

# Technical Report

## The Deduction System of Matching Logic

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### Abstract

Abstract goes here.

## 1 Syntax

Formulas of matching logic, called *patterns*, are written in a formal language, denoted as  $\mathcal{L}$ , whose grammar is listed in (1). The language  $\mathcal{L}$  is many-sorted. A signature of  $\mathcal{L}$  contains not only a finite set  $\Sigma$  of *symbols*, but also a finite nonempty set  $S$  of *sorts*. Each symbol  $\sigma \in \Sigma$  is, of course, sorted, with a fixed nonempty arity. We write  $\sigma \in \Sigma_{s_1, \dots, s_n, s}$  to emphasize that  $\sigma$  takes  $n$  arguments (with argument sorts  $s_1, \dots, s_n$ ) and returns a pattern in sort  $s$ , but we hope in most cases sorting is clear from context.

The grammar for  $\mathcal{L}$ , as defined below, is almost identical to first-order logic, except that in  $\mathcal{L}$  there is no difference between relational (predicate) and functional symbols, and we accept first-order terms as patterns in matching logic.

$$\begin{aligned} P ::= & x \\ & | P_1 \wedge P_2 \\ & | \neg P \\ & | \forall x. P \\ & | \sigma(P_1, \dots, P_n), \end{aligned} \tag{1}$$

where the universal quantifier ( $\forall x$ ) behaves the same as in first-order logic, with alpha-renaming always assumed.

For simplicity, we did not mention sorting in the grammar definition, and assume it should be clear to all readers. For example, in  $P_1 \wedge P_2$ , both patterns  $P_1$  and  $P_2$  should have the same sort, and that sort is the sort of  $P_1 \wedge P_2$ . The sort of  $\forall x. P$  is the sort of  $P$ , while the sort of variable  $x$  does not matter. To see why it is the case, consider the pattern  $\exists x. \text{list}(x, 1 \cdot 3 \cdot 5)$ , which is the set of all memory configurations that has a list  $(1, 3, 5)$  in it.

Propositional connectives are always assumed, including disjunction ( $\vee$ ), implication ( $\rightarrow$ ), and equivalence ( $\leftrightarrow$ ). Existential quantifier ( $\exists x$ ) is defined by universal quantifier ( $\forall x$ ) in the normal way. The bottom pattern ( $\perp_s$ ) and the top pattern ( $\top_s$ ) in sort  $s$  are given by  $x \wedge \neg x$  and  $\neg \perp_s$ , respectively, where  $x$  is a variable in sort  $s$ . It does not matter which variable we pick.

## 1.1 Extended Syntax

The formal language  $\mathcal{L}$  is often extended with *definedness* symbols. For  $s_1, s_2$  are two sorts, the definedness symbol  $\llbracket \_ \rrbracket_{s_1}^{s_2} \in \Sigma_{s_1, s_2}$  is a unary symbol with one argument sort  $s_1$  and the result sort  $s_2$ . For a pattern  $P$  who has sort  $s_1$ , the pattern  $\llbracket \_ \rrbracket_{s_1}^{s_2}(P)$  is often written as  $\llbracket P \rrbracket_{s_1}^{s_2}$ , or simply  $\llbracket P \rrbracket$ .

Definedness symbols carry specific intended semantics. For each definedness symbol  $\llbracket \_ \rrbracket_{s_1}^{s_2}$ , we add the pattern  $\llbracket x \rrbracket_{s_1}^{s_2}$  as an axiom to the deductive system, where  $x$  is a variable who has sort  $s_1$ . It does not matter which variable we pick.

