Chapter 2, Section 9

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April 23, 2025

1. Let X be a Noetherian scheme, Y a closed subscheme, and $\hat{X}$ the completion of X along Y. We call the ring $\Gamma(\hat{X}, \mathscr{O}_{\hat{X}})$ the ring of *formal-regular functions* on X along Y. In this exercise we show that if Y is a connected, nonsingular, positive dimensional subvariety of $X = \mathbb{P}^n_k$ over an algebraically closed field k, then $\Gamma(\hat{X}, \mathscr{O}_{\hat{X}}) = k$.

(a) Let $\mathscr{J}$ be the ideal sheaf of Y. Use (8.13) and (8.17) to show that here is an inclusion of sheaves on Y, $\mathscr{J}/\mathscr{J}^2 \hookrightarrow \mathscr{O}_Y(-1)^{n+1}$.

(b) Show that for any $r \geq 1$, $\Gamma(Y, \mathscr{J}^r/\mathscr{J}^{r+1}) = 0$.

(c) Use the exact sequences
$$0 \to \mathscr{J}^r/\mathscr{J}^{r+1} \to \mathscr{O}_X/\mathscr{J}^{r+1} \to \mathscr{O}_X/\mathscr{J}^r \to 0$$
and induction on r to show that $\Gamma(Y, \mathscr{O}_X/\mathscr{J}^r) = k$ for all $r \geq 1$.

(d) Conclude that $\Gamma(\hat{X}, \mathscr{O}_{\hat{X}}) = k$.

2. Use the result of (Ex. 9.1) to prove the following geometric result. Let $Y \subseteq X = \mathbb{P}^n_k$ be as above, and let $f : X \to Z$ be a morphism of k-varieties. Suppose that $f(Y)$ is a single closed point $P \in Z$. Then $f(X) = P$ also.

3. Prove the analogue of (5.6) for formal schemes, which says, if $\mathfrak{X}$ is an affine formal scheme, and if
$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$
is an exact sequence of $\mathscr{O}_{\mathfrak{X}}$-modules, and if $\mathfrak{F}'$ is coherent, then the sequence of global sections
$$0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$$
is exact. For the proof, proceed in the following steps.

(a) Let $\mathfrak{J}$ be an ideal of definition for $\mathfrak{X}$, and for each $n > 0$ consider the exact sequence
$$0 \to \mathfrak{F}'/\mathfrak{J}^n\mathfrak{F}' \to \mathfrak{F}/\mathfrak{J}^n\mathfrak{F} \to \mathfrak{F}'' \to 0.$$
Use (5.6), slightly modified, to show that for every open affine subset $\mathfrak{U} \subseteq \mathfrak{X}$, the sequence
$$0 \to \Gamma(\mathfrak{U}, \mathfrak{F}'/\mathfrak{J}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}/\mathfrak{J}^n\mathfrak{J}) \to \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0$$
is exact.

(b) Now pass to the limit, using (9.1), (9.2), and (9.6). Conclude that $\mathfrak{F} \cong \varprojlim \mathfrak{F}/\mathfrak{J}\mathscr{F}$ and that the sequence of global sections above is exact.

4. Use (Ex. 9.3) to prove that if
$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$
is an exact sequence of $\mathscr{O}_{\mathfrak{X}}$-modules on a Noetherian formal scheme $\mathscr{X}$, and if $\mathfrak{F}', \mathfrak{F}''$ are coherent, then $\mathscr{F}$ is coherent.

5. If $\mathscr{F}$ is a coherent sheaf on a Noetherian formal scheme $\mathscr{X}$, which can be generated by global sections, show in fact that it can be generated by a finite number of its global sections.

6. Let $\mathfrak{X}$ be a Noetherian formal scheme, let $\mathfrak{J}$ be an ideal of definition, and for each n, let Y_n be the scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathfrak{J}^n)$. Assume that the inverse system of groups $(\Gamma(Y_n, \mathscr{O}_{Y_n}))$ satisfies the Mittag-Leffler condition. Then prove that $\operatorname{Pic}\mathfrak{X} = \varprojlim \operatorname{Pic} Y_n$. As in the case of a scheme, we define $\operatorname{Pic}\mathfrak{X}$ to be the group of locally free $\mathscr{O}_{\mathfrak{X}}$-modules of rank 1 under the operation $\otimes$. Proceed in the following steps.

(a) Use the fact that $\ker(\Gamma(Y_{n+1}, \mathscr{O}_{Y_{n+1}})) \to \Gamma(Y_n, \mathscr{O}_{Y_n})$ is a nilpotent ideal to show that the inverse system $(\Gamma(Y_n, \mathscr{O}_{Y_n}^*))$ of units in the respective rings also satisfies (ML).

(b) Let $\mathfrak{F}$ be a coherent sheaf of $\mathscr{O}_{\mathfrak{X}}$-modules, and assume that for each n, there is some isomorphism $\varphi_n : \mathfrak{F}/\mathfrak{I}^n\mathfrak{F} \cong \mathscr{O}_{Y_n}$. Be careful, because the φ_n may not be compatible with the maps in the two inverse systems $(\mathfrak{F}/\mathfrak{I}^n\mathfrak{F})$ and $(\mathscr{O}_{Y_n})$! Then show that there is an isomorphism $\mathfrak{F} \cong \mathscr{O}_{\mathfrak{X}}$. Conclude that the natural map $\operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is injective.

(c) Given an invertible sheaf $\mathscr{L}_n$ on Y_n for each n, and given isomorphism $\mathscr{L}_{n+1} \otimes \mathscr{O}_{Y_n} \cong \mathscr{L}_n$, construct maps $\mathscr{L}_{n'} \to \mathscr{L}_n$ for each $n' \geq n$ so as to make an inverse system, and show that $\mathfrak{L} = \varprojlim \mathscr{L}_n$ is a coherent sheaf on $\mathfrak{X}$. Then show that $\mathfrak{L}$ is locally free of rank 1, and thus conclude that the map $\operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is surjective. Again be careful, because even though each $\mathscr{L}_n$ is locally free of rank 1, the open sets needed to make them free might get smaller and smaller with n.

(d) Show that the hypothesis "$(\Gamma(Y_n, \mathscr{O}_{Y_n}))$ satisfies (ML)" is satisfied if either $\mathfrak{X}$ is affine, or each Y_n is projective over a field k.