Chapter 3, Section 2

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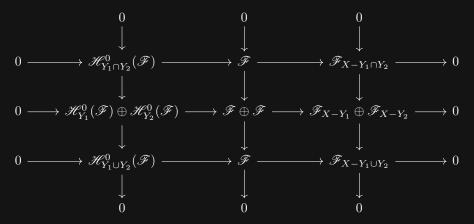
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4. Mayer-Vietoris Sequence. Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\cdots \longrightarrow H^{i}_{Y_{1} \cap Y_{2}}(X, \mathscr{F}) \longrightarrow H^{i}_{Y_{1}}(X, \mathscr{F}) \oplus H^{i}_{Y_{2}}(X, \mathscr{F}) \longrightarrow H^{i}_{Y_{1} \cup Y_{2}}(X, \mathscr{F}) \longrightarrow \cdots$$

$$\longrightarrow H^{i+1}_{Y_{1} \cap Y_{2}}(X, \mathscr{F}) \longrightarrow \cdots$$

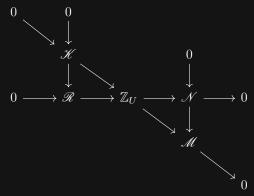
Proof. There is an exact sequence of sheaves



which induces the desired long sequence of cohomology with supports by (1.1A).

6. Let X be a Noetherian topological space, and let $\{\mathscr{I}_{\alpha}\}_{{\alpha}\in A}$ be a direct system of injective sheaves of Abelian groups on X. Then $\lim \mathscr{I}_{\alpha}$ is also injective.

Proof. We follow the hint. One direction is clear. Conversely, let $i: \mathcal{N} \to \mathcal{M}$ be an injective morphism of sheaves. By the proof of (2.7) we can write $\mathcal{N} = \varinjlim \mathcal{N}_{\beta}$ where \mathcal{N}_{β} is generated by the sections on some open U_{β} , and similarly for $\mathcal{M} = \varinjlim \mathcal{M}_{\beta}$. Notice that we can assume \mathcal{N} and \mathcal{M} are defined over the same direct system so that they belong to the same Abelian category. Thus, the inclusion map $i: \mathcal{N} \to \mathcal{M}$ can be broken down into inclusion maps $i_{\beta}: \mathcal{N}_{\beta} \to \mathcal{M}_{\beta}$. A direct system of morphisms $\mathcal{N}_{\beta} \to \mathcal{M}$ induces the same inclusion morphism $\mathcal{N} = \varinjlim \mathcal{N}_{\beta} \to \mathcal{M}$, so we reduce to the case when \mathcal{N} and \mathcal{M} are generated by a single section over some open set U. We have an exact sequence



where all the maps are natural, and \mathcal{R}, \mathcal{K} are kernels of the quotients \mathcal{N}, \mathcal{M} , respectively. It is not hard to see from above that any $f: \mathcal{N} \to \mathcal{I}$ naturally extends to \mathcal{M} , which is what we wanted to show.

Next, we show any subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$ such that \mathbb{Z}_U/\mathscr{R} is generated by a single section must be finitely generated. Indeed, fix some $x \in X$. Following the proof of (2.7), there exists some open neighborhood $x \in V \subseteq U$ such that $\mathscr{R}|_{V} \cong d \cdot \mathbb{Z}|_{V}$ for some positive integer d. Since X is noetherian, we can cover U by finite number of such V, say V_i for $i = 1, \ldots, n$. Therefore, there is an exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \bigoplus_{i=1}^n d_i \cdot \mathbb{Z}_{V_i} \longrightarrow \bigoplus_{i,j,k} d_{ijk} \cdot \mathbb{Z}_{V_i \cap V_j \cap V_k}$$

where d_{ijk} is the minimum of d_i, d_j, d_k . The terms on the right are finitely generated. Thus, \mathscr{R} is finitely generated, and any $\mathscr{R} \to \varinjlim \mathscr{I}_{\alpha}$ must factor through one of the \mathscr{I}_{α} (each generator s_i of \mathscr{R} factors through one of the \mathscr{I}_{α_i} , so take any $\beta > \alpha_i$, which exists by definition of a direct system).

