Towards an Efficient and Economic Deductive System of Matching Logic

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We aim for a Hilbert style deductive system which has a relatively large number of axioms but only a few inference rules.

1 Grammar and extended grammar

The formal language $\mathcal{L}$ we use to write matching logic patterns is defined as follows.

$$
\begin{aligned}
P ::=\ & x \\
& \mid P_1 \to P_2 \\
& \mid \neg P \\
& \mid \forall x.P \\
& \mid \sigma(P_1, \ldots, P_n) \\
(***\ & \texttt{extended} ***) \\
& \mid P_1 \vee P_2 \\
& \mid P_1 \wedge P_2 \\
& \mid P_1 \leftrightarrow P_2 \\
& \mid \exists x.P \\
& \mid \lceil P \rceil \\
& \mid \lfloor P \rfloor \\
& \mid P_1 = P_2 \\
& \mid P_1 \neq P_2 \\
& \mid \top \\
& \mid \bot \\
& \mid P_1 \subseteq P_2 \\
& \mid x \in P
\end{aligned}
$$

with the extended grammar defined as

$$
\begin{aligned}
&P_1 \vee P_2 \coloneqq \neg P_2 \to P_1 \\
&P_1 \wedge P_2 \coloneqq \neg(\neg P_1 \vee \neg P_2) \\
&P_1 \leftrightarrow P_2 \coloneqq (P_1 \to P_2) \wedge (P_2 \to P_1) \\
&\exists x.P \coloneqq \neg\forall x.\neg P \\
&\lfloor P \rfloor \coloneqq \neg\lceil \neg P \rceil \\
&P_1 = P_2 \coloneqq \lfloor P_1 \leftrightarrow P_2 \rfloor \\
&P_1 \neq P_2 \coloneqq \neg(P_1 = P_2) \\
&\bot \coloneqq x_1 \wedge \neg x_1 \\
&\top \coloneqq \neg\bot \\
&P_1 \subseteq P_2 \coloneqq \lfloor P_1 \to P_2 \rfloor \\
&x \in P \coloneqq \lceil x \wedge P \rceil
\end{aligned}
$$

We will extend the grammar to a many-sorted one in the future.

2 Hilbert proof system

Axioms in $\mathcal{L}$ are given by the following nine axiom schemata where P, Q, R are arbitrary patterns and x, y are variables.

- (K1) $P \to (Q \to P)$
- (K2) $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$
- (K3) $(\neg P \to \neg Q) \to (Q \to P)$
- (K4) $\forall x.(P \to Q) \to (P \to \forall x.Q)$ if x does not occur free in P
- (K5) $\exists y.x = y$
- (K6) $\exists y.Q = y \to (\forall x.P(x) \to P[Q/x])$ if Q is free for x in P
- (K7) $P_1 = P_2 \to (Q[P_1/x] \to Q[P_2/x])$
- (M1) $x \in y = (x = y)$
- (M2) $x \in \neg P = \neg(x \in P)$
- (M3) $x \in P \wedge Q = (x \in P) \wedge (x \in Q)$
- (M4) $x \in \exists y.P = \exists y.x \in P$ where x is distinct from y
- (M5) $x \in \sigma(\ldots, P_i, \ldots) = \exists y.y \in P_i \wedge x \in \sigma(\ldots, y, \ldots)$

Inference rules include

- (Modus Ponens) From P and $P \to Q$, deduce Q.

- (Universal Generalization) From P, deduce $\forall x.P$.
- (Membership Introduction) From P, deduce $\forall x.x \in P$, where x occurs free in P.
- (Membership Elimination) From $\forall x.x \in P$, deduce P, where x occurs free in P.

Theorem 1 (Soundness of $K_{\mathcal{L}}$)**.** *Theorems of $K_{\mathcal{L}}$ are valid.*

Proof. Trivial. □

We provide some metatheorems of $K_{\mathcal{L}}$.

Proposition 2 (Tautology)**.** *For any propositional tautology $\mathcal{A}(p_1, \ldots, p_n)$ where $p_1, \ldots, p_n$ are propositional variables,*

$$\vdash \mathcal{A}(P_1, \ldots, P_n).$$

Proof. Omit proof here. □

Remark Proposition 2 makes any metatheorem of propositional logic a metatheorem of $K_{\mathcal{L}}$.

Proposition 3 (Variable Substitution)**.** $\vdash \forall x.P \to P[y/x]$.

Proposition 4 (Functional Substitution)**.** $\vdash \exists y.(Q = y) \to (P[Q/x] \to \exists x.P(x))$.

Proposition 5 ($\vee$-Introduction)**.** $\vdash P$ *implies* $\vdash P \vee Q$.

Proof. Use Proposition 2 and Modus Ponens. Note that in general, $\vdash P \vee Q$ does not imply $\vdash P$ or $\vdash Q$. □

Proposition 6 ($\wedge$-Introduction and Elimination)**.** $\vdash P$ *and* $\vdash Q$ *iff* $\vdash P \wedge Q$.

