

# Chapter 3, Section 3

James Lee

April 23, 2025

1. Let  $X$  be a Noetherian scheme. Show that  $X$  is affine if and only if  $X_{\text{red}}$  (II, Ex. 2.3) is affine.

*Proof.* One direction is clear. Suppose  $X_{\text{red}} = \text{Spec } A$  where  $A$  is a Noetherian ring with no nilpotent elements, let  $f : X_{\text{red}} \rightarrow X$  be the natural map, and let  $\mathcal{F}$  be any quasi-coherent sheaf on  $X$ . Following the hint, consider the filtration

$$\mathcal{F} \supseteq \mathcal{N} \cdot \mathcal{F} \supseteq \mathcal{N}^2 \cdot \mathcal{F} \supseteq \dots,$$

where  $\mathcal{N}$  is the sheaf of nilpotent elements on  $X$ . Note that  $X \cong X_{\text{red}}$  as topological space, and the associated morphism of sheaves  $\mathcal{O}_X \rightarrow f_* \mathcal{O}_{X_{\text{red}}}$  is surjective with kernel  $\mathcal{N}$ . Thus, each of the quotients of this filtration can be naturally viewed as  $A$ -modules. In particular, we have a natural isomorphism (2.10)

$$H^i(X, \mathcal{N}^r \cdot \mathcal{F} / \mathcal{N}^{r+1} \cdot \mathcal{F}) \cong H^i(X_{\text{red}}, f^*(\mathcal{N}^r \cdot \mathcal{F} / \mathcal{N}^{r+1} \cdot \mathcal{F})).$$

Also, the nilradical of a Noetherian ring is nilpotent, so there exists a positive integer  $r > 0$  such that  $\mathcal{N}^r = 0$  (A.M. 7.15). Using our hypothesis and (3.7), we climb up the filtration and deduce that  $H^1(X, \mathcal{F}) = 0$ . Hence,  $X$  is affine by (3.7).  $\square$

2. Let  $X$  be a reduced Noetherian scheme. Show that  $X$  is affine if and only if each irreducible component is affine.

*Proof.* Suppose  $X = \text{Spec } A$  is affine for some reduced Noetherian ring  $A$ . The irreducible components of  $X$  correspond to the minimal prime ideals  $\mathfrak{p}$  of  $A$  (A.M. Ex. 1.20). In particular, the irreducible components of  $X$  are precisely  $\text{Spec } A/\mathfrak{p}$ . Conversely, suppose each irreducible component of  $X$  is affine.  $\square$

6. Let  $X$  be a Noetherian scheme.

- (a) Show that the sheaf  $\mathcal{G}$  constructed in the proof of (3.6) is an injective object in the category  $\mathfrak{Qco}(X)$  of quasi-coherent sheaves on  $X$ . Thus,  $\mathfrak{Qco}(X)$  has enough injectives.
- (b) Show that any injective object of  $\mathfrak{Qco}(X)$  is flasque.
- (c) Conclude that one can compute cohomology as the derived functors of  $\Gamma(X, \cdot)$ , considered as a functor  $\mathfrak{Qco}(X)$  to  $\mathfrak{Ab}$ .

*Proof.*

- (a) The Hom functor commutes with finite direct sums in the second argument, so we can assume  $\mathcal{G} = j_* \tilde{I}$ , where  $j : U = \text{Spec } A \rightarrow X$  is an open immersion, and  $I$  is an injective  $A$ -module. Suppose  $\mathcal{N} \rightarrow \mathcal{M}$  is an injective map of  $\mathcal{O}_X$ -modules, and we are given any  $f : \mathcal{N} \rightarrow j_* \tilde{I}$ . Since  $j^*$  is left exact, the induced map of  $A$ -modules  $j^* \mathcal{N} \rightarrow j^* \mathcal{M}$  is also injective. For any such  $f$  there is an associated morphism of  $A$ -modules  $g : j^* \mathcal{N} \rightarrow \tilde{I}$  by adjointness of  $j_*$ , so there exists an extension of  $g$  to  $j^* \mathcal{M}$  by injectivity of  $\tilde{I}$ . By adjointness of  $j^*$  again, we obtain a morphism  $\mathcal{M} \rightarrow j_* \tilde{I}$  that naturally extends  $f : \mathcal{N} \rightarrow j_* \tilde{I}$ , which is what we wanted to show.
- (b) Let  $\mathcal{I}$  be an injective object of  $\mathfrak{Qco}(X)$ .
- (c)

$\square$

7. Let  $A$  be a Noetherian ring, let  $X = \text{Spec } A$ , let  $\mathfrak{a} \subseteq A$  be an ideal, and let  $U \subseteq X$  be the open set  $X - V(\mathfrak{a})$ .

(a) For any  $A$ -module  $M$ , establish the following formula of Deligne:

$$\Gamma(U, \tilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M).$$

(b) Apply this in the case of an injective  $A$ -module  $I$ , to give another proof of (3.4).

*Proof.*

(a)

(b)

□