Chapter 2, Section 4

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1. Show that a finite morphism is proper.

Lemma 1. *If $\varphi : A \to B$ is an integral homomorphism of rings, then the corresponding morphism between affine schemes $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed mapping.*

Proof. We can decompose φ into $A \to A/\ker\varphi \hookrightarrow B$, and $\operatorname{Spec}(A/\ker\varphi)$ is homeomorphic to a closed subspace of $\operatorname{Spec} A$, so we reduce to the case when φ is injective. Let $\mathfrak{b}$ be any ideal of B, then we want to show $\varphi^*(V(\mathfrak{b})) = V(\mathfrak{b} \cap A)$ (note that in general, we only have $\overline{\varphi(V(\mathfrak{b}))} = V(\varphi^{-1}(\mathfrak{b}))$). This follows from (A.M. 5.10), which states if $A \subseteq B$ are rings, B integral over A, and $\mathfrak{p}$ prime ideal of A, then there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$. $\square$

Lemma 2. *Let $f : B \to B'$ be a homomorphism of A-algebras, and let C be an A-algebra. If f is integral, prove that $f \otimes 1 : B \otimes_A C \to B' \otimes_A C$ is integral.*

Proof. It suffices to show all pure tensors $b' \otimes c$ in $B' \otimes_A C$ have an equation of integral dependence over $B \otimes_A C$. Since B' is an integral B-algebra, we have

$$b'^n + d_1 b'^{n-1} + \cdots + d_n = 0$$

for some $d_i \in B$, $n > 0$, then

$$(b' \otimes c)^n + (d_1 \otimes 1)(b' \otimes c)^{n-1} + \cdots + d_n \otimes c = 0.$$

$\square$

Proof. It follows from these lemmas that if $f : A \to B$ is integral and C is any A-algebra, then the mapping $(f \otimes 1)^* : \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec} C$ is a closed map. Let $f : X \to Y$ be a finite morphism of schemes. A finite morphism is an affine morphism (Ex. 3.4), so by (4.6f) we reduce to the case when $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, where B is a finite A-module (hence integral over A), and f is induced by a ring homomorphism $A \to B$. Morphisms between affine schemes are separated (4.1), and finite morphisms are of finite type, so it remains to show f is universally closed. If $Y' \to Y$ is any morphism, then we want to show $X \times_Y Y' \to Y'$ is a closed mapping. There is an open cover of Y' by spectra of A-algebras C_i so that the fiber product $X \times_Y Y'$ is covered by spectra of $B \otimes_A C_i$. By the above remarks, the morphisms $\operatorname{Spec}(B \otimes_A C_i) \to \operatorname{Spec} C_i$ are closed, hence $X \times_Y Y' \to Y'$ is closed. $\square$

2. Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either (a) X is nonreduced, or (b) Y is nonseparated.

Proof. Let U be an open dense subset of X such that f and g agree on U, let $h : X \to Y \times_S Y$ be the map obtained from f and g, and let $\Delta : Y \to Y \times_S Y$ be the diagonal morphism. By hypothesis, $h(U) \subseteq \Delta(Y)$. But U is dense in X, and $\Delta(Y)$ is closed since Y is separated over S, so $h(X) \subseteq \Delta(Y)$. This says that f and g agree topologically, so $f_*\mathscr{O}_X = g_*\mathscr{O}_X$. Set $\mathscr{O} = f_*\mathscr{O}_X = g_*\mathscr{O}_X$. Let V be open subset of Y, and let $t \in \Gamma(V, \mathscr{O}_Y)$. We want to show $f^\#(t) = g^\#(t) \in \mathscr{O}(V)$. By hypothesis, we have $W = (f^{-1}(V))_{f^\#(t) - g^\#(t)} \subseteq X - U$, but W is an open subset and U is dense, which implies $W = \emptyset$. This implies $f^\#(t) - g^\#(t)$ is nilpotent, but X is reduced, hence $f^\#(t) - g^\#(t) = 0$.

(a) Let k be a field, let $X = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ (Ex. 2.8), and let Y be any scheme over k. Giving a k-morphism $X \to Y$ is equivalent to giving a point in $y \in Y$ rational over k, and an element of $\mathfrak{m}_y/\mathfrak{m}_y^2$.

(b) Let X be the affine line, and let Y be the affine line with the origin doubled. We have two possible open immersions of X into Y with each one having either origin in its image, and the open immersions agree on the complement of the origin of X, which is an open dense subset of X.

$\square$

6. Let $f : X \to Y$ be a proper morphism of affine varieties over k. Then f is a finite morphism.

Proof. Let $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$, where A and B are finitely generated k-algebras that are integral domains. Let $\varphi : A \to B$ be a k-algebra homomorphism such that B is a finitely generated A-algebra, and f is induced by φ. Closed immersions are proper, so we reduce to the case when φ is injective. We want to show B is a finite A-module, which is equivalent to B being finitely generated and integral over A, so it suffices to show B is integral over A. Let K be the field of fractions of B so that A and B are subrings of K. By (4.11A), the integral closure of A in K is the intersection of all valuation rings of K which contains A, so it suffices to show B is contained in all such subrings. This is an easy consequence of the valuative criterion of properness: given any valuation ring R containing A, we have inclusions $A \to R$ and $B \to K$ forming a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & R \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & K \end{tikzcd}$$

and the valuation criterion of properness implies there exists a unique homomorphism $B \to R$ making the whole diagram commute. All homomorphisms are inclusions, so $B \to R$ is an inclusion, which is what we wanted to show. (See an alternative proof of this result that uses the universally closed property instead of the valuative criterion in (A.M. Ex. 5.35).) $\square$

8. Let $\mathscr{P}$ be a property of morphisms of schemes such that:

(a) a closed immersion has $\mathscr{P}$;

(b) a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$;

(c) $\mathscr{P}$ is stable under base extension.

