Chapter 3, Section 12

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1. Let Y be a scheme of finite type over an algebraically closed field k. Show that the function

$$\varphi(y) = \dim_k(\mathfrak{m}_y/\mathfrak{m}_y^2)$$

is upper semicontinuous on the set of closed points of Y.

2. Let $\{X_t\}$ be a family of hypersurfaces of the same degree in $\mathbb{P}^n_k$. Show that for each i, the function $h^i(X_t, \mathscr{O}_{X_t})$ is a constant function of t.

3. Let $X_1 \subseteq \mathbb{P}^4_k$ be the *rational normal quartic curve* (which is the 4-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^4$). Let $X_0 \subseteq \mathbb{P}^3_k$ be a nonsingular rational quartic curve, such as the one in (I, Ex. 3.18b). Use (9.8.3) to construct a flat family $\{X_t\}$ of curves in $\mathbb{P}^4$, parameterized by $T = \mathbb{A}^1$, with the given fibers X_1 and X_0 for $t = 1$ and $t = 0$.

Let $\mathscr{I} \subseteq \mathscr{O}_{\mathbb{P}^4 \times T}$ be the ideal sheaf of the total family $X \subseteq \mathbb{P}^4 \times T$. Show that $\mathscr{I}$ is flat over T. Then show that

$$h^0(t, \mathscr{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

and also

$$h^1(t, \mathscr{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0. \end{cases}$$

This gives another example of cohomology groups jumping at a special point.

4. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $f : X \to Y$ be a flat projective morphism whose fibers are all integral schemes. Let $\mathscr{L}, \mathscr{M}$ be invertible sheaves on X, and assume for each $y \in Y$ that $\mathscr{L}_y \cong \mathscr{M}_y$ on the fiber X_y. Then show that there is an invertible sheaf $\mathscr{N}$ on Y such that $\mathscr{L} \cong \mathscr{M} \otimes f^*\mathscr{N}$.

5. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $\mathscr{E}$ be a locally free sheaf on Y, and let $X = \mathbb{P}(\mathscr{E})$. Then show that $\operatorname{Pic} X \cong (\operatorname{Pic} Y) \times \mathbb{Z}$. This strengthens (II, Ex. 7.9).

6. Let X be an integral projective scheme over an algebraically closed field k, and assume that $H^1(X, \mathscr{O}_X) = 0$. Let T be a connected scheme of finite type over k.

(a) If $\mathscr{L}$ is an invertible sheaf on $X \times T$, show that the invertible sheaves $\mathscr{L}_t$ on $X = X \times \{t\}$ are isomorphic, for all closed points $t \in T$.

(b) Show that $\operatorname{Pic}(X \times T) = \operatorname{Pic} X \times \operatorname{Pic} T$.