Chapter 1, Section 3

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April 23, 2025

2. A morphism whose underlying map on the topological space is a homeomorphism need not be an isomorphism.

(a) For example, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.

(b) For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism*.

Proof.

(a) Let Y be the curve $y^2 = x^3$. Then φ clearly maps into Y and is injective since

$$(t^2, t^3) = (u^2, u^3) \implies tu^2 = t^2 u \implies t = u.$$

If $P = (a_1, a_2) \in Y$, then clearly $\varphi(\sqrt[3]{a_2}) = P$ since k is algebraically closed, hence φ is bijective onto Y. To show φ is bicontinuous it is enough to show φ and φ^{-1} of closed sets in $\mathbb{A}^1$ and Y, respectively, is closed. A closed set in either $\mathbb{A}^1$ or Y is a finite set of points, so by the bijectivity of φ, it is also bicontinuous.

To show it is not an isomorphism, it suffices to show the coordinate rings of the affine line $\mathbb{A}^1$ and the curve Y are not isomorphic. Indeed, $A(\mathbb{A}^1) = k[x]$ and $A(Y) = k[x, y]/(y^2 - x^3)$, and while $k[x]$ is factorial, $A(Y)$ is not since $y^2 = x^3$.

(b) It is bijective and bicontinuous for the same reason in (a). If φ is an isomorphism, then it must induce an automorphism $\varphi^* : k[x] \to k[x]$ of the affine coordinate ring $k[x]$ of $\mathbb{A}^1$. However, φ^* is defined by $x \mapsto x^p$, which is not even surjective.

□

3. (a) Let $\varphi : X \to Y$ be a morphism. Then for each $P \in X$, φ induces a homomorphism of local rings $\varphi_P^* : \mathscr{O}_{\varphi(P),Y} \to \mathscr{O}_{P,X}$.

(b) Show that a morphism φ is an isomorphism if and only if φ is a homeomorphism, and that the induced map φ_P^* on local rings is an isomorphism for all $P \in X$.

(c) Show that if $\varphi(X)$ is dense in Y, then the map φ_P^* is *injective* for all $P \in X$.

Proof.

(a) We have already seen that φ induces a homomorphism $\varphi^* : \mathscr{O}(Y) \to \mathscr{O}(X)$. By (2.3), there exists an open affine variety V in Y containing $\varphi(P)$. Then $U = \varphi^{-1}(V)$ contains P, so $\mathscr{O}_{P,X} \simeq \mathscr{O}_{P,U}$ and $\mathscr{O}_{\varphi(P),Y} \simeq \mathscr{O}_{\varphi(P),V}$. By definition f is a regular function defined in some open neighborhood of $\varphi(P)$ in V such that $f(\varphi(P))$ is nonzero if and only if $\varphi^* f$ is nonzero at P. Thus, $\varphi^* f$ is a unit in $\mathscr{O}_{P,U}$, so by the universal property of localization there exists a unique ring homeomorphism $\varphi_P^* : \mathscr{O}_{\varphi(P),Y} \to \mathscr{O}_{P,X}$.

(b) The only if direction is obvious. Conversely, if φ_P^* on local rings is an isomorphism for all $P \in X$, then since isomorphism is a local property (A.M. p. 40), we have an isomorphism $\varphi^* : \mathscr{O}_V \to \mathscr{O}_U$, where V is any affine variety in Y and $U = \varphi^{-1}(V)$. Hence, $\varphi : U \to V$ is an isomorphism.

(c) We want to show for any $P \in X$ that if f, g are elements of $\mathscr{O}_{\varphi(P),Y}$ such that $\varphi_P^* f = \varphi_P^* g$ on some open neighborhood U of P, then $f = g$ in some open neighborhood V of $\varphi(P)$. Let V be an open neighborhood of $\varphi(P)$ with both f and g are defined so that $\varphi_P^* f$ and $\varphi_P^* g$ are defined on $U = \varphi^{-1}(V)$. Then, $\varphi(U) = V \cap \varphi(X)$ is a closed and dense subset of V such that $f = g$, hence $f = g$ on V.