Then show that:

(d) a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$;

(e) if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$.

(f) if $f : X \to Y$ has $\mathscr{P}$, then $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ has $\mathscr{P}$.

Proof.

(d) Let $X \to Y$ and $X' \to Y'$ be two morphisms having $\mathscr{P}$. By (c) $X \times_Y (Y \times Y') \to Y \times Y'$ and $X' \times'_Y (Y \times Y') \to Y \times Y'$ have $\mathscr{P}$. Hence, $X \times X' = (X \times_Y (Y \times Y')) \times_{Y \times Y'} (X' \times_{Y'} (Y \times Y')) \to Y \times Y'$ has $\mathscr{P}$.

(e) We can base extend $g \circ f : X \to Z$ by $g : Y \to Z$ so that $h : X \times_Z Y \to Y$ has $\mathscr{P}$. Then f factors through h, so by (b) it suffices to show $\Gamma_f : X \to X \times_Z Y$ has $\mathscr{P}$. By hypothesis, the diagonal morphism $\Delta : Y \to Y \times_Z Y$ has $\mathscr{P}$. We can obtain Γ_f by base extension of Δ by $(f, 1_Y) : X \times_Z Y \to Y \times_Z Y$ since $(X \times_Z Y) \times_{Y \times_Z Y} Y \cong X \times_{Y \times_Z Y} (Y \times_Z Y) \cong X$. Hence, by (b) Γ_f has $\mathscr{P}$.

(f) By the universal property of the reduced scheme associated to Y, f_{red} is the unique morphism that makes the diagram

$$\begin{tikzcd} & & Y_{\text{red}} \arrow[d] \\ X_{\text{red}} \arrow[r] \arrow[urr, dashed, "f_{\text{red}}"] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

commute. The associated morphisms $X_{\text{red}} \to X, Y_{\text{red}} \to Y$ are closed immersions; in particular, $X_{\text{red}} \to Y$ has $\mathscr{P}$ and $Y_{\text{red}} \to Y$ is separated. Hence, by (e) f_{red} has $\mathscr{P}$.

Remark. In the affine case, we can translate the above to statements about rings. Let $\mathscr{Q}$ be a property of homomorphisms of rings such that

(a$'$) a surjective homomorphism has $\mathscr{Q}$;

(b$'$) a composition of two homomorphisms having $\mathscr{Q}$ has $\mathscr{Q}$;

(c$'$) if $A \to B$ has $\mathscr{Q}$ and C is any A-algebra, then $C \to B \otimes_A C$ has $\mathscr{Q}$.

Then

(d′) a product of homomorphisms having $\mathscr{Q}$ has $\mathscr{Q}$;

(e′) if $\varphi : A \to B$ and $\psi : B \to C$ are two homomorphisms, and if $\psi \circ \varphi$ has $\mathscr{Q}$, then ψ has $\mathscr{Q}$;

(f′) if $\varphi : A \to B$ has $\mathscr{Q}$, then $\varphi_{\text{red}} : A_{\text{red}} \to B_{\text{red}}$ has $\mathscr{Q}$.

Note that we can ignore the condition of g separated in (e′) since any morphism between affine schemes is separable. Indeed, (e') can be proved from the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

by tensoring with B and extending the sequence as follows

$$B \cong A \otimes_A B \xrightarrow{f \otimes 1_B} B \otimes_A B \xrightarrow{g \otimes 1_B} C \otimes_A B \longrightarrow C$$

where $C \otimes_A B \to C$ is defined by $c \otimes b \mapsto g(b)c$. By (c′), $B \to C \otimes_A B$ has $\mathscr{Q}$, and $C \otimes_A B \to C$ is surjective, so it has $\mathscr{Q}$ by (a′). Notice that composing the homomorphisms give g, hence by (b′) g has $\mathscr{Q}$. □

9. Show that a composition of projective morphisms is projective. Conclude that projective morphisms have properties (a)-(f) of (Ex. 4.8) above.

Proof. By the results of (Ex. 3.13), (4.6), and (4.8), it suffices to show if $f : X \to \mathbb{P}^r$ is a projective morphism, then X is projective over $\operatorname{Spec} \mathbb{Z}$. If f is projective, then there exists a closed embedding $i : X \to \mathbb{P}^r \times \mathbb{P}^s$ such that f factors through $\mathbb{P}^r \to \mathbb{P}^s$. The Segre embedding (§1, 2.14) $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{rs+r+s}$ is a closed embedding, so $\psi \circ i : X \to \mathbb{P}^{rs+r+s}$ is a closed embedding such that $X \to \operatorname{Spec} \mathbb{Z}$ factors through $\mathbb{P}^{rs+r+s} \to \operatorname{Spec} \mathbb{Z}$, which is what we wanted to show. □