

# COMPUTER SCIENCE

An Interdisciplinary Approach

**ROBERT SEDGEWICK  
KEVIN WAYNE**

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# Computer Science

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# Computer Science

*An Interdisciplinary Approach*

Robert Sedgewick  
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Princeton University

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*To Adam, Andrew, Brett, Robbie,  
Henry, Iona, Rose, Peter,  
and especially Linda*

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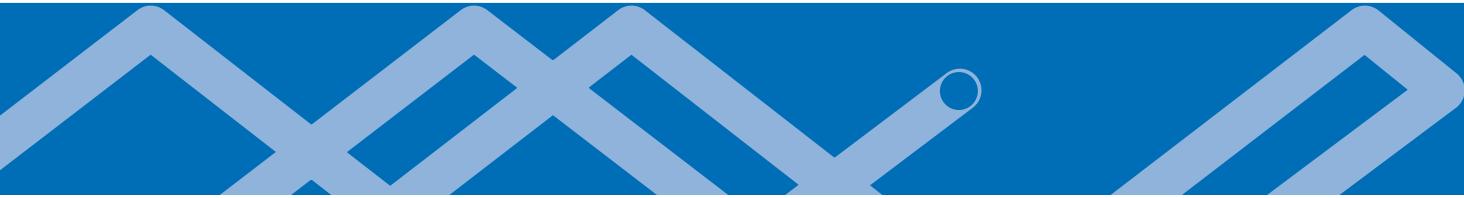
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*To Jackie, Alex, and Michael*

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# Contents

<i>Preface . . . . .</i>	<i>xiii</i>
<i>1—Elements of Programming . . . . .</i>	<i>1</i>
1.1 Your First Program	2
1.2 Built-in Types of Data	14
1.3 Conditionals and Loops	50
1.4 Arrays	90
1.5 Input and Output	126
1.6 Case Study: Random Web Surfer	170
<i>2—Functions and Modules . . . . .</i>	<i>191</i>
2.1 Defining Functions	192
2.2 Libraries and Clients	226
2.3 Recursion	262
2.4 Case Study: Percolation	300
<i>3—Object-Oriented Programming . . . . .</i>	<i>329</i>
3.1 Using Data Types	330
3.2 Creating Data Types	382
3.3 Designing Data Types	428
3.4 Case Study: N-Body Simulation	478
<i>4—Algorithms and Data Structures . . . . .</i>	<i>493</i>
4.1 Performance	494
4.2 Sorting and Searching	532
4.3 Stacks and Queues	566
4.4 Symbol Tables	624
4.5 Case Study: Small-World Phenomenon	670



<b>5—Theory of Computing . . . . .</b>	<b>715</b>
5.1 Formal Languages	718
5.2 Turing Machines	766
5.3 Universality	786
5.4 Computability	806
5.5 Intractability	822
<b>6—A Computing Machine . . . . .</b>	<b>873</b>
6.1 Representing Information	874
6.2 TOY Machine	906
6.3 Machine-Language Programming	930
6.4 TOY Virtual Machine	958
<b>7—Building a Computing Device . . . . .</b>	<b>985</b>
7.1 Boolean Logic	986
7.2 Basic Circuit Model	1002
7.3 Combinational Circuits	1012
7.4 Sequential Circuits	1048
7.5 Digital Devices	1070
<b>Context . . . . .</b>	<b>1093</b>
<b>Glossary . . . . .</b>	<b>1097</b>
<b>Index . . . . .</b>	<b>1107</b>
<b>APIs . . . . .</b>	<b>1139</b>

# Programs

## Elements of Programming

### Your First Program

1.1.1	Hello, World . . . . .	4
1.1.2	Using a command-line argument . . . . .	7

### Built-in Types of Data

1.2.1	String concatenation . . . . .	20
1.2.2	Integer multiplication and division . . . . .	23
1.2.3	Quadratic formula . . . . .	25
1.2.4	Leap year . . . . .	28
1.2.5	Casting to get a random integer . . . . .	34

### Conditionals and Loops

1.3.1	Flipping a fair coin . . . . .	53
1.3.2	Your first while loop. . . . .	55
1.3.3	Computing powers of 2 . . . . .	57
1.3.4	Your first nested loops . . . . .	63
1.3.5	Harmonic numbers . . . . .	65
1.3.6	Newton's method . . . . .	66
1.3.7	Converting to binary . . . . .	68
1.3.8	Gambler's ruin simulation . . . . .	71
1.3.9	Factoring integers. . . . .	73

### Arrays

1.4.1	Sampling without replacement . . . . .	98
1.4.2	Coupon collector simulation . . . . .	102
1.4.3	Sieve of Eratosthenes . . . . .	104
1.4.4	Self-avoiding random walks . . . . .	113

### Input and Output

1.5.1	Generating a random sequence . . . . .	128
1.5.2	Interactive user input . . . . .	136
1.5.3	Averaging a stream of numbers . . . . .	138
1.5.4	A simple filter . . . . .	140
1.5.5	Standard input-to-drawing filter . . . . .	147
1.5.6	Bouncing ball . . . . .	153
1.5.7	Digital signal processing . . . . .	158

### Case Study: Random Web Surfer

1.6.1	Computing the transition matrix . . . . .	173
1.6.2	Simulating a random surfer . . . . .	175
1.6.3	Mixing a Markov chain . . . . .	182

## Functions and Modules

### Defining Functions

2.1.1	Harmonic numbers (revisited) . . . . .	194
2.1.2	Gaussian functions . . . . .	203
2.1.3	Coupon collector (revisited) . . . . .	206
2.1.4	Play that tune (revisited) . . . . .	213

### Libraries and Clients

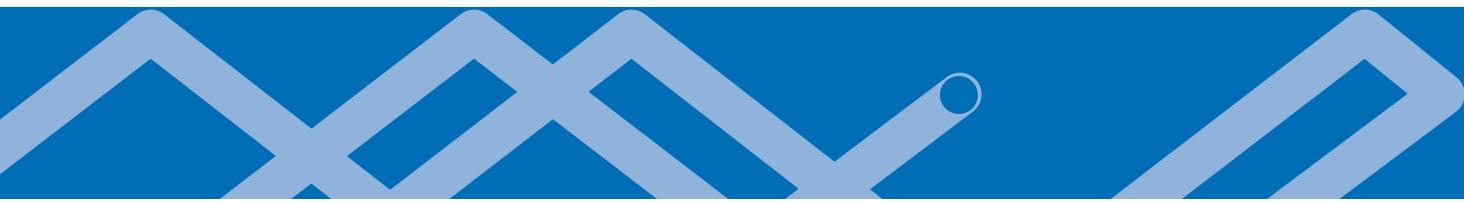
2.2.1	Random number library . . . . .	234
2.2.2	Array I/O library . . . . .	238
2.2.3	Iterated function systems. . . . .	241
2.2.4	Data analysis library. . . . .	245
2.2.5	Plotting data values in an array . . . . .	247
2.2.6	Bernoulli trials . . . . .	250

### Recursion

2.3.1	Euclid's algorithm. . . . .	267
2.3.2	Towers of Hanoi . . . . .	270
2.3.3	Gray code . . . . .	275
2.3.4	Recursive graphics . . . . .	277
2.3.5	Brownian bridge . . . . .	279
2.3.6	Longest common subsequence . . . . .	287

### Case Study: Percolation

2.4.1	Percolation scaffolding. . . . .	304
2.4.2	Vertical percolation detection. . . . .	306
2.4.3	Visualization client . . . . .	309
2.4.4	Percolation probability estimate . . . . .	311
2.4.5	Percolation detection . . . . .	313
2.4.6	Adaptive plot client . . . . .	316



## *Object-Oriented Programming*

### *Using Data Types*

3.1.1	Identifying a potential gene . . . . .	337
3.1.2	Albers squares . . . . .	342
3.1.3	Luminance library . . . . .	345
3.1.4	Converting color to grayscale . . . . .	348
3.1.5	Image scaling . . . . .	350
3.1.6	Fade effect . . . . .	352
3.1.7	Concatenating files . . . . .	356
3.1.8	Screen scraping for stock quotes	359
3.1.9	Splitting a file . . . . .	360

### *Creating Data Types*

3.2.1	Charged particle . . . . .	387
3.2.2	Stopwatch . . . . .	391
3.2.3	Histogram . . . . .	393
3.2.4	Turtle graphics . . . . .	396
3.2.5	Spira mirabilis . . . . .	399
3.2.6	Complex number . . . . .	405
3.2.7	Mandelbrot set . . . . .	409
3.2.8	Stock account . . . . .	413

### *Designing Data Types*

3.3.1	Complex number (alternate) . . . . .	434
3.3.2	Counter . . . . .	437
3.3.3	Spatial vectors . . . . .	444
3.3.4	Document sketch . . . . .	461
3.3.5	Similarity detection . . . . .	463

### *Case Study: N-Body Simulation*

3.4.1	Gravitational body . . . . .	482
3.4.2	N-body simulation . . . . .	485

## *Algorithms and Data Structures*

### *Performance*

4.1.1	3-sum problem . . . . .	497
4.1.2	Validating a doubling hypothesis	499

### *Sorting and Searching*

4.2.1	Binary search (20 questions) . . . . .	534
4.2.2	Bisection search . . . . .	537
4.2.3	Binary search (sorted array) . . . . .	539
4.2.4	Insertion sort . . . . .	547
4.2.5	Doubling test for insertion sort	549
4.2.6	Mergesort . . . . .	552
4.2.7	Frequency counts . . . . .	557

### *Stacks and Queues*

4.3.1	Stack of strings (array) . . . . .	570
4.3.2	Stack of strings (linked list) . . . . .	575
4.3.3	Stack of strings (resizing array)	579
4.3.4	Generic stack . . . . .	584
4.3.5	Expression evaluation . . . . .	588
4.3.6	Generic FIFO queue (linked list)	594
4.3.7	M/M/1 queue simulation . . . . .	599
4.3.8	Load balancing simulation . . . . .	607

### *Symbol Tables*

4.4.1	Dictionary lookup . . . . .	631
4.4.2	Indexing . . . . .	633
4.4.3	Hash table . . . . .	638
4.4.4	Binary search tree . . . . .	646
4.4.5	Dedup filter . . . . .	653

### *Case Study: Small-World Phenomenon*

4.5.1	Graph data type . . . . .	677
4.5.2	Using a graph to invert an index	681
4.5.3	Shortest-paths client . . . . .	685
4.5.4	Shortest-paths implementation	691
4.5.5	Small-world test . . . . .	696
4.5.6	Performer–performer graph . . . . .	698

## *Theory of Computing*

### *Formal Languages*

5.1.1	RE recognition . . . . .	729
5.1.2	Generalized RE pattern match	736
5.1.3	Universal virtual DFA . . . . .	743

### *Turing Machines*

5.2.1	Virtual Turing machine tape . .	776
5.2.2	Universal virtual TM . . . . .	777

### *Universality*

### *Computability*

### *Intractability*

5.5.1	SAT solver . . . . .	855
-------	----------------------	-----

## *A Computing Machine*

### *Representing Information*

6.1.1	Number conversion . . . . .	881
6.1.2	Floating-point components . .	893

### *TOY Machine*

6.2.1	Your first TOY program . . . .	915
6.2.2	Conditionals and loops . . . .	921
6.2.3	Self-modifying code. . . . .	923

### *Machine-Language Programming*

6.3.1	Calling a function . . . . .	933
6.3.2	Standard output . . . . .	935
6.3.3	Standard input . . . . .	937
6.3.4	Array processing . . . . .	939
6.3.5	Linked structures . . . . .	943

### *TOY Virtual Machine*

6.4.1	TOY virtual machine . . . . .	967
-------	-------------------------------	-----

# Circuits

## *Building a Computing Device*

### *Boolean Logic*

### *Basic Circuit Model*

### *Combinational Circuits*

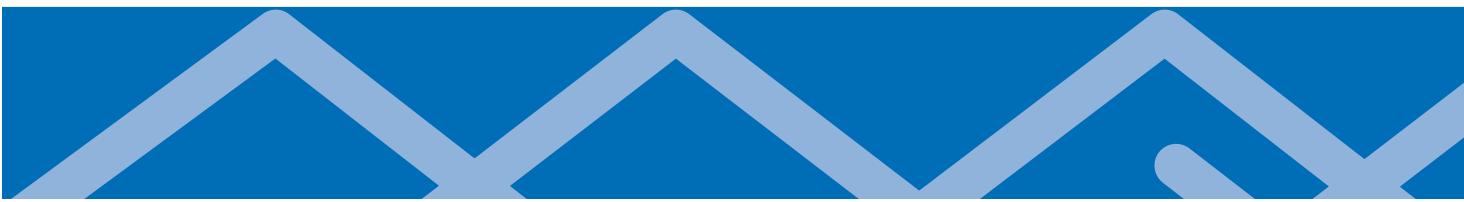
Basic logic gates. . . . .	1014
Selection multiplexer . . . . .	1024
Decoder . . . . .	1021
Demultiplexer . . . . .	1022
Multiplexer . . . . .	1023
XOR . . . . .	1024
Majority. . . . .	1025
Odd parity. . . . .	1026
Adder . . . . .	1029
ALU . . . . .	1033
Bus multiplexer. . . . .	1036

### *Sequential Circuits*

SR flip-flop . . . . .	1050
Register bit. . . . .	1051
Register . . . . .	1052
Memory bit . . . . .	1056
Memory. . . . .	1057
Clock . . . . .	1061

### *Digital Devices*

Program counter . . . . .	1074
Control . . . . .	1081
CPU . . . . .	1086



# Preface

THE BASIS FOR EDUCATION IN THE last millennium was “reading, writing, and arithmetic”; now it is reading, writing, and *computing*. Learning to program is an essential part of the education of every student in the sciences and engineering. Beyond direct applications, it is the first step in understanding the nature of computer science’s undeniable impact on the modern world. This book aims to teach programming to those who need or want to learn it, in a scientific context.

Our primary goal is to *empower* students by supplying the experience and basic tools necessary to use computation effectively. Our approach is to teach students that composing a program is a natural, satisfying, and creative experience. We progressively introduce essential concepts, embrace classic applications from applied mathematics and the sciences to illustrate the concepts, and provide opportunities for students to write programs to solve engaging problems. We seek also to *demystify* computation for students and to *build awareness* about the substantial intellectual underpinnings of the field of computer science.

We use the Java programming language for all of the programs in this book. The first part of the book teaches basic skills for computational problem solving that are applicable in many modern computing environments, and it is a self-contained treatment intended for people with no previous experience in programming. It is about *fundamental concepts in programming*, not Java per se. The second part of the book demonstrates that there is much more to computer science than programming, but we do often use Java programs to help communicate the main ideas.

This book is an *interdisciplinary* approach to the traditional CS1 curriculum, in that we highlight the role of computing in other disciplines, from materials science to genomics to astrophysics to network systems. This approach reinforces for students the essential idea that mathematics, science, engineering, and computing are intertwined in the modern world. While it is a CS1 textbook designed for any first-year college student, the book also can be used for self-study.

**Coverage** The first part of the book is organized around three stages of learning to program: basic elements, functions, object-oriented programming, and algorithms. We provide the basic information that readers need to build confidence in composing programs at each level before moving to the next level. An essential feature of our approach is the use of example programs that solve intriguing problems, supported with exercises ranging from self-study drills to challenging problems that call for creative solutions.

*Elements of programming* include variables, assignment statements, built-in types of data, flow of control, arrays, and input/output, including graphics and sound.

*Functions and modules* are the students' first exposure to modular programming. We build upon students' familiarity with mathematical functions to introduce Java functions, and then consider the implications of programming with functions, including libraries of functions and recursion. We stress the fundamental idea of dividing a program into components that can be independently debugged, maintained, and reused.

*Object-oriented programming* is our introduction to data abstraction. We emphasize the concept of a data type and its implementation using Java's class mechanism. We teach students how to *use*, *create*, and *design* data types. Modularity, encapsulation, and other modern programming paradigms are the central concepts of this stage.

The second part of the book introduces advanced topics in computer science: algorithms and data structures, theory of computing, and machine architecture.

*Algorithms and data structures* combine these modern programming paradigms with classic methods of organizing and processing data that remain effective for modern applications. We provide an introduction to classical algorithms for sorting and searching as well as fundamental data structures and their application, emphasizing the use of the scientific method to understand performance characteristics of implementations.

*Theory of computing* helps us address basic questions about computation, using simple abstract models of computers. Not only are the insights gained invaluable, but many of the ideas are also directly useful and relevant in practical computing applications.

*Machine architecture* provides a path to understanding what computation actually looks like in the real world—a link between the abstract machines of the theory of computing and the real computers that we use. Moreover, the study of

machine architecture provides a link to the past, as the microprocessors found in today's computers and mobile devices are not so different from the first computers that were developed in the middle of the 20th century.

*Applications in science and engineering* are a key feature of the text. We motivate each programming concept that we address by examining its impact on specific applications. We draw examples from applied mathematics, the physical and biological sciences, and computer science itself, and include simulation of physical systems, numerical methods, data visualization, sound synthesis, image processing, financial simulation, and information technology. Specific examples include a treatment in the first chapter of Markov chains for web page ranks and case studies that address the percolation problem,  $n$ -body simulation, and the small-world phenomenon. These applications are an integral part of the text. They engage students in the material, illustrate the importance of the programming concepts, and provide persuasive evidence of the critical role played by computation in modern science and engineering.

*Historical context* is emphasized in the later chapters. The fascinating story of the development and application of fundamental ideas about computation by Alan Turing, John von Neumann, and many others is an important subtext.

Our primary goal is to teach the specific mechanisms and skills that are needed to develop effective solutions to any programming problem. We work with complete Java programs and encourage readers to use them. We focus on programming by individuals, not programming in the large.

**Use in the Curriculum** This book is intended for a first-year college course aimed at teaching computer science to novices in the context of scientific applications. When such a course is taught from this book, college student will learn to program in a familiar context. Students completing a course based on this book will be well prepared to apply their skills in later courses in their chosen major and to recognize when further education in computer science might be beneficial.

Prospective computer science majors, in particular, can benefit from learning to program in the context of scientific applications. A computer scientist needs the same basic background in the scientific method and the same exposure to the role of computation in science as does a biologist, an engineer, or a physicist.

Indeed, our interdisciplinary approach enables colleges and universities to teach prospective computer science majors and prospective majors in other fields in the *same* course. We cover the material prescribed by CS1, but our focus on

applications brings life to the concepts and motivates students to learn them. Our interdisciplinary approach exposes students to problems in many different disciplines, helping them to choose a major more wisely.

Whatever the specific mechanism, the use of this book is best positioned early in the curriculum. First, this positioning allows us to leverage familiar material in high school mathematics and science. Second, students who learn to program early in their college curriculum will then be able to use computers more effectively when moving on to courses in their specialty. Like reading and writing, programming is certain to be an essential skill for any scientist or engineer. Students who have grasped the concepts in this book will continually develop that skill throughout their lifetimes, reaping the benefits of exploiting computation to solve or to better understand the problems and projects that arise in their chosen field.

**Prerequisites** This book is suitable for typical first-year college students. That is, we do not expect preparation beyond what is typically required for other entry-level science and mathematics courses.

*Mathematical maturity* is important. While we do not dwell on mathematical material, we do refer to the mathematics curriculum that students have taken in high school, including algebra, geometry, and trigonometry. Most students in our target audience automatically meet these requirements. Indeed, we take advantage of their familiarity with the basic curriculum to introduce basic programming concepts.

*Scientific curiosity* is also an essential ingredient. Science and engineering students bring with them a sense of fascination with the ability of scientific inquiry to help explain what goes on in nature. We leverage this predilection with examples of simple programs that speak volumes about the natural world. We do not assume any specific knowledge beyond that provided by typical high school courses in mathematics, physics, biology, or chemistry.

*Programming experience* is not necessary, but also is not harmful. Teaching programming is one of our primary goals, so we assume no prior programming experience. But composing a program to solve a new problem is a challenging intellectual task, so students who have written numerous programs in high school can benefit from taking an introductory programming course based on this book. The book can support teaching students with varying backgrounds because the applications appeal to both novices and experts alike.

*Experience using a computer* is not necessary, but also is not a problem. College students use computers regularly—for example, to communicate with friends and relatives, listen to music, process photos, and as part of many other activities. The realization that they can harness the power of their own computer in interesting and important ways is an exciting and lasting lesson.

In summary, virtually all college students are prepared to take a course based on this book as a part of their first-semester curriculum.

**Goals** What can *instructors* of upper-level courses in science and engineering expect of students who have completed a course based on this book?

We cover the CS1 curriculum, but anyone who has taught an introductory programming course knows that expectations of instructors in later courses are typically high: each instructor expects all students to be familiar with the computing environment and approach that he or she wants to use. A physics professor might expect some students to design a program over the weekend to run a simulation; an engineering professor might expect other students to use a particular package to numerically solve differential equations; or a computer science professor might expect knowledge of the details of a particular programming environment. Is it realistic for a single entry-level course to meet such diverse expectations? Should there be a different introductory course for each set of students?

Colleges and universities have been wrestling with such questions since computers came into widespread use in the latter part of the 20th century. Our answer to them is found in this common introductory treatment of programming, which is analogous to commonly accepted introductory courses in mathematics, physics, biology, and chemistry. *Computer Science* strives to provide the basic preparation needed by all students in science and engineering, while sending the clear message that there is much more to understand about computer science than programming. Instructors teaching students who have studied from this book can expect that they will have the knowledge and experience necessary to enable those students to adapt to new computational environments and to effectively exploit computers in diverse applications.

What can *students* who have completed a course based on this book expect to accomplish in later courses?

Our message is that programming is not difficult to learn and that harnessing the power of the computer is rewarding. Students who master the material in this book are prepared to address computational challenges wherever they might

appear later in their careers. They learn that modern programming environments, such as the one provided by Java, help open the door to any computational problem they might encounter later, and they gain the confidence to learn, evaluate, and use other computational tools. Students interested in computer science will be well prepared to pursue that interest; students in science and engineering will be ready to integrate computation into their studies.

**Online lectures** A complete set of studio-produced videos that can be used in conjunction with this text are available at

<http://www.informit.com/title/9780134493831>

As with traditional live lectures, the purpose of these videos is to *inform* and *inspire*, motivating students to study and learn from the text. Our experience is that student engagement with the material is significantly better than with live lectures because of the ability to play the lectures at a chosen speed and to replay and review the lectures at any time.

**Booksit** An extensive amount of other information that supplements this text may be found on the web at

<http://introcs.cs.princeton.edu/java>

For economy, we refer to this site as the *booksit* throughout. It contains material for instructors, students, and casual readers of the book. We briefly describe this material here, though, as all web users know, it is best surveyed by browsing. With a few exceptions to support testing, the material is all publicly available.

One of the most important implications of the booksite is that it empowers instructors and students to use their own computers to teach and learn the material. Anyone with a computer and a browser can begin learning to program by following a few instructions on the booksite. The process is no more difficult than downloading a media player or a song. As with any website, our booksite is continually evolving. It is an essential resource for everyone who owns this book. In particular, the supplemental materials are critical to our goal of making computer science an integral component of the education of all scientists and engineers.

For *instructors*, the booksite contains information about teaching. This information is primarily organized around a teaching style that we have developed over the past decade, where we offer two lectures per week to a large audience, supplemented by two class sessions per week where students meet in small groups

with instructors or teaching assistants. The booksite has presentation slides for the lectures, which set the tone.

For *teaching assistants*, the booksite contains detailed problem sets and programming projects, which are based on exercises from the book but contain much more detail. Each programming assignment is intended to teach a relevant concept in the context of an interesting application while presenting an inviting and engaging challenge to each student. The progression of assignments embodies our approach to teaching programming. The booksite fully specifies all the assignments and provides detailed, structured information to help students complete them in the allotted time, including descriptions of suggested approaches and outlines for what should be taught in class sessions.

For *students*, the booksite contains quick access to much of the material in the book, including source code, plus extra material to encourage self-learning. Solutions are provided for many of the book's exercises, including complete program code and test data. A wealth of information is associated with the programming assignments, including suggested approaches, checklists, FAQs, and test data.

For *casual readers*, the booksite is a resource for accessing all manner of extra information associated with the book's content. All of the booksite content provides web links and other routes to pursue more information about the topic under consideration. There is far more information accessible than any individual could fully digest, but our goal is to provide enough to whet any reader's appetite for more information about the book's content.

**Acknowledgments** This project has been under development since 1992, so far too many people have contributed to its success for us to acknowledge them all here. Special thanks are due to Anne Rogers, for helping to start the ball rolling; to Dave Hanson, Andrew Appel, and Chris van Wyk, for their patience in explaining data abstraction; to Lisa Worthington and Donna Gabai, for being the first to truly relish the challenge of teaching this material to first-year students; and to Doug Clark for his patience as we learned about building Turing machines and circuits. We also gratefully acknowledge the efforts of /dev/126; the faculty, graduate students, and teaching staff who have dedicated themselves to teaching this material over the past 25 years here at Princeton University; and the thousands of undergraduates who have dedicated themselves to learning it.

*Robert Sedgewick  
Kevin Wayne*

*May 2016*



## *Chapter Two*

# Functions and Modules

2.1	Defining Functions . . . . .	192
2.2	Libraries and Clients . . . . .	226
2.3	Recursion . . . . .	262
2.4	Case Study: Percolation . . . . .	300

**T**HIS CHAPTER CENTERS ON A CONSTRUCT that has as profound an impact on control flow as do conditionals and loops: the *function*, which allows us to transfer control back and forth between different pieces of code. Functions (which are known as *static methods* in Java) are important because they allow us to clearly separate tasks within a program and because they provide a general mechanism that enables us to reuse code.

We group functions together in *modules*, which we can compile independently. We use modules to break a computational task into subtasks of a reasonable size. You will learn in this chapter how to build modules of your own and how to use them, in a style of programming known as *modular programming*.

Some modules are developed with the primary intent of providing code that can be reused later by many other programs. We refer to such modules as *libraries*. In particular, we consider in this chapter libraries for generating random numbers, analyzing data, and providing input/output for arrays. Libraries vastly extend the set of operations that we use in our programs.

We pay special attention to functions that transfer control to themselves—a process known as *recursion*. At first, recursion may seem counterintuitive, but it allows us to develop simple programs that can address complex tasks that would otherwise be much more difficult to carry out.

*Whenever you can clearly separate tasks within programs, you should do so.* We repeat this mantra throughout this chapter, and end the chapter with a case study showing how a complex programming task can be handled by breaking it into smaller subtasks, then independently developing modules that interact with one another to address the subtasks.

## 2.1 Defining Functions

THE JAVA CONSTRUCT FOR IMPLEMENTING A function is known as the *static method*. The modifier `static` distinguishes this kind of method from the kind discussed in CHAPTER 3—we will apply it consistently for now and discuss the difference then. You have actually been using static methods since the beginning of this book, from mathematical functions such as `Math.abs()` and `Math.sqrt()` to all of the methods in `StdIn`, `StdOut`, `StdDraw`, and `StdAudio`. Indeed, every Java program that you have written has a static method named `main()`. In this section, you will learn how to *define* your own static methods.

In mathematics, a *function* maps an input value of one type (the *domain*) to an output value of another type (the *range*). For example, the function  $f(x) = x^2$  maps 2 to 4, 3 to 9, 4 to 16, and so forth. At first, we work with static methods that implement mathematical functions, because they are so familiar. Many standard mathematical functions are implemented in Java’s `Math` library, but scientists and engineers work with a broad variety of mathematical functions, which cannot all be included in the library. At the beginning of this section, you will learn how to implement such functions on your own.

Later, you will learn that we can do more with static methods than implement mathematical functions: static methods can have strings and other types as their range or domain, and they can produce side effects such as printing output. We also consider in this section how to use static methods to organize programs and thus to simplify complicated programming tasks.

Static methods support a key concept that will pervade your approach to programming from this point forward: *whenever you can clearly separate tasks within programs, you should do so*. We will be overemphasizing this point throughout this section and reinforcing it throughout this book. When you write an essay, you break it up into paragraphs; when you write a program, you will break it up into methods. Separating a larger task into smaller ones is much more important in programming than in writing, because it greatly facilitates *debugging, maintenance, and reuse*, which are all critical in developing good software.

2.1.1	Harmonic numbers (revisited)	194
2.1.2	Gaussian functions	203
2.1.3	Coupon collector (revisited)	206
2.1.4	Play that tune (revisited)	213

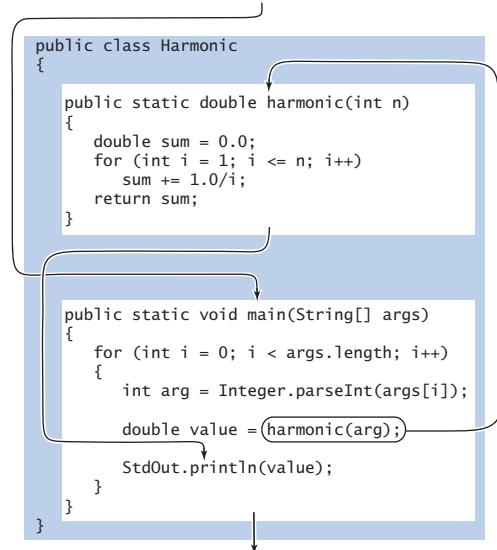
*Programs in this section*

**Static methods** As you know from using Java's Math library, the use of static methods is easy to understand. For example, when you write `Math.abs(a-b)` in a program, the effect is as if you were to replace that code with the *return value* that is produced by Java's `Math.abs()` method when passed the expression `a-b` as an *argument*. This usage is so intuitive that we have hardly needed to comment on it. If you think about what the system has to do to create this effect, you will see that it involves changing a program's *control flow*. The implications of being able to change the control flow in this way are as profound as doing so for conditionals and loops.

You can *define* static methods other than `main()` in a `.java` file by specifying a method signature, followed by a sequence of statements that constitute the method. We will consider the details shortly, but we begin with a simple example—`Harmonic` (PROGRAM 2.1.1)—that illustrates how methods affect control flow. It features a static method named `harmonic()` that takes an integer argument `n` and returns the *n*th harmonic number (see PROGRAM 1.3.5).

PROGRAM 2.1.1 is superior to our original implementation for computing harmonic numbers (PROGRAM 1.3.5) because it clearly separates the two primary tasks performed by the program: calculating harmonic numbers and interacting with the user. (For purposes of illustration, PROGRAM 2.1.1 takes several command-line arguments instead of just one.) *Whenever you can clearly separate tasks within programs, you should do so.*

**Control flow.** While `Harmonic` appeals to our familiarity with mathematical functions, we will examine it in detail so that you can think carefully about what a static method is and how it operates. `Harmonic` comprises two static methods: `harmonic()` and `main()`. Even though `harmonic()` appears first in the code, the first statement that Java executes is, as usual, the first statement in `main()`. The next few statements operate as usual, except that the code `harmonic(arg)`, which is known as a *call* on the static method `harmonic()`, causes a *transfer of control* to the first line of code in `harmonic()`, each time that it is encountered. Moreover, Java



Flow of control for a call on a static method

### Program 2.1.1 Harmonic numbers (revisited)

```
public class Harmonic
{
    public static double harmonic(int n)
    {
        double sum = 0.0;
        for (int i = 1; i <= n; i++)
            sum += 1.0/i;
        return sum;
    }

    public static void main(String[] args)
    {
        for (int i = 0; i < args.length; i++)
        {
            int arg = Integer.parseInt(args[i]);
            double value = harmonic(arg);
            StdOut.println(value);
        }
    }
}
```

sum | cumulated sum

arg  
value | argument  
value return value

This program defines two static methods, one named `harmonic()` that has integer argument `n` and computes the  $n$ th harmonic numbers (see PROGRAM 1.3.5) and one named `main()`, which tests `harmonic()` with integer arguments specified on the command line.

```
% java Harmonic 1 2 4
1.0
1.5
2.083333333333333
```

```
% java Harmonic 10 100 1000 10000
2.9289682539682538
5.187377517639621
7.485470860550343
9.787606036044348
```

initializes the *parameter variable* `n` in `harmonic()` to the value of `arg` in `main()` at the time of the call. Then, Java executes the statements in `harmonic()` as usual, until it reaches a *return statement*, which transfers control back to the statement in `main()` containing the call on `harmonic()`. Moreover, the method call `harmonic(arg)` produces a value—the value specified by the `return` statement, which is the value of the variable `sum` in `harmonic()` at the time that the `return`

statement is executed. Java then assigns this *return value* to the variable `value`. The end result exactly matches our intuition: The first value assigned to `value` and printed is 1.0—the value computed by code in `harmonic()` when the parameter variable `n` is initialized to 1. The next value assigned to `value` and printed is 1.5—the value computed by `harmonic()` when `n` is initialized to 2. The same process is repeated for each command-line argument, transferring control back and forth between `harmonic()` and `main()`.

