Chapter 3, Section 7

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1. Let X be an integral projective scheme of dimension ≥ 1 over a field k, and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $H^0(X, \mathscr{L}^{-1}) = 0$. (this is an easy special case of Kodaira's vanishing theorem.)

2. Let $f : X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y° be a dualizing sheaf for Y.

(a) Show that $f^! \omega_Y^\circ$ is a dualizing sheaf for X, where f^1 is defined as in (Ex. 6.10).

(b) If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t : f_* \omega_X \to \omega_Y$.

3. Let $X = \mathbb{P}_k^n$. Show that $H^q(X, \Omega_X^p) = 0$ for $p \neq q$, k for $p = q$, $0 \leq p, q \leq n$.

4. *The Cohomology Class of a Subvariety.* Let X be a nonsingular projective variety of dimension n over an algebraically closed field k. Let Y be a nonsingular subvariety of codimension p (hence dimension $n - p$). From the natural map $\Omega_X \otimes \mathscr{O}_Y \to \Omega_Y$ of (II, 8.12) we deduce a map $\Omega_X^{n-p} \to \Omega_Y^{n-p}$. This induces a map on cohomology $H^{n-p}(X, \Omega_X^{n-p}) \to H^{n-p}(Y, \Omega_Y^{n-p})$. Now $\Omega_Y^{n-p} = \omega_Y$ is a dualizing sheaf for Y, so we have the trace map $t_Y : H^{n-p}(Y, \Omega_Y^{n-p}) \to k$. Composing, we obtain a linear map $H^{n-p}(X, \Omega_X^{n-p}) \to k$. By (7.13) this corresponds to an element $\eta(Y) \in H^p(X, \Omega_X^p)$, which we call the *cohomology class* of Y.

(a) If $P \in X$ is a closed point, show that $t_X(\eta(P)) =$, where $\eta(P) \in H^n(X, \Omega^n)$ and t_X is the trace map.

(b) If $X = \mathbb{P}^n$, identify $H^p(X, \Omega^p)$ with k by (Ex. 7.3), and show that $\eta(Y) = (\deg Y) \cdot 1$, where $\deg Y$ is its *degree* as a projective variety (I, §7).

(c) For any scheme X of finite type over k, we define a homomorphism of sheaves of Abelian groups $\operatorname{dlog} : \mathscr{O}_X^* \to \Omega_X$ by $\operatorname{dlog}(f) = f^{-1} df$. Here $\mathscr{O}^*$ is a group under multiplication, and Ω_X is a group under addition. This induces a map on cohomology $\operatorname{Pic} X = H^1(X, \mathscr{O}_X^*) \to H^1(X, \Omega_X)$ which we denote by c.

(d) Returning to the hypotheses above, suppose $p = 1$. Show that $\eta(Y) = c(\mathscr{L}(Y))$, where $\mathscr{L}(Y)$ is the invertible sheaf corresponding to the divisor Y.