Chapter 2, Section 5

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1. Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of finite rank. We define the *dual* of $\mathscr{E}$, denoted $\mathscr{E}^\vee$ to be the sheaf $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)$.

(a) Show that $(\mathscr{E}^\vee)^\vee \cong \mathscr{E}$.

(b) For any $\mathscr{O}_X$-module $\mathscr{F}$, $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F}) \cong \mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F}$.

(c) For any $\mathscr{O}_X$-modules $\mathscr{F}, \mathscr{G}$, $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{G}))$.

(d) (*Projection Formula*). If $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces, if $\mathscr{F}$ is an $\mathscr{O}_X$-module, and if $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E}) \cong f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{E}$.

Lemma 1. *Let $(X, \mathscr{O})$ be a ringed space, let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}$-modules, and let $\mathscr{F}$ be locally free of finite rank. For any $x \in X$, $\mathcal{H}om(\mathscr{F}, \mathscr{G})_x \cong \mathrm{Hom}_{\mathscr{O}_{x,X}}(\mathscr{F}_x, \mathscr{G}_x)$.*

Proof. An element of $\mathcal{H}om(\mathscr{F}, \mathscr{G})_x$ can be represented as a pair $\langle U, f \rangle$, where U is an open neighborhood of x and $f : \mathscr{F}\big|_U \to \mathscr{G}\big|_U$ is a morphism of sheaves on U. Since $(\mathscr{F}\big|_U)_x \cong \mathscr{F}_x, (\mathscr{G}\big|_U)_x \cong \mathscr{G}_x$, there is a natural map

$$\begin{aligned} \alpha : \mathcal{H}om(\mathscr{F}, \mathscr{G})_x &\to \mathrm{Hom}_{\mathscr{O}_{x,X}}(\mathscr{F}_x, \mathscr{G}_x) \\ \langle U, f \rangle &\mapsto f_x. \end{aligned}$$

We show α is a bijection. If $\mathscr{F}$ has rank n, then $\mathscr{F}_x \cong \mathscr{O}_x^{\oplus n}$ since colimits commute and the image of f_x is a finitely generated $\mathscr{O}_x$-module. Thus, we can represent f_x as an $n \times n$-matrix, and there are only a finite number of entries, so we can further assume the entries lie in $\Gamma(W, \mathscr{O})$ for a smaller open neighborhood $W \subseteq U$. It is immediate that α is injective. Also, by this representation any $\varphi \in \mathrm{Hom}_{\mathscr{O}_x}(\mathscr{F}_x, \mathscr{G}_x)$ defines a $\mathscr{O}_X(W)$-module homomorphism $\mathscr{F}(W) = \mathscr{O}_X(W)^{\oplus n} \to \mathscr{G}(W)$, and by sheafification it ascends to a morphism of sheaves $\mathscr{F}\big|_W \to \mathscr{G}\big|_W$. Hence, α is surjective. □

Lemma 2. *Let $(X, \mathscr{O}_X)$ be a ringed space, let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules, and let $x \in X$. Then $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x \cong \mathscr{F}_x \otimes_{\mathscr{O}_{x,X}} \mathscr{G}_x$.*

Proof. (A.M. Ex. 2.20). □

Lemma 3. *If $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces, and $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_Y$-modules, then $f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \cong f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} f^*(\mathscr{G})$.*

Proof. Check locally at each point and use Lemma 2. □

Proof.

(a) The question is local, so assume $\mathscr{E}$ is free of rank n. Then $\mathscr{E}_x = \mathscr{O}_{x,X}^{\oplus n}$, and by lemma 1, we have

$$(\mathscr{E}^\vee)_x^\vee = \mathrm{Hom}_{\mathscr{O}_{x,X}}(\mathrm{Hom}_{\mathscr{O}_{x,X}}(\mathscr{O}_{x,X}^{\oplus n}, \mathscr{O}_{x,X}), \mathscr{O}_{x,X}) = (\mathscr{O}_{x,X}^{\oplus n \vee})^\vee \cong \mathscr{O}_{x,X}^{\oplus n} = \mathscr{E}_x,$$

where the last isomorphism is a basic fact of finite free modules. The sheaves $(\mathscr{E}^\vee)^\vee$ and $\mathscr{E}$ are isomorphic at the level of stalks, hence they are isomorphic.

