Towards an Efficient and Economic Deductive System of Matching Logic

FSL group

January 18, 2017

We aim for a Hilbert style deductive system which has a relatively large number of axioms but only a few inference rules.

1 Grammar and extended grammar

The formal language $\mathcal{L}$ we use to write matching logic patterns is defined as follows.

$$
\begin{aligned}
P ::=~ & x \\
& |~ P_1 \wedge P_2 \\
& |~ \neg P \\
& |~ \forall x.P \\
& |~ \sigma(P_1, \ldots, P_n) \\
(* * * ~ & \mathtt{extended} * *) \\
& |~ P_1 \vee P_2 \\
& |~ P_1 \wedge P_2 \\
& |~ P_1 \leftrightarrow P_2 \\
& |~ \exists x.P \\
& |~ \lceil P \rceil \\
& |~ \lfloor P \rfloor \\
& |~ P_1 = P_2 \\
& |~ P_1 \neq P_2 \\
& |~ \top \\
& |~ \bot \\
& |~ P_1 \subseteq P_2 \\
& |~ x \in P
\end{aligned}
$$

with the extended grammar defined as

$$P_1 \lor P_2 := \neg P_2 \to P_1$$
$$P_1 \land P_2 := \neg(\neg P_1 \lor \neg P_2)$$
$$P_1 \leftrightarrow P_2 := (P_1 \to P_2) \land (P_2 \to P_1)$$
$$\exists x.P := \neg \forall x.\neg P$$
$$\lfloor P \rfloor := \neg \lceil \neg P \rceil$$
$$P_1 = P_2 := \lfloor P_1 \leftrightarrow P_2 \rfloor$$
$$P_1 \neq P_2 := \neg(P_1 = P_2)$$
$$\bot := x_1 \land \neg x_1$$
$$\top := \neg\bot$$
$$P_1 \subseteq P_2 := \lfloor P_1 \to P_2 \rfloor$$
$$x \in P := \lceil x \land P \rceil$$

We will extend the grammar to a many-sorted one in the future.

2   Hilbert proof system

Axioms in $\mathcal{L}$ are given by the following nine axiom schemata where P, Q, R are arbitrary patterns and x, y are variables.

- (K1) $P \to (Q \to P)$

- (K2) $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

- (K3) $(\neg P \to \neg Q) \to (Q \to P)$

- (K4) $\forall x.(P \to Q) \to (P \to \forall x.Q)$ if x does not occur free in P

- (K5) $\exists y.x = y$

- (K6) $\exists y.Q = y \to (\forall x.P(x) \to P[Q/x])$ if Q is free for x in P

- (K7) $P_1 = P_2 \to (Q[P_1/x] \to Q[P_2/x])$

- (M1) $x \in y = (x = y)$

- (M2) $x \in \neg P = \neg(x \in P)$

- (M3) $x \in P \land Q = (x \in P) \land (x \in Q)$

- (M4) $x \in \exists y.P = \exists y.x \in P$ where x is distinct from y

- (M5) $x \in \sigma(\ldots, P_i, \ldots) = \exists y.y \in P_i \land x \in \sigma(\ldots, y, \ldots)$

Inference rules include

- (Modus Ponens) From P and $P \to Q$, deduce Q.

- (Universal Generalization) From P, deduce $\forall x.P$.

- (Membership Introduction) From P, deduce $\forall x.(x \in P)$, where x does not occur free in P.

- (Membership Elimination) From $\forall x.(x \in P)$, deduce P, where x does not occur free in P.

Theorem 1 (Soundness of $K_{\mathcal{L}}$). *Theorems of $K_{\mathcal{L}}$ are valid.*

Proof. Trivial. □

We provide some metatheorems of $K_{\mathcal{L}}$.

Proposition 2 (Tautology). *For any propositional tautology $\mathcal{A}(p_1, \ldots, p_n)$ where $p_1, \ldots, p_n$ are propositional variables,*

$$\vdash \mathcal{A}(P_1, \ldots, P_n).$$

Proof. Omit proof here. □

Remark Proposition 2 makes any metatheorem of propositional logic a metatheorem of $K_{\mathcal{L}}$.

Proposition 3 (Variable Substitution). $\vdash \forall x.P \rightarrow P[y/x]$.

Proposition 4 (Functional Substitution). $\vdash \exists y.(Q = y) \rightarrow (P[Q/x] \rightarrow \exists x.P(x))$.