With definedness symbols, we extend the formal language  $\mathcal{L}$  with

$$\begin{aligned} \llbracket P \rrbracket_{s_1}^{s_2} &:= \neg \llbracket \neg P \rrbracket_{s_1}^{s_2} \\ P_1 =_{s_1}^{s_2} P_2 &:= \llbracket P_1 \leftrightarrow P_2 \rrbracket_{s_1}^{s_2} \\ P_1 \neq_{s_1}^{s_2} P_2 &:= \neg(P_1 =_{s_1}^{s_2} P_2) \\ P_1 \subseteq_{s_1}^{s_2} P_2 &:= \llbracket P_1 \rightarrow P_2 \rrbracket_{s_1}^{s_2} \\ x \in_{s_1}^{s_2} P &:= x \subseteq_{s_1}^{s_2} P. \end{aligned}$$

**Remark 1.** To prevent writing tangled subscripts and superscripts that indicate sorts of variables and patterns all the time, we omit them as much as possible, unless there is a chance of confusing things. A statement with sorting subscripts and superscripts omitted is treated as (possibly many) statements with the omitting sorting subscripts and superscripts completed in all possible well-formed ways.

## 2 Deductive system

A deductive system is a recursive set of patterns as *axioms* and a finite set of *inference rules*. The deductive system of matching logic that we introduce in this section has been proved *sound* and *complete*.

### 2.1 The Deductive System

Axioms are given by the following axiom schemata where  $P, Q, R$  are arbitrary patterns and  $x, y$  are logic variables.

- (K1)  $P \rightarrow (Q \rightarrow P)$
- (K2)  $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$
- (K3)  $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$

- (K4)  $\forall x.P \rightarrow P[y/x]$
- (K5)  $\forall x.(P \rightarrow Q) \rightarrow (P \rightarrow \forall x.Q)$  if  $x$  does not occur free in  $P$
- (K6)  $P_1 = P_2 \rightarrow (Q[P_1/x] \rightarrow Q[P_2/x])$
- (M1)  $x \in y = (x = y)$
- (M2)  $x \in P \wedge Q = (x \in P) \wedge (x \in Q)$
- (M3)  $x \in \neg P = \neg(x \in P)$
- (M4)  $x \in \forall y.P = \forall y.x \in P$  where  $x$  is distinct from  $y$
- (M5)  $x \in \sigma(\dots, P_i, \dots) = \exists y.y \in P_i \wedge x \in \sigma(\dots, y, \dots)$  where  $y$  occurs free in the left hand side of the equation.

**Remark 2.** Substitution is denoted as  $Q[P/x]$ . Alpha-renaming is always assumed in order to avoid free variables capturing.

Inference rules include

- (Modus Ponens) From  $P$  and  $P \rightarrow Q$ , deduce  $Q$ .
- (Universal Generalization) From  $P$ , deduce  $\forall x.P$ .
- (Membership Introduction) From  $P$ , deduce  $\forall x.(x \in P)$ , where  $x$  does not occur free in  $P$ .
- (Membership Elimination) From  $\forall x.(x \in P)$ , deduce  $P$ , where  $x$  does not occur free in  $P$ .

**Theorem 3.** The proof system is sound and complete.

*Proof.* No proof. □

## 2.2 Metatheorems of the deductive system

**Proposition 4** (Tautology). For any propositional tautology  $\mathcal{A}(p_1, \dots, p_n)$  where  $p_1, \dots, p_n$  are all propositional variables in  $\mathcal{A}$ , and for any patterns  $P_1, \dots, P_n$ ,

$$\vdash \mathcal{A}(P_1, \dots, P_n).$$

*Proof.* No proof. □

**Corollary 5.**  $\vdash \top$ .

*Proof.* By definition,  $\top = \neg \perp = \neg(x \wedge \neg x)$ , where  $x$  is a matching logic variable who has the same sort with  $\top$ . Let proposition  $\mathcal{A} = \neg(p \wedge \neg p)$  with  $p$  is a propositional variable. Then  $\mathcal{A}$  is a propositional tautology. By Proposition 4,  $\top = \mathcal{A}[x/p]$  is derivable in the proof system, i.e.,  $\vdash \top$ . □

**Proposition 6** ( $\vee$ -Introduction).  $\vdash P$  implies  $\vdash P \vee Q$ .

*Proof.* Use Proposition 4 and Modus Ponens. Note that in general,  $\vdash P \vee Q$  does not imply  $\vdash P$  or  $\vdash Q$ .  $\square$

**Proposition 7** (Functional Substitution).  $\vdash \exists y.(Q = y) \rightarrow (P[Q/x] \rightarrow \exists x.P(x))$ .