Proof. Use Proposition 2 and Modus Ponens. □

Proposition 7 (Equality Introduction)**.** $\vdash P = P$.

Proof. Use Membership Introduction and Proposition 2. □

Proposition 8 (Equality Replacement)**.** $\vdash P_1 = P_2$ *and* $\vdash Q[P_1/x]$ *implies* $\vdash Q[P_2/x]$.

Proof. Use Axiom (K7) and Modus Ponens. □

Proposition 9 (Equality Establishment)**.** $\vdash P \leftrightarrow Q$ *implies* $\vdash P = Q$.

Proof. Use Membership Axoims and $\vee$-Introduction. □

Corollary 10. $\vdash P$ *implies* $\vdash P = \top$.

Proposition 11. $\vdash x \in \lceil y \rceil$.

Proof.

$$
\begin{aligned}
&\vdash x \in \lceil y \rceil \\
\text{if } &\vdash \forall x.(x \in \lceil y \rceil) && (K5, K6, \text{and Modus Ponens}) \\
\text{iff } &\vdash \lceil y \rceil.
\end{aligned}
$$

□

Proposition 12. $\vdash P \to \lceil P \rceil$.

Proof.

$$
\begin{aligned}
&\vdash P \to \lceil P \rceil \\
\text{iff } &\vdash \forall x.(x \in P \to \lceil P \rceil) \\
\text{if } &\vdash x \in P \to \lceil P \rceil \\
\text{iff } &\vdash x \in P \to x \in \lceil P \rceil \\
\text{iff } &\vdash x \in P \to \exists y.(y \in P \land x \in \lceil y \rceil) \\
\text{iff } &\vdash x \in P \to \neg\forall y.(y \notin P \lor x \notin \lceil y \rceil) \\
\text{iff } &\vdash \forall y.(y \notin P \lor x \notin \lceil y \rceil) \to x \notin P \\
\text{if } &\vdash x \notin P \lor x \notin \lceil x \rceil \to x \notin P \\
\text{iff } &\vdash x \in P \to x \in P \land x \in \lceil x \rceil \\
\text{iff } &\vdash x \in P \to x \in \lceil x \rceil \\
\text{if } &\vdash x \in \lceil x \rceil
\end{aligned}
$$

Remark Similarly we can show $\vdash \lfloor P \rfloor \to P$. □

Proposition 13. $\vdash \forall x.(x \in P) = \lfloor P \rfloor$, *where x occurs free in P.*

Proof. By Proposition 9 and 6, it suffices to show

$$\vdash \forall x.(x \in P) \to \lfloor P \rfloor \tag{1}$$

and

$$\vdash \lfloor P \rfloor \to \forall x.(x \in P). \tag{2}$$

To show (1),

$$
\begin{aligned}
&\vdash \forall x.(x \in P) \to \lfloor P \rfloor \\
\text{iff } &\vdash \forall x.\lceil x \wedge P \rceil \to \neg\lceil \neg P \rceil \\
\text{iff } &\vdash \lceil \neg P \rceil \to \exists x.\neg\lceil x \wedge P \rceil \\
\text{iff } &\vdash \forall y.(y \in (\lceil \neg P \rceil \to \exists x.\neg\lceil x \wedge P \rceil)) \\
\text{if } &\vdash y \in (\lceil \neg P \rceil \to \exists x.\neg\lceil x \wedge P \rceil \\
\text{iff } &\vdash \exists z_1.(z_1 \notin P \wedge y \in \lceil z_1 \rceil) \to \\
&\qquad \exists x.\neg(\exists z_2.(z_2 = x \wedge z_2 \in P \wedge y \in \lceil z_2 \rceil)) \\
\text{iff } &\vdash \exists z_1.(z_1 \notin P \wedge \top) \to \qquad \text{(Proposition 11, 8, and Corollary 10)} \\
&\qquad \exists x.\neg(\exists z_2.(z_2 = x \wedge z_2 \in P \wedge \top)) \\
\text{iff } &\vdash \exists z_1.(z_1 \notin P) \to \exists x.\neg(\exists z_2.(z_2 = x \wedge z_2 \in P)) \\
\text{iff } &\vdash \forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \to \forall z_1.(z_1 \in P) \\
\text{if } &\vdash \forall z_1.(\forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \to (z_1 \in P)) \\
\text{if } &\vdash \forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \to (z_1 \in P).
\end{aligned}
$$

Since $\vdash \forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \to \exists z_2.(z_2 = z_1 \wedge z_2 \in P)$, it suffices to show

$$
\begin{aligned}
&\vdash \exists z_2.(z_2 = z_1 \wedge z_2 \in P) \to (z_1 \in P) \\
\text{iff } &\vdash z_1 \notin P \to \forall z_2.(z_2 \neq z_1 \vee z_2 \notin P) \\
\text{if } &\vdash \forall z_2.(z_1 \notin P \to z_2 \neq z_1 \vee z_2 \notin P) \\
\text{if } &\vdash z_1 \notin P \to z_2 \neq z_1 \vee z_2 \notin P \\
\text{if } &\vdash z_2 = z_1 \wedge z_2 \in P \to z_1 \in P.
\end{aligned}
$$

And we proved (1).

