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Търсене и извличане на информация. Приложение на дълбоко машинно обучение.

Домашно задание 2

Функцията на загубата $J_{w,c,\bar{c}_1,\dots,\bar{c}_n}(U,V,W)$ в точка:

$$J_{w,c,\bar{c}_1,\dots,\bar{c}_n}(U,V,W) = -\left(\log\sigma(\mathbf{v}_{c_i}^T W\mathbf{u}_w) + \sum_{j=1}^{n}\log\sigma(-\mathbf{v}_{\bar{c}_j}^T W\mathbf{u}_w)\right)$$

където $\mathbf{v}_c, \mathbf{v}_{\bar{c}_j}, \mathbf{u}_w \in \mathbb{R}^M$ и $W \in \mathbb{R}^{M\times M}$.

Задача 1. Изразете частичните производни

1. $\frac{\partial J_{w,c,\bar{c}_1,\dots,\bar{c}_n}(U,V,W)}{\partial \mathbf{u}_w}$
2. $\frac{\partial J_{w,c,\bar{c}_1,\dots,\bar{c}_n}(U,V,W)}{\partial \mathbf{v}_c}$
3. $\frac{\partial J_{w,c,\bar{c}_1,\dots,\bar{c}_n}(U,V,W)}{\partial \mathbf{v}_{\bar{c}_j}}$
4. $\frac{\partial J_{w,c,\bar{c}_1,\dots,\bar{c}_n}(U,V,W)}{\partial W}$

Решение.
1.

$$\begin{aligned}
&\frac{\partial}{\partial \mathbf{u}_w} - \log\sigma(\mathbf{v}_{c_i}^T W\mathbf{u}_w) = \\
&\overset{\log\sigma(z)'=1-\sigma(z)}{=} -\left(1-\sigma(\mathbf{v}_{c_i}^T W\mathbf{u}_w)\right)\frac{\partial}{\partial \mathbf{u}_w}\mathbf{v}_{c_i}^T W\mathbf{u}_w = \\
&\overset{\frac{\partial \mathbf{a}^T\mathbf{b}}{\partial \mathbf{b}}=\mathbf{a}}{=} -\left(1-\sigma(\mathbf{v}_{c_i}^T W\mathbf{u}_w)\right) W^T\mathbf{v}_{c_i}
\end{aligned} \tag{1}$$

$$\begin{aligned}
&\frac{\partial}{\partial \mathbf{u}_w} - \sum_{j=1}^{n}\log\sigma(-\mathbf{v}_{\bar{c}_j}^T W\mathbf{u}_w) = \\
&= -\sum_{j=1}^{n}\left(1-\sigma(-\mathbf{v}_{\bar{c}_j}^T W\mathbf{u}_w)\right)\frac{\partial}{\partial \mathbf{u}_w} - \mathbf{v}_{\bar{c}_j}^T W\mathbf{u}_w = \\
&= -\sum_{j=1}^{n}\left(1-\sigma(-\mathbf{v}_{\bar{c}_j}^T W\mathbf{u}_w)\right)\left(-W^T\mathbf{v}_{\bar{c}_j}\right) = \\
&= \sum_{j=1}^{n}\left(1-\sigma(-\mathbf{v}_{\bar{c}_j}^T W\mathbf{u}_w)\right) W^T\mathbf{v}_{\bar{c}_j}
\end{aligned} \tag{2}$$

От (1) и (2):

$$\boxed{\frac{\partial J_{w,c,\bar{c}_1,\dots,\bar{c}_n}(U,V,W)}{\partial \mathbf{u}_w} = -\left(\sigma(-\mathbf{v}_{c_i}^T W\mathbf{u}_w)\right) W^T\mathbf{v}_{c_i} + \sum_{j=1}^{n}\sigma(\mathbf{v}_{\bar{c}_j}^T W\mathbf{u}_w) W^T\mathbf{v}_{\bar{c}_j}}$$

2. При диференциране по $\mathbf{v}_c$ второто събираемо $-\sum_{j=1}^{n} \log \sigma(-\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w)$ се занулява и остава само

$$\begin{aligned} &\frac{\partial}{\partial \mathbf{v}_c} - \log \sigma(\mathbf{v}_{c_i}^T W \mathbf{u}_w) = \\ &= -\left(1 - \sigma(\mathbf{v}_{c_i}^T W \mathbf{u}_w)\right) \frac{\partial}{\partial \mathbf{v}_c} \mathbf{v}_{c_i}^T W \mathbf{u}_w = \\ &\overset{\frac{\partial \mathbf{b}^T \mathbf{a}}{\partial \mathbf{b}} = \mathbf{a}}{=} -\left(1 - \sigma(\mathbf{v}_{c_i}^T W \mathbf{u}_w)\right) W \mathbf{u}_w \end{aligned} \tag{3}$$

