Chapter 3, Section 8

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1. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf of Abelian groups on X, and assume that $R^i f_*(\mathscr{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geqslant 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F}).$$

(This is a degenerated case of the Leray spectral sequence.)

2. Let $f : X \to Y$ be an affine morphism of schemes (II, Ex. 5.17) with X Noetherian, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Show that the hypotheses of (Ex. 8.1) are satisfies, and hence that $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$ for each $i \geqslant 0$. (This gives another proof of (Ex. 4.1).)

3. Let $f : X \to Y$ be a morphism of ringed spaces, let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of finite rank. Prove the *projection formula* (cf. (II, Ex. 5.1))

$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}.$$

4. Let Y be a Noetherian scheme, and let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of rank $n+1$, $n \geqslant 1$. Let $X = \mathbb{P}(\mathscr{E})$ (II, §7), with the invertible sheaf $\mathscr{O}_X(1)$ and the projection morphism $\pi : X \to Y$.

(a) Then $\pi_*(\mathscr{O}(l)) \cong S^l(\mathscr{E})$ for $l \geqslant 0$, $\pi_*(\mathscr{O}(l)) = 0$ for $l < 0$ (II, 7.11); $R^i\pi_*(\mathscr{O}(l)) = 0$ for $0 < i < n$ and all $l \in \mathbb{Z}$; and $R^n\pi_*(\mathscr{O}(l)) = 0$ for $l > -n-1$.

(b) Show there is a natural exact sequence

$$0 \to \Omega_{X/Y} \to (\pi^*\mathscr{E})(-1) \to \mathscr{O} \to 0,$$

cf. (II, 8.13), and conclude that the *relative canonical sheaf* $\omega_{X/Y} = \bigwedge^n \Omega_{X/Y}$ is isomorphic to $(\pi^* \bigwedge^{n+1} \mathscr{E})(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathscr{O}_Y$ (cf. (7.1.1)).

(c) Now show, for any $l \in \mathbb{Z}$, that

$$R^n\pi_*(\mathscr{O}(l)) \cong \pi_*(\mathscr{O}(-l-n-1)) \otimes \bigwedge^{n+1} \mathscr{E}$$

(d) Show that $p_a(X) = (-1)^n p_a(Y)$ (use (Ex. 8.1)) and $p_g(X) = 0$ (use II, 8.11).

(e) In particular, if Y is a nonsingular projective curve of genus g, and $\mathscr{E}$ a locally free sheaf of rank 2, then X is a projective surface with $p_a = -g$, $p_g = 0$, and irregularity g (7.12.3). This kind of surface is called a *geometrically ruled surface* (V, §2).