Chapter 2, Section 1

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1. Let A be an Abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \emptyset$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.

Proof. Let $\mathscr{C}^+$ be the sheaf associated to the constant presheaf $\mathscr{C}$ defined above. It suffices to show $\mathscr{C}^+$ and $\mathscr{A}$ are isomorphic at the level of stalks. Fix $P \in X$, then the stalk of $\mathscr{C}^+$ at P is the same as the stalk of $\mathscr{C}$ at P, and since the restriction morphisms of $\mathscr{C}$ are identity maps, we must have $\mathscr{C}^+_P = \mathscr{C}_P = A$. The constant sheaf $\mathscr{A}$ is defined as $U \mapsto \{\text{continuous maps } U \to A\}$, where A has the discrete topology. Then, we can take the stalk of $\mathscr{A}$ at x to be the direct limit of $\mathscr{A}(U)$ where U is a connected open neighborhood of x, and if U is connected, $\mathscr{A}(U) = A$ since the image of a connected set must be connected, i.e. $U \to A$ is continuous if and only if it is constant. Therefore, we have $\mathscr{A}_P = A$. Hence, $\mathscr{C}^+$ and $\mathscr{A}$ are isomorphic. □

2. (a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, show that for each point P, $(\ker\varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im}\varphi)_P = \operatorname{im}(\varphi_P)$.

(b) Show that φ is injective (respectively, surjective) if and only if the induced maps on the stalks φ_P is injective (respectively, surjective) for all P.

(c) Show that a sequence $\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \dots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof.

(a) It is true in general that kernels commute with limits and cokernels commmute with colimits in any abelian category, since kernels and cokernels can be realized as limits and colimits, respectively.

(b) φ is injective $\iff \ker\varphi = 0 \iff (\ker\varphi)_P = 0 \iff \ker\varphi_P = 0$.
φ is surjective $\iff \operatorname{im}\varphi = \mathscr{G} \iff \operatorname{im}(\varphi_P) = (\operatorname{im}\varphi)_P = \mathscr{G}_P$.

(c) The sequence $\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \dots$ is exact $\iff \operatorname{im}\varphi^{i-1} = \ker\varphi^i$ for all $i \iff (\operatorname{im}\varphi^{i-1})_P = (\ker\varphi^i)_P$ for all i and $P \in X \iff \operatorname{im}(\varphi_P^{i-1}) = \ker(\varphi_P^i)$ for all i and $P \in X$ by part (a) above.

□

3. (a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s\big|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, and an open set U such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Proof.

(a) Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a surjective morphism of sheaves. Then for any $P \in U$, the induced morphism of stalks $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is surjective. The elements of $\mathscr{G}_P$ (and similarly $\mathscr{F}_P$) are an equivalence class of ordered pairs (U, s) with U an open neighborhood of P and $s \in \mathscr{G}(U)$ and the equivalence relation $(U, s) \sim (V, t)$ if and only if there exists some open $W \subseteq U \cap V$ containing P such that $s\big|_W = t\big|_W$. If φ_P is surjective, then for every $(U, s) \in \mathscr{G}_P$ there exists some $(V, f) \in \mathscr{G}_P$ such that

$$\varphi_P(V, f) = (V, \varphi f), \quad \varphi : \mathscr{F}(V) \to \mathscr{G}(V),$$

so there exists some open subset $W \subseteq U \cap V$ such that $\varphi f\big|_W = s\big|_W$. We can repeat this process for all $x \in U$ to obtain an open cover with the desired properties, that is $\{W_x\}_{x \in U}$ of U where W_x is defined as above with an associated $f_x \in \mathscr{F}(W_x)$ such that by definition $\varphi f_x = s\big|_{W_x}$.

Conversely, it suffices to show $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is surjective for all $P \in X$. Fix $P \in X$ and consider an arbitrary element $(U, s) \in \mathscr{F}_P$. We want to show there exists some $(V, f) \in \mathscr{G}_P$ such that $(V, \varphi_V(f)) \sim (U, s)$. To that end, by assumption there exists some open subset $W \subseteq U$ and $g \in \mathscr{F}(W)$ such that $\varphi_W(g) = s\big|_W$, hence $\varphi_P(W, g) = (U, s)$.

