Chapter 1, Section 5

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April 23, 2025

1. Locate the singular points and sketch the following curves in $\mathbb{A}^2$ (assume char $k \neq 2$):

(a) $x^2 = x^4 + y^4$;
(b) $xy = x^6 + y^6$;
(c) $x^3 = y^2 + x^4 + y^4$;
(d) $x^2y + xy^2 = x^4 + y^4$.

Proof.

f	$\partial f/\partial x$	$\partial f/\partial y$	$\langle \partial f/\partial x, \partial f/\partial y\rangle = \langle 0, 0\rangle$	Sing f
$x^4 - x^2 + y^4$	$4x^3 - 2x$	$4y^3$	$(0,0)$, $(\pm 1/\sqrt{2}, 0)$	$(0,0)$
$x^6 + y^6 - xy$	$6x^5 - y$	$6y^5 - x$	$(0,0)$, $(6^{-1/4}, 6^{-1/4})$	$(0,0)$
$x^4 + y^4 - x^3 + y^2$	$4x^3 - 3x^2$	$4y^3 - 2y$	$(0,0), (3/4, 0), (0, \pm 1/\sqrt{2}), (3/4, \pm 1/\sqrt{2})$	$(0,0)$
$x^4 + y^4 - x^2y - xy^2$	$4x^3 - 2xy - y^2$	$4y^3 - 2xy - x^2$	$(0,0)$, $(3/4, 3/4)$	$(0,0)$

(a) is the tacnode, (b) is the node, (c) is the cusp, and (d) is the triple point. □

2. Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^3$ (assume char $k \neq 2$).

(a) $xy^2 = z^2$;
(b) $x^2 + y^2 = z^2$;
(c) $xy + x^3 + y^3 = 0$.

Proof.

f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$	Sing f
$xy^2 - z^2$	y^2	$2xy$	$-2z$	$(t, 0, 0),\ t \in k$
$x^2 + y^2 - z^2$	$2x$	$2y$	$-2z$	$(0,0,0)$
$x^3 + y^3 + xy$	$3x^2 + y$	$3y^2 + x$	0	$(0, 0, t),\ t \in k$

(a) is the pinch point, (b) is the conical double point, and (c) is the double line. □

3. *Multiplicities.* Let $Y \subseteq \mathbb{A}^2$ be a curve defined by the equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \cdots + f_d$, where f_i is a homogenous polynomial of degree i in x and y. Then we define the *multiplicity* of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y \iff \mu_P(Y) > 0$.) The linear factors of f_r are called the *tangent directions at* P.

(a) Show that $\mu_P(Y) = 1 \iff P$ is a nonsingular point of Y.
(b) Find the multiplicity of each of the singular points in Exercise 1 above.

Proof.

(a) Let f' be the polynomial $f(x,y)$ after change of coordinates so that P becomes the origin and let $O=(0,0)$ be the origin. Then, P is a nonsingular point of $Y \iff (\partial f/\partial x)(P),\ (\partial f/\partial y)(P) \neq 0 \iff (\partial f'/\partial x)(O),\ (\partial f'/\partial y)(O) \neq 0 \iff \partial f'/\partial x,\ \partial f'/\partial y = \text{higher degree terms} + \text{constant} \iff f'$ has linear terms $\iff \mu_P(Y)=1$.

(b) (a), (b), and (c) has multiplicity 2, and (d) has multiplicity 3 at O.

□

6. *Blowing Up Curve Singularities.*

(a) Let Y be the cusp or node of Exercise 1. Show that the curve $\widetilde{Y}$ obtained by blowing up Y at $O=(0,0)$ is nonsingular.

(b) We define a *node* (also called *ordinary double point*) to be a double point (i.e., a point of multiplicity 2) of a plane curve with distinct tangent directions. If P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular points on the blown-up curve $\widetilde{Y}$. We say that "blowing up P resolves the singularity at P".

(c) Let $P \in Y$ be the tacnode of Exercise 1. If $\varphi : \widetilde{Y} \to Y$ is the blowing-up at P, show that $\varphi^{-1}(P)$ is a node. Using (b) we see that the tacnode can be resolved by two successive blowing-ups.

(d) Let Y be the plane curve $y^3 = x^5$, which has a "high order cusp" at O. Show that O is a triple point; that blowing up O gives rise to a double point (what kind?) and that one further blowing up resolves the singularity.

