

# Chapter 3, Section 3

James Lee

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1. Let  $X$  be a Noetherian scheme. Show that  $X$  is affine if and only if  $X_{\text{red}}$  (II, Ex. 2.3) is affine.

*Proof.* One direction is clear. Suppose  $X_{\text{red}} = \text{Spec } A$  where  $A$  is a Noetherian ring with no nilpotent elements, let  $f : X_{\text{red}} \rightarrow X$  be the natural map, and let  $\mathcal{F}$  be any quasi-coherent sheaf on  $X$ . Following the hint, consider the filtration

$$\mathcal{F} \supseteq \mathcal{N} \cdot \mathcal{F} \supseteq \mathcal{N}^2 \cdot \mathcal{F} \supseteq \dots,$$

where  $\mathcal{N}$  is the sheaf of nilpotent elements on  $X$ . Note that  $X \cong X_{\text{red}}$  as topological space, and the associated morphism of sheaves  $\mathcal{O}_X \rightarrow f_* \mathcal{O}_{X_{\text{red}}}$  is surjective with kernel  $\mathcal{N}$ . Thus, each of the quotients of this filtration can be naturally viewed as  $A$ -modules. In particular, we have a natural isomorphism (2.10)

$$H^i(X, \mathcal{N}^r \cdot \mathcal{F} / \mathcal{N}^{r+1} \cdot \mathcal{F}) \cong H^i(X_{\text{red}}, f^*(\mathcal{N}^r \cdot \mathcal{F} / \mathcal{N}^{r+1} \cdot \mathcal{F})).$$

Also, the nilradical of a Noetherian ring is nilpotent, so there exists a positive integer  $r > 0$  such that  $\mathcal{N}^r = 0$  (A.M. 7.15). Using our hypothesis and (3.7), we climb up the filtration and deduce that  $H^1(X, \mathcal{F}) = 0$ . Hence,  $X$  is affine by (3.7).  $\square$

2. Let  $X$  be a reduced Noetherian scheme. Show that  $X$  is affine if and only if each irreducible component is affine.

*Proof.* Suppose  $X = \text{Spec } A$  is affine for some reduced Noetherian ring  $A$ . The irreducible components of  $X$  correspond to the minimal prime ideals  $\mathfrak{p}$  of  $A$  (A.M. Ex. 1.20). In particular, the irreducible components of  $X$  are precisely  $\text{Spec } A/\mathfrak{p}$ . Conversely, let  $X_i$  be the irreducible components of  $X$ , and let  $\phi : \mathcal{F} \rightarrow \bigoplus_i j_* \mathcal{F}|_{X_i}$  be the natural map of  $\mathcal{O}_X$ -modules, where  $j : X_i \hookrightarrow X$  is the inclusion. Since  $X$  is Noetherian,  $X_i \cap X_j$  is quasi-compact, so we can cover it with a finite number of open affine subsets  $X_{ijk}$ . Because  $X$  is reduced,  $\phi$  is injective, so we can extend  $\phi$  by the following exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_i j_* \mathcal{F}|_{X_i} \longrightarrow \bigoplus_{i,j} j_* \mathcal{F}|_{X_{ijk}}.$$

Each  $j_* \mathcal{F}|_{X_i}, j_* \mathcal{F}|_{X_{ijk}}$  has vanishing cohomology for  $i > 0$  by (2.10), (3.5), and (3.7). While the sequence above is not surjective on the right, the image is still a quasi-coherent sheaf, so using the long exact sequence of cohomology, we deduce that  $H^i(X, \mathcal{F}) = 0$  for  $i > 0$ . Hence,  $X$  is affine by (3.7).  $\square$

6. Let  $X$  be a Noetherian scheme.

- (a) Show that the sheaf  $\mathcal{G}$  constructed in the proof of (3.6) is an injective object in the category  $\mathfrak{Qco}(X)$  of quasi-coherent sheaves on  $X$ . Thus,  $\mathfrak{Qco}(X)$  has enough injectives.
- (b) Show that any injective object of  $\mathfrak{Qco}(X)$  is flasque.
- (c) Conclude that one can compute cohomology as the derived functors of  $\Gamma(X, \cdot)$ , considered as a functor  $\mathfrak{Qco}(X) \rightarrow \mathfrak{Ab}$ .

