Towards an Efficient and Economic Deductive System of Matching Logic

FSL group

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We aim for a Hilbert style deductive system which has a relatively large number of axioms but only a few inference rules.

1 Grammar and extended grammar

$$
\begin{aligned}
P ::= {} & x \\
& \mid P_1 \to P_2 \\
& \mid \neg P \\
& \mid \forall x.P \\
& \mid \sigma(P_1, \ldots, P_n) \\
& \mid P_1 = P_2 \\
(*** & \ \texttt{extended} ***) \\
& \mid P_1 \vee P_2 \\
& \mid P_1 \wedge P_2 \\
& \mid P_1 \leftrightarrow P_2 \\
& \mid \exists x.P \\
& \mid \top \\
& \mid \bot \\
& \mid \lceil P \rceil \\
& \mid \lfloor P \rfloor \\
& \mid P_1 \subseteq P_2
\end{aligned}
$$

2 Hilbert proof system

- (K1) $P \to (Q \to P)$
- (K2) $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

- (K3) $(\neg P \to \neg Q) \to (Q \to P)$
- (K4) $\forall x.(P \to Q) \to (P \to \forall x.Q)$ if x does not occur free in P
- (K5) $\forall x.P \to P$ if x does not occur free in P
- (K6) $\forall x.P(x) \to P(y)$
- (K7) $P = P$
- (K8) $P_1 = P_2 \to (Q[P_1/x] \to Q[P_2/x])$
- (K9) $\exists y.Q = y \to (\forall x.P(x) \to P[Q/x])$ if Q is free for x in P

  Inference rules include

- (Modus Ponens) From P and $P \to Q$, deduce Q.
- (Universal Generalization) From P, deduce $\forall x.P$.

Proposition 1 (Deduction Theorem)**.** *If $\Gamma \cup \{P\} \vdash Q$ and the proof does not use $\forall x$-Generalization where x is free in P, then $\Gamma \vdash P \to Q$.*

3 Inference rules

Axioms

$$\frac{\cdot}{\Gamma \vdash A}$$

where A is an axiom.

Modus Ponens

$$\frac{\Gamma \vdash Q \to P \quad \Gamma \vdash Q}{\Gamma \vdash P}$$

Universal Generalization

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P}\ (\forall x)$$

Closed-Form Deduction Theorem

$$\frac{\Gamma \cup \{P\} \vdash Q}{\Gamma \vdash P \to Q}$$

where P is closed.

Inclusion

$$\frac{\cdot}{\Gamma \vdash P}$$

where $P \in \Gamma$.

Conjunction Introduction

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$$

Examples of proof.sty

`\infer` draws beautiful proof figures easily:

(1)

$$\frac{\dfrac{B11 \mathbin{\&} B12 \mathbin{\&} B13 \quad B21 \mathbin{\&} B22 \mathbin{\&} B23}{B} \qquad C}{A}$$

(2)

$$\frac{\dfrac{B11 \mathbin{\&} B12 \mathbin{\&} B13 \quad B21 \mathbin{\&} B22 \mathbin{\&} B23}{B} \qquad C}{A1 \mathbin{\&} A2 \mathbin{\&} A3 \mathbin{\&} A4 \mathbin{\&} A5 \mathbin{\&} A6}$$

(3)

$$\frac{C \qquad \dfrac{B11 \mathbin{\&} B12 \mathbin{\&} B13 \quad B21 \mathbin{\&} B22 \mathbin{\&} B23}{B}}{A1 \mathbin{\&} A2 \mathbin{\&} A3 \mathbin{\&} A4 \mathbin{\&} A5 \mathbin{\&} A6}$$

You can use also some variations:

(4)

$$\frac{\begin{array}{c} B11 \mathbin{\&} B12 \mathbin{\&} B13 \quad B21 \mathbin{\&} B22 \mathbin{\&} B23 \\ \vdots \\ B \end{array} \qquad C}{A}\ (1)$$

(5)

$$\begin{array}{c} \begin{array}{c} B11 \mathbin{\&} B12 \mathbin{\&} B13 \quad B21 \mathbin{\&} B22 \mathbin{\&} B23 \\ \sum \\ B \end{array} \qquad \begin{array}{c} (2) \\ C \end{array} \\ \vdots\ (1) \\ A1 \mathbin{\&} A2 \mathbin{\&} A3 \mathbin{\&} A4 \mathbin{\&} A5 \mathbin{\&} A6 \end{array}$$

(6)

$$\begin{array}{c} A \mathbin{\&} B \mathbin{\&} C \\ \hline\hline A \end{array}$$

Here are more practical examples:

(7)

$$\frac{A \quad B}{A \mathbin{\&} B}\ (\&I) \qquad \frac{A \mathbin{\&} B}{A}\ (\&E_l) \qquad \frac{A \mathbin{\&} B}{B}\ (\&E_r)$$

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \to B}\ (\to I) \qquad \frac{A \to B \quad A}{B}\ (\to E)$$

Some techniques: Use `\vcenter` for an equation of proofs.

(8)

$$\pi = \dfrac{A \qquad \dfrac{B \quad C}{D}}{E}$$

Use `\kern` to adjust the form of a proof.

(9)

$$\dfrac{A \qquad \dfrac{B \quad C}{D}}{E}$$