**Function-call trace.** One simple approach to following the control flow through function calls is to imagine that each function prints its name and argument value(s) when it is called and its return value just before returning, with indentation added on calls and subtracted on returns. The result enhances the process of tracing a program by printing the values of its variables, which we have been using since SECTION 1.2. The added indentation exposes the flow of the control, and helps us check that each function has the effect that we expect. Generally, adding calls on `StdOut.println()` to trace *any* program’s control flow in this way is a fine way to begin to understand what it is doing. If the return values match our expectations, we need not trace the function code in detail, saving us a substantial amount of work.

FOR THE REST OF THIS CHAPTER, your programming will center on creating and using static methods, so it is worthwhile to consider in more detail their basic properties. Following that, we will study several examples of function implementations and applications.

**Terminology.** It is useful to draw a distinction between abstract concepts and Java mechanisms to implement them (the Java `if` statement implements the conditional, the `while` statement implements the loop, and so forth). Several concepts are rolled up in the idea of a mathematical function, and there are Java constructs corresponding to each, as summarized in the table at the top of the next page. While these formalisms have served mathematicians well for centuries (and have served programmers well for decades), we will refrain from considering in detail all of the implications of this correspondence and focus on those that will help you learn to program.

```
i = 1
arg = 1
harmonic(1)
    sum = 0.0
    sum = 1.0
    return 1.0
value = 1.0
i = 2
arg = 2
harmonic(2)
    sum = 0.0
    sum = 1.0
    sum = 1.5
    return 1.5
value = 1.5
i = 3
arg = 4
harmonic(4)
    sum = 0.0
    sum = 1.0
    sum = 1.5
    sum = 1.833333333333333
    sum = 2.083333333333333
    return 2.083333333333333
value = 2.083333333333333
```

*Function-call trace for  
java Harmonic 1 2 4*

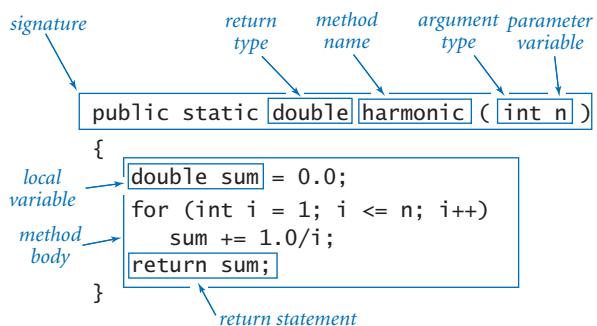
<i>concept</i>	<i>Java construct</i>	<i>description</i>
<i>function</i>	static method	mapping
<i>input value</i>	argument	input to function
<i>output value</i>	return value	output from function
<i>formula</i>	method body	function definition
<i>independent variable</i>	parameter variable	symbolic placeholder for input value

When we use a symbolic name in a formula that defines a mathematical function (such as  $f(x) = 1 + x + x^2$ ), the symbol  $x$  is a placeholder for some input value that will be substituted into the formula to determine the output value. In Java, we use a *parameter variable* as a symbolic placeholder and we refer to a particular input value where the function is to be evaluated as an *argument*.

**Static method definition.** The first line of a static method definition, known as the *signature*, gives a name to the method and to each parameter variable. It also specifies the type of each parameter variable and the return type of the method. The signature consists of the keyword `public`; the keyword `static`; the *return type*; the *method name*; and a sequence of zero or more parameter variable types and names, separated by commas and enclosed in parentheses. We will discuss the meaning of the `public` keyword in the next section and the meaning of the `static` keyword in CHAPTER 3. (Technically, the signature in Java includes only the method name and parameter types, but we leave that distinction for experts.) Following the signature is the *body* of the method, enclosed in curly braces. The body consists of the kinds of statements we discussed in CHAPTER 1. It also can contain a *return statement*, which transfers control back to the point where the static method was called and returns the result of the computation or *return value*. The body may declare *local variables*, which are variables that are available only inside the method in which they are declared.

```
public static double harmonic (int n)
{
    double sum = 0.0;
    for (int i = 1; i <= n; i++)
        sum += 1.0/i;
    return sum;
}
```

Anatomy of a static method



## *Anatomy of a static method*

**Function calls.** As you have already seen, a static method call in Java is nothing more than the method name followed by its arguments, separated by commas and enclosed in parentheses, in precisely the same form as is customary for mathematical functions. As noted in SECTION 1.2, a method call is an expression, so you can use it to build up more complicated expressions. Similarly, an argument is an expression—Java evaluates the expression and passes the resulting value to the method. So, you can write code like `Math.exp(-x*x/2) / Math.sqrt(2*Math.PI)` and Java knows what you mean.

```
for (int i = 0; i < args.length; i++)
{
    arg = Integer.parseInt(args[i]);
    double value = harmonic(arg);
    StdOut.println(value);
}
```

Anatomy of a function call

**Multiple arguments.** Like a mathematical function, a Java static method can take on more than one argument, and therefore can have more than one parameter variable. For example, the following static method computes the length of the hypotenuse of a right triangle with sides of length `a` and `b`:

```
public static double hypotenuse(double a, double b)
{   return Math.sqrt(a*a + b*b); }
```

Although the parameter variables are of the same type in this case, in general they can be of different types. The type and the name of each parameter variable are declared in the function signature, with the declarations for each variable separated by commas.

**Multiple methods.** You can define as many static methods as you want in a `.java` file. Each method has a body that consists of a sequence of statements enclosed in curly braces. These methods are independent and can appear in any order in the file. A static method can call any other static method in the same file or any static method in a Java library such as `Math`, as illustrated with this pair of methods:

```
public static double square(double a)
{   return a*a; }

public static double hypotenuse(double a, double b)
{   return Math.sqrt(square(a) + square(b)); }
```

Also, as we see in the next section, a static method can call static methods in other `.java` files (provided they are accessible to Java). In SECTION 2.3, we consider the ramifications of the idea that a static method can even call *itself*.

**Overloading.** Static methods with different signatures are different static methods. For example, we often want to define the same operation for values of different numeric types, as in the following static methods for computing absolute values:

```
public static int abs(int x)
{
    if (x < 0) return -x;
    else        return x;
}

public static double abs(double x)
{
    if (x < 0.0) return -x;
    else        return x;
}
```

These are two different methods, but are sufficiently similar so as to justify using the same name (`abs`). Using the same name for two static methods whose signatures differ is known as *overloading*, and is a common practice in Java programming. For example, the Java Math library uses this approach to provide implementations of `Math.abs()`, `Math.min()`, and `Math.max()` for all primitive numeric types. Another common use of overloading is to define two different versions of a method: one that takes an argument and another that uses a default value for that argument.

**Multiple return statements.** You can put `return` statements in a method wherever you need them: control goes back to the calling program as soon as the first `return` statement is reached. This *primality-testing* function is an example of a function that is natural to define using multiple `return` statements:

```
public static boolean isPrime(int n)
{
    if (n < 2) return false;
    for (int i = 2; i <= n/i; i++)
        if (n % i == 0) return false;
    return true;
}
```

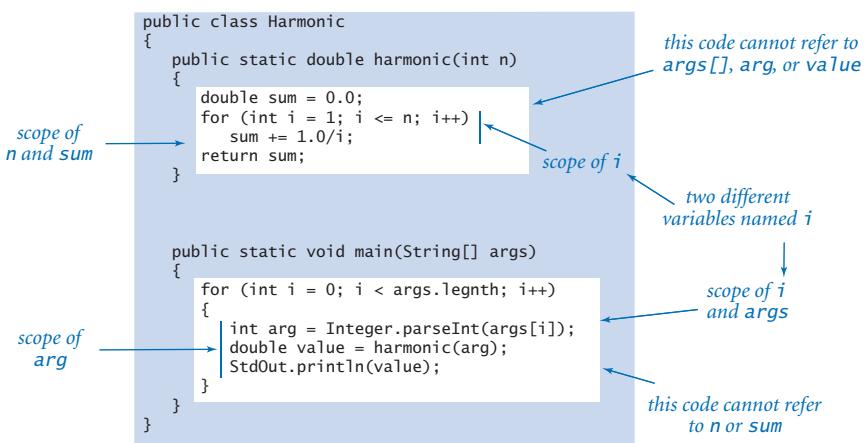
Even though there may be multiple `return` statements, any static method returns a single value each time it is invoked: the value following the first `return` statement encountered. Some programmers insist on having only one `return` per method, but we are not so strict in this book.

<i>absolute value of an int value</i>	<pre>public static int abs(int x) {     if (x &lt; 0) return -x;     else        return x; }</pre>
<i>absolute value of a double value</i>	<pre>public static double abs(double x) {     if (x &lt; 0.0) return -x;     else          return x; }</pre>
<i>primality test</i>	<pre>public static boolean isPrime(int n) {     if (n &lt; 2) return false;     for (int i = 2; i &lt;= n/i; i++)         if (n % i == 0) return false;     return true; }</pre>
<i>hypotenuse of a right triangle</i>	<pre>public static double hypotenuse(double a, double b) {   return Math.sqrt(a*a + b*b); }</pre>
<i>harmonic number</i>	<pre>public static double harmonic(int n) {     double sum = 0.0;     for (int i = 1; i &lt;= n; i++)         sum += 1.0 / i;     return sum; }</pre>
<i>uniform random integer in <math>[0, n]</math></i>	<pre>public static int uniform(int n) {   return (int) (Math.random() * n); }</pre>
<i>draw a triangle</i>	<pre>public static void drawTriangle(double x0, double y0,                                double x1, double y1,                                double x2, double y2 ) {     StdDraw.line(x0, y0, x1, y1);     StdDraw.line(x1, y1, x2, y2);     StdDraw.line(x2, y2, x0, y0); }</pre>

*Typical code for implementing functions (static methods)*

**Single return value.** A Java method provides only one return value to the caller, of the type declared in the method signature. This policy is not as restrictive as it might seem because Java data types can contain more information than the value of a single primitive type. For example, you will see later in this section that you can use arrays as return values.

**Scope.** The *scope* of a variable is the part of the program that can refer to that variable by name. The general rule in Java is that the scope of the variables declared in a block of statements is limited to the statements in that block. In particular, the scope of a variable declared in a static method is limited to that method's body. Therefore, you cannot refer to a variable in one static method that is declared in another. If the method includes smaller blocks—for example, the body of an `if` or a `for` statement—the scope of any variables declared in one of those blocks is limited to just the statements within that block. Indeed, it is common practice to use the same variable names in independent blocks of code. When we do so, we are declaring different independent variables. For example, we have been following this practice when we use an index `i` in two different `for` loops in the same program. A guiding principle when designing software is that each variable should be declared so that its scope is as small as possible. One of the important reasons that we use static methods is that they ease debugging by limiting variable scope.



Scope of local and parameter variables

*Side effects.* In mathematics, a function maps one or more input values to some output value. In computer programming, many functions fit that same model: they accept one or more arguments, and their only purpose is to return a value. A *pure function* is a function that, given the same arguments, always returns the same value, without producing any observable *side effects*, such as consuming input, producing output, or otherwise changing the state of the system. The functions `harmonic()`, `abs()`, `isPrime()`, and `hypotenuse()` are examples of pure functions.

However, in computer programming it is also useful to define functions that do produce side effects. In fact, we often define functions whose *only* purpose is to produce side effects. In Java, a static method may use the keyword `void` as its return type, to indicate that it has no return value. An explicit `return` is not necessary in a `void` static method: control returns to the caller after Java executes the method's last statement.

For example, the static method `StdOut.println()` has the side effect of printing the given argument to standard output (and has no return value). Similarly, the following static method has the side effect of drawing a triangle to standard drawing (and has no specified return value):

```
public static void drawTriangle(double x0, double y0,
                                double x1, double y1,
                                double x2, double y2)
{
    StdDraw.line(x0, y0, x1, y1);
    StdDraw.line(x1, y1, x2, y2);
    StdDraw.line(x2, y2, x0, y0);
}
```

It is generally poor style to write a static method that both produces side effects and returns a value. One notable exception arises in functions that read input. For example, `StdIn.readInt()` both returns a value (an integer) and produces a side effect (consuming one integer from standard input). In this book, we use `void` static methods for two primary purposes:

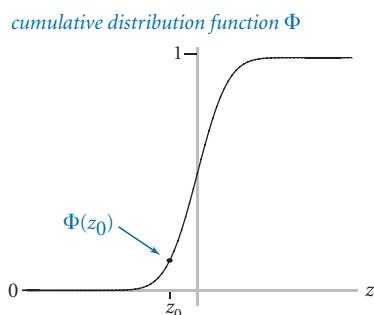
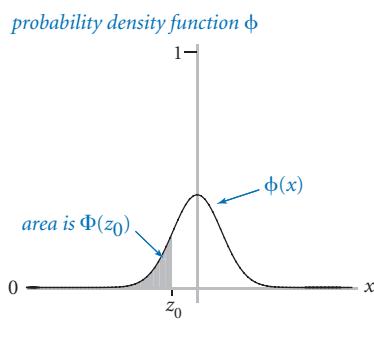
- For I/O, using `StdIn`, `StdOut`, `StdDraw`, and `StdAudio`
- To manipulate the contents of arrays

You have been using `void` static methods for output since `main()` in `HelloWorld`, and we will discuss their use with arrays later in this section. It is possible in Java to write methods that have other side effects, but we will avoid doing so until CHAPTER 3, where we do so in a specific manner supported by Java.

**Implementing mathematical functions** Why not just use the methods that are defined within Java, such as `Math.sqrt()`? The answer to this question is that we *do* use such implementations when they are present. Unfortunately, there are an unlimited number of mathematical functions that we may wish to use and only a small set of functions in the library. When you encounter a mathematical function that is not in the library, you need to implement a corresponding static method.

As an example, we consider the kind of code required for a familiar and important application that is of interest to many high school and college students in the United States. In a recent year, more than 1 million students took a standard college entrance examination. Scores range from 400 (lowest) to 1600 (highest) on the multiple-choice parts of the test. These scores play a role in making important decisions: for example, student athletes are required to have a score of at least 820, and the minimum eligibility requirement for certain academic scholarships is 1500. What percentage of test takers are ineligible for athletics? What percentage are eligible for the scholarships?

Two functions from statistics enable us to compute accurate answers to these questions. The *Gaussian (normal) probability density function* is characterized by the familiar bell-shaped curve and defined by the formula  $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ . The *Gaussian cumulative distribution function*  $\Phi(z)$  is defined to be the area under the curve defined by  $\phi(x)$  above the  $x$ -axis and to the left of the vertical line  $x=z$ . These functions play an important role in science, engineering, and finance because they arise as accurate models throughout the natural world and because they are essential in understanding experimental error.



Gaussian probability functions

In particular, these functions are known to accurately describe the distribution of test scores in our example, as a function of the mean (average value of the scores) and the standard deviation (square root of the average of the sum of the squares of the differences between each score and the mean), which are published each year. Given the mean  $\mu$  and the standard deviation  $\sigma$  of the test scores, the percentage of students with scores less than a given value  $z$  is closely approximated by the function  $\Phi((z-\mu)/\sigma)$ . Static methods to calculate  $\phi$  and  $\Phi$  are not available in Java's `Math` library, so we need to develop our own implementations.

### Program 2.1.2 Gaussian functions

```

public class Gaussian
{ // Implement Gaussian (normal) distribution functions.
    public static double pdf(double x)
    {
        return Math.exp(-x*x/2) / Math.sqrt(2*Math.PI);
    }

    public static double cdf(double z)
    {
        if (z < -8.0) return 0.0;
        if (z > 8.0) return 1.0;
        double sum = 0.0;
        double term = z;
        for (int i = 3; sum != sum + term; i += 2)
        {
            sum = sum + term;
            term = term * z * z / i;
        }
        return 0.5 + pdf(z) * sum;
    }

    public static void main(String[] args)
    {
        double z      = Double.parseDouble(args[0]);
        double mu     = Double.parseDouble(args[1]);
        double sigma = Double.parseDouble(args[2]);
        StdOut.printf("%.3f\n", cdf((z - mu) / sigma));
    }
}

```

sum	cumulated sum
term	current term

This code implements the Gaussian probability density function (`pdf`) and Gaussian cumulative distribution function (`cdf`), which are not implemented in Java's `Math` library. The `pdf()` implementation follows directly from its definition, and the `cdf()` implementation uses a Taylor series and also calls `pdf()` (see accompanying text and EXERCISE 1.3.38).

```

% java Gaussian 820 1019 209
0.171
% java Gaussian 1500 1019 209
0.989
% java Gaussian 1500 1025 231
0.980

```

*Closed form.* In the simplest situation, we have a closed-form mathematical formula defining our function in terms of functions that are implemented in the library. This situation is the case for  $\phi$ —the Java Math library includes methods to compute the exponential and the square root functions (and a constant value for  $\pi$ ), so a static method `pdf()` corresponding to the mathematical definition is easy to implement (see PROGRAM 2.1.2).

*No closed form.* Otherwise, we may need a more complicated algorithm to compute function values. This situation is the case for  $\Phi$ —no closed-form expression exists for this function. Such algorithms sometimes follow immediately from Taylor series approximations, but developing reliably accurate implementations of mathematical functions is an art that needs to be addressed carefully, taking advantage of the knowledge built up in mathematics over the past several centuries. Many different approaches have been studied for evaluating  $\Phi$ . For example, a Taylor series approximation to the ratio of  $\Phi$  and  $\phi$  turns out to be an effective basis for evaluating the function:

$$\Phi(z) = 1/2 + \phi(z) ( z + z^3/3 + z^5/(3\cdot5) + z^7/(3\cdot5\cdot7) + \dots )$$

This formula readily translates to the Java code for the static method `cdf()` in PROGRAM 2.1.2. For small (respectively large)  $z$ , the value is extremely close to 0 (respectively 1), so the code directly returns 0 (respectively 1); otherwise, it uses the Taylor series to add terms until the sum converges.

Running `Gaussian` with the appropriate arguments on the command line tells us that about 17% of the test takers were ineligible for athletics and that only about 1% qualified for the scholarship. In a year when the mean was 1025 and the standard deviation 231, about 2% qualified for the scholarship.

COMPUTING WITH MATHEMATICAL FUNCTIONS OF ALL kinds has always played a central role in science and engineering. In a great many applications, the functions that you need are expressed in terms of the functions in Java’s Math library, as we have just seen with `pdf()`, or in terms of Taylor series approximations that are easy to compute, as we have just seen with `cdf()`. Indeed, support for such computations has played a central role throughout the evolution of computing systems and programming languages. You will find many examples on the booksite and throughout this book.

**Using static methods to organize code** Beyond evaluating mathematical functions, the process of calculating an output value on the basis of an input value is important as a general technique for organizing control flow in *any* computation. Doing so is a simple example of an extremely important principle that is a prime guiding force for any good programmer: *whenever you can clearly separate tasks within programs, you should do so.*

Functions are natural and universal for expressing computational tasks. Indeed, the “bird’s-eye view” of a Java program that we began with in SECTION 1.1 was equivalent to a function: we began by thinking of a Java program as a function that transforms command-line arguments into an output string. This view expresses itself at many different levels of computation. In particular, it is generally the case that a long program is more naturally expressed in terms of functions instead of as a sequence of Java assignment, conditional, and loop statements. With the ability to define functions, we can better organize our programs by defining functions within them when appropriate.

For example, Coupon (PROGRAM 2.1.3) is a version of CouponCollector (PROGRAM 1.4.2) that better separates the individual components of the computation. If you study PROGRAM 1.4.2, you will identify three separate tasks:

- Given  $n$ , compute a random coupon value.
- Given  $n$ , do the coupon collection experiment.
- Get  $n$  from the command line, and then compute and print the result.

Coupon rearranges the code in CouponCollector to reflect the reality that these three functions underlie the computation. With this organization, we could change `getCoupon()` (for example, we might want to draw the random numbers from a different distribution) or `main()` (for example, we might want to take multiple inputs or run multiple experiments) without worrying about the effect of any changes in `collectCoupons()`.

Using static methods isolates the implementation of each component of the collection experiment from others, or *encapsulates* them. Typically, programs have many independent components, which magnifies the benefits of separating them into different static methods. We will discuss these benefits in further detail after we have seen several other examples, but you certainly can appreciate that it is better to express a computation in a program by breaking it up into functions, just as it is better to express an idea in an essay by breaking it up into paragraphs. *Whenever you can clearly separate tasks within programs, you should do so.*

### Program 2.1.3 Coupon collector (revisited)

```

public class Coupon
{
    public static int getCoupon(int n)
    { // Return a random integer between 0 and n-1.
        return (int) (Math.random() * n);
    }

    public static int collectCoupons(int n)
    { // Collect coupons until getting one of each value
        // and return the number of coupons collected.
        boolean[] isCollected = new boolean[n];
        int count = 0, distinct = 0;
        while (distinct < n)
        {
            int r = getcoupon(n);
            count++;
            if (!isCollected[r])
                distinct++;
            isCollected[r] = true;
        }
        return count;
    }

    public static void main(String[] args)
    { // Collect n different coupons.
        int n = Integer.parseInt(args[0]);
        int count = collectCoupons(n);
        StdOut.println(count);
    }
}

```

n	<i># coupon values (0 to n-1)</i>
isCollected[i]	<i>has coupon i been collected?</i>
count	<i># coupons collected</i>
distinct	<i># distinct coupons collected</i>
r	<i>random coupon</i>

This version of PROGRAM 1.4.2 illustrates the style of encapsulating computations in static methods. This code has the same effect as CouponCollector, but better separates the code into its three constituent pieces: generating a random integer between 0 and n-1, running a coupon collection experiment, and managing the I/O.

```
% java Coupon 1000
6522
% java Coupon 1000
6481
```

```
% java Coupon 10000
105798
% java Coupon 1000000
12783771
```

**Passing arguments and returning values** Next, we examine the specifics of Java’s mechanisms for passing arguments to and returning values from functions. These mechanisms are conceptually very simple, but it is worthwhile to take the time to understand them fully, as the effects are actually profound. Understanding argument-passing and return-value mechanisms is key to learning any new programming language.

*Pass by value.* You can use parameter variables anywhere in the code in the body of the function in the same way you use local variables. The only difference between a parameter variable and a local variable is that Java evaluates the argument provided by the calling code and initializes the parameter variable with the resulting value. This approach is known as *pass by value*. The method works with the *value* of its arguments, not the arguments themselves. One consequence of this approach is that changing the value of a parameter variable within a static method has no effect on the calling code. (For clarity, we do not change parameter variables in the code in this book.) An alternative approach known as *pass by reference*, where the method works directly with the calling code’s arguments, is favored in some programming environments.

A STATIC METHOD CAN TAKE AN array as an argument or return an array to the caller. This capability is a special case of Java’s object orientation, which is the subject of CHAPTER 3. We consider it in the present context because the basic mechanisms are easy to understand and to use, leading us to compact solutions to a number of problems that naturally arise when we use arrays to help us process large amounts of data.

*Arrays as arguments.* When a static method takes an array as an argument, it implements a function that operates on an arbitrary number of values of the same type. For example, the following static method computes the mean (average) of an array of double values:

```
public static double mean(double[] a)
{
    double sum = 0.0;
    for (int i = 0; i < a.length; i++)
        sum += a[i];
    return sum / a.length;
}
```

We have been using arrays as arguments since our first program. The code

```
public static void main(String[] args)
```

defines `main()` as a static method that takes an array of strings as an argument and returns nothing. By convention, the Java system collects the strings that you type after the program name in the `java` command into an array and calls `main()` with that array as argument. (Most programmers use the name `args` for the parameter variable, even though any name at all would do.) Within `main()`, we can manipulate that array just like any other array.

*Side effects with arrays.* It is often the case that the purpose of a static method that takes an array as argument is to produce a side effect (change values of array elements). A prototypical example of such a method is one that exchanges the values at two given indices in a given array. We can adapt the code that we examined at the beginning of SECTION 1.4:

```
public static void exchange(String[] a, int i, int j)
{
    String temp = a[i];
    a[i] = a[j];
    a[j] = temp;
}
```

This implementation stems naturally from the Java array representation. The parameter variable in `exchange()` is a reference to the array, not a copy of the array values: when you pass an array as an argument to a method, the method has an opportunity to reassign values to the elements in that array. A second prototypical example of a static method that takes an array argument and produces side effects is one that randomly shuffles the values in the array, using this version of the algorithm that we examined in SECTION 1.4 (and the `exchange()` and `uniform()` methods considered earlier in this section):

```
public static void shuffle(String[] a)
{
    int n = a.length;
    for (int i = 0; i < n; i++)
        exchange(a, i, i + uniform(n-i));
}
```

<i>find the maximum of the array values</i>	<pre>public static double max(double[] a) {     double max = Double.NEGATIVE_INFINITY;     for (int i = 0; i &lt; a.length; i++)         if (a[i] &gt; max) max = a[i];     return max; }</pre>
<i>dot product</i>	<pre>public static double dot(double[] a, double[] b) {     double sum = 0.0;     for (int i = 0; i &lt; a.length; i++)         sum += a[i] * b[i];     return sum; }</pre>
<i>exchange the values of two elements in an array</i>	<pre>public static void exchange(String[] a, int i, int j) {     String temp = a[i];     a[i] = a[j];     a[j] = temp; }</pre>
<i>print a one-dimensional array (and its length)</i>	<pre>public static void print(double[] a) {     StdOut.println(a.length);     for (int i = 0; i &lt; a.length; i++)         StdOut.println(a[i]); }</pre>
<i>read a 2D array of double values (with dimensions) in row-major order</i>	<pre>public static double[][] readDouble2D() {     int m = StdIn.readInt();     int n = StdIn.readInt();     double[][] a = new double[m][n];     for (int i = 0; i &lt; m; i++)         for (int j = 0; j &lt; n; j++)             a[i][j] = StdIn.readDouble();     return a; }</pre>

*Typical code for implementing functions with array arguments or return values*

Similarly, we will consider in SECTION 4.2 methods that *sort* an array (rearrange its values so that they are in order). All of these examples highlight the basic fact that the mechanism for passing arrays in Java is *call by value* with respect to the array reference but *call by reference* with respect to the array elements. Unlike primitive-type arguments, the changes that a method makes to the elements of an array *are* reflected in the client program. A method that takes an array as its argument cannot change the array itself—the memory location, length, and type of the array are the same as they were when the array was created—but a method can assign different values to the elements in the array.

*Arrays as return values.* A method that sorts, shuffles, or otherwise modifies an array taken as an argument does not have to return a reference to that array, because it is changing the elements of a client array, not a copy. But there are many situations where it is useful for a static method to provide an array as a return value. Chief among these are static methods that create arrays for the purpose of returning multiple values of the same type to a client. For example, the following static method creates and returns an array of the kind used by StdAudio (see PROGRAM 1.5.7): it contains values sampled from a sine wave of a given frequency (in hertz) and duration (in seconds), sampled at the standard 44,100 samples per second.

```
public static double[] tone(double hz, double t)
{
    int SAMPLING_RATE = 44100;
    int n = (int) (SAMPLING_RATE * t);
    double[] a = new double[n+1];
    for (int i = 0; i <= n; i++)
        a[i] = Math.sin(2 * Math.PI * i * hz / SAMPLING_RATE);
    return a;
}
```

In this code, the length of the array returned depends on the duration: if the given duration is  $t$ , the length of the array is about  $44100 \cdot t$ . With static methods like this one, we can write code that treats a sound wave as a single entity (an array containing sampled values), as we will see next in PROGRAM 2.1.4.

**Example: superposition of sound waves** As discussed in SECTION 1.5, the simple audio model that we studied there needs to be embellished to create sound that resembles the sound produced by a musical instrument. Many different embellishments are possible; with static methods we can systematically apply them to produce sound waves that are far more complicated than the simple sine waves that we produced in SECTION 1.5. As an illustration of the effective use of static methods to solve an interesting computational problem, we consider a program that has essentially the same functionality as `PlayThatTune` (PROGRAM 1.5.7), but adds harmonic tones one octave above and one octave below each note to produce a more realistic sound.

*Chords and harmonics.* Notes like concert A have a pure sound that is not very musical, because the sounds that you are accustomed to hearing have many other components. The sound from the guitar string echoes off the wooden part of the

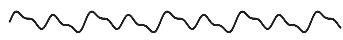
A		440.00
C♯		554.37
E		659.26

*A major chord*



A		440.00
A		220.00
A		880.00

*concert A with harmonics*



*Superposing waves to make composite sounds*

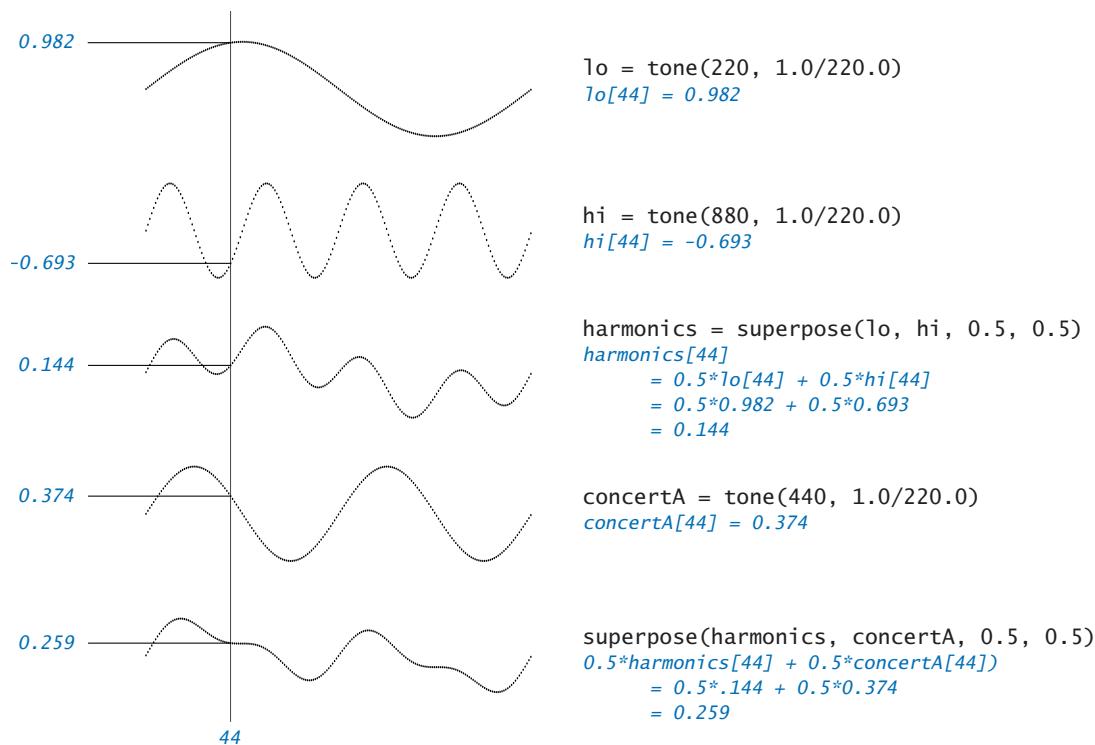
instrument, the walls of the room that you are in, and so forth. You may think of such effects as modifying the basic sine wave. For example, most musical instruments produce *harmonics* (the same note in different octaves and not as loud), or you might play chords (multiple notes at the same time). To combine multiple sounds, we use *superposition*: simply add the waves together and rescale to make sure that all values stay between  $-1$  and  $+1$ . As it turns out, when we superpose sine waves of different frequencies in this way, we can get arbitrarily

complicated waves. Indeed, one of the triumphs of 19th-century mathematics was the development of the idea that any smooth periodic function can be expressed as a sum of sine and cosine waves, known as a *Fourier series*. This mathematical idea corresponds to the notion that we can create a large range of sounds with musical instruments or our vocal cords and that all sound consists of a composition of various oscillating curves. Any sound corresponds to a curve and any curve corresponds to a sound, and we can create arbitrarily complex curves with superposition.