Proposition 5 ($\vee$-Introduction). $\vdash P$ *implies* $\vdash P \vee Q$.

Proof. Use Proposition 2 and Modus Ponens. Note that in general, $\vdash P \vee Q$ does not imply $\vdash P$ or $\vdash Q$. □

Proposition 6 ($\wedge$-Introduction and Elimination). $\vdash P$ *and* $\vdash Q$ *iff* $\vdash P \wedge Q$.

Proof. Use Proposition 2 and Modus Ponens. □

Proposition 7 (Equality Introduction). $\vdash P = P$.

Proof. Use Membership Introduction and Proposition 2. □

Proposition 8 (Equality Replacement). $\vdash P_1 = P_2$ *and* $\vdash Q[P_1/x]$ *implies* $\vdash Q[P_2/x]$.

Proof. Use Axiom (K7) and Modus Ponens. □

Proposition 9 (Equality Establishment). $\vdash P \leftrightarrow Q$ *implies* $\vdash P = Q$.

Proof. Use Membership Axoims and $\vee$-Introduction. □

Corollary 10. $\vdash P$ *implies* $\vdash P = \top$.

Proposition 11. $\vdash x \in \lceil y \rceil$.

Proof.

$$\vdash x \in \lceil y \rceil$$
$$\text{if } \vdash \forall x.(x \in \lceil y \rceil) \qquad (K5, K6, \text{ and Modus Ponens})$$
$$\text{iff } \vdash \lceil y \rceil.$$

$\square$

Proposition 12. $\vdash P \rightarrow \lceil P \rceil$.

Proof.

$$\vdash P \rightarrow \lceil P \rceil$$
$$\text{iff } \vdash \forall x.(x \in P \rightarrow \lceil P \rceil)$$
$$\text{if } \vdash x \in P \rightarrow \lceil P \rceil$$
$$\text{iff } \vdash x \in P \rightarrow x \in \lceil P \rceil$$
$$\text{iff } \vdash x \in P \rightarrow \exists y.(y \in P \land x \in \lceil y \rceil)$$
$$\text{iff } \vdash x \in P \rightarrow \neg \forall y.(y \notin P \lor x \notin \lceil y \rceil)$$
$$\text{iff } \vdash \forall y.(y \notin P \lor x \notin \lceil y \rceil) \rightarrow x \notin P$$
$$\text{if } \vdash x \notin P \lor x \notin \lceil x \rceil \rightarrow x \notin P$$
$$\text{iff } \vdash x \in P \rightarrow x \in P \land x \in \lceil x \rceil$$
$$\text{iff } \vdash x \in P \rightarrow x \in \lceil x \rceil$$
$$\text{if } \vdash x \in \lceil x \rceil$$

Remark Similarly we can show $\vdash \lfloor P \rfloor \rightarrow P$. $\square$

Proposition 13. $\vdash \forall x.(x \in P) = \lfloor P \rfloor$, *where x occurs free in P.*

Proof. By Proposition 9 and 6, it suffices to show

$$\vdash \forall x.(x \in P) \rightarrow \lfloor P \rfloor \qquad (1)$$

and

$$\vdash \lfloor P \rfloor \rightarrow \forall x.(x \in P). \qquad (2)$$

To show (1),

$$\vdash \forall x.(x \in P) \rightarrow \lfloor P \rfloor$$
$$\text{iff} \vdash \forall x.\lceil x \land P \rceil \rightarrow \neg\lceil \neg P \rceil$$
$$\text{iff} \vdash \lceil \neg P \rceil \rightarrow \exists x.\neg\lceil x \land P \rceil$$
$$\text{iff} \vdash \forall y.(y \in (\lceil \neg P \rceil \rightarrow \exists x.\neg\lceil x \land P \rceil))$$
$$\text{if} \vdash y \in (\lceil \neg P \rceil \rightarrow \exists x.\neg\lceil x \land P \rceil$$
$$\text{iff} \vdash \exists z_1.(z_1 \notin P \land y \in \lceil z_1 \rceil) \rightarrow$$
$$\exists x.\neg(\exists z_2.(z_2 = x \land z_2 \in P \land y \in \lceil z_2 \rceil))$$
$$\text{iff} \vdash \exists z_1.(z_1 \notin P \land \top) \rightarrow \qquad \text{(Proposition 11, 8, and Corollary 10)}$$
$$\exists x.\neg(\exists z_2.(z_2 = x \land z_2 \in P \land \top))$$
$$\text{iff} \vdash \exists z_1.(z_1 \notin P) \rightarrow \exists x.\neg(\exists z_2.(z_2 = x \land z_2 \in P))$$
$$\text{iff} \vdash \forall x.(\exists z_2.(z_2 = x \land z_2 \in P)) \rightarrow \forall z_1.(z_1 \in P)$$
$$\text{if} \vdash \forall z_1.(\forall x.(\exists z_2.(z_2 = x \land z_2 \in P)) \rightarrow (z_1 \in P))$$
$$\text{if} \vdash \forall x.(\exists z_2.(z_2 = x \land z_2 \in P)) \rightarrow (z_1 \in P).$$

