Chapter 2, Section 6

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1. Let X be a scheme satisfying $(*)$. Then $X \times \mathbb{P}^n$ also satisfies $(*)$, and $\mathrm{Cl}(X \times \mathbb{P}^n) \cong (\mathrm{Cl}\, X) \times \mathbb{Z}$.

Proof. Since $\mathbb{P}^n$ is a union of $n+1$ copies of $\mathbb{A}^n$, it is not hard to see from (6.6) that $X \times \mathbb{P}^n$ is noetherian and integral. It's left to show it is separated. Indeed, $X \times \mathbb{P}^n \to X$ is a projective morphism of noetherian schemes, hence it is proper by (4.9). Composition of separated morphisms is separated. Hence, $X \times \mathbb{P}^n$ satisfies $(*)$.

The projective n-space is the union of a hyperplane $\mathbb{P}^{n-1}$ and a copy of $\mathbb{A}^n$, and $\mathbb{P}^{n-1}$ is a prime divisor of $\mathbb{P}^n$. Thus, $X \times \mathbb{P}^{n-1}$ is a prime divisor of $X \times \mathbb{P}^n$Also, $X \times \mathbb{P}^n - X \times \mathbb{P}^{n-1} = X \times \mathbb{A}^n$, and $\mathrm{Cl}(X \times \mathbb{A}^n) \cong \mathrm{Cl}\, X$. Hence, we have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto X \times \mathbb{P}^{n-1}} \mathrm{Cl}(X \times \mathbb{P}^n) \to \mathrm{Cl}\, X \to 0.$$

This exact sequence splits via $Y \in \mathrm{Cl}\, X \mapsto Y \times \mathbb{P}^n \in \mathrm{Cl}(X \times \mathbb{P}^n)$. Injectivity of this map follows from $\mathbb{P}^n$ being birational to $\mathbb{A}^n$ and (6.6). Hence, $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{Z}$. $\square$

2. *Varieties in Projective Space.* Let k be an algebraically closed field, and let X be a closed subvariety of $\mathbb{P}^n_k$ which is nonsingular in codimension one (hence satisfies $(*)$). For any divisor $D = \sum n_i Y_i$ on X, we define the *degree* of D to be $\sum n_i \deg Y_i$, where $\deg Y_i$ is the degree of Y_i, considered as a projective variety itself.

(a) Let V be an irreducible hypersurface in $\mathbb{P}^n$ which does not contain X, and let Y_i be the irreducible components of $V \cap X$. They all have codimension 1 by (I, Ex. 1.8). For each i, let f_i be a local equation for V on some open set U_i of $\mathbb{P}^n$ for which $Y_i \cap U_i \neq \emptyset$, and let $n_i = v_{Y_i}(\bar{f}_i)$, where $\bar{f}_i$ is the restriction of f_i to $U_i \cap X$. Then we define the *divisor* $V.X$ to be $\sum n_i Y_i$. Extend by linearity and show that this gives a well-defined homomorphism from the subgroup of $\mathrm{Div}\, \mathbb{P}^n$ consisting of divisors, none of whose components contain X, to $\mathrm{Div}\, X$.

(b) If D is a principal divisor on $\mathbb{P}^n$, for which $D.X$ is defined as in (a), show that $D.X$ is principal on X. Thus, we get a homomorphism $\mathrm{Cl}\, \mathbb{P}^n \to \mathrm{Cl}\, X$.

(c) Show that the integer n_i defined in (a) is the same as the intersection multiplicity $i(X, V; Y_i)$ defined in (I, §7). Then use the generalized Bézout's theorem (I, 7.7) to show that for any divisor D on $\mathbb{P}^n$, none of whose components contain X,

$$\deg(D.X) = (\deg D) \cdot (\deg X).$$

(d) If D is a principal divisor on X, show that there is a rational function f on $\mathbb{P}^n$ such that $D = (f).X$. Conclude that $\deg D = 0$. Thus, the degree function defines a homomorphism $\deg : \mathrm{Cl}\, X \to \mathbb{Z}$. Finally, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}\,\mathbb{P}^n & \longrightarrow & \mathrm{Cl}\, X \\ {\scriptstyle\cong}\downarrow{\scriptstyle \deg} & & \downarrow{\scriptstyle \deg} \\ \mathbb{Z} & \xrightarrow[\cdot(\deg X)]{} & \mathbb{Z} \end{array}$$

and in particular, we see that the map $\mathrm{Cl}\, \mathbb{P}^n \to \mathrm{Cl}\, X$ is injective.

Proof.