- 7. Let S^1 be the circle (with its usual topology), and let \mathbb{Z} be the constant sheaf \mathbb{Z} .
 - (a) Show that $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$, using our definition of cohomology.
 - (b) Now let \mathscr{R} be the sheaf of germs of continuous real-valued functions on S^1 . Show that $H^1(S^1,\mathscr{R})=0$.

Proof.

(a) We remark that cohomology commutes with colimits on paracompact Hausdorff spaces. In particular, the statements of (II, Ex.1.11), (2.9) hold for S^1 . Let A, B be closed subsets of S^1 homeomorphic to the unit interval such that $A \cup B = S^1$ and $A \cap B = \{P, Q\}$ for two distinct points P, Q in S^1 (in the obvious way...). From now on, for any closed subset C of S^1 , denote $\mathbb{Z}_C = i_*\mathbb{Z}$, where $i: C \hookrightarrow S^1$ is the inclusion map and \mathbb{Z} is the constant sheaf on C. Without ambiguity \mathbb{Z} will denote the constant sheaf on the ambient space. We claim the following sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}_A \oplus \mathbb{Z}_B \xrightarrow{\tau} \mathbb{Z}_{A \cap B} \longrightarrow 0$$

defined by $\Delta(a) = (a,a)$ and $\tau(a,b) = b-a$ is exact. In particular, there exists natural maps $i_A^\#, i_B^\# : \mathbb{Z} \to \mathbb{Z}_A, \mathbb{Z}_B$ associated to the inclusion maps $i_A, i_B : A, B \to S^1$, so that $\Delta = (i_A^\#, i_B^\#)$. In the same way, the associated morphism of sheaves of the inclusion maps $j_A, j_B : A \cap B \to A, B$ ascends to naturally defined maps $j_A^\#, j_B^\# : \mathbb{Z}_A, \mathbb{Z}_B \to \mathbb{Z}_{A \cap B}$ in the form of a restriction morphism. Thus, $\tau = j_B^\# - j_A^\#$. Exactness can be checked at the level of stalks. Suppose $R \notin A \cap B$. Then either $R \in A$ or $R \in B$, say $R \in A$. Then the stalks are $(\mathbb{Z})_R = \mathbb{Z}, (\mathbb{Z}_A)_R = \mathbb{Z}, (\mathbb{Z}_B)_R = \mathbb{Z}$, which is an exact sequence. If $R \in A \cap B$, then the stalks are $(\mathbb{Z})_R = \mathbb{Z}, (\mathbb{Z}_A)_R = \mathbb{Z}, (\mathbb{Z}_B)_R = \mathbb{Z}$ defined by Δ and τ , which is clearly exact. Hence, the sequence is exact at all points, so the sequence is exact.

Taking cohomology, we get a long exact sequence of cohomology groups

$$0 \longrightarrow H^0(S^1, \mathbb{Z}) \longrightarrow H^0(S^1, \mathbb{Z}_A) \oplus H^0(S^1, \mathbb{Z}_B) \xrightarrow{\tau_0} H^0(S^1, \mathbb{Z}_{A \cap B}) \longrightarrow$$

$$\longrightarrow H^1(S^1, \mathbb{Z}) \longrightarrow H^1(S^1, \mathbb{Z}_A) \oplus H^1(S^1, \mathbb{Z}_B) \longrightarrow H^1(S^1, \mathbb{Z}_{A \cap B}) \longrightarrow \cdots$$

where $H^i(\mathbb{Z}_A \oplus \mathbb{Z}_B) \cong H^i(\mathbb{Z}_A) \oplus H^i(\mathbb{Z}_B)$ by (2.9). By (2.10), we have

$$H^0(S^1, \mathbb{Z}), H^0(S^1, \mathbb{Z}_A), H^0(S^1, \mathbb{Z}_B) = \mathbb{Z},$$