Since $\vdash \forall x.(\exists z_2.(z_2 = x \land z_2 \in P)) \rightarrow \exists z_2.(z_2 = z_1 \land z_2 \in P)$, it suffices to show

$$\vdash \exists z_2.(z_2 = z_1 \land z_2 \in P) \rightarrow (z_1 \in P)$$
$$\text{iff} \vdash z_1 \notin P \rightarrow \forall z_2.(z_2 \neq z_1 \lor z_2 \notin P)$$
$$\text{if} \vdash \forall z_2.(z_1 \notin P \rightarrow z_2 \neq z_1 \lor z_2 \notin P)$$
$$\text{if} \vdash z_1 \notin P \rightarrow z_2 \neq z_1 \lor z_2 \notin P$$
$$\text{if} \vdash z_2 = z_1 \land z_2 \in P \rightarrow z_1 \in P.$$

And we proved (1).

Similarly, to show (2),

$$\vdash \lfloor P \rfloor \rightarrow \forall x.(x \in P)$$
$$\text{iff} \vdash \exists x.\neg\lceil x \land P \rceil \rightarrow \lceil \neg P \rceil$$
$$\text{iff} \vdash \forall y.(y \in \exists x.\neg\lceil x \land P \rceil \rightarrow \lceil \neg P \rceil)$$
$$\text{if} \vdash y \in \exists x.\neg\lceil x \land P \rceil \rightarrow \lceil \neg P \rceil$$
$$\text{iff} \vdash \exists x.\neg\exists z_2.(z_2 = x \land z_2 \in P) \rightarrow \exists z_1.(z_1 \notin P)$$
$$\text{iff} \vdash \forall z_1.(z_1 \in P) \rightarrow \exists z_2.(z_2 = x \land z_2 \in P)$$
$$\text{iff} \vdash x \in P \rightarrow \exists z_2.(z_2 = x \land z_2 \in P).$$

We proved (2).

Remark If x occurs free in P, the result does not hold. For example, let P be $upto(x)$ where $upto(\cdot)$ is interpreted to $upto(n) = \{0, 1, \dots, n\}$ on $\mathbb{N}$. □

Remark From Membership Introduction and Elimination inference rules and Proposition 13, $\vdash P$ iff $\vdash \lfloor P \rfloor$.

Proposition 14 (Classification Reasoning). *For any P and Q, from $\vdash P \rightarrow Q$ and $\vdash \neg P \rightarrow Q$ deduce $\vdash Q$.*

Proof. From $\vdash \neg P \rightarrow Q$ deduce $\vdash \neg Q \rightarrow P$. Notice that $\vdash P \rightarrow Q$, so we have $\vdash \neg Q \rightarrow Q$, i.e., $\vdash \neg\neg Q \lor Q$ which concludes the proof. $\qquad\square$

Corollary 15. *For any P_1, P_2, and Q are patterns with $\vdash P_1 \lor P_2$, from $\vdash P_1 \rightarrow Q$ and $\vdash P_2 \rightarrow Q$, deduce $\vdash Q$.*

Definition 16 (Predicate Pattern). *A pattern P is called a predicate pattern or a predicate if $\vdash (P = \top) \lor (P = \bot)$.*

Remark Predicate patterns are closed under all logic connectives.

Remark For any P, $\lceil P \rceil$ is a predicate pattern.

Proposition 17. $\vdash (\lfloor P \rfloor = \neg\top) = (P = \bot)$.

