Chapter 1, Section 2

James Lee

April 23, 2025

1. Prove the "homogenous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogenous ideal, and if $f \in S$ is a homogenous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.

Proof. We can apply the usual Nullstellensatz by viewing f as a polynomial over $\mathbb{A}^{n+1}$ and $Z(\mathfrak{a})$ as the set $Z(\mathfrak{a})'$ consisting of all $(n+1)$-uples $Q = (a_0, \ldots, a_{n+1}) \in \mathbb{A}^{n+1}$ such that Q is the homogenous coordinate of some point $P \in Z(\mathfrak{a})$. Since f is homogenous, $f(P) = 0$ for $P \in \mathbb{P}^n$ if and only if $f(Q) = 0$ where P has Q as homogenous coordinates. Thus, $f^q \in \mathfrak{a}$ by the Nullstellensatz. $\square$

2. For a homogenous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(i) $Z(\mathfrak{a}) = \emptyset$;

(ii) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;

(iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Proof. (i) $\implies$ (ii) Let $S = k[x_0, \ldots, x_n]$. If $Z(\mathfrak{a})$, then all homogenous polynomials in S with degree > 0 are in $\sqrt{\mathfrak{a}}$ by the Nullstellensatz, so we have $S_+ \subseteq \sqrt{\mathfrak{a}}$. If $\sqrt{\mathfrak{a}}$ contains any element in $S - S_+$, then $\sqrt{\mathfrak{a}}$ contains an element in S_0, which is a unit. Hence, $\sqrt{\mathfrak{a}} = S$.

(ii) $\implies$ (iii) Either case $S_+ \subseteq \sqrt{\mathfrak{a}}$, so $x_i^{d_i} \in \mathfrak{a}$ for some $d_i \geq 0$ for each i, which implies $S_d \subseteq \mathfrak{a}$ where $d = \max d_i$.

(iii) $\implies$ (i) Let $d > 0$ be the smallest integer such that $\mathfrak{a} \supseteq S_d$, then we have $\mathfrak{a} \supseteq \bigoplus_{l \geq d} S_l$ so that $x_i^l \in \mathfrak{a}$ for all $0 \leq i \leq n$ and $l \geq d$. If there exists $P = (a_0, \ldots, a_n) \in Z(\mathfrak{a})$, then $a_i^d = a_i^{d+1} = 0$, which implies $a_i = 0$ for all $0 \leq i \leq n$, which is impossible. $\square$

4. (a) There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogenous radicals of S not equal S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note:* Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S.

(b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

(c) Show that $\mathbb{P}^n$ itself is irreducible.

Proof.

(a) Only the last part that states S_+ does not occur in this correspondence is new. Indeed, $I(Y) \supseteq S_+$ for some algebraic set $Y = Z(\mathfrak{a})$, then $Z(\mathfrak{a}) = \emptyset$ b (I, Ex. 2.2), and any constant polynomial $f \in S_0$ vacuously satisfies $f(P) = 0$ for all $P \in Z(\mathfrak{a})$. Hence, $I(Y) = S$.

(b) By the 1-1 correspondence from (a), $I(Y)$ is a homogenous ideal, so it is sufficient to show for any two homogenous elements f, g, that $fg \in I(Y)$ implies $f \in I(Y)$ or $g(Y)$. Indeed, if $fg \in I(Y)$, then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$, thus $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$ both being closed subsets of Y. Since Y is irreducible, we have either $Y = Y \cap Z(f)$, in which case $Y \subseteq Z(f)$, or $Y \subseteq Z(g)$. Hence, either $f \in I(Y)$ or $g \in I(Y)$.

Conversely, let $\mathfrak{p}$ be a homogenous prime ideal, and suppose that $Z(\mathfrak{p}) = Y_1 \cup Y_2$. Then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$, so either $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. Thus, $Z(\mathfrak{p}) = Y_1$ or Y_2, hence it is irreducible.

(c) $I(\mathbb{P}^n) = (0)$, which is a prime ideal.

