# Chapter 2, Section 3

James Lee

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**1.** Show that a morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of $Y$, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each $A_j$ is a finitely generated $B$-algebra.

*Proof.* We reduce to proving the following statement: let $f : X \to Y$ be a morphism of schemes with $Y = \operatorname{Spec} B$, which can be covered by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each $i$, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each $A_{ij}$ is a finitely generated $B_i$-algebra. Then $X$ can be covered by open affine subsets $U_k = \operatorname{Spec} A_k$, where each $A_k$ is a finitely generated $B$-algebra.

We show each $B_i$ can be chosen to be a finitely generated $B$-algebra, then by restriction of scalars each $A_{ij}$ is a finitely generated $B$-algebra, which proves the statement. For each $i$, there exists $g_i \in B$ such that $D(g_i) \subseteq V_i$, where $D(g_i) \cong \operatorname{Spec} B_{g_i}$. Then $f^{-1}(D(g_i)) \subseteq f^{-1}(V_i)$, so we can cover $f^{-1}(D(g_i))$ by open affines of the form $\operatorname{Spec}((A_{ij})_{h_k})$ with $h_k \in A_{ij}$. Since each $(A_{ij})_{h_k}$ is a finitely generated $A_{ij}$-algebra, and each $(A_{ij})_{h_k}$ is a finitely generated $B_{g_i} \cong (B_i)_{\overline{g}_i}$, where each $\overline{g}_i$ is the image of $g_i$ in $B_i$, what we have shown is each $f^{-1}(D(g_i))$ can be covered by open affine subsets $U'_{ij} = \operatorname{Spec} A'_{ij}$, where each $A'_{ij}$ is a finitely generated $B_{g_i}$-algebra. Each $B_{g_i}$ is a finitely generated $B$-algebra, generated by 1 and $1/g_i$, which is what we wanted to show. □

**2.** A morphism $f : X \to Y$ of schemes is *quasi-compact* if there is a cover of $Y$ by open affines $V_i$ such that $f^{-1}(V_i)$ is quasi-compact for each $i$. Show that $f$ is quasi-compact if and only if for *every* open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

*Proof.* We reduce to proving the following statement: let $f : X \to Y$ be a morphism of schemes with $Y$ affine, which can be covered by open subsets $V_i$ such that $f^{-1}(V_i)$ is quasi-compact in $X$. Then $X$ is quasi-compact. Indeed, $Y$ is quasi-compact, so a finite number of $i$ will do, and a finite union of quasi-compact sets is quasi-compact, and $X = f^{-1}(Y) = f^{-1}(\bigcup_{i=1}^n V_i) = \bigcup_{i=1}^n f^{-1}(V_i)$, hence $X$ is quasi-compact. □

**3.** (a) Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

(b) Conclude from this that $f$ is of finite type if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of $Y$, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each $A_j$ is a finitely generated $B$-algebra.

(c) Show also if $f$ is of finite type, then for *every* open affine subset $V = \operatorname{Spec} B \subseteq Y$, and for *every* open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, $A$ is a finitely generated $B$-algebra.

*Proof.*

(a) Suppose $f : X \to Y$ is of finite type. Then by definition it is also locally of finite type, so there exists a covering of $Y$ by open affine subsets $V_i = \operatorname{Spec} B_i$, such that each $i$, $f^{-1}(V_i)$ can be covered by a finite number of open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each $A_{ij}$ is a finitely generated $B_i$-algebra. Each $f^{-1}(V_i)$ is quasi-compact since every affine scheme is quasi-compact and $f^{-1}(V_i)$ is a finite union of quasi-compact sets. Conversely, if $f : X \to Y$ is locally of finite type and quasi-compact, then each $f^{-1}(V_i)$ is quasi-compact, so it can be covered by a finite number of open affines $U_{ij} = \operatorname{Spec} A_{ij}$, where each $A_{ij}$ is a finitely generated $B_i$-algebra.

(b) A morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact if and only if for every open affine subset $V = \operatorname{Spec} B$ of $Y$, $f^{-1}(V)$ is quasi-compact and can be covered by open affines $U_j = \operatorname{Spec} A_j$, where each $A_j$ is a finitely generated $B$-algebra if and only if for every open affine subset $V = \operatorname{Spec} B$ of $Y$, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each $A_j$ is a finitely generated $B$-algebra.

(c) By restricting the domain of $f$, it suffices to prove the following: if $X$ is a scheme over a ring $B$ that can be covered by a finite number of open subsets that are the spectra of finitely generated $B$-algebras, then for every open affine $U = \operatorname{Spec} A \subseteq X$, $A$ is a finitely generated $B$-algebra. By Nike's trick, we can cover $U$ by the distinguished basic sets $\operatorname{Spec} A_f$, where each $A_f$ is a finitely generated $B$-algebra. Since $U$ is quasi-compact, a finite number will do.

