Chapter 2, Section 8

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April 23, 2025

1. Here we will strengthen the results of the text to include information about the sheaf of differentials at a not necessarily closed point of a scheme X.

(a) Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then the exact sequence of (8.4A),

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \longrightarrow \Omega_{k(B)/k} \longrightarrow 0$$

is exact on the left also.

(b) Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B + \operatorname{tr.d.} k(B)/k$.

(c) Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necessarily closed, show that the local ring $\mathscr{O}_{x,X}$ is a regular local ring if and only if the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at x is free of rank n.

(d) Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \{x \in X \mid \mathscr{O}_x \text{ is regular}\}$ is an open dense subset of X.

Proof.

(a) In copying the proof of (8.7), we want to show the map

$$\delta^\vee : \operatorname{Der}_{k(B)}(B, k(B)) \to \operatorname{Hom}_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$$

of dual vector spaces is surjective. If $d : B \to k(B)$ is a derivation, then the $\delta^\vee(d)$ is obtained by restricting to $\mathfrak{m}$. This is well-defined, since $\mathfrak{m} = 0$ in $k(B)$, so $d\mathfrak{m}^2 = \mathfrak{m}d\mathfrak{m} = 0 \subset k(B)$. Now to show $\delta^\vee$ is surjective, let $h \in \operatorname{Hom}_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$. Since $B/\mathfrak{m}^2$ is a complete local ring with residue field $k(B)$, there exists a field of representatives $K \subseteq B$ for B (8.25A). Thus, for any $b \in B$, $\bar{b} \in B/\mathfrak{m}^2$, the image of b, can be written as $\bar{b} = \lambda + \bar{c}$, $\lambda \in K, \bar{c} \in \mathfrak{m}/\mathfrak{m}^2$, uniquely. Define $db = h(\bar{c})$. Let $b, b' \in B$, and write $\bar{b} = \lambda + \bar{c}, \bar{b}' = \lambda' + \bar{c}'$ for some $\lambda, \lambda' \in K, \bar{c}', \bar{c}' \in \mathfrak{m}/\mathfrak{m}^2$. Note that $\bar{b} = \lambda, \bar{b}' = \lambda'$ and $d\bar{b} = d\bar{c}, d\bar{b}' = d\bar{c}'$ in $k(B)$, $bb' = \lambda\bar{c}' + \lambda'\bar{c} \in \mathfrak{m}/\mathfrak{m}^2$. Hence, $dbb' = d(\lambda'\bar{c} + \lambda\bar{c}') = \lambda' d\bar{c} + \lambda d\bar{c}' = b'db + bdb'$, so d is a well-defined $k(B)$-derivation.

(b) Immediate by the exact sequence of (a), (8.6A), and (8.8).

(c) If $x \in X$ is any point, then the local ring $B = \mathscr{O}_{x,X}$ has dimension n, residue field some finitely generated, hence separable, extension $k(B)$ (since k is perfect), and is a localization of a k-algebra of finite type. Furthermore, the module $\Omega_{B/k}$ of differentials of B over k is equal to the stalk $(\Omega_{X/k})_x$ of the sheaf $\Omega_{X/k}$. Thus, we can apply (b) and we see that $(\Omega_{X/k})_x$ is free of rank n if and only if B is a regular local ring.

(d) Follows from (c) and (Ex. 5.7a).

□

2. Let X be a variety of dimension n over k. Let $\mathscr{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subseteq \Gamma(X, \mathscr{E})$ be a vector space of global sections which generate $\mathscr{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x \mathscr{E}_x$. Conclude that there is a morphism $\mathscr{O}_X \to \mathscr{E}$ giving rise to an exact sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}' \longrightarrow 0$$

where $\mathscr{E}'$ is also locally free.

