Chapter 2, Section 7

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1. Let $(X, \mathscr{O}_X)$ be a locally ringed space, and let $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism.

Proof. We reduce to the following algebraic problem: Any A-linear surjective map $\phi : A \to A$ is an isomorphism. Indeed, ϕ is determined by $r = \phi(1)$, and ϕ surjective implies there exists $a \in A$ such that $ar = \phi(a) = 1$. Thus, r is a unit, and ϕ is injective. □

2. Let X be a scheme over a field k. Let $\mathscr{L}$ be an invertible sheaf on X, and let $\{s_0, \dots, s_n\}$ and $\{t_0, \dots, t_m\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathscr{L})$, and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \le m$. Show that the corresponding morphisms $\varphi : X \to \mathbb{P}^n_k$ and $\psi : X \to \mathbb{P}^m_k$ differ by a suitable linear projection $\mathbb{P}^m - L \to \mathbb{P}^n$ and an automorphism of $\mathbb{P}^n$, where L is a linear subspace of $\mathbb{P}^m$ of dimension $m - n - 1$.

3. Let $\varphi : \mathbb{P}^n_k \to \mathbb{P}^m_k$ be a morphism. Then:

(a) either $\varphi(\mathbb{P}^n) = pt$ or $m \ge n$ and $\dim \varphi(\mathbb{P}^n) = n$;

(b) in the second case, φ can be obtained as the composition of (1) a d-uple embedding $\mathbb{P}^n \to \mathbb{P}^N$ for a uniquely determined $d \ge 1$, (2) a linear projection $\mathbb{P}^N - L \to \mathbb{P}^m$, and an automorphism of $\mathbb{P}^m$. Also, φ has finite fibers.

Proof.

(a) Let $\mathscr{L}$ be the invertible sheaf on $\mathbb{P}^n$ associated to the morphism $\varphi : \mathbb{P}^n \to \mathbb{P}^m$, which is generated by global sections $s_i = \varphi^*(x_i)$, $i = 0, 1, \dots, m$. Pick $P \in \varphi^{-1}((1, 0, \dots, 0))$ (we can assume such P always exists by applying an automorphism of $\mathbb{P}^m$). Identifying $\mathscr{L}_P \cong \mathscr{O}_{P,\mathbb{P}^n}$, we see that $(s_i)_P$ for $i \ne 0$ is not a unit, and they generate the local ring of $\mathscr{O}_{P,\mathbb{P}^n}$. Since $\mathbb{P}^n$ is smooth, the minimal number of generators of the maximal ideal of $\mathscr{O}_{P,\mathbb{P}^n}$ is equal to the dimension n. The sections s_i generate $\mathscr{O}_{P,\mathbb{P}^n}$. Hence, $m \ge n$. Our proof also shows the induced homomorphism of local rings $\varphi_P : \mathscr{O}_{\varphi(P),\mathbb{P}^m} \to \mathscr{O}_{P,\mathbb{P}^n}$ is surjective, which implies $\mathscr{O}_{\varphi(P),\varphi(\mathbb{P}^n)} = \mathscr{O}_{\varphi(P),\mathbb{P}^m} / \ker \varphi_P \cong \mathscr{O}_{P,\mathbb{P}^n}$. Hence, $\varphi(P)$ has dimension n.

□

5. Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathscr{L}, \mathscr{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.

(a) If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes M$ is ample.

(b) If $\mathscr{L}$ is ample and $\mathscr{M}$ is arbitrary, then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.

(c) If $\mathscr{L}, \mathscr{M}$ are both ample, so is $\mathscr{L} \otimes \mathscr{M}$.

(d) If $\mathscr{L}$ is very ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is very ample.

(e) If $\mathscr{L}$ is ample, then there is an $n_0 > 0$ such that $\mathscr{L}^n$ is very ample for all $n \ge n_0$.

Proof.

(a) Fix a coherent sheaf $\mathscr{F}$ on X for the remainder of this problem. Let $n \gg 0$ such that $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections. If $\mathscr{M}$ is generated by global sections, then so is $\mathscr{M}^n$ for any $n > 0$. More generally, the tensor product of two invertible sheaves generated by global sections is generated by global sections. Indeed, if t_i and r_j are generating global sections of $\mathscr{L}$ and $\mathscr{M}$, then $\mathscr{L} \otimes \mathscr{M}$ is generated by $t_i \otimes r_j$ for all pairs (i, j) (tensor products commutes with colimits). hus, $\mathscr{F} \otimes \mathscr{L}^n \otimes \mathscr{M}^n \cong \mathscr{F} \otimes (\mathscr{L} \otimes \mathscr{M})^n$ is generated by global sections. Hence, $\mathscr{L} \otimes \mathscr{M}$ is ample.

(b) Let $n > 0$ such that $\mathscr{M} \otimes \mathscr{L}^{n-1}$ is generated by global sections. By (a), $\mathscr{L} \otimes (\mathscr{M} \otimes \mathscr{L}^{n-1}) = \mathscr{M} \otimes \mathscr{L}^n$ is ample.

(c) If $\mathscr{M}$ is ample, then $\mathscr{M}^n$ is generated by global sections for large enough n. Also, $\mathscr{L}^n$ is ample by (II, 7.5), so $\mathscr{L}^n \otimes \mathscr{M}^n$ is ample. Hence, $\mathscr{L} \otimes \mathscr{M}$ is ample by (II, 7.5) again.

(d) Let $i : X \to \mathbb{P}^n_A$ be an immersion such that $\mathscr{L} \cong i^*\mathscr{O}_{\mathbb{P}^n_A}(1)$, and let $\varphi : X \to \mathbb{P}^m_A$ be the unique A-morphism corresponding to $\mathscr{M}$ (II, 7.1). By the universal property of the fiber project $\mathbb{P}^n_A \times_A \mathbb{P}^m_A$, there exists a unique A-morphism $\phi : X \to \mathbb{P}^n_A \times_A \mathbb{P}^m_A$ such that $i = p_n \circ \phi$ and $\varphi = p_m \circ \phi$, where $p_n : \mathbb{P}^n_A \times_A \mathbb{P}^m_A \to \mathbb{P}^n_A$ and $p_m : \mathbb{P}^n_A \times_A \mathbb{P}^m_A \to \mathbb{P}^m_A$ are the natural projection maps. It is not hard to see ϕ is an immersion, and composing it with a Segre embedding (Ex. 5.12), we obtain an immersion $\phi' : X \to \mathbb{P}^N_A$, where $N = nm + n + m$, such that $\mathscr{L} \otimes \mathscr{M} \cong \phi'^*\mathscr{O}(1)$.

(e) If $\mathscr{L}$ is ample, then there exists $n_0 > 0$ such that $\mathscr{L}^{n-1}$ is generated by global sections for all $n \geq n_0$. By (d), $\mathscr{L}^n = \mathscr{L} \otimes \mathscr{L}^{n-1}$ is very ample for all $n \geq n_0$.

$\square$