*Weighted superposition.* Since we represent sound waves by arrays of numbers that represent their values at the same sample points, superposition is simple to implement: we add together the values at each sample point to produce the combined result and then rescale. For greater control, we specify a relative weight for each of the two waves to be added, with the property that the weights are positive and sum to 1. For example, if we want the first sound to have three times the effect of the second, we would assign the first a weight of 0.75 and the second a weight of 0.25. Now, if one wave is in an array `a[]` with relative weight `awt` and the other is in an array `b[]` with relative weight `bwt`, we compute their weighted sum with the following code:

```
double[] c = new double[a.length];
for (int i = 0; i < a.length; i++)
    c[i] = a[i]*awt + b[i]*bwt;
```

The conditions that the weights are positive and sum to 1 ensure that this operation preserves our convention of keeping the values of all of our waves between  $-1$  and  $+1$ .



*Adding harmonics to concert A (1/220 second at 44,100 samples/second)*

### Program 2.1.4 Play that tune (revisited)

```

public class PlayThatTuneDeluxe
{
    public static double[] superpose(double[] a, double[] b,
                                     double awt, double bwt)
    { // Weighted superposition of a and b.
        double[] c = new double[a.length];
        for (int i = 0; i < a.length; i++)
            c[i] = a[i]*awt + b[i]*bwt;
        return c;
    }

    public static double[] tone(double hz, double t)
    { /* see text */ }

    public static double[] note(int pitch, double t)
    { // Play note of given pitch, with harmonics.
        double hz = 440.0 * Math.pow(2, pitch / 12.0);
        double[] a = tone(hz, t);
        double[] hi = tone(2*hz, t);
        double[] lo = tone(hz/2, t);
        double[] h = superpose(hi, lo, 0.5, 0.5);
        return superpose(a, h, 0.5, 0.5);
    }

    public static void main(String[] args)
    { // Read and play a tune, with harmonics.
        while (!StdIn.isEmpty())
        { // Read and play a note, with harmonics.
            int pitch = StdIn.readInt();
            double duration = StdIn.readDouble();
            double[] a = note(pitch, duration);
            StdAudio.play(a);
        }
    }
}

```

hz	frequency
a[]	pure tone
hi[]	upper harmonic
lo[]	lower harmonic
h[]	tone with harmonics

This code embellishes the sounds produced by PROGRAM 1.5.7 by using static methods to create harmonics, which results in a more realistic sound than the pure tone.

```
% more elise.txt
7 0.25
6 0.25
7 0.25
6 0.25
...

```

```
% java PlayThatTuneDeluxe < elise.txt
```

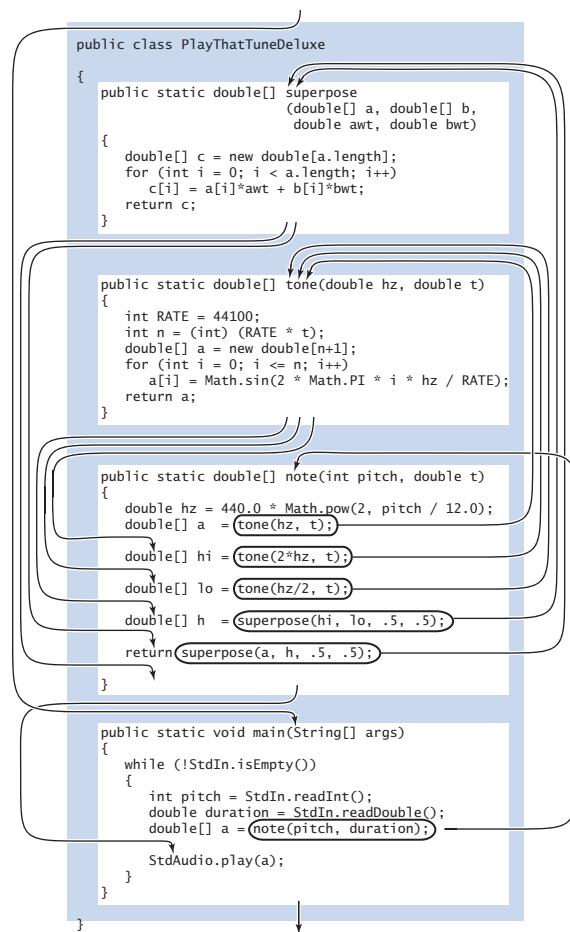


PROGRAM 2.1.4 IS AN IMPLEMENTATION THAT applies these concepts to produce a more realistic sound than that produced by PROGRAM 1.5.7. To do so, it makes use of functions to divide the computation into four parts:

- Given a frequency and duration, create a pure tone.
- Given two sound waves and relative weights, superpose them.
- Given a pitch and duration, create a note with harmonics.
- Read and play a sequence of pitch/duration pairs from standard input.

These tasks are each amenable to implementation as a function, with all of the functions then depending on one another. Each function is well defined and straightforward to implement. All of them (and StdAudio) represent sound as a sequence of floating-point numbers kept in an array, corresponding to sampling a sound wave at 44,100 samples per second.

Up to this point, the use of functions has been somewhat of a notational convenience. For example, the control flow in PROGRAM 2.1.1–2.1.3 is simple—each function is called in just one place in the code. By contrast, `PlayThatTuneDeluxe` (PROGRAM 2.1.4) is a convincing example of the effectiveness of defining functions to organize a computation because the functions are each called multiple times. For example, the function `note()` calls the function `tone()` three times and the function `sum()` twice. Without functions methods, we would need multiple copies of the code in



*Flow of control among several static methods*

`tone()` and `sum()`; with functions, we can deal directly with concepts close to the application. Like loops, functions have a simple but profound effect: one sequence of statements (those in the method definition) is executed multiple times during the execution of our program—once for each time the function is called in the control flow in `main()`.

FUNCTIONS (STATIC METHODS) ARE IMPORTANT BECAUSE they give us the ability to *extend* the Java language within a program. Having implemented and debugged functions such as `harmonic()`, `pdf()`, `cdf()`, `mean()`, `abs()`, `exchange()`, `shuffle()`, `isPrime()`, `uniform()`, `superpose()`, `note()`, and `tone()`, we can use them almost as if they were built into Java. The flexibility to do so opens up a whole new world of programming. Before, you were safe in thinking about a Java program as a sequence of statements. Now you need to think of a Java program as a *set of static methods* that can call one another. The statement-to-statement control flow to which you have been accustomed is still present within static methods, but programs have a higher-level control flow defined by static method calls and returns. This ability enables you to think in terms of operations called for by the application, not just the simple arithmetic operations on primitive types that are built into Java.

*Whenever you can clearly separate tasks within programs, you should do so.* The examples in this section (and the programs throughout the rest of the book) clearly illustrate the benefits of adhering to this maxim. With static methods, we can

- Divide a long sequence of statements into independent parts.
- Reuse code without having to copy it.
- Work with higher-level concepts (such as sound waves).

This produces code that is easier to understand, maintain, and debug than a long program composed solely of Java assignment, conditional, and loop statements. In the next section, we discuss the idea of using static methods defined in *other* programs, which again takes us to another level of programming.

## Q&A

**Q.** What happens if I leave out the keyword `static` when defining a static method?

**A.** As usual, the best way to answer a question like this is to try it yourself and see what happens. Here is the result of omitting the `static` modifier from `harmonic()` in `Harmonic`:

```
Harmonic.java:15: error: non-static method harmonic(int)
    cannot be referenced from a static context
        double value = harmonic(arg);
                           ^
1 error
```

Non-static methods are different from static methods. You will learn about the former in CHAPTER 3.

**Q.** What happens if I write code after a `return` statement?

**A.** Once a `return` statement is reached, control immediately returns to the caller, so any code after a `return` statement is useless. Java identifies this situation as a compile-time error, reporting `unreachable code`.

**Q.** What happens if I do not include a `return` statement?

**A.** There is no problem, if the return type is `void`. In this case, control will return to the caller after the last statement. When the return type is not `void`, Java will report a `missing return statement` compile-time error if there is *any* path through the code that does not end in a `return` statement.

**Q.** Why do I need to use the return type `void`? Why not just omit the return type?

**A.** Java requires it; we have to include it. Second-guessing a decision made by a programming-language designer is the first step on the road to becoming one.

**Q.** Can I return from a `void` function by using `return`? If so, which return value should I use?

**A.** Yes. Use the statement `return;` with no return value.



**Q.** This issue with side effects and arrays passed as arguments is confusing. Is it really all that important?

**A.** Yes. Properly controlling side effects is one of a programmer’s most important tasks in large systems. Taking the time to be sure that you understand the difference between passing a value (when arguments are of a primitive type) and passing a reference (when arguments are arrays) will certainly be worthwhile. The very same mechanism is used for all other types of data, as you will learn in CHAPTER 3.

**Q.** So why not just eliminate the possibility of side effects by making all arguments pass by value, including arrays?

**A.** Think of a huge array with, say, millions of elements. Does it make sense to copy all of those values for a static method that is going to exchange just two of them? For this reason, most programming languages support passing an array to a function without creating a copy of the array elements—Matlab is a notable exception.

**Q.** In which order does Java evaluate method calls?

**A.** Regardless of operator precedence or associativity, Java evaluates subexpressions (including method calls) and argument lists from left to right. For example, when evaluating the expression

$$f1() + f2() * f3(f4(), f5())$$

Java calls the methods in the order `f1()`, `f2()`, `f4()`, `f5()`, and `f3()`. This is most relevant for methods that produce side effects. As a matter of style, we avoid writing code that depends on the order of evaluation.

## Exercises

**2.1.1** Write a static method `max3()` that takes three `int` arguments and returns the value of the largest one. Add an overloaded function that does the same thing with three `double` values.

**2.1.2** Write a static method `odd()` that takes three `boolean` arguments and returns `true` if an odd number of the argument values are `true`, and `false` otherwise.

**2.1.3** Write a static method `majority()` that takes three `boolean` arguments and returns `true` if at least two of the argument values are `true`, and `false` otherwise. Do not use an `if` statement.

**2.1.4** Write a static method `eq()` that takes two `int` arrays as arguments and returns `true` if the arrays have the same length and all corresponding pairs of elements are equal, and `false` otherwise.

**2.1.5** Write a static method `areTriangular()` that takes three `double` arguments and returns `true` if they could be the sides of a triangle (none of them is greater than or equal to the sum of the other two). See EXERCISE 1.2.15.

**2.1.6** Write a static method `sigmoid()` that takes a `double` argument  $x$  and returns the `double` value obtained from the formula  $1 / (1 + e^{-x})$ .

**2.1.7** Write a static method `sqrt()` that takes a `double` argument and returns the square root of that number. Use Newton's method (see PROGRAM 1.3.6) to compute the result.

**2.1.8** Give the function-call trace for `java Harmonic 3 5`

**2.1.9** Write a static method `lg()` that takes a `double` argument  $n$  and returns the base-2 logarithm of  $n$ . You may use Java's `Math` library.

**2.1.10** Write a static method `lg()` that takes an `int` argument  $n$  and returns the largest integer not larger than the base-2 logarithm of  $n$ . Do *not* use the `Math` library.

**2.1.11** Write a static method `signum()` that takes an `int` argument  $n$  and returns `-1` if  $n$  is less than 0, `0` if  $n$  is equal to 0, and `+1` if  $n$  is greater than 0.

**2.1.12** Consider the static method `duplicate()` below.

```
public static String duplicate(String s)
{
    String t = s + s;
    return t;
}
```

What does the following code fragment do?

```
String s = "Hello";
s = duplicate(s);
String t = "Bye";
t = duplicate(duplicate(duplicate(t)));
StdOut.println(s + t);
```

**2.1.13** Consider the static method `cube()` below.

```
public static void cube(int i)
{
    i = i * i * i;
}
```

How many times is the following `for` loop iterated?

```
for (int i = 0; i < 1000; i++)
    cube(i);
```

*Answer:* Just 1,000 times. A call to `cube()` has no effect on the client code. It changes the value of its local parameter variable `i`, but that change has no effect on the `i` in the `for` loop, which is a different variable. If you replace the call to `cube(i)` with the statement `i = i * i * i;` (maybe that was what you were thinking), then the loop is iterated five times, with `i` taking on the values 0, 1, 2, 9, and 730 at the beginning of the five iterations.

**2.1.14** The following *checksum* formula is widely used by banks and credit card companies to validate legal account numbers:

$$d_0 + f(d_1) + d_2 + f(d_3) + d_4 + f(d_5) + \dots = 0 \pmod{10}$$

The  $d_i$  are the decimal digits of the account number and  $f(d)$  is the sum of the decimal digits of  $2d$  (for example,  $f(7) = 5$  because  $2 \times 7 = 14$  and  $1 + 4 = 5$ ). For example, 17,327 is valid because  $1 + 5 + 3 + 4 + 7 = 20$ , which is a multiple of 10. Implement the function  $f$  and write a program to take a 10-digit integer as a command-line argument and print a valid 11-digit number with the given integer as its first 10 digits and the checksum as the last digit.

**2.1.15** Given two stars with angles of *declination* and *right ascension* ( $d_1, a_1$ ) and ( $d_2, a_2$ ), the angle they subtend is given by the formula

$$2 \arcsin((\sin^2(d/2) + \cos(d_1)\cos(d_2)\sin^2(a/2))^{1/2})$$

where  $a_1$  and  $a_2$  are angles between  $-180$  and  $180$  degrees,  $d_1$  and  $d_2$  are angles between  $-90$  and  $90$  degrees,  $a = a_2 - a_1$ , and  $d = d_2 - d_1$ . Write a program to take the declination and right ascension of two stars as command-line arguments and print the angle they subtend. *Hint:* Be careful about converting from degrees to radians.

**2.1.16** Write a static method `scale()` that takes a `double` array as its argument and has the side effect of scaling the array so that each element is between 0 and 1 (by subtracting the minimum value from each element and then dividing each element by the difference between the minimum and maximum values). Use the `max()` method defined in the table in the text, and write and use a matching `min()` method.

**2.1.17** Write a static method `reverse()` that takes an array of strings as its argument and returns a new array with the strings in reverse order. (Do not change the order of the strings in the argument array.) Write a static method `reverseInplace()` that takes an array of strings as its argument and produces the side effect of reversing the order of the strings in the argument array.



**2.1.18** Write a static method `readBoolean2D()` that reads a two-dimensional boolean matrix (with dimensions) from standard input and returns the resulting two-dimensional array.

**2.1.19** Write a static method `histogram()` that takes an `int` array `a[]` and an integer `m` as arguments and returns an array of length `m` whose `i`th element is the number of times the integer `i` appeared in `a[]`. Assuming the values in `a[]` are all between 0 and `m-1`, the sum of the values in the returned array should equal `a.length`.

**2.1.20** Assemble code fragments in this section and in SECTION 1.4 to develop a program that takes an integer command-line argument `n` and prints `n` five-card hands, separated by blank lines, drawn from a randomly shuffled card deck, one card per line using card names like Ace of Clubs.

**2.1.21** Write a static method `multiply()` that takes two square matrices of the same dimension as arguments and produces their product (another square matrix of that same dimension). *Extra credit:* Make your program work whenever the number of columns in the first matrix is equal to the number of rows in the second matrix.

**2.1.22** Write a static method `any()` that takes a `boolean` array as its argument and returns `true` if any of the elements in the array is `true`, and `false` otherwise. Write a static method `all()` that takes an array of `boolean` values as its argument and returns `true` if all of the elements in the array are `true`, and `false` otherwise.

**2.1.23** Develop a version of `getCoupon()` that better models the situation when one of the coupons is rare: choose one of the `n` values at random, return that value with probability  $1/(1,000n)$ , and return all other values with equal probability. *Extra credit:* How does this change affect the expected number of coupons that need to be collected in the coupon collector problem?

**2.1.24** Modify `PlayThatTune` to add harmonics two octaves away from each note, with half the weight of the one-octave harmonics.

## Creative Exercises

**2.1.25** *Birthday problem.* Develop a class with appropriate static methods for studying the birthday problem (see EXERCISE 1.4.38).

**2.1.26** *Euler's totient function.* Euler's totient function is an important function in number theory:  $\varphi(n)$  is defined as the number of positive integers less than or equal to  $n$  that are relatively prime with  $n$  (no factors in common with  $n$  other than 1). Write a class with a static method that takes an integer argument  $n$  and returns  $\varphi(n)$ , and a `main()` that takes an integer command-line argument, calls the method with that argument, and prints the resulting value.

**2.1.27** *Harmonic numbers.* Write a program `Harmonic` that contains three static methods `harmonic()`, `harmonicSmall()`, and `harmonicLarge()` for computing the harmonic numbers. The `harmonicSmall()` method should just compute the sum (as in PROGRAM 1.3.5), the `harmonicLarge()` method should use the approximation  $H_n = \log_e(n) + \gamma + 1/(2n) - 1/(12n^2) + 1/(120n^4)$  (the number  $\gamma = 0.577215664901532\dots$  is known as *Euler's constant*), and the `harmonic()` method should call `harmonicSmall()` for  $n < 100$  and `harmonicLarge()` otherwise.

**2.1.28** *Black–Scholes option valuation.* The Black–Scholes formula supplies the theoretical value of a European call option on a stock that pays no dividends, given the current stock price  $s$ , the exercise price  $x$ , the continuously compounded risk-free interest rate  $r$ , the volatility  $\sigma$ , and the time (in years) to maturity  $t$ . The Black–Scholes value is given by the formula  $s \Phi(a) - xe^{-rt} \Phi(b)$ , where  $\Phi(z)$  is the Gaussian cumulative distribution function,  $a = (\ln(s/x) + (r + \sigma^2/2)t) / (\sigma\sqrt{t})$ , and  $b = a - \sigma\sqrt{t}$ . Write a program that takes  $s$ ,  $r$ ,  $\sigma$ , and  $t$  from the command line and prints the Black–Scholes value.

**2.1.29** *Fourier spikes.* Write a program that takes a command-line argument  $n$  and plots the function

$$(\cos(t) + \cos(2t) + \cos(3t) + \dots + \cos(nt)) / n$$

for 500 equally spaced samples of  $t$  from  $-10$  to  $10$  (in radians). Run your program for  $n = 5$  and  $n = 500$ . Note: You will observe that the sum converges to a spike (0 everywhere except a single value). This property is the basis for a proof that *any* smooth function can be expressed as a sum of sinusoids.

**2.1.30** *Calendar.* Write a program `Calendar` that takes two integer command-line arguments `m` and `y` and prints the monthly calendar for month `m` of year `y`, as in this example:

```
% java Calendar 2 2009
February 2009
S M Tu W Th F S
1 2 3 4 5 6 7
8 9 10 11 12 13 14
15 16 17 18 19 20 21
22 23 24 25 26 27 28
```

*Hint:* See `LeapYear` (PROGRAM 1.2.4) and EXERCISE 1.2.29.

**2.1.31** *Horner's method.* Write a class `Horner` with a method `evaluate()` that takes a floating-point number `x` and array `p[]` as arguments and returns the result of evaluating the polynomial whose coefficients are the elements in `p[]` at `x`:

$$p(x) = p_0 + p_1x^1 + p_2x^2 + \dots + p_{n-2}x^{n-2} + p_{n-1}x^{n-1}$$

Use *Horner's method*, an efficient way to perform the computations that is suggested by the following parenthesization:

$$p(x) = p_0 + x(p_1 + x(p_2 + \dots + x(p_{n-2} + xp_{n-1}) \dots ))$$

Write a test client with a static method `exp()` that uses `evaluate()` to compute an approximation to  $e^x$ , using the first  $n$  terms of the Taylor series expansion  $e^x = 1 + x + x^2/2! + x^3/3! + \dots$ . Your client should take a command-line argument `x` and compare your result against that computed by `Math.exp(x)`.

**2.1.32** *Chords.* Develop a version of `PlayThatTune` that can handle songs with chords (including harmonics). Develop an input format that allows you to specify different durations for each chord and different amplitude weights for each note within a chord. Create test files that exercise your program with various chords and harmonics, and create a version of *Für Elise* that uses them.

**2.1.33 Benford's law.** The American astronomer Simon Newcomb observed a quirk in a book that compiled logarithm tables: the beginning pages were much grubbier than the ending pages. He suspected that scientists performed more computations with numbers starting with 1 than with 8 or 9, and postulated that, under general circumstances, the leading digit is much more likely to be 1 (roughly 30%) than the digit 9 (less than 4%). This phenomenon is known as *Benford's law* and is now often used as a statistical test. For example, IRS forensic accountants rely on it to discover tax fraud. Write a program that reads in a sequence of integers from standard input and tabulates the number of times each of the digits 1–9 is the leading digit, breaking the computation into a set of appropriate static methods. Use your program to test the law on some tables of information from your computer or from the web. Then, write a program to foil the IRS by generating random amounts from \$1.00 to \$1,000.00 with the same distribution that you observed.

**2.1.34 Binomial distribution.** Write a function

```
public static double binomial(int n, int k, double p)
```

to compute the probability of obtaining exactly  $k$  heads in  $n$  biased coin flips (heads with probability  $p$ ) using the formula

$$f(n, k, p) = p^k(1-p)^{n-k} n! / (k!(n-k)!)$$

*Hint:* To stave off overflow, compute  $x = \ln f(n, k, p)$  and then return  $e^x$ . In `main()`, take  $n$  and  $p$  from the command line and check that the sum over all values of  $k$  between 0 and  $n$  is (approximately) 1. Also, compare every value computed with the normal approximation

$$f(n, k, p) \approx \phi(np, np(1-p))$$

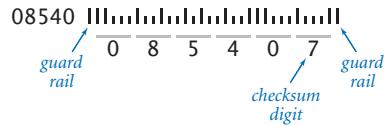
(see EXERCISE 2.2.1).

**2.1.35 Coupon collecting from a binomial distribution.** Develop a version of `getCoupon()` that uses `binomial()` from the previous exercise to return coupon values according to the binomial distribution with  $p = 1/2$ . *Hint:* Generate a uniformly random number  $x$  between 0 and 1, then return the smallest value of  $k$  for which the sum of  $f(n, j, p)$  for all  $j < k$  exceeds  $x$ . *Extra credit:* Develop a hypothesis for describing the behavior of the coupon collector function under this assumption.

**2.1.36 Postal bar codes.** The barcode used by the U.S. Postal System to route mail is defined as follows: Each decimal digit in the ZIP code is encoded using a sequence of three half-height and two full-height bars. The barcode starts and ends with a full-height bar (the guard rail) and includes a checksum digit (after the five-digit ZIP code or ZIP+4), computed by summing up the original digits modulo 10. Implement the following functions

- Draw a half-height or full-height bar on StdDraw.
- Given a digit, draw its sequence of bars.
- Compute the checksum digit.

Also implement a test client that reads in a five- (or nine-) digit ZIP code as the command-line argument and draws the corresponding postal bar code.



## 2.2 Libraries and Clients

EACH PROGRAM THAT YOU HAVE WRITTEN so far consists of Java code that resides in a single .java file. For large programs, keeping all the code in a single file in this way is restrictive and unnecessary. Fortunately, it is very easy in Java to refer to a method in one file that is defined in another. This ability has two important consequences on our style of programming.

First, it enables *code reuse*. One program can make use of code that is already written and debugged, not by copying the code, but just by referring to it. This ability to define code that can be reused is an essential part of modern programming. It amounts to extending Java—you can define and use your own operations on data.

Second, it enables *modular programming*. You can not only divide a program up into static methods, as just described in SECTION 2.1, but also keep those methods in different files, grouped together according to the needs of the application. Modular programming is important because it allows us to *independently* develop, compile, and debug parts of big programs one piece at a time, leaving each finished piece in its own file for later use without having to worry about its details again. We develop libraries of static methods for use by any other program, keeping each library in its own file and using its methods in any other program. Java’s Math library and our Std\* libraries for input/output are examples that you have already used. More importantly, you will soon see that it is very easy to define libraries of your own. The ability to define libraries and then to use them in multiple programs is a critical aspect of our ability to build programs to address complex tasks.

Having just moved in SECTION 2.1 from thinking of a Java program as a sequence of statements to thinking of a Java program as a class comprising a set of static methods (one of which is `main()`), you will be ready after this section to think of a Java program as a set of *classes*, each of which is an independent module consisting of a set of methods. Since each method can call a method in another class, all of your code can interact as a network of methods that call one another, grouped together in classes. With this capability, you can start to think about managing complexity when programming by breaking up programming tasks into classes that can be implemented and tested independently.

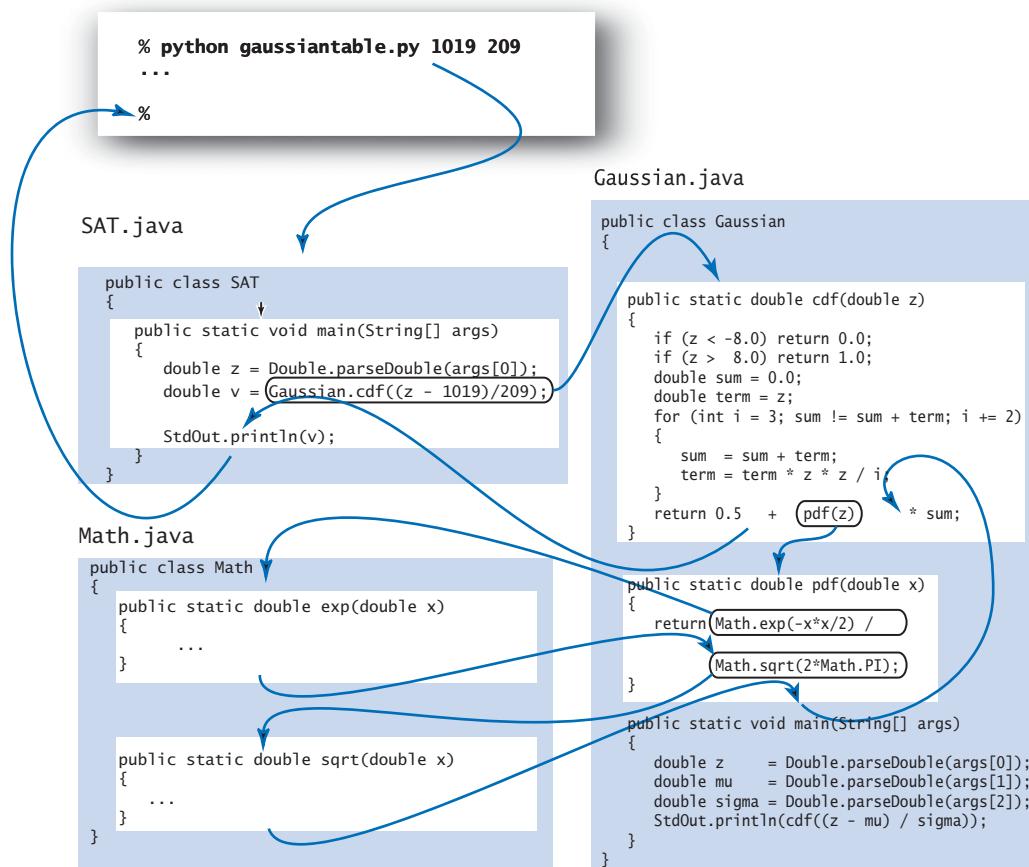
2.2.1	Random number library . . . . .	234
2.2.2	Array I/O library. . . . .	238
2.2.3	Iterated function systems . . . . .	241
2.2.4	Data analysis library. . . . .	245
2.2.5	Plotting data values in an array. . . . .	247
2.2.6	Bernoulli trials. . . . .	250

*Programs in this section*

**Using static methods in other programs** To refer to a static method in one class that is defined in another, we use the same mechanism that we have been using to invoke methods such as `Math.sqrt()` and `StdOut.println()`:

- Make both classes accessible to Java (for example, by putting them both in the same directory in your computer).
- To call a method, prepend its class name and a period separator.

For example, we might wish to write a simple client `SAT.java` that takes an SAT score  $z$  from the command line and prints the percentage of students scoring less than  $z$  in a given year (in which the mean score was 1,019 and its standard deviation was 209). To get the job done, `SAT.java` needs to compute  $\Phi((z - 1,019) / 209)$ , a



Flow of control in a modular program

task perfectly suited for the `cdf()` method in `Gaussian.java` (PROGRAM 2.1.2). All that we need to do is to keep `Gaussian.java` in the same directory as `SAT.java` and prepend the class name when calling `cdf()`. Moreover, any other class in that directory can make use of the static methods defined in `Gaussian`, by calling `Gaussian.pdf()` or `Gaussian.cdf()`. The `Math` library is always accessible in Java, so any class can call `Math.sqrt()` and `Math.exp()`, as usual. The files `Gaussian.java`, `SAT.java`, and `Math.java` implement Java classes that interact with one another: `SAT` calls a method in `Gaussian`, which calls another method in `Gaussian`, which then calls two methods in `Math`.

The potential effect of programming by defining multiple files, each an independent class with multiple methods, is another profound change in our programming style. Generally, we refer to this approach as *modular programming*. We independently develop and debug methods for an application and then utilize them at any later time. In this section, we will consider numerous illustrative examples to help you get used to the idea. However, there are several details about the process that we need to discuss before considering more examples.

***The public keyword.*** We have been identifying every static method as `public` since `HelloWorld`. This modifier identifies the method as available for use by any other program with access to the file. You can also identify methods as `private` (and there are a few other categories), but you have no reason to do so at this point. We will discuss various options in SECTION 3.3.

***Each module is a class.*** We use the term *module* to refer to all the code that we keep in a single file. In Java, by convention, each module is a Java `class` that is kept in a file with the same name of the class but has a `.java` extension. In this chapter, each `class` is merely a set of static methods (one of which is `main()`). You will learn much more about the general structure of the Java `class` in CHAPTER 3.

***The .class file.*** When you compile the program (by typing `javac` followed by the class name), the Java compiler makes a file with the class name followed by a `.class` extension that has the code of your program in a language more suited to your computer. If you have a `.class` file, you can use the module's methods in another program even without having the source code in the corresponding `.java` file (but you are on your own if you discover a bug!).

*Compile when necessary.* When you compile a program, Java typically compiles everything that needs to be compiled in order to run that program. If you call `Gaussian.cdf()` in `SAT`, then, when you type `javac SAT.java`, the compiler will also check whether you modified `Gaussian.java` since the last time it was compiled (by checking the time it was last changed against the time `Gaussian.class` was created). If so, it will also compile `Gaussian.java`! If you think about this approach, you will agree that it is actually quite helpful. After all, if you find a bug in `Gaussian.java` (and fix it), you want all the classes that call methods in `Gaussian` to use the new version.

*Multiple `main()` methods.* Another subtle point is to note that more than one class might have a `main()` method. In our example, both `SAT` and `Gaussian` have their own `main()` method. If you recall the rule for executing a program, you will see that there is no confusion: when you type `java` followed by a class name, Java transfers control to the machine code corresponding to the `main()` method defined in that class. Typically, we include a `main()` method in every class, to test and debug its methods. When we want to run `SAT`, we type `java SAT`; when we want to debug `Gaussian`, we type `java Gaussian` (with appropriate command-line arguments).

IF YOU THINK OF EACH PROGRAM that you write as something that you might want to make use of later, you will soon find yourself with all sorts of useful tools. Modular programming allows us to view every solution to a computational problem that we may develop as adding value to our computational environment.

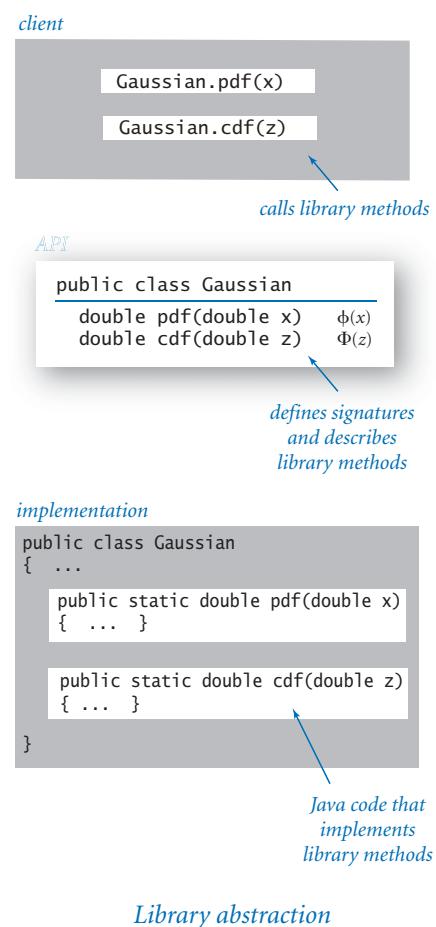
For example, suppose that you need to evaluate  $\Phi$  for some future application. Why not just cut and paste the code that implements `cdf()` from `Gaussian`? That would work, but would leave you with two copies of the code, making it more difficult to maintain. If you later want to fix or improve this code, you would need to do so in both copies. Instead, you can just call `Gaussian.cdf()`. Our implementations and uses of our methods are soon going to proliferate, so having just one copy of each is a worthy goal.