*Proof.*

- (a) The Hom functor commutes with finite direct sums in the second argument, so we can assume  $\mathcal{G} = j_* \tilde{I}$ , where  $j : U = \text{Spec } A \rightarrow X$  is the inclusion, and  $I$  is an injective  $A$ -module. Suppose  $\mathcal{N} \rightarrow \mathcal{M}$  is an injective map of  $\mathcal{O}_X$ -modules, and we are given any  $f : \mathcal{N} \rightarrow j_* \tilde{I}$ . Since  $j^*$  is left exact when  $j$  is an open immersion, the induced map of  $A$ -modules  $j^* \mathcal{N} \rightarrow j^* \mathcal{M}$  is also injective. For any such  $f$  there is an associated morphism of  $A$ -modules  $g : j^* \mathcal{N} \rightarrow \tilde{I}$  by adjointness of  $j_*$ , so there exists an extension of  $g$  to  $j^* \mathcal{M}$  by injectivity of  $\tilde{I}$ . By adjointness of  $j^*$  again, we obtain a morphism  $\mathcal{M} \rightarrow j_* \tilde{I}$  that naturally extends  $f$ , which is what we wanted to show.

- (b) Essentially imitating (a) but replacing  $i^*$  with  $i_*$  and vice versa, we deduce that  $\mathcal{S}|_U$  is an injective object of  $\mathfrak{Qco}(U)$ . Covering  $X$  with finite number of open affines  $U_i = \text{Spec } A_i$ , we have  $\mathcal{S}|_{U_i} \cong \tilde{I}_i$  for some injective  $A_i$ -module  $I_i$  for each  $i$  by (II, 5.5). Each  $\tilde{I}_i$  is flasque by (3.4), so  $\mathcal{S}$  is flasque on a local basis. Hence,  $\mathcal{S}$  is flasque.
- (c) Considering  $\Gamma(X, \cdot)$  as a functor from  $\mathfrak{Qco}(X)$  to  $\mathfrak{Ab}$ , we calculate its derived functors by taking injective resolutions in the category  $\mathfrak{Qco}(X)$ . But any injective is flasque (b), and flasques are acyclic (2.5), so this resolution gives the usual cohomology functors (1.2A).

□

7. Let  $A$  be a Noetherian ring, let  $X = \text{Spec } A$ , let  $\mathfrak{a} \subseteq A$  be an ideal, and let  $U \subseteq X$  be the open set  $X - V(\mathfrak{a})$ .

(a) For any  $A$ -module  $M$ , establish the following formula of Deligne:

$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \text{Hom}_A(\mathfrak{a}^n, M).$$

(b) Apply this in the case of an injective  $A$ -module  $I$ , to give another proof of (3.4).

*Proof.*

(a) To define a map  $\phi : \varinjlim_n \text{Hom}_A(\mathfrak{a}^n, M) \rightarrow \Gamma(U, \widetilde{M})$ , it suffices to define  $A$ -homomorphisms

$$\phi_n : \text{Hom}_A(\mathfrak{a}^n, M) \rightarrow \Gamma(U, \widetilde{M})$$

that respect the direct system

$$M \cong \text{Hom}_A(A, M) \xrightarrow{\mu_0} \text{Hom}_A(\mathfrak{a}, M) \xrightarrow{\mu_1} \text{Hom}_A(\mathfrak{a}^2, M) \xrightarrow{\mu_2} \dots,$$

i.e.,  $\phi_n = \phi_{n+1} \circ \mu_n$  for all  $n$ . By the Noetherian hypothesis,  $\mathfrak{a}$  is generated by finitely many elements, say  $f_1, \dots, f_r$ . Consider the localization of any  $\alpha \in \text{Hom}_A(\mathfrak{a}^n, M)$  with respect to  $f_i$  for any  $i$ , that is  $\alpha_i : (\mathfrak{a}^n)_{f_i} \rightarrow M_{f_i}$ . Then  $(\mathfrak{a}^n)_{f_i} \cong A_{f_i}$  since  $f_i^n \in \mathfrak{a}^n$  and  $f_i$  is a unit in  $A_{f_i}$  (we exclude the case when  $f_i$  is nilpotent, since  $A_{f_i} = 0$ ). Let  $\phi_n(\alpha)$  be the section equal to  $\alpha_i(1)$  on  $U_i$ , where  $U_i$  is the distinguished open set associated to  $f_i$ . We remark that  $U = \bigcup_{i=1}^r U_i$ , so this definition is well-defined, and it respects the direct system of above since the localization of the inclusion  $\mathfrak{a}^n \hookrightarrow \mathfrak{a}^{n+1}$  is the identity map on  $A_{f_i}$ . Thus, to show  $\phi_n(\alpha)$  is a well-defined section, it is sufficient to show  $\alpha_i(1)$  and  $\alpha_j(1)$  agree on  $U_i \cap U_j \cong \text{Spec } A_{f_i f_j}$  for all  $i, j$ . Indeed,  $\alpha(f_i^n) = \alpha_i(f_i^n)$  where we naturally view  $\alpha(f_i^n)$  as an element of  $M_{f_i}$ ,  $\alpha_i$  is  $A_{f_i}$ -linear, and the restriction map  $U_i \rightarrow U_i \cap U_j$  is given by  $m/f_i^n \mapsto f_j^n/(f_i f_j)^n$ . Thus, we have

