Chapter 3, Section 4

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1. Let $f : X \to Y$ be an affine morphism of Noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geq 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F}).$$

Proof. Let $\mathfrak{V}$ be an open affine cover of Y so that $\mathfrak{U} = (f^{-1}(V))_{V \in \mathfrak{V}}$ is an open affine cover of X. We can compute $H^i(X, \mathscr{F})$ using the Čech complex defined by the open covering $\mathfrak{U}$. Also, $f_*\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Y$-module by (II, 5.8), so the cohomology of $f_*\mathscr{F}$ can be computed via the Čech complex defined by $\mathfrak{V}$ (4.5). Lastly, $f_*\mathscr{F}(V) = \mathscr{F}(U)$ for all $V \in \mathfrak{V}$ and corresponding $U = f^{-1}(V) \in \mathfrak{U}$, so the Čech cohomology of $\mathscr{F}$ and $f_*\mathscr{F}$ with respect to $\mathfrak{U}$ and $\mathfrak{V}$, respectively, are isomorphic. Hence, $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$ for all $i \geq 0$ by (4.5). *Is this natural?* □

2. Prove Chevalley's theorem: Let $f : X \to Y$ be a finite surjective morphism of Noetherian separated schemes, with X affine. Then Y is affine.

(a) Let $f : X \to Y$ be a finite surjective morphism of integral Noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha : \mathscr{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

(b) For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta : f_*\mathscr{G} \to \mathscr{F}^r$ which is an isomorphism at the generic point of Y.

(c) Now prove Chevalley's theorem.

Proof.

(a) Let $\mathscr{L}$ and $\mathscr{K}$ be the sheaf of total quotient rings of X and Y, respectively. By hypothesis, the function field of X is a finite field extension of that of Y, so there exists an isomorphism $\varphi : \mathscr{K}^r \to f_*\mathscr{L}$ for some $r > 0$. Let $\alpha : \mathscr{O}_Y^r \to f_*\mathscr{L}$ be the compositon of the natural map $\mathscr{O}_Y^r \to \mathscr{K}^r$ with φ. Then α is an ismorphism at the generic point of Y. Replacing $\mathscr{L}$ by the sub-$\mathscr{O}_X$-module spanned by the image of $\mathscr{K}^r$, we obtain the desired $\mathscr{M}$.

(b) Let $\mathscr{F}$ be any coherent sheaf $\mathscr{F}$ on Y, let $\mathscr{A} = f_*\mathscr{O}_X$, and let $\alpha : \mathscr{O}_Y^r \to f_*\mathscr{M}$ as in (a) for some coherent sheaf $\mathscr{M}$ on X. Applying $\mathscr{H}om(\cdot, \mathscr{F})$ induces a morphism of sheaves

$$\beta : \mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \to \mathscr{H}om(\mathscr{O}_Y^r, \mathscr{F}) \cong \mathscr{F}^r.$$

It is clearly an isomorphism at the generic point of Y. Consider the $\mathscr{O}_Y$-module $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F})$. It is naturally a coherent $\mathscr{A}$-module, and finite morpshims are also affine, so by the equivalence of categories of quasi-coherent $\mathscr{O}_X$-modules and $\mathscr{A}$-modules via the$\tilde{\ }$-functor of (II, 5.17e), there exists a coherent sheaf $\mathscr{G}$ on X such that $f_*\mathscr{G} \cong \mathscr{H}om(f_*\mathscr{M}, \mathscr{F})$.

(c) By (II, 3.1), (Ex. 3.1), and (Ex. 3.2), we reduce to the case when X and Y are integral Noetherian separated schemes. Let $\mathscr{F}$ be a coherent sheaf on Y. By (2.9) and (3.7), it suffices to show $H^i(Y, \mathscr{F}^r) = 0$ for all $i > 0$ and some $r > 0$. Let $\beta : f_*\mathscr{G} \to \mathscr{F}^r$ be a morphism of coherent sheaves on X as in (b), where $\mathscr{G}$ is a coherent sheaf on X.

Consider $\ker \beta$ and $\operatorname{coker} \beta$. We proceed by notherian induction

□

3. Let $X = \mathbf{A}_k^2 = \operatorname{Spec} k[x, y]$, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathscr{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j \mid i, j < 0\}$. In particular, it is infinite-dimensional.

Proof. Let $\mathfrak{U}$ be the open covering by the two open sets $V = X - \{x = 0\}$ and $U = X - \{y = 0\}$, with affine coordinates obtained by restricting the ones from X. Then the Čech complex has only two terms:

$$\begin{aligned} C^0 &= \Gamma(V, \mathscr{O}_V) \times \Gamma(W, \mathscr{O}_W), \\ C^1 &= \Gamma(V \cap W, \mathscr{O}_{V \cap W}). \end{aligned}$$

Now

$$\begin{aligned} \Gamma(V, \Omega) &= k\left[x, \frac{1}{x}, y\right] \\ \Gamma(W, \Omega) &= k\left[x, y, \frac{1}{y}\right] \\ \Gamma(V, W, \Omega) &= k\left[x, y, \frac{1}{x}, \frac{1}{y}\right] \end{aligned}$$

and the map $d : C^0 \to C^1$ is given by $(f, g) \mapsto f - g$. To compute H^1, the image of d is the set of all expressions $x^i y^j$ where at least one of i, j is non-negative. Hence, $H^1(U, \mathscr{O}_U)$ is spanned by $\{x^i y^j \mid i, j < 0\}$. □

4. On an arbitrary topological space X with an arbitrary Abelian sheaf $\mathscr{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{V} = (V_j)_{j \in J}$, together with a map $\lambda : J \to I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology for any Abelian sheaf $\mathscr{F}$, and for each i,

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathscr{F}).$$

(b) For any Abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$$

are compatible with the refinement maps above.

(c) Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of Abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is an isomorphism.

Proof.

(a)

(b)

(c)

□

5. For any ringed space $(X, \mathscr{O}_X)$, let $\operatorname{Pic} X$ be the group of isomorphism classes of invertible sheaves (II, §6). Show that $\operatorname{Pic} X \cong H^1(X, \mathscr{O}_X^*)$ where $\mathscr{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathscr{O}_X)$, with multiplication as the group operation.

Proof. □