(a) Let f_i' be another local equation for V on some open set U_i' of $\mathbb{P}^n$ for which $Y_i \cap U_i' \neq \emptyset$. Since X is irreducible, $U_i \cap U_i' \cap X \neq \emptyset$, and $f_i / f_i' \in \Gamma(U_i \cap U_i', \mathscr{O}^*_{\mathbb{P}^n})$. Hence, $\bar{f}_i / \bar{f}_i' \in \Gamma(U_i \cap U_i' \cap X, \mathscr{O}^*_X)$. Hence, n_i is independent of the choice of U_i.

(b) Suppose $D = (f)$, and let $i : X \hookrightarrow \mathbb{P}^n$ be a closed immersion. Then $D.X = (i^{-1} f)$.

(c) We recall the definition of the intersection multiplicity:

$$i(V,X;Y_i) = \mathrm{length}_{S_{\mathfrak{p}_i}}(S/(I_X+I_V))_{\mathfrak{p}_i},$$

where $S = k[x_0,\ldots,x_n]$, and I_X, I_V are the homogeneous ideals defining X and V, respectively. The homogeneous prime ideal $\mathfrak{p}_i$ defines Y_i. Note that $\mathrm{length}_{S_\mathfrak{p}} M_\mathfrak{p} = \mathrm{length}_{S_{(\mathfrak{p})}} M_{(\mathfrak{p})}$. Let $\mathfrak{P}_i$ be the image of $\mathfrak{p}_i$ in S/I_X, and let $A = (S/I_X)_{(\mathfrak{P}_i)}$. Then A is precisely the valuation ring of Y_i as a prime divisor of X. If we denote I'_V the extension of the ideal I_V in S, it is not hard to see that $\mathrm{length}_A A/I'_V = v_{Y_i}(\overline{f})$, where f is any local equation defining V. Let $D = \sum n_j V_j$ be any $\mathbb{P}^n$ with each V_j an irreducible hypersurface not containing X. By the generalized Bézout's theorem,

$$\begin{aligned}\deg(D.X) &= \sum n_j \deg(V_j.X)\\ &= \sum n_j \sum i(V_j,X;V_{jk}) \deg V_{jk}\\ &= \sum n_j(\deg V_j \cdot \deg X)\\ &= (\deg D)\cdot(\deg X).\end{aligned}$$

(d) Let $D = (f)$ for some $f \in K(X)$, where $K(X)$ is the function field of X. If X is locally defined by an ideal $I \subset k[y_1,\ldots,y_n]$ on some open subset $\operatorname{Spec} k[y_1,\ldots,y_n]$ of $\mathbb{P}^n$ where f is regular, then we can write $f = g/h$ for some $g,h \in k[y_1,\ldots,y_n]$ such that h is nowhere zero on some open set in X, which is what we wanted to show. Since any principal divisor has degree zero, by (c), $\deg(D) = \deg((f).X) = 0\cdot\deg X = 0$.

□

4. Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1,\ldots,x_n]$ be a *square-free* nonconstant polynomial, i.e., in the unique factorization of f into irreducible polynomials, there are no repeated factors. Let $A = k[x_1,\ldots,x_n,z]/(z^2-f)$. Show that A is an integrally closed ring.

Proof. The quotient field K of A is just $k(x_1,\ldots,x_n)[\sqrt{f}]$. It is a Galois extension of $k[x_1,\ldots,x_n]$ with Galois group $\mathbb{Z}/2\mathbb{Z}$ generated by $\sqrt{f}\mapsto-\sqrt{f}$. If $\alpha = g+h\sqrt{f}\in K$, where $g,h\in k(x_1,\ldots,x_n)$, then the minimal polynomial of α is $X^2-2gX+(g^2-h^2f)$. Suppose α is integral over $k[x_1,\ldots,x_n]$. Since $k[x_1,\ldots,x_n]$ is integrally closed, the coefficients of p_α lie in $k[x_1,\ldots,x_n]$ (A.M. 5.15). It follows $g,h\in k[x_1,\ldots,x_n]$. Hence, the integral closure of $k[x_1,\ldots,x_n]$ in K lies in A. The converse is immediate by the formula for the minimal polynomial of any $\alpha\in K$. Hence, A is an integrally closed ring. □

7. Let X be the nodal cubic curve $y^2z = x^3+x^2z$ in $\mathbb{P}^2$. Imitate (6.11.4) and show that the group of Cartier divisors of degree 0, $\mathrm{CaCl}^\circ X$, is naturally isomorphic to the multiplicative group $\mathbb{G}_m$.