(b) We provide an example of a sheaf from open sets to sets that satisfies the condition. Let $X, Y = S^1$. Let $\pi_X : X \to S^1$ be the identity map, and let $\tau : Y \to X$ be defined by $z \mapsto z^2$ where X, Y is identified with the unit circle in the complex plane, and let $\pi_Y := \tau \circ \pi_X$. Define a sheaf $\mathscr{F}$ on S^1 for a nonempty open subset U of S^1 as

$$\begin{aligned} \mathscr{F}(U) &= \{\text{sections } s : U \to X \text{ with respect to } \pi_X\} \\ &= \{\text{continuous maps } s : U \to X \text{ such that } \pi_X \circ s = \mathrm{id}_U\}, \end{aligned}$$

and similarly define $\mathscr{G}$ as the map that maps U to the set of sections from U to Y with respect to π_Y. If $s \in \mathscr{G}(U)$ so that $\pi_Y \circ s = \mathrm{id}_U$, then we can compose s with τ so that

$$\pi_X \circ (\tau \circ s) = (\pi_X \circ \tau) \circ s = \pi_Y \circ s = \mathrm{id}_U,$$

which shows $\tau \circ s$ is a section from to X, so we can define a morphism $\tau_\# : \mathscr{G} \to \mathscr{F}$ induced by τ as

$$\begin{aligned} \tau_\#(U) : \mathscr{G}(U) &\to \mathscr{F}(U) \\ s &\mapsto \tau \circ s. \end{aligned}$$

Let U be the entire space S^1, then $\mathscr{G}(S^1) = \emptyset$ and $\mathscr{F}(S^1) = \{\mathrm{id} : S^1 \to X\}$, so $\mathscr{G}(S^1) \to \mathscr{F}(S^1)$ cannot be surjective. However, for any proper open subset U of S^1, the map $\tau_\# : \mathscr{G}(U) \to \mathscr{F}(U)$ is surjective, thus by part (a) above $\tau_\#$ is a surjective morphism.

□

5. Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

Proof. $\varphi : \mathscr{F} \to \mathscr{G}$ is an isomorphism $\iff$ $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism for all $P \in X$ $\iff$ φ is injective and surjective by Exercise 2. □

6. (a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus, there is an exact sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{F}' \longrightarrow 0.$$

(b) Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient $\mathscr{F}$ by this subsheaf.

Proof.

(a) The map $\mathscr{F}(U) \to \mathscr{F}(U)/\mathscr{F}'(U)$ is surjective for all open $U \subseteq X$, so $\pi : \mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is surjective. To show π has kernel $\mathscr{F}'$, by Exercise 2 it suffices to show $(\ker \pi)_P = \mathscr{F}'_P$ for all $P \in X$. Each map $\mathscr{F}(U) \to \mathscr{F}(U)/\mathscr{F}'(U)$ has kernel $\mathscr{F}'(U)$, hence we have $(\ker \pi)_P = \varinjlim \mathscr{F}'(U) = \mathscr{F}'_P$.

(b) By Exercise 2, We have the following equivalent statements:

$$\begin{aligned} &0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0 \text{ is exact.} \\ \iff\ &0 \to \mathscr{F}'_P \to \mathscr{F}_P \to \mathscr{F}''_P \to 0 \text{ is exact.} \\ \iff\ &\mathscr{F}'_P \subseteq \mathscr{F}_P \text{ and } \mathscr{F}''_P \simeq \mathscr{F}_P/\mathscr{F}'_P \simeq (\mathscr{F}/\mathscr{F}')_P \\ \iff\ &\mathscr{F}' \text{ is isomorphic to a subsheaf of } \mathscr{F} \text{ and } \mathscr{F}'' \cong \mathscr{F}/\mathscr{F}'. \end{aligned}$$

□

7. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.

(a) Show that $\operatorname{im}\varphi \cong \mathscr{F}/\ker\varphi$.

(b) Show that $\operatorname{coker}\varphi \cong \mathscr{G}/\operatorname{im}\varphi$.

Proof.

