Chapter 3, Section 10

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1. Let $f : X \to Y$ be a property, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fiber X_y is smooth over $k(y)$. Then show that there is an open neighborhood U of y in Y such that $f : f^{-1}(U) \to U$ is smooth.

2. A morphism $f : X \to Y$ of schemes of finite type over k is *étale* if it is smooth of relative dimension 0. It is *unramified* if for every $x \in X$, letting $y = f(x)$, we have $\mathfrak{m}_y \cdot \mathscr{O}_x = \mathfrak{m}_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i) f is étale;

(ii) f is flat, and $\Omega_{X/Y} = 0$;

(iii) f is flat and unramified.

3. Show that a morphism $f : X \to Y$ of schemes of finite type over k is étale if and only if the following condition is satisfied: for each $x \in X$, let $y = f(x)$. Let $\hat{\mathscr{O}}_x$ and $\hat{\mathscr{O}}_y$ be the completions of the local rings at x and y. Choose fields of representatives (II, 8.25A) $k(x) \subseteq \hat{\mathscr{O}}_x$ and $k(y) \subseteq \hat{\mathscr{O}}_y$ so that $k(y) \subseteq k(x)$ via the natural map $\hat{\mathscr{O}}_y \to \hat{\mathscr{O}}_x$. Then our condition is that for every $x \in X$, $k(x)$ is a separable algebraic extension of $k(y)$, and the natural map

$$\hat{\mathscr{O}}_y \otimes_{k(y)} k(x) \to \hat{\mathscr{O}}_x$$

is an isomorphism.

4. If x is a point of a scheme X, we define an *étale neighborhood* of x to be an étale morphism $f : U \to X$, together with a point $x' \in U$, such that $f(x') = x$. As an example of the use of étale neighborhoods, prove the following: if $\mathscr{F}$ is a coherent sheaf on X, and if every point of X has an étale neighborhood $f : U \to X$ for which $f^*\mathscr{F}$ is a free $\mathscr{O}_U$-module, then $\mathscr{F}$ is locally free on X.

5. Let Y be the plane nodal cubic curve $y^2 = x^2(x+1)$. Show that Y has a finite étale covering X of degree 22, where X is a union of two irreducible components, each one isomorphic to the normalization of Y.

6. *(Serre). A linear system with moving singularities.* Let k be an algebraically closed field of characteristic 2. Let $P_1, \ldots, P_7 \in \mathbb{P}^2_k$ be the seven points of the projective plane over the prime field $\mathbb{F}_2 \subseteq k$. Let $\mathfrak{d}$ be the linear system of all cubic curves in X passing through $P_1, \ldots, P_7$.

(a) $\mathfrak{d}$ is a linear system of dimension 2 with base points $P_1, \ldots, P_7$, which determines an inseparable morphism of degree 2 from $X - \{P_i\}$ to $\mathbb{P}^2$.

(b) Every curve $C \in \mathfrak{d}$ is singular. More precisely, either C consists of 3 lines all passing through one of the P_i, or C is an irreducible cuspidal cubic with cusp $P \neq$ any P_i. Furthermore, the correspondence $C \mapsto$ the singular point of C is a one-to-one correspondence between $\mathfrak{d}$ and $\mathbb{P}^2$. Thus, the singular points of elements of $\mathfrak{d}$ move all over.

7. *A linear system with moving singularities contained in the base locus (any characteristic).* In affine 3-space with coordinates x, y, z, let C be the conic $(x-1)^2 + y^2 = 1$ in the xy-plane, and let P be the point $(0, 0, t)$ on the z-axis. Let Y_t be the closure in $\mathbb{P}^3$ of the cone over C with vertex P. Show that as t varies, the surfaces $\{Y_t\}$ form a linear system of dimension 1, with a moving singularity at P. The base locus of this linear system is the conic C plus the z-axis.

8. Let $f : X \to Y$ be a morphism of varieties over k. Assume that Y is regular, X is Cohen-Macaulay, and that every fiber of f has dimension equal to $\dim X - \dim Y$. Then f is flat.