So now, we have reduced to the following purely algebraic problem: $A$ is a $B$-algebra, $f_1, \ldots, f_r$ are a finite number of elements of $A$, which generate the unit ideal, and each localization $A_{f_i}$ is a finitely generated $B$-algebra. Then $A$ is finitely generated over $B$. Write $\sum_{i=1}^r g_i f_i$. Let $S$ be the union of $\{g_1, \ldots, g_r, f_1, \ldots, f_r\}$ and a finite subset of $A$ such that the image of it under the natural map $A \to A_{f_i}$ generates $A_{f_i}$ as a $B$-algebra for all $i$, and let $B[S]$ be the $B$-subalgebra of $A$ generated by $S$. Localization is left-exact, so $B[S]_{f_i} \to A_{f_i}$ is injective for all $i$, and it is surjective as well by construction of $S$. Thus, the inclusion $B[S] \to A$ viewed as a $B[S]$-module homomorphism is an isomorphism by the following statement from commutative algebra: let $R$ be any ring, let $M$ by any $R$-module, and let $h_1, \ldots, h_n$ be elements of $R$ that generate the unit ideal. If $M_{h_i} = 0$ for all $i$, then $M = 0$. Indeed, if $\mathfrak{p}$ is any prime ideal of $R$, then $h_i \notin \mathfrak{p}$ for some $i$. Denote $\bar{h}_i$ the image of $h_i$ in $A_\mathfrak{p}$. Then, $M_\mathfrak{p} = (M_\mathfrak{p})_{\bar{h}_i} = (M_{h_i})_{\mathfrak{p}R_{h_i}} = 0$ (A.M. §3).

□

**Theorem 1** (Nike's Trick). *Let $X$ be a scheme, and let $U_i = \operatorname{Spec} A_i$, $i = 1, 2$, be open affine subsets of $X$. Then there is an open cover of $U_1 \cap U_2$ consisting of open sets that are distinguished basic open sets in both $U_i$.*

**Lemma 1.** *If $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of locally ringed spaces, then $f^{-1}(Y_g) = X_{f^\#(g)}$ for any $g \in \Gamma(Y, \mathscr{O}_Y)$.*

*Proof.* Pick $g \in A_1$ such that $\operatorname{Spec}(A_1)_g = (U_1)_g \subseteq U_1 \cap U_2$, and set $U_1 = (U_1)_g$, so we have an open immersion $\iota : U_1 \hookrightarrow U_2$, which induces a ring homomorphism $\varphi : A_2 \to (A_1)_g$. Pick $h \in A_2$ such that $(U_2)_h \subseteq U_1$. By lemma 1, $\iota^{-1}((U_2)_h) = (U_1)_{\varphi(h)}$, and since $\iota$ is an open immersion, $(U_1)_{\varphi(h)} \cong (U_2)_h$ (we can assume $\varphi(h) \in A$). □

**4.** Show that a morphism $f : X \to Y$ is finite if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of $Y$, $f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where $A$ is a finite $B$-module.

*Proof.* Let $f : X \to Y$ be a finite morphism of schemes. Then there exists an open covering of $Y$ by sets $V_i = \operatorname{Spec} B_i$, such that for each $i$, $f^{-1}(V_i)$ is affine, equal to $\operatorname{Spec} A_i$, where $A_i$ is a finite $B_i$-module. Let $V = \operatorname{Spec} B$ be an open subset of $Y$. By Nike's trick, we can cover $V$ by open affines that are distinguished open set in $V$ and some $V_i$, i.e., open sets of the form $\operatorname{Spec} B_g = \operatorname{Spec}(B_i)_h$ for some $g \in B, h \in B_i$. Since $V$ is quasi-compact, a finite number will do. By lemma 1, $f^{-1}(\operatorname{Spec}(B_i)_h) = \operatorname{Spec}(A_i)_h \subseteq f^{-1}(V)$. Since $(A_i)_h$ is a finite $(B_i)_h$-module and $B_g \cong (B_i)_h$, we deduce that there exists a finite cover of $V$ by basic open sets $\operatorname{Spec} B_{g_j}$ for some $g_j \in B$ such that $f^{-1}(\operatorname{Spec} B_{g_j}) = \operatorname{Spec} C_j$, where each $C_j$ is a finite $B_{g_j}$-module. We show $X' = f^{-1}(V)$ is affine using the criterion in (Ex. 2.17b). The restriction of $f$ to $X' \to \operatorname{Spec} B$ induces a ring homomorphism $B \to A = \Gamma(X', \mathscr{O}_X)$. Denote $\bar{g}_i$ the image of $g_i$ in $A$. Since $g_i$ generate the unit ideal in $B$, its image in $A$ also generate the unit ideal. Also, by lemma 1 $X'_{\bar{g}_j} = f^{-1}(\operatorname{Spec} B_{g_j}) = \operatorname{Spec} C_j$, hence $X' = \operatorname{Spec} A$. It remains to show $A$ is a finite $B$-module, which we reduce to the following algebraic problem: let $A$ be a $B$-algebra, let $g_i \in B$ $(1 \le i \le n)$ generate the unit ideal, and suppose $A_{g_i}$ is a finite $B_{g_i}$-module. Then $A$ is a finite $B$-module. The proof is identical to (Ex. 3.3c), so we omit this part. □

