Chapter 4, Section 1

James Lee

May 1, 2025

1. Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

Proof. Pick another closed point $Q \neq P \in X$. By (1.3.2), there exists $n > 0$ such that $\dim |n(-P+2Q)| > 0$. Hence, there exists a rational function with a pole at P of order n and regular everywhere else. □

2. Again let X be a curve, and let $P_1, \ldots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i, and regular elsewhere.

Proof. Imitate the proof of the previous exercise for the divisor $D = -\sum_{i=1}^{r} P_i + (r+1)Q$. This time, such a Q always exists because k is algebraically, so it is infinite. □

3. Let X be an integral, separated, regular, one-dimensional scheme of finite type over k, which is *not* proper over k. Then X is affine.

4. Show that a separated, one-dimensional scheme of finite type over k, none of whose irreducible components is proper over k, is affine.

5. For an effective divisor D on a curve X of genus g, show that $\dim |D| \leq \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

Proof. By definition $\dim |D| = \ell(D) - 1$. Rearranging the Riemann-Roch Theorem gives

$$\dim |D| = \ell(K - D) + \deg D - g,$$

so we want to show $\ell(K-D) \leq g$. But D is effective, so $\mathscr{L}(K-D) \to \mathscr{L}(K)$ is injective, and $g = \ell(K) = \dim H^0(X, \mathscr{L}(K))$ by definition. □

6. Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leq g+1$.

Proof. □

7. A curve X is called *hyperelliptic* if $g \geq 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

(a) If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. Use (II, 7.8.1) to conclude that X is hyperelliptic.

(b) Show that the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbb{P}^1$. Thus, there exist hyperelliptic curves of any genus $g \geq 2$.

Proof.

(a) In general, $|K|$ has no base points for $g \geq 2$ (5.1). If $g = 2$, then $\dim |K| = g - 1 = 1$ and $\deg K = 2g - 2 = 2$. Thus, $|K|$ defines a finite morphism $f : X \to \mathbb{P}^1$ of degree 2 by (II, 7.8.1).

(b)

□

8. *p_a of a Singular Curve.* Let X be an integral projective scheme of dimension 1 over k, and let $\widetilde{X}$ be its normalization (II, Ex. 3.8). Then there is an exact sequence of sheaves on X,

$$0 \longrightarrow \mathscr{O}_X \longrightarrow f_*\mathscr{O}_{\widetilde{X}} \longrightarrow \sum_{P\in X} \widetilde{\mathscr{O}}_P/\mathscr{O}_P \longrightarrow 0$$

where $\widetilde{\mathscr{O}}_P$ is the integral closure of $\mathscr{O}_P$. For each $P \in X$, let $\delta_P = \operatorname{length}(\widetilde{\mathscr{O}}_P/\mathscr{O}_P)$.

(a) Show that $p_a(X) = p_a(\widetilde{X}) + \sum_{P\in X} \delta_P$.

(b) If $p_a(X) = 0$, show that X is already nonsingular and in fact isomorphic to $\mathbb{P}^1$.

(c) If P is a node or an ordinary cusp (I, Ex. 5.6, Ex. 5.14), show that $\delta_P = 1$.

9. *Riemann-Roch for Singular Curves.* Let X be an integral projective scheme of dimension 1 over k. Let X_{reg} be the set of regular points of X.

(a) Let $D = \sum n_i P_i$ be a divisor with support in X_{reg}, i.e., all $P_i \in X_{\mathrm{reg}}$. Then define $\deg D = \sum n_i$. Let $\mathscr{L}(D)$ be the associated invertible sheaf on X, and show that

$$\chi(\mathscr{L}(D)) = \deg D + 1 - p_a.$$

(b) Show that any Cartier divisor on X is the difference of two very ample Cartier divisors.

(c) Conclude that every invertible sheaf $\mathscr{L}$ on X is isomorphic to $\mathscr{L}(D)$ for some divisor D with support in X_{reg}.

(d) Assume Furthermore that X is locally complete intersection in some projective space. Then by (III, 7.11) the dualizing sheaf ω_X° is an invertible sheaf on X, so we can define the *canonical divisor* K to be a divisor with support in X_{reg} corresponding to ω_X°. Then the formula of (a) becomes

$$\ell(D) = \ell(K - D) = \deg D + 1 - p_a.$$

(e) Let X be an integral projective scheme of dimension 1 over k, which is locally complete intersection, and has $p_a = 1$. Fix a point $P_0 \in X_{\mathrm{reg}}$. Imitate (1.3.7) to show that the map $P \to \mathscr{L}(P - P_0)$ gives a one-to-cone correspondence between the pints of X_{reg} and the elements of the group $\operatorname{Pic}^\circ X$. This generalizes (II, 6.11.4) and (II, Ex. 6.7).