Chapter 3, Section 3

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1. Let X be a Noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine.

   Proof. One direction is clear. Suppose $X_{\text{red}} = \operatorname{Spec} A$ where A is a Noetherian ring with no nilpotent elements, let $f : X_{\text{red}} \to X$ be the natural map, and let $\mathscr{F}$ be any quasi-coherent sheaf on X. Following the hint, consider the filtration
   $$\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \cdots,$$
   where $\mathscr{N}$ is the sheaf of nilpotent elements on X. Note that $X \cong X_{\text{red}}$ as topological space, and the associated morphism of sheaves $\mathscr{O}_X \to f_* \mathscr{O}_{X_{\text{red}}}$ is surjective with kernel $\mathscr{N}$. Thus, each of the quotients of this filtration can be naturally viewed as A-modules. In particular, we have a natural isomorphism (2.10)
   $$H^i(X, \mathscr{N}^r \cdot \mathscr{F}/\mathscr{N}^{r+1}\mathscr{F}) \cong H^i(X_{\text{red}}, f^*(\mathscr{N}^r \cdot \mathscr{F}/\mathscr{N}^{r+1} \cdot \mathscr{F})).$$
   Also, the nilradical of a Noetherian ring is nilpotent, so there exists a positive integer $r > 0$ such that $\mathscr{N}^r = 0$ (A.M. 7.15). Using our hypothesis and (3.7), we climb up the filtration and deduce that $H^1(X, \mathscr{F}) = 0$. Hence, X is affine by (3.7). □

2. Let X be a reduced Noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

   Proof. Suppose $X = \operatorname{Spec} A$ is affine for some reduced Noetherian ring A. The irreducible components of X correspond to the minimal prime ideals $\mathfrak{p}$ of A (A.M. Ex. 1.20). In particular, the irreducible components of X are precisely $\operatorname{Spec} A/\mathfrak{p}$. Conversely, let X_i be the irreducible components of X, and let $\phi : \mathscr{F} \to \bigoplus_i j_* \mathscr{F}\big|_{X_i}$ be the natural map of $\mathscr{O}_X$-modules, where $j : X_i \hookrightarrow X$ is the inclusion. Since X is Noetherian, $X_i \cap X_j$ is quasi-compact, so we can cover it with a finite number of open affine subsets X_{ijk}. Because X is reduced, ϕ is injective, so we can extend ϕ by the following exact sequence
   $$0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_i j_* \mathscr{F}\big|_{X_i} \longrightarrow \bigoplus_{i,j} j_* \mathscr{F}\big|_{X_{ijk}}.$$
   Each $j_* \mathscr{F}\big|_{X_i}, j_* \mathscr{F}\big|_{X_{ijk}}$ has vanishing cohomology for $i > 0$ by (2.10), (3.5), and (3.7). While the sequence above is not surjective on the right, the image is still a quasi-coherent sheaf, so using the long exact sequence of cohomology, we deduce that $H^i(X, \mathscr{F}) = 0$ for $i > 0$. Hence, X is affine by (3.7). □

6. Let X be a Noetherian scheme.

   (a) Show that the sheaf $\mathscr{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus, $\mathfrak{Qco}(X)$ has enough injectives.

   (b) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque.

   (c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

   Proof.

(a) The Hom functor commutes with finite direct sums in the second argument, so we can assume $\mathscr{G} = j_* \tilde{I}$, where $j : U = \operatorname{Spec} A \to X$ is the inclusion, and I is an injective A-module. Suppose $\mathscr{N} \to \mathscr{M}$ is an injective map of $\mathscr{O}_X$-modules, and we are given any $f : \mathscr{N} \to j_* \tilde{I}$. Since j^* is left exact when j is an open immersion, the induced map of A-modules $j^* \mathscr{N} \to j^* \mathscr{M}$ is also injective. For any such f there is an associated morphism of A-modules $g : j^* \mathscr{N} \to \tilde{I}$ by adjointness of j_*, so there exists an extension of g to $j^* \mathscr{M}$ by injectivity of $\widetilde{I}$. By adjointness of j^* again, we obtain a morphism $\mathscr{M} \to j_* \tilde{I}$ that naturally extends f, which is what we wanted to show.

(b) Essentially imitating (a) but replacing i^* with i_* and vice versa, we deduce that $\mathscr{I}\big|_U$ is an injective object of $\mathfrak{Qco}(U)$. Covering X with finite number of open affines $U_i = \operatorname{Spec} A_i$, we have $\mathscr{I}\big|_{U_i} \cong \tilde{I}_i$ for some injective A_i-module I_i for each i by (II, 5.5). Each $\tilde{I}_i$ is flasque by (3.4), so $\mathscr{I}$ is flasque on a local basis. Hence, $\mathscr{I}$ is flasque.

(c) Considering $\Gamma(X, \cdot)$ as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$, we calculate its derived funcotrs by taking injective resolutions in the category $\mathfrak{Qco}(X)$. But any injective is flasque (b), and flasques are acyclic (2.5), so this resolution gives the usual cohomology functors (1.2A).

□

7. Let A be a Noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

    (a) For any A-module M, establish the following formula of Deligne:

$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M).$$

    (b) Apply this in the case of an injective A-module I, to give another proof of (3.4).

Proof.

