Towards an Efficient and Economic Deductive System of Matching Logic

FSL group

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We aim for a Hilbert style deductive system which has a relatively large number of axioms but only a few inference rules.

1 Grammar and extended grammar

The formal language $\mathcal{L}$ we use to write matching logic patterns is defined as follows.

$$
\begin{aligned}
P ::= {} & x \\
& \mid P_1 \to P_2 \\
& \mid \neg P \\
& \mid \forall x.P \\
& \mid \sigma(P_1, \ldots, P_n) \\
& \mid P_1 = P_2 \\
(*** & \texttt{ extended} ***) \\
& \mid P_1 \vee P_2 \\
& \mid P_1 \wedge P_2 \\
& \mid P_1 \leftrightarrow P_2 \\
& \mid \exists x.P \\
& \mid P_1 \neq P_2 \\
& \mid \top \\
& \mid \bot \\
& \mid \lceil P \rceil \\
& \mid \lfloor P \rfloor \\
& \mid P_1 \subseteq P_2 \\
& \mid x \in P
\end{aligned}
$$

with the extended grammar defined as

$$
\begin{aligned}
&P_1 \vee P_2 := \neg P_2 \to P_1 \\
&P_1 \wedge P_2 := \neg(\neg P_1 \vee \neg P_2) \\
&P_1 \leftrightarrow P_2 := (P_1 \to P_2) \wedge (P_2 \to P_1) \\
&\exists x.P := \neg \forall x.\neg P \\
&P_1 \neq P_2 := \neg(P_1 = P_2) \\
&\top := x_1 = x_1 \\
&\bot := x_1 \neq x_1 \\
&\lceil P \rceil := P \neq \bot \\
&\lfloor P \rfloor := P = \top \\
&P_1 \subseteq P_2 := \lfloor P_1 \to P_2 \rfloor \\
&x \in P := x \subseteq P
\end{aligned}
$$

We will extend the grammar to a many-sorted one in the future.

2 Hilbert proof system

Axioms in $\mathcal{L}$ are given by the following nine axiom schemata where P, Q, R are arbitrary patterns and x, y are variables.

- (K1) $P \to (Q \to P)$
- (K2) $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$
- (K3) $(\neg P \to \neg Q) \to (Q \to P)$
- (K4) $\forall x.(P \to Q) \to (P \to \forall x.Q)$ if x does not occur free in P
- (K5) $\forall x.P \to P$ if x does not occur free in P
- (K6) $\forall x.P(x) \to P(y)$
- (K7) $P = P$
- (K8) $P_1 = P_2 \to (Q[P_1/x] \to Q[P_2/x])$
- (K9) $\exists y.Q = y \to (\forall x.P(x) \to P[Q/x])$ if Q is free for x in P
- (K10) $\exists x.x$

Inference rules include

- (Modus Ponens) From P and $P \to Q$, deduce Q.
- (Universal Generalization) From P, deduce $\forall x.P$.

Theorem 1 (Soundness of $K_{\mathcal{L}}$)**.** *Theorems of $K_{\mathcal{L}}$ are valid.*

Proposition 2 (Deduction Theorem)**.** *If $\Gamma \cup \{P\} \vdash Q$ and the proof does not use $\forall x$-Generalization where x is free in P, then $\Gamma \vdash P \to Q$. In particular, when P is closed, $\Gamma \cup \{P\} \vdash Q$ implies $\Gamma \vdash P \to Q$.*

Proposition 3 (Tautology)**.** *For any tautology $\mathcal{A}(p_1, \ldots, p_n)$ where $p_1, \ldots, p_n$ are propositional variables,*

$$\vdash \mathcal{A}(P_1, \ldots, P_n).$$

Proposition 4 (More Theorems in $\mathcal{L}$)**.**

$$\begin{aligned}
&\vdash \exists x.x \\
&\vdash \lceil x \rceil \\
&\vdash \exists y.x = y \\
&\vdash P_1 = P_2 \to Q[P_1/x] = Q[P_2/x]
\end{aligned}$$

Proof. (1)

$$\dfrac{\dfrac{\cdot}{\vdash \neg(x \to \neg x)}\ \text{(TAUT)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall x.\neg x \vdash \neg x \quad \forall x.\neg x \vdash \neg x}{\forall x.\neg x \vdash \neg x} \qquad \dfrac{\cdot}{\forall x.\neg x \vdash \neg x \to (x \to \neg x)}\ \text{(K1)}}{\forall x.\neg x \vdash x \to \neg x}\ \text{(MP)}}{\vdash (\forall x.\neg x) \to (x \to \neg x)}\ \text{(DEDUCT)}}{\vdash \neg(x \to \neg x) \to \neg\forall x.\neg x}\ \text{(MP\&K3)}}{\dfrac{\vdash \neg\forall x.\neg x}{\vdash \exists x.x}\ \text{(DEF)}}\ \text{(MP)}$$

(3)

$$\dfrac{\dfrac{\dfrac{\cdot}{x = x}\ \text{(K7)}}{\vdash \neg\neg(x = x)}\ \text{(MP\&TAUT)} \qquad \dfrac{\dfrac{\cdot}{\vdash \forall y.\neg(x = y) \to \neg(x = x)}\ \text{(K6)}}{\vdash \neg\neg(x = x) \to \neg\forall y.\neg(x = y)}\ \text{(MP\&K3)}}{\dfrac{\vdash \neg\forall y.\neg(x = y)}{\vdash \exists y.x = y}\ \text{(DEF)}}\ \text{(MP)}$$

□

Before we prove the adequacy theorem of $K_{\mathcal{L}}$, we prove some lemmas.

Proposition 5. *If a pattern is valid, then its closure is valid.*

Proposition 6. *If a pattern's closure is a theorem of $K_{\mathcal{L}}$, then itself is a theorem of $K_{\mathcal{L}}$, too.*

Proposition 7. *If P is not a theorem of $K_{\mathcal{L}}$, then $K_{\mathcal{L}}$ extended with adding $\neg P$ as an axiom is consistent.*

Proposition 8. *If S is a consistent extended system of $K_{\mathcal{L}}$, then for any theorem P of S, there exists a model M and an M-evaluation ρ such that $M, \rho \vDash P$.*

3 Inference rules

Axioms

$$\frac{\cdot}{\Gamma \vdash A}$$

where A is an axiom.

Inclusion

$$\frac{\cdot}{\Gamma \vdash P}$$

where $P \in \Gamma$.

Modus Ponens

$$\frac{\Gamma \vdash Q \to P \quad \Gamma \vdash Q}{\Gamma \vdash P}$$

Closed-Form Deduction Theorem

$$\frac{\Gamma \cup \{P\} \vdash Q}{\Gamma \vdash P \to Q}$$

where P is closed.

Universal Generalization

$$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P}\ (\forall x)$$

Conjunction Splitting

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \land Q}$$