Towards an Efficient and Economic Deductive System of Matching Logic

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We aim for a Hilbert style deductive system which has a relatively large number of axioms but only a few inference rules.

1 Grammar and extended grammar

The formal language $\mathcal{L}$ we use to write matching logic patterns is defined as follows.

$$
\begin{aligned}
P ::= {} & x \\
& \mid P_1 \to P_2 \\
& \mid \neg P \\
& \mid \forall x.P \\
& \mid \sigma(P_1, \ldots, P_n) \\
& \mid P_1 = P_2 \\
(***\ & \texttt{extended} ***) \\
& \mid P_1 \vee P_2 \\
& \mid P_1 \wedge P_2 \\
& \mid P_1 \leftrightarrow P_2 \\
& \mid \exists x.P \\
& \mid P_1 \neq P_2 \\
& \mid \top \\
& \mid \bot \\
& \mid \lceil P \rceil \\
& \mid \lfloor P \rfloor \\
& \mid P_1 \subseteq P_2 \\
& \mid x \in P
\end{aligned}
$$

with the extended grammar defined as

$$
\begin{aligned}
&P_1 \vee P_2 := \neg P_2 \to P_1 \\
&P_1 \wedge P_2 := \neg(\neg P_1 \vee \neg P_2) \\
&P_1 \leftrightarrow P_2 := (P_1 \to P_2) \wedge (P_2 \to P_1) \\
&\exists x.P := \neg \forall x.\neg P \\
&P_1 \neq P_2 := \neg(P_1 = P_2) \\
&\top := x_1 = x_1 \\
&\bot := x_1 \neq x_1 \\
&\lceil P \rceil := P \neq \bot \\
&\lfloor P \rfloor := P = \top \\
&P_1 \subseteq P_2 := \lfloor P_1 \to P_2 \rfloor \\
&x \in P := x \subseteq P
\end{aligned}
$$

2 Hilbert proof system

Axioms in $\mathcal{L}$ are given by the following nine axiom schemata where P, Q, R are arbitrary patterns and x, y are variables.

- (K1) $P \to (Q \to P)$
- (K2) $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$
- (K3) $(\neg P \to \neg Q) \to (Q \to P)$
- (K4) $\forall x.(P \to Q) \to (P \to \forall x.Q)$ if x does not occur free in P
- (K5) $\forall x.P \to P$ if x does not occur free in P
- (K6) $\forall x.P(x) \to P(y)$
- (K7) $P = P$
- (K8) $P_1 = P_2 \to (Q[P_1/x] \to Q[P_2/x])$
- (K9) $\exists y.Q = y \to (\forall x.P(x) \to P[Q/x])$ if Q is free for x in P

Inference rules include

- (Modus Ponens) From P and $P \to Q$, deduce Q.
- (Universal Generalization) From P, deduce $\forall x.P$.

Proposition 1 (Deduction Theorem)**.** *If $\Gamma \cup \{P\} \vdash Q$ and the proof does not use $\forall x$-Generalization where x is free in P, then $\Gamma \vdash P \to Q$.*

Proposition 2 (Tautology)**.** *For any tautology $\mathcal{A}(p_1, \ldots, p_n)$ where $p_1, \ldots, p_n$ are propositional variables,*

$$\vdash \mathcal{A}(P_1, \ldots, P_n).$$

Proposition 3 (More Theorems in $\mathcal{L}$)**.**

$$\vdash \exists x.x$$
$$\vdash \lceil x \rceil$$
$$\vdash \exists y.x = y$$
$$\vdash P_1 = P_2 \rightarrow Q[P_1/x] = Q[P_2/x]$$

3 Inference rules

Axioms

$$\frac{.}{\Gamma \vdash A}$$

where A is an axiom.

Inclusion

$$\frac{.}{\Gamma \vdash P}$$

where $P \in \Gamma$.

Modus Ponens

$$\frac{\Gamma \vdash Q \rightarrow P \quad \Gamma \vdash Q}{\Gamma \vdash P}$$

Closed-Form Deduction Theorem

$$\frac{\Gamma \cup \{P\} \vdash Q}{\Gamma \vdash P \rightarrow Q}$$

where P is closed.

Universal Generalization

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P}\ (\forall x)$$

Conjunction Splitting

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$$

Examples of proof.sty

`\infer` draws beautiful proof figures easily:

(1)

$$\frac{\dfrac{B11\ \&\ B12\ \&\ B13 \quad B21\ \&\ B22\ \&\ B23}{B} \qquad C}{A}$$

(2)

$$\frac{\dfrac{B11 \mathbin{\&} B12 \mathbin{\&} B13 \qquad B21 \mathbin{\&} B22 \mathbin{\&} B23}{B} \qquad C}{A1 \mathbin{\&} A2 \mathbin{\&} A3 \mathbin{\&} A4 \mathbin{\&} A5 \mathbin{\&} A6}$$

(3)

$$\frac{C \qquad \dfrac{B11 \mathbin{\&} B12 \mathbin{\&} B13 \qquad B21 \mathbin{\&} B22 \mathbin{\&} B23}{B}}{A1 \mathbin{\&} A2 \mathbin{\&} A3 \mathbin{\&} A4 \mathbin{\&} A5 \mathbin{\&} A6}$$

You can use also some variations:

(4)

$$\frac{\begin{matrix} B11 \mathbin{\&} B12 \mathbin{\&} B13 \quad B21 \mathbin{\&} B22 \mathbin{\&} B23 \\ \vdots \\ B \end{matrix} \qquad C}{A}\ (1)$$

(5)

$$\begin{array}{c} \begin{matrix} B11 \mathbin{\&} B12 \mathbin{\&} B13 \quad B21 \mathbin{\&} B22 \mathbin{\&} B23 \\ \sum \\ B \end{matrix} \qquad \begin{matrix} (2) \\ C \end{matrix} \\ \vdots\ (1) \\ A1 \mathbin{\&} A2 \mathbin{\&} A3 \mathbin{\&} A4 \mathbin{\&} A5 \mathbin{\&} A6 \end{array}$$

(6)

$$\frac{\underline{A \mathbin{\&} B \mathbin{\&} C}}{A}$$

Here are more practical examples:

(7)

$$\frac{A \quad B}{A \mathbin{\&} B}\ (\&I) \qquad \frac{A \mathbin{\&} B}{A}\ (\&E_l) \qquad \frac{A \mathbin{\&} B}{B}\ (\&E_r)$$

$$\frac{\begin{matrix} [A] \\ \vdots \\ B \end{matrix}}{A \to B}\ (\to I) \qquad \frac{A \to B \quad A}{B}\ (\to E)$$

Some techniques: Use `\vcenter` for an equation of proofs.

(8)

$$\pi = \frac{A \qquad \dfrac{B \quad C}{D}}{E}$$

Use `\kern` to adjust the form of a proof.

(9)

$$\frac{A \quad \dfrac{B \quad C}{D}}{E}$$