From this point forward, you should write *every* program by identifying a reasonable way to divide the computation into separate parts of a manageable size and implementing each part as if someone will want to use it later. Most frequently, that someone will be you, and you will have yourself to thank for saving the effort of rewriting and re-debugging code.

**Libraries** We refer to a module whose methods are primarily intended for use by many other programs as a *library*. One of the most important characteristics of programming in Java is that thousands of libraries have been predefined for your use. We reveal information about those that might be of interest to you throughout the book, but we will postpone a detailed discussion of the scope of Java libraries, because many of them are designed for use by experienced programmers. Instead, we focus in this chapter on the even more important idea that we can build *user-defined libraries*, which are nothing more than classes that contain a set of related methods for use by other programs. No Java library can contain all the methods that we might need for a given computation, so this ability to create our own library of methods is a crucial step in addressing complex programming applications.

**Clients.** We use the term *client* to refer to a program that calls a given library method. When a class contains a method that is a client of a method in another class, we say that the first class is a client of the second class. In our example, Gaussian is a client of SAT. A given class might have multiple clients. For example, all of the programs that you have written that call Math.sqrt() or Math.random() are clients of Math.

**APIs.** Programmers normally think in terms of a *contract* between the client and the implementation that is a clear specification of what the method is to do. When you are writing both clients and implementations, you are making contracts with yourself, which by itself is helpful because it provides extra help in debugging. More important, this approach enables code reuse. You have been able to write programs that are clients of Std\* and Math and other built-in Java classes because of an informal contract (an English-



language description of what they are supposed to do) along with a precise specification of the signatures of the methods that are available for use. Collectively, this information is known as an *application programming interface* (API). This same mechanism is effective for user-defined libraries. The API allows any client to use the library without having to examine the code in the implementation, as you have been doing for Math and Std\*. The guiding principle in API design is to *provide to clients the methods they need and no others*. An API with a huge number of methods may be a burden to implement; an API that is lacking important methods may be unnecessarily inconvenient for clients.

**Implementations.** We use the term *implementation* to describe the Java code that implements the methods in an API, kept by convention in a file with the library name and a .java extension. Every Java program is an implementation of some API, and no API is of any use without some implementation. Our goal when developing an implementation is to honor the terms of the contract. Often, there are many ways to do so, and separating client code from implementation code gives us the freedom to substitute new and improved implementations.

FOR EXAMPLE, CONSIDER THE GAUSSIAN DISTRIBUTION functions. These do not appear in Java's Math library but are important in applications, so it is worthwhile for us to put them in a library where they can be accessed by future client programs and to articulate this API:

```
public class Gaussian
```

---

double pdf(double x)	$\phi(x)$
double pdf(double x, double mu, double sigma)	$\phi(x, \mu, \sigma)$
double cdf(double z)	$\Phi(z)$
double cdf(double z, double mu, double sigma)	$\Phi(z, \mu, \sigma)$

*API for our library of static methods for Gaussian distribution functions*

The API includes not only the one-argument Gaussian distribution functions that we have previously considered (see PROGRAM 2.1.2) but also three-argument versions (in which the client specifies the mean and standard deviation of the distribution) that arise in many statistical applications. Implementing the three-argument Gaussian distribution functions is straightforward (see EXERCISE 2.2.1).

How much information should an API contain? This is a gray area and a hotly debated issue among programmers and computer-science educators. We might try to put as much information as possible in the API, but (as with any contract!) there are limits to the amount of information that we can productively include. In this book, we stick to a principle that parallels our guiding design principle: *provide to client programmers the information they need and no more*. Doing so gives us vastly more flexibility than the alternative of providing detailed information about implementations. Indeed, any extra information amounts to implicitly extending the contract, which is undesirable. Many programmers fall into the bad habit of checking implementation code to try to understand what it does. Doing so might lead to client code that depends on behavior not specified in the API, which would not work with a new implementation. Implementations change more often than you might think. For example, each new release of Java contains many new implementations of library functions.

Often, the implementation comes first. You might have a working module that you later decide would be useful for some task, and you can just start using its methods in other programs. In such a situation, it is wise to carefully articulate the API at some point. The methods may not have been designed for reuse, so it is worthwhile to use an API to do such a design (as we did for `Gaussian`).

The remainder of this section is devoted to several examples of libraries and clients. Our purpose in considering these libraries is twofold. First, they provide a richer programming environment for your use as you develop increasingly sophisticated client programs of your own. Second, they serve as examples for you to study as you begin to develop libraries for your own use.

**Random numbers** We have written several programs that use `Math.random()`, but our code often uses particular idioms that convert the random `double` values between 0 and 1 that `Math.random()` provides to the type of random numbers that we want to use (random `boolean` values or random `int` values in a specified range, for example). To effectively reuse our code that implements these idioms, we will, from now on, use the `StdRandom` library in PROGRAM 2.2.1. `StdRandom` uses overloading to generate random numbers from various distributions. You can use any of them in the same way that you use our standard I/O libraries (see the first Q&A at the end of SECTION 2.1). As usual, we summarize the methods in our `StdRandom` library with an API:

---

<code>public class StdRandom</code>	
<code>void setSeed(long seed)</code>	<i>set the seed for reproducible results</i>
<code>int uniform(int n)</code>	<i>integer between 0 and n-1</i>
<code>double uniform(double lo, double hi)</code>	<i>floating-point number between lo and hi</i>
<code>boolean bernoulli(double p)</code>	<i>true with probability p, false otherwise</i>
<code>double gaussian()</code>	<i>Gaussian, mean 0, standard deviation 1</i>
<code>double gaussian(double mu, double sigma)</code>	<i>Gaussian, mean mu, standard deviation sigma</i>
<code>int discrete(double[] p)</code>	<i>i with probability p[i]</i>
<code>void shuffle(double[] a)</code>	<i>randomly shuffle the array a[]</i>

*API for our library of static methods for random numbers*

These methods are sufficiently familiar that the short descriptions in the API suffice to specify what they do. By collecting all of these methods that use `Math.random()` to generate random numbers of various types in one file (`StdRandom.java`), we concentrate our attention on generating random numbers to this one file (and reuse the code in that file) instead of spreading them through every program that uses these methods. Moreover, each program that uses one of these methods is clearer than code that calls `Math.random()` directly, because its purpose for using `Math.random()` is clearly articulated by the choice of method from `StdRandom`.

**API design.** We make certain assumptions about the values passed to each method in `StdRandom`. For example, we assume that clients will call `uniform(n)` only for positive integers `n`, `bernoulli(p)` only for `p` between 0 and 1, and `discrete()` only for an array whose elements are between 0 and 1 and sum to 1. All of these assumptions are part of the contract between the client and the implementation. We strive to design libraries such that the contract is clear and unambiguous and to avoid getting bogged down with details. As with many tasks in programming, a good API design is often the result of several iterations of trying and living with various possibilities. We always take special care in designing APIs, because when we change an API we might have to change all clients and all implementations. Our goal is to articulate what clients can expect *separate from the code* in the API. This practice frees us to change the code, and perhaps to use an implementation that achieves the desired effect more efficiently or with more accuracy.

### Program 2.2.1 Random number library

```
public class StdRandom
{
    public static int uniform(int n)
    { return (int) (Math.random() * n); }

    public static double uniform(double lo, double hi)
    { return lo + Math.random() * (hi - lo); }

    public static boolean bernoulli(double p)
    { return Math.random() < p; }

    public static double gaussian()
    { /* See Exercise 2.2.17. */ }

    public static double gaussian(double mu, double sigma)
    { return mu + sigma * gaussian(); }

    public static int discrete(double[] probabilities)
    { /* See Program 1.6.2. */ }

    public static void shuffle(double[] a)
    { /* See Exercise 2.2.4. */ }

    public static void main(String[] args)
    { /* See text. */ }
}
```

---

The methods in this library compute various types of random numbers: random nonnegative integer less than a given value, uniformly distributed in a given range, random bit (Bernoulli), standard Gaussian, Gaussian with given mean and standard deviation, and distributed according to a given discrete distribution.

```
% java StdRandom 5
90 26.36076 false 8.79269 0
13 18.02210 false 9.03992 1
58 56.41176 true 8.80501 0
29 16.68454 false 8.90827 0
85 86.24712 true 8.95228 0
```

**Unit testing.** Even though we implement StdRandom without reference to any particular client, it is good programming practice to include a *test client* main() that, although not used when a client class uses the library, is helpful when debugging and testing the methods in the library. *Whenever you create a library, you should include a main() method for unit testing and debugging.* Proper unit testing can be a significant programming challenge in itself (for example, the best way of testing whether the methods in StdRandom produce numbers that have the same characteristics as truly random numbers is still debated by experts). At a minimum, you should always include a main() method that

- Exercises all the code
- Provides some assurance that the code is working
- Takes an argument from the command line to allow more testing

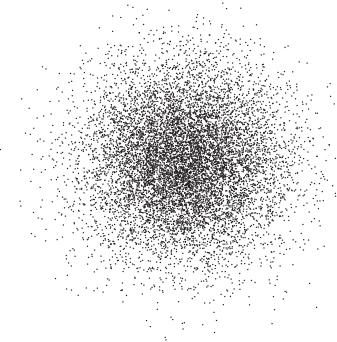
Then, you should refine that main() method to do more exhaustive testing as you use the library more extensively. For example, we might start with the following code for StdRandom (leaving the testing of shuffle() for an exercise):

```
public static void main(String[] args)
{
    int n = Integer.parseInt(args[0]);
    double[] probabilities = { 0.5, 0.3, 0.1, 0.1 };
    for (int i = 0; i < n; i++)
    {
        StdOut.printf(" %2d " , uniform(100));
        StdOut.printf("%8.5f ", uniform(10.0, 99.0));
        StdOut.printf("%5b " , bernoulli(0.5));
        StdOut.printf("%7.5f ", gaussian(9.0, 0.2));
        StdOut.printf("%2d " , discrete(probabilities));
        StdOut.println();
    }
}
```

When we include this code in StdRandom.java and invoke this method as illustrated in PROGRAM 2.2.1, the output includes no surprises: the integers in the first column might be equally likely to be any value from 0 to 99; the numbers in the second column might be uniformly spread between 10.0 and 99.0; about half of the values in the third column are true; the numbers in the fourth column seem to average about 9.0, and seem unlikely to be too far from 9.0; and the last column seems to be not far from 50% 0s, 30% 1s, 10% 2s, and 10% 3s. If something seems

amiss in one of the columns, we can type `java StdRandom 10` or `100` to see many more results. In this particular case, we can (and should) do far more extensive testing in a separate client to check that the numbers have many of the same properties as truly random numbers drawn from the cited distributions (see EXERCISE 2.2.3). One effective approach is to write test clients that use `StdDraw`, as data visualization can be a quick indication that a program is behaving as intended. For example, a plot of a large number of points whose  $x$ - and  $y$ -coordinates are

```
public class RandomPoints
{
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        for (int i = 0; i < n; i++)
        {
            double x = StdRandom.gaussian(.5, .2);
            double y = StdRandom.gaussian(.5, .2);
            StdDraw.point(x, y);
        }
    }
}
```



*A `StdRandom` test client*

both drawn from various distributions often produces a pattern that gives direct insight into the important properties of the distribution. More important, a bug in the random number generation code is likely to show up immediately in such a plot.

**Stress testing.** An extensively used library such as `StdRandom` should also be subjected to *stress testing*, where we make sure that it does not crash when the client does not follow the contract or makes some assumption that is not explicitly covered. Java libraries have already been subjected to such stress testing, which requires carefully examining each line of code and questioning whether some condition might cause a problem. What should `discrete()` do if the array elements do not sum to exactly 1? What if the argument is an array of length 0? What should the two-argument

`uniform()` do if one or both of its arguments is `Nan`? `Infinity`? Any question that you can think of is fair game. Such cases are sometimes referred to as *corner cases*. You are certain to encounter a teacher or a supervisor who is a stickler about corner cases. With experience, most programmers learn to address them early, to avoid an unpleasant bout of debugging later. Again, a reasonable approach is to implement a stress test as a separate client.

**Input and output for arrays** We have seen—and will continue to see—many examples where we wish to keep data in arrays for processing. Accordingly, it is useful to build a library that complements StdIn and StdOut by providing static methods for reading arrays of primitive types from standard input and printing them to standard output. The following API provides these methods:

```
public class StdArrayIO
```

double[] readDouble1D()	<i>read a one-dimensional array of double values</i>
double[][] readDouble2D()	<i>read a two-dimensional array of double values</i>
void print(double[] a)	<i>print a one-dimensional array of double values</i>
void print(double[][] a)	<i>print a two-dimensional array of double values</i>

*Note 1. 1D format is an integer  $n$  followed by  $n$  values.*

*Note 2. 2D format is two integers  $m$  and  $n$  followed by  $m \times n$  values in row-major order.*

*Note 3. Methods for int and boolean are also included.*

*API for our library of static methods for array input and output*

The first two notes at the bottom of the table reflect the idea that we need to settle on a *file format*. For simplicity and harmony, we adopt the convention that all values appearing in standard input include the dimension(s) and appear in the order indicated. The `read*`() methods expect input in this format; the `print()` methods produce output in this format. The third note at the bottom of the table indicates that `StdArrayIO` actually contains 12 methods—four each for `int`, `double`, and `boolean`. The `print()` methods are overloaded (they all have the same name `print()` but different types of arguments), but the `read*`() methods need different names, formed by adding the type name (capitalized, as in `StdIn`) followed by `1D` or `2D`.

Implementing these methods is straightforward from the array-processing code that we have considered in SECTION 1.4 and in SECTION 2.1, as shown in `StdArrayIO` (PROGRAM 2.2.2). Packaging up all of these static methods into one file—`StdArrayIO.java`—allows us to easily reuse the code and saves us from having to worry about the details of reading and printing arrays when writing client programs later on.

### Program 2.2.2 Array I/O library

```

public class StdArrayIO
{
    public static double[] readDouble1D()
    { /* See Exercise 2.2.11. */ }

    public static double[][] readDouble2D()
    {
        int m = StdIn.readInt();
        int n = StdIn.readInt();
        double[][] a = new double[m][n];
        for (int i = 0; i < m; i++)
            for (int j = 0; j < n; j++)
                a[i][j] = StdIn.readDouble();
        return a;
    }

    public static void print(double[] a)
    { /* See Exercise 2.2.11. */ }

    public static void print(double[][] a)
    {
        int m = a.length;
        int n = a[0].length;
        System.out.println(m + " " + n);
        for (int i = 0; i < m; i++)
        {
            for (int j = 0; j < n; j++)
                StdOut.printf("%9.5f ", a[i][j]);
            StdOut.println();
        }
        StdOut.println();
    }

    // Methods for other types are similar (see booksite).

    public static void main(String[] args)
    { print(readDouble2D()); }
}

```

```

% more tiny2D.txt
4 3
  .000  .270  .000
  .246  .224 - .036
  .222  .176  .0893
 - .032  .739  .270

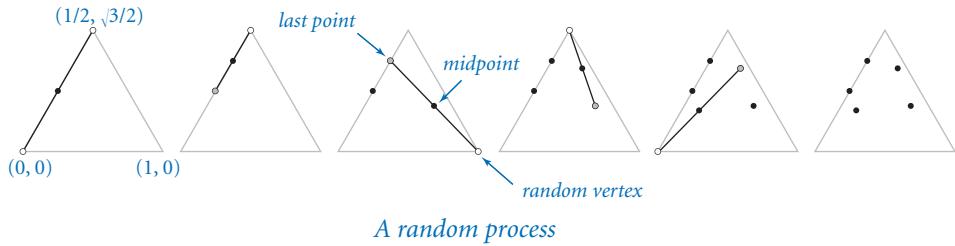
% java StdArrayIO < tiny.txt
4 3
  0.00000  0.27000  0.00000
  0.24600  0.22400 -0.03600
  0.22200  0.17600  0.08930
 -0.03200  0.73900  0.27000

```

*This library of static methods facilitates reading one-dimensional and two-dimensional arrays from standard input and printing them to standard output. The file format includes the dimensions (see accompanying text). Numbers in the output in the example are truncated.*

**Iterated function systems** Scientists have discovered that complex visual images can arise unexpectedly from simple computational processes. With `StdRandom`, `StdDraw`, and `StdArrayIO`, we can study the behavior of such systems.

*Sierpinski triangle.* As a first example, consider the following simple process: Start by plotting a point at one of the vertices of a given equilateral triangle. Then pick one of the three vertices at random and plot a new point halfway between the point just plotted and that vertex. Continue performing this same operation. Each time, we are pick a random vertex from the triangle to establish the line whose midpoint will be the next point plotted. Since we make random choices, the set of points should have some of the characteristics of random points, and that does seem to be the case after the first few iterations:



We can study the process for a large number of iterations by writing a program to plot `trials` points according to the rules:

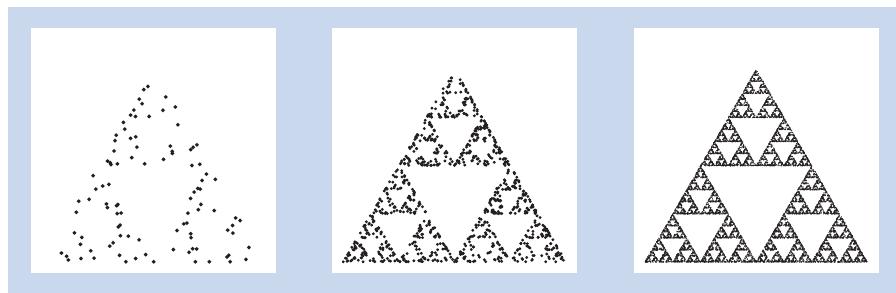
```

double[] cx = { 0.000, 1.000, 0.500 };
double[] cy = { 0.000, 0.000, 0.866 };
double x = 0.0, y = 0.0;
for (int t = 0; t < trials; t++)
{
    int r = StdRandom.uniform(3);
    x = (x + cx[r]) / 2.0;
    y = (y + cy[r]) / 2.0;
    StdDraw.point(x, y);
}

```

We keep the  $x$ - and  $y$ -coordinates of the triangle vertices in the arrays `cx[]` and `cy[]`, respectively. We use `StdRandom.uniform()` to choose a random index `r` into

these arrays—the coordinates of the chosen vertex are  $(\text{cx}[r], \text{cy}[r])$ . The  $x$ -coordinate of the midpoint of the line from  $(x, y)$  to that vertex is given by the expression  $(x + \text{cx}[r])/2.0$ , and a similar calculation gives the  $y$ -coordinate. Adding a call to `StdDraw.point()` and putting this code in a loop completes the implementation. Remarkably, despite the randomness, the same figure always emerges after a large number of iterations! This figure is known as the *Sierpinski triangle* (see EXERCISE 2.3.27). Understanding why such a regular figure should arise from such a random process is a fascinating question.



*A random process?*

**Barnsley fern.** To add to the mystery, we can produce pictures of remarkable diversity by playing the same game with different rules. One striking example is known as the *Barnsley fern*. To generate it, we use the same process, but this time driven by the following table of formulas. At each step, we choose the formulas to use to update  $x$  and  $y$  with the indicated probability (1% of the time we use the first pair of formulas, 85% of the time we use the second pair of formulas, and so forth).

<i>probability</i>	<i>x-update</i>		<i>y-update</i>	
1%	$x =$	0.500	$y =$	$0.16y$
85%	$x =$	$0.85x + 0.04y + 0.075$	$y =$	$-0.04x + 0.85y + 0.180$
7%	$x =$	$0.20x - 0.26y + 0.400$	$y =$	$0.23x + 0.22y + 0.045$
7%	$x =$	$-0.15x + 0.28y + 0.575$	$y =$	$0.26x + 0.24y - 0.086$

### Program 2.2.3 Iterated function systems

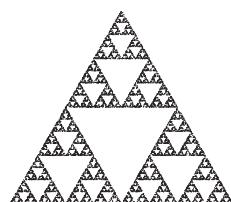
```
public class IFS
{
    public static void main(String[] args)
    { // Plot trials iterations of IFS on StdIn.
        int trials = Integer.parseInt(args[0]);
        double[] dist = StdArrayIO.readDouble1D();
        double[][] cx = StdArrayIO.readDouble2D();
        double[][] cy = StdArrayIO.readDouble2D();
        double x = 0.0, y = 0.0;
        for (int t = 0; t < trials; t++)
        { // Plot 1 iteration.
            int r = StdRandom.discrete(dist);
            double x0 = cx[r][0]*x + cx[r][1]*y + cx[r][2];
            double y0 = cy[r][0]*x + cy[r][1]*y + cy[r][2];
            x = x0;
            y = y0;
            StdDraw.point(x, y);
        }
    }
}
```

trials	<i>iterations</i>
dist[]	<i>probabilities</i>
cx[][]	<i>x coefficients</i>
cy[][]	<i>y coefficients</i>
x, y	<i>current point</i>

This data-driven client of `StdArrayIO`, `StdRandom`, and `StdDraw` iterates the function system defined by a 1-by- $m$  vector (`probabilities`) and two  $m$ -by-3 matrices (`coefficients` for updating `x` and `y`, respectively) on standard input, plotting the result as a set of points on standard drawing. Curiously, this code does not need to know the value of  $m$ , as it uses separate methods to create and process the matrices.

```
% more sierpinski.txt
3
.33 .33 .34
3 3
.50 .00 .00
.50 .00 .50
.50 .00 .25
3 3
.00 .50 .00
.00 .50 .00
.00 .50 .433
```

```
% java IFS 10000 < sierpinski.txt
```



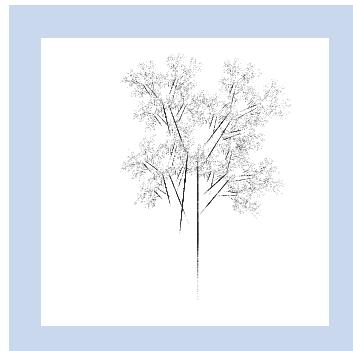
```
% more barnsley.txt
4
  .01  .85  .07  .07
4 3
  .00   .00   .500
  .85   .04   .075
  .20  -.26   .400
 -.15   .28   .575
4 3
  .00   .16   .000
 -.04   .85   .180
  .23   .22   .045
  .26   .24  -.086
```

```
% java IFS 20000 < barnsley.txt
```



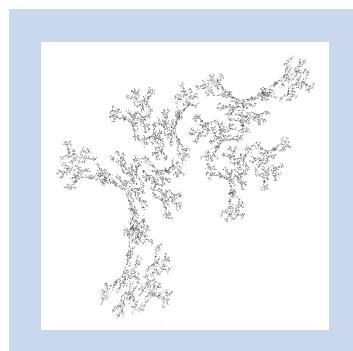
```
% more tree.txt
6
  .1  .1  .2  .2  .2  .2
6 3
  .00   .00   .550
 -.05   .00   .525
  .46  -.15   .270
  .47  -.15   .265
  .43   .26   .290
  .42   .26   .290
6 3
  .00   .60   .000
 -.50   .00   .750
  .39   .38   .105
  .17   .42   .465
 -.25   .45   .625
 -.35   .31   .525
```

```
% java IFS 20000 < tree.txt
```



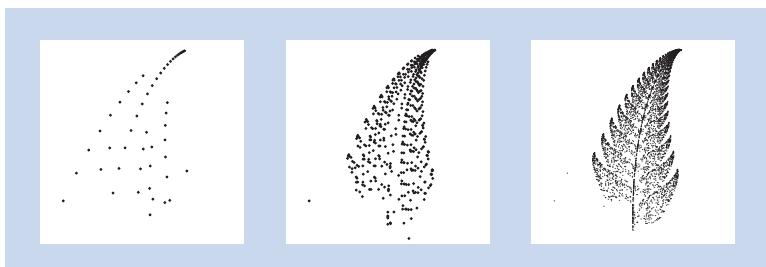
```
% more coral.txt
3
  .40  .15  .45
3 3
  .3077  -.5315   .8863
  .3077  -.0769   .2166
  .0000   .5455   .0106
3 3
 -.4615  -.2937  1.0962
  .1538  -.4476   .3384
  .6923  -.1958   .3808
```

```
% java IFS 20000 < coral.txt
```



*Examples of iterated function systems*

We could write code just like the code we just wrote for the Sierpinski triangle to iterate these rules, but matrix processing provides a uniform way to generalize that code to handle any set of rules. We have  $m$  different transformations, chosen from a 1-by- $m$  vector with `StdRandom.discrete()`. For each transformation, we have an equation for updating  $x$  and an equation for updating  $y$ , so we use two  $m$ -by-3 matrices for the equation coefficients, one for  $x$  and one for  $y$ . IFS (PROGRAM 2.2.3) implements this data-driven version of the computation. This program enables limitless exploration: it performs the iteration for any input containing a vector that defines the probability distribution and the two matrices that define the coefficients, one for updating  $x$  and the other for updating  $y$ . For the coefficients just given, again, even though we choose a random equation at each step, the same figure emerges every time that we do this computation: an image that looks remarkably similar to a fern that you might see in the woods, not something generated by a random process on a computer.



*Generating a Barnsley fern*

That the same short program that takes a few numbers from standard input and plots points on standard drawing can (given different data) produce both the Sierpinski triangle and the Barnsley fern (and many, many other images) is truly remarkable. Because of its simplicity and the appeal of the results, this sort of calculation is useful in making synthetic images that have a realistic appearance in computer-generated movies and games.

Perhaps more significantly, the ability to produce such realistic diagrams so easily suggests intriguing scientific questions: What does computation tell us about nature? What does nature tell us about computation?

**Statistics** Next, we consider a library for a set of mathematical calculations and basic visualization tools that arise in all sorts of applications in science and engineering and are not all implemented in standard Java libraries. These calculations relate to the task of understanding the statistical properties of a set of numbers. Such a library is useful, for example, when we perform a series of scientific experiments that yield measurements of a quantity. One of the most important challenges facing modern scientists is proper analysis of such data, and computation is playing an increasingly important role in such analysis. These basic data analysis methods that we will consider are summarized in the following API:

---

public class StdStats	
double max(double[] a)	<i>largest value</i>
double min(double[] a)	<i>smallest value</i>
double mean(double[] a)	<i>average</i>
double var(double[] a)	<i>sample variance</i>
double stddev(double[] a)	<i>sample standard deviation</i>
double median(double[] a)	<i>median</i>
void plotPoints(double[] a)	<i>plot points at (i, a[i])</i>
void plotLines(double[] a)	<i>plot lines connecting points at (i, a[i])</i>
void plotBars(double[] a)	<i>plot bars to points at (i, a[i])</i>

*Note: Overloaded implementations are included for other numeric types.*

*API for our library of static methods for data analysis*

**Basic statistics.** Suppose that we have  $n$  measurements  $x_0, x_1, \dots, x_{n-1}$ . The average value of those measurements, otherwise known as the *mean*, is given by the formula  $\mu = (x_0 + x_1 + \dots + x_{n-1}) / n$  and is an estimate of the value of the quantity. The minimum and maximum values are also of interest, as is the median (the value that is smaller than and larger than half the values). Also of interest is the *sample variance*, which is given by the formula

$$\sigma^2 = ((x_0 - \mu)^2 + (x_1 - \mu)^2 + \dots + (x_{n-1} - \mu)^2) / (n-1)$$

### Program 2.2.4 Data analysis library

```
public class StdStats
{
    public static double max(double[] a)
    { // Compute maximum value in a[].
        double max = Double.NEGATIVE_INFINITY;
        for (int i = 0; i < a.length; i++)
            if (a[i] > max) max = a[i];
        return max;
    }

    public static double mean(double[] a)
    { // Compute the average of the values in a[].
        double sum = 0.0;
        for (int i = 0; i < a.length; i++)
            sum = sum + a[i];
        return sum / a.length;
    }

    public static double var(double[] a)
    { // Compute the sample variance of the values in a[].
        double avg = mean(a);
        double sum = 0.0;
        for (int i = 0; i < a.length; i++)
            sum += (a[i] - avg) * (a[i] - avg);
        return sum / (a.length - 1);
    }

    public static double stddev(double[] a)
    { return Math.sqrt(var(a)); }

    // See Program 2.2.5 for plotting methods.

    public static void main(String[] args)
    { /* See text. */ }
}
```

This code implements methods to compute the maximum, mean, variance, and standard deviation of numbers in a client array. The method for computing the minimum is omitted; plotting methods are in PROGRAM 2.2.5; see EXERCISE 4.2.20 for median().

```
% more tiny1D.txt
5
3.0 1.0 2.0 5.0 4.0
```

```
% java StdStats < tiny1D.txt
min      1.000
mean     3.000
max      5.000
std dev   1.581
```

and the *sample standard deviation*, the square root of the sample variance. `StdStats` (PROGRAM 2.2.4) shows implementations of static methods for computing these basic statistics (the median is more difficult to compute than the others—we will consider the implementation of `median()` in SECTION 4.2). The `main()` test client for `StdStats` reads numbers from standard input into an array and calls each of the methods to print the minimum, mean, maximum, and standard deviation, as follows:

```
public static void main(String[] args)
{
    double[] a = StdArrayIO.readDouble1D();
    StdOut.printf("      min %7.3f\n", min(a));
    StdOut.printf("      mean %7.3f\n", mean(a));
    StdOut.printf("      max %7.3f\n", max(a));
    StdOut.printf("  std dev %7.3f\n", stddev(a));
}
```

As with `StdRandom`, a more extensive test of the calculations is called for (see EXERCISE 2.2.3). Typically, as we debug or test new methods in the library, we adjust the unit testing code accordingly, testing the methods one at a time. A mature and widely used library like `StdStats` also deserves a stress-testing client for extensively testing everything after any change. If you are interested in seeing what such a client might look like, you can find one for `StdStats` on the booksite. Most experienced programmers will advise you that any time spent doing unit testing and stress testing will more than pay for itself later.

**Plotting.** One important use of `StdDraw` is to help us visualize data rather than relying on tables of numbers. In a typical situation, we perform experiments, save the experimental data in an array, and then compare the results against a model, perhaps a mathematical function that describes the data. To expedite this process for the typical case where values of one variable are equally spaced, our `StdStats` library contains static methods that you can use for plotting data in an array. PROGRAM 2.2.5 is an implementation of the `plotPoints()`, `plotLines()`, and `plotBars()` methods for `StdStats`. These methods display the values in the argument array at evenly spaced intervals in the drawing window, either connected together by line segments (`lines`), filled circles at each value (`points`), or bars from the  $x$ -axis to the value (`bars`). They all plot the points with  $x$ -coordinate  $i$  and  $y$ -coordinate  $a[i]$  using filled circles, lines through the points, and bars, respectively. In addition,

**Program 2.2.5 Plotting data values in an array**

```
public static void plotPoints(double[] a)
{ // Plot points at (i, a[i]).
    int n = a.length;
    StdDraw.setXscale(-1, n);
    StdDraw.setPenRadius(1/(3.0*n));
    for (int i = 0; i < n; i++)
        StdDraw.point(i, a[i]);
}

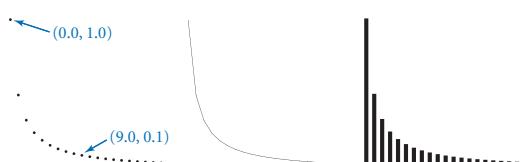
public static void plotLines(double[] a)
{ // Plot lines through points at (i, a[i]).
    int n = a.length;
    StdDraw.setXscale(-1, n);
    StdDraw.setPenRadius();
    for (int i = 1; i < n; i++)
        StdDraw.line(i-1, a[i-1], i, a[i]);
}

public static void plotBars(double[] a)
{ // Plot bars from (0, a[i]) to (i, a[i]).
    int n = a.length;
    StdDraw.setXscale(-1, n);
    for (int i = 0; i < n; i++)
        StdDraw.filledRectangle(i, a[i]/2, 0.25, a[i]/2);
}
```

This code implements three methods in `StdStats` (PROGRAM 2.2.4) for plotting data. They plot the points  $(i, a[i])$  with filled circles, connecting line segments, and bars, respectively.

```
plotPoints(a);  plotLines(a);  plotBars(a);

int n = 20;
double[] a = new double[n];
for (int i = 0; i < n; i++)
    a[i] = 1.0/(i+1);
```



they all rescale  $x$  to fill the drawing window (so that the points are evenly spaced along the  $x$ -coordinate) and leave to the client scaling of the  $y$ -coordinates.

These methods are not intended to be a general-purpose plotting package, but you can certainly think of all sorts of things that you might want to add: different types of spots, labeled axes, color, and many other artifacts are commonly found in modern systems that can plot data. Some situations might call for more complicated methods than these.