**Proposition 8** ( $\wedge$ -Introduction and Elimination).  $\vdash P$  and  $\vdash Q$  iff  $\vdash P \wedge Q$ .

*Proof.* Use Proposition 4 and Modus Ponens.  $\square$

**Proposition 9** (Equality Introduction).  $\vdash P = P$ .

*Proof.* Use Membership Introduction and Proposition 4.  $\square$

**Proposition 10** (Equality Replacement).  $\vdash P_1 = P_2$  and  $\vdash Q[P_1/x]$  implies  $\vdash Q[P_2/x]$ .

*Proof.* Use Axiom (K7) and Modus Ponens.  $\square$

**Proposition 11** (Equality Establishment).  $\vdash P \leftrightarrow Q$  implies  $\vdash P = Q$ .

*Proof.* Use Membership Axioms and  $\vee$ -Introduction.  $\square$

**Corollary 12.**  $\vdash P$  implies  $\vdash P = \top$ .

**Proposition 13.**  $\vdash x \in [y]$ .

*Proof.*

$$\begin{aligned} & \vdash x \in [y] \\ & \text{if } \vdash \forall x.(x \in [y]) & (K5, K6, \text{ and Modus Ponens}) \\ & \text{iff } \vdash [y]. \end{aligned}$$

$\square$

**Proposition 14.**  $\vdash P \rightarrow [P]$ .

*Proof.*

$$\begin{aligned} & \vdash P \rightarrow [P] \\ & \text{iff } \vdash \forall x.(x \in P \rightarrow [P]) \\ & \text{if } \vdash x \in P \rightarrow [P] \\ & \text{iff } \vdash x \in P \rightarrow x \in [P] \\ & \text{iff } \vdash x \in P \rightarrow \exists y.(y \in P \wedge x \in [y]) \\ & \text{iff } \vdash x \in P \rightarrow \neg \forall y.(y \notin P \vee x \notin [y]) \\ & \text{iff } \vdash \forall y.(y \notin P \vee x \notin [y]) \rightarrow x \notin P \\ & \text{if } \vdash x \notin P \vee x \notin [x] \rightarrow x \notin P \\ & \text{iff } \vdash x \in P \rightarrow x \in P \wedge x \in [x] \\ & \text{iff } \vdash x \in P \rightarrow x \in [x] \\ & \text{if } \vdash x \in [x] \end{aligned}$$

**Remark** Similarly we can show  $\vdash \lfloor P \rfloor \rightarrow P$ . □

**Proposition 15.**  $\vdash \forall x.(x \in P) = \lfloor P \rfloor$ , where  $x$  occurs free in  $P$ .

*Proof.* By Proposition 11 and 8, it suffices to show

$$\vdash \forall x.(x \in P) \rightarrow \lfloor P \rfloor \quad (2)$$

and

$$\vdash \lfloor P \rfloor \rightarrow \forall x.(x \in P). \quad (3)$$

To show (2),

$$\begin{aligned} & \vdash \forall x.(x \in P) \rightarrow \lfloor P \rfloor \\ \text{iff } & \vdash \forall x.[x \wedge P] \rightarrow \neg[\neg P] \\ \text{iff } & \vdash [\neg P] \rightarrow \exists x.\neg[x \wedge P] \\ \text{iff } & \vdash \forall y.(y \in ([\neg P] \rightarrow \exists x.\neg[x \wedge P])) \\ \text{if } & y \in ([\neg P] \rightarrow \exists x.\neg[x \wedge P]) \\ \text{iff } & \vdash \exists z_1.(z_1 \notin P \wedge y \in [z_1]) \rightarrow \\ & \quad \exists x.\neg(\exists z_2.(z_2 = x \wedge z_2 \in P \wedge y \in [z_2])) \\ \text{iff } & \vdash \exists z_1.(z_1 \notin P \wedge \top) \rightarrow \quad (\text{Proposition 13, 10, and Corollary 12}) \\ & \quad \exists x.\neg(\exists z_2.(z_2 = x \wedge z_2 \in P \wedge \top)) \\ \text{iff } & \vdash \exists z_1.(z_1 \notin P) \rightarrow \exists x.\neg(\exists z_2.(z_2 = x \wedge z_2 \in P)) \\ \text{iff } & \vdash \forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \rightarrow \forall z_1.(z_1 \in P) \\ \text{if } & \vdash \forall z_1.(\forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \rightarrow (z_1 \in P)) \\ \text{if } & \vdash \forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \rightarrow (z_1 \in P). \end{aligned}$$