(a) To define a map $\phi : \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M) \to \Gamma(U, \widetilde{M})$, it suffices to define A-homomoprhisms

$$\phi_n : \operatorname{Hom}_A(\mathfrak{a}^n, M) \to \Gamma(U, \widetilde{M})$$

that respect the direct system

$$M \cong \operatorname{Hom}_A(A, M) \xrightarrow{\mu_0} \operatorname{Hom}_A(\mathfrak{a}, M) \xrightarrow{\mu_1} \operatorname{Hom}_A(\mathfrak{a}^2, M) \xrightarrow{\mu_2} \cdots,$$

i.e., $\phi_n = \phi_{n+1} \circ \mu_n$ for all n. By the Noetherian hypothesis, $\mathfrak{a}$ is generated by finitely many elements, say $f_1, \dots, f_r$. Consider the localization of any $\alpha \in \operatorname{Hom}_A(\mathfrak{a}^n, M)$ with respect to f_i for any i, that is $\alpha_i : (\mathfrak{a}^n)_{f_i} \to M_{f_i}$. Then $(\mathfrak{a}^n)_{f_i} \cong A_{f_i}$ since $f_i^n \in \mathfrak{a}^n$ and f_i is a unit in A_{f_i} (we exclude the case when f_i is nilpotent, since $A_{f_i} = 0$). Let $\phi_n(\alpha)$ be the section equal to $\alpha_i(1)$ on U_i, where U_i is the distinguished open set associated to f_i. We remark that $U = \bigcup_{i=1}^r U_i$, so this definition is well-defined, and it respects the direct system of above since the localization of the inclusion $\mathfrak{a}^n \hookrightarrow \mathfrak{a}^{n-1}$ is the identity map on A_{f_i}. Thus, to show $\phi_n(\alpha)$ is a well-defined section, it is sufficient to show $\alpha_i(1)$ and $\alpha_j(1)$ agree on $U_i \cap U_j \cong \operatorname{Spec} A_{f_i f_j}$ for all i, j. Indeed, $\alpha(f_i^n) = \alpha_i(f_i^n)$ where we naturally view $\alpha(f_i^n)$ as an element of M_{f_i}, α_i is A_{f_i}-linear, and the restriction map $U_i \to U_i \cap U_j$ is given by $m/f_i^n \mapsto f_j^n/(f_i f_j)^n$. Thus, we have

$$\begin{aligned}
\alpha_i(1)\big|_{U_i \cap U_j} &= \alpha_i(f_i^{-n} f_i^n)\big|_{U_i \cap U_j} \\
&= \frac{\alpha(f_i^n)}{f_i^n}\bigg|_{U_i \cap U_j} \\
&= \frac{f_j^n \alpha(f_i^n)}{(f_i f_j)^n} \\
&= \frac{f_i^n \alpha(f_j^n)}{(f_i f_j)^n} \\
&= \alpha_j(1)\big|_{U_i \cap U_j}.
\end{aligned}$$

Also, each ϕ_n is injective, so the induced map $\phi : \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M) \to \Gamma(U, \widetilde{M})$ is injective. It remains to show ϕ is surjective. Let $s \in \Gamma(U, \widetilde{M})$ be any section. For each i, we can express $s\big|_{U_i}$ as $m_i/f_i^{n_i}$ for some $n_i > 0$ and $m_i \in M$ such that $m_i/f_i^{n_i}\big|_{U_i \cap U_j} = m_j/f_j^{n_j}\big|_{U_i \cap U_j}$, where $U_i \cap U_j = \operatorname{Spec} A_{f_i f_j}$ for all i, j. Choose $n = n_i$ that works for all i, so we have

$$\frac{f_j^n m_i}{(f_i f_j)^n} = \frac{f_i^n m_j}{(f_i f_j)^n}.$$

We want to show there exists an A-homomoprhism $\alpha : \mathfrak{a}^N \to M$ for some $N \geq n$ such that $\alpha_i(1) = f_i^{N-n} m_i / f_i^N$. Imitating the proof of (3.3), let $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$ be the sequence of annihilators of $\cdots \subseteq \mathfrak{a}^2 \subseteq \mathfrak{a}$. Since A is Noetherian, there is an r such that $\mathfrak{b}_r = \mathfrak{b}_{r+1} = \cdots$. Define $\sigma : \mathfrak{a}^{n+r} \to M$ by sending f_i^{n+r} to $f_i^r m_i$ for all i and extending by zero. This is a well-defined homomorphism because the annihilator of f_i^{n+r} is $\mathfrak{b}_{n+r} = \mathfrak{b}_r$, and $\mathfrak{b}_r$ annihilates $f_i^r m_i$. Hence, $\phi_{n+r}(\sigma) = s$.

(b) Let I be an injective A-module. It will be sufficient to show for any open set $U \subseteq X$, where $U = X - V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A, that $\Gamma(X, \tilde{I}) \to \Gamma(U, \tilde{I})$ is surjective. Indeed, d$\Gamma(X, \tilde{I}) \to \Gamma(U, \tilde{I})$ is induced by the natural maps $\mathrm{Hom}_A(A, I) \to \mathrm{Hom}_A(\mathfrak{a}^n, I)$, which is surjective since I is injective. Hence, $\Gamma(X, I) \to \Gamma(U, I)$ is surjective.

$\square$