$\square$

4. Show that the d-uple embedding of $\mathbb{P}^n$ (I, Ex. 2.12) is an isomorphism onto its image.

Proof. Let $\rho : \mathbb{P}^n \to Y \subseteq \mathbb{P}^N$ be the d-uple embedding of $\mathbb{P}^n$ with $N = \binom{n+d}{n} - 1$ and image restricted so that ρ is a homeomorphism. By Exercise 2, to show ρ is an isomorphism it suffices to show it is an isomorphism locally. Let $P = (a_0, \ldots, a_n) \in \mathbb{P}^n$ and $\rho(P) = (b_0, \ldots, b_N) \in Y$ and assume $a_0 \neq 0$. By (3.4), we have the following identifications of local rings

$$\mathscr{O}_{P,\mathbb{P}^n} = S(\mathbb{P}^n)_{(\mathfrak{m}_P)}, \quad \mathscr{O}_{\rho(P),Y} = S(Y)_{(\mathfrak{m}_{\rho(P)})},$$

where $\mathfrak{m}_P$ (respectively, $\mathfrak{m}_{\rho(P)}$) is the ideal generated by the set of homogenous $f \in S(\mathbb{P}^n)$ (respectively, $f \in S(Y)$) such that $f(P) = 0$ (respectively, $f(\rho(P)) = 0$). We want to show the induced map $\rho_P^* : \mathscr{O}_{\rho(P),Y} \to \mathscr{O}_{P,\mathbb{P}^n}$ is an isomorphism. By the identification above, the map ρ_P^* can be defined by the canonical projection map $\overline{\theta} : k[y_0, \ldots, y_N]/\mathfrak{a} \to k[x_0, \ldots, x_n]$ induced by θ in Exercise 2.12 as

$$\rho_P^*\left(\frac{f}{g}\right) = \frac{\overline{\theta}(f)}{\overline{\theta}(g)},$$

where f and g are homogenous elements of same degree in $S(Y)$ such that $g(\rho(P)) \neq 0$ (since we are only concerned with homogenous elements of degree 0 in $S(Y)_{\mathfrak{m}_P}$). Clearly ρ_P^* is injective since $\overline{\theta}$ is injective, and ρ_P^* maps $S(Y)_{(\mathfrak{m}_{\rho(P)})}$ into $S(\mathbb{P}^n)_{(\mathfrak{m}_P)}$ since $\theta(y_i)$ is of the same degree for all $0 \leq i \leq N$ and $\mathfrak{a}$ is a homogenous ideal. To show it is surjective, suppose $h/k \in \mathscr{O}_{P,\mathbb{P}^n}$ with $k(P) \neq 0$ and h, k both homogenous of degree e. We can assume e is a multiple d since x_0 is a unit in $\mathscr{O}_{P,\mathbb{P}^n}$, and we can multiply h/k by a sufficent power of x_0/x_0 to obtain the desired degrees in the numerator and denominator. Then, each term in h and k is the product of monomials of degree d, and since $\overline{\theta}$ is surjective onto such elements, ρ_P^* is surjective. Hence, ρ_P^* is an isomorphism. $\square$

6. There are quasi-affine varieties which are not affine. For example, show that $X = \mathbb{A}^2 - \{(0,0)\}$ is not affine.

Proof. The ideal $I(X)$ consists of all polynomials in $k[x,y]$ that vanish at X. If $f \in I(X)$, then viewing f as a regular function on $\mathbb{A}^2$, $f^{-1}(0)$ is a closed and dense subset of $\mathbb{A}^2$, hence it equals to $\mathbb{A}^2$, hence $I(X) = (0)$. Thus, if X is affine, then

$$\mathscr{O}(X) = k[x,y]/I(X) \simeq k[x,y] = \mathscr{O}(\mathbb{A}^2),$$

so by (3.5) and (3.7) $X \cong \mathbb{A}^2$ under the inclusion map, which is ridiculous. $\square$

12. If P is a point on a variety X, then $\dim \mathscr{O}_P = \dim X$.

Proof. We can find an open affine variety Y in X containing P, and since local rings behave well under open subsets, i.e. $\mathscr{O}_{P,Y} \simeq \mathscr{O}_{P,X}$, we can reduce to the affine case. Then, the statement follows from (3.2). $\square$

15. *Products of Affine Varieties.* Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.

(a) Show that $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible. The affine variety $X \times Y$ is called the *product* of X and Y. Note that its topology is in general not equal to the product topology.

(b) Show that $A(X \times Y) \simeq A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties.

(d) Show that $\dim X \times Y = \dim X + \dim Y$.

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times Y \arrow[r] \arrow[d] & Y \\
& X &
\end{tikzcd}$$

Proof.