(a) It suffices to show $(\operatorname{im}\varphi)_P \simeq \mathscr{F}_P/(\ker\varphi)_P$, which again follows from the fact that for all open $U \subseteq X$, $\operatorname{im}(\varphi(U)) \simeq \mathscr{F}(U)/\ker(\varphi(U))$.

(b) In the same vain, $\operatorname{coker}\varphi \simeq \mathscr{G}/\operatorname{im}\varphi$ follows from the fact that $\operatorname{coker}(\varphi(U)) \simeq \mathscr{G}(U)/\operatorname{im}(\varphi(U))$.

$\square$

8. For any open subset $U \subseteq X$, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e. if $0 \to \mathscr{F}' \xrightarrow{f} \mathscr{F} \xrightarrow{g} \mathscr{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U, \mathscr{F}') \xrightarrow{f_U} \Gamma(U, \mathscr{F}) \xrightarrow{g_U} \Gamma(U, \mathscr{F}'')$ is an exact sequence of groups.

Proof. Exactness at $\Gamma(U, \mathscr{F}')$ follows from the fact that $\mathscr{F}' \to \mathscr{F}$ is an injective morphism of sheaves if and only if $\Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F})$ is injective for all U. To show exactness at $\Gamma(U, \mathscr{F})$, the sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is exact if and only if $\mathscr{F}'_P \to \mathscr{F}_P \to \mathscr{F}''_P$ is an exact sequence for all $P \in X$

If $s \in \operatorname{im} f_U$ then $(U, s) \in \ker g_P$ for all $P \in X$, so $(U, g(s))$ equals to 0 everywhere locally, hence $s \in g_U$. Conversely, if $s \in \ker g_U$, then by Exercise 3 there exists an open cover $\{U_i\}$ and elements $t_i \in \Gamma(U_i, \mathscr{F}')$ such that $f_U(t_i) = s\big|_{U_i}$. For any i, j, consider $t_i\big|_{U_i \cap U_j}$, $t_j\big|_{U_i \cap U_j}$. Since $f_{U_i \cap U_j} : \Gamma(U_i \cap U_j, \mathscr{F}') \to \Gamma(U_i \cap U_j, \mathscr{F})$ is injective and $f_{U_i \cap U_j}(t_i\big|_{U_i \cap U_j}) = f_{U_i \cap U_j}(t_j\big|_{U_i \cap U_j}) = s\big|_{U_i \cap U_j}$, we have must $t_i\big|_{U_i \cap U_j} = t_j\big|_{U_i \cap U_j}$, so by the sheaf property there exists $t \in \Gamma(U, \mathscr{F}')$ such that $t\big|_{U_i} = t_i$, so $f_U(t) = s$, hence $s \in \operatorname{im} f_U$. $\square$

9. *Direct Sum.* Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathscr{F}$ and $\mathscr{G}$, and is denotes by $\mathscr{F} \oplus \mathscr{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups.

Proof. Since the category of sheaves on X is a full subcategory of presheaves on X, it suffices to show it satisfies the universal property of a direct sum and of direct product in the category of presheaves of abelian groups on X, then show it satisfies the sheaf property. Let $\pi_1 : \mathscr{F} \oplus \mathscr{G} \to \mathscr{F}$, $\pi_2 : \mathscr{F} \oplus \mathscr{G}_{\to}\mathscr{G}$ be the canonical projection morphisms, and let $\mathscr{H} \in \mathfrak{PreSh}_X$ with morphisms $\tau_1 : \mathscr{H} \to \mathscr{F}$ and $\tau_2 : \mathscr{H} \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{H} \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "\tau_2"] \arrow[ddr, bend right, "\tau_1"'] & & \\
& \mathscr{F} \oplus \mathscr{G} \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & \mathscr{G} \\
& \mathscr{F} &
\end{tikzcd}$$

we find the find a unique morphism $\alpha : \mathscr{H} \to \mathscr{F} \oplus \mathscr{G}$ such that the diagram above commmutes. For any open $U \subseteq X$, define $\alpha(U) : \mathscr{H}(U) \to (\mathscr{F} \oplus \mathscr{G})(U)$ as $\alpha(h) = (\tau_1(h), \tau_2(h))$. Since $\alpha(U)$ is unique by the same universal property for abelian groups, α must be the desired unique morphism, and since the direct sum and direct product are essentially the same for finite collections, we have shown $\mathscr{F} \oplus \mathscr{G}$ is a direct sum and product in the category of presheaves.

