Chapter 4, Section 2

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1. Use (2.5.3) to show that $\mathbb{P}^n$ is simply connected.

2. *Classification of Curves of Genus 2.* Fix an algebraically closed field k of characteristic $\neq 2$

(a) If X is a curve of genus 2 over k, the canonical linear system $|K|$ determines a finite morphism $f : X\mathbb{P}^1$ of degree 2 (Ex. 1.7). Show that it is ramified at exactly 6 points, with ramification index 2 at each one. Note that f is uniquely determined up to an automorphism of $\mathbb{P}^1$, so X determines an (unordered) set of 6 points of $\mathbb{P}^1$, up to an automorphism of $\mathbb{P}^1$.

(b) Conversely, given six distinct elements $a_1, \dots, a_6 \in k$, let K be the extension of $k(x)$ determined by the equation $z^2 = (x - a_1) \cdots (x - a_6)$. Let $f : X \to \mathbb{P}^1$ be the corresponding morphism of curves. Show that $g(X) = 2$, the map f is the same as the one determined by the canonical linear system, and f is ramified over the six points $x = a_i$ of $\mathbb{P}^1$, and nowhere else. (Cf. (II, Ex. 6.4).)

(c) Using (I, Ex. 6.6), show that if P_1, P_2, P_3 are three distinct points of $\mathbb{P}^1$, then there exists a unique $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\varphi(P_1) = 0$, $\varphi(P_2) = 1$, $\varphi(P_3) = \infty$. Thus in (a), if we order the six points of $\mathbb{P}^1$ and then normalize by sending the first three to $0, 1, \infty$ respectively, we may assume that X is ramified over $0, 1, \infty, \beta_1, \beta_2, \beta_3$, where $\beta_1, \beta_2, \beta_3$ are three distinct elements of $k \setminus \{0, 1\}$.

(d) Let S_6 be the symmetric group on 6 letters. Define an action of S_6 on sets of three distinct elements $\beta_1, \beta_2, \beta_3$ of $k, \neq 0, 1$, as follows: reorder the set $\{0, 1, \infty, \beta_1, \beta_2, \beta_3\}$ according to a given element $\sigma \in S_6$, then renormalize as in (c) so that the first three become $0, 1, \infty$ again. Then the last three are the new $\beta_1', \beta_2', \beta_3'$.

(e) Summing up, conclude that there is a one-to-one correspondence between the set of isomorphism classes of curves of genus 2 over k, and triples of distinct elements $\beta_1, \beta_2, \beta_3$ of $k, \neq 0, 1$, modulo the action of Σ_6 described in (d). In particular, there are many non-isomorphic curves of genus 2. We say that curves of genus 2 depend on three parameters, since they correspond to the points of an open subset of $\mathbb{A}^3$ modulo a finite group.

3. *Plane Curves.* Let X be a curve of degree d in $\mathbb{P}^2$. For each point $P \in X$, let $T_P(X)$ be the tangent line to X at P (I, Ex. 7.3). Considering $T_P(X)$ as a point of the dual projective plane $(\mathbb{P}^2)^*$, the map $P \mapsto T_P(X)$ gives a morphism $X \to X^* \subset (\mathbb{P}^2)^*$. Assume $\mathrm{char}(k) = 0$.

(a) Fix a line $L \subset \mathbb{P}^2$ which is not tangent to X. Define a morphism $\varphi : X \to L$ by $\varphi(P) = T_P(X) \cap L$. Show that φ is ramified at P if and only if either:

i. $P \in L$, or

ii. P is an inflection point of X, i.e., the intersection multiplicity of $T_P(X)$ with X at P is ≥ 3.

Conclude that X has only finitely many inflection points.

(b) A line in $\mathbb{P}^2$ is a multiple tangent of X if it is tangent to X at more than one point. It is a bitangent if it is tangent to X at exactly two points. If L is a multiple tangent to X at points $P_1, \dots, P_r$ and none of the P_i is an inflection point, show that the corresponding point of the dual curve X^* is an ordinary r-fold point. Conclude that X has only finitely many multiple tangents.

(c) Let $O \in \mathbb{P}^2$ be a point not on X, nor on any inflectional or multiple tangent of X. Let L be a line not containing O. Define $\psi : X \to L$ by projection from O. Show that ψ is ramified at $P \in X$ iff line OP is tangent to X at P, and then the ramification index is 2. Use Hurwitz's theorem to conclude that there are exactly $d(d-1)$ tangents of X passing through O. Hence, $\deg(X^*) = d(d-1)$.

(d) Show that for all but a finite number of points on X, a point lies on exactly $(d+1)(d-2)$ tangents of X, not counting the tangent at that point.

