi/x])$$
$$(\varphi_1 \wedge \varphi_2)[\psi/x] = \varphi_1[\psi/x] \wedge \varphi_2[\psi/x] \quad \sigma(\varphi_1, \ldots, \varphi_n)[\psi/x] = \sigma(\varphi_1[\psi/x], \ldots, \varphi_n[\psi/x])$$
$$(\exists x.\varphi)[\psi/x] = \exists x.\varphi \quad (\exists y.\varphi)[\psi/x] = \exists y.(\varphi[\psi/x]) \text{ if } y \text{ is distinct from } x \text{ and } y \notin FV(\psi)$$

Finally, alpha-renaming allows to avoid variable capture:

$$\exists x.\varphi \equiv \exists y.(\varphi[y/x]) \quad \text{ if } y \notin FV(\varphi).$$

Example 1.3 (Variable-capture-free substitution). Let x, y, and z be distinct variables.

$$
\begin{aligned}
(\exists x.(x \wedge y))[x/y] \quad &\equiv (\exists z.((x \wedge y)[z/x]))[x/y] \quad && \text{// alpha-renaming} \\
&= (\exists z.(z \wedge y))[x/y] \quad && \text{// variable-capture-free substitution} \\
&= \exists z.((z \wedge y)[x/y]) \quad && \text{// variable-capture-free substitution} \\
&= \exists z.(z \wedge x) \quad && \text{// variable-capture-free substitution}
\end{aligned}
$$

As a convention, we consider patterns that are alpha-equivalent the same, so $\exists x.(x \wedge y)$ and $\exists z.(z \wedge y)$ are the same pattern.

Definition 1.4 (Models, valuations, and satisfaction). Let $\overline{\Sigma} = (S, \Sigma)$ be a signature. A $\overline{\Sigma}$-model is a pair $\mathcal{M} = (M, \_\mathcal{M})$ with $M = \{M_s\}_{s \in S}$ an S-indexed family of nonempty sets called the carrier sets, and $\_\mathcal{M} = \{\sigma_\mathcal{M}\}_{\sigma \in \Sigma}$ a Σ-indexed interpretation mapping:

- each $\sigma \in \Sigma_{s_1 \ldots s_n, s}$ maps to a function $\sigma_\mathcal{M}: M_{s_1} \times \cdots \times M_{s_n} \to 2^{M_s}$,

where 2^{M_s} means the set of all subsets of M_s. In particular, each constant symbol $\sigma \in \Sigma_{\lambda, s}$ maps to a subset $\sigma_\mathcal{M} \subseteq M_s$; An $\mathcal{M}$-valuation is a mapping $\rho: \text{VAR} \to M$ such that for every $x \in \text{VAR}_s$, $\rho(x) \in M_s$ for every sort $s \in S$. Two $\mathcal{M}$-valuations ρ_1 and ρ_2 are called *equal modulo x*, denoted as $\rho_1 \overset{x}{\sim} \rho_2$, if $\rho_1(y) = \rho_2(y)$ for every y distinct from x. A valuation ρ can be extended to a mapping $\bar{\rho}: \text{PATTERN} \to 2^M$ such that $\bar{\rho}(\varphi_s) \subseteq M_s$. The extended valuation $\bar{\rho}$ is inductively defined as follows:

- $\bar{\rho}(x) = \{\rho(x)\}$, for every $x \in \text{VAR}_s$;
- $\bar{\rho}(\sigma(\varphi_1, \ldots, \varphi_n)) = \sigma_\mathcal{M}(\bar{\rho}(\varphi_1), \ldots, \bar{\rho}(\varphi_n))$, for every $\sigma \in \Sigma_{s_1 \ldots s_n, s}$ and appropriate $\varphi_1, \ldots, \varphi_n$
- $\bar{\rho}(\neg\varphi) = M_s \setminus \bar{\rho}(\varphi)$, for every $\varphi \in \text{PATTERN}_s$;
- $\bar{\rho}(\varphi_1 \wedge \varphi_2) = \bar{\rho}(\varphi_1) \cap \bar{\rho}(\varphi_2)$, for every φ_1, φ_2 of the same sort;
- $\bar{\rho}(\exists x.\varphi) = \bigcup\{\overline{\rho'}(\varphi) \mid \text{for every } \rho' \overset{x}{\sim} \rho\}$.

Given a model $\mathcal{M}$ and a valuation ρ, we say $\mathcal{M}$ and ρ satisfy a pattern φ_s, denoted as $\mathcal{M}, \rho \vDash \varphi_s$, if $\bar{\rho}(\varphi) = M_s$. We say $\mathcal{M}$ satisfies φ_s or φ_s holds in $\mathcal{M}$, denoted as $\mathcal{M} \vDash \varphi_s$, if $\mathcal{M}, \rho \vDash \varphi_s$ for every valuation ρ. We say φ_s is valid if it holds in every model. Let Γ be a pattern set. We say $\mathcal{M}$ satisfies Γ, if $\mathcal{M} \vDash \varphi$ for every $\varphi \in \Gamma$. We say Γ semantically entails φ, denoted as $\Gamma \vDash \varphi$, if for every model $\mathcal{M}$ such that $\mathcal{M} \vDash \Gamma$, $\mathcal{M} \vDash \varphi$. When Γ is the empty set, we abbreviate $\emptyset \vDash \varphi$ as just $\vDash \varphi$, which is equivalent to φ is valid.

Various basic properties about the semantics of matching logic can be found in [?]. Here we only introduce the Substitution Lemma, which is needed in the proof of the completeness theorems. Substitution Lemma connects the semantic valuation modification and the syntactic substitution.

Let ρ be a valuation. For any $x \in \text{Var}_s$ and $v \in M_s$, the modified valuation $\rho' = \rho[v/x]$ is defined as follows

$$\rho'(y) = \begin{cases} \rho(y) & \text{if } y \text{ is distinct from } x \\ v & \text{if } y \text{ is } x \end{cases}$$

LEMMA 1.5 (SUBSTITUTION LEMMA). *For any pattern φ and valuation ρ,*

$$\bar{\rho}(\varphi[y/x]) = \overline{\rho'}(\varphi)$$

where $\rho' = \rho[\rho(y)/x]$.

PROOF. The proof is by induction on the structure of φ. The only nontrivial case is when $\varphi \equiv \exists z.\psi$. Without loss of generality, let us assume z is distinct from x and y. If not, apply alpha-renaming to make them different. Then

$$
\begin{aligned}
& \bar{\rho}((\exists z.\psi)[y/x]) \\
\equiv\ & \bar{\rho}(\exists z.(\psi[y/x])) && \text{// variable-capture-free substitution} \\
\equiv\ & \textstyle\bigcup\{\overline{\rho_1}(\psi[y/x]) \mid \rho_1 \overset{z}{\sim} \rho\} && \text{// Definition 1.4} \\
\equiv\ & \textstyle\bigcup\{\overline{\rho_1'}(\psi) \mid \rho_1 \overset{z}{\sim} \rho \text{ and } \rho_1' = \rho_1[\rho_1(y)/x]\} && \text{// Induction hypothesis} \\
\equiv\ & \textstyle\bigcup\{\overline{\rho_1'}(\psi) \mid \rho_1 \overset{z}{\sim} \rho \text{ and } \rho_1' = \rho_1[\rho(y)/x]\} && \text{// } \rho_1(y) = \rho(y) \\
\equiv\ & \textstyle\bigcup\{\overline{\rho_1'}(\psi) \mid \rho_1 \overset{z}{\sim} \rho[\rho(y)/x]\} && \text{// } \rho_1(x) = \rho(y) \\
\equiv\ & \textstyle\bigcup\{\overline{\rho_1'}(\psi) \mid \rho_1 \overset{z}{\sim} \rho'\} && \text{// Definition of } \rho' \\
\equiv\ & \overline{\rho'}(\exists z.\psi) && \text{// Definition 1.4}
\end{aligned}
$$

□

2 PROOF SYSTEM OF MATCHING LOGIC

In this section, we study the proof system of matching logic. Firstly, let us define the notions of a context and context application.

Definition 2.1 (Contexts, Symbol Contexts, and Context Application). A context C is a pattern with a placeholder variable $\square \in \text{Var}$ that has exactly one free occurrence and no bound occurrence in C. For any pattern φ of the same sort as $\square$, context application $C[\varphi]$ is the result of replacing $\square$ for φ in C (without any alpha-renaming). Symbol contexts are a special family of contexts defined as follows

- The pattern $\square$ is a symbol context called the identity context;
- The pattern $\sigma(\varphi_1, \ldots, \varphi_{i-1}, \square, \varphi_{i+1}, \ldots, \varphi_n)$ is a symbol context. When patterns $\varphi_1, \ldots, \varphi_{i-1}$, $\varphi_{i+1}, \ldots, \varphi_n$ are not of interest, we just write the pattern as $C_{\sigma,i}$ or simply C_σ;
- The pattern of the form $C_{\sigma_1}[C_{\sigma_2}[\ldots C_{\sigma_n}[\square]\ldots]]$ is a symbol context where $\sigma_1, \ldots, \sigma_n$ are symbols (not necessarily distinct).

Remark 1. Unlike variable-capture-free substitution, context application $C[\varphi]$ may bind free variables in φ. For example, let C be $\exists x.(x \wedge \square)$ and φ be x, then $C[\varphi]$ is $\exists x.(x \wedge x)$, where the free variable x in φ is bound in $C[\varphi]$. Compare this example with Example 1.3 to get a better understanding of the difference between variable-capture-free substitution and context application.

Figure 1 is a sound and complete Hilbert-style proof system of matching logic. It has 13 proof rules in total. A proof rule has the form

$$\frac{\varphi_1 \quad \cdots \quad \varphi_n}{\varphi}$$

where patterns above the bar are called premises of the rule and the pattern below is called the conclusion. A proof rule with no premises is called an axiom, denoted as

$$\frac{\cdot}{\varphi} \quad \text{or simply} \quad \varphi$$

(PROPOSITION$_1$)	$\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_1)$
(PROPOSITION$_2$)	$(\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3)) \rightarrow (\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_3)$
(PROPOSITION$_3$)	$(\neg\varphi_1 \rightarrow \neg\varphi_2) \rightarrow (\varphi_2 \rightarrow \varphi_1)$
(MODUS PONENS)	$\dfrac{\varphi_1 \quad \varphi_1 \rightarrow \varphi_2}{\varphi_2}$
(VARIABLE SUBSTITUTION)	$\forall x.\varphi \rightarrow \varphi[y/x]$
($\forall$)	$\forall x.(\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \forall x.\varphi_2) \quad$ if $x \notin FV(\varphi_1)$
(UNIVERSAL GENERALIZATION)	$\dfrac{\varphi}{\forall x.\varphi}$
(PROPAGATION$_\perp$)	$C_\sigma[\perp] \rightarrow \perp$
(PROPAGATION$_\vee$)	$C_\sigma[\varphi_1 \vee \varphi_2] \rightarrow C_\sigma[\varphi_1] \vee C_\sigma[\varphi_2]$
(PROPAGATION$_\exists$)	$C_\sigma[\exists x.\varphi] \rightarrow \exists x.C_\sigma[\varphi] \quad$ if $x \notin FV(C_\sigma[\exists x.\varphi])$
(FRAMING)	$\dfrac{\varphi_1 \rightarrow \varphi_2}{C_\sigma[\varphi_1] \rightarrow C_\sigma[\varphi_2]}$
(EXISTENCE)	$\exists x.x$
(SINGLETON VARIABLE)	$\neg(C_1[x \wedge \varphi] \wedge C_2[x \wedge \neg\varphi])$
	where C_1 and C_2 are symbol contexts.

Fig. 1. Sound and Complete Proof System of Matching Logic

Definition 2.2 (Hilbert-Style Proofs). Let Γ be pattern set called the hypothesis set. A Γ-proof (or simply a proof when Γ is clear from the context) is a finite sequence of patterns $\varphi_1, \ldots, \varphi_n$ for some $n \geq 1$, such that for every $1 \leq k \leq n$, φ_k is either an axiom, or a member in Γ, or the conclusion of a rule with $\varphi_1, \ldots, \varphi_{k-1}$ as available premises. We say Γ syntactically entails φ or φ is provable from Γ, denoted as $\Gamma \vdash \varphi$, if there is a Γ-proof of φ. When Γ is the empty set, we abbreviate $\emptyset \vdash \varphi$ as $\vdash \varphi$.

We can prove a few useful results about the proof system. In the following, let Γ be any set of patterns.

PROPOSITION 2.3 (SOUND FOL REASONING). *Let $\overline{\Sigma} = (S, \Sigma)$ be a matching logic signature. Let (S, Π, F) be any first-order logic signature with $\Pi = \{\Pi_s\}_{s \in S}$ a set of constant predicate symbols and $F = \emptyset$ a set of function symbols. For any predicate logic formula $\Psi(\pi_1, \ldots, \pi_n)$ where $\pi_1, \ldots, \pi_n \in \Pi$, if $\Psi(\pi_1, \ldots, \pi_n)$ is derivable in FOL, then $\Psi(\varphi_1, \ldots, \varphi_n)$ is derivable in matching logic, where φ_i has the same sort as π_i for $1 \leq i \leq n$.*

PROOF. Notice that $F = \emptyset$, so the only FOL terms are variables. Under that condition, the first seven rules in Figure 1 forms a complete FOL proof system as in [?]. □

Remark 2. In literature, FOL with no function symbols is often called the "pure predicate logic", so Proposition 2.3 may better be called "sound predicate logic reasoning". Recently, in [?] the author established a more general result and proved that full FOL reasoning is also sound in matching logic. In this paper, however, we stick to our less general result Proposition 2.3 and call it "sound FOL reasoning", because it is good enough for our purposes and it requires fewer words to say than "sound predicate logic reasoning".

Remark 3. As a special case, propositional reasoning is sound in matching logic.

Matching logic enjoys frame reasoning. Even though rule (FRAMING) only allows basic frame reasoning at one argument of one symbol, it is not hard to extend it to multiple arguments or multiple symbol applications.

PROPOSITION 2.4 (SOUND FRAME REASONING). *For any $\sigma \in \Sigma_{s_1 \ldots s_n, s}$ and $\varphi_i, \varphi_i' \in$ PATTERN$_{s_i}$ such that $\Gamma \vdash \varphi_i \rightarrow \varphi_i'$ for any $1 \leq i \leq n$, then $\Gamma \vdash \sigma(\varphi_1, \ldots, \varphi_n) \rightarrow \sigma(\varphi_1', \ldots, \varphi_n')$. For any symbol context C and φ_i, φ_i' such that $\Gamma \vdash \varphi_i \rightarrow \varphi_i'$, then $\Gamma \vdash C[\varphi] \rightarrow C[\varphi_i']$.*

PROOF. For the first case, it suffices to show that

$$\Gamma \vdash \sigma(\varphi_1, \varphi_2, \ldots, \varphi_n) \rightarrow \sigma(\varphi_1', \varphi_2, \ldots, \varphi_n)$$
$$\Gamma \vdash \sigma(\varphi_1', \varphi_2, \ldots, \varphi_n) \rightarrow \sigma(\varphi_1', \varphi_2', \ldots, \varphi_n)$$
$$\cdots$$
$$\Gamma \vdash \sigma(\varphi_1', \varphi_2', \ldots, \varphi_n) \rightarrow \sigma(\varphi_1', \varphi_2', \ldots, \varphi_n')$$

which directly follow by (FRAMING).

For the second case, the proof is by structure induction on C. If C is the identity context, the conclusion is obvious. If C has the form $C_\sigma[C']$, the conclusion follows from induction hypothesis and (FRAMING).   □

PROPOSITION 2.5 (PROPAGATION OF SYMBOL APPLICATION). *For any symbol context C and patterns $\varphi_1, \varphi_2, \varphi$, the following propositions hold.*

- $\Gamma \vdash C[\bot] \leftrightarrow \bot$
- $\Gamma \vdash C[\varphi_1 \lor \varphi_2] \leftrightarrow C[\varphi_1] \lor C[\varphi_2]$
- $\Gamma \vdash C[\exists x.\varphi] \leftrightarrow \exists x.C[\varphi]$ *if $x \notin FV(C[\exists x.\varphi])$*

The following results are often useful in practice, whose proofs can be obtained by standard propositional reasoning with the above propositions:

- $\Gamma \vdash C[\varphi_1 \lor \varphi_2]$ *iff* $\Gamma \vdash C[\varphi_1] \lor C[\varphi_2]$
- $\Gamma \vdash C[\exists x.\varphi]$ *iff* $\Gamma \vdash \exists x.C[\varphi]$ *if $x \notin FV(C[\exists x.\varphi])$*

PROOF. The proof is by structure induction on the symbol context C. If C is the identity context then the conclusion is obvious. Now assume $C = C_\sigma[C']$ where C' is a symbol context for which the conclusion holds.

