# Chapter 3, Section 3

James Lee

April 24, 2025

**1.** Let $X$ be a Noetherian scheme. Show that $X$ is affine if and only if $X_{\mathrm{red}}$ (II, Ex. 2.3) is affine.

*Proof.* One direction is clear. Suppose $X_{\mathrm{red}} = \operatorname{Spec} A$ where $A$ is a Noetherian ring with no nilpotent elements, let $f : X_{\mathrm{red}} \to X$ be the natural map, and let $\mathscr{F}$ be any quasi-coherent sheaf on $X$. Following the hint, consider the filtration

$$\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \cdots,$$

where $\mathscr{N}$ is the sheaf of nilpotent elements on $X$. Note that $X \cong X_{\mathrm{red}}$ as topological space, and the associated morphism of sheaves $\mathscr{O}_X \to f_*\mathscr{O}_{X_{\mathrm{red}}}$ is surjective with kernel $\mathscr{N}$. Thus, each of the quotients of this filtration can be naturally viewed as $A$-modules. In particular, we have a natural isomorphism (2.10)

$$H^i(X, \mathscr{N}^r \cdot \mathscr{F}/\mathscr{N}^{r+1}\mathscr{F}) \cong H^i(X_{\mathrm{red}}, f^*(\mathscr{N}^r \cdot \mathscr{F}/\mathscr{N}^{r+1} \cdot \mathscr{F})).$$

Also, the nilradical of a Noetherian ring is nilpotent, so there exists a positive integer $r > 0$ such that $\mathscr{N}^r = 0$ (A.M. 7.15). Using our hypothesis and (3.7), we climb up the filtration and deduce that $H^1(X, \mathscr{F}) = 0$. Hence, $X$ is affine by (3.7). $\square$

**2.** Let $X$ be a reduced Noetherian scheme. Show that $X$ is affine if and only if each irreducible component is affine.

*Proof.* Suppose $X = \operatorname{Spec} A$ is affine for some reduced Noetherian ring $A$. The irreducible components of $X$ correspond to the minimal prime ideals $\mathfrak{p}$ of $A$ (A.M. Ex. 1.20). In particular, the irreducible components of $X$ are precisely $\operatorname{Spec} A/\mathfrak{p}$. Conversely, let $X_i = \operatorname{Spec} A_i$ be the irreducible components of $X$, where $A_i$ is a reduced Noetherian ring for each $i = 1, \ldots, n$, and let $\mathscr{F}$ be any quasi-coherent sheaf on $X$. We have $\mathscr{F}\big|_{U_i} \cong \widetilde{M}_i$ for some $A_i$-module $M_i$, and we have a natural map $\mathscr{F} \to j_*\mathscr{F}\big|_{X_i} = j_*j^*\mathscr{F}$ (I, Ex. 1.18). Thus, we have a natural map of $\mathscr{O}_X$-modules $\phi : \mathscr{F} \to \bigoplus_{i=1}^n j_*\mathscr{F}\big|_{X_i}$. Each $j_*\mathscr{F}\big|_{X_i}$ has vanishing cohomology for $i > 0$ by (2.10) and (3.5). Cohomology commutes with direct sums (2.9), so it suffices to show $\phi$ is an isomorphism. It is clearly injective. Conversely, let $(f_1, \ldots, f_n) \in \bigoplus_{i=1}^n j_*\mathscr{F}\big|_{X_i}$, where $f_i$ is an element belonging to some localization of $M_i$.

$\square$

**6.** Let $X$ be a Noetherian scheme.

(a) Show that the sheaf $\mathscr{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on $X$. Thus, $\mathfrak{Qco}(X)$ has enough injectives.

(b) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque.

(c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

*Proof.*

(a) The Hom functor commutes with finite direct sums in the second argument, so we can assume $\mathscr{G} = j_*\tilde{I}$, where $j : U = \operatorname{Spec} A \to X$ is an open immersion, and $I$ is an injective $A$-module. Suppose $\mathscr{N} \to \mathscr{M}$ is an injective map of $\mathscr{O}_X$-modules, and we are given any $f : \mathscr{N} \to j_*\tilde{I}$. Since $j^*$ is left exact, the induced map of $A$-modules $j^*\mathscr{N} \to j^*\mathscr{M}$ is also injective. For any such $f$ there is an associated morphism of $A$-modules $g : j^*\mathscr{N} \to \tilde{I}$ by adjointness of $j_*$, so there exists an extension of $g$ to $j^*\mathscr{M}$ by injectivity of $\tilde{I}$. By adjointness of $j^*$ again, we obtain a morphism $\mathscr{M} \to j_*\tilde{I}$ that naturally extends $f : \mathscr{N} \to j_*\tilde{I}$, which is what we wanted to show.

(b) Let $\mathscr{I}$ be an injective object of $\mathfrak{Qco}(X)$.

(c)

$\square$

**7.** Let $A$ be a Noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

(a) For any $A$-module $M$, establish the following formula of Deligne:

$$\Gamma(U, \tilde{M}) \cong \varinjlim_{n} \operatorname{Hom}_A(\mathfrak{a}^n, M).$$

(b) Apply this in the case of an injective $A$-module $I$, to give another proof of (3.4).

*Proof.*

(a)

(b)

$\square$