Chapter 3, Section 2

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4. *Mayer-Vietoris Sequence.* Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\cdots \longrightarrow H^i_{Y_1\cap Y_2}(X,\mathscr{F}) \longrightarrow H^i_{Y_1}(X,\mathscr{F}) \oplus H^i_{Y_2}(X,\mathscr{F}) \longrightarrow H^i_{Y_1\cup Y_2}(X,\mathscr{F}) \longrightarrow$$

$$\longrightarrow H^{i+1}_{Y_1\cap Y_2}(X,\mathscr{F}) \longrightarrow \cdots.$$

Proof. There is an exact sequence of sheaves

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{H}^0_{Y_1\cap Y_2}(\mathscr{F}) \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}_{X-Y_1\cap Y_2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{H}^0_{Y_1}(\mathscr{F}) \oplus \mathscr{H}^0_{Y_2}(\mathscr{F}) \arrow[r] \arrow[d] & \mathscr{F} \oplus \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}_{X-Y_1} \oplus \mathscr{F}_{X-Y_2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{H}^0_{Y_1\cup Y_2}(\mathscr{F}) \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}_{X-Y_1\cup Y_2} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

which induces the desired long sequence of cohomology with supports by (1.1A). $\square$

6. Let X be a Noetherian topological space, and let $\{\mathscr{I}_\alpha\}_{\alpha\in A}$ be a direct system of injective sheaves of Abelian groups on X. Then $\varinjlim \mathscr{I}_\alpha$ is also injective.

Proof. We follow the hint. One direction is clear. Conversely, let $i : \mathscr{N} \to \mathscr{M}$ be an injective morphism of sheaves. By the proof of (2.7) we can write $\mathscr{N} = \varinjlim \mathscr{N}_\beta$ where $\mathscr{N}_\beta$ is generated by the sections on some open U_β, and similarly for $\mathscr{M} = \varinjlim \mathscr{M}_\beta$. Notice that we can assume $\mathscr{N}$ and $\mathscr{M}$ are defined over the same direct system so that they belong to the same Abelian category. Thus, the inclusion map $i : \mathscr{N} \to \mathscr{M}$ can be broken down into inclusion maps $i_\beta : \mathscr{N}_\beta \to \mathscr{M}_\beta$. A direct system of morphisms $\mathscr{N}_\beta \to \mathscr{M}$ induces the same inclusion morphism $\mathscr{N} = \varinjlim \mathscr{N}_\beta \to \mathscr{M}$, so we reduce to the case when $\mathscr{N}$ and $\mathscr{M}$ are generated by a single section over some open set U. We have an exact sequence

$$\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[d] & & & \\
 & \mathscr{K} \arrow[d] \arrow[dr] & & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{R} \arrow[r] & \mathbb{Z}_U \arrow[r] \arrow[dr] & \mathscr{N} \arrow[r] \arrow[d] & 0 \\
 & & & \mathscr{M} \arrow[dr] & \\
 & & & & 0
\end{tikzcd}$$

where all the maps are natural, and $\mathscr{R}, \mathscr{K}$ are kernels of the quotients $\mathscr{N}, \mathscr{M}$, respectively. It is not hard to see from above that any $f : \mathscr{N} \to \mathscr{I}$ naturally extends to $\mathscr{M}$, which is what we wanted to show.

Next, we show any subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$ such that $\mathbb{Z}_U/\mathscr{R}$ is generated by a single section must be finitely generated. Indeed, fix some $x \in X$. Following the proof of (2.7), there exists some open neighborhood $x \in V \subseteq U$ such that $\mathscr{R}\big|_V \cong d \cdot \mathbb{Z}\big|_V$ for some positive integer d. Since X is noetherian, we can cover U by finite number of such V, say V_i for $i = 1, \dots, n$. Therefore, there is an exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \bigoplus_{i=1}^n d_i \cdot \mathbb{Z}_{V_i} \longrightarrow \bigoplus_{i,j,k} d_{ijk} \cdot \mathbb{Z}_{V_i \cap V_j \cap V_k}$$

where d_{ijk} is the minimum of d_i, d_j, d_k. The terms on the right are finitely generated. Thus, $\mathscr{R}$ is finitely generated, and any $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ must factor through one of the $\mathscr{I}_\alpha$ (each generator s_i of $\mathscr{R}$ factors through one of the $\mathscr{I}_{\alpha_i}$, so take any $\beta > \alpha_i$, which exists by definition of a direct system). $\square$

