Chapter 1, Section 1

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2. Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t$, $y = t^2$, $z = t^3$.

Proof. We have $I(Y) = (x^2 - y, x^3 - z)$, and Y is irreducible since it is homeomorphic to the affine line $\mathbb{A}^1$ under the map $t \mapsto (t, t^2, t^3)$, thus Y is an affine variety of dimension 1. Consider the map $f : k[x, y, z] \to k[t]$ defined by

$$x \mapsto t, \quad y \mapsto t^2, \quad z \mapsto t^3.$$

It is clearly surjective, and $k[t]$ is an integral domain, which implies $\ker f$ is a prime ideal. Since $\dim k[t] = 1$, the set Y is an affine variety of dimension 1 provided $\ker f = I(Y)$. We obviously have $I(Y) \subseteq \ker f$. By (1.8A), we have

$$\operatorname{height} \ker f + \dim k[t] = \dim k[x, y, z] \implies \operatorname{height} \ker f = 2,$$

and $(x^2 - y)$ is a prime ideal contained in $I(Y)$, thus $I(Y)$ cannot be properly contained in $\ker f$, hence $\ker f = I(Y)$. $\square$

3. Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

Proof. We claim that

$$Y = Z((x, y)) \cup Z((x, z)) \cup Z((x^2 - y, z - 1)).$$

It is clear the subsets $Z((x, y))$ and $Z((x, z))$ are irreducible. What is less obvious is the irreducibility of $Z((x^2 - y, z - 1))$. Observe that

$$\frac{k[x, y, z]}{(x^2 - y, z - 1)} \simeq \frac{k[x, y]}{(x^2 - y)} \simeq k[x, x^2] \simeq k[x],$$

hence $(x^2 - y, z - 1)$ is a prime ideal, hence $Z((x^2 - y, z - 1))$ is irreducible. If $P = (u, v, w) \in Y$, then $u^2 - vw = 0$ and $uw - u = 0$, so by the second equation either $u = 0$ or $u = 1$. If $u = 0$, either $v = 0$ or $w = 0$, which implies $P \in Z((x, y))$ or $P \in Z((x, z))$. If $u = 1$, then we have $P \in Z((x^2 - y, z - 1))$. The converse direction follows in similar fashion. $\square$

6. Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Proof. Let X be an irreducible topological space and let U be a non-empty proper open subset of X. Then $C = X - U$ is a proper closed subset of X, so we have $X = \overline{U} \cup C$, hence $\overline{U} = X$. Conversely, if every open subset of a topological space is dense, and we have $X = C_1 \cup C_2$ for two closed subsets of X with C_1 proper, then $U_1 = X - C_1$ is an open subset contained in C_2, which implies $X = \overline{U_1} \subseteq \overline{C_2} = C_2$, hence $C_2 = X$. It follows immediately that any open subset of an irreducible space is irreducible: if $V \subset U$ are open subsets, then the closure of V as a subspace of U is the intersection of U and its closure as a subspace of X, and V is dense in X, hence it is dense in U. Now, if U is an open subset of $\overline{Y}$ for an irreducible subset Y of any topological space X, then it must meet Y by definition of the closure of a subset, then the closure of U as a subspace of X contains Y since U is dense in Y, hence $\overline{U} \cap \overline{Y} = \overline{Y}$. $\square$

7. (a) Show that the following conditions are equivalent for a topological space X:

(i) X is noetherian;

(ii) every nonempty family of closed subsets has a minimal element;

(iii) X satisfies the ascending chain condition for open subsets;

(vi) every nonempty family of open subsets has a maximal element.

(b) A noetherian topological space is *quasi-compact*.

(c) A subset of a noetherian topological space is noetherian in its induced topology.

(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Proof.

(a) (i) $\implies$ (ii) If (ii) is false there is a non-empty collection T of closed subsets with no minimal element, and we can construct inductively a non-terminating strictly decreasing sequence in T.

(ii) $\implies$ (iii) Let $U_1 \subset U_2 \subset \cdots$ be an ascending chain of open subsets of X and let $C_i = X - U_i$. Then $\{C_i\}_{i \geq 1}$ is a non-empty collection of closed subsets of X, hence has a minimal element, say C_m. Hence, we have $U_m = U_{m+1} = \cdots$.

(iii) $\implies$ (iv) If (iv) is false, then there is a non-empty collection S of open subsets with no maximal element, and we can construct inductively a non-terminating strictly increasing sequence in S.

(iv) $\implies$ (i) Let $C_1 \supseteq C_2 \supseteq \cdots$ be a descending chain of closed subsets of X and let $U_i = X - C_i$. Then $\{U_i\}_{i \geq 1}$ is a non-empty collection of open subsets of X, hence has a maximal element, say U_m. Hence, we have $C_m = C_{m+1} = \cdots$.

(b) Suppose X is not quasi-compact so that the set Σ of non-quasi-compact closed subsets of X is non-empty. Let Y be a minimal element in Σ. If U is an open subset of Y, then $A = \overline{U}$ and $B = Y - U$ are closed subsets of X contained in Y such that $Y = A \cup B$. Since Y is minimal amongst the set of non-quasi-compact closed subsets in X, A and B must be compact; however, a finite union of compact sets is compact, a contradiction.

(c) Let Y be a subspace of X. Then any open subset of Y is of the form $V = U \cap Y$ for some open subset U of X, so if $V_1 \subseteq V_2 \subseteq \cdots$ is an ascending chain of open sets in Y with $V_i = U_i \cap Y$, then let $U_i' = \bigcup_{j \leq i} U_i$ so that we have the ascending chain $U_1' \subseteq U_2' \subseteq \cdots$ and

$$U_i' \cap Y = \bigcup_{j \leq i} U_i \cap Y = \bigcup_{j \leq i} V_i = V_i.$$

If X is noetherian, then this chain eventually terminates, say at $i = n$, which implies the chain $V_1 \subseteq V_2 \subseteq \cdots$ terminates at $V_n = U_n' \cap Y$. Hence, Y is noetherian.

