Chapter 4, Section 1

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1. Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

Proof. Pick another point $Q \neq P \in X$, which exists since we always assume the ground field k is algebraically closed. By the Riemann-Roch Theorem, there exists $n > 0$ such that $\dim |n(-P + 2Q)| > 0$. Hence, there exists a rational function with pole at P of order n, and regular everywhere else. $\square$

2. Again let X be a curve, and let $P_1, \ldots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i, and regular elsewhere.

Proof. Pick another point $Q \neq P_i \in X$, and repeat (Ex. 1.1) with $n(-\sum P_i + 2rQ)$. $\square$

3. Let X be an integral, separated, regular, one-dimensional scheme of finite type over k, which is *not* proper over k. Then X is affine.

4. Show that a separated, one-dimensional scheme of finite type over k, none of whose irreducible components is proper over k, is affine.

5. For an effective divisor D on a curve X of genus g, show that $\dim |D| \leq \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

Proof. By definition $\dim |D| = \ell(D) - 1$. Rearranging the Riemann-Roch Theorem gives

$$\dim |D| = \ell(K - D) + \deg D - g,$$

so we want to show $\ell(K - D) \leq g$. But D is effective, so $\mathscr{L}(K - D) \to \mathscr{L}(K)$ is injective, and $g = \ell(K) = \dim H^0(X, \mathscr{L}(K))$ by definition. $\square$

6. Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leq g + 1$.

Proof. By the Riemann-Roch Theorem, it suffices to show there exists a divisor D such that $2 \leq \ell(D) \leq \ell(K - D)$. Indeed, take $D = -P$ $\square$

7. A curve X is called *hyperelliptic* if $g \geq 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

(a) If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. Use (II, 7.8.1) to conclude that X is hyperelliptic.

(b) Show taht the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbb{P}^1$. Thus there exist hyperelliptic curves of any genus $g \geq 2$.