$$\begin{aligned} \alpha_i(1)|_{U_i \cap U_j} &= \alpha_i(f_i^{-n} f_i^n)|_{U_i \cap U_j} \\ &= \frac{\alpha(f_i^n)}{f_i^n}|_{U_i \cap U_j} \\ &= \frac{f_j^n \alpha(f_i^n)}{(f_i f_j)^n} \\ &= \frac{f_i^n \alpha(f_j^n)}{(f_i f_j)^n} \\ &= \alpha_j(1)|_{U_i \cap U_j}. \end{aligned}$$

Also, each  $\phi_n$  is injective, so the induced map  $\phi : \varinjlim_n \text{Hom}_A(\mathfrak{a}^n, M) \rightarrow \Gamma(U, \widetilde{M})$  is injective. It remains to show  $\phi$  is surjective. Let  $s \in \Gamma(U, \widetilde{M})$  be any section. For each  $i$ , we can express  $s|_{U_i}$  as  $m_i/f_i^{n_i}$  for some  $n_i > 0$  and  $m_i \in M$  such that  $m_i/f_i^{n_i}|_{U_i \cap U_j} = m_j/f_j^{n_j}|_{U_i \cap U_j}$ , where  $U_i \cap U_j = \text{Spec } A_{f_i f_j}$  for all  $i, j$ . Choose  $n = n_i$  that works for all  $i$ , so we have

$$\frac{f_j^n m_i}{(f_i f_j)^n} = \frac{f_i^n m_j}{(f_i f_j)^n}.$$

We want to show there exists an  $A$ -homomorphism  $\alpha : \mathfrak{a}^N \rightarrow M$  for some  $N \geq n$  such that  $\alpha_i(1) = f_i^{N-n} m_i / f_i^N$ . Imitating the proof of (3.3), let  $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \dots$  be the sequence of annihilators of  $\dots \subseteq \mathfrak{a}^2 \subseteq \mathfrak{a}$ . Since  $A$  is Noetherian, there is an  $r$  such that  $\mathfrak{b}_r = \mathfrak{b}_{r+1} = \dots$ . Define  $\sigma : \mathfrak{a}^{n+r} \rightarrow M$  by sending  $f_i^{n+r}$  to  $f_i^r m_i$  for all  $i$  and extending by zero. This is a well-defined homomorphism because the annihilator of  $f_i^{n+r}$  is  $\mathfrak{b}_{n+r} = \mathfrak{b}_r$ , and  $\mathfrak{b}_r$  annihilates  $f_i^r m_i$ . Hence,  $\phi_{n+r}(\sigma) = s$ .

- (b) Let  $I$  be an injective  $A$ -module. It will be sufficient to show for any open set  $U \subseteq X$ , where  $U = X - V(\mathfrak{a})$  for some ideal  $\mathfrak{a}$  of  $A$ , that  $\Gamma(X, \tilde{I}) \rightarrow \Gamma(U, \tilde{I})$  is surjective. Indeed,  $d\Gamma(X, \tilde{I}) \rightarrow \Gamma(U, \tilde{I})$  is induced by the natural maps  $\mathrm{Hom}_A(A, I) \rightarrow \mathrm{Hom}_A(\mathfrak{a}^n, I)$ , which is surjective since  $I$  is injective. Hence,  $\Gamma(X, I) \rightarrow \Gamma(U, I)$  is surjective.

□