Since  $\vdash \forall x.(\exists z_2.(z_2 = x \wedge z_2 \in P)) \rightarrow \exists z_2.(z_2 = z_1 \wedge z_2 \in P)$ , it suffices to show

$$\begin{aligned} & \vdash \exists z_2.(z_2 = z_1 \wedge z_2 \in P) \rightarrow (z_1 \in P) \\ \text{iff } & \vdash z_1 \notin P \rightarrow \forall z_2.(z_2 \neq z_1 \vee z_2 \notin P) \\ \text{if } & \vdash \forall z_2.(z_1 \notin P \rightarrow z_2 \neq z_1 \vee z_2 \notin P) \\ \text{if } & \vdash z_1 \notin P \rightarrow z_2 \neq z_1 \vee z_2 \notin P \\ \text{if } & \vdash z_2 = z_1 \wedge z_2 \in P \rightarrow z_1 \in P. \end{aligned}$$

And we proved (2).

Similarly, to show (3),

$$\begin{aligned} & \vdash \lfloor P \rfloor \rightarrow \forall x.(x \in P) \\ \text{iff } & \vdash \exists x.\neg[x \wedge P] \rightarrow [\neg P] \\ \text{iff } & \vdash \forall y.(y \in \exists x.\neg[x \wedge P] \rightarrow [\neg P]) \\ \text{if } & y \in \exists x.\neg[x \wedge P] \rightarrow [\neg P] \\ \text{iff } & \vdash \exists x.\neg\exists z_2.(z_2 = x \wedge z_2 \in P) \rightarrow \exists z_1.(z_1 \notin P) \\ \text{iff } & \vdash \forall z_1.(z_1 \in P) \rightarrow \exists z_2.(z_2 = z_1 \wedge z_2 \in P) \\ \text{iff } & \vdash x \in P \rightarrow \exists z_2.(z_2 = x \wedge z_2 \in P). \end{aligned}$$

We proved (3).

**Remark** If  $x$  occurs free in  $P$ , the result does not hold. For example, let  $P$  be  $upto(x)$  where  $upto(\cdot)$  is interpreted to  $upto(n) = \{0, 1, \dots, n\}$  on  $\mathbb{N}$ .  $\square$

**Remark** From Membership Introduction and Elimination inference rules and Proposition 15,  $\vdash P$  iff  $\vdash \lfloor P \rfloor$ .

**Proposition 16** (Classification Reasoning). *For any  $P$  and  $Q$ , from  $\vdash P \rightarrow Q$  and  $\vdash \neg P \rightarrow Q$  deduce  $\vdash Q$ .*

*Proof.* From  $\vdash \neg P \rightarrow Q$  deduce  $\vdash \neg Q \rightarrow P$ . Notice that  $\vdash P \rightarrow Q$ , so we have  $\vdash \neg Q \rightarrow Q$ , i.e.,  $\vdash \neg\neg Q \vee Q$  which concludes the proof.  $\square$

**Corollary 17.** *For any  $P_1, P_2$ , and  $Q$  are patterns with  $\vdash P_1 \vee P_2$ , from  $\vdash P_1 \rightarrow Q$  and  $\vdash P_2 \rightarrow Q$ , deduce  $\vdash Q$ .*

**Definition 18** (Predicate Pattern). *A pattern  $P$  is called a predicate pattern or a predicate if  $\vdash (P = \top) \vee (P = \perp)$ .*

**Remark** Predicate patterns are closed under all logic connectives.

**Remark** For any  $P$ ,  $\lceil P \rceil$  is a predicate pattern.

**Proposition 19.**  $\vdash (\lceil P \rceil = \perp) = (P = \perp)$  and  $\vdash (\lfloor P \rfloor = \top) = (P = \top)$ .