Proof. Take $Z = (0:0:1)$ and $P_0 = (0:1:0)$. There is a bijection with the closed points of $X-Z$, the set of non-singular points of X, and $\mathrm{CaCl}^\circ X$. It remains to show $X - Z\cong\mathbb{G}_m$, where $\mathbb{G}_m = \mathbb{A}^1-\{0\}$ is the multiplicative group defined to be the spectrum of the ring $k[t,t^{-1}]$. Looking at the affine subset $z\neq 1$, we see that any line through the singular point Z, which has affine coordinates $(0,0)$, intersects the curve one more time. Plugging in $y=\lambda x$ into the defining equation of X for some $\lambda\in k^*$, we have $x=\lambda^2-1$. Thus, we have a map $k^*-\{\pm1\}\to X-Z$ defined by $\lambda\mapsto(\lambda^2-1,(\lambda^2-1)^3+(\lambda^2-1)^2)$. Extend this map to all of k^* via $1\mapsto P_0$ and $-1\mapsto(-1,0)$, and we are done. □

8. (a) Let $f: X\to Y$ be a morphism of schemes. Show that $\mathscr{L}\mapsto f^*\mathscr{L}$ induces a homomorphism of Picard groups, $f^*:\operatorname{Pic} Y\to\operatorname{Pic} X$.

(b) If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^*:\operatorname{Cl} Y\to\operatorname{Cl} X$ defined in the text, via the isomorphisms of (6.16).

(c) If X is a locally factorial integral closed subscheme of $\mathbb{P}^n_k$, and if $f: X\to\mathbb{P}^n$ is the inclusion map, then f^* on Pic agrees with the homomorphism on divisor class groups defined in (Ex. 6.2) via the isomorphisms of (6.16).

Proof.

(a) If V is an open set in Y such that $\mathscr{L}\big|_V\cong\mathscr{O}_V$, then $f^*\mathscr{L}\big|_{f^{-1}V}\cong\mathscr{O}_{f^{-1}V}$.

(b) Let D be a Weil divisor on Y. We want to show $f^*(\mathscr{O}_Y(D)) \cong \mathscr{O}_X(f^*D)$. By the isomorphism $\operatorname{CaCl} X \cong \operatorname{Cl} X$, there exists a representation $\{(U_i, s_i)\}$ of D as a Cartier divisor. A finite morphism of nonsingular curves induces an inclusion of function fields $K(Y) \to K(X)$. Since the s_i's are elements of $K(Y)$, by (a), $f^*(\mathscr{O}_X(D))$ is the Cartier divisor defined by $\{(f^{-1}U_i, f^{-1}s_i)\}$. It is clear that the Weil divisor associated to $\{(f^{-1}U_i, f^{-1}s_i)\}$ is f^*D.

(c) The map defined in (Ex. 6.2) is precisely $f^* : \operatorname{Cl} Y \to \operatorname{Cl} X$ as defined in the text.

□

10. *The Grothendiek Group $K(X)$.* Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all expressions $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$, whenever there is an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves on X. If $\mathscr{F}$ is a coherent sheaf, we denote by $\gamma(\mathscr{F})$ its image in $K(X)$.

(a) If $X = \mathbb{A}^1_k$, then $K(X) \cong \mathbb{Z}$.

(b) If X is any integral scheme, and $\mathscr{F}$ a coherent sheaf, we define the *rank* of $\mathscr{F}$ to be $\dim_K \mathscr{F}_\xi$, where ξ is the generic point of X, and $K = \mathscr{O}_\xi$ is the function field of X. Show that the rank function defines a surjective homomorphism $\operatorname{rank} : K(X) \to \mathbb{Z}$.

(c) If Y is a closed subcheme of X, there is an exact sequence

$$K(Y) \longrightarrow K(X) \longrightarrow K(X - Y) \longrightarrow 0,$$

where the first map is extension by zero, and the second map is restriction.

Proof.

(a) Suppose $X = \operatorname{Spec} A$ for some principle ideal domain A. Coherent sheaves on X correspond to finitely generated A-modules. Any such module can be decomposed as a direct sum of a free and torsion submodule. Any torsion module is a direct sum of $A/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A. Since A is a principle ideal domain, all prime ideals are rank one A-modules, so $\gamma(A) = \gamma(\mathfrak{p})$. We have the exact sequence

$$0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0,$$

which implies $\gamma(A/\mathfrak{p}) = 0$. Hence, we can identify elements of $K(X)$ to the rank of their free parts.

(b) Take $\mathscr{F} = \mathscr{O}_X^{\oplus n}$ for each n, then $\operatorname{rank} \mathscr{F} = n$.

□

11. *The Grothendiek Group of a Nonsingular Curve.* Let X be a nonsingular curve over an algebraically closed field k. We will show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$, in several steps.

(a) For any divisor $D = \sum n_i P_i$ on X, let $\psi(D) = \sum n_i \gamma(k(P_i)) \in K(X)$, where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathscr{O}_D$ be the structure sheaf of the associated subscheme of codimension 1, and show that $\psi(D) = \gamma(\mathscr{O}_D)$. Then use (6.18) to show that for any D, $\psi(D)$ depends only on the linear equivalence class of D, so ψ defines a homomorphism $\psi : \operatorname{Cl} X \to K(X)$.

