Chapter 3, Section 5

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1. Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum(-1)^i \dim_k H^i(X, \mathscr{F}).$$

If

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

2. (a) Let X be a projective scheme over a field k, let $\mathscr{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$. We call P the *Hilbert polynomial* of $\mathscr{F}$ with respect to the sheaf $\mathscr{O}_X(1)$.

(b) Now let $X = \mathbb{P}^r_k$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0, \ldots, x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathscr{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

3. *Arithmetic Genus.* Let X be a projective scheme of dimension r over a field k. We define the *arithmetic genus* p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathscr{O}_X) - 1).$$

Note that it depends only on X, not on any projective embedding.

(a) If X is integral, and k algebraically closed, show that $H^0(X, \mathscr{O}_X) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1}(-1)^i \dim_k H^{r-i}(X, \mathscr{O}_X).$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X, \mathscr{O}_X).$$

(b) If X is a closed subvariety of $\mathbb{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

(c) if X is a nonsingular projective curve over an algebraically closed field k¡ show that $p_a(X)$ is in fact a *birational* invariant. Conclude that a nonsingular plane curve of degree $d \geq 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used the geometric genus.)

4. Recall from (II, Ex. 6.10) the definition of the Grothendieck group $K(X)$ of a noetherian scheme X.

(a) Let X be a projective scheme over a field k, and let $\mathscr{O}_X(1)$ be a very ample invertible sheaf on X. Show that there is a (unique) additive homomorphism

$$P : K(X) \to \mathbb{Q}[z]$$

such that for each coherent sheaf $\mathscr{F}$ on X, $P(\gamma(\mathscr{F}))$ is the Hilbert polynomial of $\mathscr{F}$ (Ex. 5.2).

(b) Now let $X = \mathbb{P}^r_k$. For each $i = 0, \ldots, r$, let L_i be a linear space of dimension i in X. Then show that

(1) $K(X)$ is the free Abelian group generated by $\{\gamma(\mathscr{O}_{K_i}) \mid i = 0, \ldots, r\}$, and

(2) the map $P : K(X) \to \mathbb{Q}[z]$ is injective.

5. Let k be a field, let $X = \mathbb{P}^r_k$, and let Y be a closed subscheme of dimension $q \geq 1$, which is a complete intersection (II, Ex. 8.4). Then:

(a) for all $n \in \mathbb{Z}$, the natural map
$$H^0(X, \mathscr{O}_X(n)) \to H^0(Y, \mathscr{O}_Y(n))$$
is surjective. (This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.)

(b) Y is connected;

(c) $H^i(Y, \mathscr{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbb{Z}$;

(d) $p_a(Y) = \dim_k H^q(Y, \mathscr{O}_Y)$.

6. *Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbb{P}^3_k$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that $\operatorname{Pic} Q \cong \mathbb{Z} \oplus \mathbb{Z}$, so we can talk about the *type* (a, b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathscr{L}(Y)$ by $\mathscr{O}_Q(a, b)$. Thus for any $n \in \mathbb{Z}$, $\mathscr{O}_Q(n) = \mathscr{O}_Q(n, n)$.

(a) Use the special cases $(q, 0)$ and $(0, q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbb{P}^1$ in Q, to show:
  (1) if $|a - b| \leq 1$, then $H^1(Q, \mathscr{O}_Q(a, b)) = 0$;
  (2) if $a, b < 0$, then $H^1(Q, \mathscr{O}_Q(a, b)) = 0$;
  (3) if $a \leq -2$, then $H^1(Q, \mathscr{O}_Q(a, 0)) \neq 0$.

(b) Now use these results to show:
  (1) if Y is a locally principal closed subscheme of type (a, b) with $a, b > 0$, the Y is connected;
  (2) now assume k is algebraically closed. Then for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a, b). Use (II, 7.6.2) and (II, 8.18).
  (3) an irreducible nonsingular curve Y of type (a, b), $a, b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a - b| \leq 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbb{P}^3$. The simplest is the one of type $(1, 3)$, which is just the rational quartic curve (I, Ex. 3.18).