(b) The question is local, so assume $\mathscr{E}$ is free of rank n. For any open set $U \subseteq X$, we have

$$\begin{aligned} \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})(U) &= \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{E}\big|_U, \mathscr{F}\big|_U), \\ (\mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F})(U) &= \mathscr{E}^\vee(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}(U) = \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{E}\big|_U, \mathscr{O}_X\big|_U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}(U), \end{aligned}$$

and since $\mathscr{E}$ is free, we have $\mathscr{E}(U) = \mathscr{O}_X(U)^{\oplus n}$. Let $f_i : \mathscr{E} \to \mathscr{O}_X$ be the projection morphisms onto the ith component, let $e_i\big|_U$ be the standard basis of $\mathscr{E}(U)$, and let $f : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X) \to \mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F}$ be the morphism of sheaves defined by

$$\begin{aligned} f(U) : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})(U) &\to (\mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F})(U) \\ g &\mapsto \sum_{i=1}^n f_i\big|_U \otimes g(e_i\big|_U) \end{aligned}$$

It is isomorphism of sheaves from the fact that $\mathscr{E}$ is free and g is $\mathscr{O}_X$-linear.

(c) Let $f \in \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G})$, which for every open $U \subseteq X$ defines a homomorphism of abelian groups

$$f(U) : (\mathscr{E} \otimes \mathscr{F})(U) \to \mathscr{G}(U).$$

We want to show f naturally defines an element of $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{G}))$. Define $\tilde{f} : \mathscr{F} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{G})$ as the following: for each $s \in \mathscr{F}(U)$, let $\tilde{f}(s) : \mathscr{E}\big|_U \to \mathscr{G}\big|_U$ be the morphism defined by

$$\begin{gathered} \tilde{f}(s) : \mathscr{E}\big|_U \to \mathscr{G}\big|_U \\ t \in \Gamma(V, \mathscr{E}\big|_U) \mapsto f(t \otimes s\big|_V) \end{gathered}$$

where V is any open subset of U. The fact that this defines an isomorphism of abelian groups follows the corresponding algebraic fact about finite free modules over rings.

(d) The question is local on Y, so assume $\mathscr{E}$ is free of finite rank. Also, if the statement is true for two sheaves of $\mathscr{O}_Y$-modules, then it true for their direct sum, so we can assume $\mathscr{E} = \mathscr{O}_Y$. Thus, we have the following composite of canonical isomorphisms

$$f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y \cong f_*\mathscr{F} \cong f_*(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X) \cong f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{O}_Y).$$

The last isomorphism is because structure sheaves pullback to structure sheaves, i.e.,

$$f^*\mathscr{O}_Y = f^{-1}\mathscr{O}_Y \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \cong \mathscr{O}_X.$$

Alternatively, by adjointness of f_* and f^*, it suffices to show

$$\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E} \cong f^*(f_*(\mathscr{F})) \otimes_{\mathscr{O}_X} f^*\mathscr{E}$$

We assume the basic fact that the pullback distributes with tensor products, i.e., $f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \cong f^*\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}$. Again, by adjointness of f_* and f^*, we have $f^*f_*\mathscr{F} \cong \mathscr{F}$, which is what we wanted to show.

□

7. Let X be a noetherian scheme, and let $\mathscr{F}$ be a coherent sheaf.

(a) If the stalk $\mathscr{F}_x$ is a free $\mathscr{O}_x$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathscr{F}\big|_U$ is free.

(b) $\mathscr{F}$ is locally free if and only if its stalks $\mathscr{F}_x$ are free $\mathscr{O}_x$-modules for all $x \in X$.

(c) $\mathscr{F}$ is invertible (i.e. locally free of rank 1) if and only if there is a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \cong \mathscr{O}_X$.

Proof.