Proof.

(a) Let Y be the curve defined by the equation $xy = x^6 + y^6$. It is the node of Exercise 1. We consider the equations $xy = x^6+y^6$ and $xu = ty$ in $\mathbb{A}^2 \times \mathbb{P}^1$, where t, u are homogenous coordinates for $\mathbb{P}^1$. Now $\mathbb{P}^1$ is covered by open sets $t \neq 0$ and $u \neq 0$, which we can consider separately. If $t \neq 0$, we can set $t=1$, and use u as an affine parameter. Then we have the equations

$$\begin{aligned} xy &= x^6 + y^6 \\ y &= xu \end{aligned}$$

in $\mathbb{A}^3$ with coordinates x, y, u. Substituting, we get $x^2u = x^6 + x^6u^6$, which factors. Thus, we obtain two irreducible components, one defined by $x=0$, $y=0$, u arbitrary, which is the exceptional curve E, and the other defined by $u = x^4 + x^4u^6$. This is $\widetilde{Y}$, which meets E at the point $u=0$. This curve is nonsingular since if we set $f(x,y,u) = x^4(1+u^6) - u$, then

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial u} = -1,$$

so $\widetilde{Y}$ is nonsingular for all points $t \neq 0$. The curve Y is symmetric with respect to $x=y$, so $\widetilde{Y}$ is also nonsingular for all points $u \neq 0$, hence $\widetilde{Y}$ is nonsingular.

(b) We can assume P is the origin so that Y is defined by $f = f_2 + f_3 + \cdots + f_r$. Thus, we can write $f_2 = (ax+by)(cx+dy)$, where $(a,b) \neq (c,d)$ as points in $\mathbb{P}^1$. Take t, u to be homogenous coordinates for $\mathbb{P}^1$. If $t \neq 0$, substituting $y = xu$ we have the equation

$$F_u = (a+bu)(c+du) + f_3' + \cdots + f_r' = 0,$$

where f_i' is obtained by substituting $y = xu$ into f_i and dividing by x^2, which is possible since f_i is homogenous of degree $i > 2$. This equation defines $\widetilde{Y}$, which meets the exceptional curve at $(b,-a)$ and $(d,-c)$. Similarly, if $u \neq 0$, then we obtain the equation

$$F_t = (at+b)(ct+d) + \hat{f}_3 + \cdots + \hat{f}_r = 0,$$

thus $(b,-a)$ and $(d,-c)$ are the only points $\widetilde{Y}$ meets the exceptional curve, that is $\varphi^{-1}(P) = \{(a,b),(c,d)\}$. Also, $\widetilde{Y}$ is nonsingular at both points since

$$\frac{\partial F_u}{\partial u} = b\left(c - \frac{ad}{b}\right) \neq 0, \quad \frac{\partial F_t}{\partial u} = b\left(a - \frac{bc}{d}\right) \neq 0.$$

(c) Let Y be the plane curve defined by $x^2 = x^4 + y^4$ and let t, u, be homogenous coordinates for $\mathbb{P}^1$. If $t \neq 0$, then we have the equations

$$\begin{aligned} x^2 &= x^4 + y^4 \\ y &= xu \end{aligned}$$

in $\mathbb{A}^3$ with coordinates x, y, u. Substituting, we get $x^2 = x^4 + x^4u^4$, thus $\widetilde{Y}$ is defined by $1 = x^2 + x^2u^4$, which intersects the exceptional curve at $u = 0$. Similarly, if $u \neq 0$, then $\widetilde{Y}$ is defined by $t^2 = t^2y^2 + y^2$ and intersects the exceptional curve at $t = 0$. These two points correspond to the slopes of the two branches of Y at P, which are the two distinct tangent directions on the node P.

(d) If $t \neq 0$, then we have the equation $x^2 = u^3$, which is a cusp. If $u \neq 0$, then we have the equation $1 = t^5y^2$, which does not meet the exceptional curve.

□

8. Let $Y \subseteq \mathbb{P}^n$ be a projective variety of dimension r. Let $f_1, \ldots, f_t \in S = k[x_0, \ldots, x_n]$ be homogenous polynomials which generate the ideal of Y. Let $P \in Y$ be a point, with homogenous coordinates $P = (a_0, \ldots, a_n)$. Show that P is nonsingular on Y if and only if the rank of the matrix $\|(\partial f_i/\partial x_j)(a_0, \ldots, a_n)\|$ is $n - r$.

