

Towards an Efficient and Economic Deductive System of Matching Logic

FSL group

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We aim for a Hilbert style deductive system which has a relatively large number of axioms but only a few inference rules.

1 Grammar and extended grammar

The formal language \mathcal{L} we use to write matching logic patterns is defined as follows.

$$\begin{aligned} P ::= & x \\ & | P_1 \rightarrow P_2 \\ & | \neg P \\ & | \forall x.P \\ & | \sigma(P_1, \dots, P_n) \\ & | P_1 = P_2 \\ (** \text{ extended } **) \\ & | P_1 \vee P_2 \\ & | P_1 \wedge P_2 \\ & | P_1 \leftrightarrow P_2 \\ & | \exists x.P \\ & | P_1 \neq P_2 \\ & | \top \\ & | \perp \\ & | [P] \\ & | \lfloor P \rfloor \\ & | P_1 \subseteq P_2 \\ & | x \in P \end{aligned}$$

with the extended grammar defined as

$$\begin{aligned}
P_1 \vee P_2 &:= \neg P_2 \rightarrow P_1 \\
P_1 \wedge P_2 &:= \neg(\neg P_1 \vee \neg P_2) \\
P_1 \leftrightarrow P_2 &:= (P_1 \rightarrow P_2) \wedge (P_2 \rightarrow P_1) \\
\exists x.P &:= \neg \forall x. \neg P \\
P_1 \neq P_2 &:= \neg(P_1 = P_2) \\
\top &:= x_1 = x_1 \\
\perp &:= x_1 \neq x_1 \\
[P] &:= P \neq \perp \\
\lfloor P \rfloor &:= P = \top \\
P_1 \subseteq P_2 &:= \lfloor P_1 \rightarrow P_2 \rfloor \\
x \in P &:= x \subseteq P
\end{aligned}$$

2 Hilbert proof system

Axioms in \mathcal{L} are given by the following nine axiom schemata where P, Q, R are arbitrary patterns and x, y are variables.

- (K1) $P \rightarrow (Q \rightarrow P)$
- (K2) $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$
- (K3) $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$
- (K4) $\forall x.(P \rightarrow Q) \rightarrow (P \rightarrow \forall x.Q)$ if x does not occur free in P
- (K5) $\forall x.P \rightarrow P$ if x does not occur free in P
- (K6) $\forall x.P(x) \rightarrow P(y)$
- (K7) $P = P$
- (K8) $P_1 = P_2 \rightarrow (Q[P_1/x] \rightarrow Q[P_2/x])$
- (K9) $\exists y.Q = y \rightarrow (\forall x.P(x) \rightarrow P[Q/x])$ if Q is free for x in P

Inference rules include

- (Modus Ponens) From P and $P \rightarrow Q$, deduce Q .
- (Universal Generalization) From P , deduce $\forall x.P$.

Proposition 1 (Deduction Theorem). *If $\Gamma \cup \{P\} \vdash Q$ and the proof does not use $\forall x$ -Generalization where x is free in P , then $\Gamma \vdash P \rightarrow Q$.*

Proposition 2 (Tautology). *For any tautology $\mathcal{A}(p_1, \dots, p_n)$ where p_1, \dots, p_n are propositional variables,*

$$\vdash \mathcal{A}(P_1, \dots, P_n).$$

Proposition 3 (More Theorems in \mathcal{L}).

$$\begin{aligned} &\vdash \exists x.x \\ &\vdash [x] \\ &\vdash \exists y.x = y \\ &\vdash P_1 = P_2 \rightarrow Q[P_1/x] = Q[P_2/x] \end{aligned}$$

3 Inference rules

Axioms

$$\frac{\cdot}{\Gamma \vdash A}$$

where A is an axiom.

Inclusion

$$\frac{\cdot}{\Gamma \vdash P}$$

where $P \in \Gamma$.

Modus Ponens

$$\frac{\Gamma \vdash Q \rightarrow P \quad \Gamma \vdash Q}{\Gamma \vdash P}$$

Closed-Form Deduction Theorem

$$\frac{\Gamma \cup \{P\} \vdash Q}{\Gamma \vdash P \rightarrow Q}$$

where P is closed.

Universal Generalization

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P} (\forall x)$$

Conjunction Splitting

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$$

Examples of proof.sty

`\infer` draws beautiful proof figures easily:

(1)

$$\frac{\frac{B11 \ \& \ B12 \ \& \ B13 \quad B21 \ \& \ B22 \ \& \ B23}{B} \quad C}{A}$$

(2)

$$\frac{\frac{B11 \& B12 \& B13 \quad B21 \& B22 \& B23}{B} \quad C}{A1 \& A2 \& A3 \& A4 \& A5 \& A6}$$

(3)

$$\frac{C \quad \frac{B11 \& B12 \& B13 \quad B21 \& B22 \& B23}{B}}{A1 \& A2 \& A3 \& A4 \& A5 \& A6}$$

You can use also some variations:

(4)

$$\frac{B11 \& B12 \& B13 \quad \vdots \quad B21 \& B22 \& B23}{\frac{B}{A} \quad C} \quad (1)$$

(5)

$$\frac{B11 \& B12 \& B13 \quad B21 \& B22 \& B23}{\sum_B \quad C} \quad (2)$$

$$\frac{\vdots \quad (1)}{A1 \& A2 \& A3 \& A4 \& A5 \& A6}$$

(6)

$$\frac{A \& B \& C}{A}$$

Here are more practical examples:

(7)

$$\frac{A \quad B}{A \& B} \quad (\&I) \quad \frac{A \& B}{A} \quad (\&E_l) \quad \frac{A \& B}{B} \quad (\&E_r)$$

$$\frac{\frac{[A]}{\vdots} \quad B}{A \rightarrow B} \quad (\rightarrow I) \quad \frac{A \rightarrow B \quad A}{B} \quad (\rightarrow E)$$

Some techniques: Use `\vcenter` for an equation of proofs.

(8)

$$\pi = A \frac{B \quad C}{D} \quad E$$

Use `\kern` to adjust the form of a proof.

(9)

$$A \frac{B \quad C}{D} \quad E$$