**6.** Let $X$ be an integral scheme. Show that the local ring $\mathscr{O}_\xi$ of the generic point $\xi$ of $X$ is a field. It is called the *function field* of $X$, and is denoted by $K(X)$. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of $X$, then $K(X)$ is isomorphic to the quotient field of $A$.

*Proof.* Any nonempty open set $U$ of $X$ must contain $\xi$ since $X - U$ is a proper closed subset of $X$. In particular, any element of $\mathscr{O}_\xi$ can be represented as a pair $(\operatorname{Spec} A, f)$ where $\operatorname{Spec} A$ is an open affine set in $X$ and $f \in A$. We further assume $\operatorname{Spec} A$ is connected, so $A$ is an integral domain. If $f$ is a nonzero element of $A$, then $(\operatorname{Spec} A, f)$ is a nonzero element of $\mathscr{O}_\xi$ with inverse $(\operatorname{Spec} A, f)^{-1} = (\operatorname{Spec} A_f, 1/f)$, hence $\mathscr{O}_\xi$ is a field. Lastly, the distinguished open sets $\operatorname{Spec} A_f$ form a neighborhood basis of $\xi$, so any element of $\mathscr{O}_\xi$ can be written as $(\operatorname{Spec} A_f, a/f^n)$, which is the quotient field of $A$. □

**7.** A morphism $f : X \to Y$, with $Y$ irreducible, is *generically finite* if $f^{-1}(\eta)$ is a finite set, where $\eta$ is the generic point of $Y$. A morphism $f : X \to Y$ is *dominant* if $f(X)$ is dense in $Y$. Now let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite.

**8.** *Normalization.* A scheme is *normal* if all of its local rings are integrally closed domains. Let $X$ be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of $X$, let $\tilde{A}$ be the integral closure of $A$ in its quotient field, and let $\tilde{U} = \operatorname{Spec} \tilde{A}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the *normalization* of $X$. Show also that there is a morphism $\tilde{X} \to X$, having the following universal property: for every normal integral scheme $Z$, and for every dominant morphisms $f : Z \to X$, $f$ factors uniquely through $\tilde{X}$. If $X$ is of finite type over a field $k$, then the morphism $\tilde{X} \to X$ is a finite morphism.

**Lemma 2.** *If $f : Z \to X$ is a dominant morphism of schemes with $Z$ reduced, then $f^{\#} : \mathscr{O}_X \to f_* \mathscr{O}_Z$ is injective.*

*Proof.* Let $U$ be any open subset of $X$. We want to show if $g \in \Gamma(U, \mathscr{O}_X)$ such that $f^{\#}(g) = 0 \in \Gamma(f^{-1}(U), \mathscr{O}_Z)$, then $g = 0$. By lemma 1, we have $f^{-1}(U_g) = (f^{-1}(U))_{f^{\#}(g)}$, and if $f^{\#}(g) = 0$, then $(f^{-1}(U))_{f^{\#}(g)} = \emptyset$, so $U_g$ must not meet the $f(Z)$. But $f(Z)$ is dense in $X$, and $U_g$ is an open set in $X$ by (Ex. 2.16a), so $U_g = \emptyset$, which implies $g$ is nilpotent (this result corresponds to the algebraic fact that the intersection of all prime ideals of a ring is the nilradical of the ring), hence $g = 0$. □