(e) Show that the degree of the morphism φ in (a) is $d(d-1)$. Conclude that if $d \geq 2$, then X has $3d(d-2)$ inflection points, properly counted (if $T_P(X)$ has intersection multiplicity r, then P is counted $r-2$ times). Show that an ordinary inflection point corresponds to an ordinary cusp of X^*.

(f) Let X be a plane curve of degree $d \geq 2$, and suppose X^* has only nodes and ordinary cusps. Then show that X has exactly $\frac{1}{2}d(d-2)(d-3)(d+3)$ bitangents. [Hint: Use normalization and compute $p_a(X^*)$ in two ways.]

(g) For example, a plane cubic curve has exactly 9 inflection points, all ordinary. The line joining any two of them intersects the curve in a third one

(h) A plane quartic has exactly 28 bitangents. (This holds even if the curve has a tangent with four-fold contact, in which case the dual curve X^* has a tacnode.)

4. Let X be the plane quartic curve $x^3y + y^3 + z^3x = 0$ over a field of characteristic 3. Show that X is nonsingular, every point of X is an inflection point, $X^* \cong X$, but the natural map $X \to X^*$ is purely inseparable.

5. *Automorphismss of a Curve of Genus* ≥ 2. Prove Hurwitz's theorem: A curve X of genus $g \geq 2$ over a field of char 0 has at most $84(g-1)$ automorphisms. Let $G = \mathrm{Aut}(X)$, $|G| = n$. Then G acts on $K(X)$, let $L = K(X)^G$, corresponding to a morphism $f : X \to Y$ of degree n.

(a) For a ramification point $P \in X$ with index r, show that $f^{-1}(f(P))$ has n/r points with ramification index r. Let $P_1, \ldots, P_s$ be ramification points over distinct points of Y with indices r_i. Then Hurwitz's formula implies:

$$\frac{2g-2}{n} = 2g(Y) - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right)$$

(b) Since $g \geq 2$, the LHS > 0. Show the RHS has minimum $1/42$, so $n \leq 84(g-1)$. **Note:** This bound is sharp for infinitely many g (Macbeath). In characteristic $p > 0$, same bound holds if $p > g+1$, with one exception: $y^2 = x^p - x$, $p = 2g+1$, which has $2p(p^2-1)$ automorphisms (Roquette).

6. Let $f : X \to Y$ be a finite morphism of curves of degree n.

(a) Define $f_* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ by $f_*(\sum n_i P_i) = \sum n_i f(P_i)$.
For any locally free sheaf $\mathscr{F}$ on Y of rank r, define $\det \mathscr{G} = \bigwedge^r \mathscr{F} \in \mathrm{Pic}(Y)$. For invertible sheaf $\mathcal{L}$ on X, $f_*\mathscr{L}$ is locally free of rank n on Y. Show

$$\det(f_*\mathscr{O}_X(D)) \cong \det(f_*\mathscr{O}_X) \otimes \mathscr{O}_Y(f_*D)$$

(b) Conclude that f_*D depends only on the linear equivalence class of D. Then f_* induces a homomorphism $f_* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. Show that $f^* \circ f_* : \mathrm{Pic}(Y) \to \mathrm{Pic}(Y)$ is multiplication by n.

(c) Use duality for finite flat morphisms to show:

$$\det(f_*\omega_X) \cong \det(f_*\mathscr{O}_X)^{-1} \otimes \omega_Y$$

(d) If f is separable with ramification divisor R, define the branch divisor $B = f_*R$. Show:

$$\det(f_*\mathscr{O}_X)^2 \cong \mathscr{O}_Y(-B)$$

7. Let Y be a curve over a field of char $\neq 2$. There is a one-to-one correspondence between finite étale covers $f : X \to Y$ of degree 2 and 2-torsion elements of $pic(Y)$.

(a) Given $f : X \to Y$ étale of degree 2, there is a natural map $\mathscr{O}_Y \to f_*\mathscr{O}_X$, with cokernel $\mathscr{L}$. Then fL is invertible on Y, and $\mathscr{L}^2 \cong \mathscr{O}_Y$.

(b) Conversely, given a 2-torsion line bundle $\mathscr{L}$, define $\mathscr{O}_Y$-algebra structure on $\mathscr{O}_Y \oplus \mathscr{L}$ by:

$$(a, b) \cdot (a', b') = (aa' + \varphi(b \otimes b'), ab' + a'b)$$

where $\varphi : \mathscr{L} \otimes \mathscr{L} \to \mathscr{O}_Y$ is an isomorphism. Let $X = \mathrm{Spec}(\mathscr{O}_Y \oplus \mathscr{L})$.

(c) Show these two constructions are inverses of each other. Use the involution on X and the trace map $a \mapsto a + \tau(a)$ to split the exact sequence.