Proof. It suffices to prove

$$\vdash (\lfloor P \rfloor \rightarrow \neg\top) = (P = \bot) \tag{3}$$

and

$$\vdash (P = \bot) \rightarrow (\lfloor P \rfloor = \neg\top) \tag{4}$$

Let us first prove (4). $\qquad\square$

Proposition 18. *For any predicate P, $\vdash (P \neq \top) = (P = \bot)$ and $\vdash (P \neq \bot) = (P = \top)$.*

Proposition 19. *For any pattern Q and any predicate pattern P, $\vdash P \lor Q$ iff $\vdash P \lor \lfloor Q \rfloor$.*

Proof. ($\Rightarrow$) is obtained immediately by the remark of Proposition 12. We now prove ($\Leftarrow$).

Because $\vdash Q = \top \lor Q \neq \top$, it suffices to show

$$\vdash Q = \top \rightarrow (P \lor \lfloor Q \rfloor = \top) \tag{5}$$

and

$$\vdash Q \neq \top \rightarrow (P \lor \lfloor Q \rfloor = \top) \tag{6}$$

by Corollary 15, and the fact that $\vdash P \lor \lfloor Q \rfloor = \top$ and $\vdash \top$ imply $\vdash P \lor \lfloor Q \rfloor$.

The proof of (5) is straightforward as follows.

$$
\begin{aligned}
&\quad \vdash Q = \top \rightarrow (P \lor \lfloor Q \rfloor = \top) \\
\text{if}\quad &\quad \vdash Q = \top \rightarrow (P \lor \lfloor \top \rfloor = \top) \\
\text{if}\quad &\quad \vdash Q = \top \rightarrow (\top = \top) \\
\text{if}\quad &\quad \vdash \top.
\end{aligned}
$$

The proof of (6) needs more effort. We first show that $\vdash (Q \neq \top) = (\lfloor Q \rfloor = \bot)$. Since $\lfloor Q \rfloor$ is a predicate pattern, it suffices to show $\vdash (Q = \top) = (\lfloor Q \rfloor = \top)$.

It is trivial to show $\vdash (Q = \top) \to (\lfloor Q \rfloor = \top)$. We show the other direction $\vdash (\lfloor Q \rfloor = \top) \to (Q = \top)$ through the following backward reasoning.
have the backward reasoning as follows.

$$\vdash Q \neq \top \to (P \vee \lfloor Q \rfloor = \top)$$
$$\text{if} \quad \vdash$$

$\square$

Proposition 20 (Deduction Theorem). *If $\Gamma \cup \{P\} \vdash Q$ and the derivation does not use $\forall x$-Generalization where x is free in P, then $\Gamma \vdash \lfloor P \rfloor \to Q$.*

Proof. The proof is by induction on n, the length of the derivation of Q from $\Gamma \cup \{P\}$.

Base step: $n = 1$, and Q is an axiom, or P, or a member of Γ. If Q is an axiom or a member of Γ, then $\Gamma \vdash Q$ and as a result, $\Gamma \vdash \lfloor P \rfloor \to Q$. If Q is P, then $\Gamma \vdash \lfloor P \rfloor \to Q$ by Proposition 12.

Induction step: Let $n > 1$. Suppose that if P' can be deduced from $\Gamma \cup \{P\}$ without using $\forall x$-Generalization where x is free in P, in a derivation containing fewer than n steps, then $\Gamma \vdash \lfloor P \rfloor \to P'$.

Case 1: Q is an axiom, or P, or a member of Γ. Precisely as in the Base step, we show that $\vdash \lfloor P \rfloor \to Q$.

Case 2: Q follows from two previous patterns in the derivation by an application of Modus Ponens. These two patterns must have the forms Q_1 and $Q_1 \to Q$, and each one can certainly be deduced from $\Gamma \cup \{P\}$ by a derivation with fewer than n steps, by just omitting the subsequent members from the original derivation from $\Gamma \cup \{P\} \vdash Q$. So we have $\Gamma \cup \{P\} \vdash Q_1$ and $\Gamma \cup \{P\} \vdash Q_1 \to Q$, and, applying the hypothesis of induction, $\Gamma \vdash \lfloor P \rfloor \to Q_1$ and $\Gamma \vdash \lfloor P \rfloor \to (Q_1 \to Q)$. It follows immediately that $\Gamma \vdash \lfloor P \rfloor \to Q$.