*Proof.* It is easy to prove one derivation from the other, so we only prove the first one. By Proposition 11, it suffices to prove

$$\vdash (\lceil P \rceil = \perp) \rightarrow (P = \perp) \quad (4)$$

and

$$\vdash (P = \perp) \rightarrow (\lceil P \rceil = \perp) \quad (5)$$

The proof of (5) is trivial and we left it as an exercise. We now prove (4) through the following backward reasoning.

$$\begin{aligned} & \vdash (\lceil P \rceil = \perp) \rightarrow (P = \perp) \\ \text{iff } & \vdash \forall y. (y \in ((\lceil P \rceil = \perp) \rightarrow (P = \perp))) \\ \text{if } & \vdash y \in ((\lceil P \rceil = \perp) \rightarrow (P = \perp)) \\ \text{iff } & \vdash (y \in (\lceil P \rceil = \perp) \rightarrow (y \in (P = \perp))). \end{aligned} \quad (6)$$

While for any pattern  $Q$ ,

$$\begin{aligned} & \vdash y \in (Q = \perp) \\ \text{iff } & \vdash y \in \neg[\neg(Q \leftrightarrow \perp)] \\ \text{iff } & \vdash y \in \neg\lceil Q \rceil \\ \text{iff } & \vdash \neg\exists z. (z \in Q \wedge y \in \lceil z \rceil) \\ \text{iff } & \vdash \neg\exists z. (z \in Q) \end{aligned}$$

So we continue to prove (6) by showing

$$\begin{aligned}
& \vdash (y \in ([P] = \perp)) \rightarrow (y \in (P = \perp)) \\
\text{iff } & \vdash \neg \exists z. (z \in [P]) \rightarrow \neg \exists z. (z \in P) \\
\text{iff } & \vdash \exists z. (z \in P) \rightarrow \exists z. (z \in [P]) \\
\text{iff } & \vdash \exists z. (z \in P) \rightarrow \exists z. (\exists z_1. (z_1 \in P \wedge z \in [z_1])) \\
\text{iff } & \vdash \exists z. (z \in P) \rightarrow \exists z. \exists z_1. (z_1 \in P) \\
\text{iff } & \vdash \exists z_1. (z_1 \in P) \rightarrow \exists z. \exists z_1. (z_1 \in P).
\end{aligned}$$

And we finish the proof by noticing the fact that for any pattern  $Q$  and variable  $x$ ,

$$\vdash Q \rightarrow \exists x. Q.$$

□

**Proposition 20.** *For any predicate  $P$ ,  $\vdash (P \neq \top) = (P = \perp)$  and  $\vdash (P \neq \perp) = (P = \top)$ .*

*Proof.* We only prove the first derivation, by showing both

$$\vdash (P \neq \top) \rightarrow (P = \perp) \tag{7}$$

and

$$\vdash (P = \perp) \rightarrow (P \neq \top). \tag{8}$$

Proving (8) is trivial. We now prove (7), which is also trivial by transforming disjunction to implication. □

**Proposition 21.** *For any pattern  $Q$  and any predicate pattern  $P$ ,  $\vdash P \vee Q$  iff  $\vdash P \vee [Q]$ .*

*Proof.*  $(\Leftarrow)$  is obtained immediately by the remark of Proposition 14. We now prove  $(\Rightarrow)$ .

Because  $\vdash Q = \top \vee Q \neq \top$ , it suffices to show

$$\vdash Q = \top \rightarrow (P \vee [Q] = \top) \tag{9}$$

and

$$\vdash Q \neq \top \rightarrow (P \vee [Q] = \top) \tag{10}$$

by Corollary 17, and the fact that  $\vdash P \vee [Q] = \top$  and  $\vdash \top$  imply  $\vdash P \vee [Q]$ .

