Chapter 3, Section 3

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1. Let X be a Noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine.

   Proof. One direction is clear. Suppose $X_{\text{red}} = \operatorname{Spec} A$ where A is a Noetherian ring with no nilpotent elements, let $f : X_{\text{red}} \to X$ be the natural map, and let $\mathscr{F}$ be any quasi-coherent sheaf on X. Following the hint, consider the filtration
   $$\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \cdots,$$
   where $\mathscr{N}$ is the sheaf of nilpotent elements on X. Note that $X \cong X_{\text{red}}$ as topological space, and the associated morphism of sheaves $\mathscr{O}_X \to f_*\mathscr{O}_{X_{\text{red}}}$ is surjective with kernel $\mathscr{N}$. Thus, each of the quotients of this filtration can be naturally viewed as A-modules. In particular, we have a natural isomorphism (2.10)
   $$H^i(X, \mathscr{N}^r \cdot \mathscr{F}/\mathscr{N}^{r+1}\mathscr{F}) \cong H^i(X_{\text{red}}, f^*(\mathscr{N}^r \cdot \mathscr{F}/\mathscr{N}^{r+1} \cdot \mathscr{F})).$$
   Also, the nilradical of a Noetherian ring is nilpotent, so there exists a positive integer $r > 0$ such that $\mathscr{N}^r = 0$ (A.M. 7.15). Using our hypothesis and (3.7), we climb up the filtration and deduce that $H^1(X, \mathscr{F}) = 0$. Hence, X is affine by (3.7). $\square$

2. Let X be a reduced Noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

   Proof. Suppose $X = \operatorname{Spec} A$ is affine for some reduced Noetherian ring A. The irreducible components of X correspond to the minimal prime ideals $\mathfrak{p}$ of A (A.M. Ex. 1.20). In particular, the irreducible components of X are precisely $\operatorname{Spec} A/\mathfrak{p}$. Conversely, let X_i be the irreducible components of X, and let $\phi : \mathscr{F} \to \bigoplus_i j_*\mathscr{F}\big|_{X_i}$ be the natural map of $\mathscr{O}_X$-modules, where $j : X_i \hookrightarrow X$ is the inclusion. Since X is Noetherian, $X_i \cap X_j$ is quasi-compact, so we can cover it with a finite number of open affine subsets X_{ijk}. Because X is reduced, ϕ is injective, so we can extend ϕ by the following exact sequence
   $$0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_i j_*\mathscr{F}\big|_{X_i} \longrightarrow \bigoplus_{i,j} j_*\mathscr{F}\big|_{X_{ijk}}.$$
   Each $j_*\mathscr{F}\big|_{X_i}, j_*\mathscr{F}\big|_{X_{ijk}}$ has vanishing cohomology for $i > 0$ by (2.10), (3.5), and (3.7). While the sequence above is not surjective on the right, the image is still a quasi-coherent sheaf, so using the long exact sequence of cohomology, we deduce that $H^i(X, \mathscr{F}) = 0$ for $i > 0$. Hence, X is affine by (3.7). $\square$

6. Let X be a Noetherian scheme.

   (a) Show that the sheaf $\mathscr{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus, $\mathfrak{Qco}(X)$ has enough injectives.

   (b) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque.

   (c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

   Proof.

(a) The Hom functor commutes with finite direct sums in the second argument, so we can assume $\mathscr{G} = j_*\tilde{I}$, where $j : U = \operatorname{Spec} A \to X$ is the inclusion, and I is an injective A-module. Suppose $\mathscr{N} \to \mathscr{M}$ is an injective map of $\mathscr{O}_X$-modules, and we are given any $f : \mathscr{N} \to j_*\tilde{I}$. Since j^* is left exact when j is an open immersion, the induced map of A-modules $j^*\mathscr{N} \to j^*\mathscr{M}$ is also injective. For any such f there is an associated morphism of A-modules $g : j^*\mathscr{N} \to \tilde{I}$ by adjointness of j_*, so there exists an extension of g to $j^*\mathscr{M}$ by injectivity of $\tilde{I}$. By adjointness of j^* again, we obtain a morphism $\mathscr{M} \to j_*\tilde{I}$ that naturally extends f, which is what we wanted to show.

(b) Essentially imitating (a) replacing i^* with i_* and vice versa, we deduce that $\mathscr{I}\big|_U$ is an injective object of $\mathfrak{Qco}(U)$. Covering X with finite number of open affines $U_i = \operatorname{Spec} A_i$, we have $\mathscr{I}\big|_{U_i} \cong \tilde{I}_i$ for some injective A_i-module I_i for each i by (II, 5.5). Each $\tilde{I}_i$ is flasque by (3.4), so $\mathscr{I}$ is flasque on a local basis. Hence, $\mathscr{I}$ is flasque.

(c) Considering $\Gamma(X, \cdot)$ as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$, we calculate its derived funcotrs by taking injective resolutions in the category $\mathfrak{Qco}(X)$. But any injective is flasque (b), and flasques are acyclic (2.5), so this resolution gives the usual cohomology functors (1.2A).

□

7. Let A be a Noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

   (a) For any A-module M, establish the following formula of Deligne:

$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M).$$

   (b) Apply this in the case of an injective A-module I, to give another proof of (3.4).

Proof.

(a) Conssider the case when $\mathfrak{a} = (f)$ is principal for some non-zerodivisor $f \in A$. The left-hand side of above is simply $M_f \cong A_f \otimes_A M$ (II, 5.1). Then $\mathfrak{a}^n$ is principal for all n, generated by f^n, so the A-module $\operatorname{Hom}_A((f^n), M)$ is naturally isomorphic to $(f^n)^\vee \otimes_A M$, where $^\vee$ means the dual. It is not hard to see $(f^n)^\vee$ is naturally ismorphic to the sub A-module $Af^{-n} \subset A_f$ generated by f^{-n}. Indeed, any A-module homomorphism $\alpha : (f^n) \to A$ in $(f^n)^\vee$ is defined by $\alpha(f^n)$, so there is a natural ismorphism $\alpha \mapsto \alpha(f^n)f^{-n}$. Tensor products commute with colimits (A.M., Ex. 2.20), and another result (A.M., Ex. 2.17) shows

$$\begin{aligned}\varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M) &\cong \varinjlim_n Af^{-n} \otimes_A M \\ &\cong M \otimes_A \varinjlim_n Af^{-n} \\ &\cong M \otimes_A A_f.\end{aligned}$$

In the general case, by the Noetherian hypothesis, $\mathfrak{a}$ is a finitely generated A-module, say $\mathfrak{a} = (f_1, \ldots, f_r)$. We have $U = \bigcup_{i=1}^r U_i$, where U_i is the distinguished open set in X associated to $f_i \in A$. In the category of open sets, unions can be expressed as colmits. In particular, $U = \varinjlim_i U_i$. By passing to a direct system of

$$\begin{aligned}\Gamma(U, \widetilde{M}) &= \Gamma(\varinjlim_i U_i, \widetilde{M}) \\ &\cong \varinjlim_i \Gamma(U_i, \widetilde{M}) \\ &\cong \varinjlim_i M_{f_i}.\end{aligned}$$

(b) Let I be an injective A-module. It will be sufficeint to show for any open set $U \subseteq X$, where $U = X - V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A, that $\Gamma(X, \tilde{I}) \to \Gamma(U, \tilde{I})$ is surjective.

□