(a) This is a local question, so assume $X = \operatorname{Spec} A$ and $\mathscr{F} = \tilde{M}$ for a noetherian ring A and a finitely generated A-module M. We reduce to the following algebraic problem: if there exists a prime ideal $\mathfrak{p}$ such that $M_\mathfrak{p} \cong M \otimes_A A_\mathfrak{p}$ is a finite free $A_\mathfrak{p}$-module, then there exists $f \in A - \mathfrak{p}$ such that $M_f \cong M \otimes_A A_f$ is a finite free A_f-module. Let $m_1, \dots, m_r \in M$ such that the image of m_i form a basis in $M_\mathfrak{p}$, and let $\phi : A^{\oplus r} \to M$ be an A-module homomorphism defined by $e_i \mapsto m_i$, where e_i is the standard basis of $A^{\oplus r}$. We have an exact sequence

$$0 \to \ker\phi \to A^{\oplus r} \to M \to M/\operatorname{im}\phi \to 0,$$

and localization is an exact functor, so we have an induced exact sequence of $A_\mathfrak{p}$-modules

$$0 \to (\ker\phi)_\mathfrak{p} \to A_\mathfrak{p}^{\oplus r} \to M_\mathfrak{p} \to (M/\operatorname{im}\phi)_\mathfrak{p} \to 0.$$

Since $A_\mathfrak{p}^{\oplus r} \to M_\mathfrak{p}$ is an isomorphism, we have $(\ker\phi)_\mathfrak{p}, (M/\operatorname{im}\phi)_\mathfrak{p} = 0$. Submodules and quotients of noetherian modules are noetherian; in particular they are finitely generated, so by (A.M. Ex. 2.1) we can find $f \in A - \mathfrak{p}$ such that $(\ker\phi)_f, (M/\operatorname{im}\phi)_f = 0$. Hence, $M_f \cong A_f^{\oplus r}$.

(b) The if direction follows from (a). The converse direction follows from the following facts: we can realize finite free modules as a colimit, the stalk of a sheaf is defined as a colimit, and colimits commute.

(c) By the previous parts and lemma 2, we reduce to the following algebraic problem: let A be a noetherian local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated A-module. Then $M \cong A$ if and only if there exists a finitely generated A-module N such that $M \otimes_A N \cong A$. One direction is clear. Conversely, suppose M, N are finitely generated A-modules such that $M \otimes_A N \cong A$. Let $k = A/\mathfrak{m}$ be the residue field of A. Tensoring with k gives

$$(M \otimes_A k) \otimes_k (N \otimes_A k) \cong (M \otimes_A N) \otimes_A k \cong A \otimes_A k \cong k,$$

which implies $M \otimes_A k \cong k$. By Nakayama's lemma, M is generated by a single element, which implies $M \cong A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of A, and similarly $N = A/\mathfrak{b}$. We have

$$A \cong M \otimes_A N \cong A/\mathfrak{a} \otimes_A A/\mathfrak{b} \cong A/(\mathfrak{a} + \mathfrak{b}),$$

which implies $\mathfrak{a} + \mathfrak{b} = (0)$. Hence, $M, N \cong A$.

□

8. Again let X be a noetherian scheme, and $\mathscr{F}$ a coherent sheaf on X. We will consider the function

$$\varphi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathscr{O}_x} k(x),$$

where $k(x) = \mathscr{O}_x/\mathfrak{m}_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results:

(a) The function φ is *upper semi-continuous*, i.e. for any $n \in \mathbb{Z}$, the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.

(b) If $\mathscr{F}$ is locally free, and X is connected, then φ is a constant function.

(c) Conversely, if X is reduced, and φ is constant, then $\mathscr{F}$ is locally free.

Lemma 4 (Nakayama)**.** *Let A be a local ring with residue field k, and let M be a finitely generated A-module. Then any k-basis of M/mM lifts to a minimal set of generators of M.*

Proof.

(a) We show the set $\{x \in X : \varphi(x) < n\}$ is open. By (5.4), we reduce to the following algebraic problem: let A be a noetherian ring, let M be a finitely generated A-module, and let $\mathfrak{p}$ be a prime ideal of A with residue field $k(\mathfrak{p})$. If $\dim_{k(\mathfrak{p})} M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) < n$ for some $n \in \mathbb{Z}$, there exists a basic open neighborhood $\operatorname{Spec} A_s$ of $\mathfrak{p}$ such that $\dim_{k(\mathfrak{q})} M_\mathfrak{q} \otimes_{A_\mathfrak{q}} k(\mathfrak{q}) < n$ for all $\mathfrak{q} \in \operatorname{Spec} A_s$, where $s \in A - \mathfrak{p}$. By Nakayama's lemma, there exists $m_1, \ldots, m_r \in M$ with $r < n$ such that their image in $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p})$ is a $k(\mathfrak{p})$-basis. Let $f : A^{\oplus r} \to M$ be the A-module homomorphism defined by $e_i \mapsto m_i$. We have an exact sequence