(d) Every subspace of a noetherian space is compact since every subspace is noetherian, and in a Hausdorff space every compact set is closed, hence every subset is closed, therefore a noetherian Hausdorff space must be discrete. Finiteness follows from quasi-compactness.

□

8. Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

Proof. Let $Y = Z(\mathfrak{p})$ and $H = Z(f)$ for some prime ideal $\mathfrak{p}$ and irreducible polynomial f in $k[x_1, \ldots, x_n]$. If $Y \not\subseteq H$, then $f \notin \mathfrak{p}$, so the image of f in $A(Y) = k[x_1, \ldots, x_n]/\mathfrak{p}$ is not a zero-divisor. If f is a unit in $A(Y)$, then f does not vanish at any points in Y, which implies Y and H does not intersect, so $Y \cap H$ has no irreducible components (the empty set is defined to be not irreducible). Otherwise, the coordinate ring of an irreducible component of $Y \cap H$ corresponds to a minimal prime ideal $\mathfrak{q}$ in $A(Y)$ which contains f. By Krull's Hauptidealsatz, any such prime ideal must have height 1, and since $A(Y)$ has dimension r, it follows that $A(Y)/\mathfrak{q}$ has dimension $r - 1$ by (1.8A). □

9. Let $\mathfrak{a} \subseteq A = k[x_1, \ldots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n - r$.

Proof. We proceed by induction on r. If $r = 1$ and $\mathfrak{a} = (f)$, then the irreducible components of $Z(\mathfrak{a})$ correspond to the irreducible factors of f, which are hypersurfaces in $\mathbb{A}^n$ and therefore has dimension $n - 1$. Let $r > 1$ and assume the statement to be true for $r - 1$ and suppose $\mathfrak{a} = (f_1, \ldots, f_{r-1}, f_r)$. An irreducible component of $Z(f_1, \ldots, f_{r-1}, f_r)$ corresponds to a minimal prime ideal $\mathfrak{p}$ in $k[x_1, \ldots, x_n]$ containing $(f_1, \ldots, f_{r-1}, f_r)$, and by the inductive hypothesis a minimal prime ideal containing $(f_1, \ldots, f_{r-1})$ have height at most $r - 1$, so by Krull's Hauptidealsatz, $\mathfrak{p}$ can have height at most r, hence $A/\mathfrak{p}$ has dimension at least $n - r$, hence an irreducible component of $Z(f_1, \ldots, f_{r-1}, f_r)$ have dimension at least $n - r$. □

10. (a) If Y is any subset of a topological space X, then $\dim Y \leq \dim X$.

(b) If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$

(c) Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$.

(d) If Y is a closed subset of an irreducible finite dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.

(e) Give an example of a noetherian topological space of infinite dimension.

Proof.

(a) If $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ is a chain of distinct irreducible closed subsets of Y, then their closures as subsets of X are also irreducible by Exercise 1.6, that is $\overline{Z_0} \subset \overline{Z_1} \subset \cdots \subset \overline{Z_n}$ is a chain of distinct irreducible closed subsets of X, hence $\dim Y \leq \dim X$. It is indeed distinct since if $x \in Z_{i+1}$ and $x \in \overline{Z_i} = \overline{C_i \cap Y} \subseteq C_i \cap \overline{Y}$ for some closed subset C_i of X, then x must be in $Z_{i+1} \cap C_i \cap \overline{Y} = Z_i$.

(b) By (a) we have $\sup \dim U_i \leq \dim X$. Conversely, if $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ is a chain of distinct irreducible closed subsets of X, then there exists $U \in \{U_i\}$ such that $V_0 = Z_0 \cap U \neq \emptyset$, thus $V_i = Z_i \cap U \neq \emptyset$, so by Exercise 1.6 $V_0 \subset V_1 \subset \cdots \subset V_n$ is a distinct chain of irreducible closed subsets in U, therefore $n \leq \dim U$, hence taking the supremum we have $\dim X \leq \sup \dim U_i$.

(c) Let $X = \{a, b\}$ with topology defined by the open subsets $\{\emptyset, \{a\}, X\}$. Then $U = \{a\}$ is dense since the only closed subset containing U is X, and it has dimension 0, but X has dimension 1.

(d) If Y itself is irreducible, then any chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of irreducible closed subsets of Y can be extended to $Z_0 \subset Z_1 \subset \cdots \subset Z_n \subset Y$ unless $Z_n = Y$, so if $n = \dim Y$ then we must have $Z_n = Y$, and since $\dim Y = \dim X$, Y cannot be a proper subset of X. Otherwise, if Y is not irreducible then Y and therefore Z_n are proper subsets of X, and by the proof of (a) any chain in Y induces a chain of same length in X. Since $\overline{Z}_n$ is a proper subset of X, the chain can be extended to $\overline{Z_0} \subset \overline{Z_1} \subset \cdots \subset \overline{Z_n} \subset X$, contradicting the dimensions of X and Y. Hence, $Y = X$.

(e) Let $X = \mathbb{N}$ with topology defined by the closed sets $C_n = \{1, \ldots, n\}$. If $\{C_{n_\alpha}\}$ is any collection of closed subsets, then the minimal element is C_{n_β} for $n_\beta = \min n_\alpha$, so X is noetherian. Also, every closed set is irreducible, so we have an infinite chain of irreducible closed subsets $C_1 \subset C_2 \subset \cdots$.

$\square$