Firstly, let us prove $\Gamma \vdash C_\sigma[C'[\bot]] \leftrightarrow \bot$. The implication from right to left is by simple propositional reasoning. For the other direction, notice by induction hypothesis $\Gamma \vdash C'[\bot] \rightarrow \bot$ and by (FRAMING) $\Gamma \vdash C_\sigma[C'[\bot]] \rightarrow C_\sigma[\bot]$. In addition by (PROPAGATION$_\bot$), $\Gamma \vdash C_\sigma[\bot] \rightarrow \bot$, and the rest of the proof is by standard propositional reasoning.

Secondly, let us prove $\Gamma \vdash C_\sigma[C'[\varphi_1 \lor \varphi_2]] \leftrightarrow C_\sigma[C'[\varphi_1]] \lor C_\sigma[C'[\varphi_2]]$. For the implication from right to left, it suffices to prove $\Gamma \vdash C_\sigma[C'[\varphi_i]] \rightarrow C_\sigma[C'[\varphi_1 \lor \varphi_2]]$ for $i = 1, 2$. By (FRAMING), it suffices to prove $\Gamma \vdash C'[\varphi_i] \rightarrow C'[\varphi_1 \lor \varphi_2]$, which follows from the induction hypothesis. For the implication from left to right, the proof is the same as how we proved $\Gamma \vdash C_\sigma[C'[\bot]] \rightarrow \bot$, while instead of (PROPAGATION$_\bot$) we use (PROPAGATION$_\lor$).

Finally, let us prove $\Gamma \vdash C_\sigma[C'[\exists x.\varphi]] \leftrightarrow \exists x.C_\sigma[C'[\varphi]]$ for $x \notin FV(C_\sigma[C'[\exists x.\varphi]])$. In fact the proof is the same as above, while instead of (PROPAGATION$_\lor$) we use (PROPAGATION$_\exists$).   □

PROPOSITION 2.6 (CONGRUENCE OF PROVABLY EQUIVALENCE). *For any context C (not necessarily just symbol context), $\Gamma \vdash \varphi_1 \leftrightarrow \varphi_2$ implies $\Gamma \vdash C[\varphi_1] \leftrightarrow C[\varphi_2]$.*

Proof. The proof is by induction on the structure of C. If C is the identity context the conclusion is obvious. If C is of the form $\neg C'$, $\psi \wedge C'$, or $C' \wedge \psi$ where C' is a context and ψ is a pattern (notice ψ does not have the placeholder variable $\square$ in it), the conclusion is by standard propositional reasoning. If C has the form $\exists x.C'$, the conclusion follows from standard FOL reasoning. If C has the form $C_\sigma[C']$, the conclusion follows from Proposition 2.4. □

2.1 The Definedness Symbols and Completeness

In [?], the author proposed a sound and complete proof system of matching logic, which is shown in Figure 2. That proof system, however, assumes the existence of a special family of symbols in the signature called the definedness symbols. Using definedness symbols, we can easily define equality and membership as derived constructs, and proved they have the expected semantics. We refer to [?] for more details, and only summarize the definitions of definedness symbols, equality, and membership as follows.

Definition 2.7. For any sorts $s_1, s_2 \in S$, the definedness symbol $\lceil \_ \rceil_{s_1}^{s_2} \in \Sigma_{s_1, s_2}$ is a unary symbol which has an axiom $\lceil x{:}s_1 \rceil_{s_1}^{s_2}$. For any $\varphi, \varphi' \in \text{Pattern}_{s_1}$, define the equality construct $\varphi =_{s_1}^{s_2} \varphi' \equiv \neg \lceil \neg(\varphi \leftrightarrow \varphi') \rceil_{s_1}^{s_2}$ and the membership construct $x{:}s_1 \in_{s_1}^{s_2} \varphi \equiv \lceil x{:}s_1 \wedge \varphi \rceil_{s_1}^{s_2}$. We feel free to drop the sorts when they are clear from the context.

(Propositional Tautology)	φ, if φ is a proposition tautology
(Modus Ponens)	From φ_1 and $\varphi_1 \to \varphi_2$, deduce φ_2.
(Functional Substitution)	$(\forall x.\varphi) \wedge (\exists y.\varphi' = y) \to \varphi[\varphi'/x]$ if $y \notin FV(\varphi')$
($\forall$)	$\forall x.(\varphi_1 \to \varphi_2) \to (\varphi_1 \to \forall x.\varphi_2)$ if $x \notin FV(\varphi_1)$
(Universal Generalization)	From φ, deduce $\forall x.\varphi$.
(Equality Introduction)	$\varphi = \varphi$
(Equality Elimination)	$(\varphi_1 = \varphi_2) \wedge \psi[\varphi_1/x] \to \psi[\varphi_2/x]$
(Membership Introduction)	From φ, deduce $\forall x.(x \in \varphi)$, if $x \notin FV(\varphi)$.
(Membership Elimination)	From $\forall x.(x \in \varphi)$, deduce φ, if $x \notin FV(\varphi)$.
(Membership Variable)	$(x \in y) = (x = y)$
(Membership$_\neg$)	$(x \in \neg\varphi) = \neg(x \in \varphi)$
(Membership$_\wedge$)	$(x \in \varphi_1 \wedge \varphi_2) = (x \in \varphi_1) \wedge (x \in \varphi_2)$
(Membership$_\exists$)	$(x \in \exists y.\varphi) = \exists y.(x \in \varphi)$, where x and y distinct.
(Membership Symbol)	$x \in C_\sigma[\varphi] = \exists y.(y \in \varphi) \wedge (x \in C_\sigma[y])$ if $y \notin FV(C_\sigma[\varphi])$

Fig. 2. Sound and Complete Proof System with Definedness Symbols

The proof system in Figure 2 contains rules that use equality and membership constructs, so it cannot be used when definedness symbols are not in the signature. Its completeness is given through a reduction to predicate logic. In contrary, the proof system in Figure 1 is generic and does not depend on definedness symbols. The technique we use to prove its completeness (in Section ??) is also different from [?].

The main purpose of this section is to show that all rules in Figure 2 are in fact derivable using our proof system in Figure 1 plus the definedness axioms. Here are some quick ones. Rule (Propositional Tautology) follows from Proposition 2.3. Rules (Modus Ponens), ($\forall$), and (Universal Generalization) are themselves rules in Figure 1. To prove the rest, we need some lemmas and a form of the Deduction Theorem. To ease our notation, we refer to Proposition 2.3 as "by FOL reasoning", Proposition 2.4 as "by Frame reasoning", and Proposition 2.5 as "by Propagation".

We will see that many rules in modal logic have their counterparts in matching logic. Let us first define the compliment of a symbol.

Definition 2.8 (Compliment of Symbols). For any $\sigma \in \Sigma_{s_1 \ldots s_n, s}$, define its compliment $\bar{\sigma}$ as follows:

$$\bar{\sigma}(\varphi_1, \ldots, \varphi_n) \equiv \neg\sigma(\neg\varphi_1, \ldots, \neg\varphi_n)$$

In particular, the compliment of the definedness symbol is denoted as $\lfloor \varphi \rfloor \equiv \neg\lceil \neg\varphi \rceil$. Using this notation, the definition of equality can also be written as $(\varphi_1 =_{s_1}^{s_2} \varphi_2) \equiv \lfloor \varphi_1 \leftrightarrow \varphi_2 \rfloor_{s_1}^{s_2}$.

We will see that the compliments of symbols are the counterparts of modalities in model logic.

LEMMA 2.9. *For any symbol context C, $\Gamma \vdash \varphi$ implies $\Gamma \vdash \neg C[\neg\varphi]$. In particular, $\Gamma \vdash \varphi$ implies $\Gamma \vdash \bar{\sigma}(\ldots, \varphi, \ldots)$ for any symbol σ.*

PROOF.

$$
\begin{array}{l|ll}
1 & \varphi & \text{hypothesis} \\
2 & \neg\varphi \rightarrow \bot & \text{by 1, FOL reasoning} \\
3 & C[\neg\varphi] \rightarrow C[\bot] & \text{by 2, Frame reasoning} \\
4 & C[\bot] \rightarrow \bot & \text{by Propagation} \\
5 & C[\neg\varphi] \rightarrow \bot & \text{by 3 and 4, FOL reasoning} \\
6 & \neg C[\neg\varphi] & \text{by 5, FOL reasoning}
\end{array}
$$

□

Remark 4. When σ takes exactly one argument, $\Gamma \vdash \varphi$ implies $\Gamma \vdash \bar{\sigma}(\varphi)$ is known as the (NECESSITATION) rule in modal logic.

The next lemma is useful in establishing an equality.

LEMMA 2.10. $\Gamma \vdash \varphi_1 \leftrightarrow \varphi_2$ *implies* $\Gamma \vdash \varphi_1 = \varphi_2$.

PROOF.

$$
\begin{array}{l|ll}
1 & \varphi_1 \leftrightarrow \varphi_2 & \text{hypothesis} \\
2 & \neg\lceil\neg(\varphi_1 \leftrightarrow \varphi_2)\rceil & \text{by 1, Lemma 2.9} \\
3 & \varphi_1 = \varphi_2 & \text{by 2, definition of equality}
\end{array}
$$

□

LEMMA 2.11. *(EQUALITY INTRODUCTION) is derivable.*

PROOF.

$$
\begin{array}{l|ll}
1 & \varphi \leftrightarrow \varphi & \text{propositional tautology} \\
2 & \varphi = \varphi & \text{by 1, Lemma 2.10}
\end{array}
$$

□

LEMMA 2.12. *(MEMBERSHIP INTRODUCTION) is derivable.*

PROOF.

$$
\begin{array}{r|ll}
1 & \varphi & \text{hypothesis} \\
2 & \varphi \rightarrow (x \rightarrow \varphi) & (\text{Proposition}_1) \\
3 & x \rightarrow \varphi & \text{by 1 and 2, (Modus Ponens)} \\
4 & x \rightarrow x & \text{propositional tautology} \\
5 & x \rightarrow x \wedge \varphi & \text{by 3 and 4, FOL reasoning} \\
6 & \lceil x \rceil \rightarrow \lceil x \wedge \varphi \rceil & \text{by 5, (Framing)} \\
7 & \lceil x \rceil & \text{definedness axiom} \\
8 & \lceil x \wedge \varphi \rceil & \text{by 6 and 7, (Modus Ponens)} \\
9 & x \in \varphi & \text{by 8, definition of membership} \\
10 & \forall x.(x \in \varphi) & \text{by 9, (Universal Generalization)}
\end{array}
$$

□

Lemma 2.13. *(Membership Elimination) is derivable.*

Proof.

$$
\begin{array}{r|ll}
1 & \forall x.(x \in \varphi) & \text{hypothesis} \\
2 & (\forall x.(x \in \varphi)) \rightarrow x \in \varphi & (\text{Variable Substitution}) \\
3 & x \in \varphi & \text{by 1 and 2, (Modus Ponens)} \\
4 & \lceil x \wedge \varphi \rceil & \text{by 3, definition of membership} \\
5 & \neg(\lceil x \wedge \varphi \rceil \wedge (x \wedge \neg\varphi)) & (\text{Singleton Variable}) \\
6 & \lceil x \wedge \varphi \rceil \rightarrow (x \rightarrow \varphi) & \text{by 5, FOL reasoning} \\
7 & x \rightarrow \varphi & \text{by 4 and 6, (Modus Ponens)} \\
8 & \forall x.(x \rightarrow \varphi) & \text{by 7, (Universal Generalization)} \\
9 & (\exists x.x) \rightarrow \varphi & \text{by 8, FOL reasoning} \\
10 & \exists x.x & (\text{Existence}) \\
11 & \varphi & \text{by 10 and 9, (Modus Ponens)}
\end{array}
$$

□

Lemma 2.14. *(Membership Variable) is derivable.*

Proof. By Lemma 2.10, it suffices to show that both $(x \in y) \rightarrow (x = y)$ and $(x = y) \rightarrow (x \in y)$ are derivable. Let us first prove $(x = y) \rightarrow (x \in y)$ is derivable.

$$
\begin{array}{r|ll}
1 & \lceil x \rceil & \text{definedness axiom} \\
2 & \lceil x \rceil \vee \lceil y \rceil & \text{by 1, FOL reasoning} \\
3 & \lceil x \vee y \rceil & \text{by 2, Proposition 2.5} \\
4 & \lceil \neg(x \leftrightarrow y) \vee (x \wedge y) \rceil & \text{by 3, FOL reasoning} \\
5 & \lceil \neg(x \leftrightarrow y) \rceil \vee \lceil x \wedge y \rceil & \text{by 4, Proposition 2.5} \\
6 & \neg\lceil \neg(x \leftrightarrow y) \rceil \rightarrow \lceil x \wedge y \rceil & \text{by 5, FOL reasoning} \\
7 & (x = y) \rightarrow (x \in y) & \text{by 6, definition of equality and membership}
\end{array}
$$

Let us now prove $(x \in y) \rightarrow (x = y)$ is derivable.

$$
\begin{array}{rll}
1 & \neg(\lceil x \wedge y\rceil \wedge \lceil x \wedge \neg y\rceil) & \text{by (Singleton Variable)}\\
2 & \neg(\lceil x \wedge y\rceil \wedge \lceil \neg x \wedge y\rceil) & \text{by (Singleton Variable)}\\
3 & \lceil x \wedge y\rceil \rightarrow \neg\lceil x \wedge \neg y\rceil & \text{by 1, FOL reasoning}\\
4 & \lceil x \wedge y\rceil \rightarrow \neg\lceil \neg x \wedge y\rceil & \text{by 2, FOL reasoning}\\
5 & \lceil x \wedge y\rceil \rightarrow \neg\lceil x \wedge \neg y\rceil \wedge \neg\lceil \neg x \wedge y\rceil & \text{by 3 and 4, FOL reasoning}\\
6 & \lceil x \wedge y\rceil \rightarrow \neg(\lceil x \wedge \neg y\rceil \vee \lceil \neg x \wedge y\rceil) & \text{by 5, FOL reasoning}\\
7 & \lceil x \wedge y\rceil \rightarrow \neg\lceil (x \wedge \neg y) \vee (\neg x \wedge y)\rceil & \text{by 6, Proposition 2.5}\\
8 & \lceil x \wedge y\rceil \rightarrow \neg\lceil \neg(x \leftrightarrow y)\rceil & \text{by 7, FOL reasoning}\\
9 & (x \in y) \rightarrow (x = y) & \text{by 8, definition of equality and membership}
\end{array}
$$

$\square$

LEMMA 2.15. *(MEMBERSHIP$_\neg$) is derivable.*

PROOF. We first prove $(x \in \neg\varphi) \rightarrow \neg(x \in \varphi)$ is derivable.

$$
\begin{array}{rll}
1 & \neg(\lceil x \wedge \varphi\rceil \wedge \lceil x \wedge \neg\varphi\rceil) & \text{by (Singleton Variable)}\\
2 & \lceil x \wedge \neg\varphi\rceil \rightarrow \neg\lceil x \wedge \varphi\rceil & \text{by 1, FOL reasoning}\\
3 & (x \in \neg\varphi) \rightarrow \neg(x \in \varphi) & \text{by 2, definition of membership}
\end{array}
$$

We then prove $\neg(x \in \varphi) \rightarrow (x \in \neg\varphi)$ is derivable.

$$
\begin{array}{rll}
1 & \lceil x\rceil & \text{definedness axiom}\\
2 & \lceil (x \wedge \varphi) \vee (x \wedge \neg\varphi)\rceil & \text{by 1, FOL reasoning}\\
3 & \lceil x \wedge \varphi\rceil \vee \lceil x \wedge \neg\varphi\rceil & \text{by 2, Proposition 2.5}\\
4 & \neg\lceil x \wedge \varphi\rceil \rightarrow \lceil x \wedge \neg\varphi\rceil & \text{by 3, FOL reasoning}\\
5 & \neg(x \in \varphi) \rightarrow (x \in \neg\varphi) & \text{by 4, definition of membership}
\end{array}
$$

$\square$

LEMMA 2.16. $(x \in (\varphi_1 \vee \varphi_2)) \leftrightarrow (x \in \varphi_1) \vee (x \in \varphi_2)$ *is derivable.*

PROOF. Use (PROPAGATION$_\vee$) and FOL reasoning. $\square$

LEMMA 2.17. *(MEMBERSHIP$_\wedge$) is derivable.*

PROOF. Use Lemma 2.15 and 2.16, and the fact that $\varphi_1 \wedge \varphi_2 \leftrightarrow \neg(\neg\varphi_1 \vee \neg\varphi_2)$ is derivable. $\square$

LEMMA 2.18. *(MEMBERSHIP$_\exists$) is derivable.*

PROOF. Use (PROPAGATION$_\exists$) and FOL reasoning. $\square$

The following is a useful lemma about the definedness symbols.