$$A^{\oplus r} \to M \to M/\operatorname{im} f.$$

Localizing gives

$$A_\mathfrak{p}^{\oplus r} \to M_\mathfrak{p} \to (M/\operatorname{im} f)_\mathfrak{p},$$

where $(M/\operatorname{im} f)_\mathfrak{p} = 0$. Quotients of noetherian modules are noetherian, so by (A.M. Ex. 2.1), we can find $s \in A - \mathfrak{p}$ such that $(M/\operatorname{im} f)_s = 0$. Thus, we have an exact sequence

$$A_s^{\oplus r} \to M_s \to 0,$$

which implies $M_\mathfrak{q} \otimes_{A_\mathfrak{q}} k(\mathfrak{q})$ has rank at most r for any $\mathfrak{q} \in \operatorname{Spec} A_s$.

(b) Let U_i for $i = 1, \ldots, n$ be a finite open cover of X such that $\mathscr{F}\big|_{U_i}$ is a free $\mathscr{O}_X\big|_{U_i}$-module. If X is connected, then for all U_i, there exists U_j with $j \neq i$ such that $U_i \cap U_j$. Clearly φ is constant on each U_i, so let r_i be the rank of $\mathscr{F}\big|_{U_i}$. Choose any $x \in U_i \cap U_j$, then $r_i = r_j = \varphi(x)$.

(c) Let $x \in X$, let $\operatorname{Spec} A$ be an open affine neighborhood of x for some noetherian reduced ring A, and let $\mathfrak{p}$ be the prime ideal corresponding to $x \in \operatorname{Spec} A$. Set $r = \varphi(x)$. By (5.4), there exists a finitely generated A-module M such that $\mathscr{F}\big|_{\operatorname{Spec} A} = \tilde{M}$. Let $m_1, \ldots, m_r \in \mathscr{F}_x \cong M_\mathfrak{p}$ such that their image in $\mathscr{F}_x \otimes_{\mathscr{O}_x} k(x)$ form a $k(x)$-basis. By Nakayama's lemma, the m_i's generate $M_\mathfrak{p}$ over $A_\mathfrak{p}$, so they generate $M_\mathfrak{q}$ for all prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$. Suppose $\sum_{i=1}^r a_i m_i = 0$ for $a_i \in A_\mathfrak{p}$. Since the image of the m_i in $M_\mathfrak{q} \otimes_{A_\mathfrak{q}} k(\mathfrak{q})$ for all $i = 1, \ldots, r$ form a $k(\mathfrak{q})$-basis, the images of a_i in $k(\mathfrak{q})$ must be zero for all i. This implies a_i is contained in all prime ideals of $A_\mathfrak{p}$. However, $A_\mathfrak{p}$ is reduced, so $a_i = 0$. Hence, m_i are linearly independent.

□

9. Let S be a graded ring, generated by S_1 as an S_0-algebra, let M be a graded S-module, and let $X = \operatorname{Proj} S$.

(a) Show that there is a natural homomorphism $\alpha : M \to \Gamma_*(\tilde{M})$.

(b) Assume now that $S_0 = A$ is a finitely generated k-algebra for some field k, that S_1 is a finitely generated A-module, and that M is a finitely generated S-module. Show that the map α is an isomorphism in all large enough degrees, i.e. there is a $d_0 \in \mathbb{Z}$ such that for all $d \geq d_0$, $\alpha_d : M_d \to \Gamma(X, \tilde{M}(d))$ is an isomorphism.

(c) With the same hypotheses, we define an equivalence relation $\approx$ on graded S-modules by saying $M \approx M'$ is there is an integer d such that $M_{\geq d} \cong M'_{\geq d}$. Here $M_{\geq d} = \bigoplus_{n \geq d} M_n$. We will say that a graded S-module M is *quasi-finitely generated* if it is equivalent to a finitely generated module. Now show that the functors $\tilde{}$ and Γ_* induce an equivalence of categories between the category of quasi-finitely generated graded S-modules modulo the equivalence relation $\approx$, and the category of coherent $\mathscr{O}_X$-modules.

Proof.

(a) Write $M = \bigoplus_{d=0}^{\infty} M_d$. If $s \in M_d$, then s determines in a natural way a global section $s \in \Gamma(X, \tilde{M}(d))$, so define $\alpha_d : M_d \to \Gamma(X, \tilde{M}(d))$ in this way. Define β by extending this map linearly for all d.