*Proof.* Let $X = \bigcup \operatorname{Spec} A_i$ be an open affine covering of $X$ where each $A_i$ is an integral domain. For each $i \neq j$, we have an identification $\varphi_{ij} : U_i \to U_j$, which is an isomorphism of open subschemes $U_i \subseteq \operatorname{Spec} A_i, U_j \subseteq \operatorname{Spec} A_j$. By Nike's trick, there exists an open covering of $U_i$ by basic open sets $\operatorname{Spec}(A_i)_{f_k}$ with $f_k \in A_i$ such that $\varphi_{ij}(\operatorname{Spec}(A_i)_{f_k}) = \operatorname{Spec}(A_j)_{g_k}$ with $g_k \in A_j$. For each $i$, let $\tilde{A}_i$ be the integral closure of $A_i$ in its quotient field (note by (Ex. 6), every $A_i$ has the same quotient field), and let $\pi_i : \operatorname{Spec} \tilde{A}_i \to \operatorname{Spec} A_i$ be the morphism induced by the inclusion $A_i \hookrightarrow \tilde{A}_i$. We have $\pi_i^{-1}(\operatorname{Spec}(A_i)_{f_k}) = \operatorname{Spec}(\tilde{A}_i)_{f_k}$ and $(\tilde{A}_i)_{f_k} \cong (\tilde{A}_j)_{g_k}$, so we can naturally glue open subsets of $X'$ that are of the form $\pi_i^{-1}(U_i)$ using $\varphi_{ij}$ to obtain $\tilde{X}$. The morphism $\pi : \tilde{X} \to X$ is obtained by glueing $\pi_i$ accordingly.

Now suppose $Z$ is a normal integral scheme, and let $f : Z \to X$ be a dominant morphism of schemes. It is clear we can assume $X = \operatorname{Spec} A$, where $A$ is an integral domain. Then $f$ induces a ring homomorphism $\varphi : A \to B = \Gamma(Z, \mathscr{O}_Z)$. Let $\tilde{X} = \operatorname{Spec} \tilde{A}$ be the normalization of $X$ with associated morphism $\pi : \tilde{X} \to X$ induced by the inclusion homomorphism $\iota : A \to \tilde{A}$, where $\tilde{A}$ is the integral closure of $A$ in its quotient field. We want to show there exists a unique morphism $\tilde{f} : Z \to \tilde{X}$ such that $f = \pi \circ \tilde{f}$. Being integrally closed is a local property (A.M. 5.13), so $B$ is integrally closed (it is automatically an integral domain by definition of an integral scheme). Also, $\varphi$ is injective by lemma 2. Since $\tilde{X}$ is affine, by the bijection in (Ex. 24), we have reduced to proving the following universal property for the integral closure of a domain $A \xrightarrow{\iota} \tilde{A}$: for any injective homomorphism $\varphi : A \to B$ where $B$ is an integrally closed domain, there exists a unique homomorphism $\psi : \tilde{A} \to B$ such that $\varphi = \psi \circ \iota$. Any injective homomorphism between integral domains induces an inclusion of fraction fields, so let $\Phi : \operatorname{Frac}(A) \to \operatorname{Frac}(B)$ be induced by $\varphi$, where $\Phi\big|_A = \varphi$. Note that we have inclusions $A \subseteq \tilde{A} \subseteq \operatorname{Frac}(A)$, thus we claim $\psi = \Phi\big|_{\tilde{A}} : \tilde{A} \to \operatorname{Frac}(B)$ is the desired ring homomorphism. It suffices to show the image of $\psi$ is contained in $B$. If $f \in \tilde{A}$, then there exists an equation of integral dependence $f^n + a_1 f^{n-1} + \cdots + a_n = 0$ where $a_i \in A$. Then $\Phi(f)$ has an equation of integral dependence $\Phi(f)^n + \varphi(a_1)\Phi(f)^{n-1} + \cdots + \varphi(a_n) = 0$, and since $B$ is integrally closed, $\Phi(f)$ must be an element of $B$. Since any other $\psi' : \tilde{A} \to B$ such that $\varphi = \psi' \circ \iota$ must agree with $\Phi$ on $\tilde{A}$, $\psi$ is unique by construction. □

**10.** *Fibers of a Morphism.*

(a) If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $\operatorname{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

(b) Let $X = \operatorname{Spec} k[s,t]/(s - t^2)$, let $Y = \operatorname{Spec} k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \to s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fiber $X_y$, consists of two points, with residue field $k$. If $y \in Y$ corresponds to $0 \in k$, show that the fiber $X_y$ is a nonreduced one-point scheme. If $\eta$ is the generic point of $Y$, show that $X_\eta$ is a one-point scheme, whose residue field is an extension of degree two of the residue field of $\eta$. (Assume $k$ is algebraically closed.)