LEMMA 2.19. *For any symbol context C, $C[\varphi] \rightarrow \lceil\varphi\rceil$ is derivable.*

PROOF. Let x be a variable that does not occur free in φ and $C[\varphi]$ in the following proof.

1	$\lceil x \rceil$	definedness axiom
2	$\lceil x \rceil \vee \lceil \varphi \rceil$	by 1, FOL reasoning
3	$\lceil x \vee \varphi \rceil$	by 2, Proposition 2.5
4	$\lceil x \wedge \neg \varphi \vee \varphi \rceil$	by 3, FOL reasoning and Proposition 2.6
5	$\lceil x \wedge \neg \varphi \rceil \vee \lceil \varphi \rceil$	by 4, Proposition 2.5
6	$C[x \wedge \varphi] \rightarrow \neg \lceil x \wedge \neg \varphi \rceil$	by (Singleton Variable) and FOL reasoning
7	$\neg \lceil x \wedge \neg \varphi \rceil \rightarrow \lceil \varphi \rceil$	by 5, FOL reasoning
8	$C[x \wedge \varphi] \rightarrow \lceil \varphi \rceil$	by 6 and 7, FOL reasoning
9	$\forall x.(C[x \wedge \varphi] \rightarrow \lceil \varphi \rceil)$	by 8, (Universal Generalization)
10	$(\exists x.C[x \wedge \varphi]) \rightarrow \lceil \varphi \rceil$	by 9, FOL reasoning
11	$\varphi \rightarrow (\exists x.x) \wedge \varphi$	by (Existence) and FOL reasoning
12	$\varphi \rightarrow \exists x.(x \wedge \varphi)$	by 11, FOL reasoning
13	$C[\varphi] \rightarrow C[\exists x.(x \wedge \varphi)]$	by 12, Frame reasoning
14	$C[\exists x.(x \wedge \varphi)] \rightarrow \lceil \varphi \rceil$	by 10, Proposition 2.5
15	$C[\varphi] \rightarrow \lceil \varphi \rceil$	by 13 and 14, FOL reasoning

$\square$

COROLLARY 2.20. *For any symbol σ, $C_\sigma[\varphi] \rightarrow \lceil \varphi \rceil$ and $\lfloor \varphi \rfloor \rightarrow \neg C_\sigma[\neg \varphi]$ is derivable. In particular, $\varphi \rightarrow \lceil \varphi \rceil$ and $\lfloor \varphi \rfloor \rightarrow \varphi$ is derivable.*

Definedness symbols are important. Corollary 2.20 suggests that definedness symbols are the "maximal" among all symbols. In fact, this observation is endorsed by its semantics. In any model $\mathcal{M} = (M, \_\mathcal{M})$, the definedness symbol $\lceil \_ \rceil$ has to be interpreted to the constant function $\lceil \_ \rceil_\mathcal{M} \colon M \rightarrow 2^M$ where $\lceil a \rceil = M$ for every $a \in M$.

Definedness symbols also give us a nice form of deduction theorem as follows.

THEOREM 2.21 (DEDUCTION THEOREM). *For any set Γ of patterns and any patterns φ and ψ, if $\Gamma \cup \{\psi\} \vdash \varphi$ and the proof does not use (Universal Generalization) on variables that occur free in ψ, then $\Gamma \vdash \lfloor \psi \rfloor \rightarrow \varphi$. In particular, if ψ is closed, then $\Gamma \cup \{\psi\} \vdash \varphi$ implies $\Gamma \vdash \lfloor \psi \rfloor \rightarrow \varphi$.*

The verbose condition about (Universal Generalization) is known to be necessary for the deduction theorem in FOL, too [?]. We will use deduction theorem to prove that (Equality Elimination) is derivable, and in there the premise pattern ψ might not be a close pattern. Therefore, we need this more general deduction theorem as stated above. The case when ψ is actually closed is a special case.

Readers familiar with modal logics should not find the conclusion $\Gamma \vdash \lfloor \psi \rfloor \rightarrow \varphi$ unusual. For example in [?], the author gave a similar form of the deduction theorem for finite-trace linear temporal logic.

PROOF. The proof is based on the length of the derivation $\Gamma \cup \{\psi\} \vdash \varphi$. If the length is one, then the derivation consists of just φ itself. Therefore, either φ is an axiom of the proof system, or $\varphi \in \Gamma$, or φ is ψ. In either case, it is obvious that $\Gamma \vdash \lfloor \psi \rfloor \rightarrow \varphi$ (noticing Corollary 2.20 for the case φ is ψ).

Now assume the conclusion holds for every pattern whose derivation is less than or equal to n steps for some $n \geq 1$. Assume φ has a derivation $\Gamma \cup \{\psi\} \vdash \varphi$ of length $n + 1$:

$$\varphi_1, \ldots, \varphi_n, \varphi.$$

If φ is an axiom, or $\varphi \in \Gamma$, or φ is ψ, for the same reason as above, $\Gamma \vdash \lfloor \psi \rfloor \rightarrow \varphi$. If φ is derived by applying (Modus Ponens) on φ_i and φ_j for some $1 \leq i, j \leq n$ such that φ_j has the form $\varphi_i \rightarrow \varphi$, by induction hypothesis, $\Gamma \vdash \lfloor \psi \rfloor \rightarrow \varphi_i$ and $\Gamma \vdash \lfloor \psi \rfloor \rightarrow (\varphi_i \rightarrow \varphi)$. By FOL reasoning, $\Gamma \vdash \lfloor \psi \rfloor \rightarrow \varphi$. If φ is derived by applying (Universal Generalization) on φ_i for some $1 \leq i \leq n$, then φ must have

the form $\forall x.\varphi_i$ where x does not occur free in ψ. By induction hypothesis, $\Gamma \vdash \lfloor \psi \rfloor \to \varphi_i$. By FOL reasoning, $\Gamma \vdash \lfloor \psi \rfloor \to \forall x.\varphi_i$.

If φ is derived by applying (FRAMING) on φ_i for some $1 \le i \le n$, then φ_i must have the form $\varphi'_i \to \varphi''_i$, and φ must have the form $C_\sigma[\varphi'_i] \to C_\sigma[\varphi''_i]$ for some symbol σ. By induction hypothesis, $\Gamma \vdash \lfloor \psi \rfloor \to (\varphi'_i \to \varphi''_i)$. We now prove $\lfloor \psi \rfloor \to (C_\sigma[\varphi'_i] \to C_\sigma[\varphi''_i])$ is derivable.

1	$\lfloor \psi \rfloor \to (\varphi'_i \to \varphi''_i)$	hypothesis
2	$\varphi'_i \to \varphi''_i \vee \lceil \neg \psi \rceil$	by 1, FOL reasoning
3	$C_\sigma[\lceil \neg \psi \rceil] \to \lceil \neg \psi \rceil$	Corollary 2.20
4	$C_\sigma[\varphi'_i] \to C_\sigma[\varphi''_i \vee \lceil \neg \psi \rceil]$	by 2, Frame reasoning
5	$C_\sigma[\varphi'_i] \to C_\sigma[\varphi''_i] \vee C_\sigma[\lceil \neg \psi \rceil]$	by 4, Proposition 2.5
6	$C_\sigma[\varphi''_i] \vee C_\sigma[\lceil \neg \psi \rceil] \to C_\sigma[\varphi''_i] \vee \lceil \neg \psi \rceil$	by 3, FOL reasoning
7	$C_\sigma[\varphi'_i] \to C_\sigma[\varphi''_i] \vee \lceil \neg \psi \rceil$	by 5 and 6, FOL reasoning
8	$\lfloor \psi \rfloor \to (C_\sigma[\varphi'_i] \to C_\sigma[\varphi''_i])$	by 7, FOL reasoning

$\square$

The main difference between the deduction theorem in matching logic and the one in FOL is that when we move a pattern ψ from the hypothesis set Γ to the right of the derivation, we should use $\lfloor \psi \rfloor$ instead of just ψ. In fact, $\Gamma \cup \{\psi\} \vdash \varphi$ does not always imply $\Gamma \vdash \psi \to \varphi$. Consider the following simple example. By Lemma 2.9, it is easy to show that for every pattern ψ, $\{\psi\} \vdash \lfloor \psi \rfloor$. However, $\psi \to \lfloor \psi \rfloor$ is not a valid pattern. According to the soundness of our proof system (see Section 3.1), $\psi \to \lfloor \psi \rfloor$ must not be provable.

LEMMA 2.22. *(EQUALITY ELIMINATION) is derivable.*

PROOF. Recall the definition of equality $(\varphi_1 = \varphi_2) \equiv \lfloor \varphi_1 \leftrightarrow \varphi_2 \rfloor$. Theorem 2.21 together with Proposition 2.6 give us a nice way to deal with equality premises. To show $(\varphi_1 = \varphi_2) \to (\psi[\varphi_1/x] \to \psi[\varphi_2/x])$ is derivable, it suffices to show $\{\varphi_1 \leftrightarrow \varphi_2\} \vdash \psi[\varphi_1/x] \to \psi[\varphi_2/x]$ and the proof does not use (UNIVERSAL GENERALIZATION) on any variable that occurs free in φ_1 or φ_2. Fortunately, the proof given by Proposition 2.6 does not use (UNIVERSAL GENERALIZATION) at all, and here ends the proof. $\square$

LEMMA 2.23. *(FUNCTIONAL SUBSTITUTION) is derivable.*

PROOF. Let z be a fresh variable that does not occur free in φ and φ', and is distinct from x. Notice the side condition that y does not occur free in φ'.

1	$\varphi' = z \leftrightarrow z = \varphi'$	definition of equality, Proposition 2.6, and FOL reasoning
2	$z = \varphi' \to (\varphi[z/x] \to \varphi[\varphi'/x])$	(EQUALITY ELIMINATION) and Lemma 2.22
3	$(\forall x.\varphi) \to \varphi[z/x]$	(VARIABLE SUBSTITUTION)
4	$\varphi' = z \to ((\forall x.\varphi) \to \varphi[z/x])$	by 3, FOL reasoning
5	$\varphi' = z \to (\varphi[z/x] \to \varphi[\varphi'/x])$	by 2 and 1, FOL reasoning
6	$\varphi' = z \to ((\forall x.\varphi) \to \varphi[\varphi'/x])$	by 4 and 5, FOL reasoning
7	$\forall z.(\varphi' = z \to ((\forall x.\varphi) \to \varphi[\varphi'/x]))$	by 6, (UNIVERSAL GENERALIZATION)
8	$(\exists z.\varphi' = z) \to ((\forall x.\varphi) \to \varphi[\varphi'/x])$	by 7, FOL reasoning
9	$(\forall x.\varphi) \wedge (\exists z.\varphi' = z) \to \varphi[\varphi'/x]$	by 8, FOL reasoning
10	$(\forall x.\varphi) \wedge (\exists y.\varphi' = y) \to \varphi[\varphi'/x]$	by 9, FOL reasoning

$\square$

The rest of the section is devoted to show that (Membership Symbol) is derivable. We first establish some lemmas.

LEMMA 2.24. $C_\sigma[\varphi_1 \wedge (x \in \varphi_2)] = C_\sigma[\varphi_1] \wedge (x \in \varphi_2)$ is derivable.

PROOF. We first derive $C_\sigma[\varphi_1 \wedge (x \in \varphi_2)] \to C_\sigma[\varphi_1] \wedge (x \in \varphi_2)$. By FOL reasoning, it suffices to show $C_\sigma[\varphi_1 \wedge (x \in \varphi_2)] \to C_\sigma[\varphi_1]$ and $C_\sigma[\varphi_1 \wedge (x \in \varphi_2)] \to (x \in \varphi_2)$ are both derivable. The first follows immediately by (Framing) and FOL reasoning. The second can be derived in the following way.

1	$\lceil x \rceil$	definedness axiom
2	$\lceil (x \wedge \neg\varphi_2) \vee (x \wedge \varphi_2) \rceil$	by 1, FOL reasoning and Proposition 2.6
3	$\lceil x \wedge \neg\varphi_2 \rceil \vee \lceil x \wedge \varphi_2 \rceil$	by 2, (Propagation$_\vee$) and FOL reasoning
4	$\neg\lceil x \wedge \neg\varphi_2 \rceil \to \lceil x \wedge \varphi_2 \rceil$	by 3, FOL reasoning
5	$C_\sigma[\lceil x \wedge \varphi_2 \rceil] \to \neg\lceil x \wedge \neg\varphi_2 \rceil$	by (Singleton Variable) and FOL reasoning
6	$C_\sigma[\lceil x \wedge \varphi_2 \rceil] \to \lceil x \wedge \varphi_2 \rceil$	by 5 and 4, FOL reasoning
7	$C_\sigma[\varphi_1 \wedge \lceil x \wedge \varphi_2 \rceil] \to C_\sigma[\lceil x \wedge \varphi_2 \rceil]$	by (Framing) and FOL reasoning
8	$C_\sigma[\varphi_1 \wedge \lceil x \wedge \varphi_2 \rceil] \to \lceil x \wedge \varphi_2 \rceil$	definedness axiom
9	$C_\sigma[\varphi_1 \wedge (x \in \varphi_2)] \to (x \in \varphi_2)$	by 8, definition of membership

We now derive $C_\sigma[\varphi_1] \wedge (x \in \varphi_2) \to C_\sigma[\varphi_1 \wedge (x \in \varphi_2)]$.

□

LEMMA 2.25. $\exists y.((x = y) \wedge \varphi) = \varphi[x/y]$ is derivable, if x and y are distinct variables.

PROOF. The proof is by induction on the structural of φ and using Lemma 2.24. □

LEMMA 2.26. $\varphi = \exists y.(\lceil y \wedge \varphi \rceil \wedge y)$ is derivable, if y does not occur free in φ.

PROOF. We first derive $\exists y.(\lceil y \wedge \varphi \rceil \wedge y) \to \varphi$.

1	$\neg(\lceil y \wedge \varphi \rceil \wedge (y \wedge \neg\varphi))$	(Singleton Variable)
2	$\lceil y \wedge \varphi \rceil \wedge y \to \varphi$	by 1, FOL reasoning
3	$\forall y.(\lceil y \wedge \varphi \rceil \wedge y \to \varphi)$	by 2, (Universal Generalization)
4	$\exists y.(\lceil y \wedge \varphi \rceil \wedge y) \to \varphi$	by 3, FOL reasoning

We then derive $\varphi \to \exists y.(\lceil y \wedge \varphi \rceil \wedge y)$. Let x be a fresh variable distinct from y.

1	$x \in \varphi \to x \in \varphi$	propositional tautology
2	$x \in \varphi \to \lceil x \wedge \varphi \rceil$	by 1, definition of membership
3	$x \in \varphi \to \lceil x \wedge \lceil x \wedge \varphi \rceil \rceil$	by 2, Lemma 2.24 and FOL reasoning
4	$x \in \varphi \to x \in \lceil x \wedge \varphi \rceil$	by 3, definition of membership
5	$x \in \varphi \to \exists y.(x = y \wedge x \in \lceil y \wedge \varphi \rceil)$	by 4, Lemma 2.25 and FOL reasoning
6	$x \in \varphi \to \exists y.(x \in y \wedge x \in \lceil y \wedge \varphi \rceil)$	by 5, (Membership Variable) and FOL reasoning
7	$x \in \varphi \to \exists y.(x \in (y \wedge \lceil y \wedge \varphi \rceil))$	by 6, (Membership$_\wedge$) and FOL reasoning
8	$x \in \varphi \to x \in \exists y.(y \wedge \lceil y \wedge \varphi \rceil)$	by 7, (Membership$_\exists$) and FOL reasoning
9	$x \in (\varphi \to \exists y.(y \wedge \lceil y \wedge \varphi \rceil))$	by 8, (Membership$_\neg$), (Membership$_\wedge$), and FOL reasoning
10	$\forall x.(x \in (\varphi \to \exists y.(y \wedge \lceil y \wedge \varphi \rceil)))$	by 9, (Universal Generalization)
11	$\varphi \to \exists y.(y \wedge \lceil y \wedge \varphi \rceil)$	by 10, (Membership)

□

LEMMA 2.27. (Membership Symbol) is derivable.