$\square$

5. (a) $\mathbb{P}^n$ is a noetherian topological space.

(b) Every algebraic set in $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components.*

Proof.

(a) $\mathbb{P}^n$ is covered by the open set U_i defined the non-vanshing of the ith homogenous coordinate, each of which is homeomorphic to the affine plane $\mathbb{A}^n$. The affine plane is a noetherian topological space, so it suffices to show a finite union of noetherian topological spaces is also noetherian. An equivalent condition for a space to be noetherian is if every subset is quasi-compact by (A.M. p. 79). If Y is any subset of $\mathbb{P}^n$, then $Y = Y \cap \bigcup U_i$, each of which is quasi-compact in the induced topology of U_i, so Y is a finite union of quasi-compact sets. Hence, Y is quasi-compact.

(b) Follows from (a), (1.5), and (I, Ex. 2.4b).

$\square$

6. If Y is a projective variety with homogenous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$.

Proof. Let $\varphi_i : U_i \to \mathbb{A}^n$ be the homeomoprhism of (2.2), let Y_i be the affine variety $\varphi_i(Y \cap U_i)$, and let $A(Y_i)$ be its affine coordinate ring. If $g(y_1, \ldots, y_n)$ is an element of $A(Y_i)$, then define the map $A(Y_i) \to S(Y)_{x_i}$ as $g(x_0/x_i, \ldots, x_n/x_i)$, or equivalently $g \mapsto \varphi_i^* g = g \circ \varphi_i$; thus we can identify $A(Y_i)$ with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then, $S(Y)_{x_i} \simeq A(Y_i)[x_i, x_i^{-1}]$ since every monomial in $k[x_0, \ldots, x_n]$ can be written as

$$x_0^{d_0} \cdots x_i^{d_i} \cdots x_n^{d_n} = \frac{x_0^{d_0} \cdots \widehat{x_i^{d_i}} \cdots x_n^{d_n}}{x_i^{d_0 + \cdots + \widehat{d_i} + \cdots + d_n}} x_i^{d_0 + \cdots + d_n},$$

where $\widehat{\ }$ denotes omission, and the quotient is in the image of $A(Y_i) \to S(Y)_{x_i}$. By (1.8A), the dimension of $S(Y)$ is equal to the transcendence degree of the quotient field of $S(Y)$, which is isomorphic to the quotient field of $S(Y)_{x_i}$. The dimension of $A(Y_i)$ is equal to the dimension to Y_i by (1.7). Therefore, we have

$$\begin{aligned} \dim S(Y) &= \dim S(Y)_{x_i} \\ &= \dim A(Y_i)[x_i, x_i^{-1}] \\ &= \dim A(Y_i) + 1 \\ &= \dim Y_i + 1. \end{aligned}$$

Since the Y_i cover Y, $\dim Y = \sup \dim Y_i$, so $\dim S(Y) = \dim Y + 1$ By (I, Ex. 1.10). $\square$

7. (a) $\dim \mathbb{P}^n = n$.

(b) If $Y \subseteq \mathbb{P}^n$ is quasi-projective variety, then $\dim Y = \dim \overline{Y}$.

Proof.

(a) $\dim \mathbb{P}^n = \dim k[x_0, \ldots, x_n] - 1 = n$.

(b) By Exercise 2.6, we have $\dim Y = \dim Y_i$ if Y_i is non-empty, so by (1.10) $\dim \overline{Y} = \dim \overline{Y}_i = \dim Y_i = \dim Y$.

$\square$

8. A projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n-1$ if and only if it is the zero set of a single irreducible homogenous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbb{P}^n$.