(b) By (§1, 7.4), there is a finite filtration

$$0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$$

of M by graded submodules, where for each i, $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$ for some homogenous prime ideal $\mathfrak{p}_i \subseteq S$, and some integer n_i. This filtration gives a filtration of $\tilde{M}$ and short exact sequences

$$0 \to \tilde{M}^{i-1} \to \tilde{M}^i \to \widetilde{M^i/M^{i-1}} \to 0.$$

Twisting by d and taking global sections, all maps are natural, so we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M_d^{i-1} \arrow[r] \arrow[d] & M_d^i \arrow[r] \arrow[d] & (M^i/M^{i-1})_d \arrow[d] \\
0 \arrow[r] & \Gamma(X, \tilde{M}^{i-1}(d)) \arrow[r] & \Gamma(X, \tilde{M}^i(d)) \arrow[r] & \Gamma(X, \widetilde{M^i/M^{i-1}}(d))
\end{tikzcd}$$

where the vertical arrows are maps defined in (a). By the five lemma, to show that $M_d^i \to \Gamma(X, \tilde{M}^i(d))$ is surjective for large enough d, it will be sufficient to show that $(S/\mathfrak{p})_d \to \Gamma(X, \widetilde{S/\mathfrak{p}}(d))$ is surjective for large enough d, for each $\mathfrak{p}$ and n. Thus, we have reduced to the following special case: Let S be a graded integral domain, finitely generated by S_1 as an S_0-algebra, where $S_0 = A$ is a finitely generated domain over k, and let $X = \operatorname{Proj} S$. Then the map $\alpha : S \to S' = \bigoplus_{d \geq 0} \Gamma(X, \mathscr{O}_X(d))$ is an isomorphism in all large enough degrees.
Let $x_0, \ldots, x_r \in S_1$ be a set of generators of S_1 as an A-module. Following the argument in (5.13), S' is a ring, containing S, and contained in the intersection $\bigcap S_{x_i}$ of the localizations of S at the elements $x_0, \ldots, x_r$. By the proof of (5.19), since S'_d is a finitely generated A-module for every d, there exists $d_0 \geq 0$ such that $S_n S'_d \subseteq S_{n+d} \subseteq S'_{n+d}$ for some large enough n and all $d \geq d_0$. It follows by $S_n S'_d = S'_{n+d}$ that $S_{n+d} = S'_{n+d}$ for all $d \geq d_0$.

(c) The natural homomorphism α viewed as a morphism between equivalence classes of quasi-finitely generated graded S-modules is an isomorphism by (b). Hence, the two categories are equivalent with β from (5.15).

□

12. (a) Let X be a scheme over a scheme Y, and let $\mathcal{L}$, $\mathcal{M}$ be two very ample invertible sheaves on X. Show that $\mathcal{L} \otimes \mathcal{M}$ is also very ample.

(b) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes. Let $\mathcal{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathcal{M}$ be a very ample invertible sheaf on Y relative to Z. Show that $\mathcal{L} \otimes f^*\mathcal{M}$ is a very ample invertible sheaf on X relative to Z.

Proof.

(a) Let $g : X \to \mathbb{P}^r_Y, h : X \to \mathbb{P}^s_Y$ be immersions such that $\mathscr{L} \cong g^*(\mathscr{O}_{\mathbb{P}^r_Y}(1)), \mathscr{M} \cong h^*(\mathscr{O}_{\mathbb{P}^s_Y}(1))$ respectively, let $\psi : \mathbb{P}^r_Y \times \mathbb{P}^s_Y \to \mathbb{P}^N_Y$ be a Segre embedding with $N = rs + r + s$, let $\pi_r, \pi_s : \mathbb{P}^r_Y \times \mathbb{P}^s_Y \to \mathbb{P}^s_Y, \mathbb{P}^s_Y$ be the natural projection morphisms, and let $f : X \to \mathbb{P}^r_Y \times \mathbb{P}^s_Y$ be the unique morphism such that $g = \pi_r \circ f, h = \pi_s \circ f$. A Segre embedding is an immersion, and compositions of immersions are immersions. In particular, $\psi \circ f$ is an immersion, and we have the following isomorphisms