(c) If Y is a locally principal subscheme of type (a, b) in Q, show that $p_a(Y) = ab - a - b + 1 = (a - 1)(b - 1)$.

7. Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathscr{L}$ an invertible sheaf.

(a) If $\mathscr{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathscr{L}$ is ample on Y, where $i : Y \to X$ is the inclusion.

(b) $\mathscr{L}$ is ample on X if and only if $\mathscr{L}_{\text{red}} = \mathscr{L} \otimes \mathscr{O}_{X_{\text{red}}}$ is ample on X.

(c) Suppose X is reduced. Then $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes \mathscr{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

(d) Let $f : X \to Y$ be a finite surjective morphism, and let $\mathscr{L}$ be an invertible sheaf on Y. Then $\mathscr{L}$ is ample on Y if and only if $f^*\mathscr{L}$ is ample on X.

8. Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.

(a) If X is irreducible and nonsingular, then X is projective by (II, 6.7).

(b) If X is integral, let $\tilde{X}$ be its normalization (II, Ex. 3.8). Show that $\tilde{X}$ is complete and nonsingular, hence projective by (a). Let $f : \tilde{X} \to X$ be the projection. Let $\mathscr{L}$ be a very ample invertible sheaf on $\tilde{X}$. Show there is an effective divisor $D = \sum P_i$ on $\tilde{X}$ with $\mathscr{L}(D) \cong \mathscr{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i. Conclude that there is an invertible sheaf $\mathscr{L}_0$ on X with $f^*\mathscr{L}_0 \cong \mathscr{L}$. Then use (Ex. 5.7d), (II, 7.6) and (II, 5.16.1) to show that X is projective.

(c) If X is reduced, but not necessarily irreducible, let $X_1, \ldots, X_r$ be the irreducible components of X. Use (Ex. 4.5) to show $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is surjective. Then use (Ex. 5.7c) to show X is projective.

(d) Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that $\operatorname{Pic} X \to \operatorname{Pic} X_{\text{red}}$ is surjective. Then use (Ex. 5.7b) to show X is projective.

9. *A Nonprojective scheme.* We show the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0, and let $X = \mathbb{P}^2_k$. Let ω be the sheaf of differential 2-forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X, \omega \otimes \mathscr{T})$ defined as follows (Ex. 4.10). Let x_0, x_1, x_2 be the homogenous coordinates of X, let U_0, U_2, U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)d(x_i/x_j)$. This gives a Čech 1-cocycle with values in Ω^1_X, and since $\dim X = 2$, we have $\omega \otimes \mathscr{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence

$$\cdots \to H^1(X, \omega) \to \operatorname{Pic} X' \to \operatorname{Pic} X \xrightarrow{\delta} H^2(X, \omega) \to \cdots$$

of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathscr{O}_X(-3)$ by (II, 8.20.1), so $H^2(X, \omega) \cong k$. Since char $k = 0$, you need only show that $\delta(\mathscr{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology. Since $H^1(X, \omega) = 0$, we see that $\operatorname{Pic} X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.

Note. In fact, this result can be generalized to show that for any nonsingular projective surface X over an algebraically closed field k of characteristic 0, there is an infinitesimal extension X' of X by ω, such that X' is not projective over k. Indeed, let D be an ample divisor on X. Then D determines an element $c_1(D) \in H^1(X, \Omega^1)$ which we use to define X', as above. Then for any divisor E on X one can show that $\delta(\mathscr{L}(E)) = (D.E)$, where $(D.E)$ is the intersection number (Chapter V), considered as an element of k. Hence, if E is ample, $\delta(\mathscr{L}(E)) \neq 0$. Therefore, X' has no ample divisors.

On the other hand, over a field of characteristic $p > 0$, a proper scheme X is projective if and only if X_{red} is!

10. Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \cdots \to \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geq n_0$, the sequence of global sections

$$\Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \cdots \to \Gamma(X, \mathscr{F}^r(n))$$

is exact.