Now we show $\mathscr{F} \oplus \mathscr{G}$ is a sheaf. Let U be an open subset of X, and let $\{V_i\}$ be a cover of U by open sets. If $(s, t) \in \Gamma(U, \mathscr{F} \oplus \mathscr{G})$, then $(s\big|_{U_i}, t\big|_{U_i}) = (s, t)\big|_{U_i} = 0$ by universal property nonsense, so $s = 0$ and $t = 0$. Also, if we have elements $(s_i, t_i) \in \Gamma(V_i, \mathscr{F} \oplus \mathscr{G})$ such that for all i, j, $(s_i, t_i)\big|_{V_i \cap V_j} = (s_j, t_j)\big|_{V_i \cap V_j}$, then again $(s_i\big|_{U_i \cap U_j}, t_i\big|_{U_i \cap U_j}) = (s_j\big|_{U_i \cap U_j}, t_j\big|_{U_i \cap U_j})$ if and only if $s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$ and $t_i\big|_{U_i \cap U_j} = t_j\big|_{U_i \cap U_j}$, so by the sheaf properties of $\mathscr{F}$ and $\mathscr{G}$, there exists $s \in \Gamma(U, \mathscr{F})$ and $t \in \Gamma(U, \mathscr{G})$ such that $(s, t)\big|_{U_i} = (s_i, t_i)$. $\square$

10. *Direct Limit.* Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the *direct limit* of the system $\{\mathscr{F}_i\}$, denoted $\varinjlim \mathscr{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X.

Proof. Let $\mathscr{F}$ denote the presheaf defined $U \mapsto \varinjlim \mathscr{F}_i$, and let $\mathscr{F}^+$ denote the sheaf associated to $\mathscr{F}$ with associated morphism $\theta : \mathscr{F} \to \mathscr{F}^+$. From the direct system, we have morphisms $\mu_{ij} : \mathscr{F}_i \to \mathscr{F}_j$ and $\mu_i : \mathscr{F}_i \to \mathscr{F}$ such that for all i, j, $\mu_i = \mu_j \circ \mu_{ij}$. To show $\mathscr{F}^+$ is a direct limit in the category-theoretic sense, we must provide the data of morphisms (between sheaves) $\alpha_i : \mathscr{F}_i \to \mathscr{F}^+$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$ for all i, j satisfying the following universal property: if $\mathscr{G}$ is any sheaf on X with the data of morphisms $\{\phi_i : \mathscr{F}_i \to \mathscr{G}\}$ satisfying for all i, j, $\phi_i = \phi_j \circ \mu_{ij}$, then there exists a unique morphism $\phi : \mathscr{F}^+ \to \mathscr{G}$ such that $\phi_i = \phi \circ \alpha_i$. Observe the following commutative diagram:

$$\begin{tikzcd}
\mathscr{F}_i \arrow[r, "\mu_i"] \arrow[rr, bend left, "\phi_i"] \arrow[dr, "\alpha_i:=\theta\circ\mu_i"'] & \mathscr{F} \arrow[r, dashed, "\psi"] \arrow[d, "\theta"'] & \mathscr{G} \\
& \mathscr{F}^+ \arrow[ur, dashed, "\phi"'] &
\end{tikzcd}$$

We claim $\{\alpha_i := \theta \circ \mu_i\}$ are the desired morphisms. By the universal property of the direct limit which defines $\mathscr{F}$ and the direct system of morphisms $\{\psi_i\}$, there exists a unique morphism between presheaves $\phi : \mathscr{F} \to \mathscr{G}$ such that $\phi_i = \psi \circ \mu_i$. Then, by the universal property of the sheaf $\mathscr{F}^+$ associated to the presheaf $\mathscr{F}$, there exists a unique morphism between sheaves $\phi : \mathscr{F}^+ \to \mathscr{G}$ such that $\psi = \phi \circ \theta$. It remains to check $\phi_i = \phi \circ \alpha_i$. Indeed, we have

$$\phi_i = \psi \circ \mu_i = (\phi \circ \theta) \circ \mu_i = \phi \circ (\theta \circ \mu_i) = \phi \circ \alpha_i,$$

and ϕ is unique by the universal properties above. $\square$

11. Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$.