Similarly, to show (2),

$$
\begin{aligned}
&\vdash \lfloor P \rfloor \to \forall x.(x \in P) \\
\text{iff } &\vdash \exists x.\neg\lceil x \wedge P \rceil \to \lceil \neg P \rceil \\
\text{iff } &\vdash \forall y.(y \in \exists x.\neg\lceil x \wedge P \rceil \to \lceil \neg P \rceil) \\
\text{if } &\vdash y \in \exists x.\neg\lceil x \wedge P \rceil \to \lceil \neg P \rceil \\
\text{iff } &\vdash \exists x.\neg\exists z_2.(z_2 = x \wedge z_2 \in P) \to \exists z_1.(z_1 \notin P) \\
\text{iff } &\vdash \forall z_1.(z_1 \in P) \to \exists z_2.(z_2 = x \wedge z_2 \in P) \\
\text{iff } &\vdash x \in P \to \exists z_2.(z_2 = x \wedge z_2 \in P).
\end{aligned}
$$

We proved (2).

Remark If x occurs free in P, the result does not hold. For example, let P be $upto(x)$ where $upto(\cdot)$ is interpreted to $upto(n) = \{0, 1, \ldots, n\}$ on $\mathbb{N}$. □

Proposition 14 (Deduction Theorem)**.** *If $\Gamma \cup \{P\} \vdash Q$ and the proof does not use $\forall x$-Generalization where x is free in P, then $\Gamma \vdash P \to Q$. In particular, when P is closed, $\Gamma \cup \{P\} \vdash Q$ implies $\Gamma \vdash P \to Q$.*

Proposition 15 (More Theorems in $\mathcal{L}$)**.**

$$\vdash \exists x.x$$
$$\vdash \lceil x \rceil$$
$$\vdash \exists y.x = y$$
$$\vdash P_1 = P_2 \to Q[P_1/x] = Q[P_2/x]$$

Proof. (1)

$$\dfrac{\dfrac{\cdot}{\vdash \neg(x \to \neg x)}\,(\text{TAUT}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x.\neg x \vdash \neg x \quad \forall x.\neg x \vdash \neg x}{\forall x.\neg x \vdash \neg x} \qquad \dfrac{\cdot}{\forall x.\neg x \vdash \neg x \to (x \to \neg x)}\,(\text{K1})}{\forall x.\neg x \vdash x \to \neg x}\,(\text{MP})}{\vdash (\forall x.\neg x) \to (x \to \neg x)}\,(\text{DEDUCT})}{\vdash \neg(x \to \neg x) \to \neg\forall x.\neg x}\,(\text{MP\&K3})}{\dfrac{\vdash \neg\forall x.\neg x}{\vdash \exists x.x}\,(\text{DEF})}\,(\text{MP})$$

(3)

$$\dfrac{\dfrac{\dfrac{\cdot}{x = x}\,(\text{K7})}{\vdash \neg\neg(x = x)}\,(\text{MP\&TAUT}) \qquad \dfrac{\dfrac{\cdot}{\vdash \forall y.\neg(x = y) \to \neg(x = x)}\,(\text{K6})}{\vdash \neg\neg(x = x) \to \neg\forall y.\neg(x = y)}\,(\text{MP\&K3})}{\dfrac{\vdash \neg\forall y.\neg(x = y)}{\vdash \exists y.x = y}\,(\text{DEF})}\,(\text{MP})$$

□

Before we prove the adequacy theorem of $K_{\mathcal{L}}$, we prove some lemmas.

Proposition 16. *If a pattern is valid, then its closure is valid.*

Proposition 17. *If a pattern's closure is a theorem of $K_{\mathcal{L}}$, then itself is a theorem of $K_{\mathcal{L}}$, too.*

Proposition 18. *If P is not a theorem of $K_{\mathcal{L}}$, then $K_{\mathcal{L}}$ extended with adding $\neg P$ as an axiom is consistent.*

Proposition 19. *If S is a consistent extended system of $K_{\mathcal{L}}$, then for any theorem P of S, there exists a model M and an M-evaluation ρ such that $M, \rho \vDash P$.*

3 Inference rules

Axioms

$$\dfrac{\cdot}{\Gamma \vdash A}$$

where A is an axiom.

Inclusion

$$\dfrac{\cdot}{\Gamma \vdash P}$$

where $P \in \Gamma$.

Modus Ponens

$$\frac{\Gamma \vdash Q \to P \quad \Gamma \vdash Q}{\Gamma \vdash P}$$

Closed-Form Deduction Theorem

$$\frac{\Gamma \cup \{P\} \vdash Q}{\Gamma \vdash P \to Q}$$

where P is closed.

Universal Generalization

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P}\ (\forall x)$$

Conjunction Splitting

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$$