7. Let S^1 be the circle (with its usual topology), and let $\mathbb{Z}$ be the constant sheaf $\mathbb{Z}$.

(a) Show that $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$, using our definition of cohomology.

(b) Now let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1, \mathscr{R}) = 0$.

Proof.

(a) We remark that cohomology commutes with colimits on paracompact Hausdorff spaces. In particular, the statements of (II, Ex.1.11), (2.9) hold for S^1. Let A, B be closed subsets of S^1 homeomorphic to the unit interval such that $A \cup B = S^1$ and $A \cap B = \{P, Q\}$ for two distinct points P, Q in S^1 (in the obvious way...). From now on, for any closed subset C of S^1, denote $\mathbb{Z}_C = i_* \mathbb{Z}$, where $i : C \hookrightarrow S^1$ is the inclusion map and $\mathbb{Z}$ is the constant sheaf on C. Without ambiguity $\mathbb{Z}$ will denote the constant sheaf on the ambient space. We claim the following sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}_A \oplus \mathbb{Z}_B \xrightarrow{\tau} \mathbb{Z}_{A \cap B} \longrightarrow 0$$

defined by $\Delta(a) = (a, a)$ and $\tau(a, b) = b - a$ is exact. In particular, there exists natural maps $i_A^\#, i_B^\# : \mathbb{Z} \to \mathbb{Z}_A, \mathbb{Z}_B$ associated to the inclusion maps $i_A, i_B : A, B \hookrightarrow S^1$, so that $\Delta = (i_A^\#, i_B^\#)$. In the same way, the associated morphism of sheaves of the inclusion maps $j_A, j_B : A \cap B \to A, B$ ascends to naturally defined maps $j_A^\#, j_B^\# : \mathbb{Z}_A, \mathbb{Z}_B \to \mathbb{Z}_{A \cap B}$ in the form of a restriction morphism. Thus, $\tau = j_B^\# - j_A^\#$. Exactness can be checked at the level of stalks. Suppose $R \notin A \cap B$. Then either $R \in A$ or $R \in B$, say $R \in A$. Then the stalks are $(\mathbb{Z})_R = \mathbb{Z}$, $(\mathbb{Z}_A)_R = \mathbb{Z}$, $(\mathbb{Z}_B)_R = 0$, which is an exact sequence. If $R \in A \cap B$, then the stalks are $(\mathbb{Z})_R = \mathbb{Z}$, $(\mathbb{Z}_A)_R = \mathbb{Z}$, $(\mathbb{Z}_B)_R = \mathbb{Z}$, $\mathbb{Z}_{A \cap B} = \mathbb{Z}$ defined by Δ and τ, which is clearly exact. Hence, the sequence is exact at all points, so the sequence is exact.
Taking cohomology, we get a long exact sequence of cohomology groups

$$0 \longrightarrow H^0(S^1, \mathbb{Z}) \longrightarrow H^0(S^1, \mathbb{Z}_A) \oplus H^0(S^1, \mathbb{Z}_B) \xrightarrow{\tau_0} H^0(S^1, \mathbb{Z}_{A \cap B}) \longrightarrow$$

$$\longrightarrow H^1(S^1, \mathbb{Z}) \longrightarrow H^1(S^1, \mathbb{Z}_A) \oplus H^1(S^1, \mathbb{Z}_B) \longrightarrow H^1(S^1, \mathbb{Z}_{A \cap B}) \longrightarrow \cdots$$

where $H^i(\mathbb{Z}_A \oplus \mathbb{Z}_B) \cong H^i(\mathbb{Z}_A) \oplus H^i(\mathbb{Z}_B)$ by (2.9). By (2.10), we have

$$\begin{aligned} H^0(S^1, \mathbb{Z}), H^0(S^1, \mathbb{Z}_A), H^0(S^1, \mathbb{Z}_B) &= \mathbb{Z}, \\ H^0(\mathbb{Z}_{A \cap B}) &= \mathbb{Z} \oplus \mathbb{Z}, \\ H^1(\mathbb{Z}_{A \cap B}) &= 0 \end{aligned}$$