Proof. Let m be the rank of $\mathscr{E}$, and let $r = \dim_k V$. For any closed point $x \in X$, we can define a map of k-vector spaces $\varphi_x : V \to \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ in the obvious way. It is surjective by hypothesis, and $\dim_k \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x = m$, which shows $r \geq m$. Now considering the vector space V as an affine space over k, consider the subset $B \subseteq X \times V$ consisting of all pairs (x, s) such that $x \in X$ is a closed point and $s \in \ker \varphi_x$. Clearly B is the set of closed points of a closed subset of $X \times V$, which we denote by B, and which we give a reduced induced structure. Consider the first projection $p_1 : B \to X$. It is surjective, with fiber an affine space of dimension $r - m$ (in particular, each fiber is a linear subspace of V). Hence, B is irreducible, and has dimension $r - m + n$. By hypothesis $n < m$, so $\dim B \leq r - 1$. Therefore, considering the second projection $p_2 : B \to V$, we have $\dim p_2(B) \leq r - 1$. Since $\dim V = r$, we conclude that $p_2(B) \subset V$. Pick any $s \in V - p_2(B)$, then $X \times \{s\} \subset X \times V - B$, which is what we wanted to show. For the conclusion, the morphism $\mathscr{O}_X \to \mathscr{E}$ defined by $1 \mapsto s$ gives the desired exact sequence. $\square$

6. *The Infinitesimal Lifting Property.* The following result is very important in studying deformations of nonsingular varieties. Let k be an algebraically closed field, let A be a finitely generated k-algebra such that $\operatorname{Spec} A$ is a nonsingular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \to B$. Then there exists a k-algebra homomorphism $g : A \to B'$ making a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & I \arrow[d] \\
 & B' \arrow[d] \\
A \arrow[ur, dashed, "g"] \arrow[r, "f"'] & B \arrow[d] \\
 & 0
\end{tikzcd}$$

We call this result the *Infinitesimal lifting property* for A. We prove this result in several steps.

(a) First suppose that $g : A \to B'$ is a given homomorphism lifting f. If $g' : A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, which we can consider as an element of $\operatorname{Hom}_A(\Omega_{A/k}, I)$. Note that since $I^2 = 0$, I has a natural structure of B-module and hence also of A-module. Conversely, for any $\theta \in \operatorname{Hom}_A(\Omega_{A/k}, I)$, $g' = g + \theta$ is another homomorphism lifting f. (For this step, you do not need the hypothesis about $\operatorname{Spec} A$ being nonsingular.)

(b) Now let $P = k[x_1, \ldots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exist a homomorphism $h : P \to B'$ making a commutative diagram,

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
J \arrow[d] & I \arrow[d] \\
P \arrow[r, "h"] \arrow[d] & B' \arrow[d] \\
A \arrow[r, "f"'] \arrow[d] & B \arrow[d] \\
0 & 0
\end{tikzcd}$$

and show that h induces an A-linear map $\bar{h} : J/J^2 \to I$.

(c) Now use the hypothesis $\operatorname{Spec} A$ nonsingular and (8.17) to obtain an exact sequence

$$0 \longrightarrow J/J^2 \longrightarrow \Omega_{P/k} \otimes A \longrightarrow \Omega_{A/k} \longrightarrow 0.$$

Show furthermore that applying the functor $\operatorname{Hom}_A(\cdot, I)$ gives an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(\Omega_{A/k}, I) \longrightarrow \operatorname{Hom}_P(\Omega_{P/k}, I) \longrightarrow \operatorname{Hom}_A(J/J^2, I) \longrightarrow 0.$$

Let $\theta \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\overline{h} \in \operatorname{Hom}_A(J/J^2, I)$. Consider θ as a derivation of P to B'. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that $h'(J) = 0$. Thus, h' induces the desired homomorphism $g : A \to B'$.

Proof.