Proof. If Y has dimension $n-1$, then by (1.13) Y is the union of affine varieties Y_i of dimension $n-1$, so by the map β in the proof of (2.2) and (1.13) each Y_i is the zero set of an irreducible homogenous polynomial f of positive degree, hence $Y = \bigcup Y_i = \bigcup_{Y_i \neq \emptyset} Z(f_i) = Z(f_0 \cdots f_n)$. Conversely, if $Y = Z(f)$ for some homogenous polynomial f of positive degree, then $Y_i = Z(\alpha(f))$ where α is the map defined in the proof of (2.2), hence by (1.13) and Exercise 6 we have $\dim Y = \dim Y_i = n - 1$. $\square$

12. *The d-Uple Embedding.* For given $n, d > 0$, let $M_0, M_1, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ by sending the point $P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1$, $d = 2$, then $N = 2$, and the image of Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.

(a) Let $\theta : k[y_0, \dots, y_N] \to k[x_0, \dots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogenous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$.

(c) Now show that ρ_d is a homeomoprhism of $\mathbb{P}^n$ onto the projective varieety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbb{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

Proof.

(a) The image of θ is an integral domain, so $\mathfrak{a}$ is a prime ideal, and it is clearly homogenous since each y_i is sent to a polynomial of same degree.

(b) If $Q \in \operatorname{im} \rho_d$, then $Q = \rho_d(P)$ for some $P \in \mathbb{P}^n$, for any $f \in \mathfrak{a}$ we have

$$f(Q) = f(\rho_d(P)) = \theta(f)(P) = 0 \implies Q \in Z(\mathfrak{a}).$$

Before proving the converse direction, consider the case when $n = 1$ and $d = 2$, so that $\rho_d : \mathbb{P}^1 \to \mathbb{P}^2$ is defined as $(b_0, b_1) \mapsto (b_0^2, b_0 b_1, b_1^2)$. Then, the polynomial $y_0 y_2 - y_1^2$ is in $\mathfrak{a}$. If $Q = (a_0, a_1, a_2) \in Z(\mathfrak{a})$, then since k is algebraically closed we have

$$a_0 a_2 - a_1^2 = 0 \implies a_1 = \pm\sqrt{a_0 a_2} \implies \rho_d(\sqrt{a_0}, \sqrt{a_2}) = (a_0, a_1, a_2).$$

Returning to the general case, if $Q = (a_0, \dots, a_N) \in Z(\mathfrak{a})$, indexing M_i using the stars and bars method, we have $\rho_d(\sqrt[d]{a_0}, \sqrt[d]{a_d}, \dots, \sqrt[d]{a_N}) = Q$.

(c) ρ_d is clearly bijective, so it will be sufficient to show that the closed sets of $\mathbb{P}^n$ are identified with the closed sets of $Z(\mathfrak{a})$ by ρ_d. Let $Y \subseteq \mathbb{P}^n$ be a closed subset, so $Y = Z(T)$ for some subset $T \subseteq k[x_0, \dots, x_n]$, then it is easy to see that $\rho_d(Y) = Z(\theta^{-1}(T)) \cap Z(\mathfrak{a})$. Conversely, let W be a closed subset of $Z(\mathfrak{a})$. Let $\overline{W}$ be its closure in $\mathbb{P}^N$. This is an algebraic set, so $\overline{W} = Z(T')$ for some $T' \subseteq k[y_0, \dots, y_N]$, hence $\rho_d^{-1}(W) = \rho_d^{-1}(\overline{W}) = Z(\theta(T'))$.

(d) Let (S, T) and (X, Y, Z, W) be homogenous coordinates of $\mathbb{P}^1$ and $\mathbb{P}^3$, respectively. Then, the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$ is given by

$$(S, T) \mapsto (S^3, S^2 T, S T^2, T^3).$$

Let Y be the twisted cubic curve in $\mathbb{A}^3$ and let $\overline{Y}$ be its projective closure in $\mathbb{P}^3$, then we have

$$\varphi_3^{-1}(Y) = \left\{ \left(\frac{u}{v}, \frac{u^2}{v^2}, \frac{u^3}{v^3}, 1 \right) \middle| u, v \in k,\ v \neq 0 \right\} \implies \overline{Y} = \{(u^3, u^2 v, u v^2, v^3) \mid u, v \in k\}.$$

□

14. *Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_i b_j, \dots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a *subvariety* of $\mathbb{P}^N$.