*Proof.*

(a) We can assume $Y$ to be affine by taking any open affine neighborhood of $y \in Y$ and restricting $f$ to its preimage in $X$. Also, if $U_i = \operatorname{Spec} A_i$ is an open affine cover of $X$, then $f^{-1}(y) = \bigcup_i (\operatorname{Spec} A_i \cap f^{-1}(y))$, and $\operatorname{sp}(X_y) = \bigcup_i \operatorname{sp}(U_i \times_Y k(y))$, so we can also assume $X$ to be affine. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then $f$ induces a ring homomorphism $f^{\#} : B \to A$. A point $y \in Y$ corresponds to a prime ideal $\mathfrak{q}$ in $B$, where $k(y) = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$. Thus, $f^{-1}(y)$ is the set of all prime ideals $\mathfrak{p}$ in $A$ such that $f^{\#-1}(\mathfrak{p}) = \mathfrak{q}$. Next, we look at $\operatorname{sp}(X_y)$. Notice that $X_y = \operatorname{Spec} A \otimes_B k(y)$, and $A \otimes_B k(y) = A \otimes_B B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} = A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$. The prime ideals of $A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$ correspond to prime ideals of $A$ that contain the image of $\mathfrak{q}$ and does not contain meet the image of $B - \mathfrak{q}$, which are precisely the prime ideals of $A$ such that $f^{\#-1}(\mathfrak{p}) = \mathfrak{q}$.

(b) $X_y = \operatorname{Spec} k[t]/(a-t^2)$, $X_\eta = \operatorname{Spec} k(s)[\sqrt{s}]$. Any element of $k(s)[\sqrt{s}]$ can be written as $F+G\sqrt{s}$, where $F, G \in k(s)$. It is a field, since $(F+G\sqrt{s})^{-1} = (F-G\sqrt{s})/(F^2-sG^2)$.

$\square$

**11.** *Closed Subschemes.*

(a) Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.

(b) If $Y$ is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then $Y$ is also affine, and in fact $Y$ is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$.

(c) Let $Y$ be a closed subset of a scheme $X$, and give $Y$ the reduced induced subscheme structure. If $Y'$ is any other closed subscheme of $X$ with the same underlying topological space, show that the closed immersion $Y \to X$ factors through $Y'$. We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.

(d) Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme $Y$ of $X$ with the following property: the morphism $f$ factors through $Y$, and if $Y'$ is any other closed subscheme of $X$ through which $f$ factors, then $Y \to X$ factors through $Y'$ also. We call $Y$ the *scheme-theoretic image* of $f$. If $Z$ is a reduced scheme, then $Y$ is just the reduced induced structure on the closure of the image $f(Z)$.

*Proof.*

(a) Consider the special case when $X = \operatorname{Spec} A, Y = \operatorname{Spec} A/\mathfrak{a}$, and $X' = \operatorname{Spec} B$ where $\mathfrak{a}$ is an ideal of $A$ and $B$ is any $A$-algebra. The natural map $A \to A/\mathfrak{a}$ induces a closed immersion $Y \to X$, and the structure homomorphism $A \to B$ induces a morphism of schemes $X' \to X$. The fiber product $Y \times_X X'$ is equal to the spectra of the tensor product $A/\mathfrak{a} \otimes_A B$, which is isomorphic to $B/\mathfrak{a}B$. The induced structure morphism $Y \times_X X' \to X'$ corresponds to the canonical homomorphism $B \to B/\mathfrak{a}B$, thus $Y \times_X X' \to X'$ is a closed immersion. In other words, this property of closed immersions corresponds to the algebraic fact that the tensor operation is a right exact functor. In the general case, we can still assume $X$ to be affine by taking an open affine cover of $X$. Then $Y$ is an affine scheme by part (b), and if $U$ is any open subset of $X'$, $f'^{-1}(U) = Y \times_X U$, so by taking an open affine cover of $X'$, we can reduce to the case when $X'$ is affine, which is just the special case as in above.

(b) Let $Y$ be a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, and let $\varphi : A \to B = \Gamma(Y, \mathscr{O}_Y)$ be the induced ring homomorphism of global sections. Fix $y \in Y$, let $V = \operatorname{Spec} C$ be an open affine neighborhood of $y$ as a subspace of $Y$, and let $\rho : B \to C$ be the restriction homomorphism. By definition of the subspace topology, there exists an open set $U$ of $X$ such that $V = U \cap Y$. We can cover $U$ by distinguished open sets, and at least one of them must contain $y$, so let $X_f$ be such set with $f \in A$. By lemma 1, $X_f \cap Y = Y_{\varphi(f)} = \operatorname{Spec} C_{\rho(\varphi(f))}$. Thus, we can cover $Y$ by open affines of the form $X_{f_i} \cap Y$. Since $Y$ is homeomorphic to a closed subset of a quasi-compact set, it is also quasi-compact, so a finite number will do. By adding some more $f_i$ with $D(f_i) \cap Y = \emptyset$ by taking an open cover of $X - Y$, we assume $X_{f_i}$ cover $X$. Such collection of $X_{f_i}$ cover $X$ if and only if $f_i$ generate the unit ideal in $A$, so $\varphi(f_i)$ must generate the unit ideal of $B$. Hence, $Y$ is affine by (Ex. 2.17b). The quotient ring $A/\ker\varphi$ is a subring of $B$, so the affine scheme $X' = \operatorname{Spec} A/\ker\varphi$ contains $Y$ as a dense subset. $X'$ is also homeomorphic to a closed subset of $X$, hence $Y = \operatorname{Spec} A/\ker\varphi$.