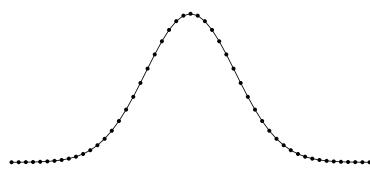
Our intent with `StdStats` is to introduce you to data analysis while showing you how easy it is to define a library to take care of useful tasks. Indeed, this library has already proved useful—we use these plotting methods to produce the figures in this book that depict function graphs, sound waves, and experimental results. Next, we consider several examples of their use.

*Plotting function graphs.* You can use the `StdStats.plot*` methods to draw a plot of the function graph for any function at all: choose an  $x$ -interval where you want to plot the function, compute function values evenly spaced through that interval and store them in an array, determine and set the  $y$ -scale, and then call `StdStats.plotLines()` or another `plot*` method. For example, to plot a sine function, rescale the  $y$ -axis to cover values between  $-1$  and  $+1$ . Scaling the  $x$ -axis is automatically handled by the `StdStats` methods. If you do not know the range, you can handle the situation by calling:

```
StdDraw.setScale(StdStats.min(a), StdStats.max(a));
```

The smoothness of the curve is determined by properties of the function and by the number of points plotted. As we discussed when first considering `StdDraw`, you have to be careful to sample enough points to catch fluctuations in the function. We will consider another approach to plotting functions based on sampling values that are not equally spaced in SECTION 2.4.

```
int n = 50;
double[] a = new double[n+1];
for (int i = 0; i <= n; i++)
    a[i] = Gaussian.pdf(-4.0 + 8.0*i/n);
StdStats.plotPoints(a);
StdStats.plotLines(a);
```



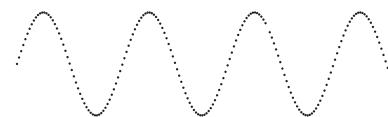
*Plotting a function graph*

**Plotting sound waves.** Both the `StdAudio` library and the `StdStats` plot methods work with arrays that contain sampled values at regular intervals. The diagrams of sound waves in SECTION 1.5 and at the beginning of this section were all produced by first scaling the  $y$ -axis with `StdDraw.setScale(-1, 1)`, then plotting the points with `StdStats.plotPoints()`. As you have seen, such plots give direct insight into processing audio. You can also produce interesting effects by plotting sound waves as you play them with `StdAudio`, although this task is a bit challenging because of the huge amount of data involved (see EXERCISE 1.5.23).

**Plotting experimental results.** You can put multiple plots on the same drawing. One typical reason to do so is to compare experimental results with a theoretical model. For example, `Bernoulli` (PROGRAM 2.2.6) counts the number of heads found when a fair coin is flipped  $n$  times and compares the result with the predicted Gaussian probability density function. A famous result from probability theory is that the distribution of this quantity is the *binomial distribution*, which is extremely well approximated by the Gaussian distribution with mean  $n/2$  and standard deviation  $\sqrt{n}/2$ . The more trials we perform, the more accurate the approximation. The drawing produced by `Bernoulli` is a succinct summary of the results of the experiment and a convincing validation of the theory. This example is prototypical of a scientific approach to applications programming that we use often throughout this book and that you should use whenever you run an experiment. If a theoretical model that can explain your results is available, a visual plot comparing the experiment to the theory can validate both.

THESE FEW EXAMPLES ARE INTENDED TO suggest what is possible with a well-designed library of static methods for data analysis. Several extensions and other ideas are explored in the exercises. You will find `StdStats` to be useful for basic plots, and you are encouraged to experiment with these implementations and to modify them or to add methods to make your own library that can draw plots of your own design. As you continue to address an ever-widening circle of programming tasks, you will naturally be drawn to the idea of developing tools like these for your own use.

```
StdDraw.setScale(-1.0, 1.0);
double[] hi;
hi = PlayThatTune.tone(880, 0.01);
StdStats.plotPoints(hi);
```



*Plotting a sound wave*

### Program 2.2.6 Bernoulli trials

```

public class Bernoulli
{
    public static int binomial(int n)
    { // Simulate flipping a coin n times; return # heads.
        int heads = 0;
        for (int i = 0; i < n; i++)
            if (StdRandom.bernoulli(0.5)) heads++;
        return heads;
    }
    public static void main(String[] args)
    { // Perform Bernoulli trials, plot results and model.
        int n = Integer.parseInt(args[0]);
        int trials = Integer.parseInt(args[1]);
        int[] freq = new int[n+1];
        for (int t = 0; t < trials; t++)
            freq[binomial(n)]++;
        double[] norm = new double[n+1];
        for (int i = 0; i <= n; i++)
            norm[i] = (double) freq[i] / trials;
        StdStats.plotBars(norm);

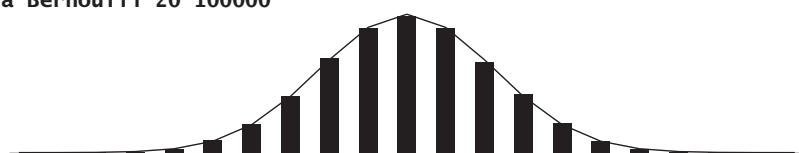
        double mean   = n / 2.0;
        double stddev = Math.sqrt(n) / 2.0;
        double[] phi  = new double[n+1];
        for (int i = 0; i <= n; i++)
            phi[i] = Gaussian.pdf(i, mean, stddev);
        StdStats.plotLines(phi);
    }
}

```

n	<i>number of flips per trial</i>
trials	<i>number of trials</i>
freq[]	<i>experimental results</i>
norm[]	<i>normalized results</i>
phi[]	<i>Gaussian model</i>

This `StdStats`, `StdRandom`, and `Gaussian` client provides visual evidence that the number of heads observed when a fair coin is flipped  $n$  times obeys a Gaussian distribution.

```
% java Bernoulli 20 100000
```



**Modular programming** The library implementations that we have developed illustrate a programming style known as *modular programming*. Instead of writing a new program that is self-contained within its own file to address a new problem, we break up each task into smaller, more manageable subtasks, then implement and independently debug code that addresses each subtask. Good libraries facilitate modular programming by allowing us to define and provide solutions for important subtasks for future clients. *Whenever you can clearly separate tasks within a program, you should do so.* Java supports such separation by allowing us to independently debug and later use classes in separate files. Traditionally, programmers use the term *module* to refer to code that can be compiled and run independently; in Java, each *class* is a module.

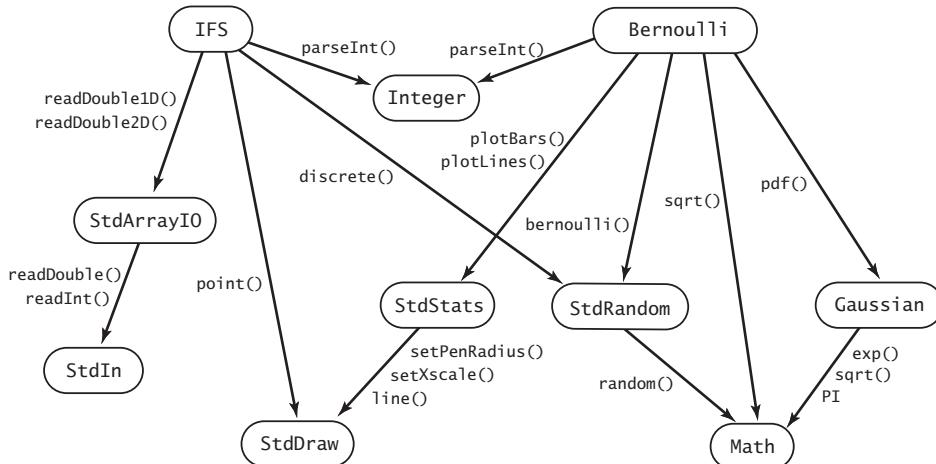
IFS (PROGRAM 2.2.3) exemplifies modular programming. This relatively sophisticated computation is implemented with several relatively small modules, developed independently. It uses StdRandom and StdArrayIO, as well as the methods from Integer and StdDraw that we are accustomed to using. If we were to put all of the code required for IFS in a single file, we would have a large amount of code on our hands to maintain and debug; with modular programming, we can study iterated function systems with some confidence that the arrays are read properly and that the random number generator will produce properly distributed values, because we already implemented and tested the code for these tasks in separate modules.

Similarly, Bernoulli (PROGRAM 2.2.6) exemplifies modular programming. It is a client of Gaussian, Integer, Math, StdRandom, and StdStats. Again, we can have some confidence that the methods in these modules produce the expected results because they are system libraries or libraries that we have tested, debugged, and used before.

API	description
Gaussian	Gaussian distribution functions
StdRandom	random numbers
StdArrayIO	input and output for arrays
IFS	client for iterated function systems
StdStats	functions for data analysis
Bernoulli	client for Bernoulli trials

*Summary of classes in this section*

To describe the relationships among modules in a modular program, we often draw a *dependency graph*, where we connect two class names with an arrow labeled with the name of a method if the first class contains a method call and the second class contains the definition of the method. Such diagrams play an important role because understanding the relationships among modules is necessary for proper development and maintenance.



Dependency graph (partial) for the modules in this section

We emphasize modular programming throughout this book because it has many important advantages that have come to be accepted as essential in modern programming, including the following:

- We can have programs of a reasonable size, even in large systems.
- Debugging is restricted to small pieces of code.
- We can reuse code without having to re-implement it.
- Maintaining (and improving) code is much simpler.

The importance of these advantages is difficult to overstate, so we will expand upon each of them.

**Programs of a reasonable size.** No large task is so complex that it cannot be divided into smaller subtasks. If you find yourself with a program that stretches to more than a few pages of code, you must ask yourself the following questions: Are there subtasks that could be implemented separately? Could some of these subtasks be

logically grouped together in a separate library? Could other clients use this code in the future? At the other end of the range, if you find yourself with a huge number of tiny modules, you must ask yourself questions such as these: Is there some group of subtasks that logically belong in the same module? Is each module likely to be used by multiple clients? There is no hard-and-fast rule on module size: one implementation of a critically important abstraction might properly be a few lines of code, whereas another library with a large number of overloaded methods might properly stretch to hundreds of lines of code.

*Debugging.* Tracing a program rapidly becomes more difficult as the number of statements and interacting variables increases. Tracing a program with hundreds of variables requires keeping track of hundreds of values, as any statement might affect or be affected by any variable. To do so for hundreds or thousands of statements or more is untenable. With modular programming and our guiding principle of keeping the scope of variables local to the extent possible, we severely restrict the number of possibilities that we have to consider when debugging. Equally important is the idea of a contract between client and implementation. Once we are satisfied that an implementation is meeting its end of the bargain, we can debug all its clients under that assumption.

*Code reuse.* Once we have implemented libraries such as `StdStats` and `StdRandom`, we do not have to worry about writing code to compute averages or standard deviations or to generate random numbers again—we can simply reuse the code that we have written. Moreover, we do not need to make copies of the code: any module can just refer to any public method in any other module.

*Maintenance.* Like a good piece of writing, a good program can always be improved, and modular programming facilitates the process of continually improving your Java programs because improving a module improves all of its clients. For example, it is normally the case that there are several different approaches to solving a particular problem. With modular programming, you can implement more than one and try them independently. More importantly, suppose that while developing a new client, you find a bug in some module. With modular programming, fixing that bug essentially fixes bugs in all of the module's clients.

IF YOU ENCOUNTER AN OLD PROGRAM (or a new program written by an old programmer!), you are likely to find one huge module—a long sequence of statements, stretching to several pages or more, where any statement can refer to any variable in the program. Old programs of this kind are found in critical parts of our computational infrastructure (for example, some nuclear power plants and some banks) precisely because the programmers charged with maintaining them cannot even understand them well enough to rewrite them in a modern language! With support for modular programming, modern languages like Java help us avoid such situations by separately developing libraries of methods in independent classes.

The ability to share static methods among different files fundamentally extends our programming model in two different ways. First, it allows us to reuse code without having to maintain multiple copies of it. Second, by allowing us to organize a program into files of manageable size that can be independently debugged and compiled, it strongly supports our basic message: *whenever you can clearly separate tasks within a program, you should do so.*

In this section, we have supplemented the `Std*` libraries of SECTION 1.5 with several other libraries that you can use: `Gaussian`, `StdArrayIO`, `StdRandom`, and `StdStats`. Furthermore, we have illustrated their use with several client programs. These tools are centered on basic mathematical concepts that arise in any scientific project or engineering task. Our intent is not just to provide tools, but also to illustrate that it is easy to create your own tools. The first question that most modern programmers ask when addressing a complex task is “Which tools do I need?” When the needed tools are not conveniently available, the second question is “How difficult would it be to implement them?” To be a good programmer, you need to have the confidence to build a software tool when you need it and the wisdom to know when it might be better to seek a solution in a library.

After libraries and modular programming, you have one more step to learn a complete modern programming model: *object-oriented programming*, the topic of CHAPTER 3. With object-oriented programming, you can build libraries of functions that use side effects (in a tightly controlled manner) to vastly extend the Java programming model. Before moving to object-oriented programming, we consider in this chapter the profound ramifications of the idea that any method can call itself (in SECTION 2.3) and a more extensive case study (in SECTION 2.4) of modular programming than the small clients in this section.



## Q&A

**Q.** I tried to use `StdRandom`, but got the error message `Exception in thread "main" java.lang.NoClassDefFoundError: StdRandom`. What's wrong?

**A.** You need to make `StdRandom` accessible to Java. See the first Q&A at the end of SECTION 1.5.

**Q.** Is there a keyword that identifies a class as a library?

**A.** No, any set of public methods will do. There is a bit of a conceptual leap in this viewpoint because it is one thing to sit down to create a `.java` file that you will compile and run, quite another thing to create a `.java` file that you will rely on much later in the future, and still another thing to create a `.java` file for *someone else* to use in the future. You need to develop some libraries for your own use before engaging in this sort of activity, which is the province of experienced systems programmers.

**Q.** How do I develop a new version of a library that I have been using for a while?

**A.** With care. Any change to the API might break any client program, so it is best to work in a separate directory. When you use this approach, you are working with a copy of the code. If you are changing a library that has a lot of clients, you can appreciate the problems faced by companies putting out new versions of their software. If you just want to add a few methods to a library, go ahead: that is usually not too dangerous, though you should realize that you might find yourself in a situation where you have to support that library for years!

**Q.** How do I know that an implementation behaves properly? Why not automatically check that it satisfies the API?

**A.** We use informal specifications because writing a detailed specification is not much different from writing a program. Moreover, a fundamental tenet of theoretical computer science says that doing so does not even solve the basic problem, because generally there is no way to check that two different programs perform the same computation.

## Exercises

**2.2.1** Add to `Gaussian` (PROGRAM 2.1.2) an implementation of the three-argument static method `pdf(x, mu, sigma)` specified in the API that computes the Gaussian probability density function with a given mean  $\mu$  and standard deviation  $\sigma$ , based on the formula  $\phi(x, \mu, \sigma) = \phi((x - \mu) / \sigma) / \sigma$ . Also add an implementation of the associated cumulative distribution function `cdf(z, mu, sigma)`, based on the formula  $\Phi(z, \mu, \sigma) = \Phi((z - \mu) / \sigma)$ .

**2.2.2** Write a library of static methods that implements the *hyperbolic* functions based on the definitions  $\sinh(x) = (e^x - e^{-x}) / 2$  and  $\cosh(x) = (e^x + e^{-x}) / 2$ , with  $\tanh(x)$ ,  $\coth(x)$ ,  $\operatorname{sech}(x)$ , and  $\operatorname{csch}(x)$  defined in a manner analogous to standard trigonometric functions.

**2.2.3** Write a test client for both `StdStats` and `StdRandom` that checks that the methods in both libraries operate as expected. Take a command-line argument  $n$ , generate  $n$  random numbers using each of the methods in `StdRandom`, and print their statistics. *Extra credit:* Defend the results that you get by comparing them to those that are to be expected from analysis.

**2.2.4** Add to `StdRandom` a method `shuffle()` that takes an array of `double` values as argument and rearranges them in random order. Implement a test client that checks that each permutation of the array is produced about the same number of times. Add overloaded methods that take arrays of integers and strings.

**2.2.5** Develop a client that does stress testing for `StdRandom`. Pay particular attention to `discrete()`. For example, do the probabilities sum to 1?

**2.2.6** Write a static method that takes `double` values `ymin` and `ymax` (with `ymin` strictly less than `ymax`), and a `double` array `a[]` as arguments and uses the `StdStats` library to linearly scale the values in `a[]` so that they are all between `ymin` and `ymax`.

**2.2.7** Write a `Gaussian` and `StdStats` client that explores the effects of changing the mean and standard deviation for the Gaussian probability density function. Create one plot with the Gaussian distributions having a fixed mean and various standard deviations and another with Gaussian distributions having a fixed standard deviation and various means.



**2.2.8** Add a method `exp()` to `StdRandom` that takes an argument  $\lambda$  and returns a random number drawn from the *exponential distribution* with rate  $\lambda$ . *Hint:* If  $x$  is a random number uniformly distributed between 0 and 1, then  $-\ln x / \lambda$  is a random number from the exponential distribution with rate  $\lambda$ .

**2.2.9** Add to `StdRandom` a static method `maxwellBoltzmann()` that returns a random value drawn from a *Maxwell–Boltzmann distribution* with parameter  $\sigma$ . To produce such a value, return the square root of the sum of the squares of three random numbers drawn from the Gaussian distribution with mean 0 and standard deviation  $\sigma$ . The speeds of molecules in an ideal gas obey a Maxwell–Boltzmann distribution.

**2.2.10** Modify `Bernoulli` (PROGRAM 2.2.6) to animate the bar graph, replotting it after each experiment, so that you can watch it converge to the Gaussian distribution. Then add a command-line argument and an overloaded `binomial()` implementation to allow you to specify the probability  $p$  that a biased coin comes up heads, and run experiments to get a feeling for the distribution corresponding to a biased coin. Be sure to try values of  $p$  that are close to 0 and close to 1.

**2.2.11** Develop a full implementation of `StdArrayIO` (implement all 12 methods indicated in the API).

**2.2.12** Write a library `Matrix` that implements the following API:

```
public class Matrix
```

---

<code>double dot(double[] a, double[] b)</code>	<i>vector dot product</i>
<code>double[][] multiply(double[][] a, double[][] b)</code>	<i>matrix–matrix product</i>
<code>double[][] transpose(double[][] a)</code>	<i>transpose</i>
<code>double[] multiply(double[][] a, double[] x)</code>	<i>matrix–vector product</i>
<code>double[] multiply(double[] x, double[][] a)</code>	<i>vector–matrix product</i>

(See SECTION 1.4.) As a test client, use the following code, which performs the same calculation as `Markov` (PROGRAM 1.6.3):

```
public static void main(String[] args)
{
    int trials = Integer.parseInt(args[0]);
    double[][] p = StdArrayIO.readDouble2D();
    double[] ranks = new double[p.length];
    rank[0] = 1.0;
    for (int t = 0; t < trials; t++)
        ranks = Matrix.multiply(ranks, p);
    StdArrayIO.print(ranks);
}
```

Mathematicians and scientists use mature libraries or special-purpose matrix-processing languages for such tasks. See the booksite for details on using such libraries.

**2.2.13** Write a `Matrix` client that implements the version of `Markov` described in SECTION 1.6 but is based on squaring the matrix, instead of iterating the vector-matrix multiplication.

**2.2.14** Rewrite `RandomSurfer` (PROGRAM 1.6.2) using the `StdArrayIO` and `StdRandom` libraries.

*Partial solution.*

```
...
double[][] p = StdArrayIO.readDouble2D();
int page = 0; // Start at page 0.
int[] freq = new int[n];
for (int t = 0; t < trials; t++)
{
    page = StdRandom.discrete(p[page]);
    freq[page]++;
}
...
```

## Creative Exercises

**2.2.15 Sicherman dice.** Suppose that you have two six-sided dice, one with faces labeled 1, 3, 4, 5, 6, and 8 and the other with faces labeled 1, 2, 2, 3, 3, and 4. Compare the probabilities of occurrence of each of the values of the sum of the dice with those for a standard pair of dice. Use `StdRandom` and `StdStats`.

**2.2.16 Craps.** The following are the rules for a *pass bet* in the game of *craps*. Roll two six-sided dice, and let  $x$  be their sum.

- If  $x$  is 7 or 11, you win.
- If  $x$  is 2, 3, or 12, you lose.

Otherwise, repeatedly roll the two dice until their sum is either  $x$  or 7.

- If their sum is  $x$ , you win.
- If their sum is 7, you lose.

Write a modular program to estimate the probability of winning a pass bet. Modify your program to handle loaded dice, where the probability of a die landing on 1 is taken from the command line, the probability of landing on 6 is  $1/6$  minus that probability, and 2–5 are assumed equally likely. Hint: Use `StdRandom.discrete()`.

**2.2.17 Gaussian random values.** Implement the no-argument `gaussian()` function in `StdRandom` (PROGRAM 2.2.1) using the Box–Muller formula (see EXERCISE 1.2.27). Next, consider an alternative approach, known as *Marsaglia’s method*, which is based on generating a random point in the unit circle and using a form of the Box–Muller formula (see the discussion of `do-while` at the end of SECTION 1.3).

```
public static double gaussian()
{
    double r, x, y;
    do
    {
        x = uniform(-1.0, 1.0);
        y = uniform(-1.0, 1.0);
        r = x*x + y*y;
    } while (r >= 1 || r == 0);
    return x * Math.sqrt(-2 * Math.log(r) / r);
}
```

For each approach, generate 10 million random values from the Gaussian distribution, and measure which is faster.

**2.2.18** *Dynamic histogram.* Suppose that the standard input stream is a sequence of `double` values. Write a program that takes an integer `n` and two `double` values `lo` and `hi` from the command line and uses `StdStats` to plot a histogram of the count of the numbers in the standard input stream that fall in each of the  $n$  intervals defined by dividing  $(lo, hi)$  into  $n$  equal-sized intervals. Use your program to add code to your solution to EXERCISE 2.2.3 to plot a histogram of the distribution of the numbers produced by each method, taking `n` from the command line.

**2.2.19** *Stress test.* Develop a client that does stress testing for `StdStats`. Work with a classmate, with one person writing code and the other testing it.

**2.2.20** *Gambler's ruin.* Develop a `StdRandom` client to study the gambler's ruin problem (see PROGRAM 1.3.8 and EXERCISE 1.3.24–25). *Note:* Defining a static method for the experiment is more difficult than for `Bernoulli` because you cannot return two values.

**2.2.21** *IFS.* Experiment with various inputs to `IFS` to create patterns of your own design like the Sierpinski triangle, the Barnsley fern, or the other examples in the table in the text. You might begin by experimenting with minor modifications to the given inputs.

**2.2.22** *IFS matrix implementation.* Write a version of `IFS` that uses the static method `multiply()` from `Matrix` (see EXERCISE 2.2.12) instead of the equations that compute the new values of `x0` and `y0`.

**2.2.23** *Library for properties of integers.* Develop a library based on the functions that we have considered in this book for computing properties of integers. Include functions for determining whether a given integer is prime; determining whether two integers are relatively prime; computing all the factors of a given integer; computing the greatest common divisor and least common multiple of two integers; Euler's totient function (EXERCISE 2.1.26); and any other functions that you think might be useful. Include overloaded implementations for `long` values. Create an API, a client that performs stress testing, and clients that solve several of the exercises earlier in this book.



**2.2.24** *Music library.* Develop a library based on the functions in `PlayThatTune` (PROGRAM 2.1.4) that you can use to write client programs to create and manipulate songs.

**2.2.25** *Voting machines.* Develop a `StdRandom` client (with appropriate static methods of its own) to study the following problem: Suppose that in a population of 100 million voters, 51% vote for candidate *A* and 49% vote for candidate *B*. However, the voting machines are prone to make mistakes, and 5% of the time they produce the wrong answer. Assuming the errors are made independently and at random, is a 5% error rate enough to invalidate the results of a close election? What error rate can be tolerated?

**2.2.26** *Poker analysis.* Write a `StdRandom` and `StdStats` client (with appropriate static methods of its own) to estimate the probabilities of getting one pair, two pair, three of a kind, a full house, and a flush in a five-card poker hand via simulation. Divide your program into appropriate static methods and defend your design decisions. *Extra credit:* Add straight and straight flush to the list of possibilities.

**2.2.27** *Animated plots.* Write a program that takes a command-line argument  $m$  and produces a bar graph of the  $m$  most recent `double` values on standard input. Use the same animation technique that we used for `BouncingBall` (PROGRAM 1.5.6): erase, redraw, show, and wait briefly. Each time your program reads a new number, it should redraw the whole bar graph. Since most of the picture does not change as it is redrawn slightly to the left, your program will produce the effect of a fixed-size window dynamically sliding over the input values. Use your program to plot a huge time-variant data file, such as stock prices.

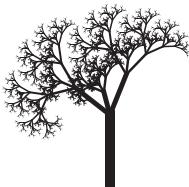
**2.2.28** *Array plot library.* Develop your own plot methods that improve upon those in `StdStats`. Be creative! Try to make a plotting library that you think will be useful for some application in the future.

## 2.3 Recursion

THE IDEA OF CALLING ONE FUNCTION from another immediately suggests the possibility of a function calling *itself*. The function-call mechanism in Java and most modern programming languages supports this possibility, which is known as *recursion*. In this section, we will study examples of elegant and efficient recursive solutions to a variety of problems. Recursion is a powerful programming technique that we use often in this book. Recursive programs are often more compact and easier to understand than their nonrecursive counterparts. Few programmers become sufficiently comfortable with recursion to use it in everyday code, but solving a problem with an elegantly crafted recursive program is a satisfying experience that is certainly accessible to every programmer (even you!).

2.3.1	Euclid's algorithm . . . . .	267
2.3.2	Towers of Hanoi . . . . .	270
2.3.3	Gray code . . . . .	275
2.3.4	Recursive graphics . . . . .	277
2.3.5	Brownian bridge . . . . .	279
2.3.6	Longest common subsequence . . . . .	287

*Programs in this section*



A recursive model  
of the natural world

Recursion is much more than a programming technique. In many settings, it is a useful way to describe the natural world. For example, the recursive tree (to the left) resembles a real tree, and has a natural recursive description. Many, many phenomena are well explained by recursive models. In particular, recursion plays a central role in computer science. It provides a simple computational model that embraces everything that can be computed with any computer; it helps us to organize and to analyze programs; and it is the key to numerous critically important computational applications, ranging from combinatorial search to tree data structures that support information processing to the fast Fourier transform for signal processing.

One important reason to embrace recursion is that it provides a straightforward way to build simple mathematical models that we can use to prove important facts about our programs. The proof technique that we use to do so is known as *mathematical induction*. Generally, we avoid going into the details of mathematical proofs in this book, but you will see in this section that it is worthwhile to understand that point of view and make the effort to convince yourself that recursive programs have the intended effect.



**Your first recursive program** The “Hello, World” for recursion is the *factorial* function, defined for positive integers  $n$  by the equation

$$n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$$

In other words,  $n!$  is the product of the positive integers less than or equal to  $n$ . Now,  $n!$  is easy to compute with a `for` loop, but an even easier method is to use the following recursive function:

```
public static long factorial(int n)
{
    if (n == 1) return 1;
    return n * factorial(n-1);
}
```

This function calls itself. The implementation clearly produces the desired effect. You can persuade yourself that it does so by noting that `factorial()` returns  $1 = 1!$  when  $n$  is 1 and that if it properly computes the value

$$(n-1)! = (n-1) \times (n-2) \times \dots \times 2 \times 1$$

then it properly computes the value

$$\begin{aligned} n! &= n \times (n-1)! \\ &= n \times (n-1) \times (n-2) \times \dots \times 2 \times 1 \end{aligned}$$

To compute `factorial(5)`, the recursive function multiplies 5 by `factorial(4)`; to compute `factorial(4)`, it multiplies 4 by `factorial(3)`; and so forth. This process is repeated until calling `factorial(1)`, which directly returns the value 1. We can trace this computation in precisely the same way that we trace any sequence of function calls. Since we treat all of the calls as being independent copies of the code, the fact that they are recursive is immaterial.

Our `factorial()` implementation exhibits the two main components that are required for every recursive function. First, the *base case* returns a value without making any subsequent recursive calls. It does this for one or more special input values for which the function can be evaluated without recursion. For `factorial()`, the base case is  $n = 1$ . Second, the *reduction step* is the central part of a recursive

```
factorial(5)
  factorial(4)
    factorial(3)
      factorial(2)
        factorial(1)
          return 1
        return 2*1 = 2
      return 3*2 = 6
    return 4*6 = 24
  return 5*24 = 120
```

*Function-call trace for `factorial(5)`*

function. It relates the function at one (or more) arguments to the function evaluated at one (or more) other arguments. For `factorial()`, the reduction step is `n * factorial(n-1)`. All recursive functions must have these two components. Furthermore, the sequence of argument values must converge to the base case. For `factorial()`, the value of `n` decreases by 1 for each call, so the sequence of argument values converges to the base case `n = 1`.

Tiny programs such as `factorial()` perhaps become slightly clearer if we put the reduction step in an `else` clause. However, adopting this convention for every recursive program would unnecessarily complicate larger programs because it would involve putting most of the code (for the reduction step) within curly braces after the `else`. Instead, we adopt the convention of always putting the base case as the first statement, ending with a `return`, and then devoting the rest of the code to the reduction step.

The `factorial()` implementation itself is not particularly useful in practice because  $n!$  grows so quickly that the multiplication will overflow a `long` and produce incorrect answers for  $n > 20$ . But the same technique is effective for computing all sorts of functions. For example, the recursive function

```
public static double harmonic(int n)
{
    if (n == 1) return 1.0;
    return harmonic(n-1) + 1.0/n;
}
```

computes the  $n$ th harmonic numbers (see PROGRAM 1.3.5) when  $n$  is small, based on the following equations:

$$\begin{aligned} H_n &= 1 + 1/2 + \dots + 1/n \\ &= (1 + 1/2 + \dots + 1/(n-1)) + 1/n \\ &= H_{n-1} + 1/n \end{aligned}$$

Indeed, this same approach is effective for computing, with only a few lines of code, the value of *any* finite sum (or product) for which you have a compact formula. Recursive functions like these are just loops in disguise, but recursion can help us better understand the underlying computation.

1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800
11	39916800
12	479001600
13	6227020800
14	87178291200
15	1307674368000
16	20922789888000
17	355687428096000
18	6402373705728000
19	121645100408832000
20	2432902008176640000

*Values of  $n!$  in `long`*

**Mathematical induction** Recursive programming is directly related to *mathematical induction*, a technique that is widely used for proving facts about the natural numbers.

Proving that a statement involving an integer  $n$  is true for infinitely many values of  $n$  by mathematical induction involves the following two steps:

- The *base case*: prove the statement true for some specific value or values of  $n$  (usually 0 or 1).
- The *induction step* (the central part of the proof): assume the statement to be true for all positive integers less than  $n$ , then use that fact to prove it true for  $n$ .

Such a proof suffices to show that the statement is true for *infinitely* many values of  $n$ : we can start at the base case, and use our proof to establish that the statement is true for each larger value of  $n$ , one by one.

Everyone's first induction proof is to demonstrate that the sum of the positive integers less than or equal to  $n$  is given by the formula  $n(n+1)/2$ . That is, we wish to prove that the following equation is valid for all  $n \geq 1$ :

$$1 + 2 + 3 \dots + (n-1) + n = n(n+1)/2$$

The equation is certainly true for  $n=1$  (base case) because  $1 = 1(1+1)/2$ . If we assume it to be true for all positive integers less than  $n$ , then, in particular, it is true for  $n-1$ , so

$$1 + 2 + 3 \dots + (n-1) = (n-1)n/2$$

and we can add  $n$  to both sides of this equation and simplify to get the desired equation (induction step).

Every time we write a recursive program, we need mathematical induction to be convinced that the program has the desired effect. The correspondence between induction and recursion is self-evident. The difference in nomenclature indicates a difference in outlook: in a recursive program, our outlook is to get a computation done by reducing to a smaller problem, so we use the term *reduction step*; in an induction proof, our outlook is to establish the truth of the statement for larger problems, so we use the term *induction step*.

When we write recursive programs we usually do not write down a full formal proof that they produce the desired result, but we are always dependent upon the existence of such a proof. We often appeal to an informal induction proof to convince ourselves that a recursive program operates as expected. For example, we just discussed an informal proof to become convinced that `factorial()` computes the product of the positive integers less than or equal to  $n$ .

