

Chapter 3, Section 4

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1. Let $f : X \rightarrow Y$ be an affine morphism of Noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf \mathcal{F} on X , there are natural isomorphisms for all $i \geq 0$,

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F}).$$

Proof. Ahhh!

□

2. Prove Chevalley's theorem: Let $f : X \rightarrow Y$ be a finite surjective morphism of Noetherian separated schemes, with X affine. Then Y is affine.
 - (a) Let $f : X \rightarrow Y$ be a finite surjective morphism of integral Noetherian schemes. Show that there is a coherent sheaf \mathcal{M} on X , and a morphism of sheaves $\alpha : \mathcal{O}_Y^r \rightarrow f_*\mathcal{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y .
 - (b) For any coherent sheaf \mathcal{F} on Y , show that there is a coherent sheaf \mathcal{G} on X , and a morphism $\beta : f_*\mathcal{G} \rightarrow \mathcal{F}^r$ which is an isomorphism at the generic point of Y .
 - (c) Now prove Chevalley's theorem.

Proof.

(a)

□

3. Let $X = \mathbb{A}_k^2 = \text{Spec } k[x, y]$, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k -vector space spanned by $\{x^i y^j \mid i, j < 0\}$. In particular, it is infinite-dimensional.
4. On an arbitrary topological space X with an arbitrary Abelian sheaf \mathcal{F} , Čech cohomology may not give the same result as the derived functor cohomology. But here we show that H^1 , there is an isomorphism if one takes the limit over all coverings.
 - (a) Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of the topological space X . A *refinement* of \mathfrak{U} is a covering $\mathfrak{B} = (V_j)_{j \in J}$, together with a map $\lambda : J \rightarrow I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If \mathfrak{B} is a refinement of \mathfrak{U} , show that there is a natural induced map on Čech cohomology for any Abelian sheaf \mathcal{F} , and for each i ,

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathcal{F}) \rightarrow \check{H}^i(\mathfrak{B}, \mathcal{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathcal{F}).$$

- (b) For any Abelian sheaf \mathcal{F} on X , show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U}, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$$

are compatible with the refinement maps above.

- (c) Now prove the following theorem. Let X be a topological space, \mathcal{F} a sheaf of Abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \tilde{H}^1(\mathfrak{U}, \mathcal{F}) \rightarrow H^1(X, \mathcal{F})$$

is an isomorphism.

5. For any ringed space (X, \mathcal{O}_X) , let $\text{Pic } X$ be the group of isomorphism classes of invertible sheaves (II, §6). Show that $\text{Pic } X \cong H^1(X, \mathcal{O}_X^*)$ where \mathcal{O}_X^* denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$, with multiplication as the group operation.