Fixpoints in Matching Logic

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Fixpoints are solutions to equations of the form

$$X = F(X)$$

where X is the variable of the equation and $F(X)$ is an expression about X. Depending on what X is, fixpoints are categorized as

- element fixpoints, if X ranges over elements;
- relation fixpoints, if X ranges over relations;
- function fixpoints, if X ranges over functions;

In addition, if there is a partial order $\leq$ on the domain where X ranges, we can define least fixpoints w.r.t. the order $\leq$ as follows

$$X = F(X) \text{ and for any } Y \text{ such that } Y = F(Y), \text{ we have } X \leq Y.$$

Element fixpoints are definable in first order logic. Given an equation

$$x = f(x)$$

where the right-hand side is a term in which the variable x occurs free. The following formula

$$\exists x.x = f(x)$$

is true in precisely those models where the interpretation of f has an fixpoint. The following formula

$$\exists x.(x = f(x) \wedge \forall y.y = f(y) \to x \leq y)$$

is true in precisely those models where the interpretation of f has a least fixpoint. Let e be a constant symbol, the following formula

$$e = f(e) \wedge \forall y.y = f(y) \to e \leq y$$

is true in precisely those models where the interpretation of f has a least fixpoint, which is the interpretation of e.

Relation fixpoints are definable in first order logic.