(a) Let $\pi : B' \to B$ be the natural projection homomorphism. If $\pi \circ g = \pi \circ g'$, then $\pi \circ \theta = 0$. Hence, $\theta(A) \subseteq \ker \pi = I$. Let a, a' be elements of A. We have $g(a) = a, g'(a') = a' \in B$, so the natural A-module structure of I gives

$$\begin{aligned} \theta(aa') &= g(a)g(a') - g'(a)g'(a') \\ &= g(a)g(a') - g(a)g'(a') + g(a)g'(a') - g'(a)g'(a') \\ &= g(a)(g(a') - g'(a')) + g'(a')(g(a) - g'(a)) \\ &= a\theta(a') - a'\theta(a). \end{aligned}$$

Also, g, g' are k-linear, so $\theta(\lambda) = 0$ for all $\lambda \in k$. Hence, θ is a k-derivation.

In the converse direction, since $\operatorname{im}\theta \subseteq I$, $\pi \circ g = \pi \circ g'$, it is enough to check that g' is indeed a homomorphism. It is clear it is additive. For any $a, a' \in A$, we have $\theta(a)\theta(a') \in I^2 = 0$ and $g(a)\theta(a') = a\theta(a'), g(a')\theta(a) = a'\theta(a)$. It follows that

$$\begin{aligned} g'(aa') &= g(aa') + \theta(aa') \\ &= g(a)g(a') + a\theta(a') + a'\theta(a) \\ &= g(a)g(a') + a\theta(a') + a'\theta(a) + \theta(a)\theta(a') \\ &= (g(a) + \theta(a))(g(a') + \theta(a')) \\ &= g'(a)g'(a'). \end{aligned}$$

(b) Let $y_i \in B$ be the image of $x_i \in P$ for all $i = 1, \ldots, n$. Then $f(A) = k[y_i, \ldots, y_n]$, and $\pi : B' \to B$ is surjective, so there exists $z_i \in B'$ such that $\pi(z_i) = y_i$. Let $A' = k[z_1, \ldots, z_n] \subseteq B'$. We have $\pi(A') = f(A)$, There is a natural map $h : P \to A' \hookrightarrow B'$ defined by $x_i \mapsto z_i$. It satisfies the conditions by construction.

To show h induces an A-lienar map $\bar{h} : J/J^2 \to I$, we need to show $h(J) \subseteq I$ and $h(J^2) = 0$. Indeed, the diagram above commutes with h, and J gets mapped to 0 in B. Hence, $h(J) \subseteq \ker \pi = I$. Let $cc' \in J^2$ for some $c, c' \in J$. Then $h(cc') = h(c)h(c') \in I^2 = 0$ since $h(c), h(c) \in I$. Hence, we can obtain $\bar{h}$ by restricting h to J and passing to the quotient J/J^2.

(c) Let $X = \mathbb{A}^n = \operatorname{Spec} P$, and let $X = \operatorname{Spec} A$. By (8.17), (5.5), and (5.10), taking global sections of the exact sequence in (8.17) gives the desired exact sequence.

In general, $\operatorname{Hom}_A(\cdot, I)$ is only left exact. In particular, the sequence

$$0 \longrightarrow \operatorname{Hom}_A(\Omega_{A/k}, I) \longrightarrow \operatorname{Hom}_A(\Omega_{P/k} \otimes A, I) \longrightarrow \operatorname{Hom}_A(J/J^2, I) \longrightarrow 0$$

is not necessarily exact on the right. However, the middle term is isomorphic to $\operatorname{Hom}_P(\Omega_{P/k}, I)$, which by definition can be identified with $\operatorname{Der}_k(P, I)$, the set of all k-derivations of P to I. Noting that P/J^2 has dimension $1 + \dim A$ as a k-vector space by non-singularity of X and k being algebraically closed, we can uniquely write any element of P/J^2 as a sum $\lambda + c$, where $\lambda \in k, c \in J/J^2$. We conclude that the sequence above is exact on the right as well.

Imitating the proof of (a), it remains to show $h'(J) = 0$. For any θ, $\theta(J^2) = 0$ Passing to the quotient P/J^2 gives us $h'(J) = (\bar{h} - \bar{h})(J + J^2) = 0$.

□