Chapter 1, Section 4

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1. If f and g are regular functions on open subsets U and V of a variety X, and if $f = g$ on $U \cap V$, show that the function which is f on U and g on V is a regular function on $U \cup V$. Conclude that if f is a *rational* function on X, then there is a largest open subset U of X on which f is represented by a regular function. We say that f is *defined* at the points of U.

Proof. Denote the function which is f on U and g on V as F. It suffices to show for all $P \in U \cap V$ that there exists an open neighborhood W of P contained in $U \cap V$ such that F is a rational function on W. Since $U \cap V$ is an open subset of U and V, for all $P \in U \cap V$ there exists open subsets A of U and B of W such that F is rational with $h(Q), h(Q) \neq 0$ for all $Q \in A \cap B$ and $F = f = g$ on $A \cap B$, hence $W = A \cap B \subseteq U \cap V$ is the desired open subset.

If f is a rational function on X, then the set of open subsets of X on which f is represented by regular function is nonempty, and since X is a noetherian space, there is a maximal element in this set. Then, the largest open subset U of X on which f is represented by a regular function is the union of maximal elements in this subset of open subsets. $\square$

2. Same problem for rational maps. If φ is a rational map of X to Y, show there is a largest open set on which φ is represented by a morphism. We say the rational map is *defined* at the points of that open set.

Proof. Let φ and ψ be morphisms on open subsets U and V of a variety X to a variety Y, and suppose $\varphi = \psi$ on $U \cap V$. If f is a regular function on Y, then $\varphi^* f$ and $\psi^* f$ are regular functions on the open subsets U and V which agree on $U \cap V$, so by Exercise 1 the function which is $\varphi^* f$ on U and $\psi^* f$ on V is a regular function $U \cap V$, i.e. the map that is φ on U and ψ on V is indeed a rational map.

If $\varphi : X \to Y$ is a rational map, then the set of open subsets of X on which φ is represented by morphism function is nonempty, and since X is a noetherian space, there is a maximal element in this set. Then, the largest open subset U of X on which φ is represented by a morphism is the union of maximal elements in this subset of open subsets. $\square$

4. A variety Y is *rational* if it is birationally equivalent to $\mathbb{P}^n$ for some n (or, equivalently by (4.5), if $K(Y)$ is a pure transcendental extension of k).

(a) Any conic in $\mathbb{P}^2$ is a rational curve.

(b) The cuspidal cubic $y^2 = x^3$ is a rational curve.

(c) Let Y be the nodal cubic curve $y^2 z = x^2(x + z)$ in $\mathbb{P}^2$. Show that the projection φ from the point $P = (0, 0, 1)$ to the line $z = 0$ induces a birational map from Y to $\mathbb{P}^1$. Thus, Y is a rational curve.

Proof.

(a) A conic in $\mathbb{P}^2$ can be covered by open affine varieties that are either isomorphic to $y = x^2$ or $xy = 1$. The former is isomorphic to $\mathbb{A}^1$, hence it is isomorphic to an open subset of $\mathbb{P}^1$, hence it is birationally equivalent to $\mathbb{P}^1$. The latter has function field isomorphic to $k(x)$, hence it is birationally equivalent to $\mathbb{A}^1$, hence it is also birationally equivalent to $\mathbb{P}^1$.

(b) The cuspidal cubic has coordinate ring $k[t^2, t^3]$, so its function field is $k(t)$, hence it is birationally equivalent to $\mathbb{A}^1$, hence it is birationally equivalent to $\mathbb{P}^1$.

(c) The line $z = 0$ in $\mathbb{P}^2$ corresponds to a hyperplane isomorphic to $\mathbb{P}^1$, so the projection $\varphi : \mathbb{P}^2 - \{P\} \to \mathbb{P}^1$ is a morphism. In coordinates, φ is defined as $(x_0, x_1, x_2) \mapsto (x_0, x_1)$, so φ induces a morphism from $Y - P$ to $\mathbb{P}^1$. Thus, $\varphi(Y - P)$ is the set of all lines in $\mathbb{A}^2$ that pass through the origin and a point in the affine nodal curve $y^2 = x^3 + x^2$.