### Program 2.3.1 Euclid's algorithm

```
public class Euclid
{
    public static int gcd(int p, int q)
    {
        if (q == 0) return p;
        return gcd(q, p % q);
    }
    public static void main(String[] args)
    {
        int p = Integer.parseInt(args[0]);
        int q = Integer.parseInt(args[1]);
        int divisor = gcd(p, q);
        StdOut.println(divisor);
    }
}
```

p, q | arguments  
divisor | greatest common divisor

```
% java Euclid 1440 408
24
% java Euclid 314159 271828
1
```

This program prints the greatest common divisor of its two command-line arguments, using a recursive implementation of Euclid's algorithm.

**Euclid's algorithm** The *greatest common divisor* (gcd) of two positive integers is the largest integer that divides evenly into both of them. For example, the greatest common divisor of 102 and 68 is 34 since both 102 and 68 are multiples of 34, but no integer larger than 34 divides evenly into 102 and 68. You may recall learning about the greatest common divisor when you learned to reduce fractions. For example, we can simplify  $68/102$  to  $2/3$  by dividing both numerator and denominator by 34, their gcd. Finding the gcd of huge numbers is an important problem that arises in many commercial applications, including the famous RSA cryptosystem.

We can efficiently compute the gcd using the following property, which holds for positive integers  $p$  and  $q$ :

If  $p > q$ , the gcd of  $p$  and  $q$  is the same as the gcd of  $q$  and  $p \% q$ .

To convince yourself of this fact, first note that the gcd of  $p$  and  $q$  is the same as the gcd of  $q$  and  $p-q$ , because a number divides both  $p$  and  $q$  if and only if it divides both  $q$  and  $p-q$ . By the same argument,  $q$  and  $p-2q$ ,  $q$  and  $p-3q$ , and so forth have the same gcd, and one way to compute  $p \% q$  is to subtract  $q$  from  $p$  until getting a number less than  $q$ .

The static method `gcd()` in `Euclid` (PROGRAM 2.3.1) is a compact recursive function whose reduction step is based on this property. The base case is when  $q$  is 0, with  $\text{gcd}(p, 0) = p$ . To see that the reduction step converges to the base case, observe that the second argument value strictly decreases in each recursive call since  $p \% q < q$ . If  $p < q$ , the first recursive call effectively switches the order of the two arguments. In fact, the second argument value decreases by at least a factor of 2 for every second recursive call, so the sequence of argument values quickly converges to the base case (see EXERCISE 2.3.11). This recursive solution to the problem of computing the greatest common divisor is known as *Euclid's algorithm* and is one of the oldest known algorithms—it is more than 2,000 years old.

```

gcd(1440, 408)
gcd(408, 216)
gcd(216, 192)
gcd(192, 24)
gcd(24, 0)
return 24
return 24
return 24
return 24
return 24

```

## Function-call trace for `gcd()`

# Towers of Hanoi

**Towers of Hanoi** No discussion of recursion would be complete without the ancient *towers of Hanoi* problem. In this problem, we have three poles and  $n$  discs that fit onto the poles. The discs differ in size and are initially stacked on one of the poles, in order from largest (disc  $n$ ) at the bottom to smallest (disc 1) at the top. The task is to move all  $n$  discs to another pole, while obeying the following rules:

- Move only one disc at a time.
  - Never place a larger disc on a smaller one.

One legend says that the world will end when a certain group of monks accomplishes this task in a temple with 64 golden discs on three diamond needles. But how can the monks accomplish the task at all, playing by the rules?

To solve the problem, our goal is to issue a sequence of instructions for moving the discs. We assume that the poles are arranged in a row, and that each instruction to move a disc specifies its number and whether to move it left or right. If a disc is on the left pole, an instruction to move left means to wrap to the right pole; if a disc is on the right pole, an instruction to move right means to wrap to the left pole. When the discs are all on one pole, there are two possible moves (move the smallest disc left or right); otherwise, there are three possible moves

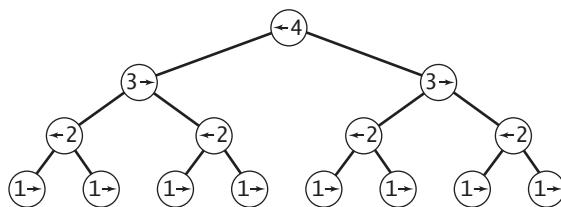
(move the smallest disc left or right, or make the one legal move involving the other two poles). Choosing among these possibilities on each move to achieve the goal is a challenge that requires a plan. Recursion provides just the plan that we need, based on the following idea: first we move the top  $n-1$  discs to an empty pole, then we move the largest disc to the other empty pole (where it does not interfere with the smaller ones), and then we complete the job by moving the  $n-1$  discs onto the largest disc.

`TowersOfHanoi` (PROGRAM 2.3.2) is a direct implementation of this recursive strategy. It takes a command-line argument  $n$  and prints the solution to the towers of Hanoi problem on  $n$  discs. The recursive function `moves()` prints the sequence of moves to move the stack of discs to the left (if the argument `left` is `true`) or to the right (if `left` is `false`). It does so exactly according to the plan just described.

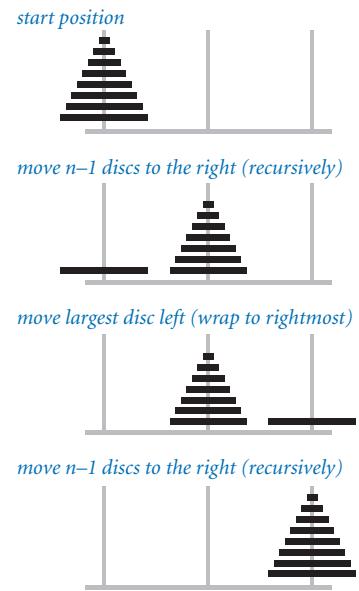
**Function-call trees** To better understand the behavior of modular programs that have multiple recursive calls (such as `TowersOfHanoi`), we use a visual representation known as a *function-call tree*. Specifically, we represent each method call as a *tree node*, depicted as a circle labeled with the values of the arguments for that call. Below each tree node, we draw the tree nodes corresponding to each call in that use of the method (in order from left to right) and lines connecting to them. This diagram contains all the information we need to understand the behavior of the program. It contains a tree node for each function call.

We can use function-call trees to understand the behavior of any modular program, but they are particularly useful in exposing the behavior of recursive

programs. For example, the tree corresponding to a call to `move()` in `TowersOfHanoi` is easy to construct. Start by drawing a tree node labeled with the values of the command-line arguments. The first argument is the number



Function-call tree for `moves(4, true)` in `TowersOfHanoi`



Recursive plan for towers of Hanoi

### Program 2.3.2 Towers of Hanoi

```
public class TowersOfHanoi
{
    public static void moves(int n, boolean left)
    {
        if (n == 0) return;
        moves(n-1, !left);
        if (left) StdOut.println(n + " left");
        else      StdOut.println(n + " right");
        moves(n-1, !left);
    }
    public static void main(String[] args)
    { // Read n, print moves to move n discs left.
        int n = Integer.parseInt(args[0]);
        moves(n, true);
    }
}
```

n	number of discs
left	direction to move pile

The recursive method `moves()` prints the moves needed to move  $n$  discs to the left (if `left` is `true`) or to the right (if `left` is `false`).

```
% java TowersOfHanoi 1
1 left
% java TowersOfHanoi 2
1 right
2 left
1 right
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

of discs in the pile to be moved (and the label of the disc to actually be moved); the second is the direction to move the disc. For clarity, we depict the direction (a boolean value) as an arrow that points left or right, since that is our interpretation of the value—the direction to move the piece. Then draw two tree nodes below with the number of discs decremented by 1 and the direction switched, and continue doing so until only nodes with labels corresponding to a first argument value 1 have no nodes below them. These nodes correspond to calls on `moves()` that do not lead to further recursive calls.

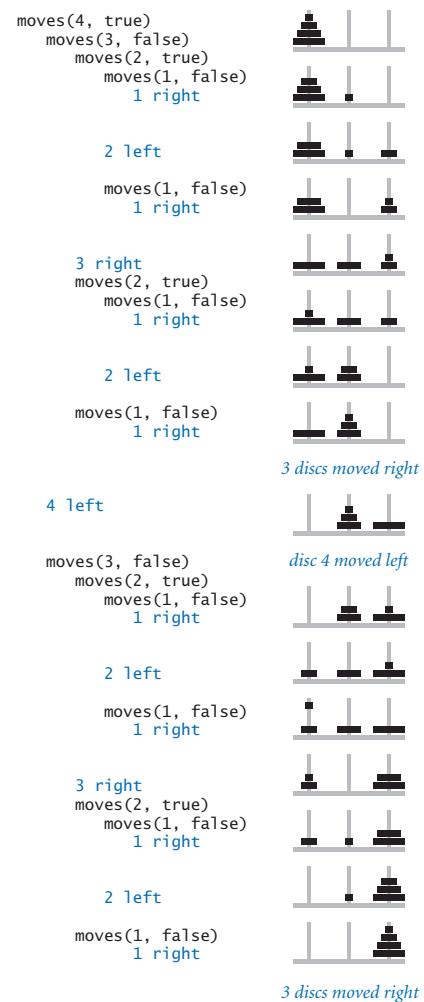
Take a moment to study the function-call tree depicted earlier in this section and to compare it with the corresponding function-call trace depicted at right. When you do so, you will see that the recursion tree is just a compact representation of the trace. In particular, reading the node labels from left to right gives the moves needed to solve the problem.

Moreover, when you study the tree, you probably notice several patterns, including the following two:

- Alternate moves involve the smallest disc.
- That disc always moves in the same direction.

These observations are relevant because they give a solution to the problem that does not require recursion (or even a computer): every other move involves the smallest disc (including the first and last), and each intervening move is the only legal move at the time not involving the smallest disc. We can *prove* that this approach produces the same outcome as the recursive program, using induction. Having started centuries ago without the benefit of a computer, perhaps our monks are using this approach.

Trees are relevant and important in understanding recursion because the tree is a quintessential recursive object. As an abstract mathematical model, trees play an essential role in many applications, and in CHAPTER 4, we will consider the use of trees as a computational model to structure data for efficient processing.



Function-call trace for `moves(4, true)`

**Exponential time** One advantage of using recursion is that often we can develop mathematical models that allow us to prove important facts about the behavior of recursive programs. For the towers of Hanoi problem, we can estimate the amount of time until the end of the world (assuming that the legend is true). This exercise is important not just because it tells us that the end of the world is quite far off (even if the legend is true), but also because it provides insight that can help us avoid writing programs that will not finish until then.

The mathematical model for the towers of Hanoi problem is simple: if we define the function  $T(n)$  to be the number of discs moved by `TowersOfHanoi` to solve an  $n$ -disc problem, then the recursive code implies that  $T(n)$  must satisfy the following equation:

$$T(n) = 2 T(n-1) + 1 \text{ for } n > 1, \text{ with } T(1) = 1$$

Such an equation is known in discrete mathematics as a *recurrence relation*. Recurrence relations naturally arise in the study of recursive programs. We can often use them to derive a closed-form expression for the quantity of interest. For  $T(n)$ , you may have already guessed from the initial values  $T(1) = 1$ ,  $T(2) = 3$ ,  $T(3) = 7$ , and  $T(4) = 15$  that  $T(n) = 2^n - 1$ . The recurrence relation provides a way to *prove* this to be true, by mathematical induction:

- *Base case*:  $T(1) = 2^1 - 1 = 1$
- *Induction step*: if  $T(n-1) = 2^{n-1} - 1$ ,  $T(n) = 2 (2^{n-1} - 1) + 1 = 2^n - 1$

Therefore, by induction,  $T(n) = 2^n - 1$  for all  $n > 0$ . The minimum possible number of moves also satisfies the same recurrence (see EXERCISE 2.3.11).

Knowing the value of  $T(n)$ , we can estimate the amount of time required to perform all the moves. If the monks move discs at the rate of one per second, it would take more than one week for them to finish a 20-disc problem, more than 34 years to finish a 30-disc problem, and more than 348 *centuries* for them to finish a 40-disc problem (assuming that they do not make a mistake). The 64-disc problem would take more than 5.8 *billion* centuries. The end of the world is likely to be even further off than that because those monks presumably never have had the benefit of using PROGRAM 2.3.2, and might not be able to move the discs so rapidly or to figure out so quickly which disc to move next.

Even computers are no match for exponential growth. A computer that can do a billion operations per second will still take centuries to do  $2^{64}$  operations, and no computer will ever do  $2^{1,000}$  operations, say. The lesson is profound: with recursion, you can easily write simple short programs



Exponential growth

that take exponential time, but they simply will not run to completion when you try to run them for large  $n$ . Novices are often skeptical of this basic fact, so it is worth your while to pause now to think about it. To convince yourself that it is true, take the print statements out of `TowersOfHanoi` and run it for increasing values of  $n$  starting at 20. You can easily verify that each time you increase the value of  $n$  by 1, the running time doubles, and you will quickly lose patience waiting for it to finish. If you wait for an hour for some value of  $n$ , you will wait more than a day for  $n + 5$ , more than a month for  $n + 10$ , and more than a century for  $n + 20$  (no one has *that* much patience). Your computer is just not fast enough to run every short Java program that you write, no matter how simple the program might seem! *Beware of programs that might require exponential time.*

We are often interested in predicting the running time of our programs. In SECTION 4.1, we will discuss the use of the same process that we just used to help estimate the running time of other programs.

**Gray codes** The towers of Hanoi problem is no toy. It is intimately related to basic algorithms for manipulating numbers and discrete objects. As an example, we consider *Gray codes*, a mathematical abstraction with numerous applications.

The playwright Samuel Beckett, perhaps best known for *Waiting for Godot*, wrote a play called *Quad* that had the following property: starting with an empty stage, characters enter and exit one at a time so that each subset of characters on the stage appears exactly once. How did Beckett generate the stage directions for this play?

One way to represent a subset of  $n$  discrete objects is to use a string of  $n$  bits. For Beckett's problem, we use a 4-bit string, with bits numbered from right to left and a bit value of 1 indicating the character onstage. For example, the string 0 1 0 1 corresponds to the scene with characters 3 and 1 onstage. This representation gives a quick proof of a basic fact: *the number of different subsets of  $n$  objects is exactly  $2^n$ .* *Quad* has four characters, so there are  $2^4 = 16$  different scenes. Our task is to generate the stage directions.

An  $n$ -bit *Gray code* is a list of the  $2^n$  different  $n$ -bit binary numbers such that each element in the list differs in precisely one bit from its predecessor. Gray codes directly apply to Beckett's problem because changing the value of a bit from 0 to 1

code	subset	move
0 0 0 0	empty	
0 0 0 1	1	enter 1
0 0 1 1	2 1	enter 2
0 0 1 0	2	exit 1
0 1 1 0	3 2	enter 3
0 1 1 1	3 2 1	enter 1
0 1 0 1	3 1	exit 2
0 1 0 0	3	exit 1
1 1 0 0	4 3	enter 4
1 1 0 1	4 3 1	enter 1
1 1 1 1	4 3 2 1	enter 2
1 1 1 0	4 3 2	exit 1
1 0 1 0	4 2	exit 3
1 0 1 1	4 2 1	enter 1
1 0 0 1	4 1	exit 2
1 0 0 0	4	exit 1

Gray code representations

corresponds to a character entering the subset onstage; changing a bit from 1 to 0 corresponds to a character exiting the subset.

How do we generate a Gray code? A recursive plan that is very similar to the one that we used for the towers of Hanoi problem is effective. The  $n$ -bit *binary-reflected Gray code* is defined recursively as follows:

- The  $(n-1)$  bit code, with 0 prepended to each word, followed by
- The  $(n-1)$  bit code *in reverse order*, with 1 prepended to each word

The 0-bit code is defined to be empty, so the 1-bit code is 0 followed by 1. From this recursive definition, we can verify by induction that the  $n$ -bit binary reflected Gray code has the required property: adjacent codewords differ in one bit position. It is true by the inductive hypothesis, except possibly for the last codeword in the first half and the first codeword in the second half: this pair differs only in their first bit.

The recursive definition leads, after some careful thought, to the implementation in Beckett (PROGRAM 2.3.3) for printing Beckett's stage directions. This program is remarkably similar to TowersOfHanoi. Indeed, except for nomenclature, the only difference is in the values of the second arguments in the recursive calls!

As with the directions in TowersOfHanoi, the enter and exit directions are redundant in Beckett, since `exit` is issued only when an actor is onstage, and `enter` is issued only when an actor is not onstage. Indeed, both Beckett and TowersOfHanoi directly involve the `ruler` function that we considered in one of our first programs (PROGRAM 1.2.1). Without the printing instructions, they both implement a simple recursive function that could allow `Ruler` to print the values of the `ruler` function for any value given as a command-line argument.

Gray codes have many applications, ranging from analog-to-digital converters to experimental design. They have been used in pulse code communication, the minimization of logic circuits, and hypercube architectures, and were even proposed to organize books on library shelves.

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2-, 3-, and 4-bit Gray codes

### Program 2.3.3 Gray code

```
public class Beckett
{
    public static void moves(int n, boolean enter)
    {
        if (n == 0) return;
        moves(n-1, true);
        if (enter) StdOut.println("enter " + n);
        else       StdOut.println("exit   " + n);
        moves(n-1, false);
    }

    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        moves(n, true);
    }
}
```

n	number of actors
enter	stage direction

This recursive program gives Beckett's stage instructions (the bit positions that change in a binary-reflected Gray code). The bit position that changes is precisely described by the ruler function, and (of course) each actor alternately enters and exits.

```
% java Beckett 1
enter 1
% java Beckett 2
enter 1
enter 2
exit 1
% java Beckett 3
enter 1
enter 2
exit 1
enter 3
enter 1
exit 2
exit 1
```

```
% java Beckett 4
enter 1
enter 2
exit 1
enter 3
enter 1
exit 2
exit 1
enter 4
enter 1
enter 2
exit 1
exit 3
enter 1
exit 2
exit 1
```

**Recursive graphics** Simple recursive drawing schemes can lead to pictures that are remarkably intricate. Recursive drawings not only relate to numerous applications, but also provide an appealing platform for developing a better understanding of properties of recursive functions, because we can watch the process of a recursive figure taking shape.

As a first simple example, consider `Htree` (PROGRAM 2.3.4), which, given a command-line argument  $n$ , draws an *H-tree* of order  $n$ , defined as follows: The base case is to draw nothing for  $n = 0$ . The reduction step is to draw, within the unit square

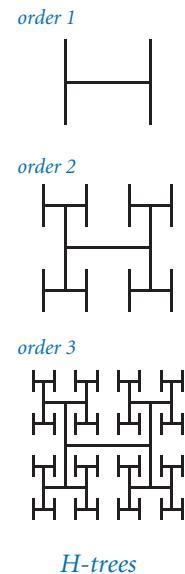
- three lines in the shape of the letter H
- four H-trees of order  $n - 1$ , one centered at each tip of the H

with the additional proviso that the H-trees of order  $n - 1$  are halved in size.

Drawings like these have many practical applications. For example, consider a cable company that needs to run cable to all of the homes distributed throughout its region. A reasonable strategy is to use an H-tree to get the signal to a suitable number of centers distributed throughout the region, then run cables connecting each home to the nearest center. The same problem is faced by computer designers who want to distribute power or signal throughout an integrated circuit chip.

Though every drawing is in a fixed-size window, H-trees certainly exhibit exponential growth. An H-tree of order  $n$  connects  $4^n$  centers, so you would be trying to plot more than a million lines with  $n = 10$ , and more than a billion with  $n = 15$ . The program will certainly not finish the drawing with  $n = 30$ .

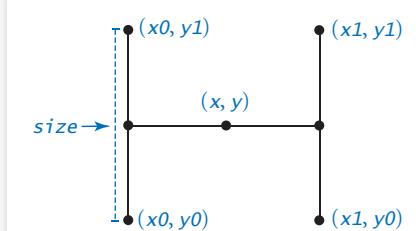
If you take a moment to run `Htree` on your computer for a drawing that takes a minute or so to complete, you will, just by watching the drawing progress, have the opportunity to gain substantial insight into the nature of recursive programs, because you can see the *order* in which the H figures appear and how they form into H-trees. An even more instructive exercise, which derives from the fact that the same drawing results no matter in which order the recursive `draw()` calls and the `StdDraw.line()` calls appear, is to observe the effect of rearranging the order of these calls on the order in which the lines appear in the emerging drawing (see EXERCISE 2.3.14).



### Program 2.3.4 Recursive graphics

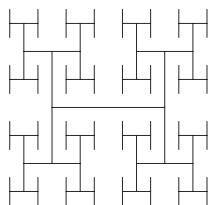
```
public class Htree
{
    public static void draw(int n, double size, double x, double y)
    { // Draw an H-tree centered at x, y
        // of depth n and given size.
        if (n == 0) return;
        double x0 = x - size/2, x1 = x + size/2;
        double y0 = y - size/2, y1 = y + size/2;
        StdDraw.line(x0, y, x1, y);
        StdDraw.line(x0, y0, x0, y1);
        StdDraw.line(x1, y0, x1, y1);
        draw(n-1, size/2, x0, y0);
        draw(n-1, size/2, x0, y1);
        draw(n-1, size/2, x1, y0);
        draw(n-1, size/2, x1, y1);
    }
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        draw(n, 0.5, 0.5, 0.5);
    }
}
```

$n$	$depth$
$size$	$line\ length$
$x, y$	$center$

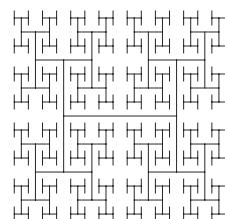


The function `draw()` draws three lines, each of length `size`, in the shape of the letter H, centered at  $(x, y)$ . Then, it calls itself recursively for each of the four tips, halving the `size` argument in each call and using an integer argument `n` to control the depth of the recursion.

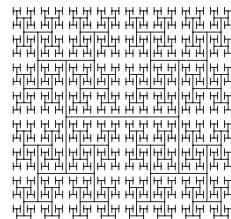
% java Htree 3



% java Htree 4



% java Htree 5



**Brownian bridge** An H-tree is a simple example of a *fractal*: a geometric shape that can be divided into parts, each of which is (approximately) a reduced-size copy of the original. Fractals are easy to produce with recursive programs, although scientists, mathematicians, and programmers study them from many different points of view. We have already encountered fractals several times in this book—for example, IFS (PROGRAM 2.2.3).

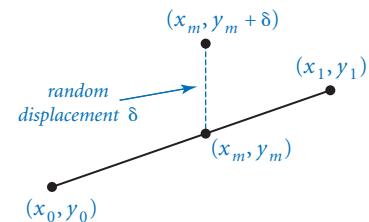
The study of fractals plays an important and lasting role in artistic expression, economic analysis, and scientific discovery. Artists and scientists use fractals to build compact models of complex shapes that arise in nature and resist description using conventional geometry, such as clouds, plants, mountains, riverbeds, human skin, and many others. Economists use fractals to model function graphs of economic indicators.

*Fractional Brownian motion* is a mathematical model for creating realistic fractal models for many naturally rugged shapes. It is used in computational finance and in the study of many natural phenomena, including ocean flows and nerve membranes. Computing the exact fractals specified by the model can be a difficult challenge, but it is not difficult to compute approximations with recursive programs.

Brownian (PROGRAM 2.3.5) produces a function graph that approximates a simple example of fractional Brownian motion known as a *Brownian bridge* and closely related functions. You can think of this graph as a random walk that connects the two points  $(x_0, y_0)$  and  $(x_1, y_1)$ , controlled by a few parameters. The implementation is based on the *midpoint displacement method*, which is a recursive plan for drawing the plot within the  $x$ -interval  $[x_0, x_1]$ . The base case (when the length of the interval is smaller than a given tolerance) is to draw a straight line connecting the two endpoints. The reduction case is to divide the interval into two halves, proceeding as follows:

- Compute the midpoint  $(x_m, y_m)$  of the interval.
- Add to the  $y$ -coordinate  $y_m$  of the midpoint a random value  $\delta$ , drawn from the Gaussian distribution with mean 0 and a given variance.
- Recur on the subintervals, dividing the variance by a given scaling factor  $s$ .

The shape of the curve is controlled by two parameters: the *volatility* (initial value of the variance) controls the distance the function graph strays from the straight



*Brownian bridge calculation*

### Program 2.3.5 Brownian bridge

```

public class Brownian
{
    public static void curve(double x0, double y0,
                            double x1, double y1,
                            double var, double s)
    {
        if (x1 - x0 < 0.01)
        {
            StdDraw.line(x0, y0, x1, y1);
            return;
        }
        double xm = (x0 + x1) / 2;
        double ym = (y0 + y1) / 2;
        double delta = StdRandom.gaussian(0, Math.sqrt(var));
        curve(x0, y0, xm, ym+delta, var/s, s);
        curve(xm, ym+delta, x1, y1, var/s, s);
    }
    public static void main(String[] args)
    {
        double hurst = Double.parseDouble(args[0]);
        double s = Math.pow(2, 2*hurst);
        curve(0, 0.5, 1.0, 0.5, 0.01, s);
    }
}

```

x0, y0	<i>left endpoint</i>
x1, y1	<i>right endpoint</i>
xm, ym	<i>middle</i>
delta	<i>displacement</i>
var	<i>variance</i>
hurst	<i>Hurst exponent</i>

By adding a small, random Gaussian to a recursive program that would otherwise plot a straight line, we get fractal curves. The command-line argument *hurst*, known as the Hurst exponent, controls the smoothness of the curves.

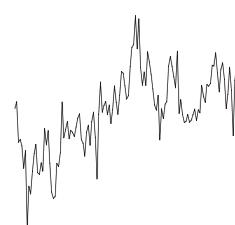
% java Brownian 1



% java Brownian 0.5



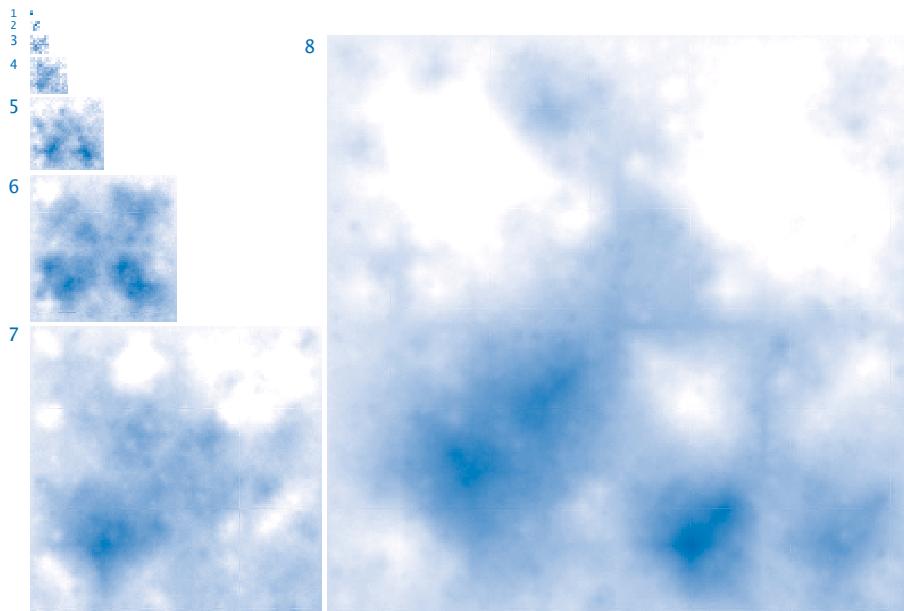
% java Brownian 0.05



line connecting the points, and *the Hurst exponent* controls the smoothness of the curve. We denote the Hurst exponent by  $H$  and divide the variance by  $2^{2H}$  at each recursive level. When  $H$  is  $1/2$  (halved at each level), the curve is a Brownian bridge—a continuous version of the gambler’s ruin problem (see PROGRAM 1.3.8). When  $0 < H < 1/2$ , the displacements tend to increase, resulting in a rougher curve. Finally, when  $2 > H > 1/2$ , the displacements tend to decrease, resulting in a smoother curve. The value  $2 - H$  is known as the *fractal dimension* of the curve.

The volatility and initial endpoints of the interval have to do with scale and positioning. The `main()` test client in `Brownian` allows you to experiment with the Hurst exponent. With values larger than  $1/2$ , you get plots that look something like the horizon in a mountainous landscape; with values smaller than  $1/2$ , you get plots similar to those you might see for the value of a stock index.

Extending the midpoint displacement method to two dimensions yields fractals known as *plasma clouds*. To draw a rectangular plasma cloud, we use a recursive plan where the base case is to draw a rectangle of a given color and the reduction step is to draw a plasma cloud in each of the four quadrants with colors that are perturbed from the average with a random Gaussian. Using the same volatility and smoothness controls as in `Brownian`, we can produce synthetic clouds that are remarkably realistic. We can use the same code to produce synthetic terrain, by interpreting the color value as the altitude. Variants of this scheme are widely used in the entertainment industry to generate background scenery for movies and games.



*Plasma clouds*

**Pitfalls of recursion** By now, you are perhaps persuaded that recursion can help you to write compact and elegant programs. As you begin to craft your own recursive programs, you need to be aware of several common pitfalls that can arise. We have already discussed one of them in some detail (the running time of your program might grow exponentially). Once identified, these problems are generally not difficult to overcome, but you will learn to be very careful to avoid them when writing recursive programs.

*Missing base case.* Consider the following recursive function, which is supposed to compute harmonic numbers, but is missing a base case:

```
public static double harmonic(int n)
{
    return harmonic(n-1) + 1.0/n;
}
```

If you run a client that calls this function, it will repeatedly call itself and never return, so your program will never terminate. You probably already have encountered infinite loops, where you invoke your program and nothing happens (or perhaps you get an unending sequence of printed output). With infinite recursion, however, the result is different because the system keeps track of each recursive call (using a mechanism that we will discuss in SECTION 4.3, based on a data structure known as a *stack*) and eventually runs out of memory trying to do so. Eventually, Java reports a `StackOverflowError` at run time. When you write a recursive program, you should always try to convince yourself that it has the desired effect by an informal argument based on mathematical induction. Doing so might uncover a missing base case.

*No guarantee of convergence.* Another common problem is to include within a recursive function a recursive call to solve a subproblem that is not smaller than the original problem. For example, the following method goes into an infinite recursive loop for any value of its argument (except 1) because the sequence of argument values does not converge to the base case:

```
public static double harmonic(int n)
{
    if (n == 1) return 1.0;
    return harmonic(n) + 1.0/n;
}
```

Bugs like this one are easy to spot, but subtle versions of the same problem can be harder to identify. You may find several examples in the exercises at the end of this section.

*Excessive memory requirements.* If a function calls itself recursively an excessive number of times before returning, the memory required by Java to keep track of the recursive calls may be prohibitive, resulting in a `StackOverflowError`. To get an idea of how much memory is involved, run a small set of experiments using our recursive function for computing the harmonic numbers for increasing values of  $n$ :

```
public static double harmonic(int n)
{
    if (n == 1) return 1.0;
    return harmonic(n-1) + 1.0/n;
}
```

The point at which you get `StackOverflowError` will give you some idea of how much memory Java uses to implement recursion. By contrast, you can run PROGRAM 1.3.5 to compute  $H_n$  for huge  $n$  using only a tiny bit of memory.

*Excessive recomputation.* The temptation to write a simple recursive function to solve a problem must always be tempered by the understanding that a function might take exponential time (unnecessarily) due to excessive recomputation. This effect is possible even in the simplest recursive functions, and you certainly need to learn to avoid it. For example, the *Fibonacci sequence*

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

is defined by the recurrence  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$  with  $F_0 = 0$  and  $F_1 = 1$ . The Fibonacci sequence has many interesting properties and arise in numerous applications. A novice programmer might implement this recursive function to compute numbers in the Fibonacci sequence:

```
// Warning: this function is spectacularly inefficient.
public static long fibonacci(int n)
{
    if (n == 0) return 0;
    if (n == 1) return 1;
    return fibonacci(n-1) + fibonacci(n-2);
}
```

```
fibonacci(8)
    fibonacci(7)
        fibonacci(6)
            fibonacci(5)
                fibonacci(4)
                    fibonacci(3)
                        fibonacci(2)
                            fibonacci(1)
                                return 1
                            fibonacci(0)
                                return 0
                            return 1
                        fibonacci(1)
                            return 1
                        return 2
                    fibonacci(2)
                        fibonacci(1)
                            return 1
                        fibonacci(0)
                            return 0
                        return 1
                    return 3
                fibonacci(3)
                    fibonacci(2)
                        fibonacci(1)
                            return 1
                        fibonacci(0)
                            return 0
                        return 1
                    fibonacci(1)
                        return 1
                    return 2
                return 5
            fibonacci(4)
                fibonacci(3)
                    fibonacci(2)
                    :
                    :
            :
```

Wrong way to compute Fibonacci numbers

However, this function is spectacularly inefficient! Novice programmers often refuse to believe this fact, and run code like this expecting that the computer is certainly fast enough to crank out an answer. Go ahead; see if your computer is fast enough to use this function to compute  $\text{fibonacci}(50)$ . To see why it is futile to do so, consider what the function does to compute  $\text{fibonacci}(8) = 21$ . It first computes  $\text{fibonacci}(7) = 13$  and  $\text{fibonacci}(6) = 8$ . To compute  $\text{fibonacci}(7)$ , it recursively computes  $\text{fibonacci}(6) = 8$  again and  $\text{fibonacci}(5) = 5$ . Things rapidly get worse because both times it computes  $\text{fibonacci}(6)$ , it ignores the fact that it already computed  $\text{fibonacci}(5)$ , and so forth. In fact, the number of times this program computes  $\text{fibonacci}(1)$  when computing  $\text{fibonacci}(n)$  is precisely  $F_n$  (see EXERCISE 2.3.12). The mistake of recomputation is compounded exponentially. As an example,  $\text{fibonacci}(200)$  makes  $F_{200} > 10^{43}$  recursive calls to  $\text{fibonacci}(1)$ ! No imaginable computer will ever be able to do this many calculations. *Beware of programs that might require exponential time.* Many calculations that arise and find natural expression as recursive functions fall into this category. Do not fall into the trap of implementing and trying to run them.

NEXT, WE CONSIDER A SYSTEMATIC TECHNIQUE known as *dynamic programming*, an elegant technique for avoiding such problems. The idea is to avoid the excessive recomputation inherent in some recursive functions by saving away the previously computed values for later reuse, instead of constantly recomputing them.

**Dynamic programming** A general approach to implementing recursive programs, known as *dynamic programming*, provides effective and elegant solutions to a wide class of problems. The basic idea is to recursively divide a complex problem into a number of simpler subproblems; store the answer to each of these subproblems; and, ultimately, use the stored answers to solve the original problem. By solving each subproblem only once (instead of over and over), this technique avoids a potential exponential blow-up in the running time.

For example, if our original problem is to compute the  $n$ th Fibonacci number, then it is natural to define  $n + 1$  subproblems, where subproblem  $i$  is to compute the  $i$ th Fibonacci number for each  $0 \leq i \leq n$ . We can solve subproblem  $i$  easily if we already know the solutions to smaller subproblems—specifically, subproblems  $i-1$  and  $i-2$ . Moreover, the solution to our original problem is simply the solution to one of the subproblems—subproblem  $n$ .

*Top-down dynamic programming.* In *top-down* dynamic programming, we store or *cache* the result of each subproblem that we solve, so that the next time we need to solve the same subproblem, we can use the cached values instead of solving the subproblem from scratch. For our Fibonacci example, we use an array  $f[]$  to store the Fibonacci numbers that have already been computed. We accomplish this in Java by using a *static* variable, also known as a *class variable* or *global variable*, that is declared outside of any method. This allows us to save information from one function call to the next.