(c) By taking an affine cover of $X$, we reduce to the case when $X = \operatorname{Spec} A$ is affine. Let $\mathfrak{a}$ be the ideal in $A$ that corresponds to the reduced induced structured of $Y$. By part (b), there exists an ideal $\mathfrak{b}$ in $A$ such that $Y' = \operatorname{Spec} A/\mathfrak{b}$. A morphism of schemes $Y \to Y'$ corresponds to a ring homomorphism $A/\mathfrak{b} \to A/\mathfrak{a}$. Recall that $\mathfrak{a}$ is the largest ideal of $A$ such that $V(\mathfrak{a}) = \operatorname{sp}(Y) = \operatorname{sp}(Y')$; in particular, $\mathfrak{b} \subseteq \mathfrak{a}$, so the natural projection map $A/\mathfrak{b} \to A/\mathfrak{a}$ is well-defined and is the desired ring homomorphism.

(d) Again, reduce to the affine case. A morphism of affine schemes $f : \operatorname{Spec} B \to \operatorname{Spec} A$ correspond to a ring homomorphism $\varphi : A \to B$. Then $Y = \operatorname{Spec} A/\ker\varphi$ is the desired closed subscheme. If $Z$ is reduced, then $B$ is a reduced ring. Then $\ker\varphi$ contains the nilradical of $A$, so $A/\ker\varphi$ is reduced, hence the intersection of all prime ideals of $A/\ker\varphi$ is the zero ideal.

$\square$

**13.** *Properties of Morphisms of Finite Type.*

(a) A closed immersion is a morphism of finite type.

(b) A quasi-compact open immersion is of finite type.

(c) A composition of two morphisms of finite type is of finite type.

(d) Morphisms of finite type are stable under base extension.

(e) If $X$ and $Y$ are schemes of finite type over $S$, then $X \times_S Y$ is of finite type over $S$.

(f) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if $f$ is quasi-compact, and $g \circ f$ is of finite type, then $f$ is of finite type.

(g) If $f : X \to Y$ is a morphism of finite type, and if $Y$ is noetherian, then $X$ is noetherian.

*Proof.*

(a) Let $f : Y \to X$ be a closed immersion. By abuse of notation, also denote $Y$ as a closed subset of $\mathrm{sp}(X)$. If $U$ is quasi-compact in $X$, then $f^{-1}(U) = U \cap Y$, which is quasi-compact since any closed subset of a quasi-compact set is also quasi-compact, so $f$ is a quasi-compact morphism. Also, if $U = \operatorname{Spec} A$ is an open affine subset of $X$, then $f^{-1}(U) = U \cap Y$ is a closed subscheme of $U$ equal to the spectra of $A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of $A$ by (Ex. 3.11b), which is a finitely generated $A$-algebra, so $f$ is also locally of finite type. Hence, $f$ is of finite type.

(b) Let $f : Y \to X$ be an open immersion. If $U = \operatorname{Spec} A$ is any open affine set in $X$, then $f^{-1}(U) = U \cap Y$ can be covered by distinguished open sets of $U$, which are spectra of $A_f$ for some $f \in A$. Any such $A_f$ is a finitely generated $A$-algebra. Hence, an open immersion is locally of finite type, so a quasi-compact open immersion is of finite type by (Ex. 3.3a).

(c) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of finite type. Let $W = \operatorname{Spec} C$ be an open affine subset of $Z$. Since $g$ is of finite type, $g^{-1}(W)$ can be covered by a finite number of open affines $V_i = \operatorname{Spec} B_i$, where each $B_i$ is a finitely generated $C$-algebra, and similarly each $f^{-1}(V_i)$ can be covered by a finite number of open affines $U_{ij} = \operatorname{Spec} A_{ij}$, where each $A_{ij}$ is a finitely generated $B_i$-algebra. Each $A_{ij}$ is also a finitely-generated $A$-algebra since $B_i$ is a finitely generated $A$-algebra, and there are a finite number of $U_{ij}$ with $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) = \bigcup U_{ij}$, $g \circ f$ is a morphism of finite type.

(d) This corresponds to the algebraic fact that if $A$ is a finitely generated $B$-algebra and $C$ is any other $B$-algebra, then $A \otimes_A C$ is a finitely generated $C$-algebra.