This is an open set in $\mathbb{P}^1$ isomorphic to $\mathbb{A}^1$ since it contains all lines in $\mathbb{A}^2$ besides the one defined by $x = \pm y$. To further elaborate, if $(x, y) \in \mathbb{P}^1$ with $x \neq \pm y$ and $x \neq 0$, say, then write $y = \lambda x$ with $\lambda \neq \pm 1$, so we have

$$\lambda^2 x^2 = x^3 + x^2 \implies x = \lambda^2 - 1,$$

that is there exists a line in $\mathbb{A}^2$ passing through a point in $y^2 + x^3 + x^2$ with slope y/x. Rephrasing, we have shown that the map $\varphi : Y - P \to \mathbb{P}^1$ is surjective besides at the two points $(1, 1)$ and $(1, -1)$. It is also injective since the x_2-coordinate can be completely determined by the values of (x_0, x_1), that is if $x_0 \neq 0$, then setting $x_0 = 1$, we have

$$x_1^2 x_2 = 1 + x_2 \implies x_2 = \frac{1}{x_1^2 - 1} \text{ or } x_2 = 0,$$

and $x_1 \neq \pm 1$ since $x_0 \neq \pm x_1$, and $x_2 \neq 0$ since the only point on Y with $x_2 = 0$ is P, and φ is restricted $Y - P$. Hence, φ is an isomorphism of the open subset $Y - P$ of Y to the open subset $\mathbb{P}^1 - \{(1, 1), (1, -1)\}$ in $\mathbb{P}^1$, hence Y is birationally equivalent to $\mathbb{P}^1$.

□

5. Show that the quadric surface $Q : xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$.

Proof. Q is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, and $\mathbb{P}^1 \times \mathbb{P}^1$ contains a copy of $\mathbb{A}^1 \times \mathbb{A}^1$, so Q contains a copy of $\mathbb{A}^2$, hence Q is birational to $\mathbb{P}^2$. It is an axiom of projective geometry that any two lines intersect in $\mathbb{P}^2$; however, it was shown in Exercise 2.15 that there exists lines in Q that do not intersect, hence Q and $\mathbb{P}^2$ cannot be isomorphic. □

10. Let Y be the cuspidal cubic curve $y^2 = x^3$ in $\mathbb{A}^2$. Blow up the point $O = (0, 0)$, let E be the exceptional curve, and let $\widetilde{Y}$ be the strict transform of Y. Show that E meets $\widetilde{Y}$ in one point, and that $\tilde{Y} \cong \mathbb{A}^1$. In this case the morphism $\varphi : \widetilde{Y} \to Y$ is bijective and bicontinuous, but it is not an isomorphism.

Proof. Let t, u be homogenous coordinates for $\mathbb{P}^1$. Then X, the blowing-up of $\mathbb{A}^2$ at O, is defined by the equation $xu = ty$ inside $\mathbb{A}^2 \times \mathbb{P}^1$. We obtain the total inverse image of Y in X by considering the equations $y^2 = x^3$ and $xu = ty$ in $\mathbb{A}^2 \times \mathbb{P}^1$. Now $\mathbb{P}^1$ is covered by the open sets $t \neq 0$ and $u \neq 0$, which we consider separately. If $t \neq 0$, we can set $t = 1$, and use u as an affine parameter. Then we have the equations

$$y^2 = x^3, \quad y = xu$$

in $\mathbb{A}^3$ with coordinates x, y, u. Substituting, we get $x^2u^2 - x^3 = 0$, which factors. Thus, we obtain two irreducible components, one defined by $x = 0$, $y = 0$, u arbitrary, which is E, and the other defined by $u = x$ and $y = ux$. This is $\widetilde{Y}$, and $\widetilde{Y}$ meets E at the point $u = 0$. Similarly, if $u \neq 0$, then we can set $u = 1$, and use t as an affine parameter to obtain the equations

$$y^2 = x^3, \quad x = ty$$

in $\mathbb{A}^3$ with coordinates x, y, t. Substituting, we get $y^2 = t^3y^3$, which factors as well. Besides the exceptional curve, we have the component defined by $1 - t^3y = 0$ and $x = ty$, which does not meet E. Hence, E meets $\widetilde{Y}$ at only $(1, 0) \in E$. This also show $\widetilde{Y}$ is contained in the open set defined by $t \neq 0$, so it is isomorphic to the affine variety in $\mathbb{A}^3$ defined by $u = x$ and $y = ux$, which isomorphic to $y = x^2$ in $\mathbb{A}^2$, hence $\widetilde{Y} \cong \mathbb{A}^1$. □