И получаваме

$$\boxed{\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial \mathbf{v}_c} = -\left(\sigma(-\mathbf{v}_{c_i}^T W \mathbf{u}_w)\right) W \mathbf{u}_w}$$

3. Аналогично първото събираемо се занулява при диференциране по $\mathbf{v}_{\bar{c}_j}$ и остава

$$\begin{aligned} &\frac{\partial}{\partial \mathbf{v}_{\bar{c}_j}} - \sum_{i=1}^{n} \log \sigma(-\mathbf{v}_{\bar{c}_i}^T W \mathbf{u}_w) = \\ &= -\sum_{i=1}^{n} \left(1 - \sigma(-\mathbf{v}_{\bar{c}_i}^T W \mathbf{u}_w)\right) \frac{\partial}{\partial \mathbf{v}_{\bar{c}_j}} - \mathbf{v}_{\bar{c}_i}^T W \mathbf{u}_w \\ &= \left(1 - \sigma(-\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w)\right) W \mathbf{u}_w \end{aligned} \tag{4}$$

Тъй като при $i \neq j$: $\frac{\partial}{\partial \mathbf{v}_{\bar{c}_j}} - \mathbf{v}_{\bar{c}_i}^T W \mathbf{u}_w = 0$.

$$\boxed{\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial \mathbf{v}_{\bar{c}_j}} = \sigma(\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w) W \mathbf{u}_w}$$

4.

$$\begin{aligned} &\frac{\partial}{\partial W} - \log \sigma(\mathbf{v}_{c_i}^T W \mathbf{u}_w) = \\ &= -\left(1 - \sigma(\mathbf{v}_{c_i}^T W \mathbf{u}_w)\right) \frac{\partial}{\partial W} \mathbf{v}_{c_i}^T W \mathbf{u}_w = \\ &\overset{\frac{\partial \mathbf{a}^T X \mathbf{b}}{\partial X} = \mathbf{a}\mathbf{b}^T}{=} -\left(1 - \sigma(\mathbf{v}_{c_i}^T W \mathbf{u}_w)\right) \mathbf{v}_{c_i} \mathbf{u}_w^T \end{aligned} \tag{5}$$

$$\begin{aligned} &\frac{\partial}{\partial W} - \sum_{j=1}^{n} \log \sigma(-\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w) = \\ &= -\sum_{j=1}^{n} \left(1 - \sigma(-\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w)\right) \frac{\partial}{\partial W} - \mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w = \\ &= \sum_{j=1}^{n} \left(1 - \sigma(-\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w)\right) \mathbf{v}_{\bar{c}_j} \mathbf{u}_w^T \end{aligned} \tag{6}$$

$$\boxed{\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial W} = -\left(\sigma(-\mathbf{v}_{c_i}^T W \mathbf{u}_w)\right) \mathbf{v}_{c_i} \mathbf{u}_w^T + \sum_{j=1}^{n} \sigma(\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w) \mathbf{v}_{\bar{c}_j} \mathbf{u}_w^T}$$

Задача 2. Изразете частичните производни

1. $\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial \mathbf{u}_w}$
2. $\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial \tilde{V}}$
3. $\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial W}$

където $\mathbf{u}_w \in \mathbb{R}^M$, $\tilde{V} \in \mathbb{R}^{(n+1)\times M}$, $\bar{\delta}_c \in \mathbb{R}^{n+1}$, $W \in \mathbb{R}^{M\times M}$.

Решение.

$$\tilde{V}W\mathbf{u}_w = \begin{bmatrix} \mathbf{v}_c^T W\mathbf{u}_w \\ \mathbf{v}_{\bar{c}_1}^T W\mathbf{u}_w \\ \mathbf{v}_{\bar{c}_2}^T W\mathbf{u}_w \\ \vdots \\ \mathbf{v}_{\bar{c}_n}^T W\mathbf{u}_w \end{bmatrix} \in \mathbb{R}^{n+1} \tag{7}$$

1. Преработваме полученото от 1.1.

$$\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial \mathbf{u}_w} = -\left(\sigma(-\mathbf{v}_{c_i}^T W\mathbf{u}_w)\right) W^T\mathbf{v}_{c_i} + \sum_{j=1}^{n} \sigma(\mathbf{v}_{\bar{c}_j}^T W\mathbf{u}_w) W^T\mathbf{v}_{\bar{c}_j}$$

$$\Downarrow$$

$$\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial \mathbf{u}_w} = \left(\sigma(\mathbf{v}_{c_i}^T W\mathbf{u}_w) - 1\right) W^T\mathbf{v}_{c_i} + \sum_{j=1}^{n} \sigma(\mathbf{v}_{\bar{c}_j}^T W\mathbf{u}_w) W^T\mathbf{v}_{\bar{c}_j}$$