```
public class TopDownFibonacci
{
    static variable
    (declared outside
    of any method)
    private static long[] f = new long[92];
    public static long fibonacci(int n)
    {
        if (n == 0) return 0;
        if (n == 1) return 1;
        if (f[n] > 0) return f[n];
        f[n] = fibonacci(n-1) + fibonacci(n-2);
        return f[n];
    }
}
```

The diagram shows the `TopDownFibonacci` class with annotations:

- A blue box surrounds the declaration of the static variable `f` with the label "static variable (declared outside of any method)" below it.
- A blue box surrounds the assignment `f[n] = fibonacci(n-1) + fibonacci(n-2);` with the label "compute and cache value" below it.
- An arrow points from the label "cached values" to the declaration of `f`.
- An arrow points from the label "return cached value (if previously computed)" to the line `if (f[n] > 0) return f[n];`.

*Top-down dynamic programming approach for computing Fibonacci numbers*

Top-down dynamic programming is also known as *memoization* because it avoids duplicating work by *remembering* the results of function calls.

*Bottom-up dynamic programming.* In *bottom-up* dynamic programming, we compute solutions to *all* of the subproblems, starting with the “simplest” subproblems and gradually building up solutions to more and more complicated subproblems. To apply bottom-up dynamic programming, we must order the subproblems so that each subsequent subproblem can be solved by combining solutions to subproblems earlier in the order (which have already been solved). For our Fibonacci example, this is easy: solve the subproblems in the order 0, 1, and 2, and so forth. By the time we need to solve subproblem  $i$ , we have already solved all smaller subproblems—in particular, subproblems  $i-1$  and  $i-2$ .

```
public static long fibonacci(int n)
{
    int[] f = new int[n+1];
    f[0] = 0;
    f[1] = 1;
    for (int i = 2; i <= n; i++)
        f[i] = f[i-1] + f[i-2];
    return f[n];
}
```

When the ordering of the subproblems is clear, and space is available to store all the solutions, bottom-up dynamic programming is a very effective approach.

NEXT, WE CONSIDER A MORE SOPHISTICATED application of dynamic programming, where the order of solving the subproblems is not so clear (until you see it). Unlike the problem of computing Fibonacci numbers, this problem would be much more difficult to solve without thinking recursively and also applying a bottom-up dynamic programming approach.

*Longest common subsequence problem.* We consider a fundamental string-processing problem that arises in computational biology and other domains. Given two strings  $x$  and  $y$ , we wish to determine how *similar* they are. Some examples include comparing two DNA sequences for homology, two English words for spelling, or two Java files for repeated code. One measure of similarity is the length of the *longest common subsequence* (LCS). If we delete some characters from  $x$  and some characters from  $y$ , and the resulting two strings are equal, we call the resulting string a *common subsequence*. The LCS problem is to find a common subsequence of two strings that is as long as possible. For example, the LCS of GGCACCCACG and ACGGGCGGATACG is GGCAACG, a string of length 7.

Algorithms to compute the LCS are used in data comparison programs like the `diff` command in Unix, which has been used for decades by programmers wanting to understand differences and similarities in their text files. Similar algorithms play important roles in scientific applications, such as the Smith–Waterman algorithm in computational biology and the Viterbi algorithm in digital communications theory.

**LCS recurrence.** Now we describe a recursive formulation that enables us to find the LSC of two given strings  $s$  and  $t$ . Let  $m$  and  $n$  be the lengths of  $s$  and  $t$ , respectively. We use the notation  $s[i..m)$  to denote the *suffix* of  $s$  starting at index  $i$ , and  $t[j..n)$  to denote the suffix of  $t$  starting at index  $j$ . On the one hand, if  $s$  and  $t$  begin with the same character, then the LCS of  $x$  and  $y$  contains that first character. Thus, our problem reduces to finding the LCS of the suffixes  $s[1..m)$  and  $t[1..n)$ . On the other hand, if  $s$  and  $t$  begin with different characters, both characters cannot be part of a common subsequence, so we can safely discard one or the other. In either case, the problem reduces to finding the LCS of two strings—either  $s[0..m)$  and  $t[1..n)$  or  $s[1..m)$  and  $t[0..n)$ —one of which is strictly shorter. In general, if we let  $\text{opt}[i][j]$  denote the length of the LCS of the suffixes  $s[i..m)$  and  $t[j..n)$ , then the following recurrence expresses  $\text{opt}[i][j]$  in terms of the length of the LCS for shorter suffixes.

$$\text{opt}[i][j] = \begin{cases} 0 & \text{if } i = m \text{ or } j = n \\ \text{opt}[i+1, j+1] + 1 & \text{if } s[i] = t[j] \\ \max(\text{opt}[i, j+1], \text{opt}[i+1, j]) & \text{otherwise} \end{cases}$$

**Dynamic programming solution.** `LongestCommonSubsequence` (PROGRAM 2.3.6) begins with a bottom-up dynamic programming approach to solving this recurrence. We maintain a two-dimensional array  $\text{opt}[i][j]$  that stores the length of the LCS of the suffixes  $s[i..m)$  and  $t[j..n)$ . Initially, the bottom row (the values for  $i = m$ ) and the right column (the values for  $j = n$ ) are 0. These are the initial values. From the recurrence, the order of the rest of the computation is clear: we start with  $\text{opt}[m][n] = 1$ . Then, as long as we decrease either  $i$  or  $j$  or both, we know that we will have computed what we need to compute  $\text{opt}[i][j]$ , since the two options involve an  $\text{opt}[][]$  entry with a larger value of  $i$  or  $j$  or both. The method `lcs()` in PROGRAM 2.3.6 computes the elements in  $\text{opt}[][]$  by filling in values in rows from bottom to top ( $i = m-1$  to 0) and from right to left in each row ( $j = n-1$  to 0). The alternative choice of filling in values in columns from right to

### Program 2.3.6 Longest common subsequence

```

public class LongestCommonSubsequence
{
    public static String lcs(String s, String t)
    { // Compute length of LCS for all subproblems.
        int m = s.length(), n = t.length();
        int[][] opt = new int[m+1][n+1];
        for (int i = m-1; i >= 0; i--)
            for (int j = n-1; j >= 0; j--)
                if (s.charAt(i) == t.charAt(j))
                    opt[i][j] = opt[i+1][j+1] + 1;
                else
                    opt[i][j] = Math.max(opt[i+1][j], opt[i][j+1]);
        // Recover LCS itself.
        String lcs = "";
        int i = 0, j = 0;
        while(i < m && j < n)
        {
            if (s.charAt(i) == t.charAt(j))
            {
                lcs += s.charAt(i);
                i++;
                j++;
            }
            else if (opt[i+1][j] >= opt[i][j+1]) i++;
            else j++;
        }
        return lcs;
    }

    public static void main(String[] args)
    { StdOut.println(lcs(args[0], args[1])); }
}

```

s, t	two strings
m, n	lengths of two strings
opt[i][j]	length of LCS of x[i..m] and y[j..n]
lcs	longest common subsequence

The function `lcs()` computes and returns the LCS of two strings  $x$  and  $y$  using bottom-up dynamic programming. The method call `s.charAt(i)` returns character  $i$  of string  $s$ .

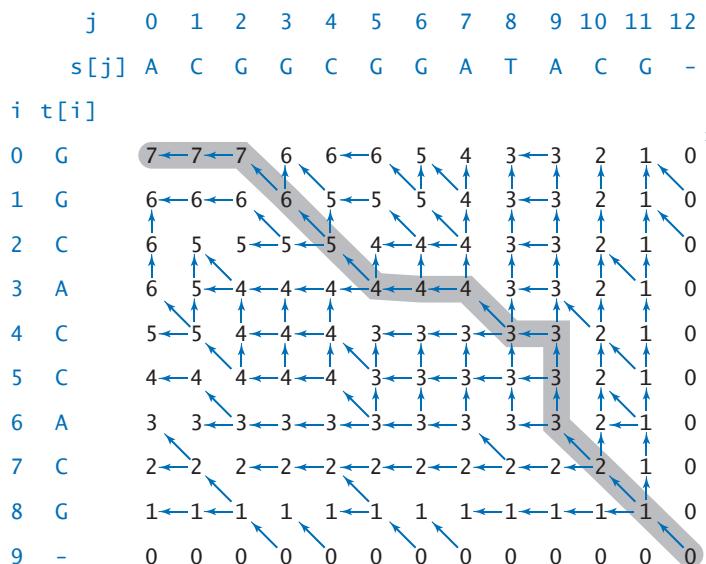
```
% java LongestCommonSubsequence GGCACCACG ACGGGGGATACG
GGCAACG
```

left and from bottom to top in each row would work as well. The above diagram has a blue arrow pointing to each entry that indicates which value was used to compute it. (When there is a tie in computing the maximum, both options are shown.)

The final challenge is to recover the longest common subsequence itself, not just its length. The key idea is to retrace the steps of the dynamic programming algorithm *backward*, rediscovering the path of choices (highlighted in gray in the diagram) from  $\text{opt}[0][0]$  to  $\text{opt}[m][n]$ . To determine the choice that led to  $\text{opt}[i][j]$ , we consider the three possibilities:

- The character  $s[i]$  equals  $t[j]$ . In this case, we must have  $\text{opt}[i][j] = \text{opt}[i+1][j+1] + 1$ , and the next character in the LCS is  $s[i]$  (or  $t[j]$ ), so we include the character  $s[i]$  (or  $t[j]$ ) in the LCS and continue tracing back from  $\text{opt}[i+1][j+1]$ .
- The LCS does not contain  $s[i]$ . In this case,  $\text{opt}[i][j] = \text{opt}[i+1][j]$  and we continue tracing back from  $\text{opt}[i+1][j]$ .
- The LCS does not contain  $t[j]$ . In this case,  $\text{opt}[i][j] = \text{opt}[i][j+1]$  and we continue tracing back from  $\text{opt}[i][j+1]$ .

We begin tracing back at  $\text{opt}[0][0]$  and continue until we reach  $\text{opt}[m][n]$ . At each step in the traceback either  $i$  increases or  $j$  increases (or both), so the process terminates after at most  $m + n$  iterations of the while loop.



Longest common subsequence of GCCACCACG and ACGGCCGGATACC

DYNAMIC PROGRAMMING IS A FUNDAMENTAL ALGORITHM design paradigm, intimately linked to recursion. If you take later courses in algorithms or operations research, you are sure to learn more about it. The idea of recursion is fundamental in computation, and the idea of avoiding recomputation of values that have been computed before is certainly a natural one. Not all problems immediately lend themselves to a recursive formulation, and not all recursive formulations admit an order of computation that easily avoids recomputation—arranging for both can seem a bit miraculous when one first encounters it, as you have just seen for the LCS problem.

**Perspective** Programmers who do not use recursion are missing two opportunities. First recursion leads to compact solutions to complex problems. Second, recursive solutions embody an argument that the program operates as anticipated. In the early days of computing, the overhead associated with recursive programs was prohibitive in some systems, and many people avoided recursion. In modern systems like Java, recursion is often the method of choice.

Recursive functions truly illustrate the power of a carefully articulated abstraction. While the concept of a function having the ability to call itself seems absurd to many people at first, the many examples that we have considered are certainly evidence that mastering recursion is essential to understanding and exploiting computation and in understanding the role of computational models in studying natural phenomena.

Recursion has reinforced for us the idea of proving that a program operates as intended. The natural connection between recursion and mathematical induction is essential. For everyday programming, our interest in correctness is to save time and energy tracking down bugs. In modern applications, security and privacy concerns make correctness an *essential* part of programming. If the programmer cannot be convinced that an application works as intended, how can a user who wants to keep personal data private and secure be so convinced?

Recursion is the last piece in a programming model that served to build much of the computational infrastructure that was developed as computers emerged to take a central role in daily life in the latter part of the 20th century. Programs built from libraries of functions consisting of statements that operate on primitive types of data, conditionals, loops, and function calls (including recursive ones) can solve important problems of all sorts. In the next section, we emphasize this point and review these concepts in the context of a large application. In CHAPTER 3 and in CHAPTER 4, we will examine extensions to these basic ideas that embrace the more expansive style of programming that now dominates the computing landscape.



## Q&A

**Q.** Are there situations when iteration is the only option available to address a problem?

**A.** No, any loop can be replaced by a recursive function, though the recursive version might require excessive memory.

**Q.** Are there situations when recursion is the only option available to address a problem?

**A.** No, any recursive function can be replaced by an iterative counterpart. In SECTION 4.3, we will see how compilers produce code for function calls by using a data structure called a *stack*.

**Q.** Which should I prefer, recursion or iteration?

**A.** Whichever leads to the simpler, more easily understood, or more efficient code.

**Q.** I get the concern about excessive space and excessive recomputation in recursive code. Anything else to be concerned about?

**A.** Be extremely wary of creating arrays in recursive code. The amount of space used can pile up very quickly, as can the amount of time required for memory management.

## Exercises

**2.3.1** What happens if you call `factorial()` with a negative value of  $n$ ? With a large value of, say, 35?

**2.3.2** Write a recursive function that takes an integer  $n$  as its argument and returns  $\ln(n!)$ .

**2.3.3** Give the sequence of integers printed by a call to `ex233(6)`:

```
public static void ex233(int n)
{
    if (n <= 0) return;
    StdOut.println(n);
    ex233(n-2);
    ex233(n-3);
    StdOut.println(n);
}
```

**2.3.4** Give the value of `ex234(6)`:

```
public static String ex234(int n)
{
    if (n <= 0) return "";
    return ex234(n-3) + n + ex234(n-2) + n;
}
```

**2.3.5** Criticize the following recursive function:

```
public static String ex235(int n)
{
    String s = ex235(n-3) + n + ex235(n-2) + n;
    if (n <= 0) return "";
    return s;
}
```

*Answer:* The base case will never be reached because the base case appears after the reduction step. A call to `ex235(3)` will result in calls to `ex235(0)`, `ex235(-3)`, `ex235(-6)`, and so forth until a `StackOverflowError`.

**2.3.6** Given four positive integers  $a$ ,  $b$ ,  $c$ , and  $d$ , explain what value is computed by  $\text{gcd}(\text{gcd}(a, b), \text{gcd}(c, d))$ .

**2.3.7** Explain in terms of integers and divisors the effect of the following Euclid-like function:

```
public static boolean gcdlike(int p, int q)
{
    if (q == 0) return (p == 1);
    return gcdlike(q, p % q);
}
```

**2.3.8** Consider the following recursive function:

```
public static int mystery(int a, int b)
{
    if (b == 0)      return 0;
    if (b % 2 == 0) return mystery(a+a, b/2);
    return mystery(a+a, b/2) + a;
}
```

What are the values of  $\text{mystery}(2, 25)$  and  $\text{mystery}(3, 11)$ ? Given positive integers  $a$  and  $b$ , describe what value  $\text{mystery}(a, b)$  computes. Then answer the same question, but replace  $+$  with  $*$  and  $\text{return } 0$  with  $\text{return } 1$ .

**2.3.9** Write a recursive program `Ruler` to plot the subdivisions of a ruler using `StdDraw`, as in PROGRAM 1.2.1.

**2.3.10** Solve the following recurrence relations, all with  $T(1) = 1$ . Assume  $n$  is a power of 2.

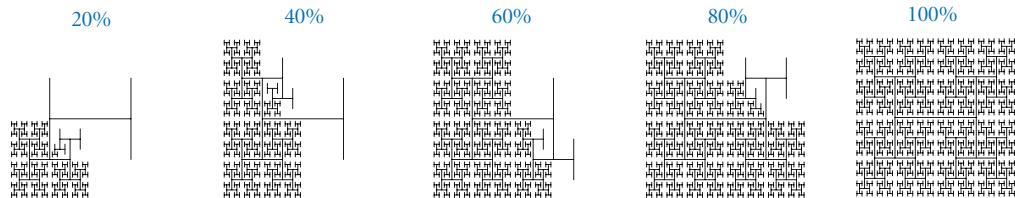
- $T(n) = T(n/2) + 1$
- $T(n) = 2T(n/2) + 1$
- $T(n) = 2T(n/2) + n$
- $T(n) = 4T(n/2) + 3$

**2.3.11** Prove by induction that the minimum possible number of moves needed to solve the towers of Hanoi satisfies the same recurrence as the number of moves used by our recursive solution.

**2.3.12** Prove by induction that the recursive program given in the text makes exactly  $F_n$  recursive calls to `fibonacci(1)` when computing `fibonacci(n)`.

**2.3.13** Prove that the second argument to `gcd()` decreases by at least a factor of 2 for every second recursive call, and then prove that `gcd(p, q)` uses at most  $2 \log_2 n + 1$  recursive calls where  $n$  is the larger of  $p$  and  $q$ .

**2.3.14** Modify `Htree` (PROGRAM 2.3.4) to animate the drawing of the H-tree. Next, rearrange the order of the recursive calls (and the base case), view the resulting animation, and explain each outcome.



## Creative Exercises

**2.3.15** *Binary representation.* Write a program that takes a positive integer  $n$  (in decimal) as a command-line argument and prints its binary representation. Recall, in PROGRAM 1.3.7, that we used the method of subtracting out powers of 2. Now, use the following simpler method: repeatedly divide 2 into  $n$  and read the remainders backward. First, write a `while` loop to carry out this computation and print the bits in the wrong order. Then, use recursion to print the bits in the correct order.

**2.3.16** *A4 paper.* The width-to-height ratio of paper in the ISO format is the square root of 2 to 1. Format A0 has an area of 1 square meter. Format A1 is A0 cut with a vertical line into two equal halves, A2 is A1 cut with a horizontal line into two halves, and so on. Write a program that takes an integer command-line argument  $n$  and uses `StdDraw` to show how to cut a sheet of A0 paper into  $2^n$  pieces.

**2.3.17** *Permutations.* Write a program `Permutations` that takes an integer command-line argument  $n$  and prints all  $n!$  permutations of the  $n$  letters starting at a (assume that  $n$  is no greater than 26). A permutation of  $n$  elements is one of the  $n!$  possible orderings of the elements. As an example, when  $n = 3$ , you should get the following output (but do not worry about the order in which you enumerate them):

```
bca cba cab acb bac abc
```

**2.3.18** *Permutations of size  $k$ .* Modify `Permutations` from the previous exercise so that it takes two command-line arguments  $n$  and  $k$ , and prints all  $P(n, k) = n! / (n-k)!$  permutations that contain exactly  $k$  of the  $n$  elements. Below is the desired output when  $k = 2$  and  $n = 4$  (again, do not worry about the order):

```
ab ac ad ba bc bd ca cb cd da db dc
```

**2.3.19** *Combinations.* Write a program `Combinations` that takes an integer command-line argument  $n$  and prints all  $2^n$  combinations of any size. A combination is a subset of the  $n$  elements, independent of order. As an example, when  $n = 3$ , you should get the following output:

```
a ab abc ac b bc c
```

Note that your program needs to print the empty string (subset of size 0).

**2.3.20** *Combinations of size k.* Modify Combinations from the previous exercise so that it takes two integer command-line arguments  $n$  and  $k$ , and prints all  $C(n, k) = n! / (k!(n-k)!)$  combinations of size  $k$ . For example, when  $n = 5$  and  $k = 3$ , you should get the following output:

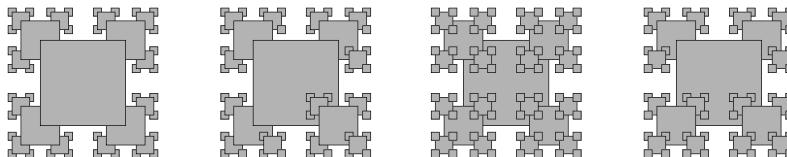
```
abc abd abe acd ace ade bcd bce bde cde
```

**2.3.21** *Hamming distance.* The Hamming distance between two bit strings of length  $n$  is equal to the number of bits in which the two strings differ. Write a program that reads in an integer  $k$  and a bit string  $s$  from the command line, and prints all bit strings that have Hamming distance at most  $k$  from  $s$ . For example, if  $k$  is 2 and  $s$  is 0000, then your program should print

```
0011 0101 0110 1001 1010 1100
```

*Hint:* Choose  $k$  of the bits in  $s$  to flip.

**2.3.22** *Recursive squares.* Write a program to produce each of the following recursive patterns. The ratio of the sizes of the squares is 2.2:1. To draw a shaded square, draw a filled gray square, then an unfilled black square.



**2.3.23** *Pancake flipping.* You have a stack of  $n$  pancakes of varying sizes on a grid-like. Your goal is to rearrange the stack in order so that the largest pancake is on the bottom and the smallest one is on top. You are only permitted to flip the top  $k$  pancakes, thereby reversing their order. Devise a recursive scheme to arrange the pancakes in the proper order that uses at most  $2n - 3$  flips.



**2.3.24** *Gray code.* Modify Beckett (PROGRAM 2.3.3) to print the Gray code (not just the sequence of bit positions that change).

**2.3.25** *Towers of Hanoi variant.* Consider the following variant of the towers of Hanoi problem. There are  $2n$  discs of increasing size stored on three poles. Initially all of the discs with odd size ( $1, 3, \dots, 2n-1$ ) are piled on the left pole from top to bottom in increasing order of size; all of the discs with even size ( $2, 4, \dots, 2n$ ) are piled on the right pole. Write a program to provide instructions for moving the odd discs to the right pole and the even discs to the left pole, obeying the same rules as for towers of Hanoi.

**2.3.26** *Animated towers of Hanoi.* Use StdDraw to animate a solution to the towers of Hanoi problem, moving the discs at a rate of approximately 1 per second.

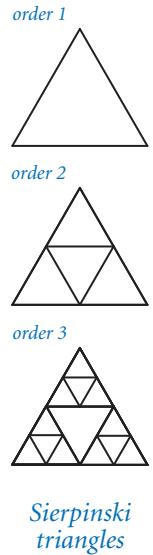
**2.3.27** *Sierpinski triangles.* Write a recursive program to draw Sierpinski triangles (see PROGRAM 2.2.3). As with Htree, use a command-line argument to control the depth of the recursion.

**2.3.28** *Binomial distribution.* Estimate the number of recursive calls that would be used by the code

```
public static double binomial(int n, int k)
{
    if ((n == 0) && (k == 0)) return 1.0;
    if ((n < 0) || (k < 0)) return 0.0;
    return (binomial(n-1, k) + binomial(n-1, k-1))/2.0;
}
```

to compute `binomial(100, 50)`. Develop a better implementation that is based on dynamic programming. *Hint:* See EXERCISE 1.4.41.

**2.3.29** *Collatz function.* Consider the following recursive function, which is related to a famous unsolved problem in number theory, known as the *Collatz problem*, or the *3n+1 problem*:





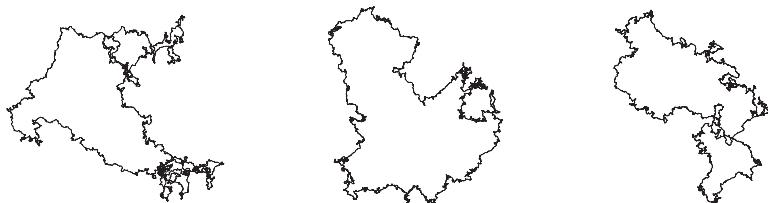
```
public static void collatz(int n)
{
    StdOut.print(n + " ");
    if (n == 1) return;
    if (n % 2 == 0) collatz(n / 2);
    else            collatz(3*n + 1);
}
```

For example, a call to `collatz(7)` prints the sequence

```
7 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1
```

as a consequence of 17 recursive calls. Write a program that takes a command-line argument  $n$  and returns the value of  $i < n$  for which the number of recursive calls for `collatz(i)` is maximized. The unsolved problem is that no one knows whether the function terminates for all integers (mathematical induction is no help, because one of the recursive calls is for a larger value of the argument).

**2.3.30** *Brownian island.* B. Mandelbrot asked the famous question *How long is the coast of Britain?* Modify Brownian to get a program `BrownianIsland` that plots Brownian islands, whose coastlines resemble that of Great Britain. The modifications are simple: first, change `curve()` to add a random Gaussian to the  $x$ -coordinate as well as to the  $y$ -coordinate; second, change `main()` to draw a curve from the point at the center of the canvas back to itself. Experiment with various values of the parameters to get your program to produce islands with a realistic look.



*Brownian islands with Hurst exponent of 0.76*



**2.3.31** *Plasma clouds.* Write a recursive program to draw plasma clouds, using the method suggested in the text.

**2.3.32** *A strange function.* Consider McCarthy's 91 function:

```
public static int mcCarthy(int n)
{
    if (n > 100) return n - 10;
    return mcCarthy(mcCarthy(n+11));
}
```

Determine the value of `mcCarthy(50)` without using a computer. Give the number of recursive calls used by `mcCarthy()` to compute this result. Prove that the base case is reached for all positive integers  $n$  or find a value of  $n$  for which this function goes into an infinite recursive loop.

**2.3.33** *Recursive tree.* Write a program `Tree` that takes a command-line argument  $n$  and produces the following recursive patterns for  $n$  equal to 1, 2, 3, 4, and 8.

1



2



3



4



8



**2.3.34** *Longest palindromic subsequence.* Write a program `LongestPalindromicSubsequence` that takes a string as a command-line argument and determines the longest subsequence of the string that is a palindrome (the same when read forward or backward). Hint: Compute the longest common subsequence of the string and its reverse.

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## 2.4 Case Study: Percolation

THE PROGRAMMING TOOLS THAT WE HAVE considered to this point allow us to attack all manner of important problems. We conclude our study of functions and modules by considering a case study of developing a program to solve an interesting scientific problem. Our purpose in doing so is to review the basic elements that we have covered, in the context of the various challenges that you might face in solving a specific problem, and to illustrate a programming style that you can apply broadly.

Our example applies a widely applicable computational technique known as *Monte Carlo simulation* to study a natural model known as *percolation*. The term “Monte Carlo simulation” is broadly used to encompass any computational technique that employs randomness to estimate an unknown quantity by performing multiple trials (known as *simulations*). We have used it in several other contexts already—for example, in the gambler’s ruin and coupon collector problems. Rather than develop a complete mathematical model or measure all possible outcomes of an experiment, we rely on the laws of probability.

In this case study we will learn quite a bit about percolation, a model which underlies many natural phenomena. Our focus, however, is on the process of developing modular programs to address computational tasks. We identify subtasks that can be independently addressed, striving to identify the key underlying abstractions and asking ourselves questions such as the following: Is there some specific subtask that would help solve this problem? What are the essential characteristics of this specific subtask? Might a solution that addresses these essential characteristics be useful in solving other problems? Asking such questions pays significant dividend, because they lead us to develop software that is easier to create, debug, and reuse, so that we can more quickly address the main problem of interest.

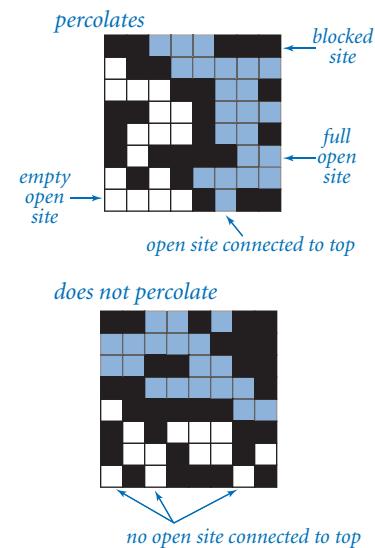
2.4.1	Percolation scaffolding . . . . .	304
2.4.2	Vertical percolation detection . . . . .	306
2.4.3	Visualization client . . . . .	309
2.4.4	Percolation probability estimate . . . . .	311
2.4.5	Percolation detection . . . . .	313
2.4.6	Adaptive plot client . . . . .	316

*Programs in this section*

**Percolation** It is not unusual for local interactions in a system to imply global properties. For example, an electrical engineer might be interested in composite systems consisting of randomly distributed insulating and metallic materials: which fraction of the materials need to be metallic so that the composite system is an electrical conductor? As another example, a geologist might be interested in a porous landscape with water on the surface (or oil below). Under which conditions will the water be able to drain through to the bottom (or the oil to gush through to the surface)? Scientists have defined an abstract process known as *percolation* to model such situations. It has been studied widely, and shown to be an accurate model in a dizzying variety of applications, beyond insulating materials and porous substances to the spread of forest fires and disease epidemics to evolution to the study of the Internet.

For simplicity, we begin by working in two dimensions and model the system as an  $n$ -by- $n$  grid of *sites*. Each site is either *blocked* or *open*; open sites are initially *empty*. A *full* site is an open site that can be connected to an open site in the top row via a chain of neighboring (left, right, up, down) open sites. If there is a full site in the bottom row, then we say that the system *percolates*. In other words, a system percolates if we fill all open sites connected to the top row and that process fills some open site on the bottom row. For the insulating/metallic materials example, the open sites correspond to metallic materials, so that a system that percolates has a metallic path from top to bottom, with full sites conducting. For the porous substance example, the open sites correspond to empty space through which water might flow, so that a system that percolates lets water fill open sites, flowing from top to bottom.

In a famous scientific problem that has been heavily studied for decades, scientists are interested in the following question: if sites are independently set to be open with *site vacancy probability*  $p$  (and therefore blocked with probability  $1-p$ ), what is the probability that the system percolates? No mathematical solution to this problem has yet been derived. Our task is to write computer programs to help study the problem.

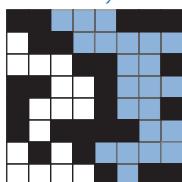


Percolation examples

**Basic scaffolding** To address percolation with a Java program, we face numerous decisions and challenges, and we certainly will end up with much more code than in the short programs that we have considered so far in this book. Our goal is to illustrate an incremental style of programming where we independently develop modules that address parts of the problem, building confidence with a small computational infrastructure of our own design and construction as we proceed.

The first step is to pick a representation of the data. This decision can have substantial impact on the kind of code that we write later, so it is not to be taken lightly. Indeed, it is often the case that we learn something while working with a chosen representation that causes us to scrap it and start all over using a new one.

*percolation system*



*blocked sites*