Proof. We directly show the presheaf $\mathscr{F}$ defined by $\mathscr{F}(U) = \varinjlim \mathscr{F}_i(U)$ is a sheaf. Let U be an open subset of X, and let $\{V_i\}$ be open cover of U. By the noetherian hypothesis, we can assume $\{V_j\}$ to be a finite set ($1 \le j \le n$), then we have the following exact sequence of abelian groups

$$0 \longrightarrow \Gamma(U, \mathscr{F}_i) \longrightarrow \textstyle\prod_{j=1}^n \Gamma(V_j, \mathscr{F}_i) \longrightarrow \prod_{j,k} \Gamma(V_j \cap V_k, \mathscr{F}_i),$$

and since both products $\prod_{j=1}^n \Gamma(V_j, \mathscr{F}_i)$ and $\prod_{j \neq k} \Gamma(V_j \cap V_k, \mathscr{F}_i)$ are finite, it is equivalent to the coproduct; in particular, colimits commute with colimits, so we have the exact sequence

$$0 \longrightarrow \varinjlim \Gamma(U, \mathscr{F}_i) \longrightarrow \textstyle\prod_{j=1}^n \varinjlim \Gamma(V_j, \mathscr{F}_i) \longrightarrow \prod_{j,k} \varinjlim \Gamma(V_j \cap V_k, \mathscr{F}_i)$$

so the sheaf properties immediately follow. $\square$

12. *Inverse Limit.* Let $\{\mathscr{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \mapsto \varprojlim \mathscr{F}_i(U)$ is a sheaf. It is called the *inverse limit* of the system $\{\mathscr{F}_i\}$, and is denoted by $\varprojlim \mathscr{F}_i$. Show that this it has the universal property of an inverse limit in the category of sheaves.

Proof. Let $\mathscr{F}$ be the presheaf $U \mapsto \varprojlim \mathscr{F}_i(U)$. We have morphisms $\pi_{ij} : \mathscr{F}_i \to \mathscr{F}_j$ and $\pi_i : \mathscr{F} \to \mathscr{F}_i$ such that for all i, j, $\pi_j = \pi_{ij} \circ \pi_i$. If U is an open set, if $\{V_\alpha\}$ is an open covering of U, then we have the exact sequence of abelian groups

$$0 \longrightarrow \Gamma(U, \mathscr{F}_i) \longrightarrow \textstyle\prod_\alpha \Gamma(V_\alpha, \mathscr{F}_i) \longrightarrow \prod_{\alpha,\beta} \Gamma(V_\alpha \cap V_\beta, \mathscr{F}_i)$$

and direct limits commute with direct products, so the fact that $\mathscr{F}$ is a sheaf follows from the exact sequence

$$0 \longrightarrow \varprojlim \Gamma(U, \mathscr{F}_i) \longrightarrow \textstyle\prod_\alpha \varprojlim \Gamma(V_\alpha, \mathscr{F}_i) \longrightarrow \prod_{\alpha,\beta} \varprojlim \Gamma(V_\alpha \cap V_\beta, \mathscr{F}_i).$$

To show $\mathscr{F}$ is an inverse limit in a category-theoretic sense, if $\mathscr{G}$ is some sheaf on X with a collection of morphisms $\tau_i : \mathscr{G} \to \mathscr{F}_i$ such that $\tau_j = \pi_{ij} \circ \tau_i$ for all i, j, then we want to show there exists a unique $\theta : \mathscr{G} \to \mathscr{F}$ such that $\tau_i = \pi_i \circ \theta$. Since for each open subset U in X we have a direct system $\tau_i(U) : \Gamma(U, \mathscr{G}) \to \Gamma(U, \mathscr{F}_i)$, by the universal property of the inverse limit $\Gamma(U, \mathscr{F}) = \varprojlim \Gamma(U, \mathscr{F}_i)$, there exists a unique morphism $\theta(U) : \Gamma(U, \mathscr{G}) \to \varprojlim \Gamma(U, \mathscr{F}_i)$ such that $\tau_i(U) = \pi_i(U) \circ \theta(U)$, so we can define θ as such, then it is unique by construction. $\square$