(a) Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$, $i = 1, 2$ and let $\pi : \mathbb{A}^{n+m} \to \mathbb{A}^m$ be the projection map. For any $x \in X$ since $x \times Y \subseteq Z_1 \cup Z_2$, we can write $x \times Y = (x \times Y \cap Z_1) \cup (x \times Y \cap Z_2)$, so let $C_i = x \times Y \cap Z_i$, then

$$Y = \pi(x \times Y) = \pi(C_1 \cup C_2) = \pi(C_1) \cup \pi(C_2).$$

We can write $Z_i = Z(f_1, \ldots, f_r)$, $i = 1, 2$, for some $f_j \in A(\mathbb{A}^{n+m})$, then we see that

$$\pi(C_i) = \bigcap_{j=1}^{r} Z(f_j(x, y_1, \ldots, y_m)),$$

where we can view $f_j(x, y_1, \ldots, y_m)$ as an element in $A(\mathbb{A}^m)$, which shows $\pi_2(C_i)$ is a closed set in $\mathbb{A}^m$. Since Y is irreducible, we have $\pi(C_2) = \emptyset$, say, hence $Y = \pi(C_1)$, hence $x \times Y = x \times Y \cap Z_1$, hence $x \times Y \subseteq Z_1$. In particular, we have $X = X_1 \cup X_2$, and we have shown $Z_i = X_i \times Y$. We can repeat the argument for $Y_i = \{y \in Y \mid X \times Y \subseteq Z_i\}$, $i = 1, 2$ so that $Z_i = X \times Y_i$. This is only possible if $X = X_1$ and $Y = Y_1$, say, hence $Z_1 = X \times Y$.

(b) We clearly have $A(\mathbb{A}^{n+m}) \simeq A(\mathbb{A}^n) \otimes_k A(\mathbb{A}^m)$, where $A(\mathbb{A}^n) = k[x_1, \ldots, x_n]$ and $A(\mathbb{A}^m) = k[y_1, \ldots, y_m]$. Then, we have

$$I(X \times Y) = I(X) \otimes_k A(\mathbb{A}^m) + A(\mathbb{A}^n) \otimes_k I(Y),$$

hence we have the isomorphism

$$\begin{aligned} A(X \times Y) &\cong \frac{A(\mathbb{A}^n) \otimes_k A(\mathbb{A}^m)}{I(X \times Y)} \\ &\cong \frac{A(\mathbb{A}^n) \otimes_k A(\mathbb{A}^m)}{I(X) \otimes_k A(\mathbb{A}^m) + A(\mathbb{A}^n) \otimes_k I(Y)} \\ &\cong \frac{A(\mathbb{A}^n)}{I(X)} \otimes_k \frac{A(\mathbb{A}^m)}{I(Y)} \\ &\cong A(X) \otimes_k A(Y). \end{aligned}$$

(c) Suppose we have morphisms $Z \to X$ and $Z \to Y$. This induces k-algebra homomorphisms $A(X) \to \mathscr{O}(Z)$ and $A(Y) \to \mathscr{O}(Z)$, which defines a k-bilinear map $A(X) \times A(Y) \to \mathscr{O}(Z)$. By the universal property of tensor products, we have a unique k-algebra homomorphism $A(X) \otimes_k A(Y) \to \mathscr{O}(W)$, hence we have a unique morphism $W \to X \times Y$.

$$\begin{tikzcd}
\mathscr{O}(Z) & & \\
& A(X) \otimes_k A(Y) \arrow[ul, dashed] & A(Y) \arrow[l] \arrow[ull, bend right] \\
& A(X) \arrow[u] \arrow[uul, bend left] &
\end{tikzcd}$$

(d) If $t_1, \ldots, t_n$ and $u_1, \ldots, u_m$ are coordinates for X and Y, respectively, then each t_i, u_j are algebraically independent in $A(X \times Y)$, that is the quotient field of $A(X \times Y)$ has transcendence degree $n + m$, hence $\dim X \times Y = \dim X + \dim Y$.

$\square$

17. *Normal Varieties.* A variety Y is *normal at a point* $P \in Y$ if $\mathscr{O}_P$ is integrally closed ring. Y is *normal* if it is normal at every point.

(a) Show that every conic in $\mathbb{P}^2$ is normal.

(b) Show that the quadric surfaces Q_1, Q_2 in $\mathbb{P}^3$ given by equations $Q_1 : xy = zw$; $Q_2 : xy = z^2$ are normal.

(c) Show that the cuspidal cubic $y^2 = x^3$ in $\mathbb{A}^2$ is not normal.

(d) If Y is affine, then Y is normal $\iff$ $A(Y)$ is integrally closed.

(e) Let Y be an affine variety. Show that there is a normal affine variety $\hat{Y}$, and a morphism $\pi : \hat{Y} \to Y$, with the property that whenever Z is normal variety, and $\varphi : Z \to Y$ is a *dominant* morphism, then there is a unique morphism $\theta : Z \to \hat{Y}$ such that $\varphi = \pi \circ \theta$. $\hat{Y}$ is called the *normalization* of Y.

