The Semantics of K

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Please feel free to contribute to this report in all ways. You could add new contents, remove redundant ones, refactor and organize the texts, and correct typos.

Definition 1 (Matching Logic Theory)**.** A matching logic theory (S, Σ, A) is a triple that contains a nonempty finite set of sorts, a finite set of symbols, and a recursive set of axioms. Two theories are *equal* if they have the same set of sorts and symbols, and they deduce the same set of theorems.

Example 2. We use serif fonts to denote matching logic theories. Some of the commonly used ones are the theory (theories) of definedness DEF, the theory of Presburger arithmetic PA, the theory of sequences of natural numbers SEQ, the theory of memory heaps HEAP, the theory of IMP programs IMP, the theory (theories) of fixed-points FIX, and the theory (theories) of contexts CTXT.

Definition 3 (The Kore Language)**.** **We haven't come to an agreement on the syntax of the Kore language yet. One could, though, refer to the Kore text representation at the wiki page at kframework repos on Github, whose link I cannot find any more, which is considered as the first step towards that direction.**

The Kore language is a language to write matching logic theories. The outcomes are called Kore definitions. Kore definitions are mainly served as the interface between a K frontend and a K backend, but a human should be able to read and write Kore definitions of simple theories, too. The Kore language is designed in a way that:

- Every Kore definition defines exactly one matching logic theory;
- Every matching logic theory can be defined as a Kore definition;
- There is no parsing ambiguity.
- The least amount of inferring is needed;
- Symbols are decorated with their argument sorts and result sort;

- There is no polymorphic or overloaded symbols, so every symbol has a unique name (reference).

  The above two points will lead to some redundancy in Kore definitions, though. For example, there will be n^2 definedness symbol if there are n sorts in the theory. Similarly, n^2 equalities are need, too.

- User-defined (matching logic) variables are allowed as in theory LAMBDA. We shouldn't fix a syntax for variables. Please refer to later sections about theory LAMBDA for more discussions.

- And more ...

Definition 4 (Frontend)**.** A K frontend is an artifact that generates Kore definitions.

Definition 5 (Backend)**.** A K backend is an artifact that consumes a Kore definition of a theory T and does some work. Whatever it does can and should be algorithmically reduced to the task of proving $\mathsf{T} \vdash \varphi$ where φ "encodes" that work. A K backend should justify its results by generating formal proofs that can be proof-checked by the oracle matching logic proof checker.

1 Object-level and meta-level

It is an aspect of life in mathematical logics to distinguish the *object-level* and *meta-level* concepts. The basic principle is to use serif fonts letters (x) for object-level concepts and normal math fonts letters for meta-level concepts (x).

Variables and metavariables for variables For any matching logic theory $\mathsf{T} = (S, \Sigma, A)$, it comes for each sort $s \in S$ a countably infinite set V_s of variables. We use $\mathsf{x}:\mathsf{s}, \mathsf{y}:\mathsf{s}, \mathsf{z}:\mathsf{s}, \ldots$ for variables in V_s, and omit their sorts when that is clear from the contexts. Different sorts have disjoint sets of variables, so $V_s \cap V_{s'} = \emptyset$ if $s \neq s'$.

Proposition 6. *Let A be a set of axioms and $\bar{A} = \forall A$ be the universal quantification closure of A, then for any pattern φ, $A \vdash \varphi$ iff $\bar{A} \vdash \varphi$.*

Remark 7 (Free variables in axioms)*.* The free variables appearing in the axioms of a theory can be regarded as implicitly universal quantified, because a theory and its universal quantification closure are equal.

Example 8.

$$
\begin{aligned}
A_1 &= \{\mathsf{mult}(\mathsf{x}, \mathsf{0}) = \mathsf{0}\} \\
A_2 &= \{\forall \mathsf{x}.\mathsf{mult}(\mathsf{x}, \mathsf{0}) = \mathsf{0}\} \\
A_3 &= \{\forall \mathsf{y}.\mathsf{mult}(\mathsf{y}, \mathsf{0}) = \mathsf{0}\} \\
A_4 &= \{\mathsf{mult}(x, \mathsf{0}) = \mathsf{0}\} \\
A_5 &= \{\forall x.\mathsf{mult}(x, \mathsf{0}) = \mathsf{0}\} \\
A_6 &= \{\forall y.\mathsf{mult}(y, \mathsf{0}) = \mathsf{0}\} \\
A_7 &= \{\mathsf{mult}(\mathsf{x}, \mathsf{0}) = \mathsf{0}, \mathsf{mult}(\mathsf{y}, \mathsf{0}) = \mathsf{0}, \mathsf{mult}(\mathsf{z}, \mathsf{0}) = \mathsf{0}, \dots\} \\
A_8 &= \{\forall \mathsf{x}.\mathsf{mult}(\mathsf{x}, \mathsf{0}) = \mathsf{0}, \forall \mathsf{y}.\mathsf{mult}(\mathsf{y}, \mathsf{0}) = \mathsf{0}, \forall \mathsf{z}.\mathsf{mult}(\mathsf{z}, \mathsf{0}) = \mathsf{0}, \dots\}
\end{aligned}
$$

