Getting Ready to Work with IPSII (Version 0.1)

(c) Dallin Durfee 2021

May 18, 2021

This document is intended to help you learn the background you need to do research with me in Interference Pattern Structured Illumination Imaging (IPSII). If you get stuck on anything or if you have any questions, be sure to let me know right away so I can help you (don't just get frustrated!).

1 Introduction to Fourier Series

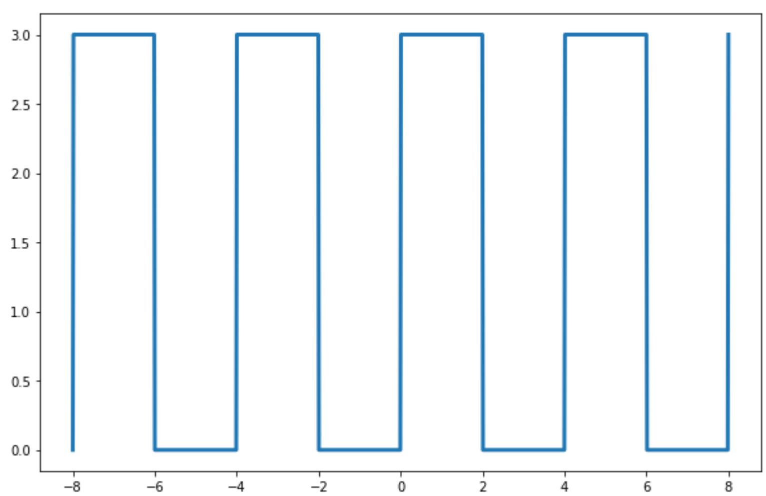
Watch the video below, and then do the exercises listed below. Note that the exercises use a more common notation for Fourier series than the one used in the video, but you should be able to adapt from one to the other.

`https://youtu.be/6MpmxZ6YB7U`

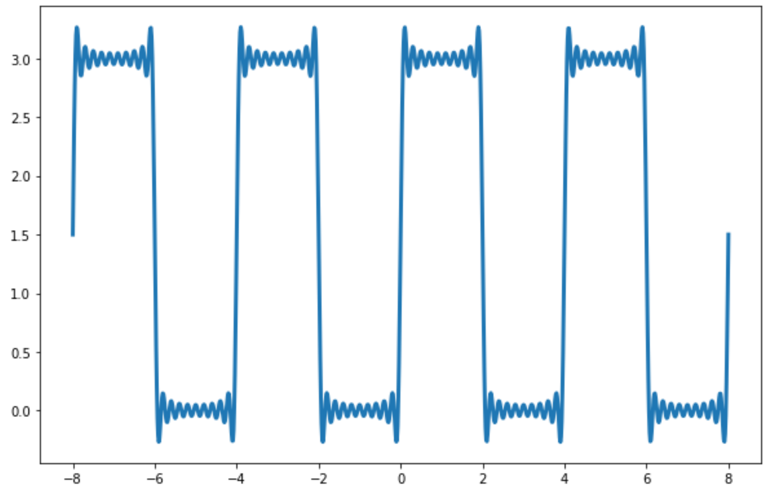
More information about Fourier series can be found at `https://en.wikipedia.org/wiki/Fourier_series`

- Figure out what k_0 should be, and then find equations for the coefficients a_n and b_n in the equation below such that when you plot the equation, you would get the plot shown below the equation. Make sure you solve the integrals and simplify your expressions for a_n and b_n.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nk_0x) + \sum_{n=1}^{\infty} b_n \sin(nk_0x)$$



- Now use Python to plot the function from $x = -8$ to $x = 8$, summing up to $n = 20$ using the equations you found for a_n and b_n. If you've done this correctly, it should look like the plot below.



1.1 Fourier series of non-repeating functions

Under construction

2 Introduction to Complex Exponentials

It turns out that exponentials are closely related to sines and cosines. Euler's formula states that

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}.$$

2.1 Proving Euler's formula

You can skip this subsection if you wish. However, if you are interested in knowing how to show that Euler's formula is true, you can prove it using a Maclaurin series (a Taylor series about $\theta = 0$):

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + ... = \sum_{n=\text{even}} \frac{(-1)^{n/2}\theta^n}{n!} + i \sum_{n=\text{odd}} \frac{(-1)^{(n-1)/2}\theta^n}{n!}$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + ... = \sum_{n=\text{even}} \frac{(-1)^{n/2}x^n}{n!} \quad \Rightarrow$$

$$\cos(\theta) = \sum_{n=\text{even}} \frac{(-1)^{n/2}\theta^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... = \sum_{n=\text{odd}} \frac{(-1)^{(n-1)/2}x^n}{n!} \quad \Rightarrow$$

$$i\sin(\theta) = i \sum_{n=\text{odd}} \frac{(-1)^{(n-1)/2}\theta^n}{n!}$$

$$\cos(\theta) + i\sin(\theta) = \sum_{n=\text{even}} \frac{(-1)^{n/2}\theta^n}{n!} + i \sum_{n=\text{odd}} \frac{(-1)^{(n-1)/2}\theta^n}{n!} = e^{i\theta}.$$

2.2 Using Euler's formula to derive trig identities

This subsection isn't essential to understand IPSII, so you can skip it if you wish. But I find it interesting, and very useful, that you can use Euler's formula to find all kinds of trig identities. For example, writing

$$e^{i(a+b)} = e^{ia}e^{ib}$$

and then applying Euler's formula to this, we get

$$\cos(a+b) + i\sin(a+b) = (\cos(a) + i\sin(a))(\cos(b) + i\sin(b))$$

$$= \cos(a)\cos(b) - \sin(a)\sin(b) + i\sin(a)\cos(b) + i\sin(b)\cos(a).$$

Then, noting that the real part of the left side must be equal to the real part of the right side, and the imaginary part of the left must equal the imaginary part of the right, we can separate this into the two equations

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b),$$

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a),$$

which are the sum identities for sine and cosine.

