Chapter 3, Section 4

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1. Let $f : X \to Y$ be an affine morphism of Noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms for all $i \geq 0$,

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F}).$$

Proof. Let $\mathfrak{V}$ be an open affine cover of Y so that $\mathfrak{U} = (f^{-1}(V))_{V \in \mathfrak{V}}$ is an open affine cover of X. We can compute $H^i(X, \mathcal{F})$ using the Čech complex defined by the open covering $\mathfrak{U}$. Also, $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module by (II, 5.8), so the cohomology of $f_*\mathcal{F}$ can be computed via the Čech complex defined by $\mathfrak{V}$ (4.5). Lastly, $f_*\mathcal{F}(V) = \mathcal{F}(U)$ for all $V \in \mathfrak{V}$ and corresponding $U = f^{-1}(V) \in \mathfrak{U}$, so the Čech cohomology of $\mathcal{F}$ and $f_*\mathcal{F}$ with respect to $\mathfrak{U}$ and $\mathfrak{V}$, respectively, are isomorphic. Hence, $H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F})$ for all $i \geq 0$ by (4.5). □

2. Prove Chevalley's theorem: Let $f : X \to Y$ be a finite surjective morphism of Noetherian separated schemes, with X affine. Then Y is affine.

(a) Let $f : X \to Y$ be a finite surjective morphism of integral Noetherian schemes. Show that there is a coherent sheaf $\mathcal{M}$ on X, and a morphism of sheaves $\alpha : \mathcal{O}_Y^r \to f_*\mathcal{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

(b) For any coherent sheaf $\mathcal{F}$ on Y, show that there is a coherent sheaf $\mathcal{G}$ on X, and a morphism $\beta : f_*\mathcal{G} \to \mathcal{F}^r$ which is an isomorphism at the generic point of Y.

(c) Now prove Chevalley's theorem.

Proof.

(a) Let $\mathcal{K}$ and $\mathcal{L}$ be the sheaf of total quotient rings of Y and X, respectively. By hypothesis, the function field of X is a finite field extension of that of Y, so there exists an isomorphism $\phi : \mathcal{K}^r \to f_*\mathcal{L}$ for some $r > 0$. Let $\alpha : \mathcal{O}_Y^r \to f_*\mathcal{L}$ be the natural map $\mathcal{O}_Y^r \to \mathcal{K}^r$ composed with ϕ. Then α is an ismorphism at the generic point of Y. While $f_*\mathcal{L}$ is not a coherent $\mathcal{O}_X$-module in general, we can replace $\mathcal{L}$ by the sub-$\mathcal{O}_X$-module of $\mathcal{L}$ generated by the images of the basis elements of $\mathcal{K}^r$ (ϕ induces a map $f^*\mathcal{K}^r \to \mathcal{L}$) to obtain the desired $\mathcal{M}$.

(b) Applying $\mathcal{H}om(\cdot, \mathcal{F})$ to α gives $\beta : \mathcal{H}om(f_*\mathcal{M}, \mathcal{F}) \to \mathcal{H}om(\mathcal{O}_Y^r, \mathcal{F})$ which is an isomorphism at the generic point of Y. Note the following ismorphisms

$$\mathcal{H}om(f_*\mathcal{M}, \mathcal{F}) \cong f_*\,\mathcal{H}om(\mathcal{M}, f^*\mathcal{F}), \quad \mathcal{H}om(\mathcal{O}_Y^r, \mathcal{F}) \cong \mathcal{F}^r,$$

where the first one follows by (II, Ex. 5.17e). Since $\mathcal{H}om(\mathcal{M}, f^*\mathcal{F})$ is coherent on X, we are done with $\mathcal{G} = \mathcal{H}om(\mathcal{M}, f^*\mathcal{F})$.

(c) By (Ex. 3.1) and (Ex. 3.2), we reduce to the case when X and Y are integral. We use the criterion of (3.7). Let $\mathcal{F}$ be a quasi-coherent sheaf on Y.

□

3. Let $X = \mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$, and let $U = X - \{(0,0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j \mid i, j < 0\}$. In particular, it is infinite-dimensional.

Proof. Let $\mathfrak{U}$ be the open covering by the two open sets $V = X - \{x = 0\}$ and $U = X - \{y = 0\}$, with affine coordinates obtained by restricting the ones from X. Then the Čech complex has only two terms:

$$C^0 = \Gamma(V, \mathcal{O}_V) \times \Gamma(W, \mathcal{O}_W),$$
$$C^1 = \Gamma(V \cap W, \mathcal{O}_{V \cap W}).$$

Now

$$\Gamma(V,\Omega) = k\left[x,\frac{1}{x},y\right]$$
$$\Gamma(W,\Omega) = k\left[x,y,\frac{1}{y}\right]$$
$$\Gamma(V,W,\Omega) = k\left[x,y,\frac{1}{x},\frac{1}{y}\right]$$

and the map $d : C^0 \to C^1$ is given by $(f,g) \mapsto f - g$. To compute H^1, the image of d is the set of all expressions $x^i y^j$ where at least one of i,j is non-negative. Hence, $H^1(U,\mathcal{O}_U)$ is spanned by $\{x^i y^j \mid i,j < 0\}$. □

4. On an arbitrary topological space X with an arbitrary Abelian sheaf $\mathcal{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{V} = (V_j)_{j\in J}$, together with a map $\lambda : J \to I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology for any Abelian sheaf $\mathcal{F}$, and for each i,

$$\lambda^i : \check{H}^i(\mathfrak{U},\mathcal{F}) \to \check{H}^i(\mathfrak{V},\mathcal{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U},\mathcal{F}).$$

(b) For any Abelian sheaf $\mathcal{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U},\mathcal{F}) \to H^i(X,\mathcal{F})$$

are compatible with the refinement maps above.

(c) Now prove the following theorem. Let X be a topological space, $\mathcal{F}$ a sheaf of Abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U},\mathcal{F}) \to H^1(X,\mathcal{F})$$

is an isomorphism.

Proof.

(a)

(b)

(c)

□

5. For any ringed space $(X,\mathcal{O}_X)$, let $\operatorname{Pic} X$ be the group of isomorphism classes of invertible sheaves (II, §6). Show that $\operatorname{Pic} X \cong H^1(X,\mathcal{O}_X^*)$ where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U,\mathcal{O}_X)$, with multiplication as the group operation.

Proof. □