Proof.

(a) A conic in $\mathbb{P}^2$ can be covered by open affine varieties isomorphic to a conic in $\mathbb{A}^2$. Thus, it suffices to show all irreducible plane algebraic curves of degree 2 are normal, that is we want to show an affine variety X in $\mathbb{A}^2$ defined by the zero set of an irreducible quadratic polynomial f in $k[x, y]$ is normal. Since being integrally closed is a local property, it suffices to show the affine coordinate ring $A(X) = k[x, y]/(f)$ is integrally closed. We first prove Exercise 1.1c, which states $A(X)$ is isomorphic to either $R = k[x, y]/(y - x^2)$ or $S = k[x, y]/(xy - 1) \simeq k[x, \frac{1}{x}]$. For simplicity assume char $k \neq 2$. Any quadratic polynomial can be written as

$$f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

We claim the following: if $B^2 - 4AC = 0$, then $A(X) \simeq R$, otherwise $A(X) \simeq S$. We proceed by showing that X can be transformed to a variety of the form $y = x^2$ or $xy = 1$ using some affine transformations.
Suppose $B^2 - 4AC = 0$, then assume $C \neq 0$ (if $C = 0$, then $B = 0$, so we already have an equation of the desired form up to some translation and stretching), so we have

$$f(x, y) = (\sqrt{A}x + \sqrt{C}y)^2 + Dx + Ey + F,$$

since k is algebraically closed. We can apply the following affine transformation

$$\begin{aligned} x &\mapsto -\sqrt{C}y \\ y &\mapsto -\sqrt{C}x + \sqrt{A}y \end{aligned}$$

which is indeed an affine transformation since it has determinant $C \neq 0$, to obtain

$$f'(x, y) = C^2x^2 - D\sqrt{C}y - E(-\sqrt{C}x + \sqrt{A}y) + F,$$

which is the case when $B = C = 0$.
Now suppose $\Delta^2 = B^2 - 4AC \neq 0$ and assume $A \neq 0$, then we have

$$f(x, y) = \left(\sqrt{A}x + \frac{B + \Delta}{2\sqrt{A}}y\right)\left(\sqrt{A}x + \frac{B - \Delta}{2\sqrt{A}}y\right) + Dx + Ey + F,$$

so we can apply the affine transformation

$$\begin{aligned} \sqrt{A}x + \frac{B + \Delta}{2\sqrt{A}}y &\mapsto x \\ \sqrt{A}x + \frac{B - \Delta}{2\sqrt{A}}y &\mapsto y \end{aligned}$$

which has determinant $1/\Delta \neq 0$, to obtain

$$f'(x, y) = xy + (\text{linear terms}).$$

Since affine coordinate rings are invariant up to affine transformations, we have either $A(X) \simeq R$ or $A(X) \simeq S$. The ring $R = k[x, y]/(y - x^2)$ is isomorphic to a polynomial ring over one variable, which is a factorial, hence it is integrally closed. The ring $S = k[x, x^{-1}]$ is a discrete valuation ring, which are integrally closed by (A.M. p. 94).

(b) The quadric surface Q_1 can be covered by open affine varieties isomoprhic to the affine surface $z = xy$. The affine coordinate ring of such surface is isomorphic to $k[x, y]$, which is integrally closed, hence Q_1 is normal. Similarily, Q_2 can be covered by open affine varieties isomoprhic to either $y = x^2$ or $xy = 1$, which was shown to be normal in (a).

(c) The affine coordinate ring $A(Y)$ is isomorphic to $k[t^2, t^3]$. The quotient field is $k(t)$, and the element $t \in k(t)$ is integral over $k[t^2, t^3]$ since $(t)^2 - t^2 = 0$ but $t \notin k[t^2, t^3]$, hence $A(Y)$ is not integrally closed, hence Y is not normal by (d).

(d) Being integrally closed is a local property, that is a ring A is integrally closed if and only if $A_\mathfrak{m}$ is integrally closed for all maximal ideals $\mathfrak{m}$ in A by (A.M. p. 63). There is a one-to-one correspondence between maximal ideals of $\mathscr{O}(Y) = A(Y)$ and points of Y, hence Y is normal if and only if $\mathscr{O}_P$ is integrally closed for all $P \in Y$ if and only if $A(Y)_\mathfrak{m}$ is integrally closed for all maximal ideals $\mathfrak{m}$ if and only if $A(Y)$ is integrally closed.