The first line follows from the fact that A, B, S^1 are all connected and locally connected. The intersection $A \cap B$ is a noetherian space of dimension zero with two irreducible components, namely the points P and Q, so its space of global sections is free of rank two. Lastly, $H^1(S^1, \mathbb{Z}_{A \cap B}) = H^1(A \cap B, \mathbb{Z}) = 0$ by (2.7). By exactness, we reduce to the following exact sequence

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/\operatorname{im} \tau_0 \longrightarrow H^1(S^1, \mathbb{Z}) \longrightarrow H^1(S^1, \mathbb{Z}_A) \oplus H^1(S^1, \mathbb{Z}_B).$$

The homomorphism τ_0 is defined by $\tau_0(a, b) = (b - a, b - a)$, which is the diagonal map. Thus, the term on the left is free of rank one. It remains to show $H^1(S^1, \mathbb{Z}_A) = H^1(S^1, \mathbb{Z}_B) = 0$. By (2.10), it suffices to show $H^1(A, \mathbb{Z}) = 0$.

From here, $\mathbb{Z}$ will denote the constant sheaf on A. Identifying A with the closed unit interval $[0,1]$, we repeat the procedure above for A. Pick any $t \in (0,1)$, say $t = 2^{-1}$. Then $X = [0,t]$ and $Y = [t,1]$ cover A, so taking cohomology groups, we get a long exact sequence

$$0 \longrightarrow H^0(A,\mathbb{Z}) \longrightarrow H^0(A,\mathbb{Z}_X) \oplus H^0(A,\mathbb{Z}_Y) \longrightarrow H^0(A,\mathbb{Z}_{X\cap Y}) \longrightarrow$$

$$\longrightarrow H^1(A,\mathbb{Z}) \longrightarrow H^1(A,\mathbb{Z}_X) \oplus H^1(A,\mathbb{Z}_Y) \longrightarrow H^1(A,\mathbb{Z}_{X\cap Y}) \longrightarrow \cdots$$

Imitating the previous calculation, the first row is exact and $H^1(A,\mathbb{Z}_{X\cap Y}) = 0$ by (2.7) and (2.10). Thus, we reduce to the following exact sequence

$$0 \longrightarrow H^1(A,\mathbb{Z}) \longrightarrow H^1(A,\mathbb{Z}_X) \oplus H^1(A,\mathbb{Z}_Y) \longrightarrow 0.$$

Since $X \cong A, B$, $H^1(A,\mathbb{Z}_X), H^1(A,\mathbb{Z}_Y) \cong H^1(A,\mathbb{Z})$ by (2.10), which is possible if and only if $H^1(A,\mathbb{Z}) = 0$.

(b) Let $\mathscr{M}$ be the sheaf of germs of measurable real-valued functions on S^1 modulo equivalence almost everywhere. It is clearly flasque, since for any measurable $f : V \to \mathbb{R}$ where $V \subseteq U \subseteq \mathbb{R}$ are open sets, the extension of f by zero on U is a measurable function. Thus, we have an exact sequence of sheaves

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}/\mathscr{R} \longrightarrow 0.$$

Any measurable function on S^1 is continuous except for on a set of measure zero in S^1, so any global section of $\mathscr{M}/\mathscr{R}$ is locally zero almost everywhere. Since we are considering equivalence classes of functions, where two functions are considered equal if they are equal except for on a measure zero set, we conclude that $\mathscr{M}/\mathscr{R}$ has no non-zero global section. Taking cohomology groups, we have $H^1(S^1,\mathscr{R}) = 0$.

$\square$