Агрументите при всички извиквания на сигмоид функцията са елементите на $\tilde{V}W\mathbf{u}_w$.
А множителите вектори са редовете на

$$\tilde{V}W = \begin{bmatrix} W^T\mathbf{v}_c \\ W^T\mathbf{v}_{\bar{c}_1}^T \\ \vdots \\ W^T\mathbf{v}_{\bar{c}_n}^T \end{bmatrix} \in \mathbb{R}^{(n+1)\times M} \tag{8}$$

като $W^T\mathbf{v}_c \in \mathbb{R}^M$, тоест $\frac{\partial J(U,V,W)}{\partial \mathbf{u}_w} \in \mathbb{R}^M$.
Неформално, поелементно умножаваме $\left(\sigma(\tilde{V}W\mathbf{u}_w) - \bar{\delta}_c\right) \in \mathbb{R}^{n+1}$ и $\tilde{V}W \in \mathbb{R}^{(n+1)\times M}$ (скалар по вектор-ред) и сумираме вектор-редовете.

$$\boxed{\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial \mathbf{u}_w} = \sum_{k=1}^{n+1} \left(\sigma(\tilde{V}W\mathbf{u}_w) - (\bar{\delta}_c)\right)_k \left((\tilde{V}W)_{k,\bullet}\right)^T}$$

където прилагаме σ поелементно върху вектора.

2.
$$\frac{\partial \mathbf{v}_c^T W \mathbf{u}_w}{\partial \tilde{V}} = \begin{bmatrix} - & W\mathbf{u}_w & - \\ & \mathbb{O}_{n\times M} & \end{bmatrix} \in \mathbb{R}^{(n+1)\times M} \tag{9}$$

Тогава
$$\frac{\partial - \log \sigma(\mathbf{v}_c^T W \mathbf{u}_w)}{\partial \tilde{V}} = \Big(\sigma(\mathbf{v}_c^T W \mathbf{u}_w) - 1\Big) \begin{bmatrix} - & W\mathbf{u}_w & - \\ & \mathbb{O}_{n\times M} & \end{bmatrix} \tag{10}$$

$$\begin{aligned} \frac{\partial - \sum_{j=1}^n \log \sigma(-\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w)}{\partial \tilde{V}} &= \sum_{j=1}^n \sigma(\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w) \begin{bmatrix} & \mathbb{O}_{j\times M} & \\ - & W\mathbf{u}_w & - \\ & \mathbb{O}_{(n-j)\times M} & \end{bmatrix} = \\ &= \begin{bmatrix} & \mathbb{O}_{1\times M} & \\ - & \sigma(\mathbf{v}_{\bar{c}_1}^T W \mathbf{u}_w) W\mathbf{u}_w & - \\ & \vdots & \\ - & \sigma(\mathbf{v}_{\bar{c}_n}^T W \mathbf{u}_w) W\mathbf{u}_w & - \end{bmatrix} \in \mathbb{R}^{(n+1)\times M} \end{aligned} \tag{11}$$

Събирайки (10) и (11):

$$\boxed{\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\tilde{V}} = \Big(\sigma(\tilde{V} W \mathbf{u}_w) - \bar{\delta}_c\Big) \otimes W\mathbf{u}_w}$$

където прилагаме σ поелементно върху вектора.

3. Преработвайки резултата от 1.4.

$$\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial W} = \Big(\sigma(\mathbf{v}_{c_i}^T W \mathbf{u}_w) - 1\Big)\, \mathbf{v}_{c_i} \mathbf{u}_w^T + \sum_{j=1}^n \sigma(\mathbf{v}_{\bar{c}_j}^T W \mathbf{u}_w) \mathbf{v}_{\bar{c}_j} \mathbf{u}_w^T \in \mathbb{R}^{M\times M}$$

Тоест събираме скалирани матрици

$$\boxed{\frac{\partial J_{w,c,\bar{c}_1,\ldots,\bar{c}_n}(U,V,W)}{\partial W} = \sum_{k=1}^{n+1} \Big(\sigma(\tilde{V} W \mathbf{u}_w) - \bar{\delta}_c\Big)_k \Big((\tilde{V}_{k,\bullet})^T \otimes \mathbf{u}_w\Big)}$$

където прилагаме σ поелементно върху вектора.