Proof. We follow the hint. The matrix $\|(\partial f_i/\partial x_i)(a_0, \ldots, a_n)\|$ has rank $n-r$ if and only if there exists a $(n-r)\times(n-r)$ submatrix with nonzero determinant. The determinant of a matrix of homogenous polynomials where entries of the same row have same degree is also a homogenous polynomial, hence the rank of $\|(\partial f_i/\partial x_j)(a_0, \ldots, a_n)\|$ is independent of the choice of homogenous coordinates of P. Assuming $a_0 \neq 0$, we can pass to an open affine $U_0 \subseteq \mathbb{P}^n$ containing P and use the affine Jacobian matrix, where U_0 is the open affine subset consisting of all points with nonzero 0th coordinate. In particular, f we set $Y_0 = Y \cap U_0$, then $g_i(y_1, \ldots, y_n) = f_i(1, y_1, \ldots, y_n)$ $(1 \leq i \leq t)$ generate the ideal of Y_0. Thus, P is nonsingular if and only if the rank of the matrix $\|(\partial g_i/\partial y_j)(a_1/a_0, \ldots, a_n/a_0)\|$ is $n - r$. Then, we have

$$\frac{\partial g_i}{\partial y_j} = \frac{\partial f_i(1, y_1, \ldots, y_n)}{\partial y_j} = \frac{\partial f_i(1, x_1, \ldots, x_n)}{\partial x_j},$$

so $||(\partial g_i/\partial y_j)(a_1/a_0, \ldots, a_n/a_0)||$ has an invertible $(n-r)\times(n-r)$ submatrix if and only if $\|(\partial f_i/\partial x_j)(a_0, \ldots, a_n)\|$ has an invertible $(n-r)\times(n-r)$ submatrix since the rank of a matrix is independent of the homogenous coordinates chosen for P. □

10. For a point P on a variety X, let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathscr{O}_P$. We define the *Zariski tangent space* $T_P(X)$ of X at P to be the dual k-vector space of $\mathfrak{m}/\mathfrak{m}^2$.

(a) For any point $P \in X$, $\dim T_P(X) \geq \dim X$, with equality if and only if P is nonsingular.

(b) For any morphism $\varphi : X \to Y$, there is a natural induced k-linear map $T_P(\varphi) : T_P(X) \to T_{\varphi(P)}(Y)$.

(c) If φ is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map $T_0(\varphi)$ of tangent spaces at the origin is the zero map.

Proof.

(a) Since $\mathfrak{m}/\mathfrak{m}^2$ is a finite dimension k-vector space, $\dim_k T_p(X) = \dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim X$ by (5.2A), and $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim \mathscr{O}_{P,X} = \dim X$ by definition of nonsingular points and Exercise 3.12.

(b) A morphism $\varphi : X \to Y$ induces a map $\varphi_P^* : \mathscr{O}_{\varphi(P),Y} \to \mathscr{O}_{P,X}$ between local rings, where the maximal ideal $\mathfrak{n}$ of $\mathscr{O}_{\varphi(P),Y}$ is mapped into the maximal ideal $\mathfrak{m}$ of $\mathscr{O}_{P,X}$. Thus, we can define $T_P(\varphi)F$ as

$$(T_P(\varphi)F)f = F(d_\varphi f) = F(f \circ \varphi),$$

where $d_\varphi : \mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is the map induced by φ_P^*. It is clearly k-linear.

(c) By the formula above, it suffices to show d_φ is the zero map at 0. Let X be the parabola $x = y^2$ and let Y by the x-axis. The maximal ideal $\mathfrak{m}$ of the local ring $\mathscr{O}_{0,X}$ is generated by y, thus $\mathfrak{m}/\mathfrak{m}^2$ is a one dimensional k-vector space spanned by y, and similarly $\mathfrak{n}/\mathfrak{n}^2$ of $\mathscr{O}_{0,Y}$ is a one dimensional k-vector space spanned by x. Thus, we have $\varphi^* x = y^2 \in \mathfrak{m}^2$, hence $d_\varphi = 0$, hence $T_0(\varphi) = 0$.

□