PROOF. We first derive $x \in C_\sigma[\varphi] \rightarrow \exists y.(y \in \varphi \wedge x \in C_\sigma[y])$. Let $\Psi \equiv \exists y.(y \in \varphi \wedge x \in C_\sigma[y])$.

$$
\begin{array}{ll}
1 & \exists y.(y \in \varphi \wedge x \in C_\sigma[y]) \rightarrow \Psi \qquad\qquad \text{propositional tautology} \\
2 & \exists y.(\lceil y \wedge \varphi \rceil \wedge x \in C_\sigma[y]) \rightarrow \Psi \qquad\quad \text{by 1, definition of membership} \\
3 & \exists y.(\lceil x \wedge \lceil y \wedge \varphi \rceil \rceil \wedge x \in C_\sigma[y]) \rightarrow \Psi \quad \text{by 2, Lemma 2.24 and FOL reasoning} \\
4 & \exists y.(x \in \lceil y \wedge \varphi \rceil \wedge x \in C_\sigma[y]) \rightarrow \Psi \quad \text{by 3, definition of membership} \\
5 & \exists y.(x \in (\lceil y \wedge \varphi \rceil \wedge C_\sigma[y])) \rightarrow \Psi \quad \text{by 4, (MEMBERSHIP$_\wedge$) and FOL reasoning} \\
6 & x \in \exists y.(\lceil y \wedge \varphi \rceil \wedge C_\sigma[y]) \rightarrow \Psi \quad \text{by 5, (MEMBERSHIP$_\exists$) and FOL reasoning} \\
7 & x \in \exists y.C_\sigma[\lceil y \wedge \varphi \rceil \wedge y] \rightarrow \Psi \quad \text{by 6, Lemma 2.24 and FOL reasoning} \\
8 & x \in C_\sigma[\exists y.\lceil y \wedge \varphi \rceil \wedge y] \rightarrow \Psi \quad \text{by 7, (PROPAGATION$_\exists$) and FOL reasoning} \\
9 & x \in C_\sigma[\varphi] \rightarrow \Psi \quad \text{by 8, Lemma 2.26 and FOL reasoning}
\end{array}
$$

□

We then derive $\exists y.(y \in \varphi \wedge x \in C[y]) \rightarrow x \in C[\varphi]$. In fact, we just need to apply the same derivation as above on $\Psi \rightarrow \exists y.(y \in \varphi \wedge x \in C[y])$.

It is known that the proof system in Figure 2 is complete. And we just showed that all its proof rules are derivable using the proof system in Figure 1 plus the definedness axioms. This leads us to the following completeness result.

PROPOSITION 2.28 (COMPLETENESS WITH DEFINEDNESS SYMBOLS). *Let Σ be a signature with definedness symbols. Let Γ be a set of Σ-patterns containing all instances of the definedness axioms. For any pattern φ such that $\Gamma \vDash \varphi$, we have $\Gamma \vdash \varphi$.*

PROOF. Since $\Gamma \vDash \varphi$, by completeness of the proof system in Figure 2, there exists a formal proof $\varphi_1, \ldots, \varphi_n$ where $n \geq 1$ and $\varphi_n \equiv \varphi$, and for any $i \in \{1, \ldots, n\}$, either φ_i is an axiom in Figure 2, or it is a conclusion of applying a proof rule in Figure 2 on some φ_j's for some $j \in \{1, \ldots, i-1\}$. On the other hand, we have proved that all proof rules in Figure 2 are derivable using the proof system in Figure 1 plus the definedness axioms. This means that we can take the formal proof $\varphi_1, \ldots, \varphi_n$ and turn it into a formal proof of φ using the proof system in Figure 1, and thus prove that $\Gamma \vdash \varphi$. We leave the details of the proof to readers. □

3 SOUNDNESS AND COMPLETENESS OF THE PROOF SYSTEM

3.1 Soundness

THEOREM 3.1 (SOUNDNESS). *For any pattern set Γ and pattern φ, $\Gamma \vdash \varphi$ implies $\Gamma \vDash \varphi$.*

The proof of Theorem 3.1 is by induction on the length of the Γ-proof of φ. Let us first prove a lemma that establishes the soundness of each rule.

LEMMA 3.2. *The following hold:*

(1) $\vDash \varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_1)$
(2) $\vDash \varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3) \rightarrow (\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_3)$
(3) $\vDash (\neg\varphi_1 \rightarrow \neg\varphi_2) \rightarrow (\varphi_2 \rightarrow \varphi_1)$
(4) $\mathcal{M}, \rho \vDash \varphi_1$ *and* $\mathcal{M}, \rho \vDash \varphi_1 \rightarrow \varphi_2$ *imply* $\mathcal{M}, \rho \vDash \varphi_2$
(5) $\vDash \forall x.\varphi \rightarrow \varphi[y/x]$
(6) $\vDash \forall x.(\varphi_1 \rightarrow \varphi_2) \rightarrow \varphi_1 \rightarrow \forall x.\varphi_2$ *if* $x \notin FV(\varphi_1)$
(7) $\mathcal{M} \vDash \varphi$ *implies* $\mathcal{M} \vDash \forall x.\varphi$
(8) $\vDash C_\sigma[\bot] \rightarrow \bot$
(9) $\vDash C_\sigma[\varphi_1 \vee \varphi_2] \rightarrow C_\sigma[\varphi_1] \vee C_\sigma[\varphi_2]$
(10) $\vDash C_\sigma[\exists x.\varphi] \rightarrow \exists x.C_\sigma[\varphi]$ *if* $x \notin FV(C_\sigma[\exists x.\varphi])$
(11) $\mathcal{M}, \rho \vDash \varphi_1 \rightarrow \varphi_2$ *implies* $\mathcal{M}, \rho \vDash C_\sigma[\varphi_1] \rightarrow C_\sigma[\varphi_2]$

(12) $\vDash \exists x.x$

(13) $\vDash \neg(C_1[x \wedge \varphi] \wedge C_2[x \wedge \neg\varphi])$

where $\varphi, \varphi_1, \varphi_2, \varphi_3$ are patterns, x, y are variables, $\sigma \in \Sigma$, C_σ is a symbol context, $\mathcal{M}$ is a model, and ρ is a valuation.

PROOF. Some of the propositions are proved in [?]. To make this paper self-contained, we present the proof of all propositions. Let $\mathcal{M}$ by a model and ρ be a valuation.

(1) $\bar\rho(\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_1)) = \bar\rho(\neg\varphi_1) \cup \bar\rho(\varphi_2 \rightarrow \varphi_1) = (M \setminus \bar\rho(\varphi_1)) \cup \bar\rho(\neg\varphi_2) \cup \bar\rho(\varphi_1) = M$.

(2) $\bar\rho(\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3) \rightarrow (\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_3)) = \bar\rho(\neg(\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3))) \cup \bar\rho((\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_3)) = (\bar\rho(\varphi_1) \cap (\bar\rho(\neg(\varphi_2 \rightarrow \varphi_3)))) \cup \bar\rho(\neg(\varphi_1 \rightarrow \varphi_2)) \cup \bar\rho(\varphi_1 \rightarrow \varphi_3) = (\bar\rho(\varphi_1) \cap \bar\rho(\varphi_2) \cap (M \setminus \bar\rho(\varphi_3))) \cup (\bar\rho(\varphi_1) \cap (M \setminus \bar\rho(\varphi_2))) \cup (M \setminus \bar\rho(\varphi_1)) \cup \bar\rho(\varphi_3) = M$.

(3) $\bar\rho(\neg\varphi_1 \rightarrow \neg\varphi_2) \rightarrow (\varphi_2 \rightarrow \varphi_1) = \bar\rho(\neg(\neg\varphi_1 \rightarrow \varphi_2)) \cup \bar\rho(\varphi_2 \rightarrow \varphi_1) = (M \setminus \bar\rho(\neg\varphi_1 \rightarrow \neg\varphi_2)) \cup (M \setminus \bar\rho(\varphi_2)) \cup \bar\rho(\varphi_1) = (M \setminus (\bar\rho(\neg\neg\varphi_1) \cup \bar\rho(\neg\varphi_2))) \cup (M \setminus \bar\rho(\varphi_2)) \cup \bar\rho(\varphi_1) = (M \setminus ((M \setminus M \setminus \bar\rho(\varphi_1)) \cup (M \setminus \bar\rho(\varphi_2)))) \cup (M \setminus \bar\rho(\varphi_2)) \cup \bar\rho(\varphi_1) = (M \setminus (\bar\rho(\varphi_1) \cup (M \setminus \bar\rho(\varphi_2)))) \cup (M \setminus \bar\rho(\varphi_2)) \cup \bar\rho(\varphi_1) = M$.

(4) $\mathcal{M}, \rho \vDash \varphi_1 \rightarrow \varphi_2$, so $\bar\rho(\varphi_1 \rightarrow \varphi_2) = (M \setminus \bar\rho(\varphi_1)) \cup \bar\rho(\varphi_2) = M$, and thus $\bar\rho(\varphi_1) \subseteq \bar\rho(\varphi_2)$. Because $\mathcal{M}, \rho \vDash \varphi_1$, $\bar\rho(\varphi_1) = M$, and thus $\bar\rho(\varphi_2) = M$.

(5) $\bar\rho(\forall x.\varphi \rightarrow \varphi[y/x]) = (M \setminus \bar\rho(\forall x.\varphi)) \cup \bar\rho(\varphi[y/x]) = (M \setminus \bigcap_{\rho'}(\overline{\rho'}(\varphi))) \cup \overline{\rho'_y}(\varphi)$ where $\rho'_y = \rho[\rho(y)/x]$ and $\rho' \overset{x}{\sim} \rho$. Notice that $\rho_y \overset{x}{\sim} \rho$. Thus $\bigcap_{\rho'}(\overline{\rho'}(\varphi)) \subseteq \overline{\rho'_y}(\varphi)$, and $(M \setminus \bigcap_{\rho'}(\overline{\rho'}(\varphi))) \cup \overline{\rho'_y}(\varphi) = M$.

(6) If suffices to show $\bar\rho(\forall x.(\varphi_1 \rightarrow \varphi_2) \subseteq \bar\rho(\varphi_1 \rightarrow \forall x.\varphi_2)$. Notice that $\bar\rho(\forall x.(\varphi_1 \rightarrow \varphi_2)) = \bigcap_{\rho'} \overline{\rho'}((\varphi_1 \rightarrow \varphi)) = \bigcap_{\rho'}((M \setminus \overline{\rho'}(\varphi_1)) \cup \overline{\rho'}(\varphi_2))$ where $\rho' \overset{x}{\sim} \rho$. Since $x \notin FV(\varphi_1)$, $\overline{\rho'}(\varphi_1) = \bar\rho(\varphi_1)$, and thus $\bigcap_{\rho'}((M \setminus \overline{\rho'}(\varphi_1)) \cup \overline{\rho'}(\varphi_2)) = \bigcap_{\rho'}((M \setminus \bar\rho(\varphi_1)) \cup \overline{\rho'}(\varphi_2)) = (M \setminus \bar\rho(\varphi_1)) \cup \bigcap_{\rho'}(\overline{\rho'}(\varphi_2)) = \bar\rho(\varphi_1 \rightarrow \forall x.\varphi_2)$.

(7) $\bar\rho(\forall x.\varphi) = \bigcap_{\rho'} \overline{\rho'}(\varphi)$ where $\rho' \overset{x}{\sim} \rho$, so it suffices to show $\overline{\rho'}(\varphi) = M$ for any ρ'. Since $\vDash \varphi$, $\mathcal{M}, \rho' \vDash \varphi$, and thus $\overline{\rho'}(\varphi) = M$.

(8) $\bar\rho(C_\sigma[\bot] \rightarrow \bot) = M \setminus \bar\rho(C_\sigma[\bot])$, so it suffices to show $\bar\rho(C_\sigma[\bot]) = \emptyset$. In fact, $\bar\rho(\sigma(\ldots \bot \ldots)) = \sigma_{\mathcal{M}}(\ldots \bar\rho(\bot) \ldots) = \sigma_{\mathcal{M}}(\ldots \emptyset \ldots) = \emptyset$.

(9) It suffices to show $\bar\rho(C_\sigma(\varphi_1 \vee \varphi_2)) \subseteq \bar\rho(C_\sigma[\varphi_1] \vee C_\sigma[\varphi_2])$. In fact, $\bar\rho(C_\sigma(\varphi_1 \vee \varphi_2)) = \bar\rho(\sigma(\ldots (\bar\rho(\varphi_1) \cup \bar\rho(\varphi_2)) \ldots)) = \sigma_{\mathcal{M}}(\ldots \bar\rho(\varphi_1) \ldots) \cup \sigma_{\mathcal{M}}(\ldots \bar\rho(\varphi_2) \ldots) = \bar\rho(C_\sigma[\varphi_1]) \cup \bar\rho(C_\sigma[\varphi_2]) = \bar\rho(C_\sigma[\varphi_1] \vee C_\sigma[\varphi_2])$.

(10) It suffices to show $\bar\rho(C_\sigma[\exists x.\varphi]) \subseteq \bar\rho(\exists x.C_\sigma[\varphi])$. In fact, $\bar\rho(C_\sigma[\exists x.\varphi]) = \bar\rho(\sigma(\ldots \exists x.\varphi \ldots)) = \sigma_{\mathcal{M}}(\ldots \bar\rho(\exists x.\varphi) \ldots) = \sigma_{\mathcal{M}}(\ldots \bigcup_{\rho'} \overline{\rho'}(\varphi) \ldots) = \bigcup_{\rho'} \sigma_{\mathcal{M}}(\ldots \overline{\rho'}(\varphi) \ldots) = \bar\rho(\exists x.C_\sigma[\varphi])$ where $\rho' \overset{x}{\sim} \rho$. Notice that we can move the big union $\bigcup_{\rho'}$ from the argument to the top without affecting other arguments because $x \notin FV(C_\sigma[\exists x.\varphi])$.

(11) It suffices to show $\bar\rho(C_\sigma[\varphi_1]) \subseteq \bar\rho(C_\sigma[\varphi_2])$. Notice that $\vDash \varphi_1 \rightarrow \varphi_2$, so $\bar\rho(\varphi_1) \subseteq \bar\rho(\varphi_2)$, and thus, $\bar\rho(C_\sigma[\varphi_1]) = \sigma_{\mathcal{M}}(\ldots \bar\rho(\varphi_1) \ldots) \subseteq \sigma_{\mathcal{M}}(\ldots \bar\rho(\varphi_2) \ldots) = \bar\rho(C_\sigma[\varphi_2])$.

(12) $\bar\rho(\exists x.x) = \bigcup_{\rho'}(\overline{\rho'}(x)) = \bigcup_{\rho'}\{\rho'(x)\}$ where $\rho' \overset{x}{\sim} \rho$. Notice $\bigcup_{\rho'}\{\rho'(x)\} = \bigcup_{a \in M}\{a\} = M$. $\quad\square$

(13) It suffices to show that either $\bar\rho(C_1[x \wedge \varphi])$ or $\bar\rho(C_2[x \wedge \neg\varphi])$ is the empty set. For every symbol context C, use the same technique in (8) and structural induction, we can prove that if $\bar\rho(\psi) = \emptyset$ then $\bar\rho(C[\psi]) = \emptyset$. Therefore, we just need to prove that either $\bar\rho(x \wedge \varphi)$ or $\bar\rho(x \wedge \neg\varphi)$ is the empty set. If $\rho(x) \notin \bar\rho(\varphi)$, then the former is empty. Otherwise, the latter is empty.

Now we are ready to prove Theorem 3.1.

THEOREM 3.1 (SOUNDNESS). *For any pattern set Γ and pattern φ, $\Gamma \vdash \varphi$ implies $\Gamma \vDash \varphi$.*

PROOF. Conduct induction on the length of proof of φ. If the length is 1, then φ is either an axiom or a member of Γ. If φ is an axiom, then $\vDash \varphi$ by Lemma 3.2. It $\varphi \in \Gamma$, then $\Gamma \vDash \varphi$ by definition.