14. *Support.* Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The *support* of s, denoted Supp s, is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that Supp s is a closed subset of U. We define the *support* of $\mathscr{F}$, Supp $\mathscr{F}$, to be $\{P \in X \mid \mathscr{F}_P \neq 0\}$. It need not be a closed subset.

Proof. We show $U -$ Supp s is an open set. If $Q \in U$ such that $s_Q = 0$, then by definition of the direct limit, there exists an open neighborhood V of Q in U such that $0 = (V, s\big|_V) \in \mathscr{F}_Q$, hence $s_R = 0$ for all $R \in V$. □

15. *Sheaf $\mathscr{H}om$.* Let $\mathscr{F}$, $\mathscr{G}$ be sheaves of abelian groups on X. For any open subset $U \subseteq X$, show that the set $\operatorname{Hom}(\mathscr{F}\big|_U, \mathscr{G}\big|_U)$ of morphisms of the restricted sheaves has a natural structure of an abelian group. Show that the presheaf $U \mapsto \operatorname{Hom}(\mathscr{F}\big|_U, \mathscr{G}\big|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and is denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

Proof. Morphisms $f, g \in \operatorname{Hom}(\mathscr{F}\big|_U, \mathscr{G}\big|_U)$ define for all open $V \subseteq U$ homomorphisms between abelian groups $f(V), g(V) : \mathscr{F}(V) \to \mathscr{G}(V)$, so we can define $f + g$ as $(f + g)(V) = f(V) + g(V)$. It is obviously an abelian group with identity 0 as the zero morphism defined by $0(V) \equiv 0$.

Let U be an open subset of X, and let $\{V_i\}$ be an open cover of U. We make some clarifying remarks. If $s \in \Gamma(U, \mathscr{H}om(\mathscr{F}, \mathscr{G})) = \operatorname{Hom}(\mathscr{F}\big|_U, \mathscr{G}\big|_U)$, then s is a natural transformation of functors $\mathscr{F}\big|_U \to \mathscr{G}\big|_U$, so $s\big|_{V_i}$ refers to the induced natural transformation of functors $\mathscr{F}\big|_{V_i} \to \mathscr{G}\big|_{V_i}$ by only considering open subsets $W \subseteq V_i$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma(W, \mathscr{F}) & \longrightarrow & \prod \Gamma(W \cap V_i, \mathscr{F}) & \longrightarrow & \prod \Gamma(W \cap V_i \cap V_j, \mathscr{F}) \\
 & & \big\downarrow{\scriptstyle s(W)} & & \big\downarrow{\scriptstyle \prod s(W \cap V_i)} & & \big\downarrow{\scriptstyle \prod s(W \cap V_i \cap V_j)} \\
0 & \longrightarrow & \Gamma(W, \mathscr{G}) & \longrightarrow & \prod \Gamma(W \cap V_i, \mathscr{G}) & \longrightarrow & \prod \Gamma(W \cap V_i \cap V_j, \mathscr{G})
\end{array}$$

hence $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf. □

17. *Skyscraper Sheaves.* Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \overline{\{P\}}$, and 0 elsewhere. Hence, the name "Skyscraper sheaf." Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\overline{\{P\}}$, and $i : \overline{\{P\}} \to X$ is the inclusion.

