

Chapter 3, Section 3

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1. Let X be a Noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine.

Proof. One direction is clear. Suppose $X_{\text{red}} = \text{Spec } A$ where A is a Noetherian ring with no nilpotent elements, let $f : X_{\text{red}} \rightarrow X$ be the natural map, and let \mathcal{F} be any quasi-coherent sheaf on X . Following the hint, consider the filtration

$$\mathcal{F} \supseteq \mathcal{N} \cdot \mathcal{F} \supseteq \mathcal{N}^2 \cdot \mathcal{F} \supseteq \dots,$$

where \mathcal{N} is the sheaf of nilpotent elements on X . Note that $X \cong X_{\text{red}}$ as topological space, and the associated morphism of sheaves $\mathcal{O}_X \rightarrow f_* \mathcal{O}_{X_{\text{red}}}$ is surjective with kernel \mathcal{N} . Thus, each of the quotients of this filtration can be naturally viewed as A -modules. In particular, we have a natural isomorphism (2.10)

$$H^i(X, \mathcal{N}^r \cdot \mathcal{F} / \mathcal{N}^{r+1} \cdot \mathcal{F}) \cong H^i(X_{\text{red}}, f^*(\mathcal{N}^r \cdot \mathcal{F} / \mathcal{N}^{r+1} \cdot \mathcal{F})).$$

Also, the nilradical of a Noetherian ring is nilpotent, so there exists a positive integer $r > 0$ such that $\mathcal{N}^r = 0$ (A.M. 7.15). Using our hypothesis and (3.7), we climb up the filtration and deduce that $H^1(X, \mathcal{F}) = 0$. Hence, X is affine by (3.7). \square

2. Let X be a reduced Noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

Proof. Suppose $X = \text{Spec } A$ is affine for some reduced Noetherian ring A . The irreducible components of X correspond to the minimal prime ideals \mathfrak{p} of A (A.M. Ex. 1.20). In particular, the irreducible components of X are precisely $\text{Spec } A/\mathfrak{p}$. Conversely, let X_i be the irreducible components of X , and let $\phi : \mathcal{F} \rightarrow \bigoplus_i j_* \mathcal{F}|_{X_i}$ be the natural map of \mathcal{O}_X -modules, where $j : X_i \hookrightarrow X$ is the inclusion. Since X is Noetherian, $X_i \cap X_j$ is quasi-compact, so we can cover it with a finite number of open affine subsets X_{ijk} . Because X is reduced, ϕ is injective, so we can extend ϕ by the following exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_i j_* \mathcal{F}|_{X_i} \longrightarrow \bigoplus_{i,j} j_* \mathcal{F}|_{X_{ijk}}.$$

Each $j_* \mathcal{F}|_{X_i}, j_* \mathcal{F}|_{X_{ijk}}$ has vanishing cohomology for $i > 0$ by (2.10), (3.5), and (3.7). While the sequence above is not surjective on the right, the image is still a quasi-coherent sheaf, so using the long exact sequence of cohomology, we deduce that $H^i(X, \mathcal{F}) = 0$ for $i > 0$. Hence, X is affine by (3.7). \square

6. Let X be a Noetherian scheme.

- (a) Show that the sheaf \mathcal{G} constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X . Thus, $\mathfrak{Qco}(X)$ has enough injectives.
- (b) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque.
- (c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor $\mathfrak{Qco}(X) \rightarrow \mathfrak{Ab}$.

Proof.

- (a) The Hom functor commutes with finite direct sums in the second argument, so we can assume $\mathcal{G} = j_* \tilde{I}$, where $j : U = \text{Spec } A \rightarrow X$ is the inclusion, and I is an injective A -module. Suppose $\mathcal{N} \rightarrow \mathcal{M}$ is an injective map of \mathcal{O}_X -modules, and we are given any $f : \mathcal{N} \rightarrow j_* \tilde{I}$. Since j^* is left exact when j is an open immersion, the induced map of A -modules $j^* \mathcal{N} \rightarrow j^* \mathcal{M}$ is also injective. For any such f there is an associated morphism of A -modules $g : j^* \mathcal{N} \rightarrow \tilde{I}$ by adjointness of j_* , so there exists an extension of g to $j^* \mathcal{M}$ by injectivity of \tilde{I} . By adjointness of j^* again, we obtain a morphism $\mathcal{M} \rightarrow j_* \tilde{I}$ that naturally extends f , which is what we wanted to show.

- (b) Essentially imitating (a) replacing i^* with i_* and vice versa, we deduce that $\mathcal{S}|_U$ is an injective object of $\mathfrak{Qco}(U)$. Covering X with finite number of open affines $U_i = \text{Spec } A_i$, we have $\mathcal{S}|_{U_i} \cong \tilde{I}_i$ for some injective A_i -module I_i for each i by (II, 5.5). Each \tilde{I}_i is flasque by (3.4), so \mathcal{S} is flasque on a local basis. Hence, \mathcal{S} is flasque.
- (c) Considering $\Gamma(X, \cdot)$ as a functor from $\mathfrak{Qco}(X)$ to \mathfrak{Ab} , we calculate its derived functors by taking injective resolutions in the category $\mathfrak{Qco}(X)$. But any injective is flasque (b), and flasques are acyclic (2.5), so this resolution gives the usual cohomology functors (1.2A).

□

7. Let A be a Noetherian ring, let $X = \text{Spec } A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

- (a) For any A -module M , establish the following formula of Deligne:

$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \text{Hom}_A(\mathfrak{a}^n, M).$$

- (b) Apply this in the case of an injective A -module I , to give another proof of (3.4).

Proof.

- (a) Consider the case when \mathfrak{a} is principal, that is $\mathfrak{a} = (f)$ for some $f \in A$. The left hand side above is isomorphism to $M_f \cong M \otimes A_f$. On the other hand, we can write M_f as the colimit $M_f \cong \varinjlim A f^{-n} \otimes M$, and $A f^{-n}$ is invertible with dual $A f^n = \mathfrak{a}^n$. Thus, we have a natural isomorphism $\text{Hom}_A(\mathfrak{a}^n, M) \cong A f^{-n} \otimes M$, which is what we wanted to show. This is a generalization of (II, 2.2) and (II, 5.1).

In the general case, by the Noetherian hypothesis, \mathfrak{a} is a finitely generated A -module, say $\mathfrak{a} = (f_1, \dots, f_r)$. Any module over a ring can be expressed as a colimit over its finitely generated submodules. In particular, $\mathfrak{a} = \varinjlim (f_i)$, and clearly $\text{Hom}_A(\cdot, M)$ commutes with colimits for any M . We can also express U as a colimit $U = \varinjlim U_{f_i}$ in the category of open sets of X , where U_{f_i} is the basic open set associated to the element $f_i \in A$. Note that $\tilde{U} = \bigcup U_{f_i}$. By naturality of $\Gamma(\cdot, \widetilde{M})$ and \sim we conclude the general case from the case when \mathfrak{a} is principal.

- (b) Let I be an injective A -module. It will be sufficient to show for any open set $U \subseteq X$, where $U = X - V(\mathfrak{a})$ for some ideal \mathfrak{a} of A , that $\Gamma(X, \tilde{I}) \rightarrow \Gamma(U, \tilde{I})$ is surjective.

□