Chapter 4, Section 2

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1. Use (2.5.3) to show that $\mathbf{P}^n$ is simply connected.

2. *Classification of Curves of Genus 2.* Fix an algebraically closed field k of characteristic $\neq 2$.

(a) If X is a curve of genus 2 over k, the canonical linear system $|K|$ determines a finite morphism $f : X \to \mathbf{P}^1$ of degree 2 (Ex. 1.7). Show that it is ramified at exactly 6 points, with ramification index 2 at each one. Note that f is uniquely determined, up to an automorphism of $\mathbf{P}^1$, so X determines an (unordered) set of 6 points of $\mathbf{P}^1$, up to an automorphism of $\mathbf{P}^1$.

(b) Conversely, given six distinct elements $\alpha_1, \ldots, \alpha_6 \in k$, let K be the extension of $k(x)$ determined by the equation $z^2 = (x - \alpha_1) \cdots (x - \alpha_6)$. Let $f : X \to \mathbf{P}^1$ be the corresponding morphism of curves. Show that $g(X) = 2$, the map f is the same as the one determined by the canonical linear system, and f is ramified over the six points $x = \alpha_i$ of $\mathbf{P}^1$, and nowhere else.