All the eight theories are equal. Theories A_4, A_5, A_6 are finite representations of theories A_7, A_8, A_8 respectively.

Remark 9. There is no need to have metavariables for variables in the Kore language, because (1) if they are used as bound variables, then replacing them with any (matching logic) variables will result in the same theories, thanks to alpha-renaming; and (2) if they are used as free variables, then it makes no difference to consider the universal quantification closure of them and we get to the case (1).

~~Given said that, there are cases when metavariables for variables make sense. In those cases we often want our metavariables to range over all variables of all sorts, in order to make our Kore definitions compact.~~ No, we do not need metavariables over variables. I was thinking of the definedness symbols. We might want to write only one axiom schema of $\lceil x \rceil$ instead many $\lceil x{:}s \rceil_s^{s'}$'s, but we cannot do that unless we allow polymorphic and overloaded symbols in Kore definitions.

Patterns and metavariables for patterns It is in practice more common to use metavariables that range over all patterns. One typical example is axiom schemata. For example, $\vdash \varphi \to \varphi$ in which φ is the metavariable that ranges all well-formed patterns.

There has been an argument on whether metavariables for patterns should be sorted or not. Here are some observations. Firstly, since all symbols are decorated and not overloaded, in most cases, the sort of a metavariable for patterns can be inferred from its context. Secondly, the only counterexample against the first point that I can think of is when they appear alone, which is not an interesting case anyway. Thirdly, we do want the least amount of reasoning and inferring in using Kore definitions, so it breaks nothing if not helping things to have metavariables for patterns carrying their sorts.

Example 10.

$$A_1 = \{\mathsf{merge}(\mathsf{h1}, \mathsf{h2}) = \mathsf{merge}(\mathsf{h2}, \mathsf{h1})\}$$
$$A_2 = \{\forall \mathsf{h1} \forall \mathsf{h2}.\mathsf{merge}(\mathsf{h1}, \mathsf{h2}) = \mathsf{merge}(\mathsf{h2}, \mathsf{h1})\}$$
$$A_3 = \{\mathsf{merge}(\varphi, \psi) = \mathsf{merge}(\psi, \varphi)\}$$

All three theories are equal. It is easier to see that fact from a model theoretic point of view, since all theories require that the interpretation of merge is commutative and nothing more. On the other hand, it is not straightforward to obtain that conclusion from a proof theoretic point of view. For example, to deduce $\mathsf{merge}(\mathsf{list}(\mathsf{one}, \mathsf{cons}(\mathsf{two}, \mathsf{epsilon})), \mathsf{top}) = \mathsf{merge}(\mathsf{top}, \mathsf{list}(\mathsf{one}, \mathsf{cons}(\mathsf{two}, \mathsf{epsilon})))$ needs only one step in A_3, but will need a lot more in either A_1 or A_2, because one cannot simply substitute any patterns for universal quantified variables in matching logic.

This is a nice example that shows the power of metavariables and how they greatly shorten formal proofs, but this is not a good example to convince people that we need metavariables for patterns in order to embrace more expressiveness. There is an example in the theory LAMBDA that clearly shows metavariables give us more expressiveness power, and we will cover that example in later sections.

2 Binders

In matching logic there is a unified representation of binders. We will be using the theory of lambda calculus LAMBDA as an example in this section. Recall that the syntax for untyped lambda calculus is

$$\Lambda ::= V \mid \lambda V.\Lambda \mid \Lambda\ \Lambda$$

where V is a countably infinite set of atomic λ-terms.

The theory LAMBDA in matching logic has one sort Exp for lambda expressions. It also has in its signature a binary symbol #lambda that builds a λ-terms, and a binary symbol app for lambda applications. To mimic the binding behavior of λ in lambda calculus, we define syntactic sugar $\mathsf{lambda}x.e = \exists x.\#\mathsf{lambda}(x, e)$ and $e_1\ e_2 = \mathsf{app}$ in theory LAMBDA.

Rewriting logic

Contexts

Fixed points