Assume every pattern which has a proof less than n steps satisfies the conclusion, and φ has a proof of $n + 1$ steps:

$$\varphi_1, \ldots, \varphi_n, \varphi_{n+1} \qquad \text{with } \varphi_{n+1} \equiv \varphi$$

By induction hypothesis, $\Gamma \vDash \varphi_1, \ldots, \Gamma \vDash \varphi_n$. If φ is an axiom or a member of Γ, then $\Gamma \vDash \varphi$ for the same reason as showed before. If φ is the conclusion of rules (Modus Ponens), (Universal Generalization), or (Framing), then $\Gamma \vDash \varphi$ by Lemma 3.2, Case (4), (7), and (11), respectively. □

3.2 Completeness when the Hypothesis Set is Empty

In this subsection and the next one, we will discuss complete deduction of matching logic. The completeness theorem that we will eventually prove says that for any hypothesis set H and any pattern φ such that $H \vDash \varphi$, under some mild assumption, there is a proof of φ from H, i.e., $H \vdash \varphi$. We call the pattern set H the hypothesis set. We will prove the theorem in two steps. Firstly, we prove a preliminary completeness theorem called the local completeness theorem, where the hypothesis set H is the empty set. Then in the next subsection, we extend local completeness theorem to global completeness theorem, where the hypothesis set H can be any pattern set. We give credits to Blackburn's previous research about hybrid and polyadic modal logic; while this paper uses novel techniques to handle many-sortedness, and to combine the hybrid and polyadic cases.

We start by stating the local completeness theorem.

Theorem 3.3 (Local completeness). *For any pattern φ such that $\emptyset \vDash \varphi$, $\emptyset \vdash \varphi$.*

Definition 3.4 (Local Provability Relation). Let s be a sort, $H_s \subseteq \textsc{Pattern}_s$ be a pattern set, and φ_s be a pattern of sort s. We write $H_s \Vdash_s \varphi_s$, if there exists a finite subset $\Delta_s \subseteq_{fin} H_s$ such that $\emptyset \vdash_s \bigwedge \Delta_s \to \varphi_s$, where $\bigwedge \Delta_s$ is the conjunction of all patterns in Δ_s. When Δ_s is the empty set, $\bigwedge \Delta_s$ is $\top_s$. Let $H = \{H_s\}_{s \in S}$ be a family set of patterns. We write $H \Vdash_s \varphi_s$ if $H_s \Vdash_s \varphi_s$. We drop sort subscripts when there is no confusion.

The provability relation $\Vdash_s$ is called a local one because it is local in the sort s. In other words, given a hypothesis set $H = \{H_s\}_{s \in S}$ and a pattern φ_s, whether $H \Vdash_s \varphi_s$ depends only on H_s. This is different from the provability relation we give in Definition 2.2, where whether $H \vdash_s \varphi_s$ depends on the hypothesis sets of all sorts. As an example, let $\varphi, \varphi' \in \textsc{Pattern}_{s_1}$ be two patterns and $\sigma \in \Sigma_{s_1, s_2}$ be a symbol. Define the hypothesis set $H = \{H_{s_1}, H_{s_2}\}$ where $H_{s_1} = \{\varphi \to \varphi'\}$ and $H_{s_2} = \emptyset$. By frame reasoning, $H \vdash_{s_2} \sigma(\varphi) \to \sigma(\varphi')$, but $H \Vdash_{s_2} \sigma(\varphi) \to \sigma(\varphi')$ does not hold.

The following proposition shows that the local provability relation is a subrelation of the provability relation defined in Definition 2.2.

Proposition 3.5. *For any pattern set H and pattern φ of the same sort, $H \Vdash \varphi$ implies $H \vdash \varphi$.*

Proof. By definition and simple propositional reasoning. □

Definition 3.6 (Consistent Sets). Let Γ_s be a pattern set of sort s. We say Γ_s is *consistent*, if $\Gamma_s \nVdash \bot_s$. Γ_s is a maximal consistent set (MCS) if any strict extension of it is inconsistent. By abuse of language, we say $\Gamma = \{\Gamma_s\}_{s \in S}$ is consistent if every Γ_s is consistent, and Γ is an MCS if every Γ_s is an MCS.

Like the local provability relation, consistency is also a local property. Pattern set Γ_s is consistent (or an MCS) only depends on itself. A useful intuition about consistent sets is that they provide consistent "views" of patterns. Recall that patterns in matching logic match elements in domain. Intuitively speaking, a pattern set Γ_s is inconsistent if it contains patterns that cannot match common elements in any models and valuations. In other words, if Γ_s is consistent, then there exist a model $\mathcal{M}$ and a valuation ρ, and an element a in the model, such that all patterns in Γ_s match a, i.e., $a \in \bar{\rho}(\varphi)$ for all pattern $\varphi \in \Gamma_s$. If Γ_s is in addition an MCS, adding any pattern

$\psi \notin \Gamma_s$ will lead to inconsistency, and thus $a \notin \bar{\rho}(\psi_s)$. Therefore, we can think of the MCS Γ_s representing that particular element a, with all patterns in Γ_s matching it while patterns outside Γ_s not. This useful intuition motivates the definition of canonical models that consist MCSs as elements (see Definition 3.10), and the Truth Lemma that says "Matching = Membership in MCSs", connecting syntax and semantics, (see Lemma 3.19). They play an important role in proving the completeness result, including both local and global completeness theorems. The rest of this and the next subsections is all about making this intuition work.

PROPOSITION 3.7 (MCS PROPERTIES). *Given an MCS Γ and patterns $\varphi, \varphi_1, \varphi_2$ of the same sort s. The following propositions hold.*

(1) *$\varphi \in \Gamma$ if and only if $\Gamma \Vdash \varphi$; In particular, if $\vdash \varphi$ then $\varphi \in \Gamma$;*
(2) *$\neg\varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$;*
(3) *$\varphi_1 \wedge \varphi_2 \in \Gamma$ if and only if $\varphi_1 \in \Gamma$ and $\varphi_2 \in \Gamma$; In general, for any finite pattern set Δ, $\bigwedge \Delta \in \Gamma$ if and only if $\Delta \subseteq \Gamma$;*
(4) *$\varphi_1 \vee \varphi_2 \in \Gamma$ if and only if $\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma$; In general, for any finite pattern set Δ, $\bigvee \Delta \in \Gamma$ if and only if $\Delta \cap \Gamma \neq \emptyset$; As a convention, when $\Delta = \emptyset$, $\bigvee \Delta$ is $\bot$;*
(5) *$\varphi_1, \varphi_1 \rightarrow \varphi_2 \in \Gamma$ implies $\varphi_2 \in \Gamma$; In particular, if $\vdash \varphi_1 \rightarrow \varphi_2$, then $\varphi_1 \in \Gamma$ implies $\varphi_2 \in \Gamma$.*

PROOF. Standard propositional reasoning. □

Definition 3.8 (Witnessed MCSs). Let Γ be an MCS of sort s. Γ is a witnessed MCS, if for any pattern $\exists x.\varphi \in \Gamma$, there is a variable y such that $(\exists x.\varphi) \rightarrow \varphi[y/x] \in \Gamma$. By abuse use of language, we say the family set $\Gamma = \{\Gamma_s\}_{s \in S}$ is a witnessed MCS if every Γ_s is a witnessed MCS.

In the following, we show any consistent set Γ can be extended to a witnessed MCS Γ^+. The extension, however, requires an extension of the set of variables. To see why such an extension is needed, consider the following example. Let $\Sigma = (\text{VAR}, S, \Sigma)$ be a signature, $s \in S$ be a sort, and $\Gamma = \{\neg x \mid x \in \text{VAR}_s\}$ be a pattern set containing all variable negations. We leave it for the readers to show that Γ is consistent. Here, we claim the consistent set Γ cannot be extended to a witnessed MCS Γ^+ in the signature Σ. The proof is by contradiction. Assume Γ^+ exists. By Proposition 3.7 and (EXISTENCE), $\exists x.x \in \Gamma^+$. Because Γ^+ is a witnessed MCS, there is a variable y such that $(\exists x.x) \rightarrow y \in \Gamma^+$, and by Proposition 3.7, $y \in \Gamma^+$. On the other hand, $\neg y \in \Gamma \subseteq \Gamma^+$. This contradicts the consistency of Γ^+.

LEMMA 3.9 (EXTENSION LEMMA). *Let $\Sigma = (\text{VAR}, S, \Sigma)$ be a signature and Γ be a consistent set of sort $s \in S$. Extend the variable set VAR to VAR^+ with countably infinitely many new variables, and denoted the extended signature as $\Sigma^+ = (\text{VAR}^+, S, \Sigma)$. There exists a pattern set Γ^+ in the extended signature Σ such that $\Gamma \subseteq \Gamma^+$ and Γ^+ is a witnessed MCS.*

PROOF. We use PATTERN_s and PATTERN_s^+ denote the set of all patterns of sort s in the original and extended signatures, respectively. Enumerate all patterns $\varphi_1, \varphi_2, \dots \in \text{PATTERN}_s^+$. For every sort s, enumerate all variables $x_1{:}s, x_2{:}s, \dots$ in $\text{VAR}_s^+ \setminus \text{VAR}_s$. We will construct a non-decreasing sequence of pattern sets $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \cdots \subseteq \text{PATTERN}_s^+$, with $\Gamma_0 = \Gamma$. Notice that Γ_0 contains variables only in VAR. Eventually, we will let $\Gamma^+ = \bigcup_{i \geq 0} \Gamma_i$ and prove it has the intended properties.

For every $n \geq 1$, we define Γ_n as follows. If $\Gamma_{n-1} \cup \{\varphi_n\}$ is inconsistent, then $\Gamma_n = \Gamma_{n-1}$. Otherwise,

$$\Gamma_n = \begin{cases} \Gamma_{n-1} \cup \{\varphi_n\} & \text{if } \varphi_n \text{ is not of the form } \exists x{:}s'.\psi \\ \Gamma_{n-1} \cup \{\varphi_n\} \cup \{\psi[x_i{:}s'/x{:}s']\} & \text{if } \varphi_n \equiv \exists x{:}s'.\psi \text{ (note that } x{:}s' \text{ is in } \text{VAR}_{s'}^+) \\ & \text{and } x_i{:}s' \text{ is the first variable in } \text{VAR}_{s'}^+ \setminus \text{VAR}_{s'} \\ & \text{that does not occur free in } \Gamma_{n-1} \text{ and } \psi \end{cases}$$

Notice that in the second case, we can always pick a variable $\varkappa_i{:}s'$ that satisfies the conditions because by construction, $\Gamma_{n-1} \cup \{\varphi_n\}$ uses at most finitely many variables in $\text{Var}^+ \setminus \text{Var}$.

We show that Γ_n is consistent for every $n \geq 0$ by induction. The base case is to show Γ_0 is consistent in the extended signature. Assume it is not. Then there exists a finite subset $\Delta_0 \subseteq_{fin} \Gamma_0$ such that $\vdash \bigwedge \Delta_0 \to \bot$. The proof of $\bigwedge \Delta_0 \to \bot$ is a finite sequence of patterns in Pattern^+. We can replace every occurrence of the variable $y \in \text{Var}^+ \setminus \text{Var}$ (y can have any sort) with a variable $y \in \text{Var}$ that has the same sort as y and does not occur (no matter bound or free) in the proof. By induction on the length of the proof, the resulting sequence is also a proof of $\bigwedge \Delta_0 \to \bot$, and it consists of only patterns in Pattern. This contradicts the consistency of Γ_0 as a subset of Pattern_s, and this contradiction finishes our proof of the base case.

Now assume Γ_{n-1} is consistent for $n \geq 1$. We will show Γ_n is also consistent. If $\Gamma_{n-1} \cup \{\varphi_n\}$ is inconsistent or φ_n does not have the form $\exists x{:}s'.\psi$, Γ_n is consistent by construction. Assume $\Gamma_{n-1} \cup \{\varphi_n\}$ is consistent, $\varphi_n \equiv \exists x{:}s'.\psi$, but $\Gamma_n = \Gamma_{n-1} \cup \{\varphi_n\} \cup \{\psi[\varkappa_i{:}s'/x{:}s']\}$ is not consistent. Then there exists a finite subset $\Delta \subseteq_{fin} \Gamma_{n-1} \cup \{\varphi_n\}$ such that $\vdash \bigwedge \Delta \to \neg\psi[\varkappa_i{:}s'/x{:}s']$. By (Universal Generalization), $\vdash \forall \varkappa_i{:}s'.(\bigwedge \Delta \to \neg\psi[\varkappa_i{:}s'/x{:}s'])$. Notice that $\varkappa_i{:}s' \notin FV(\bigwedge \Delta)$ by construction, so by FOL reasoning $\vdash \bigwedge \Delta \to \neg\exists \varkappa_i{:}s'.(\psi[\varkappa_i{:}s'/x{:}s'])$. Since $\varkappa_i{:}s' \notin FV(\psi)$, by Definition 1.2, $\exists \varkappa_i{:}s'.(\psi[\varkappa_i{:}s'/x{:}s']) \equiv \exists x{:}s'.\psi \equiv \varphi_n$, and thus $\vdash \bigwedge \Delta \to \neg\varphi_n$. This contradicts the assumption that $\Gamma_{n-1} \cup \{\varphi_n\}$ is consistent.

Since Γ_n is consistent for any $n \geq 0$, $\Gamma^+ = \bigcup_n \Gamma_n$ is also consistent. This is because the derivation that shows inconsistency would use only finitely many patterns in Γ^+. In addition, we know Γ^+ is maximal and witnessed by construction. □

We will prove that for every witnessed MCS $\Gamma = \{\Gamma_s\}_{s \in S}$, there exists a model $\mathcal{M}$ and a valuation ρ such that for every $\varphi \in \Gamma_s$, $\bar{\rho}(\varphi) \neq \emptyset$. The next definition defines the canonical model which contains all witnessed MCSs as its elements. We will construct our intended model $\mathcal{M}$ as a submodel of the canonical model.

Definition 3.10 (Canonical Model). Given a signature $\bar{\Sigma} = (S, \Sigma)$. The canonical model $\mathcal{W} = (W, \_\mathcal{W})$ consists of

- a carrier set $W_s = \{\Gamma \mid \Gamma \text{ is a witnessed MCS of sort } s\}$ for every sort $s \in S$; Let $W = \{W_s\}_{s \in S}$;
- an interpretation $\sigma_\mathcal{W} \colon W_{s_1} \times \cdots \times W_{s_n} \to 2^{W_s}$ for every symbol $\sigma \in \Sigma_{s_1 \ldots s_n, s}$, defined as

$$\Gamma \in \sigma_\mathcal{W}(\Gamma_1, \ldots, \Gamma_n) \quad \text{if and only if} \quad \text{for any } \varphi_i \in \Gamma_i, 1 \leq i \leq n, \sigma(\varphi_1, \ldots, \varphi_n) \in \Gamma;$$

  In particular, the interpretation for a constant symbol $\sigma \in \Sigma_{\lambda, s}$ is $\sigma_\mathcal{W} = \{\Gamma \in W_s \mid \sigma \in \Gamma\}$.

The carrier set W is not empty, thanks to Lemma 3.9.

The canonical model has a nontrivial property stated as the next lemma. The proof of the lemma is difficult, so we leave it to the end of the subsection.

LEMMA 3.11 (EXISTENCE LEMMA). *Let $\bar{\Sigma} = (S, \Sigma)$ be a signature and Γ be a witnessed MCS of sort $s \in S$. Given a symbol $\sigma \in \Sigma_{s_1 \ldots s_n, s}$ and patterns $\varphi_1, \ldots, \varphi_n$ of appropriate sorts. If $\sigma(\varphi_1, \ldots, \varphi_n) \in \Gamma$, then there exist n witnessed MCSs $\Gamma_1, \ldots, \Gamma_n$ of appropriate sorts such that $\varphi_i \in \Gamma_i$ for every $1 \leq i \leq n$, and $\Gamma \in \sigma_\mathcal{W}(\Gamma_1, \ldots, \Gamma_n)$.*

Definition 3.12 (Generated Models). Let $\bar{\Sigma} = (S, \Sigma)$ be a signature and $\mathcal{W} = (W, \_\mathcal{W})$ be the canonical model. Given a witnessed MCS $\Gamma = \{\Gamma_s\}_{s \in S}$. Define $Y = \{Y_s\}_{s \in S}$ be the smallest sets such that $Y_s \subseteq W_s$ for every sort s, and the following inductive properties are satisfied:

- $\Gamma_s \in Y_s$ for every sort s;
- If $\Delta \in Y_s$ and there exist a symbol $\sigma \in \Sigma_{s_1 \ldots s_n, s}$ and witnessed MCSs $\Delta_1, \ldots, \Delta_n$ of appropriate sorts such that $\Delta \in \sigma_\mathcal{W}(\Delta_1, \ldots, \Delta_n)$, then $\Delta_1 \in Y_{s_1}, \ldots, \Delta_n \in Y_{s_n}$.

