Chapter 1, Section 7

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1. (a) Find the degree of the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$.

(b) Find the degree of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$.

Proof.

(a) The homogenous coordinate ring of the d-uple embedding is the $k[y_0, \dots, y_n]$ where each y_i has degree d as a graded $k[x_0, \dots, x_n]$-module. Thus, the Hilbert polynomial of the d-uple embedding is

$$P(z) = \binom{n+dz}{n} = \frac{(dz+n)(dz+n-1)\cdots(dz)}{n!} = \frac{d^n}{n!}z^n + \cdots,$$

hence the degree is d^n.

(b) We have seen in Exercise 2.14 that the homogenous coordinate ring of the Segre embedding is isomorphic to $k[\{x_i y_j\}]$, which is a subring of $k[x_0, \dots, x_r, y_0, \dots, y_s]$. The grading is given by $\bigoplus_{k=0}^{\infty} M_k$ where M_k is the set of polynomials of degree $2k$ in $k[\{x_i y_i\}]$. Since each monomial is made up of half x_i's and half y_j's, the Hilbert polynomial is

$$P(z) = \binom{r+z}{r}\binom{s+z}{s} = \frac{1}{r!s!}z^{r+s} + \cdots,$$

hence the degree is $\binom{r+s}{r}$.

□

3. *Dual Curve.* Let $Y \subseteq \mathbb{P}^2$ be a curve. We regard the set of lines in $\mathbb{P}^2$ as another projective space, $(\mathbb{P}^2)^*$, by taking (a_0, a_1, a_2) as homogenous coordinates of the line $L : a_0x_0 + a_1x_1 + a_2x_2 = 0$. For each nonsingular point $P \in Y$, show that there is a unique line $T_P(Y)$ whose intersection multiplicity with Y at P is > 1. This is the *tangent line* to Y at P. Show that the mapping $P \mapsto T_P(Y)$ defines a *morphism* of Reg Y (the set of nonsingular points of Y) into $(\mathbb{P}^2)^*$. The closure of the image of this morphism is called the dual curve $Y^* \subseteq (\mathbb{P}^2)^*$ of Y.

Proof. Let x_0, x_1, x_2 be homogenous coordinates for $\mathbb{P}^2$, and let f be an irreducible homogenous polynomial in $S = k[x_0, x_1, x_2]$ that defines Y and let $P = (c_0, c_1, c_2)$ be a nonsingular point on Y. Then $\mathbb{P}^2$ can be covered by U_i where $U_i \cong \mathbb{A}^2$ and is defined by $x_i \neq 0$. Since every P is contained in some U_i, we find an affine line in U_i that is tangent to the affine variety $Y_i := Y \cap U_i$. We claim that the projective closure of this line is the desired unique line $T_P(Y)$.

Without loss of generality, assume $c_0 = 1$ so that $P \in U_0$. Identifying U_0 with $\mathbb{A}^2_k$ with coordinates y_1, y_2, we have $P_0 = (c_1, c_2) \in \mathbb{A}^2_k$ identified with P, and Y_0 is defined by the vanishing of $g(y_1, y_2) = f(1, y_1, y_2)$. Assume $c_1 = c_2 = 0$ for simplicity. Let $\mathfrak{m}_{P_0}$ be the maximal ideal in $S(U_0) \simeq k[y_1, y_2]$ corresponding to the point P_0. Since Y is nonsingular at P, Y_0 is nonsingular at P_0, which means $a_i = (\partial g/\partial y_i)(P_0)$ for $i = 1, 2$ are not both zero. In particular, we have $g \notin \mathfrak{m}_{P_0}^2$, so we can write

$$g = a_1y_1 + a_2y_2 + (\text{higher degree terms}),$$

thus any linear polynomial in $\mathfrak{m}_P^2 + (g)$ must be a constant multiple of $t = a_1y_1 + a_2y_2 = 0$. We define $T_P(Y)$ to be the projective closure of t. We show it is the desired unique line by showing if a line $H : h(x_0, x_1, x_2) = b_0x_0 + b_1x_1 + b_2x_2 = 0$ has intersection multiplicity with Y at $P > 1$, then $H \cap U_0 = T$, which implies $T_P(Y) = H$

since $H \cap U_0 = T$ is dense in both H and T. Setting $S(Y) = S/(f)$ and $\mathfrak{m}_P$ as the homogenous prime ideal in S corresponding to the point $P \in \mathbb{P}^2$, we have $S/(f,h) = S(Y)/(h)$, thus

$$\begin{aligned} i(Y,H;P) > 1 \quad &\iff \quad (S(Y)/(h))_{\mathfrak{m}_P} \text{ has length } k > 1 \text{ as an } S(Y)_{\mathfrak{m}_P}\text{-module} \\ &\iff \quad \mathfrak{m}_P/(h) \text{ has length } k \text{ as an } S(Y)\text{-module} \\ &\iff \quad (h) = \mathfrak{m}_P^k \text{ as an extended ideal in } S(Y)_{\mathfrak{m}_P} \end{aligned}$$

since $S(Y)_{\mathfrak{m}_P}$ is a discrete valuation ring by non-singularity of Y at P. In terms of affine coordinates, let $\ell = b_0 + b_1 y_1 + b_2 y_2$ so that $H \cap U_0$ can be identified with the affine variety defined by the vanishing of ℓ, where $b_0 = 0$ necessarily since $P_0 = (0,0)$ is a root of ℓ. Then, by the statements above, we have $\ell \in \mathfrak{m}_{P_0}^k + (g) \subseteq \mathfrak{m}_{P_0}^2 + (g)$, which is possible if and only if $\ell = \lambda t$ for some $\lambda \neq 0$, hence $H \cap U_0 = T$. $\square$

4. Given a curve Y of degree d in $\mathbb{P}^2$, show that there is a nonempty open subset U of $(\mathbb{P}^2)^*$ in its Zariski topology such that for each $L \in U$, L meets Y in exactly d points. This result shows that we could have defined the degree of Y to be the number d such that almost all lines in $\mathbb{P}^2$ meet Y in d points, where "almost all" refers to a nonempty open set of the set of lines, when this set is identified with the dual projective space $(\mathbb{P}^2)^*$.

Proof. Following the hint, we show that the set of lines in $(\mathbb{P}^2)^*$ which are either tangent to Y or pass through a singular point of Y is contained in a property closed subset. By the previous exercise, a line can meet Y at exactly d points if and only if it has intersection multiplicity equal to 1 at every point of intersection, thus the set of lines which are either tangent to Y or pass through a singular point of Y is contained in the dual curve Y^* in $(\mathbb{P}^2)^*$, hence the set of lines that meet Y at exactly d points is contained in the open subset $(\mathbb{P}^2)^* - Y^*$. $\square$