The proof of (9) is straightforward as follows.

$$\begin{aligned}
& \vdash Q = \top \rightarrow (P \vee [Q] = \top) \\
\text{if } & \vdash Q = \top \rightarrow (P \vee [\top] = \top) \\
\text{if } & \vdash Q = \top \rightarrow (\top = \top) \\
\text{if } & \vdash \top.
\end{aligned}$$

The proof of (10) needs more effort:

$$\begin{aligned}
& \vdash Q \neq \top \rightarrow (P \vee \lfloor Q \rfloor = \top) \\
\text{iff } & \vdash (Q = \top) \vee (P \vee \lfloor Q \rfloor = \top) \\
\text{iff } & \vdash (\lfloor Q \rfloor = \top) \vee (P \vee \lfloor Q \rfloor = \top) \\
\text{iff } & \vdash \lfloor Q \rfloor \neq \top \rightarrow (P \vee \lfloor Q \rfloor = \top) \\
\text{iff } & \vdash \lfloor Q \rfloor = \perp \rightarrow (P \vee \lfloor Q \rfloor = \top) \\
\text{if } & \vdash \lfloor Q \rfloor = \perp \rightarrow (P \vee \perp = \top) \\
\text{iff } & \vdash \lfloor Q \rfloor = \perp \rightarrow (P = \top) \\
\text{if } & \vdash Q = \top \vee P = \top.
\end{aligned}$$

Notice that  $P$  is a predicate pattern, so it suffices to show

$$\vdash P = \top \rightarrow (Q = \top \vee P = \top),$$

whose validity is obvious, and

$$\vdash P = \perp \rightarrow (Q = \top \vee P = \top),$$

which is proved by showing

$$\vdash P = \perp \rightarrow Q = \top. \quad (11)$$

Because  $\vdash P \vee Q$ , it suffices to show

$$\begin{aligned}
& \vdash P = \perp \rightarrow (P \vee Q) \rightarrow (Q = \top) \\
\text{if } & \vdash P = \perp \rightarrow (\perp \vee Q) \rightarrow (Q = \top) \\
\text{iff } & \vdash P = \perp \rightarrow Q \rightarrow (Q = \top) \\
\text{if } & \vdash Q \rightarrow (Q = \top) \\
\text{iff } & \vdash (Q \neq \top) \rightarrow \neg Q \\
\text{iff } & \vdash (\lfloor Q \rfloor = \perp) \rightarrow \neg Q.
\end{aligned}$$

Notice we have  $\vdash Q \rightarrow \lfloor Q \rfloor$ , which means  $\vdash \neg \lfloor Q \rfloor \rightarrow \neg Q$ , so it suffices to show

$$\begin{aligned}
& \vdash (\lfloor Q \rfloor = \perp) \rightarrow \neg \lfloor Q \rfloor \\
\text{iff } & \vdash (\lfloor Q \rfloor = \perp) \rightarrow \neg \perp \\
\text{iff } & \vdash (\lfloor Q \rfloor = \perp) \rightarrow \top \\
\text{iff } & \vdash \top.
\end{aligned}$$

And this concludes the proof.  $\square$

**Proposition 22** (Deduction Theorem). *If  $\Gamma \cup \{P\} \vdash Q$  and the derivation does not use  $\forall x$ -Generalization where  $x$  is free in  $P$ , then  $\Gamma \vdash \lfloor P \rfloor \rightarrow Q$ .*

*Proof.* The proof is by induction on  $n$ , the length of the derivation of  $Q$  from  $\Gamma \cup \{P\}$ .



Base step:  $n = 1$ , and  $Q$  is an axiom, or  $P$ , or a member of  $\Gamma$ . If  $Q$  is an axiom or a member of  $\Gamma$ , then  $\Gamma \vdash Q$  and as a result,  $\Gamma \vdash [P] \rightarrow Q$ . If  $Q$  is  $P$ , then  $\Gamma \vdash [P] \rightarrow Q$  by Proposition 14.

Induction step: Let  $n > 1$ . Suppose that if  $P'$  can be deduced from  $\Gamma \cup \{P\}$  without using  $\forall x$ -Generalization where  $x$  is free in  $P$ , in a derivation containing fewer than  $n$  steps, then  $\Gamma \vdash [P] \rightarrow P'$ .