(e) This corresponds to the algebraic fact that the tensor product of two finitely generated algebras is also finitely generated.

(f) By (Ex. 3.3a), it suffices to show $f$ is locally of finite type. Let $W = \operatorname{Spec} C$ be an open affine subset of $Z$, and let $V_i = \operatorname{Spec} B_i$ be an open affine cover of $g^{-1}(W)$. By (Ex. 3.3c), for each $i$, we can cover $f^{-1}(V_i)$ by open affines $U_{ij} = \operatorname{Spec} A_{ij}$ where each $A_{ij}$ is a finitely generated $C$-algebra. We have a composition of structure morphisms $C \to B_i \to A_{ij}$, where $A_{ij}$ is finitely generated over the image of $C$ in $A_{ij}$. Since the image of $C$ in $A_{ij}$ is contained in the image of $B_i$ in $A_{ij}$, $A_{ij}$ is also a finitely generated $B_i$-algebra.

(g) This corresponds to the fact that any finitely generated algebra over a noetherian ring is also noetherian as a ring.

□

**16.** *Noetherian Induction.* Let $X$ be a noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of $X$. Assume that for any closed subset $Y$ of $X$, if $\mathscr{P}$ holds for every proper closed subset of $Y$, then $\mathscr{P}$ holds for $Y$. (In particular, $\mathscr{P}$ must hold for the empty set.) Then $\mathscr{P}$ holds for $X$.

*Proof.* Suppose $\mathscr{P}$ does not hold for $X$, and set $X_1 = X$. Recursively define $X_{n+1}$ to be any proper closed subset of $X_n$ such that $\mathscr{P}$ does not hold. This is a nonterminating decreasing sequence of closed subsets of $X$ by construction, a contradiction. □

**20.** *Dimension.* Let $X$ be an integral scheme of finite type over a field $k$ (not necessarily algebraically closed). Use appropriate results from (I,§1) to prove the following.

(a) For any closed point $P \in X$, $\dim X = \dim \mathscr{O}_P$, where for rings, we always mean the Krull dimension.

(b) Let $K(X)$ be the function field of $X$. Then $\dim X = \text{tr.d.} K(X)/k$.

(c) If $Y$ is a closed subset of $X$, then $\operatorname{codim}(Y, X) = \inf\{\dim \mathscr{O}_{P,X} \mid P \in Y\}$.

(d) If $Y$ is a closed subset of $X$, then $\dim Y + \operatorname{codim}(Y, X) = \dim X$.

(e) If $U$ is a nonempty open subset of $X$, then $\dim U = \dim X$.

(f) If $k \subseteq k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $= \dim X$.

*Proof.*

(a) Let $V$ be a variety over $k$. The bijection between the open sets of $V$ and open sets of $t(V)$ induced by the map $\alpha$ in the proof of (2.6) implies $\dim V = \dim t(V)$. If $X$ is an integral scheme of finite type over a field $k$, then we can cover $X$ by a finite number of open affines $U_i = \operatorname{Spec} A_i$, where each $A_i$ is of the form $k[x_1, \ldots, x_{r_i}]/\mathfrak{p}_i$ for some $r_i > 0$ and prime ideal $\mathfrak{p}_i$ in $k[x_1, \ldots, x_{r_i}]$. Then $P \in U_i$ for some $i$, and if $P$ is closed, then it corresponds to a maximal ideal $\mathfrak{m}_P$ in $A_i$, so by (A.M. 11.25) and (I, 1.7), $\dim \mathscr{O}_P = \dim A_i = \dim U_i$. Also, $\dim U_i = \dim X$ for all $i$ by (I, Ex. 1.10b) and from the fact that $\dim U_i = \dim U_j$. Indeed, $X$ itself is irreducible since it is an integral scheme and therefore has a unique generic point. Then $U_i \cap U_j \neq \emptyset$ for any $i, j$, so for all $P \in U_i \cap U_j$, $\dim U_i = \dim \mathscr{O}_{U_i,P} = \dim \mathscr{O}_{U_j,P} = \dim U_j$. Hence, $\dim X = \dim \mathscr{O}_P$ for all closed $P \in X$.

(b) By part (a), (II, Ex. 3.6), (I, 1.8A), $\operatorname{tr.d.} K(X)/k = \dim U_i = \dim X$.

