Chapter 4, Section 1

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1. Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

Proof. Pick another point $Q \in X$ different from P. By the Riemann-Roch Theorem, there exists $n > 0$ such that $\dim|-P+nQ| > 0$. This means there exists an effective divisor D that has a pole at P and regular everywhere else. Thus, there exists $f \in K(X)$ such that $v_P(f) = -1$ and $v_Q(f) \geq 0$ for all $Q \neq P$. □

2. Again let X be a curve, and let $P_1, \ldots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i, and regular elsewhere.

3. Let X be an integral, separated, regular, one-dimensional scheme of finite type over k, which is *not* proper over k. Then X is affine.

4. Show that a separated, one-dimensional scheme of finite type over k, none of whose irreducible components is proper over k, is affine.

5. For an effective divisor D on a curve X of genus g, show that $\dim|D| \leq \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

Proof. By Riemann-Roch, we have

$$\dim|D| = \ell(D) - 1 = \ell(K-D) + \deg D - g,$$

so we want to show $\ell(K-D) \leq g$. But $g = \ell(K)$ by definition, and D is effective, so the result follows from the proof... □

6. Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leq g+1$.

7. A curve X is called *hyperelliptic* if $g \geq 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

(a) If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. Use (II, 7.8.1) to conclude that X is hyperelliptic.

(b) Show taht the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbb{P}^1$. Thus there exist hyperelliptic curves of any genus $g \geq 2$.