$$\begin{aligned}
\mathscr{L} \otimes \mathscr{M} &\cong g^*(\mathscr{O}_{\mathbb{P}^r_Y}(1)) \otimes h^*(\mathscr{O}_{\mathbb{P}^s_Y}(1)) \\
&\cong (p_r \circ f)^*(\mathscr{O}_{\mathbb{P}^r_Y}(1)) \otimes (p_s \circ f)^*(\mathscr{O}_{\mathbb{P}^s_Y}(1) \\
&\cong f^*(p_r^*(\mathscr{O}_{\mathbb{P}^r_Y}(1)) \otimes p_s^*(\mathscr{O}_{\mathbb{P}^s_Y}(1))) \\
&\cong f^*(\mathscr{O}_{\mathbb{P}^r_Y \times \mathbb{P}^s_Y}(1)) \\
&\cong f^*(\psi^*(\mathscr{O}_{\mathbb{P}^N_Y}(1))) \\
&\cong (\psi \circ f)^*(\mathscr{O}_{\mathbb{P}^N_Y}(1)).
\end{aligned}$$

Hence, $\mathscr{L} \otimes \mathscr{M}$ is very ample.

(b) Let $i : X \to \mathbb{P}^r_Y$ be an immersion such that $\mathscr{L} \cong i^*\mathscr{O}(1)$, let $j : Y \to \mathbb{P}^s_Z$ be an immersion such that $\mathscr{M} \cong j^*\mathscr{O}(1)$, and let $\psi : \mathbb{P}^r_{\mathbb{Z}} \times \mathbb{P}^s_{\mathbb{Z}} \to \mathbb{P}^N_Z$ be a Segre embedding, where $N = rs + r + s$. We have the following commutative diagram

$$\begin{tikzcd}
 & & \mathbb{P}^r_{\mathbb{Z}} & \\
X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^r_Y = Y \times \mathbb{P}^r_{\mathbb{Z}} \arrow[r, "j \times 1_{\mathbb{P}^r_{\mathbb{Z}}}"] \arrow[d] \arrow[ur, bend left] & \mathbb{P}^s_Z \times \mathbb{P}^r_{\mathbb{Z}} = Z \times (\mathbb{P}^r_{\mathbb{Z}} \times \mathbb{P}^s_{\mathbb{Z}}) \arrow[r, "1_Z \times \psi"] \arrow[u] \arrow[d] & \mathbb{P}^N_Z = Z \times \mathbb{P}^N_{\mathbb{Z}} \\
 & Y \arrow[r, "j"] & \mathbb{P}^s_Z &
\end{tikzcd}$$

where $p : Y \times \mathbb{P}^r_{\mathbb{Z}} \to Y, q : \mathbb{P}^s_Z \times \mathbb{P}^r_Z \to \mathbb{P}^s_Z$ are the natural projection maps. A composition of immersions is an immersion, and a product of immersions is an immersion, so $\phi = (1_Z \times \psi) \circ (j \times 1_{\mathbb{P}^r_{\mathbb{Z}}}) \circ i$ is an immersion. Hence, $\mathscr{L} \otimes f^*\mathscr{M} \cong \phi^*\mathscr{O}(1)$.

$\square$

13. Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d > 0$, let $S^{(d)}$ be the graded ring $\bigoplus_{n \geq 0} S^{(d)}_n$ where $S^{(d)}_n = S_{nd}$. Let $X = \operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)} \cong X$, and that the sheaf $\mathscr{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathscr{O}_X(d)$.

Proof. The inclusion $i : S^{(d)} \hookrightarrow S$ induces a morphism of schemes $\phi : X \to X' = \operatorname{Proj} S^{(d)}$ via $\mathfrak{p} \mapsto \mathfrak{p} \cap S^{(d)}$. If $\mathfrak{p} \in X$ so that $\mathfrak{p} \not\supseteq S_+ = \bigoplus_{n>0} S_n$, then $\mathfrak{p} \not\supseteq S^{(d)}_+$. Otherwise, $s^d \in \mathfrak{p}$ for all $s \in S_+$, which implies $S_+ \supseteq \mathfrak{p}$, a contradiction. Thus, ϕ is well-defined. Since $S^{(d)}$ is generated by $S^{(d)}_1$ as an S_0-algbera, we can cover X' by open affines $\operatorname{Spec} S^{(d)}_{(f)}$, where $f \in S^{(d)}_1 = S_d$ and $S^{(d)}_{(f)}$ consists of all degree zero elements of the form s/f^n for some $s \in S^{(d)}$. Also, $\phi^{-1}(\operatorname{Spec} S^{(d)}_{(f)}) = \operatorname{Spec} S_{(f)}$, and clearly $S_{(f)} \cong S^{(d)}_{(f)}$ as rings. Hence, ϕ is an isomorphism of schemes. Lastly, we also have an isomorphism of twisted sheaves $\mathscr{O}_{X'}(1) \cong (i_*\mathscr{O}_X)(1) = i_*(\mathscr{O}_X(d))$. $\square$