Proof. Let the homogenous coordinates of $\mathbb{P}^N$ be $\{z_{ij} \mid i = 0, \dots, r, j = 0, \dots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $\theta : k[\{z_{ij}\}] \to k[x_0, \dots, x_r, y_0, \dots, y_s]$ which sends z_{ij} to $x_i y_j$. If $P \in \operatorname{im} \psi$, then $P = \psi(Q, R)$ for some $Q \in \mathbb{P}^r$ and $R \in \mathbb{P}^s$, then for any $f \in \mathfrak{a}$ we have

$$f(P) = f(\psi(Q, R)) = \theta(f)(Q, R) = 0 \implies P \in Z(\mathfrak{a}).$$

Conversely, viewing points of $\mathbb{P}^N$ as $(r+1) \times (s+1)$-matrices, the variety $Z(\mathfrak{a})$ is defined as the vanshing of all 2×2-minors, i.e. $z_{ij} z_{kl} = z_{il} z_{jk}$ for all $0 \leq i, k \leq r$ and $0 \leq j, l \leq s$. This means $\{z_{ij}\} \in Z(\mathfrak{a})$ has rank 1, so it can be expressed as the outer product of two vectors in k^{r+1} and k^{s+1}, which is exactly the mapping defined by ψ. □

15. *Quadric Surface in $\mathbb{P}^3$.* Consider the surface Q (a *surface* is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw = 0$.

(a) Show that Q is equal to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

(b) Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}$, $\{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the proeprties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, $M_t \cap M_u = \emptyset$, and for all t, u, $L_t \cap M_u =$ one point.

(c) Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomoprhic via ψ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ (where each $\mathbb{P}^1$ has its Zariski topology).

Proof.

(a) The kernel of the mapping $k[z_{00}, z_{01}, z_{10}, z_{11}] \to k[x_0, x_1, y_0, y_1]$ as in Exercise 14 is generated by $z_{00}z_{11} - z_{01}z_{10}$. Then $Z(z_{00}z_{11} - z_{01}z_{11})$ is equal to Q for a suitable choice of coordinates.

(b) From here, we assume Q is defined by $xw - yz = 0$ (the author was too lazy to fix his mistake after realizing it at (c)). Consider the Segre embedding $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ defined by

$$(t_0, t_1) \times (u_0, u_1) \mapsto (t_0u_0, t_0u_1, t_1u_0, t_1u_1).$$

Fixing the first entry as $t = (t_0, t_1) \in \mathbb{P}^1$, we obtain an embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, which can be identified as the intersection of the zero set of the following linear polynomials

$$t_1x - t_0z = 0, \quad t_1y - t_0w = 0,$$

so we have one family of lines $\{L_t\}$, and we can obtain a second family of lines in the same manner by fixing the second entry of the map ψ, where the line M_u for $u = (u_0, u_1) \in \mathbb{P}^1$ is defined by the intersection of the zero set of the linear polynomials

$$u_1x - u_0y = 0, \quad u_1z - u_0w = 0.$$

Then $L_t \neq L_u$ implies $L_t \cap L_u = \emptyset$ follows from the fact that ψ is an embedding, in particular ψ is injective. To show the intersection of L_t and M_u is a single point for any $t, u \in \mathbb{P}^1$, it suffices to show the linear polynomials defined above intersect at exactly one point. Without loss of generality assume $t_1, u_1 \neq 0$, then setting $\lambda = t_0/t_1$ and $\mu = u_0/u_1$, we have

$$x = \lambda z, \quad y = \lambda w, \quad x = \mu y, \quad z = \mu w \implies L_t \cap M_u = \{(\lambda\mu, \lambda, \mu, 1)\}.$$

(c) Consider the curve K in Q defined by $x = w$. It is clearly not a line since it is the intersection of a nonlinear curve and a linear curve; however it is a closed subset of Q. On the other hand, $\psi^{-1}(K)$ is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, which is certainly not closed since $\mathbb{P}^1$ is not Hausdorff.

$\square$