 $H^0(\mathbb{Z}_{A \cap B}) = \mathbb{Z} \oplus \mathbb{Z},$
 $H^1(\mathbb{Z}_{A \cap B}) = 0$

The first line follows from the fact that A, B, S^1 are all connected and locally connected. The intersection $A \cap B$ is a noetherian space of dimension zero with two irreducible components, namely the points P and Q, so its space of global sections is free of rank two. Lastly, $H^1(S^1, \mathbb{Z}_{A \cap B}) = H^1(A \cap B, \mathbb{Z}) = 0$ by (2.7). By exactness, we reduce to the following exact sequence

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/\operatorname{im} \tau_0 \longrightarrow H^1(S^1, \mathbb{Z}) \longrightarrow H^1(S^1, \mathbb{Z}_A) \oplus H^1(S^1, \mathbb{Z}_B).$$

The homomorphism τ_0 is defined by $\tau_0(a,b) = (b-a,b-a)$, which is the diagonal map. Thus, the term on the left is free of rank one. It remains to show $H^1(S^1,\mathbb{Z}_A) = H^1(S^1,\mathbb{Z}_B) = 0$. By (2.10), it suffices to show $H^1(A,\mathbb{Z}) = 0$.

From here, \mathbb{Z} will denote the constant sheaf on A. Identifying A with the closed unit interval [0,1], we repeat the procedure above for A. Pick any $t \in (0,1)$, say $t=2^{-1}$. Then X=[0,t] and Y=[t,1] cover A, so taking cohomology groups, we get a long exact sequence

$$0 \longrightarrow H^0(A, \mathbb{Z}) \longrightarrow H^0(A, \mathbb{Z}_X) \oplus H^0(A, \mathbb{Z}_Y) \longrightarrow H^0(A, \mathbb{Z}_{X \cap Y}) \longrightarrow$$

$$\longrightarrow H^1(A,\mathbb{Z}) \longrightarrow H^1(A,\mathbb{Z}_X) \oplus H^1(A,\mathbb{Z}_Y) \longrightarrow H^1(A,\mathbb{Z}_{X\cap Y}) \longrightarrow \cdots$$

Imitating the previous calculation, the first row is exact and $H^1(A, \mathbb{Z}_{X \cap Y}) = 0$ by (2.7) and (2.10). Thus, we reduce to the following exact sequence

$$0 \longrightarrow H^1(A,\mathbb{Z}) \longrightarrow H^1(A,\mathbb{Z}_X) \oplus H^1(A,\mathbb{Z}_Y) \longrightarrow 0.$$

Since $X \cong A, B, H^1(A, \mathbb{Z}_X), H^1(A, \mathbb{Z}_Y) \cong H^1(A, \mathbb{Z})$ by (2.10), which is possible if and only if $H^1(A, \mathbb{Z}) = 0$.

(b) Let \mathscr{M} be the sheaf of germs of measurable real-valued functions on S^1 modulo equivalence almost everywhere. It is clearly flasque, since for any measurable $f:V\to\mathbb{R}$ where $V\subseteq U\subseteq\mathbb{R}$ are open sets, the extension of f by zero on U is a measurable function. Thus, we have an exact sequence of sheaves

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}/\mathscr{R} \longrightarrow 0.$$

Any measurable function on S^1 is continuous except for on a set of measure zero in S^1 , so any global section of \mathcal{M}/\mathcal{R} is locally zero almost everywhere. Since we are considering equivalence classes of functions, where two functions are considered equal if they are equal except for on a measure zero set, we conclude that \mathcal{M}/\mathcal{R} has no non-zero global section. Taking cohomology groups, we have $H^1(S^1, \mathcal{R}) = 0$.