Let $\mathcal{Y} = (Y, \_\mathcal{Y})$ be the model generated from Γ, where

$$\sigma_\mathcal{Y}(\Delta_1, \ldots, \Delta_n) = Y_s \cap \sigma_\mathcal{W}(\Delta_1, \ldots, \Delta_n) \quad \text{for every } \sigma \in \Sigma_{s_1 \ldots s_n, s} \text{ and } \Delta_1 \in Y_{s_1}, \ldots, \Delta_n \in Y_{s_n}.$$

We give some intuition about the generated model $\mathcal{Y} = (Y, \_\mathcal{Y})$. The interpretation $\sigma_\mathcal{Y}$ is just the restriction of the interpretation $\sigma_\mathcal{M}$ on Y. The carrier set Y is defined inductively. Firstly, Y contains Γ. Given a set $\Delta \in Y$. If sets $\Delta_1, \ldots, \Delta_n$ are "generated" from Δ by a symbol σ, meaning that $\Delta \in \sigma_\mathcal{W}(\Delta_1, \ldots, \Delta_n)$, then they are also in Y. Of course, a set Δ is in Y maybe because it is generated from a set Δ' by a symbol σ', while Δ' is generated from a set Δ'' by a symbol σ'', and so on. This generating path keeps going and eventually ends at Γ in finite number of steps. By definition, every member of Y has at least one such generating path, which we formally define as follows.

Definition 3.13 (Generating Paths). Let $\Gamma = \{\Gamma_s\}_{s \in S}$ be a witnessed MCS and $\mathcal{Y}$ be the model generated from Γ. A *generating path* π is either the empty path ϵ, or a sequence of pairs $\langle (\sigma_1, p_1), \ldots, (\sigma_k, p_k) \rangle$ where $\sigma_1, \ldots, \sigma_k$ are symbols (not necessarily distinct) and $p_1, \ldots, p_k$ are natural numbers representing positions. The *generating path relation*, denoted as GP, is a binary relation between witnessed MCSs in Y and generating paths, defined as the smallest relation that satisfies the following conditions:

- $GP(\Gamma_s, \epsilon)$ holds for every sort s;
- If $GP(\Delta, \pi)$ holds for a set $\Delta \in Y_s$ and a generating path π, and there exist a symbol $\sigma \in \Sigma_{s_1 \ldots s_n, s}$ and witnessed MCSs $\Delta_1, \ldots, \Delta_n$ such that $\Delta \in \sigma_\mathcal{W}(\Delta_1, \ldots, \Delta_n)$, then $GP(\Delta_i, \langle \pi, (\sigma, i) \rangle)$ holds for every $1 \le i \le n$.

We say that Δ has a generating path π in the generated model if $GP(\Delta, \pi)$ holds. It is easy to see that every witnessed MCS in Y has at least one generating path, and if a witnessed MCS of sort s has the empty path ϵ as its generating path, it must be Γ_s itself.

Definition 3.14 (Symbol Contexts for Generating Paths). Given a generating path π. Define the symbol context C_π inductively as follows. If $\pi = \epsilon$, then C_π is the identity context $\square$. If $\pi = \langle \pi_0, (\sigma, i) \rangle$ where $\sigma \in \Sigma_{s_1 \ldots s_n, s}$ and $1 \le i \le n$, then $C_\pi = C_{\pi_0}[\sigma(\top_{s_1}, \ldots, \top_{s_{i-1}}, \square, \top_{s_{i+1}}, \ldots, \top_{s_n})]$.

A good intuition about Definition 3.14 is given as the next lemma.

LEMMA 3.15. *Let Γ be a witnessed MCS and $\mathcal{Y}$ be the model generated from Γ. Let $\Delta \in Y$. If Δ has a generating path π, then $C_\pi[\varphi] \in \Gamma$ for any pattern $\varphi \in \Delta$.*

PROOF. The proof is by induction on the length of the generating path π. If π is the empty path ϵ, then Δ must be Γ and C_π is the identity context, and $C_\pi[\varphi] = \varphi \in \Gamma$ for any $\varphi \in \Delta$. Now assume Δ has a generating path $\pi = \langle \pi_0, (\sigma, i) \rangle$ with $\sigma \in \Sigma_{s_1 \ldots s_n, s}$. By Definition 3.13, there exist witnessed MCSs $\Delta_{s_1}, \ldots, \Delta_{s_n}, \Delta_s \in Y$ and $1 \le i \le n$ such that $\Delta = \Delta_{s_i}, \Delta_s \in \sigma_\mathcal{W}(\Delta_{s_1}, \ldots, \Delta_{s_n})$, and Δ_s has π_0 as its generating path. For every $\varphi \in \Delta = \Delta_i$, since $\top_{s_j} \in \Delta_{s_j}$ for any $j \ne i$, by Definition 3.10, $\sigma(\top_{s_1}, \ldots, \top_{s_{i-1}}, \varphi, \top_{s_{i+1}}, \ldots, \top_{s_n}) \in \Delta_s$. By induction hypothesis, $C_{\pi_0}[\sigma(\top_{s_1}, \ldots, \top_{s_{i-1}}, \varphi, \top_{s_{i+1}}, \ldots, \top_{s_n})] \in \Gamma$, while the latter is exactly $C_\pi[\varphi]$. □

LEMMA 3.16 (SINGLETON VARIABLES). *Let Γ be a witnessed MCS and $\mathcal{Y}$ be the model generated from Γ. For every $\Gamma_1, \Gamma_2 \in Y$ of the same sort and every variable x, if $x \in \Gamma_1 \cap \Gamma_2$ then $\Gamma_1 = \Gamma_2$.*

PROOF. Let π_i be a generating path of Γ_i for $i = 1, 2$. Assume $\Gamma_1 \ne \Gamma_2$. Then there exists a pattern φ such that $\varphi \in \Gamma_1$ and $\neg\varphi \in \Gamma_2$. Because $x \in \Gamma_1 \cap \Gamma_2$, we know $x \wedge \varphi \in \Gamma_1$ and $x \wedge \neg\varphi \in \Gamma_2$. By Lemma 3.15, $C_{\pi_1}[x \wedge \varphi], C_{\pi_2}[x \wedge \neg\varphi] \in \Gamma$, and thus $C_{\pi_1}[x \wedge \varphi] \wedge C_{\pi_2}[x \wedge \neg\varphi] \in \Gamma$. On the other hand, $\neg(C_{\pi_1}[x \wedge \varphi] \wedge C_{\pi_2}[x \wedge \neg\varphi])$ is an instance of (SINGLETON VARIABLE) and thus it is included in Γ. This contradicts the consistency of Γ. □

We will establish an important result about generated models in Lemma 3.19 (the Truth Lemma), which links the semantics and syntax and is essential to the completeness result. Roughly speaking, the lemma says that for any generated model $\mathcal{Y}$ and any witnessed MCS $\Delta \in Y$, a pattern φ is in Δ if and only if the interpretation of φ in $\mathcal{Y}$ contains Δ. To prove the lemma, it is important to show that every variable is interpreted to a singleton. Lemma 3.16 ensures that every variable belongs to *at most one* witnessed MCS. To make sure it is interpreted to *exactly one* MCS, we complete our model by adding a dummy element $\star$ to the carrier set, and interpreting all variables which are interpreted to none of the MCSs to the dummy element. This motivates the next definition.

Definition 3.17 (Completed Models and Completed Valuations). Let $\Gamma = \{\Gamma_s\}_{s \in S}$ be a witnessed MCS and $\mathcal{Y}$ be the Γ-generated model. We here define the Γ-completed model, denoted as $\mathcal{M} = (M, \_\mathcal{M})$. The carrier set $M = \{M_s\}_{s \in S}$ is defined as follows (for every sort $s \in S$):

- $M_s = Y_s$, if every variable $x{:}s$ belongs at least one MCS in Y_s;
- $M_s = Y_s \cup \{\star_s\}$, otherwise.

We assume $\star_s$ is an entity that is different from any MCSs, and $\star_{s_1} \neq \star_{s_2}$ if $s_1 \neq s_2$. For every $\sigma \in \Sigma_{s_1 \ldots s_n, s}$, define its interpretation

$$\sigma_\mathcal{M}(\Delta_1, \ldots, \Delta_n) = \begin{cases} \emptyset & \text{if some } \Delta_i = \star_{s_i} \\ \sigma_\mathcal{Y}(\Delta_1, \ldots, \Delta_n) \cup \{\star_s\} & \text{if for all } 1 \leq j \leq n, \Delta_j \neq \star_{s_j} \text{ and some } \Delta_i = \Gamma_{s_i} \\ \sigma_{\mathcal{Y}_{\Gamma_0}}(\Delta_1, \ldots, \Delta_n) & \text{otherwise} \end{cases}$$

The completed valuation $\rho \colon \text{Var} \to M$ is defined as

$$\rho(x{:}s) = \begin{cases} \Delta & \text{if } x{:}s \in \Delta \\ \star_s & \text{otherwise} \end{cases}$$

The valuation ρ is a well-defined function, because by Lemma 3.16, if there are two witnessed MCSs Δ_1 and Δ_2 such that $x \in \Delta_1$ and $x \in \Delta_2$, then $\Delta_1 = \Delta_2$.

Now we come back to prove Lemma 3.11. We need the following technical lemma.

LEMMA 3.18. *Let $\sigma \in \Sigma_{s_1 \ldots s_n, s}$ be a symbol, $\Phi_1, \ldots, \Phi_n, \phi$ be patterns of appropriate sorts, and $y_1, \ldots, y_n, x$ be variables of appropriate sorts such that $y_1, \ldots, y_n$ are distinct, and $y_1, \ldots, y_n \notin FV(\phi) \cup \bigcup_{1 \leq i \leq n} FV(\Phi_i)$. Then*

$$\vdash \sigma(\Phi_1, \ldots, \Phi_n) \to \exists y_1, \ldots, \exists y_n . \sigma(\Phi_1 \wedge (\exists x. \phi \to \phi[y_1/x]), \ldots, \Phi_n \wedge (\exists x. \phi \to \phi[y_n/x]))$$

PROOF. Notice that for every $1 \leq i \leq n$,

$$\vdash \exists x. \phi \to \exists y_i . (\phi[y_i/x]).$$

By easy matching logic reasoning,

$$\vdash \sigma(\Phi_1, \ldots, \Phi_n) \to \sigma(\Phi_1 \wedge (\exists x. \phi \to \exists y_1 . (\phi[y_1/x])), \ldots, \Phi_n \wedge (\exists x. \phi \to \exists y_n . (\phi[y_n/x])))$$

Then use Proposition 2.5 to move the quantifiers $\exists y_1, \ldots, \exists y_n$ to the top. □

LEMMA 3.11 (EXISTENCE LEMMA). *Let $\overline{\Sigma} = (S, \Sigma)$ be a signature and Γ be a witnessed MCS of sort $s \in S$. Given a symbol $\sigma \in \Sigma_{s_1 \ldots s_n, s}$ and patterns $\varphi_1, \ldots, \varphi_n$ of appropriate sorts. If $\sigma(\varphi_1, \ldots, \varphi_n) \in \Gamma$, then there exist n witnessed MCSs $\Gamma_1, \ldots, \Gamma_n$ of appropriate sorts such that $\varphi_i \in \Gamma_i$ for every $1 \leq i \leq n$, and $\Gamma \in \sigma_\mathcal{W}(\Gamma_1, \ldots, \Gamma_n)$.*

PROOF. Recall that $\Gamma \in \sigma_\mathcal{W}(\Gamma_1, \ldots, \Gamma_n)$ means for every $\phi_i \in \Gamma_i$, $1 \leq i \leq n$, $\sigma(\phi_1, \ldots, \phi_n) \in \Gamma$. The main technique that we will be using here is similar to Lemma 3.9. We start with the singleton sets $\{\varphi_i\}$ for every $1 \leq i \leq n$ and extend them to witnessed MCSs Γ_i, while this time we also need to make sure the results $\Gamma_1, \ldots, \Gamma_n$ satisfy the desired property $\Gamma \in \sigma_\mathcal{W}(\Gamma_1, \ldots, \Gamma_n)$. Another difference

compared to Lemma 3.9 is that this time we do not extend our set of variables, because our starting point, $\{\varphi_i\}$, contains just one pattern and uses only finitely many variables. Readers will see how these conditions play a role in the upcoming proof.

Enumerate all patterns of sorts $s_1, \ldots, s_n$ as follows $\psi_0, \psi_1, \psi_2, \cdots \in \bigcup_{1 \leq i \leq n} \text{PATTERN}_{s_i}$. Notice that $s_1, \ldots, s_n$ do not need to be all distinct. To ease our notation, we define a "choice" operator, denoted as $[\varphi_s]_{s'}$, as follows

$$[\varphi_s]_{s'} = \begin{cases} \varphi_s & \text{if } s = s' \\ \text{nothing} & \text{otherwise} \end{cases}$$

For example, $\varphi_s \wedge [\psi]_s$ means $\varphi_s \wedge \psi$ if ψ also has sort s. Otherwise, it means φ_s. The choice operator propagates with all logic connectives in the natural way. For example, $[\neg\psi]_s = \neg[\psi]_s$.

In the following, we will define a non-decreasing sequence of pattern sets $\Gamma_i^{(0)} \subseteq \Gamma_i^{(1)} \subseteq \Gamma_i^{(2)} \subseteq \cdots \subseteq \text{PATTERN}_{s_i}$ for each $1 \leq i \leq n$, such that the following conditions are true for all $1 \leq i \leq n$ and $k \geq 0$:

(1) If ψ_k has sort s_i, then either ψ_k or $\neg\psi_k$ belongs to $\Gamma_i^{(k+1)}$.

(2) If ψ_k has the form $\exists x.\phi_k$ and it belongs to $\Gamma_i^{(k+1)}$, then there exists a variable z such that $(\exists x.\phi_k) \to \phi_k[z/x]$ also belongs to $\Gamma_i^{(k+1)}$.

(3) $\Gamma_i^{(k)}$ is finite.

(4) Let $\pi_i^{(k)} = \bigwedge \Gamma_i^{(k)}$ for every $1 \leq i \leq n$. Then $\sigma(\pi_1^{(k)}, \ldots, \pi_n^{(k)}) \in \Gamma$.

(5) $\Gamma_i^{(k)}$ is consistent.

Among the above five conditions, condition (2)–(5) are like "safety" properties while condition (1) is like a "liveness" properties. We will eventually let $\Gamma_i = \bigcup_{k \geq 0} \Gamma_i^{(k)}$ and prove that Γ_i has the desired property. Before we present the actual construction, we give some hints on how to prove these conditions hold. Conditions (1)–(3) will be satisfied directly by construction, although we will put a notable effort in satisfying condition (2). Condition (4) will be proved hold by induction on k. Condition (5) is in fact a consequence of condition (4) as shown below. Assume condition (4) holds but condition (5) fails. This means that $\Gamma_i^{(k)}$ is not consistent for some $1 \leq i \leq n$, so $\vdash \pi_i^{(k)} \to \bot$. By (FRAMING)

$$\vdash \sigma(\pi_1^{(k)}, \ldots, \pi_i^{(k)}, \ldots, \pi_n^{(k)}) \to \sigma(\pi_1^{(k)}, \ldots, \bot, \ldots, \pi_n^{(k)})$$

Then by Proposition 2.5 and FOL reasoning,

$$\vdash \sigma(\pi_1^{(k)}, \ldots, \pi_i^{(k)}, \ldots, \pi_n^{(k)}) \to \bot$$

Since $\sigma(\pi_1^{(k)}, \ldots, \pi_i^{(k)}, \ldots, \pi_n^{(k)}) \in \Gamma$ by condition (4), we know $\bot \in \Gamma$ by Proposition 3.7. And this contradicts the fact that Γ is consistent.

Now we are ready to construct the sequence $\Gamma_i^{(0)} \subseteq \Gamma_i^{(1)} \subseteq \Gamma_i^{(2)} \subseteq \ldots$ for $1 \leq i \leq n$. Let $\Gamma_i^{(0)} = \{\varphi_i\}$ for $1 \leq i \leq n$. Obviously, $\Gamma_i^{(0)}$ satisfies conditions (3) and (4). Condition (5) follows as a consequence of condition (4). Conditions (1) and (2) are not applicable.