(c) Let $X = \operatorname{Spec} A$, where $A$ is an integral domain that is finitely generated over a field $k$, and let $Y$ be an irreducible, closed subset of $X$. The closed subscheme $Y$ corresponds to a prime ideal $\mathfrak{p}$ in $A$, and any irreducible closed set containing $Y$ corresponds to a prime ideal contained in $\mathfrak{p}$, so $\operatorname{codim} YX$ equals to the height of $\mathfrak{p}$. Points in $Y$ correspond to prime ideals containing $\mathfrak{p}$, so the infimum of the dimension of all local rings $\mathscr{O}_{P,X}$ over $P \in Y$ equals to the infimum of the height of all prime ideals of $A$ which contain $\mathfrak{p}$, which is just the height of $\mathfrak{p}$. If $Y$ is any closed subset, then $Y$ corresponds to an ideal $\mathfrak{a}$ in $A$, and any irreducible closed set of $Y$ contained in $Y$ corresponds to a prime ideal of $A$ that contains $\mathfrak{a}$. From the case when $Y$ is irreducible, the infimum of the codimension of all irreducible closed sets contained in $Y$ equals to the infimum of the height of prime ideals containing $\mathfrak{a}$. The height of such prime ideals is precisely the dimension of the local ring of points $P \in Y$. If $X$ is any integral scheme of finite type over a field $k$, then we can cover $X$ by a finite number of open affines $U_i = \operatorname{Spec} A_i$ where $\dim A_i = \dim A_j$. If $P \in U_i \cap U_j \cap Y$, then $\dim \mathscr{O}_{P,X} = \dim \mathscr{O}_{P,U_i} = \dim \mathscr{O}_{P,U_j}$; in particular, the prime ideals that $P$ corresponds to in $A_i, A_j$ have the same height. Thus, we can just reduce to the affine case.

(d) By the same reason as part (c), we can reduce to the affine case, which follows from (§1, 1.7) and (§1, 1.8Ab).

(e) By (§1, Ex. 1.10b), $\dim U = \sup \dim U \cap U_i = \dim U_i = \dim X$.

(f) If $U_i = \operatorname{Spec} A_i$ is an open affine cover of $X$, then $U_i' = U_i \times_k k' = \operatorname{Spec} A_i \otimes_k k'$ is an open affine cover of $X$, so by (§1, Ex. 1.10b) and part (a), we reduce to the case when $X = \operatorname{Spec} A$ is affine. Suppose $A = k[x_1, \ldots, x_n]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$. Then $X'$ is the spectra of the ring $k'[x_1, \ldots, x_n]/\mathfrak{p}'$, where $\mathfrak{p}'$ is the extension of $\mathfrak{p}$ in $k'[x_1, \ldots, x_n]$. The irreducible components of $X'$ correspond to minimal prime ideals of $\mathfrak{p}'$. Let $\mathfrak{q}$ be a minimal prime ideal of $\mathfrak{p}'$ so that $\mathfrak{p} = \mathfrak{q} \cap k[x_1, \ldots, x_n]$. We want to show $\dim k[x_1, \ldots, x_n]/\mathfrak{p} = \dim k'[x_1, \ldots, x_n]/\mathfrak{q}$. The case when $k'$ is an algebraic extension of $k$ is immediate from the going-up going-down theorems, so assume $k$ and $k'$ are algebraically closed. Let $K, K'$ be the fraction fields of $k[x_1, \ldots, x_n]/\mathfrak{p}$, $k'[x_1, \ldots, x_n]/\mathfrak{q}$, then by part (b) or (§1, 1.8Aa) it suffices to show $\text{tr.d.}K/k = \text{tr.d.}K'/k'$. Since $k, k'$ are algebraically closed, $K, K'$ are purely transcendental extensions, so we can write $K = k(y_1, \ldots, y_r), K' = k'(y_1', \ldots, y_{r'}')$, where $r = \text{tr.d.}K/k, r' = \text{tr.d.}K'/k'$. By Noether's normalization lemma, we can choose $y_i$ to be linear combinations of $x_1, \ldots, x_n$, so after a suitable $k$-algebra automorphism, we can assume $\mathfrak{p} = (x_1, \ldots, x_{n-r})$ so that $y_i = x_i$ for $i = n - r + 1, \ldots, n$. Then the extension of $\mathfrak{p}$ in $k'[x_1, \ldots, x_n]$ is also a prime ideal generated by $x_1, \ldots, x_{n-r}$, which implies $r = r'$.

Since $n = \dim k[x_1, \ldots, x_n] = \dim k'[x_1, \ldots, x_n]$, by (§1, 1.8Ab) we have the following corollary:

**Corollary 1.** *Let $k'/k$ be any field extension. If $\mathfrak{p}$ is a prime ideal in $k[x_1, \ldots, x_n]$, then any minimal prime ideal of the extension of $\mathfrak{p}$ in $k'[x_1, \ldots, x_n]$ has the same height as $\mathfrak{p}$.*

□