We can also write a pure sine or cosine in terms of complex exponentials. If

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

then

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta).$$

That means that

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta) \quad \Rightarrow \quad \boxed{\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}},$$

and

$$e^{i\theta} - e^{-i\theta} = 2i\sin(\theta) \quad \Rightarrow \quad \boxed{\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}}.$$

With Euler's formula and these relations, we can derive the trig identity that we used to understand standing waves and beats in Physics 2210:

$$\cos(a)\sin(b) = \left(\frac{e^{ia} + e^{-ia}}{2}\right)\left(\frac{e^{ib} - e^{-ib}}{2i}\right) = \frac{e^{ia}e^{ib} - e^{ia}e^{-ib} + e^{-ia}e^{ib} - e^{-ia}e^{-ib}}{4i}$$

$$= \frac{e^{ia}e^{ib} - e^{-ia}e^{-ib}}{4i} + \frac{e^{-ia}e^{ib} - e^{ia}e^{-ib}}{4i} = \frac{e^{i(a+b)} - e^{-i(a+b)}}{4i} + \frac{e^{i(b-a)} - e^{-i(b-a)}}{4i}$$

$$= \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(b-a) \quad \Rightarrow$$

$$2\cos(a)\sin(b) = \sin(a+b) + \sin(b-a).$$

Then, if we define $A = a + b$ and $B = b - a$, such that $a = (A - B)/2$ and $b = (A + B)/2$, this becomes

$$\boxed{\sin(A) + \sin(B) = 2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)}.$$

2.3 Using complex exponentials instead of sine waves

In IPSII imaging, we use illumination patterns which are sinusoidal. But exponentials are easier to work with than sines and cosines. As such, we usually represent the patterns using complex exponentials. For example, if I have a complex exponential with an argument of $\theta = \omega t + \phi$, I can separate that into the product of two exponentials, separating the part that varies in time (ωt) from the part that doesn't (ϕ):

$$e^{i(\omega t + \phi)} = e^{i\omega t}e^{i\phi}.$$

With a sine or a cosine, you can't do that:

$$\sin(\omega t + \phi) \neq \sin(\omega t)\sin(\phi).$$

One example which shows the utility of this property is adding two sound waves together. Imagine I play a sine wave through two different speakers. Because the speakers are a each a different distance from me, the sine wave from each speaker will have a different phase at my location. If I add two sine waves at the same frequency but with different phases, they can add constructively to make a large wave, or destructively to make a small one, depending on their relative phases. I can see this using the identity we derived above.

$$A\sin(\omega t+\phi_1)+A\cos(\omega t+\phi_2) = 2A\cos\left(\frac{\omega t + \phi_1 - (\omega t + \phi_2)}{2}\right)\sin\left(\frac{\omega t + \phi_1 + \omega t + \phi_2}{2}\right) = 2A\cos\left(\frac{\omega\phi_1 - \phi_2}{2}\right)\sin\left(\omega t + \frac{\phi_1\phi_2}{2}\right)$$

As such, the two waves combine to make a single sine wave at the frequency ω which has a phase equal to the average of the two phases, and which has an amplitude of

$$2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right).$$

But what if the two waves had different amplitudes?

$$A_1 \sin(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_1) = ???$$

Well, we can use complex exponentials to figure this out. First, I'm going to break this into pieces:

$$y_1(t) = A_1 \sin(\omega t + \phi_1)$$

and

$$y_2(t) = A_2 \sin(\omega t + \phi_1).$$

Now, I'm going to make up two new functions

$$\tilde{y}_1(t) = A_1 e^{i(\omega t + \phi_1)}$$

and

$$\tilde{y}_2(t) = A_2 e^{i(\omega t + \phi_2)}.$$

These are different functions from $y_1(t)$ and $y_2(t)$. But if I take the imaginary part of each of those functions, I get $y_1(t)$ and $y_2(t)$. So the imaginary part of $\tilde{y}_1(t) + \tilde{y}_2(t)$ must be equal to $y_1(t) + y_2(t)$:

$$A_1 \sin(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_1) = \Im\left[A_1 e^{i(\omega t + \phi_1)} + A_2 e^{i(\omega t + \phi_2)}\right]$$

$$= \Im\left[A_1 e^{i\omega t} e^{i\phi_1} + A_2 e^{i\omega t} e^{i\phi_2}\right].$$

Note that both terms have an $e^{i\omega t}$ in them which we can factor out:

$$A_1 \sin(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_1) = \Im\left[e^{i\omega t}\left(A_1 e^{i\phi_1} + A_2 e^{i\phi_2}\right)\right].$$

Before we analyze this any further, let's take another look at $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$. Note that I can rewrite them as

$$\tilde{y}_1(t) = A_1 e^{i(\omega t + \phi_1)} = A_1 e^{i\phi_1} e^{i\omega t} = \tilde{A}_1 e^{i\omega t}$$

and

$$\tilde{y}_2(t) = A_2 e^{i(\omega t + \phi_2)} = A_2 e^{i\phi_2} e^{i\omega t} = \tilde{A}_2 e^{i\omega t},$$

where

$$\tilde{A}_1 = A_1 e^{i\phi_1}$$

and

$$\tilde{A}_2 = A_2 e^{i\phi_2}.$$

... TO BE CONTINUED

Watch the following video, then do the exercises below
`https://youtu.be/b5hykAni1UA`

- Write code in Python that plots $\sin(x)$ from $x = -10$ to 10. Then write code that plots the imaginary part e^{ix} from $x = -10$ to 10. Note that