Suppose we have already constructed sets $\Gamma_i^{(k)}$ for every $1 \leq i \leq n$ and $k \geq 0$, which satisfy the conditions (1)–(5). We show how to construct $\Gamma_i^{(k+1)}$. In order to satisfy condition (1), we should add either ψ_k or $\neg\psi_k$ to $\Gamma_i^{(k)}$, if $\Gamma_i^{(k)}$ has the same sort as ψ_k. Otherwise, we simply let $\Gamma_i^{(k+1)}$ be the same as $\Gamma_i^{(k)}$. The question here is: if $\Gamma_i^{(k)}$ has the same sort as ψ_k, which pattern should we add to $\Gamma_i^{(k)}$, ψ_k or $\neg\psi_k$? Obviously, condition (3) will still hold no matter which one we choose to add, so we just need to make sure that we do not break conditions (2) and (4).

Let us start by satisfying condition (4). Consider pattern $\sigma(\pi_1^{(k)}, \ldots, \pi_n^{(k)})$, which, by condition (4), is in Γ. This tells us that the pattern

$$\sigma(\pi_1^{(k)} \wedge [\psi_k \vee \neg\psi_k]_{s_1}, \ldots, \pi_n^{(k)} \wedge [\psi_k \vee \neg\psi_k]_{s_n})$$

is also in Γ. Recall that $[\_]_s$ is the choice operator, so if ψ_k has sort s_i, then $\pi_i^{(k)} \wedge [\psi_k \vee \neg\psi_k]_{s_i}$ is $\pi_i^{(k)} \wedge (\psi_k \vee \neg\psi_k)$. Otherwise, it is $\pi_i^{(k)}$. Use Proposition 2.5 and FOL reasoning, and notice that the choice operator propagates with the disjunction $\vee$ and the negation $\neg$, we get

$$\sigma((\pi_1^{(k)} \wedge [\psi_k]_{s_1}) \vee (\pi_1^{(k)} \wedge \neg[\psi_k]_{s_1}), \ldots, (\pi_n^{(k)} \wedge [\psi_k]_{s_n}) \vee (\pi_n^{(k)} \wedge \neg[\psi_k]_{s_n})) \in \Gamma$$

Then we use Proposition 2.5 again and move all the disjunctions to the top, and we end up with a disjunction of 2^n patterns:

$$\bigvee \sigma(\pi_1^{(k)} \wedge [\neg]_1^{(k)}[\psi_k]_{s_1}, \ldots, \pi_n^{(k)} \wedge [\neg]_n^{(k)}[\psi_k]_{s_n}) \in \Gamma$$

where $[\neg]$ means either nothing or $\neg$. Notice that some $[\psi_k]_{s_i}$'s might be nothing, so some of these 2^n patterns may be the same.

Notice that Γ is an MCS. By proposition 3.7, among these 2^n patterns there must exists one pattern that is in Γ. We denote *that* pattern as

$$\sigma(\pi_1^{(k)} \wedge [\neg]_1^{(k)}[\psi_k]_{s_1}, \ldots, \pi_n^{(k)} \wedge [\neg]_n^{(k)}[\psi_k]_{s_n})$$

For any $1 \leq i \leq n$, if $[\neg]_i^{(k)}[\psi_k]_{s_i}$ does not have the form $\exists x.\phi$, we simply define $\Gamma_i^{(k+1)} = \Gamma_i^{(k)} \cup \{[\neg]_i^{(k)}[\psi_k]_{s_i}\}$. If $[\neg]_i^{(k)}[\psi_k]_{s_i}$ does have the form $\exists x.\phi$, we need special effort to satisfy condition (2). Without loss of generality and to ease our notation, let us assume that *for every* $1 \leq i \leq n$, the pattern $[\neg]_i^{(k)}[\psi_k]_{s_i}$ has the same form $\exists x.\phi$. We are going to find for each index i a variable z_i such that

$$\sigma(\pi_1^{(k)} \wedge \exists x.\phi \wedge (\exists x.\phi \rightarrow \phi[z_1/x]), \ldots, \pi_n^{(k)} \wedge \exists x.\phi \wedge (\exists x.\phi \rightarrow \phi[z_n/x])) \in \Gamma$$

This will allow us to define $\Gamma_i^{(k+1)} = \Gamma_i^{(k)} \cup \{\exists x.\phi\} \cup \{\exists x.\phi \rightarrow \phi[z_i/x]\}$ which satisfies conditions (2) and (4).

We find these variables z_i's by Lemma 3.18 and the fact that Γ is a witnessed set. Let $\Phi_i \equiv \pi_i^{(k)} \wedge \exists x.\phi$ for $1 \leq i \leq n$. By construction, $\sigma(\Phi_1, \ldots, \Phi_n) \in \Gamma$. Hence, by Lemma 3.18 and Proposition 3.7, for any distinct variables $y_1, \ldots, y_n \notin FV(\phi) \cup \bigcup_{1 \leq i \leq n} FV(\Phi_i)$,

$$\exists y_1 \ldots \exists y_n.\sigma(\Phi_1 \wedge (\exists x.\phi \rightarrow \phi[y_1/x]), \ldots, \Phi_n \wedge (\exists n.\phi \rightarrow \phi[y_n/x])) \in \Gamma$$

The set Γ is a witnessed set, so there exist variables $z_1, \ldots, z_n$ such that

$$\sigma(\Phi_1 \wedge (\exists x.\phi \rightarrow \phi[z_1/x]), \ldots, \Phi_n \wedge (\exists x.\phi \rightarrow \phi[z_n/x])) \in \Gamma$$

This justifies our construction $\Gamma_i^{(k+1)} = \Gamma_i^{(k)} \cup \{\exists x.\phi\} \cup \{\exists x.\phi \rightarrow \phi[z_i/x]\}$.

So far we have proved our construction of the sequences $\Gamma_i^{(0)} \subseteq \Gamma_i^{(1)} \subseteq \Gamma_i^{(2)} \subseteq \ldots$ for $1 \leq i \leq n$ satisfy the conditions (1)–(5). Let $\Gamma_i = \bigcup_{k \geq 0} \Gamma_i^{(k)}$ for $1 \leq i \leq n$. By construction, Γ_i is a witnessed MCS. It remains to prove that $\Gamma \in \sigma_W(\Gamma_1, \ldots, \Gamma_n)$. To prove it, assume $\phi_i \in \Gamma_i$ for $1 \leq i \leq n$. By construction, there exists $K > 0$ such that $\phi_i \in \Gamma_i^{(K)}$ for all $1 \leq i \leq n$. Therefore, $\vdash \pi_i^{(K)} \rightarrow \phi_i$. By condition (4), $\sigma(\pi_1^{(K)}, \ldots, \pi_n^{(K)}) \in \Gamma$, and thus by (Framing) and Proposition 3.7, $\sigma(\phi_1, \ldots, \phi_n) \in \Gamma$. □

LEMMA 3.19 (TRUTH LEMMA). *Let Γ be a witnessed MCS, $\mathcal{M} = (M, \_{\mathcal{M}})$ be its completed model, and ρ be the completed valuation. For any witnessed MCS $\Delta \in M$ and any pattern φ such that Δ and φ have the same sort,*

$$\varphi \in \Gamma \quad \text{if and only if} \quad \Gamma \in \bar{\rho}(\varphi)$$

Proof. The proof is by induction on the structure of φ. If φ is a variable the conclusion follows by Definition 3.10. If φ has the form $\psi_1 \wedge \psi_2$ or $\neg\psi_1$, the conclusion follows from Proposition 3.7. If φ has the form $\sigma(\varphi_1, \ldots, \varphi_n)$, the conclusion from left to right is given by Lemma 3.11. The conclusion from right to left follows from Definition 3.10.

Now assume φ has the form $\exists x.\psi$. If $\exists x.\psi \in \Gamma$, since Γ is a witnessed set, there is a variable y such that $\exists x.\psi \to \psi[y/x] \in \Gamma$, and thus $\psi[y/x] \in \Gamma$. By induction hypothesis, $\Gamma \in \bar\rho(\psi[y/x])$, and thus by the semantics of the logic, $\Gamma \in \bar\rho(\exists x.\psi)$.

Consider the other direction. Assume $\Gamma \in \bar\rho(\exists x.\psi)$. By definition there exists a witnessed set Γ' in $\mathcal{M}$ such that $\Gamma \in \bar{\rho'}(\psi)$ where $\rho' = \rho[\Gamma'/x]$. By Definition 3.17, every element in $\mathcal{M}$ has a variable that is assigned to it by the completed valuation ρ. Let us assume the variable y is assigned to Γ', i.e., $\rho(y) = \Gamma'$. By Lemma 1.5, $\Gamma \in \bar{\rho'}(\psi) = \bar\rho(\psi[y/x])$. By induction hypothesis, $\psi[y/x] \in \Gamma$. Finally notice that $\vdash \psi[y/x] \to \exists y.\psi[y/x]$. By Proposition 3.7, $\exists y.\psi[y/x] \in \Gamma$, i.e., $\psi \in \exists x.\psi$. □

Theorem 3.20. *For any consistent set Γ, there is a model $\mathcal{M}$ and an valuation ρ such that for any pattern $\varphi \in \Gamma$, $\bar\rho(\varphi) \neq \emptyset$.*

Proof. Use Lemma 3.9 and extend Γ to a witnessed MCS Γ^+. Let $\mathcal{M}$ and ρ be the completed model and valuation generated by Γ^+ respectively. By Lemma 3.19, for any pattern $\varphi \in \Gamma \subseteq \Gamma^+$, we get $\Gamma^+ \in \bar\rho(\varphi)$, so $\bar\rho(\varphi) \neq \emptyset$. □

We restate the completeness theorem and prove it.

Theorem 3.3 (Local completeness). *For any pattern φ such that $\emptyset \vDash \varphi$, $\emptyset \vdash \varphi$.*

Proof. Assume the opposite. If $\emptyset \nvdash \varphi$, then $\{\neg\varphi\}$ is consistent by Definition 3.6. Then there is a model $\mathcal{M}$ and an valuation ρ such that $\bar\rho(\neg\varphi) \neq \emptyset$, i.e., $\bar\rho(\varphi) \neq M$. This contradicts the fact that $\emptyset \vDash \varphi$. □

3.3 Completeness When the Hypothesis Set is Nonempty

In the previous subsection, we discussed local completeness, where the hypothesis set is empty. In this section, we take a step forward and consider global completeness, where the hypothesis set need not be empty. More specifically, we will prove the next theorem.

Theorem 3.21 (Global Completeness). *Let φ be a pattern of sort s and H be a set of patterns such that $H \nvDash \bot_{s'}$ for every sort s' different from s. Then $H \vDash \varphi$ implies $H \vdash \varphi$.*

We explain why we need the requirement $H \nvDash \bot_{s'}$. Consider the following (counter-)example. Let $\Sigma = (S, \Sigma)$ be a signature where $S = \{s, s'\}$ and $\Sigma = \emptyset$. Let $H = \{H_s, H_{s'}\}$ where $H_s = \{\bot_s\}$ and $H_{s'} = \emptyset$. Obviously, the hypothesis set H has no model, so $H \vDash \varphi$ holds for any pattern φ. On the other hand, we can prove for every pattern $\varphi_{s'}$ of sort s', $H \vdash \varphi_{s'}$ holds if and only if $\emptyset \vdash \varphi_{s'}$, because the symbol set Σ is empty. Therefore, we have $H \vDash \bot_{s'}$ and $H \nvDash \bot_{s'}$, contradicting Theorem 3.21.

Definition 3.22 (Deductive Closure). Let $H = \{H_s\}_{s \in S}$ be a hypothesis set. The *deductive closure* of H, denoted as $C(H)$, is the set of all patterns φ such that $H \vdash \varphi$. For any sort $s \in S$, define $C_s(H) = C(H) \cap \textsc{Pattern}_s$ be the set of all patterns φ_s of sort s such that $\Gamma \vdash \varphi_s$.

We immediately get the following result.

Proposition 3.23. *For any hypothesis set H and any pattern φ, $H \vdash \varphi$ if and only if $C(H) \Vdash \varphi$. In particular, if $H \nvDash \bot_s$ for every $s \in S$, then $C(H)$ is consistent.*

To prove Theorem 3.21, we assume a hypothesis set H such that $\Gamma \nvdash \bot_s$ for every sort s and a pattern φ such that $H \nvdash \varphi$, and we try to show $H \nvDash \varphi$ by looking for a model $\mathcal{M}$ and a valuation ρ such that $\mathcal{M} \vDash H$ and $\bar{\rho}(\varphi) \neq M$. We point out that it is sufficient to consider the simpler case when all patterns in H are closed. If not, we can "make it closed" by letting $\forall H = \{\forall x_1 \ldots \forall x_n.\psi \mid \psi \in H \text{ and } FV(\psi) = \{x_1, \ldots, x_n\}\}$, and notice that

$$H \vdash \psi \quad \text{iff} \quad \forall H \vdash \psi \qquad\qquad \text{for every pattern } \psi$$
$$\mathcal{M} \vDash H \quad \text{iff} \quad \mathcal{M} \vDash \forall H \qquad\qquad \text{for every model } \mathcal{M}$$

Therefore, we can assume the hypothesis set H contains only closed patterns without any loss of generality. This additional assumption is quite convenient, because when showing $\mathcal{M} \vDash H$, we just need to consider one valuation, instead of all.

The main proof technique is in fact similar to the one we used in proving local completeness. We look for a model $\mathcal{M}$ and a valuation ρ such that $\bar{\rho}(\varphi) \neq M$, but this time we also require $\mathcal{M} \vDash H$. How can we guarantee it? We can expect a form of the truth lemma, which says for every witnessed MCS $\Delta \in M$ and pattern ψ, $\psi \in \Delta$ if and only if $\Delta \in \bar{\rho}(\psi)$. To show that $\mathcal{M} \vDash H$, we need to show $\mathcal{M} \vDash \psi$ for every $\psi \in H$. Recall our assumption that all patterns in H are closed, so every $\psi \in H$ is closed. Therefore, it suffices to show that $\bar{\rho}(\psi) = M$ for every $\psi \in H$, which means for every $\Delta \in M$, $\Delta \in \bar{\rho}(\psi)$. Now by the truth lemma, $\Delta \in \bar{\rho}(\psi)$ means $\psi \in \Delta$, and thus our objective becomes to prove $\psi \in \Delta$ for every $\psi \in H$ and every $\Delta \in M$. In other words, every witnessed MCS Δ in the model $\mathcal{M}$ contains H as its subset. This motivates the following definition of the canonical model when a hypothesis pattern set H is given.

Definition 3.24 (Canonical Model II). Given a hypothesis set H such that $H \nvdash \bot_s$ for any sort s, and a pattern φ such that $H \nvdash \varphi$. The canonical model for H, denoted as $\mathcal{W} = (W, \_{\mathcal{W}})$, consists of

- for every sort $s \in S$, a carrier set $W_s = \{\Gamma \subseteq \textsc{Pattern}_s \mid \Gamma \text{ is a witnessed MCS and } C_s(H) \subseteq \Gamma\}$; let $W = \{W_s\}_{s \in S}$;
- for any symbol $\sigma \in \Sigma_{s_1 \ldots s_n, s}$, its interpretation $\sigma_{\mathcal{W}} : W_{s_1} \times \cdots \times W_{s_n} \to 2^{W_s}$ defined as
$$\Gamma \in \sigma_{\mathcal{W}}(\Gamma_1, \ldots, \Gamma_n) \quad \text{iff} \quad \text{for any } \varphi_i \in \Gamma_i, 1 \le i \le n, \sigma(\varphi_1, \ldots, \varphi_n) \in \Gamma$$

Similarly, we have a form of existence lemma.

LEMMA 3.25 (EXISTENCE LEMMA II). *Given a hypothesis set H such that $H \nvdash \bot_s$ for any sort s, and its canonical model $\mathcal{W} = (W, \_{\mathcal{W}})$. For any witnessed MCS $\Gamma \in W$, any symbol $\sigma \in \Sigma_{s_1 \ldots s_n, s}$, and any patterns $\varphi_1, \ldots, \varphi_n$, if $\sigma(\varphi_1, \ldots, \varphi_n) \in \Gamma$, then there exist witnessed MCSs $\Gamma_1, \ldots, \Gamma_n \in W$ such that $\varphi_i \in \Gamma_i$ for every $1 \le i \le n$, and $\Gamma \in \sigma_{\mathcal{W}}(\Gamma_1, \ldots, \Gamma_n)$.*

The difference between this existence lemma and the previous one (Lemma 3.11) is that the witnessed MCSs $\Gamma_1, \ldots, \Gamma_n$ must be in the canonical model $\mathcal{W}$, which means they must contain H_s (of appropriate sort s) as subsets. We will prove the lemma later.