Case 3: Q follows from a previous pattern in the derivation by an application of $\forall x_i$-Generalization where x_i does not occur free in P. So Q is $\forall x_i.Q_1$, say, and Q_1 appears previously in the derivation. Thus $\Gamma \cup \{P\} \vdash Q_1$, and the derivation has fewer than n steps, so $\Gamma \vdash \lfloor P \rfloor \to Q_1$, since there is no application of Universal Generalization involving a free variable of P. Also x_i cannot occur free in P, as it is involved in an application of Universal Generalization in the deduction of Q from $\Gamma \cup \{P\}$. So we have a derivation of $\Gamma \vdash \lfloor P \rfloor \to Q$ as follows.

$$\Gamma \vdash \lfloor P \rfloor \to Q$$
$$\text{iff} \quad \Gamma \vdash \lfloor P \rfloor \to \forall x_i.Q_1$$
$$\text{if} \quad \Gamma \vdash \forall x_i.(\lfloor P \rfloor \to Q_1)$$
$$\text{if} \quad \Gamma \vdash \lfloor P \rfloor \to Q_1.$$

So $\Gamma \vdash \lfloor P \rfloor \to Q$ as required.

Case 4: Q follows from a previous pattern in the derivation by an application of Membership Introduction. So Q is $\forall x_i.(x_i \in Q_1)$ with x_i is free in Q_1, say, and Q_1 appears previously in the derivation. Thus $\Gamma \cup \{P\} \vdash Q_1$, and the derivation has fewer

than n steps, so $\Gamma \vdash \lfloor P \rfloor \to Q_1$, since there is no application of Universal Generalization involving a free variable of P. So we have a derivation of $\Gamma \vdash \lfloor P \rfloor \to Q$ as follows.

$$\Gamma \vdash \lfloor P \rfloor \to Q$$
$$\text{iff} \quad \Gamma \vdash \lfloor P \rfloor \to \forall x_i.(x_i \in Q_1)$$
$$\text{iff} \quad \Gamma \vdash \lfloor P \rfloor \to \lfloor Q_1 \rfloor,$$

which follows by the hypothesis of induction $\Gamma \vdash \lfloor P \rfloor \to Q_1$ and the fact that $\Gamma \vdash Q_1 \to \lfloor Q_1 \rfloor$ (by the Remark in Proposition 12).

Case 5: Q follows from a previous pattern in the derivation by an application of Membership Elimination. The previous pattern must have the form $\forall x_i.(x_i \in Q)$, and can be deduced from $\Gamma \cup \{P\}$ by a derivation with fewer than n steps, by just omitting the subsequent members from the original derivation from $\Gamma \cup \{P\} \vdash Q$. So we have $\Gamma \cup \{P\} \vdash \forall x_i.(x_i \in Q)$, and, applying the hypothesis of induction, $\Gamma \vdash \lfloor P \rfloor \to \forall x_i.(x_i \in Q)$. So we have a derivation of $\Gamma \vdash \lfloor P \rfloor \to Q$ as follows.

$$\Gamma \vdash \lfloor P \rfloor \to Q$$
$$\text{iff} \quad \Gamma \vdash \neg\lfloor P \rfloor \vee Q$$
$$\text{iff} \quad \Gamma \vdash \neg\lfloor P \rfloor \vee \lfloor Q \rfloor \qquad \text{(Proposition 19)}$$
$$\text{iff} \quad \Gamma \vdash \neg\lfloor P \rfloor \vee \forall x_i.(x_i \in Q)$$
$$\text{iff} \quad \Gamma \vdash \lfloor P \rfloor \to \forall x_i.(x_i \in Q),$$

which is the hypothesis of induction. And this concludes our inductive proof. $\qquad\square$

Corollary 21 (Closed-form Deduction Theorem). *If P is closed, $\Gamma \cup \{P\} \vdash Q$ implies $\Gamma \vdash \lfloor P \rfloor \to Q$.*

3   Inference rules

Axioms

$$\frac{\cdot}{\Gamma \vdash A}$$

where A is an axiom.

Inclusion

$$\frac{\cdot}{\Gamma \vdash P}$$

where $P \in \Gamma$.

Modus Ponens

$$\frac{\Gamma \vdash Q \to P \quad \Gamma \vdash Q}{\Gamma \vdash P}$$

Closed-Form Deduction Theorem

$$\frac{\Gamma \cup \{P\} \vdash Q}{\Gamma \vdash P \to Q}$$

where P is closed.

Universal Generalization

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P} \ (\forall x)$$

Conjunction Splitting

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$$