Chapter 3, Section 2

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1. Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{F}', \mathscr{F}'' \in \mathfrak{Mod}(X)$. An *extension* of $\mathscr{F}''$ by $\mathscr{F}'$ is a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in $\mathfrak{Mod}(X)$. Two extensions are *isomorphic* if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathscr{F}'$ and $\mathscr{F}''$. Given an extension as above consider the long exact sequence arising from $\mathrm{Hom}(\mathscr{F}'', \cdot)$, in particular the map

$$\delta : \mathrm{Hom}(\mathscr{F}'', \mathscr{F}'') \to \mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}'),$$

and let $\xi \in \mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}')$ by $\delta(1_{\mathscr{F}''})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extensions of $\mathscr{F}''$ by $\mathscr{F}'$, and elements of the group $\mathrm{Ext}^1(\mathscr{F}'', \mathscr{F}')$.

2. Let $X = \mathbb{P}^1_k$, with k an infinite field.

(a) Show that there does not exist a projective object $\mathscr{P} \in \mathfrak{Mod}(X)$, together with a surjective map $\mathscr{P} \to \mathscr{O}_X \to 0$.

(b) Show that there does not exist a projective object $\mathscr{P}$ in either $\mathfrak{Qco}(X)$ or $\mathfrak{Coh}(X)$ together with a surjective $\mathscr{P} \to \mathscr{O}_X \to 0$.

3. Let X be a Noetherian scheme, and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X)$.

(a) If $\mathscr{F}, \mathscr{G}$ are both coherent, then $\mathcal{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \geq 0$.

(b) If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathcal{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent, for all $i \geq 0$.

4. Let X be a Noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. In this case we say $\mathfrak{Coh}(X)$ has *enough locally frees.* Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$, show that the δ-functor $(\mathcal{E}xt^i(\cdot, \mathscr{G}))$, from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$, is a contravariant universal δ-functor.

5. Let X be a Noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathscr{F}$ we define the *homological dimension* of $\mathscr{F}$, denoted $\mathrm{hd}(\mathscr{F})$, to be the least length of a locally free resolution of $\mathscr{F}$ (or $+\infty$ if there is no finite one). Show:

(a) $\mathscr{F}$ is locally free $\iff$ $\mathcal{E}xt^1(\mathscr{F}, \mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$;

(b) $\mathrm{hd}(\mathscr{F}) \leq n \iff \mathcal{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$;

(c) $\mathrm{hd}(\mathscr{F}) = \sup_x \mathrm{pd}_{\mathscr{O}_X} \mathscr{F}_x$.

6. Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.

(a) M is projective if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > 0$.

(b) Use (a) to show that for any n, $\mathrm{pd}\, M \leq n$ if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > n$.

7. Let $X = \mathrm{Spec}\, A$ be an affine Noetherian scheme. Let M, N be A-modules, with M finitely generated. Then

$$\mathrm{Ext}^i_X(\tilde{M}, \tilde{N}) \cong \mathrm{Ext}^i_A(M, N)$$

and

$$\mathscr{E}xt^i_X(\tilde{M}, \tilde{N}) \cong \widetilde{\mathrm{Ext}^i(M, N)}.$$

8. Prove the following theorem of Kleiman: if X is a Noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).

(a) First show that open sets of the form X_s, for various $s \in \Gamma(X, \mathscr{L})$, and various invertible sheaves $\mathscr{L}$ on X, form a base for the topology of X.

(b) Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathscr{L}_i^{n_i}$ for various invertible sheaves $\mathscr{L}_i$ and various integers n_i.

9. Let X be a noetherian, integral, separated, regular scheme. (We say a scheme is *regular* if all of its local rings are regular local rings.) Recall the definition of the *Grothendieck group* $K(X)$ from (II, Ex. 6.10). We define similarly another group $K_1(X)$ using locally free sheaves: it is the quotient of free abelian group generated by all locally free (coherent) sheaves, by the subgroup generated by all expressions of the form $\mathscr{E} - \mathscr{E}' - \mathscr{E}''$, whenever $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is a short exact sequence of locally free sheaves. Clearly there is a natural group homomorphism $\varepsilon : K_1(X) \to K(X)$. Show that ε is an isomorphism as follows.

(a) Given a coherent sheaf $\mathscr{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathscr{E} \to \mathscr{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution

$$0 \to \mathscr{E}_n \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

(b) For each $\mathscr{F}$, choose a finite locally free resolution $\mathscr{E} \to \mathscr{F} \to 0$, and let $\delta(\mathscr{F}) = \sum (-1)^i \gamma(\mathscr{E}_i)$ in $K_1(X)$. Show that $\delta(\mathscr{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K(X)$ to $K_1(X)$, and finally, that it is an inverse to ε.

10. *Duality for a Finite Flat Morphism.*

(a) Let $f : X \to Y$ be a finite morphism of Noetherian schemes. For any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$,

$$\mathscr{H}om_Y(f_*\mathscr{O}_X, \mathscr{G})$$

is a quasi-coherent $f_*\mathscr{O}_X$-module, hence corresponds to a quasi-coherent $\mathscr{O}_X$-module, which we call $f^!\mathscr{G}$ (II, Ex. 5.17e).

(b) Show that for any coherent $\mathscr{F}$ on X and any quasi-coherent $\mathscr{G}$ on Y, there is a natural isomorphism

$$f_* \mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G}).$$

(c) For each $i \geq 0$, there is a natural map

$$\varphi_i : \operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G}).$$

(d) Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathscr{O}_X$ is locally free on Y (this is equivalent to saying f flat - see §9). Show that φ_i is an isomorphism for all i, all $\mathscr{F}$ coherent on X, and all $\mathscr{G}$ quasi-coherent on Y.