(e) We want to show there exists a normal scheme $\widehat{Y}$ with morphism $\pi : \widehat{Y} \to Y$ that is universal amongst all dominant morphisms from normal schemes into Y. Let $A(Y)$ be the affine coordinate ring of Y and let B be the integral closure of $A(Y)$ in its quotient field. Then, B is a finitely generated k-algebra by (3.9A), and it is a subring of a field, so it is an integral domain, which means it defines an affine variety. Let $\widehat{Y}$ be an affine variety with coordinate ring $A(\widehat{Y}) = B$, and let $\pi : \widehat{Y} \to Y$ be the morphism induced by the inclusion map $A(Y) \hookrightarrow B$. Let Z be a normal variety, and $\varphi : Z \to Y$ a dominant morphism. Since Z can be covered by open affine varieties and local rings are perserved under open subsets, we reduce to the case when Z is an affine normal variety. We want to show there exists a ring homomorphism $\alpha : \mathscr{O}(\widehat{Y}) \to \mathscr{O}(Z)$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathscr{O}(\widehat{Y}) \arrow[dr, dashed, "\exists!\ \alpha"] & \\ \mathscr{O}(Y) \arrow[ur, hook, "\pi^*"] \arrow[rr, "\varphi^*"'] & & \mathscr{O}(Z) \end{tikzcd}$$

If $\varphi^* f = 0$ for some $f \in \mathscr{O}(Y)$, then $f = 0$ on a closed dense subset of Y, hence $f = 0$ on Y, so φ^* is injective. Thus, φ^* induces a map between the function fields $\overline{\varphi^*} : K(Y) \to K(Z)$, where we have the inclusions $\mathscr{O}(Y) \hookrightarrow \mathscr{O}(\widehat{Y}) \hookrightarrow K(Y)$ by construction. If $f \in \mathscr{O}(\widehat{Y})$, then there exists an equation of integral dependence of the form

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0, \quad a_i \in \mathscr{O}(Y),$$

and since $\overline{\varphi^*}(a_i) = \varphi^*(a_i) \in \mathscr{O}(Z)$, we can apply $\overline{\varphi^*}$ to the equation above to obtain an equation of integral dependence of $\overline{\varphi^*}(f)$ over $\mathscr{O}(Z)$

$$\overline{\varphi^*}(f)^n + b_1 \overline{\varphi^*}(f)^{n-1} + \cdots + b_n = 0, \quad b_i = \varphi^*(a_i) \in \mathscr{O}(Z),$$

and since $\mathscr{O}(Z)$ is integrally closed, we must have $\overline{\varphi^*}(f) \in \mathscr{O}(Z)$. Thus, we have $\alpha = \overline{\varphi^*}\ |_{\mathscr{O}(\widehat{Y})}$ as the desired map, and it is unique by construction. Hence, we have a unique morphism $\theta : Z \to \hat{Y}$ such that $\varphi = \pi \circ \theta$ and $\theta^* = \alpha : \mathscr{O}(\widehat{Y}) \to \mathscr{O}(Z)$ by (3.5).

□

20. Let Y be a variety of dimension ≥ 2, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

(a) Show that f extends to a regular function on Y.

(b) Show this would be false for $\dim Y = 1$.

Proof.

(a) Y can be covered by open affine varieties, so we reduce to the case when $Y \subseteq \mathbb{A}^n$ itself is an affine variety. Then by composing with the inclusion morphism $Y \to \mathbb{A}^n$, we futher reduce to the case when $Y = \mathbb{A}^n$, that is we want to show if $f = g/h$ on $\mathbb{A}^n - P$, then we can extend f to a regular function on Y. We show that if a polynomial $h \in k[x_1, \ldots, x_n]$ is nonzero on $\mathbb{A}^n - P$, then f is nonzero on P. We can also assume $P = (0, \ldots, 0)$ and $n = 2$ by induction. Suppose $h(P) = 0$, then h must have zero constant term, so we can write

$$h(x, y) = f_0(x)y^d + f_1(x)y^{d-1} + \cdots + f_d(x), \quad f_i(x) \in k[x] - k,$$

then there exists nonzero $a \in k$ such that $f_i(a) \neq 0$ for some i, we have

$$h(a, y) = a_0 y^d + a_1 y^{d-1} + \cdots a_d \neq 0, \quad a_i = f_i(a),$$

and since k is algebraically closed, $h(a, y)$ must have a solution $y = b$, so (a, b) is a solution to h that is not equal to P, a contradiction.

(b) Consider $Y = \mathbb{A}^1$ and $P = 0$, and let $f = \frac{1}{x}$. Then f is regular on $Y - P$ by definition, but it clearly cannot be extended to the entire affine line.

□