Definition 3.26 (Generated Models II). Given a hypothesis set H such that $H \nvdash \bot_s$ for any sort s and its canonical model $\mathcal{W} = (W, \_{\mathcal{W}})$. Let $\Gamma \in W$ be a witnessed MCS containing H_s (of appropriate sort). Define a subset $Y \subseteq W$ in the following inductive way:

- $\Gamma \in Y$;
- If $\Delta \in Y$ and there exist a symbol σ and witnessed MCSs $\Delta_1, \ldots, \Delta_n \in W$ such that $\Delta \in \sigma_{\mathcal{W}}(\Delta_1, \ldots, \Delta_n)$, then $\Delta_1, \ldots, \Delta_n \in Y$.

Define $\mathcal{Y} = (Y, \_{\mathcal{Y}})$ be the Γ-generated model, where

$$\sigma_{\mathcal{Y}}(\Delta_1, \ldots, \Delta_n) = Y \cap \sigma_{\mathcal{W}}(\Delta_1, \ldots, \Delta_n) \qquad \text{for any } \Delta_1, \ldots, \Delta_n \in Y$$

We could then prove a similar result as Lemma 3.16.

LEMMA 3.27 (SINGLETON VARIABLES II). *Let $\mathcal{Y} = (Y, \_y)$ be the Γ-generated model. For every $\Gamma_1, \Gamma_2 \in Y$, if $x \in \Gamma_1 \cap \Gamma_2$ then $\Gamma_1 = \Gamma_2$.*

Again, Lemma 3.27 tells us that any variable x belongs to at most one witnessed MCSs in the generated model $\mathcal{Y}$. We can then complete it by the following definition:

Definition 3.28 (Completed Model and Completed Valuation II). Given a hypothesis set H such that $H \nvdash \perp_s$ for any sort s and its canonical model $\mathcal{W} = (W, \_{\mathcal{W}})$. Let $\Gamma \in W$ be a witnessed MCS containing H_s (of appropriate sort). Let $\mathcal{Y} = (Y, \_y)$ be the Γ-generated model. The completed model of $\mathcal{Y}$, denoted as $\mathcal{M}$, is defined as follows. If every variable x belongs to at least one witnessed MCS in Y, then define $\mathcal{M}$ be the same as $\mathcal{Y}$. Otherwise, define it as follows:

- for every sort s, define $M_s = Y_s \cup \{\star_s\}$, where $\star_s$ is an arbitrary entity that is different from any MCSs; in addition, $\star_s \neq \star_{s'}$ for every $s \neq s'$;
- for every $\sigma \in \Sigma_{s_1 \ldots s_n, s}$, define

$$\sigma_{\mathcal{M}}(\Delta_1, \ldots, \Delta_n) = \begin{cases} \emptyset & \text{if there exists } 1 \leq i \leq n \text{ such that } \Delta_i = \star_{s_i} \\ \sigma_{\mathcal{Y}}(\Delta_1, \ldots, \Delta_n) \cup \{\star_s\} & \text{if for all } 1 \leq j \leq n, \Delta_j \neq \star_{s_j}, \text{ and} \\ & \text{there exists } 1 \leq i \leq n \text{ such that } \Delta_i = \Gamma_i \\ \sigma_{\mathcal{Y}}(\Delta_1, \ldots, \Delta_n) & \text{otherwise} \end{cases}$$

The completed valuation $\rho \colon \text{VAR} \to M$ is defined as

$$\rho(x{:}s) = \begin{cases} \Gamma & \text{if } x{:}s \in \Gamma \\ \star_s & \text{otherwise} \end{cases}$$

Let us come back and prove Lemma 3.25.

LEMMA 3.25 (EXISTENCE LEMMA II). *Given a hypothesis set H such that $H \nvdash \perp_s$ for any sort s, and its canonical model $\mathcal{W} = (W, \_{\mathcal{W}})$. For any witnessed MCS $\Gamma \in W$, any symbol $\sigma \in \Sigma_{s_1 \ldots s_n, s}$, and any patterns $\varphi_1, \ldots, \varphi_n$, if $\sigma(\varphi_1, \ldots, \varphi_n) \in \Gamma$, then there exist witnessed MCSs $\Gamma_1, \ldots, \Gamma_n \in W$ such that $\varphi_i \in \Gamma_i$ for every $1 \leq i \leq n$, and $\Gamma \in \sigma_{\mathcal{W}}(\Gamma_1, \ldots, \Gamma_n)$.*

PROOF. Enumerate all patterns of sorts $s_1, \ldots, s_n$ as $\psi_0, \psi_1, \cdots \in \bigcup_{1 \leq i \leq n} \text{PATTERN}_{s_i}$. We will follow the same way as in Lemma 3.11. More specifically, we will construct for every $1 \leq i \leq n$ a non-decreasing sequence of pattern sets $\Gamma_i^{(0)} \subseteq \Gamma_i^{(1)} \subseteq \ldots$, starting from $\Gamma_i^{(0)} = C_{s_i}(H) \cup \{\varphi_i\}$. As in Lemma 3.11, these sequences should satisfy the following conditions for every $k \geq 0$ and $1 \leq i \leq n$:

(1) If ψ_k has sort s_i, then either ψ_k or $\neg\psi_k$ belongs to $\Gamma_i^{(k+1)}$.
(2) If ψ_k has the form $\exists x.\phi_k$ and it belongs to $\Gamma_i^{(k+1)}$, then there exists a variable z such that $(\exists x.\phi_k) \to \phi_k[z/x]$ also belongs to $\Gamma_i^{(k+1)}$.
(3) For every $\Delta_i^{(k)} \subseteq_{fin} \Gamma_i^{(k)}$ and $\pi_i^{(k)} = \bigwedge \Delta_i^{(k)}$, $\sigma(\pi_1^{(k)}, \ldots, \pi_n^{(k)}) \in \Gamma$.
(4) $\Gamma_i^{(k)}$ is consistent.

We have seen conditions (1), (2), and (4) in the proof of Lemma 3.11. Condition (3) naturally generalizes the condition (4) in the proof of Lemma 3.11, as $\Gamma_i^{(k)}$ might not be finite anymore. It is easy to show that condition (3) implies condition (4).

We need to worry about two things. One is to show that all initial sets $\Gamma_1^{(0)}, \ldots, \Gamma_n^{(0)}$ satisfy the condition (3). The other is to construct $\Gamma_i^{(k+1)}, \ldots, \Gamma_n^{(k+1)}$ while maintaining the condition (3). We discuss these two worries one by one.

Let us first show $\Gamma_1^{(0)}, \ldots, \Gamma_n^{(0)}$ satisfy the condition (3). Let $\Delta_i^{(0)} \subseteq \Gamma_i^{(0)} = C_{s_i}(H) \cup \{\varphi_i\}$ for every $1 \leq i \leq n$. By easy matching logic reasoning, $H \vdash \varphi_i \to \bigwedge \Delta_i^{(0)}$ for any i, and then by frame

reasoning, $H \vdash \sigma(\varphi_1, \ldots, \varphi_n) \to \sigma(\bigwedge \Delta_1^{(0)}, \ldots, \bigwedge \Delta_n^{(0)})$. Since $\sigma(\varphi_1, \ldots, \varphi_n) \in \Gamma$ and Γ is a MCS, we know $\sigma(\bigwedge \Delta_1^{(0)}, \ldots, \bigwedge \Delta_n^{(0)}) \in \Gamma$, and condition (3) holds.

Now we consider the construction of $\Gamma_i^{(k+1)}, \ldots, \Gamma_n^{(k+1)}$. In Lemma 3.11, we showed if $\sigma(\pi_1^{(k)}, \ldots, \pi_n^{(k)}) \in \Gamma$, then we can always pick $[\neg]_1^{(k)}\psi_k, \ldots, [\neg]_n^{(k)}\psi_k$ where $[\neg]$ means either nothing or $\neg$, such that

$$\sigma(\pi_1^{(k)} \wedge [\neg]_1^{(k)}\psi_k, \ldots, \pi_n^{(k)} \wedge [\neg]_n^{(k)}\psi_k) \in \Gamma$$

However, here we have many $\pi_i^{(k)}$'s, one for each finite subset of $\Gamma_i^{(k)}$. How do we know if we can pick $[\neg]_1^{(k)}\psi_k, \ldots, [\neg]_n^{(k)}\psi_k$ such that they work for every finite subsets? Notice that if $[\neg]_1^{(k)}\psi_k, \ldots, [\neg]_n^{(k)}\psi_k$ work for $\pi_1^{(k)} = \bigwedge \Delta_1^{(k)}, \ldots, \pi_n^{(k)} = \bigwedge \Delta_n^{(k)}$, then they also work for every $\pi'^{(k)}_1 = \bigwedge \Delta'^{(k)}_1, \ldots, \pi'^{(k)}_n = \bigwedge \Delta'^{(k)}_n$ where $\Delta'^{(k)}_i$ is a subset of $\Delta_i^{(k)}$ for $i = 1, \ldots, n$. This is simply because that $\vdash \pi_i^{(k)} \to \pi'^{(k)}_i$ for $i = 1, \ldots, n$, and thus by frame reasoning,

$$\vdash \sigma(\pi_1^{(k)} \wedge [\neg]_1^{(k)}\psi_k, \ldots, \pi_n^{(k)} \wedge [\neg]_n^{(k)}\psi_k) \to \sigma(\pi'^{(k)}_1 \wedge [\neg]_1^{(k)}\psi_k, \ldots, \pi'^{(k)}_n \wedge [\neg]_n^{(k)}\psi_k)$$

Then by Lemma 3.7,

$$\sigma(\pi'^{(k)}_1 \wedge [\neg]_1^{(k)}\psi_k, \ldots, \pi'^{(k)}_n \wedge [\neg]_n^{(k)}\psi_k) \in \Gamma$$

Therefore, if there was a set $\Delta_i^{(k)}$ that is the "largest" finite subset of $\Gamma_i^{(k)}$ for $i = 1, \ldots, n$, we could then pick our $[\neg]_1^{(k)}\psi_k, \ldots, [\neg]_n^{(k)}\psi_k$ according to $\Delta_1^{(k)}, \ldots, \Delta_n^{(k)}$. However, since there exists no largest finite subset if $\Gamma_i^{(k)}$ is infinite, this approach does not work. Instead, we consider all possible ways of picking $[\neg]_1^{(k)}\psi_k, \ldots, [\neg]_n^{(k)}\psi_k$, and as $\Delta_i^{(k)}$ approaches towards $\Gamma_i^{(k)}$, we will converge to one particular way of picking $[\neg]_1^{(k)}\psi_k, \ldots, [\neg]_n^{(k)}\psi_k$ that work for all $\Delta_1^{(k)}, \ldots, \Delta_n^{(k)}$.

Formally, define the set of all picking ways as

$$P = \underbrace{\{\psi_k, \neg\psi_k\} \times \cdots \times \{\psi_k, \neg\psi_k\}}_{n \text{ times}}$$

Notice that P is a discrete set consisting of 2^n elements. Denote $P(\Delta_1^{(k)}, \ldots, \Delta_n^{(k)})$ as the set of all picking ways for subsets $\Delta_1^{(k)}, \ldots, \Delta_n^{(k)}$ such that $\sigma(\bigwedge \Delta_1^{(k)} \wedge [\neg]_1^{(k)}\psi_k, \ldots, \bigwedge \Delta_n^{(k)} \wedge [\neg]_n^{(k)}\psi_k) \in \Gamma$. We know $P(\Delta_1^{(k)}, \ldots, \Delta_n^{(k)})$ is nonempty, and as any $\Delta_i^{(k)}$ increases, $P(\Delta_1^{(k)}, \ldots, \Delta_n^{(k)})$ decreases. Let each $\Delta_i^{(k)}$ increases towards $\Gamma_i^{(k)}$, and we will obtain a decreasing sequence of sets of working picking ways, which converges to a nonempty set P_0. Assume $([\neg]_1^{(k)}\psi_k, \ldots, [\neg]_n^{(k)}\psi_k)$ belongs to P_0.

We now discuss condition (2). Similar to the proof of Lemma 3.11, let us assume (without loss of generality and for simplicity) that for every $1 \le i \le n$ the pattern $[\neg]_i^{(k)}[\psi_k]_{s_i}$ has the form $\exists x.\phi$, and we look for variable $z_1, \ldots, z_n$ such that

$$\sigma(\bigwedge \Delta_1^{(k)} \wedge \exists x.\phi \wedge \exists x.\phi \to \phi[z_1/x], \ldots, \bigwedge \Delta_n^{(k)} \wedge \exists x.\phi \wedge \exists x.\phi \to \phi[z_n/x]) \in \Gamma$$

for any subsets $\Delta_1^{(k)} \subseteq_{fin} \Gamma_1^{(k)}, \ldots, \Delta_n^{(k)} \subseteq_{fin} \Gamma_n^{(k)}$. The difficulty, again, is that there might be infinitely many finite subsets $\Delta_1^{(k)}, \ldots, \Delta_n^{(k)}$. We already knew from the proof of Lemma 3.11 that for any fixed subsets $\Delta_1^{(k)}, \ldots, \Delta_n^{(k)}$, we can find witnessed variables $z_1, \ldots, z_n$. In addition, by easy matching logic reasoning, if witnessed variables $z_1, \ldots, z_n$ work for subsets $\Delta_1^{(k)}, \ldots, \Delta_n^{(k)}$, they also work for subsets $\Delta'^{(k)}_1 \subseteq \Delta_1^{(k)}, \ldots, \Delta'^{(k)}_n \subseteq \Delta_n^{(k)}$. Therefore, we could consider the set of all possible witnessed variables $(z_1, \ldots, z_n)$ for subsets $\Delta_1^{(k)}, \ldots, \Delta_n^{(k)}$. Let these subsets approach $\Gamma_1^{(k)}, \ldots, \Gamma_n^{(k)}$, and we will converge to a set of witnessed variables, from which we pick anyone $(z_1, \ldots, z_n)$.

The rest of the proof is the same as the proof of Lemma 3.11.   □

We can then prove the truth lemma as Lemma 3.19.

LEMMA 3.29 (TRUTH LEMMA II). *Given a hypothesis set H such that $H \nvdash \bot_s$ for any sort s and its canonical model $\mathcal{W} = (W, \_\mathcal{W})$. Let $\Gamma \in W$ be a witnessed MCS containing H_s (of appropriate sort). Let $\mathcal{W} = (W, \_\mathcal{W})$ be the Γ-generated model. For every witnessed MCS $\Delta \in M$ and pattern φ, $\varphi \in \Delta$ if and only if $\Delta \in \bar{\rho}(\varphi)$.*

PROOF. The same as the proof of Lemma 3.19. □

THEOREM 3.21 (GLOBAL COMPLETENESS). *Let φ be a pattern of sort s and H be a set of patterns such that $H \nvDash \bot_{s'}$ for every sort s' different from s. Then $H \vDash \varphi$ implies $H \vdash \varphi$.*

PROOF. Without loss of generality, assume H contains only closed patterns. Assume $H \nvdash \varphi$, and thus $C(H) \cup \{\neg\varphi\}$ is consistent. Extend it to a witnessed MCS denoted as Γ. Let $\mathcal{W} = (W, \_\mathcal{W})$ be the canonical model for H, and $\mathcal{M} = (M, \_\mathcal{M})$ be the completed model generated by Γ. Notice that $\neg\varphi \in \Gamma$. By Lemma 3.29, $\Gamma \in \bar{\rho}(\neg\varphi)$, so $\bar{\rho}(\varphi)$ is not the total set, and $\mathcal{M} \nvDash \varphi$.

It remains to show that $\mathcal{M} \vDash H$. For every $\psi \in H$, we show that $\bar{\rho}(\psi) = M$, i.e., $\Delta \in \bar{\rho}(\psi)$ for every $\Delta \in M$. By definition, if a witnessed MCS $\Delta \in M$, then $C(H) \subseteq \Delta$. In particular, $H \subseteq \Delta$, and thus $\psi \in \Delta$. Again, by Lemma 3.29, $\Delta \in \bar{\rho}(\psi)$,

Fixme. What if Δ is $\star$?

Therefore, the completed model $\mathcal{W}$ satisfies H but not φ, and thus we conclude that $H \nvDash \varphi$. Here ends our proof. □