Case 1:  $Q$  is an axiom, or  $P$ , or a member of  $\Gamma$ . Precisely as in the Base step, we show that  $\vdash [P] \rightarrow Q$ .

Case 2:  $Q$  follows from two previous patterns in the derivation by an application of Modus Ponens. These two patterns must have the forms  $Q_1$  and  $Q_1 \rightarrow Q$ , and each one can certainly be deduced from  $\Gamma \cup \{P\}$  by a derivation with fewer than  $n$  steps, by just omitting the subsequent members from the original derivation from  $\Gamma \cup \{P\} \vdash Q$ . So we have  $\Gamma \cup \{P\} \vdash Q_1$  and  $\Gamma \cup \{P\} \vdash Q_1 \rightarrow Q$ , and, applying the hypothesis of induction,  $\Gamma \vdash [P] \rightarrow Q_1$  and  $\Gamma \vdash [P] \rightarrow (Q_1 \rightarrow Q)$ . It follows immediately that  $\Gamma \vdash [P] \rightarrow Q$ .

Case 3:  $Q$  follows from a previous pattern in the derivation by an application of  $\forall x_i$ -Generalization where  $x_i$  does not occur free in  $P$ . So  $Q$  is  $\forall x_i. Q_1$ , say, and  $Q_1$  appears previously in the derivation. Thus  $\Gamma \cup \{P\} \vdash Q_1$ , and the derivation has fewer than  $n$  steps, so  $\Gamma \vdash [P] \rightarrow Q_1$ , since there is no application of Universal Generalization involving a free variable of  $P$ . Also  $x_i$  cannot occur free in  $P$ , as it is involved in an application of Universal Generalization in the deduction of  $Q$  from  $\Gamma \cup \{P\}$ . So we have a derivation of  $\Gamma \vdash [P] \rightarrow Q$  as follows.

$$\begin{aligned} & \Gamma \vdash [P] \rightarrow Q \\ \text{iff } & \Gamma \vdash [P] \rightarrow \forall x_i. Q_1 \\ \text{if } & \Gamma \vdash \forall x_i. ([P] \rightarrow Q_1) \\ \text{if } & \Gamma \vdash [P] \rightarrow Q_1. \end{aligned}$$

So  $\Gamma \vdash [P] \rightarrow Q$  as required.

Case 4:  $Q$  follows from a previous pattern in the derivation by an application of Membership Introduction. So  $Q$  is  $\forall x_i. (x_i \in Q_1)$  with  $x_i$  is free in  $Q_1$ , say, and  $Q_1$  appears previously in the derivation. Thus  $\Gamma \cup \{P\} \vdash Q_1$ , and the derivation has fewer than  $n$  steps, so  $\Gamma \vdash [P] \rightarrow Q_1$ , since there is no application of Universal Generalization involving a free variable of  $P$ . So we have a derivation of  $\Gamma \vdash [P] \rightarrow Q$  as follows.

$$\begin{aligned} & \Gamma \vdash [P] \rightarrow Q \\ \text{iff } & \Gamma \vdash [P] \rightarrow \forall x_i. (x_i \in Q_1) \\ \text{iff } & \Gamma \vdash [P] \rightarrow [Q_1], \end{aligned}$$

which follows by the hypothesis of induction  $\Gamma \vdash [P] \rightarrow Q_1$  and the fact that  $\Gamma \vdash Q_1 \rightarrow [Q_1]$  (by the Remark in Proposition 14).

