Chapter 3, Section 11

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1. Show that the result of (11.2) is false without the projective hypothesis. For example, let $X = \mathbb{A}^n_k$, let $P = (0, \dots, 0)$, let $U = X - P$, and let $f : U \to X$ be the inclusion. Then the fibers of f all have dimension 0, but $R^{n-1} f_* \mathscr{O}_U \neq 0$.

2. Show that a projective morphism with finite fibers (= quasi-finite (II, Ex. 3.5)) is a finite morphism.

3. Let X be a normal, projective variety over an algebraically closed field k. Let $\mathfrak{d}$ be a linear system (of effective Cartier divisors) without base points, and assume that $\mathfrak{d}$ is *not composite with a pencil*, which means that if $f : X \to \mathbb{P}^n_k$ is the morphism determined by $\mathfrak{d}$, then $\dim f(X) \geq 2$. Then show that every divisor in $\mathfrak{d}$ is connected. This improves Bertini's theorem (10.9.1).

4. *Principle of Connectedness.* Let $\{X_t\}$ be a flat family of closed subschemes of $\mathbb{P}^n_k$ parameterized by an irreducible curve T of finite type over $k¿$ Suppose there is a nonempty open set $Y \subseteq T$, such that for all closed points $t \in Y$, X_t is connected. Then prove that X_t is connected for all $t \in T$.

5. Let Y be a hypersurface in $X = \mathbb{P}^N_k$ with $N \geq 4$. Let $\hat{X}$ be the formal completion of X along Y (II, §9). Prove that the natural map $\operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$ is an isomorphism.

6. Again let T be a hypersurface in $X = \mathbb{P}^N_k$, this time with $N \geq 2$.

   (a) If $\mathscr{F}$ is a locally free sheaf on X, show that the natural map

   $$H^0(X, \mathscr{F}) \to H^0(\hat{X}, \hat{\mathscr{F}})$$

   is an isomorphism.

   (b) Show that the following conditions are equivalent:

   (i) for each locally free sheaf $\mathfrak{F}$ on $\hat{X}$, there exists a coherent sheaf $\mathscr{F}$ on X such that $\mathfrak{F} \cong \hat{\mathscr{F}}$ (i.e., $\mathfrak{F}$ is *algebraizable*);

   (ii) for each locally free sheaf $\mathfrak{F}$ on $\hat{X}$, there is an integer n_0 such that $\mathfrak{F}(n)$ is generated by global sections for all $n \geq n_0$.

   (c) Show that the conditions (i) and (ii) of (b) imply that the natural map $\operatorname{Pic} X \to \operatorname{Pic} \hat{X}$ is an isomorphism.

   Note. In fact, (i) and (ii) always hold if $N \geq 3$. This fact, coupled with (Ex. 11.5) leads to Grothendieck's proof of the Lefschetz theorem which says that if Y is a hypersurface in $\mathbb{P}^N_k$ with $N \geq 4$, then $\operatorname{Pic} Y \cong \mathbb{Z}$, and it is generated by $\mathscr{O}_Y(1)$.

7. Now let Y be a curve in $X = \mathbb{P}^2_k$.

   (a) Use the method of (Ex. 11.5) to show that $\operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$ is surjective, and its kernel is an infinite-dimensional vector space over k.

   (b) Conclude that there is an invertible sheaf $\mathfrak{L}$ on $\hat{X}$ which is not algebraizable.

   (c) Conclude also that there is a locally free sheaf $\mathfrak{F}$ on $\hat{X}$ so that no twist $\mathfrak{F}(n)$ is generated by global sections.

8. Let $f : X \to Y$ be a projective morphism, let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y, and assume that $H^i(X_y, \mathscr{F}_y) = 0$ for some i and some $y \in Y$. Then show that $R^i f_*(\mathscr{F})$ is 0 in a neighborhood of y.