Proof. We first identify the restriction maps of $i_P(A)$. Let $U \subseteq V$ be open subsets of X, then $i_P(A)(V) \to i_P(A)(U)$ is the zero map if either V or U (hence V as well) does not contain P, and is the identity map if both V and U contain P. If $Q \in \overline{\{P\}}$, then every open neighborhood of Q contains P, so the direct system of open neighborhoods of Q induces the direct system consisting of a copy of A for every $U \ni Q$ with the identity map as transition maps, hence $(i_P(A))_Q = A$. If $Q \notin \overline{\{P\}}$, then there exists an open neighborhood W of Q not containing P, which means $i_P(A)(W) = 0$, hence $(i_P(A))_Q = 0$. The last statement is obvious. □

18. *Adjoint Property of f^{-1}.* Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$, and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y,

$$\operatorname{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$

Proof. Let U, V be open in X, Y, respectively. We unpack the definitions:

$$(f^{-1}f_*\mathscr{F})(U) = \varinjlim_{W \supseteq f(U)} (f_*\mathscr{F})(W) = \varinjlim_{W \supseteq f(U)} \mathscr{F}(f^{-1}(W)),$$
$$(f_*f^{-1}\mathscr{G})(V) = (f^{-1}\mathscr{G})(f^{-1}(V)) = \varinjlim_{W \supseteq f(f^{-1}(V))} \mathscr{G}(W).$$

We also have

$$\begin{aligned}
(f_*f^{-1}f_*\mathscr{F})(V) &= (f^{-1}f_*\mathscr{F})(f^{-1}(V)) \\
&= \varinjlim_{W\supseteq f(f^{-1}(V))} \mathscr{F}(f^{-1}(W)) \\
&= \mathscr{F}(f^{-1}(V)) \\
&= (f_*\mathscr{F})(V) \\
(f^{-1}f_*f^{-1}\mathscr{G})(U) &= \varinjlim_{W\supseteq f(U)} (f_*f^{-1}\mathscr{G})(W) \\
&= \varinjlim_{W\supseteq f(U)} \varinjlim_{W'\supseteq f(f^{-1}(W))} \mathscr{G}(W') \\
&= \varinjlim_{W\supseteq f(U)} \mathscr{G}(W) \\
&= (f^{-1}\mathscr{G})(U)
\end{aligned}$$

Note that $V' := f(f^{-1}(V)) = V \cap f(X)$. An element of $(f^{-1}f_*\mathscr{F})(U)$ is of the form (W, s) where $W \supseteq f(U)$ and $s \in \mathscr{F}(f^{-1}(W))$. Define $\alpha : f^{-1}f_*\mathscr{F} \to \mathscr{F}$ as

$$\begin{aligned}
\alpha(U) : (f^{-1}f_*\mathscr{F})(U) &\to \mathscr{F}(U) \\
(W, s) &\mapsto s\big|_U.
\end{aligned}$$

An element of $(f_*f^{-1}\mathscr{G})(V)$ is of the form (W, t) where $W \supseteq V'$ and $t \in \mathscr{G}(W)$. Define $\beta : \mathscr{G} \to f_*f^{-1}\mathscr{G}$ as

$$\begin{aligned}
\beta(V) : \mathscr{G}(V) &\to (f_*f^{-1}\mathscr{G})(V) \\
t &\mapsto (V, t).
\end{aligned}$$

Let $\varphi : \mathscr{F} \to \mathscr{F}'$, $\psi : \mathscr{G} \to \mathscr{G}'$, where $\mathscr{F}, \mathscr{F} \in \mathfrak{Sh}_X$, $\mathscr{G}, \mathscr{G}' \in \mathfrak{Sh}_Y$, $\varphi \in \mathrm{Hom}_X(\mathscr{F}, \mathscr{F}')$, and $\psi \in \mathrm{Hom}_Y(\mathscr{G}, \mathscr{G}')$. We explicitly describe the induced lifts with respect to f, namely $f_*\varphi : f_*\mathscr{F} \to f_*\mathscr{F}'$ and $f^{-1}\psi : f^{-1}\mathscr{G} \to f^{-1}\mathscr{G}'$. Let U, V be open sets in X, Y, respectively. Define

$$\begin{aligned}
f_*\varphi(V) : f_*\mathscr{F}(V) &\to f_*\mathscr{F}'(V) \\
s &\mapsto \varphi(f^{-1}(V))(s)
\end{aligned}$$