Case 5:  $Q$  follows from a previous pattern in the derivation by an application of Membership Elimination. The previous pattern must have the form  $\forall x_i. (x_i \in Q)$ , and can be deduced from  $\Gamma \cup \{P\}$  by a derivation with fewer than  $n$  steps, by just omitting

the subsequent members from the original derivation from  $\Gamma \cup \{P\} \vdash Q$ . So we have  $\Gamma \cup \{P\} \vdash \forall x_i.(x_i \in Q)$ , and, applying the hypothesis of induction,  $\Gamma \vdash [P] \rightarrow \forall x_i.(x_i \in Q)$ . So we have a derivation of  $\Gamma \vdash [P] \rightarrow Q$  as follows.

$$\begin{aligned}
& \Gamma \vdash [P] \rightarrow Q \\
\text{iff } & \Gamma \vdash \neg[P] \vee Q \\
\text{iff } & \Gamma \vdash \neg[P] \vee [Q] & \text{(Proposition 21)} \\
\text{iff } & \Gamma \vdash \neg[P] \vee \forall x_i.(x_i \in Q) \\
\text{iff } & \Gamma \vdash [P] \rightarrow \forall x_i.(x_i \in Q),
\end{aligned}$$

which is the hypothesis of induction. And this concludes our inductive proof.  $\square$

**Corollary 23** (Closed-form Deduction Theorem). *If  $P$  is closed,  $\Gamma \cup \{P\} \vdash Q$  implies  $\Gamma \vdash [P] \rightarrow Q$ .*

**Theorem 24** (Frame Rule). *Let  $\sigma \in \Sigma$  be a symbol in the signature. From  $P_1 \rightarrow P_2$ , deduce  $\sigma(P_1) \rightarrow \sigma(P_2)$ . In its most general form,  $P_1 \rightarrow P_2$  deduces  $\sigma(Q_1, \dots, P_1, \dots, Q_n) \rightarrow \sigma(Q_1, \dots, P_2, \dots, Q_n)$ .*

*Proof.* we write  $\sigma(Q_1, \dots, P_i, \dots, Q_n)$  as  $\sigma(P_i, \vec{Q})$  for short, for any  $i \in \{1, 2\}$ .

$$\begin{aligned}
& \vdash \sigma(P_1, \vec{Q}) \rightarrow \sigma(P_2, \vec{Q}) \\
\text{iff } & \vdash y \in (\sigma(P_1, \vec{Q}) \rightarrow \sigma(P_2, \vec{Q})) \\
\text{iff } & \vdash (y \in \sigma(P_1, \vec{Q})) \rightarrow (y \in \sigma(P_2, \vec{Q})) \\
\text{iff } & \vdash \exists z_1. \exists \vec{z}. (z_1 \in P_1 \wedge \vec{z} \in \vec{Q} \wedge y \in \sigma(z_1, \vec{z})) \\
& \rightarrow \exists z_2. \exists \vec{z}. (z_2 \in P_2 \wedge \vec{z} \in \vec{Q} \wedge y \in \sigma(z_2, \vec{z})) \\
\text{iff } & \vdash \exists z_1. \exists \vec{z}. (z_1 \in P_1 \wedge \vec{z} \in \vec{Q} \wedge y \in \sigma(z_1, \vec{z})) \\
& \rightarrow z_1 \in P_2 \wedge \vec{z} \in \vec{Q} \wedge y \in \sigma(z_1, \vec{z})) \\
\text{iff } & \vdash \exists z_1. \exists \vec{z}. (z_1 \in P_1 \rightarrow z_1 \in P_2) \\
\text{if } & \vdash \exists z_1. (z_1 \in P_1 \rightarrow z_1 \in P_2) \\
\text{if } & \vdash P_1 \rightarrow P_2.
\end{aligned}$$

$\square$

**Corollary 25** (Frame Rule as Implication).  $\vdash [P \rightarrow Q] \rightarrow (\sigma(P) \rightarrow \sigma(Q))$

### 3 Inference rules

**Axioms**

$$\frac{\cdot}{\Gamma \vdash A}$$

where  $A$  is an axiom.

**Inclusion**

$$\frac{\cdot}{\Gamma \vdash P}$$

where  $P \in \Gamma$ .

**Modus Ponens**

$$\frac{\Gamma \vdash Q \rightarrow P \quad \Gamma \vdash Q}{\Gamma \vdash P}$$

**Closed-Form Deduction Theorem**

$$\frac{\Gamma \cup \{P\} \vdash Q}{\Gamma \vdash P \rightarrow Q}$$

where  $P$  is closed.

**Universal Generalization**

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P} (\forall x)$$

**Conjunction Splitting**

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$$