16. Let $(X, \mathscr{O}_X)$ be a ringed spaces, and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We define the *tensor algebra*, *symmetric algebra*, and *exterior algebra* of $\mathscr{F}$ by taking the sheaves associated to the presheaf, which to each open set U assigns the corresponding tensor operation applied to $\mathscr{F}(U)$ as an $\mathscr{O}_X(U)$-module. The results are $\mathscr{O}_X$-algebras, and their components in each degree are $\mathscr{O}_X$-modules.

(a) Suppose that $\mathscr{F}$ is locally free of rank n. Then $T^r(\mathscr{F}), S^r(\mathscr{F})$, and $\bigwedge^r(\mathscr{F})$ are also locally free, of ranks n^r, $\binom{n+r-1}{n-1}$, and $\binom{n}{r}$ respectively.

(b) Again let $\mathscr{F}$ be locally free of rank n. Then the multiplication map $\bigwedge^r \mathscr{F} \otimes \bigwedge^{n-r} \mathscr{F} \to \bigwedge^n \mathscr{F}$ is a perfect pairing for any r, i.e. it induces an isomorphism of $\bigwedge^r \mathscr{F}$ with $(\bigwedge^{n-r} \mathscr{F})^\vee \otimes \bigwedge^n \mathscr{F}$. As a special case, note if $\mathscr{F}$ has rank 2, then $\mathscr{F} \cong \mathscr{F}^\vee \otimes \bigwedge^2 \mathscr{F}$.

(c) Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to$ be an exact sequence of locally free sheaves. Then for any r there is a finite filtration of $S^r(\mathscr{F})$,

$$S^r(\mathscr{F}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong S^p(\mathscr{F}') \otimes S^{r-p}(\mathscr{F}'')$$

for each p.

(d) Same statement as (c), with exterior powers isntead of symmetric powers. In particular, if $\mathscr{F}', \mathscr{F}, \mathscr{F}''$ have ranks n', n, n'' respectively, there is an isomorphism $\bigwedge^n \mathscr{F} \cong \bigwedge^{n'} \mathscr{F} \otimes \bigwedge^{n''} \mathscr{F}''$.

(e) Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an $\mathscr{O}_Y$-module. Then f^* commutes with all the tensor operations on $\mathscr{F}$, i.e. $f^*(S^n(\mathscr{F})) = S^n(f^*\mathscr{F})$.

Proof.

(a) The question is local, so the statements follow from the case for rings. For example, let U be an open set in X such that $\mathscr{F}\big|_U \cong \mathscr{O}_U^{\oplus n}$. Then $T^r(\mathscr{F})\big|_U \cong T^r(\mathscr{O}_U^{\oplus n}) \cong \mathscr{O}_U^{\oplus n^r}$. We remark that locally, the symmetric algebra $S(\mathscr{F})$ is isomorphic to the polynomial ring $\mathscr{O}[T_1, \ldots, T_n]$, and $S^r(\mathscr{F})$ correspond to the homogenous elements of degree r in $\mathscr{O}[T_1, \ldots, T_n]$. Thus, $S^r(\mathscr{F})$ has rank $\binom{n+r-1}{n-1}$. Let $x_1, \ldots, x_n \in \Gamma(U, \mathscr{O}_X)$ be a $\mathscr{O}_U$-basis for $\mathscr{F}\big|_U$. Then $\bigwedge^r(\mathscr{F})$ is spanned by $x_{i_1} \wedge \cdots \wedge x_{i_r}$, where $1 \le i_1 < \cdots < i_r \le n$. Hence, $\bigwedge^r(\mathscr{F})$ has rank $\binom{n}{r}$.

□