Chapter 3, Section 4

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1. Let $f : X \to Y$ be an affine morphism of Noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geq 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F}).$$

Proof. Let $\mathfrak{V} = (V_i)$ be an open affine cover of Y, where $V_i = \operatorname{Spec} A_i$ for some Noetherian ring A_i, such that $f^{-1}(V_i) = \operatorname{Spec} B_i$ is affine for A_i-algebras B_i. We can compute $H^i(X, \mathscr{F})$ using the Čech complex defined by the open covering $\mathfrak{U} = (U_i)$, and $f_*\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Y$-module by (II, 5.8), so the cohomology of $f_*\mathscr{F}$ can be computed via the Čech complex defined by $\mathfrak{V}$ (4.5). On the otherhand, $\mathscr{F}(U_i) = f_*\mathscr{F}(V_i)$ for all i, so the Čech cohomology of $\mathscr{F}$ and $f_*\mathscr{F}$ with respect to $\mathfrak{U}$ and $\mathfrak{V}$, respectively, are isomorphic. Hence, $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$. □

2. Prove Chevalley's theorem: Let $f : X \to Y$ be a finite surjective morphism of Noetherian separated schemes, with X affine. Then Y is affine.

(a) Let $f : X \to Y$ be a finite surjective morphism of integral Noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha : \mathscr{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

(b) For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta : f_*\mathscr{G} \to \mathscr{F}^r$ which is an isomorphism at the generic point of Y.

(c) Now prove Chevalley's theorem.

Proof.

(a) The question is local on Y, so assume $Y = \operatorname{Spec} A$ for some Noetherian ring A. Let $K = \operatorname{Frac}(A)$ and $L = \operatorname{Frac}(B)$. Then $X = \operatorname{Spec} B$ for some A-algebra B, where B is finitely generated as an A-module. Since the image of f is dense in Y, and Y is integral, hence reduced, the structure homomorphism $A \to B$ is injective. In particular, $K \subseteq L$. We want to find a finitely generated B module M such that $M \otimes_B L$ is a finite-dimensional K-vectorspace. But L/K is a finite extension since B is a finite A-module.

□

3. Let $X = \mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathscr{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j \mid i, j < 0\}$. In particular, it is infinite-dimensional.

Proof. Let $\mathfrak{U}$ be the open covering by the two open sets $V = X - \{x = 0\}$ and $U = X - \{y = 0\}$, with affine coordinates obtained by restricting the ones from X. Then the Čech complex has only two terms:

$$C^0 = \Gamma(V, \mathscr{O}_V) \times \Gamma(W, \mathscr{O}_W),$$
$$C^1 = \Gamma(V \cap W, \mathscr{O}_{V \cap W}).$$

Now

$$\Gamma(V, \Omega) = k\left[x, \frac{1}{x}, y\right]$$
$$\Gamma(W, \Omega) = k\left[x, y, \frac{1}{y}\right]$$
$$\Gamma(V, W, \Omega) = k\left[x, y, \frac{1}{x}, \frac{1}{y}\right]$$

and the map $d : C^0 \to C^1$ is given by $(f, g) \mapsto f - g$. To compute H^1, the image of d is the set of all expressions $x^i y^j$ where at least one of i, j is non-negative. Hence, $H^1(U, \mathscr{O}_U)$ is spanned by $\{x^i y^j \mid i, j < 0\}$. □

4. On an arbitrary topological space X with an arbitrary Abelian sheaf $\mathscr{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{V} = (V_j)_{j \in J}$, together with a map $\lambda : J \to I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology for any Abelian sheaf $\mathscr{F}$, and for each i,

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathscr{F}).$$

(b) For any Abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$$

are compatible with the refinement maps above.

(c) Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of Abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is an isomorphism.

Proof.

(a)

(b)

(c)

□

5. For any ringed space $(X, \mathscr{O}_X)$, let $\operatorname{Pic} X$ be the group of isomorphism classes of invertible sheaves (II, §6). Show that $\operatorname{Pic} X \cong H^1(X, \mathscr{O}_X^*)$ where $\mathscr{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathscr{O}_X)$, with multiplication as the group operation.

Proof. Let $\mathscr{L}$ be an invertible sheaf on X, and let $\mathfrak{U} = (U_i)$ be an open cover of X such that $\mathscr{L}\big|_{U_i} \cong \mathscr{O}_{U_i}$ for each i. Fix such an isomorphism once and for all. Let $s_i \in \mathscr{L}\big|_{U_i}$ that corresponds to $1 \in \mathscr{O}_{U_i}$ for each i. □