

Chapter 3, Section 3

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1. Let X be a Noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine.

Proof. One direction is clear. Suppose $X_{\text{red}} = \text{Spec } A$ where A is a Noetherian ring with no nilpotent elements, let $f : X_{\text{red}} \rightarrow X$ be the natural map, and let \mathcal{F} be any quasi-coherent sheaf on X . Following the hint, consider the filtration

$$\mathcal{F} \supseteq \mathcal{N} \cdot \mathcal{F} \supseteq \mathcal{N}^2 \cdot \mathcal{F} \supseteq \dots,$$

where \mathcal{N} is the sheaf of nilpotent elements on X . Note that $X \cong X_{\text{red}}$ as topological space, and the associated morphism of sheaves $\mathcal{O}_X \rightarrow f_* \mathcal{O}_{X_{\text{red}}}$ is surjective with kernel \mathcal{N} . Thus, each of the quotients of this filtration can be naturally viewed as A -modules. In particular, we have a natural isomorphism (2.10)

$$H^i(X, \mathcal{N}^r \cdot \mathcal{F} / \mathcal{N}^{r+1} \cdot \mathcal{F}) \cong H^i(X_{\text{red}}, f^*(\mathcal{N}^r \cdot \mathcal{F} / \mathcal{N}^{r+1} \cdot \mathcal{F})).$$

Also, the nilradical of a Noetherian ring is nilpotent, so there exists a positive integer $r > 0$ such that $\mathcal{N}^r = 0$ (A.M. 7.15). Using our hypothesis and (3.7), we climb up the filtration and deduce that $H^1(X, \mathcal{F}) = 0$. Hence, X is affine by (3.7). \square

2. Let X be a reduced Noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

Proof. Suppose $X = \text{Spec } A$ is affine for some reduced Noetherian ring A . The irreducible components of X correspond to the minimal prime ideals \mathfrak{p} of A (A.M. Ex. 1.20). In particular, the irreducible components of X are precisely $\text{Spec } A/\mathfrak{p}$. Conversely, suppose each irreducible component of X is affine. \square

6. Let X be a Noetherian scheme.

- (a) Show that the sheaf \mathcal{G} constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X . Thus, $\mathfrak{Qco}(X)$ has enough injectives.
- (b) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque.
- (c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor $\mathfrak{Qco}(X)$ to \mathfrak{Ab} .

Proof.

- (a) The Hom functor commutes with finite direct sums in the second argument, so we can assume $\mathcal{G} = j_* \tilde{I}$, where $j : U = \text{Spec } A \rightarrow X$ is an open immersion, and I is an injective A -module. Suppose $\mathcal{N} \rightarrow \mathcal{M}$ is an injective map of \mathcal{O}_X -modules, and we are given any $f : \mathcal{N} \rightarrow j_* \tilde{I}$. Since j^* is left exact, the induced map of A -modules $j^* \mathcal{N} \rightarrow j^* \mathcal{M}$ is also injective. For any such f there is an associated morphism of A -modules $g : j^* \mathcal{N} \rightarrow \tilde{I}$ by adjointness of j_* , so there exists an extension of g to $j^* \mathcal{M}$ by injectivity of \tilde{I} . By adjointness of j^* again, we obtain a morphism $\mathcal{M} \rightarrow j_* \tilde{I}$ that naturally extends $f : \mathcal{N} \rightarrow j_* \tilde{I}$, which is what we wanted to show.
- (b) Let \mathcal{I} be an injective object of $\mathfrak{Qco}(X)$.
- (c)

\square

7. Let A be a Noetherian ring, let $X = \text{Spec } A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

