Nonparametric statistic

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1 Introduction

This test is used to assess whether the median or distribution of a single sample is equal to a specified value.

2 Sign Test

Null Hypothesis:

$$H_0 : M_0 = m_0$$

$$H_1 : M_0 \neq m_0$$

$$H_0 : P(\text{observing }" + ") = P(\text{observing }" - ")$$

Test Statistic:
Let T be the number of positive signs in the sample.

$$T = \sum_{i=1}^{n} I(X_i < m_0)$$

where I is the indicator function. The null distribution of T is Bin (n, $p = \frac{1}{2}$).

$$n = n' - \text{numbers of } \text{"tie"}$$

where n' is the sample size.
Let

$$Y \sim \text{Bin}(n, p = \frac{1}{2})$$

To find r

$$P(Y \leq r) = \alpha_1 \approx \frac{\alpha}{2}$$

If T is less than or equal to r or is greater than or equal to $n - r$, we will reject H_0.
when $n > 25$:

$$r = \frac{1}{2}\left(n + z_{\alpha/2} \cdot \sqrt{n}\right)$$

3 Wilcoxon Signed-Rank Test

Null Hypothesis:

$$H_0 : M_0 = m_0$$

$$H_1 : M_0 \neq m_0$$

Procedure:

1. Calculate the Differences

For each observation X_i, calculate the absolute difference from m_0:

$$Z_i = |X_i - m_0|$$

2. Exclude Zero Differences

Remove any pairs where $Z_i = 0$ to avoid ties.

3. Rank the Absolute Differences

Rank the non-zero values Z_i from smallest to largest, assigning each Z_i a rank R_i. For tied ranks, assign the average rank to each tied value.

4. Assign Signs to Ranks

Assign a positive sign to the ranks of positive differences $X_i > m_0$ and a negative sign to the ranks of negative differences $X_i < m_0$.

5. Compute the Test Statistic W

Let W be the sum of the ranks with positive signs:

$$W = \sum_{X_i > m_0} R_i$$

6. Determine the Critical Value

Use the Wilcoxon signed-rank test critical value table or approximate the critical value for large n with the normal distribution.

Decision Rule:

Reject H_0 if W is less than or equal to the critical value (or, for a two-tailed test, if W is outside the acceptance region).

4 Kolmogorov-Smirnov Test

Null Hypothesis:

The sample distribution does not significantly differ from a specified theoretical distribution:

$$H_0 : F(x) = F_0(x)$$

$$H_1 : F(x) \neq F_0(x)$$

where $F(x)$ is the cumulative distribution function (CDF) of the sample, and $F_0(x)$ is the CDF of the theoretical distribution.

Test Statistic:

Calculate the empirical cumulative distribution function $S(x)$:

$$S(x) = \frac{\text{number of } X_i\text{'s that are } \leq x}{n}, \quad x \in R$$

The test statistic D is then defined as:

$$D = \max |F_0(x) - S(x)|$$

Decision Rule:

Use the observed D value and compare it to the critical value obtained from the Kolmogorov-Smirnov table. Reject H_0 if D_{obs} exceeds the $(1-\alpha)$th quantile of the distribution of D at the desired significance level α.

5 Shapiro-Wilk Test

Null Hypothesis:

The sample comes from a unknown distribution:

$$H_0 : \text{The data is normally distributed with unknown mean and variance}$$

$$H_1 : \text{The data is not normally distributed}$$

Test Statistic:

The Shapiro-Wilk test statistic W is calculated as:

$$W = \frac{\left(\sum_{i=1}^{k} a_i (X_{(n-i+1)} - X_{(i)})\right)^2}{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}$$

where:

- (i) $k = n/2$ if n is even
- (ii) $k = (n\text{-}1)/2$ if n is odd, and $X_{(k+1)}$ is not involved in the calculation of W
- $X_{(i)}$ represents the i-th ordered sample value.
- a_i are the weights based on the sample size n and the assumption of normality. The values of a_i are tabulated in Shapiro-Wilk Table1 for n = 2,3,...,50.

- $\bar{X}$ is the sample mean:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Decision Rule:

Given a significance level α, the α-th quantile W_α are tabulated in Shapiro-Wilk Table2. If the sample is drawn from a normal distribution, W should be close to 1. If W is smaller than the critical value W_α, reject the null hypothesis H_0 and conclude that the data does not follow a normal distribution.

The coefficients and critical values can be found in the Shapiro-Wilk table available at:

https://real-statistics.com/statistics-tables/shapiro-wilk-table/

For a data set with $n > 50$, we may use statistical software package.

6 Randomness (Run Test)

Null Hypothesis:

The sequence of observations is random, meaning that the number of runs is consistent with the expected distribution of runs under random conditions:

$$H_0 : \text{The sequence is random}$$

Procedure:

1. **Set a Threshold:** Choose a threshold value (e.g., mean or median) to categorize the continuous variable into two groups.

2. **Binarize the Data:** Label each observation as "success" (above threshold) or "failure" (below threshold).

3. **Count the Runs:** Calculate the number of runs R in the binary sequence.

U: The number of changes between successes and failures in the sequence.

n_1: Number of successes.

n_2: Number of failures.

4. **Determine Critical Values:** Find the lower and upper critical values r_1 and r_2 from the table for a significance level $\alpha = 0.05$.

Test Statistic:

$$U : \text{The number of changes between successes and failures in the sequence.}$$

Decision Rule:

Find the lower and upper critical values r_1 and r_2 from the table for a significance level $\alpha = 0.05$. Reject H_0 if $U \leq r_1$ or $U \geq r_2$.

When either n_1 or n_2 is greater than 20, use the test statistic:

$$Z = \frac{U - \frac{2n_1n_2}{n_1+n_2+1}}{\sqrt{\frac{2n_1n_2(2n_1n_2-n_1-n_2)}{(n_1+n_2)^2(n_1+n_2-1)}}} \sim N(0,1).$$

7 Trend Test (Cox and Stuart Test)

Null Hypothesis:

$$H_0 : \text{There is no trend in the data.}$$

$$H_1 : \text{There is either an upward or a downward trend.}$$

Procedure

Divide the data into two halves. For each pair of observations (X_i, X_{i+c}), where $c = \frac{n'}{2}$ if n' is even, and $c = \frac{n'+1}{2}$ if n' is odd. Assign to each (X_i, X_{i+c}) a "+" symbol if $X_i < X_{i+c}$, a "-" symbol if $X_i > X_{i+c}$, and a "tie" symbol otherwise. Let n be the number of non-tied pairs of data.

Test Statistic:

$$T = \text{total number of +'s}$$

and the null distribution of T is Bin (n, $p = \frac{1}{2}$). The test hypothesis and the rest of the procedure is identical to that of the sign test.

Decision Rule:

Use the binomial test to determine the significance of the count of upward trends. Reject H_0 if the number of upward or downward trends is significantly different from what would be expected by chance.

8 References

1. OpenAI. (2024). ChatGPT [Large Language Model]. https://chat.openai.com

2. Hsing-Ming Chang. (n.d.). Chapter 2: Inference on p of BIN(n,p), Test and Confidence Interval for Median, Test for Randomness and Test for Trend [Unpublished Lecture Notes]. NCKU.

3. Hsing-Ming Chang. (n.d.). Chapter 3: Test of Goodness of Fit (One Sample) [Unpublished Lecture Notes]. NCKU.