(b) For any coherent sheaf $\mathscr{F}$ on X, show that there exist locally free sheaves $\mathscr{E}_0$ and $\mathscr{E}_1$ and an exact sequence $0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$. Let $r_0 = \operatorname{rank} \mathscr{E}_0$, $r_1 = \operatorname{rank} \mathscr{E}_1$, and define $\det \mathscr{F} = (\bigwedge^{r_0} \mathscr{E}_0) \otimes (\bigwedge^{r_1} \mathscr{E}_1)^{-1} \in \operatorname{Pic} X$. Show that $\det \mathscr{F}$ is independent of the resolution chosen, and that it gives a homomorphism $\det : K(X) \to \operatorname{Pic} X$. Finally, show that if D is a divisor, then $\det(\psi(D)) = \mathscr{L}(D)$.

(c) if $\mathscr{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence $0 \to \mathscr{L}^{\oplus r}(D) \to \mathscr{F} \to \mathscr{T} \to 0$, where $\mathscr{T}$ is a torsion sheaf. Conclude that if $\mathscr{F}$ is a sheaf of rank r, then $\gamma(\mathscr{F}) - r\gamma(\mathscr{O}_X) \in \operatorname{im} \psi$.

(d) Using the maps $\psi, \det, \operatorname{rank}$, and $1 \mapsto \gamma(\mathscr{O}_X)$ from $\mathbb{Z} \to K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

Proof.

(a) Let $D = \sum n_i P_i$ be an effective divisor, and let $\{(U_i, f_i)\}$ be the effective Cartier divisor associated to D, where $f_i \in \Gamma(U_i, \mathscr{O}_{U_i})$ and $v_{P_i}(f_i) = n_i$ for all i. Thus, if t_i is an uniformizing element of $\mathscr{O}_{P_i,X}$ for each i, then $\mathscr{O}_D$ be can be identified with the direct sum of skyscraper sheaves $\bigoplus \mathscr{O}_{P_i,X}/(t_i^{n_i})$ on each P_i. Note that $\mathscr{O}_{P_i,X}/(t_i^{n_i})$ is non-zero for only finitely many P_i. Since k is algebraically closed, the residue field of $\mathscr{O}_{P_i,X}$ is k, and since each $\mathscr{O}_{P_i,X}$ is a discrete valuation ring, $\mathscr{O}_{P_i,X}/(t_i^{n_i})$ is isomorphic to $k^{\oplus n_i}$ as a k-vector space, so $\gamma(\mathscr{O}_{P_i,X}/(t_i^{n_i})) = \gamma(k(P_i)^{\oplus n_i}) = n_i \gamma(k(P_i)) = \psi(n_i P_i)$, which is what we wanted to show.

We need to show for any principal divisor $D = (f)$, where $f \in K^*$, the function field of X, $\psi(D) = 0$. Any principal divisor can be written as a difference of two effective principal divisors, so we assume D is effective. Let Y be the associated closed subscheme of D. Then the ideal sheaf $\mathscr{I}_Y$ is generated by f, so we have an isomorphism of $\mathscr{O}_X$-modules $\mathscr{O}_X \to \mathscr{I}_Y$ defined by multiplication by f. Hence, $\psi(D) = \gamma(\mathscr{O}_D) = \gamma(\mathscr{O}_X/\mathscr{I}_Y) = \gamma(\mathscr{O}_X) - \gamma(\mathscr{I}_Y) = 0$ by (6.18).

(b) Let $U_i = \operatorname{Spec} A_i$ be an open affine cover of X such that $\mathscr{F}\big|_{U_i} \cong \widetilde{M_i}$ for some finitely generated A_i-module. Since X is noetherian, a finite number will do, and let $N > 0$ such that the minimal number of generators for M_i is at most N for all i.

$\square$

12. Let X be a complete nonsingular curve. Show that there is a unique way to define the *degree* of any coherent sheaf on X, $\deg \mathscr{F} \in \mathbb{Z}$, such that:

(1) If D is a divisor, $\deg \mathscr{L}(D) = \deg D$.

(2) If $\mathscr{F}$ is a *torsion sheaf* (meaning a sheaf whose stalk at the generic point is zero), then

$$\deg \mathscr{F} = \sum_{P \in X} \operatorname{length}(\mathscr{F}_P).$$

(3) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, then $\deg \mathscr{F} = \deg \mathscr{F}' + \deg \mathscr{F}''$.

Proof. $K(X) \xrightarrow{\det} \operatorname{Pic} X \cong \operatorname{Cl} X \xrightarrow{\deg} \mathbb{Z}$. $\square$