where $s \in \Gamma(f^{-1}(V), \mathscr{F})$, and define

$$\begin{aligned}
f^{-1}\psi(U) : f^{-1}\mathscr{G}(U) &\to f^{-1}\mathscr{G}'(U) \\
(W, t) &\mapsto (W, \psi(W)(t)),
\end{aligned}$$

where $(W, t) \in \varinjlim_{W\supseteq f(U)} \mathscr{G}(W)$. For every $\varphi : f^{-1}\mathscr{G} \to \mathscr{F}$, we have the morphism $f_*\varphi : f_*f^{-1}\mathscr{G} \to f_*\mathscr{F}$, so we define $\beta^* : \mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$ as

$$\beta^*\varphi = (f_*\varphi) \circ \beta.$$

Similarly, for every $\psi : \mathscr{G} \to f_*\mathscr{F}$, we have the morphism $f^{-1}\psi : f^{-1}\mathscr{G} \to f^{-1}f_*\mathscr{F}$, so we define $\alpha_* : \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \to \mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$ as

$$\alpha_*\psi = \alpha \circ (f^{-1}\psi).$$

To show α_*, β^* are bijections with inverses to each other, it suffices to show $\alpha_* \circ \beta^* = \mathrm{id}_{\mathrm{Hom}_X(f^{-1}\mathscr{G},\mathscr{F})}$ and $\beta^* \circ \alpha_* = \mathrm{id}_{\mathrm{Hom}_Y(\mathscr{G}\to f_*\mathscr{F})}$:

$$\begin{aligned}
(\alpha_* \circ \beta^*)(\varphi) &= \alpha_*((f_*\varphi) \circ \beta) = \alpha \circ f^{-1}((f_*\varphi) \circ \beta) = \alpha \circ (f^{-1}f_*\varphi) \circ (f^{-1}\beta) \\
(\beta^* \circ \alpha^*)(\psi) &= \beta^*(\alpha \circ (f^{-1}\psi)) = f_*(\alpha \circ (f^{-1}\psi)) \circ \beta = (f_*\alpha) \circ (f_*f^{-1}\psi) \circ \beta.
\end{aligned}$$

Again, we unpack the definitions. Let $(W, s) \in f^{-1}\mathscr{G}(U)$, where $f(U) \subseteq W \subseteq Y$ with $t \in \Gamma(W, \mathscr{G})$. We want to show

$$(\alpha_*\beta^*\varphi)(U)((W, t)) = \varphi(U)((W, t)).$$

We have

$$\begin{aligned}
(\alpha_*\beta^*\varphi)(U)((W,t)) &= (\alpha(U) \circ (f^{-1}f_*\varphi))(U) \circ (f^{-1}\beta)(U))((W,t)) \\
&= (\alpha(U) \circ (f^{-1}f_*\varphi)(U))((W,t))) \\
&= \alpha(U)((f^{-1}(W), \varphi(f^{-1}(W)(t)))) \\
&= \varphi(f^{-1}(W)((W,t)))\big|_U \\
&= \varphi(U)((W,t)).
\end{aligned}$$

Here, we used the fact that the restriction map commutes with $\varphi(U)$ by definition of morphisms between sheaves. Let $t \in \Gamma(V, \mathscr{G})$. We similarly have

$$\begin{aligned}
(\beta * \alpha^*\psi)(V)(t) &= ((f_*\alpha)(V) \circ (f_*f^{-1}\psi)(V) \circ \beta(V))(t) \\
&= ((f_*\alpha)(V) \circ (f_*f^{-1}\psi(V)))((V,t)) \\
&= (f_*\alpha)(V)(\psi(V)(t)) \\
&= \psi(V)(t).
\end{aligned}$$

Hence, $\alpha*$ and β^* are bijections with inverses to each other. $\square$

22. *Glueing Sheaves.* Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij} : \mathscr{F}_i\big|_{U_i \cap U_j} \xrightarrow{\sim} \mathscr{F}_j\big|_{U_i \cap U_j}$ such that

(1) for each i, $\varphi_{ii} = \mathrm{id}$,

(2) and for each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Then there exists a unique sheaf $\mathscr{F}$ on X together with isomorphisms $\psi_i : \mathscr{F}\big|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathscr{F}$ is obtained by *glueing* the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}.