```

1 1 0 0 0 1 1 1
0 1 1 0 0 0 0 0
0 0 0 1 1 0 0 1
1 1 0 0 1 0 0 0
1 0 0 0 1 0 0 1
1 0 1 1 1 1 0 0
0 1 0 1 0 0 0 0
0 0 0 0 1 0 1 1

```

*open sites*

```

0 0 1 1 1 0 0 0
1 0 0 1 1 1 1 1
1 1 1 0 0 1 1 0
0 0 1 1 0 1 1 1
0 1 1 1 0 1 1 0
0 1 0 0 0 0 1 1
1 0 1 0 1 1 1 1
1 1 1 1 0 1 0 0

```

*full sites*

```

0 0 1 1 1 0 0 0
0 0 0 1 1 1 1 1
0 0 0 0 0 1 1 0
0 0 0 0 0 1 1 1
0 0 0 0 0 1 1 0
0 0 0 0 0 0 1 1
0 0 0 0 1 1 1 1
0 0 0 0 0 1 0 0

```

*Percolation representations*

For percolation, the path to an effective representation is clear: use an  $n$ -by- $n$  array. Which type of data should we use for each element? One possibility is to use integers, with the convention that 0 indicates an empty site, 1 indicates a blocked site, and 2 indicates a full site. Alternatively, note that we typically describe sites in terms of questions: Is the site open or blocked? Is the site full or empty? This characteristic of the elements suggests that we might use  $n$ -by- $n$  arrays in which element is either `true` or `false`. We refer to such two-dimensional arrays as *boolean* matrices. Using boolean matrices leads to code that is easier to understand than the alternative.

Boolean matrices are fundamental mathematical objects with many applications. Java does not provide direct support for operations on boolean matrices, but we can use the methods in `StdArrayIO` (see PROGRAM 2.2.2) to read and write them. This choice illustrates a basic principle that often comes up in programming: *the effort required to build a more general tool usually pays dividends*.

Eventually, we will want to work with random data, but we also want to be able to read and write to files because debugging programs with random inputs can be counterproductive. With random data, you get different input each time that you run the program; after fixing a bug, what you want to see is the *same* input that you just used, to check that the fix was effective. Accordingly, it is best to start with some specific cases that we understand, kept in files formatted compatible with `StdArrayIO` (dimensions followed by 0 and 1 values in row-major order).

When you start working on a new problem that involves several files, it is usually worthwhile to create a new folder (directory) to isolate those files from others that you may be working on. For example, we might create a folder named `percolation` to store all of the files for this case study. To get started, we can implement and debug the basic code for reading and writing percolation systems, create test files, check that the files are compatible with the code, and so forth, before worrying about percolation at all. This type of code, sometimes called *scaffolding*, is straightforward to implement, but making sure that it is solid at the outset will save us from distraction when approaching the main problem.

Now we can turn to the code for testing whether a boolean matrix represents a system that percolates. Referring to the helpful interpretation in which we can think of the task as simulating what would happen if the top were flooded with water (does it flow to the bottom or not?), our first design decision is that we will want to have a `flood()` method that takes as an argument a boolean matrix `isOpen[][]` that specifies which sites are open and returns another boolean matrix `isFull[][]` that specifies which sites are full. For the moment, we will not worry at all about how to implement this method; we are just deciding how to organize the computation. It is also clear that we will want client code to be able to use a `percolates()` method that checks whether the array returned by `flood()` has any full sites on the bottom.

`Percolation` (PROGRAM 2.4.1) summarizes these decisions. It does not perform any interesting computation, but after running and debugging this code we can start thinking about actually solving the problem. A method that performs no computation, such as `flood()`, is sometimes called a *stub*. Having this stub allows us to test and debug `percolates()` and `main()` in the context in which we will need them. We refer to code like PROGRAM 2.4.1 as *scaffolding*. As with scaffolding that construction workers use when erecting a building, this kind of code provides the support that we need to develop a program. By fully implementing and debugging this code (much, if not all, of which we need, anyway) at the outset, we provide a sound basis for building code to solve the problem at hand. Often, we carry the analogy one step further and remove the scaffolding (or replace it with something better) after the implementation is complete.

### Program 2.4.1 Percolation scaffolding

```

public class Percolation
{
    public static boolean[][] flow(boolean[][] isOpen)
    {
        int n = isOpen.length;
        boolean[][] isFull = new boolean[n][n];
        // The isFull[][] matrix computation goes here.
        return isFull;
    }

    public static boolean percolates(boolean[][] isOpen)
    {
        boolean[][] isFull = flow(isOpen);
        int n = isOpen.length;
        for (int j = 0; j < n; j++)
            if (isFull[n-1][j]) return true;
        return false;
    }

    public static void main(String[] args)
    {
        boolean[][] isOpen = StdArrayIO.readBoolean2D();
        StdArrayIO.print(flow(isOpen));
        StdOut.println(percolates(isOpen));
    }
}

```

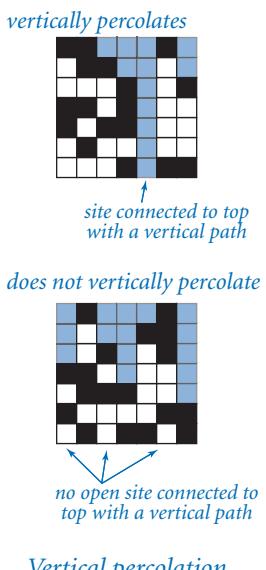
n	system size (n-by-n)
isFull[][]	full sites
isOpen[][]	open sites

To get started with percolation, we implement and debug this code, which handles all the straightforward tasks surrounding the computation. The primary function `flow()` returns a boolean matrix giving the full sites (none, in the placeholder code here). The helper function `percolates()` checks the bottom row of the returned matrix to decide whether the system percolates. The test client `main()` reads a boolean matrix from standard input and prints the result of calling `flow()` and `percolates()` for that matrix.

```
% more testEZ.txt
5 5
0 1 1 0 1
0 0 1 1 1
1 1 0 1 1
1 0 0 0 1
0 1 1 1 1
```

```
% java Percolation < testEZ.txt
5 5
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
false
```

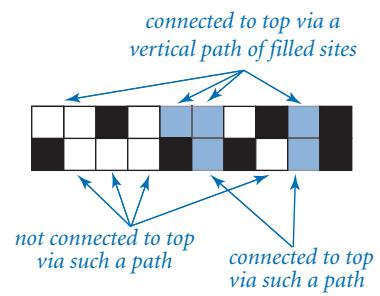
**Vertical percolation** Given a boolean matrix that represents the open sites, how do we figure out whether it represents a system that percolates? As we will see later in this section, this computation turns out to be directly related to a fundamental question in computer science. For the moment, we will consider a much simpler version of the problem that we call *vertical percolation*.



The simplification is to restrict attention to vertical connection paths. If such a path connects top to bottom in a system, we say that the system *vertically percolates* along the path (and that the system itself vertically percolates). This restriction is perhaps intuitive if we are talking about sand traveling through cement, but not if we are talking about water traveling through cement or about electrical conductivity. Simple as it is, vertical percolation is a problem that is interesting in its own right because it suggests various mathematical questions. Does the restriction make a significant difference? How many vertical percolation paths do we expect?

Determining the sites that are filled by some path that is connected vertically to the top is a simple calculation. We initialize the top row of our result array from the top row of the percolation system, with full sites corresponding to open ones. Then, moving from top to bottom, we fill in each row of the array by checking the corresponding row of the percolation system. Proceeding from top to bottom, we fill in the rows of `isFull[][]` to mark as `true` all elements that correspond to sites in `isOpen[][]` that are vertically connected to a full site on the previous row. PROGRAM 2.4.2 is an implementation of `flood()` for Percolation that returns a boolean matrix of full sites (`true` if connected to the top via a vertical path, `false` otherwise).

**Testing** After we become convinced that our code is behaving as planned, we want to run it on a broader variety of test cases and address some of our scientific questions. At this point, our initial scaffolding becomes less useful, as representing large boolean matrices with 0s and 1s on standard input and standard output and maintaining large numbers of test cases quickly becomes unwieldy. Instead,



*Vertical percolation calculation*

### Program 2.4.2 Vertical percolation detection

```
public static boolean[][] flow(boolean[][] isOpen)
{ // Compute full sites for vertical percolation.
    int n = isOpen.length;
    boolean[][] isFull = new boolean[n][n];
    for (int j = 0; j < n; j++)
        isFull[0][j] = isOpen[0][j];
    for (int i = 1; i < n; i++)
        for (int j = 0; j < n; j++)
            isFull[i][j] = isOpen[i][j] && isFull[i-1][j];
    return isFull;
}
```

$n$ $\text{isFull}[][]$ $\text{isOpen}[][]$	<i>system size (<math>n</math>-by-<math>n</math>)</i> <i>full sites</i> <i>open sites</i>
---	---

Substituting this method for the stub in PROGRAM 2.4.1 gives a solution to the vertical-only percolation problem that solves our test case as expected (see text).

```
% more test5.txt
5 5
0 1 1 0 1
0 0 1 1 1
1 1 0 1 1
1 0 0 0 1
0 1 1 1 1
```

```
% java Percolation < test5.txt
5 5
0 1 1 0 1
0 0 1 0 1
0 0 0 0 1
0 0 0 0 1
0 0 0 0 1
true
```

we want to automatically generate test cases and observe the operation of our code on them, to be sure that it is operating as we expect. Specifically, to gain confidence in our code and to develop a better understanding of percolation, our next goals are to:

- Test our code for large random boolean matrices.
- Estimate the probability that a system percolates for a given  $p$ .

To accomplish these goals, we need new clients that are slightly more sophisticated than the scaffolding we used to get the program up and running. Our modular programming style is to develop such clients in independent classes *without modifying our percolation code at all*.

**Data visualization.** We can work with much bigger problem instances if we use StdDraw for output. The following static method for Percolation allows us to visualize the contents of boolean matrices as a subdivision of the StdDraw canvas into squares, one for each site:

```
public static void show(boolean[][] a, boolean which)
{
    int n = a.length;
    StdDraw.setXscale(-1, n);
    StdDraw.setYscale(-1, n);
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            if (a[i][j] == which)
                StdDraw.filledSquare(j, n-i-1, 0.5);
}
```

The second argument which specifies which squares we want to fill—those corresponding to `true` elements or those corresponding to `false` elements. This method is a bit of a diversion from the calculation, but pays dividends in its ability to help us visualize large problem instances. Using `show()` to draw our boolean matrices representing blocked and full sites in different colors gives a compelling visual representation of percolation.

**Monte Carlo simulation.** We want our code to work properly for *any* boolean matrix. Moreover, the scientific question of interest involves random boolean matrices. To this end, we add another static method to `Percolation`:

```
public static boolean[][] random(int n, double p)
{
    boolean[][] a = new boolean[n][n];
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            a[i][j] = StdRandom.bernoulli(p);
    return a;
}
```

This method generates a random  $n$ -by- $n$  boolean matrix of any given size  $n$ , each element `true` with probability  $p$ .

Having debugged our code on a few specific test cases, we are ready to test it on random systems. It is possible that such cases may uncover a few more bugs, so some care is in order to check results. However, having debugged our code for a small system, we can proceed with some confidence. It is easier to focus on new bugs after eliminating the obvious bugs.

WITH THESE TOOLS, A CLIENT FOR testing our percolation code on a much larger set of trials is straightforward. `PercolationVisualizer` (PROGRAM 2.4.3) consists of just a `main()` method that takes `n` and `p` from the command line and displays the result of the percolation flow calculation.

This kind of client is typical. Our eventual goal is to compute an accurate estimate of percolation probabilities, perhaps by running a large number of trials, but this simple tool gives us the opportunity to gain more familiarity with the problem by studying some large cases (while at the same time gaining confidence that our code is working properly). Before reading further, you are encouraged to download and run this code from the booksite to study the percolation process. When you run `PercolationVisualizer` for moderate-size  $n$  (50 to 100, say) and various  $p$ , you will immediately be drawn into using this program to try to answer some questions about percolation. Clearly, the system never percolates when  $p$  is low and always percolates when  $p$  is very high. How does it behave for intermediate values of  $p$ ? How does the behavior change as  $n$  increases?

**Estimating probabilities** The next step in our program development process is to write code to estimate the probability that a random system (of size  $n$  with site vacancy probability  $p$ ) percolates. We refer to this quantity as the *percolation probability*. To estimate its value, we simply run a number of trials. The situation is no different from our study of coin flipping (see PROGRAM 2.2.6), but instead of flipping a coin, we generate a random system and check whether it percolates.

`PercolationProbability` (PROGRAM 2.4.4) encapsulates this computation in a method `estimate()`, which takes three arguments `n`, `p`, and `trials` and returns an estimate of the probability that an  $n$ -by- $n$  system with site vacancy probability `p` percolates, obtained by generating `trials` random systems and calculating the fraction of them that percolate.

How many trials do we need to obtain an accurate estimate? This question is addressed by basic methods in probability and statistics, which are beyond the scope of this book, but we can get a feeling for the problem with computational experience. With just a few runs of `PercolationProbability`, you can learn that if the site vacancy probability is close to either 0 or 1, then we do not need many trials, but that there are values for which we need as many as 10,000 trials to be able to estimate it within two decimal places. To study the situation in more detail, we might modify `PercolationProbability` to produce output like `Bernoulli` (PROGRAM 2.2.6), plotting a histogram of the data points so that we can see the distribution of values (see EXERCISE 2.4.9).

**Program 2.4.3 Visualization client**

```
public class PercolationVisualizer
{
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        double p = Double.parseDouble(args[1]);
        StdDraw.enableDoubleBuffering();

        // Draw blocked sites in black.
        boolean[][] isOpen = Percolation.random(n, p);
        StdDraw.setPenColor(StdDraw.BLACK);
        Percolation.show(isOpen, false);

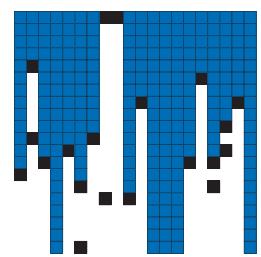
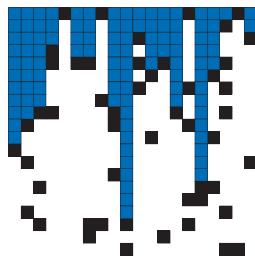
        // Draw full sites in blue.
        StdDraw.setPenColor(StdDraw.BOOK_BLUE);
        boolean[][] isFull = Percolation.flow(isOpen);
        Percolation.show(isFull, true);

        StdDraw.show();
    }
}
```

n	system size ( $n$ -by- $n$ )
p	site vacancy probability
isOpen[][]	open sites
isFull[][]	full sites

This client takes two command-line argument  $n$  and  $p$ , generates an  $n$ -by- $n$  random system with site vacancy probability  $p$ , determines which sites are full, and draws the result on standard drawing. The diagrams below show the results for vertical percolation.

```
% java PercolationVisualizer 20 0.9      % java PercolationVisualizer 20 0.95
```



Using `PercolationProbability.estimate()` represents a giant leap in the amount of computation that we are doing. All of a sudden, it makes sense to run thousands of trials. It would be unwise to try to do so without first having thoroughly debugged our percolation methods. Also, we need to begin to take the time required to complete the computation into account. The basic methodology for doing so is the topic of SECTION 4.1, but the structure of these programs is sufficiently simple that we can do a quick calculation, which we can verify by running the program. If we perform  $T$  trials, each of which involves  $n^2$  sites, then the total running time of `PercolationProbability.estimate()` is proportional to  $n^2T$ . If we increase  $T$  by a factor of 10 (to gain more precision), the running time increases by about a factor of 10. If we increase  $n$  by a factor of 10 (to study percolation for larger systems), the running time increases by about a factor of 100.

Can we run this program to determine percolation probabilities for a system with billions of sites with several digits of precision? No computer is fast enough to use `PercolationProbability.estimate()` for this purpose. Moreover, in a scientific experiment on percolation, the value of  $n$  is likely to be much higher. We can hope to formulate a hypothesis from our simulation that can be tested experimentally on a much larger system, but not to precisely simulate a system that corresponds atom-for-atom with the real world. Simplification of this sort is essential in science.

You are encouraged to download `PercolationProbability` from the book-site to get a feel for both the percolation probabilities and the amount of time required to compute them. When you do so, you are not just learning more about percolation, but are also testing the hypothesis that the models we have just described apply to the running times of our simulations of the percolation process.

What is the probability that a system with site vacancy probability  $p$  vertically percolates? Vertical percolation is sufficiently simple that elementary probabilistic models can yield an exact formula for this quantity, which we can validate experimentally with `PercolationProbability`. Since our only reason for studying vertical percolation was an easy starting point around which we could develop supporting software for studying percolation methods, we leave further study of vertical percolation for an exercise (see EXERCISE 2.4.11) and turn to the main problem.

### Program 2.4.4 Percolation probability estimate

```

public class PercolationProbability
{
    public static double estimate(int n, double p, int trials)
    { // Generate trials random n-by-n systems; return empirical
      // percolation probability estimate.
        int count = 0;
        for (int t = 0; t < trials; t++)
        { // Generate one random n-by-n boolean matrix.
          boolean[][] isOpen = Percolation.random(n, p);
          if (Percolation.percolates(isOpen)) count++;
        }
        return (double) count / trials;
    }
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        double p = Double.parseDouble(args[1]);
        int trials = Integer.parseInt(args[2]);
        double q = estimate(n, p, trials);
        StdOut.println(q);
    }
}

```

n	system size ( $n$ -by- $n$ )
p	site vacancy probability
trials	number of trials
isOpen[][]	open sites
q	percolation probability

The method `estimate()` generates `trials` random  $n$ -by- $n$  systems with site vacancy probability `p` and computes the fraction of them that percolate. This is a Bernoulli process, like coin flipping (see PROGRAM 2.2.6). Increasing the number of `trials` increases the accuracy of the estimate. If `p` is close to 0 or to 1, not many trials are needed to achieve an accurate estimate. The results below are for vertical percolation.

```

% java PercolationProbability 20 0.05 10
0.0
% java PercolationProbability 20 0.95 10
1.0
% java PercolationProbability 20 0.85 10
0.7
% java PercolationProbability 20 0.85 1000
0.564
% java PercolationProbability 40 0.85 100
0.1

```

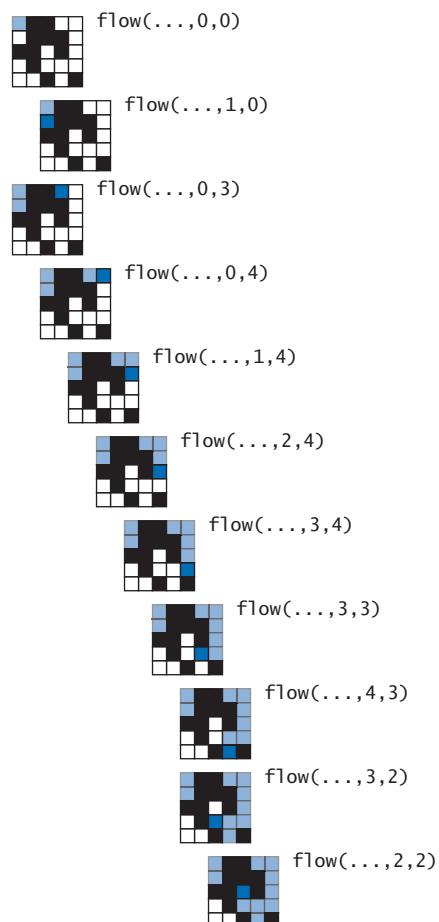
**Recursive solution for percolation** How do we test whether a system percolates in the general case when *any* path starting at the top and ending at the bottom (not just a vertical one) will do the job?

Remarkably, we can solve this problem with a compact program, based on a classic recursive scheme known as *depth-first search*. PROGRAM 2.4.5 is an implementation of `flow()` that computes the matrix `isFull[][]`, based on a recursive four-argument version of `flow()` that takes as arguments the site vacancy matrix `isOpen[][]`, the current matrix `isFull[][]`, and a site position specified by a row index *i* and a column index *j*. The base case is a recursive call that just returns (we refer to such a call as a *null call*), for one of the following reasons:

- Either *i* or *j* is outside the array bounds.
- The site is blocked (`isOpen[i][j]` is `false`).
- We have already marked the site as full (`isFull[i][j]` is `true`).

The reduction step is to mark the site as filled and issue recursive calls for the site's four neighbors: `isOpen[i+1][j]`, `isOpen[i][j+1]`, `isOpen[i][j-1]`, and `isOpen[i-1][j]`. The one-argument `flow()` calls the recursive method for every site on the top row. The recursion always terminates because each recursive call either is null or marks a new site as full. We can show by an induction-based argument (as usual for recursive programs) that a site is marked as full if and only if it is connected to one of the sites on the top row.

Tracing the operation of `flow()` on a tiny test case is an instructive exercise. You will see that it calls `flow()` for every site that can be reached via a path of open sites from the top row. This example illustrates that simple recursive programs can mask computations that otherwise are quite sophisticated. This method is a special case of the depth-first search algorithm, which has many important applications.



Recursive percolation (null calls omitted)

### Program 2.4.5 Percolation detection

```

public static boolean[][] flow(boolean[][] isOpen)
{ // Fill every site reachable from the top row.
    int n = isOpen.length;
    boolean[][] isFull = new boolean[n][n];
    for (int j = 0; j < n; j++)
        flow(isOpen, isFull, 0, j);
    return isFull;
}
public static void flow(boolean[][] isOpen,
                      boolean[][] isFull, int i, int j)
{ // Fill every site reachable from (i, j).
    int n = isFull.length;
    if (i < 0 || i >= n) return;
    if (j < 0 || j >= n) return;
    if (!isOpen[i][j]) return;
    if (isFull[i][j]) return;
    isFull[i][j] = true;
    flow(isOpen, isFull, i+1, j); // Down.
    flow(isOpen, isFull, i, j+1); // Right.
    flow(isOpen, isFull, i, j-1); // Left.
    flow(isOpen, isFull, i-1, j); // Up.
}

```

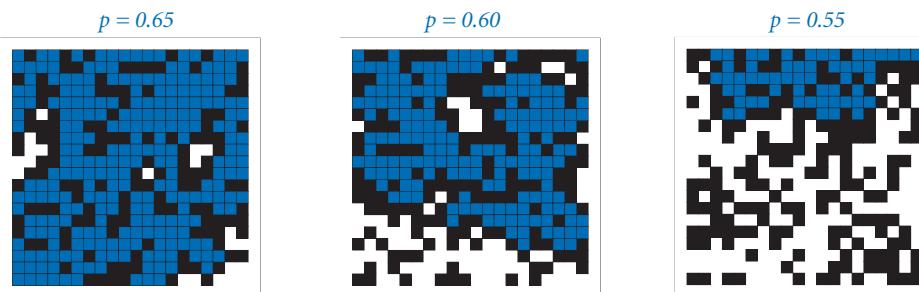
n	system size ( $n$ -by- $n$ )
isOpen[][]	open sites
isFull[][]	full sites
i, j	current site row, column

Substituting these methods for the stub in PROGRAM 2.4.1 gives a depth-first-search-based solution to the percolation problem. The recursive `flow()` sets to `true` the element in `isFull[][]` corresponding to any site that can be reached from `isOpen[i][j]` via a chain of neighboring open sites. The one-argument `flow()` calls the recursive method for every site on the top row.

```
% more test8.txt
8 8
0 0 1 1 1 0 0 0
1 0 0 1 1 1 1 1
1 1 1 0 0 1 1 0
0 0 1 1 0 1 1 1
0 1 1 1 0 1 1 0
0 1 0 0 0 0 1 1
1 0 1 0 1 1 1 1
1 1 1 1 0 1 0 0
```

```
% java Percolation < test8.txt
8 8
0 0 1 1 1 0 0 0
0 0 0 1 1 1 1 1
0 0 0 0 0 1 1 0
0 0 0 0 0 1 1 1
0 0 0 0 0 1 1 0
0 0 0 0 0 0 1 1
0 0 0 0 1 1 1 1
0 0 0 0 0 1 0 0
true
```

To avoid conflict with our solution for vertical percolation (PROGRAM 2.4.2), we might rename that class `PercolationVertical`, making another copy of `Percolation` (PROGRAM 2.4.1) and substituting the two `flow()` methods in PROGRAM 2.4.5 for the placeholder `flow()`. Then, we can visualize and perform experiments with this algorithm with the `PercolationVisualizer` and `PercolationProbability` tools that we have developed. If you do so, and try various values for  $n$  and  $p$ , you will quickly get a feeling for the situation: the systems always percolate when the site vacancy probability  $p$  is high and never percolate when  $p$  is low, and (particularly as  $n$  increases) there is a value of  $p$  above which the systems (almost) always percolate and below which they (almost) never percolate.



*Percolation is less probable as the site vacancy probability  $p$  decreases*

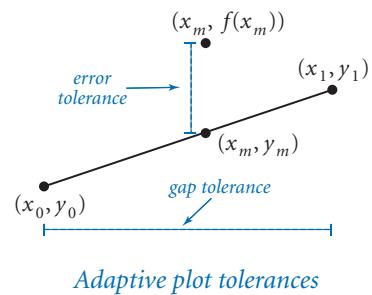
Having debugged `PercolationVisualizer` and `PercolationProbability` on the simple vertical percolation process, we can use them with more confidence to study percolation, and turn quickly to study the scientific problem of interest. Note that if we want to experiment with vertical percolation again, we would need to edit `PercolationVisualizer` and `PercolationProbability` to refer to `PercolationVertical` instead of `Percolation`, or write other clients of both `PercolationVertical` and `Percolation` that run methods in both classes to compare them.

**Adaptive plot** To gain more insight into percolation, the next step in program development is to write a program that plots the percolation probability as a function of the site vacancy probability  $p$  for a given value of  $n$ . Perhaps the best way to produce such a plot is to first derive a mathematical equation for the function, and then use that equation to make the plot. For percolation, however, no one has been able to derive such an equation, so the next option is to use the Monte Carlo method: run simulations and plot the results.

Immediately, we are faced with numerous decisions. For how many values of  $p$  should we compute an estimate of the percolation probability? Which values of  $p$  should we choose? How much precision should we aim for in these calculations? These decisions constitute an experimental design problem. Much as we might like to instantly produce an accurate rendition of the curve for any given  $n$ , the computation cost can be prohibitive. For example, the first thing that comes to mind is to plot, say, 100 to 1,000 equally spaced points, using `StdStats` (PROGRAM 2.2.5). But, as you learned from using `PercolationProbability`, computing a sufficiently precise value of the percolation probability for each point might take several seconds or longer, so the whole plot might take minutes or hours or even longer. Moreover, it is clear that a lot of this computation time is completely wasted, because we know that values for small  $p$  are 0 and values for large  $p$  are 1. We might prefer to spend that time on more precise computations for intermediate  $p$ . How should we proceed?

`PercolationPlot` (PROGRAM 2.4.6) implements a recursive approach with the same structure as `Brownian` (PROGRAM 2.3.5) that is widely applicable to similar problems. The basic idea is simple: we choose the maximum distance that we wish to allow between values of the  $x$ -coordinate (which we refer to as the *gap tolerance*), the maximum known error that we wish to tolerate in the  $y$ -coordinate (which we refer to as the *error tolerance*), and the number of trials  $T$  per point that we wish to perform. The recursive method draws the plot within a given interval  $[x_0, x_1]$ , from  $(x_0, y_0)$  to  $(x_1, y_1)$ . For our problem, the plot is from  $(0, 0)$  to  $(1, 1)$ . The base case (if the distance between  $x_0$  and  $x_1$  is less than the gap tolerance, or the distance between the line connecting the two endpoints and the value of the function at the midpoint is less than the error tolerance) is to simply draw a line from  $(x_0, y_0)$  to  $(x_1, y_1)$ . The reduction step is to (recursively) plot the two halves of the curve, from  $(x_0, y_0)$  to  $(x_m, f(x_m))$  and from  $(x_m, f(x_m))$  to  $(x_1, y_1)$ .

The code in `PercolationPlot` is relatively simple and produces a good-looking curve at relatively low cost. We can use it to study the shape of the curve for various values of  $n$  or choose smaller tolerances to be more confident that the curve is close to the actual values. Precise mathematical statements about quality of approximation can, in principle, be derived, but it is perhaps not appropriate to go into too much detail while exploring and experimenting, since our goal is simply to develop a hypothesis about percolation that can be tested by scientific experimentation.



Adaptive plot tolerances

### Program 2.4.6 Adaptive plot client

```

public class PercolationPlot
{
    public static void curve(int n,
                            double x0, double y0,
                            double x1, double y1)
    { // Perform experiments and plot results.
        double gap = 0.01;
        double err = 0.0025;
        int trials = 10000;
        double xm = (x0 + x1)/2;
        double ym = (y0 + y1)/2;
        double fxm = PercolationProbability.estimate(n, xm, trials);
        if (x1 - x0 < gap || Math.abs(ym - fxm) < err)
        {
            StdDraw.line(x0, y0, x1, y1);
            return;
        }
        curve(n, x0, y0, xm, f xm);
        StdDraw.filledCircle(xm, f xm, 0.005);
        curve(n, xm, f xm, x1, y1);
    }

    public static void main(String[] args)
    { // Plot experimental curve for n-by-n percolation system.
        int n = Integer.parseInt(args[0]);
        curve(n, 0.0, 0.0, 1.0, 1.0);
    }
}

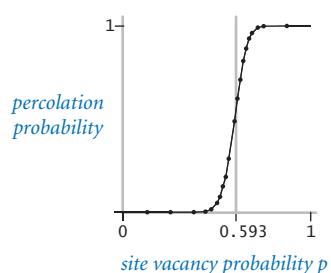
```

n	system size
x0, y0	left endpoint
x1, y1	right endpoint

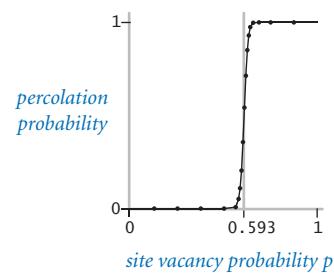
xm, ym	midpoint
fxm	value at midpoint
gap	gap tolerance
err	error tolerance
trials	number of trials

This recursive program draws a plot of the percolation probability (experimental observations) against the site vacancy probability  $p$  (control variable) for random  $n$ -by- $n$  systems.

% java PercolationPlot 20



% java PercolationPlot 100



Indeed, the curves produced by `PercolationPlot` immediately confirm the hypothesis that there is a *threshold* value (about 0.593): if  $p$  is greater than the threshold, then the system almost certainly percolates; if  $p$  is less than the threshold, then the system almost certainly does not percolate. As  $n$  increases, the curve approaches a step function that changes value from 0 to 1 at the threshold. This phenomenon, known as a *phase transition*, is found in many physical systems.

The simple form of the output of PROGRAM 2.4.6 masks the huge amount of computation behind it. For example, the curve drawn for  $n = 100$  has 18 points, each the result of 10,000 trials, with each trial involving  $n^2$  sites. Generating and testing each site involves a few lines of code, so this plot comes at the cost of executing *billions* of statements. There are two lessons to be learned from this observation. First, we need to have confidence in any line of code that might be executed billions of times, so our care in developing and debugging code incrementally is justified. Second, although we might be interested in systems that are much larger, we need further study in computer science to be able to handle larger cases—that is, to develop faster algorithms and a framework for knowing their performance characteristics.

With this reuse of all of our software, we can study all sorts of variants on the percolation problem, just by implementing different `flow()` methods. For example, if you leave out the last recursive call in the recursive `flow()` method in PROGRAM 2.4.5, it tests for a type of percolation known as *directed percolation*, where paths that go up are not considered. This model might be important for a situation like a liquid percolating through porous rock, where gravity might play a role, but not for a situation like electrical connectivity. If you run `PercolationPlot` for both methods, will you be able to discern the difference (see EXERCISE 2.4.10)?

```

PercolationPlot.curve()
PercolationProbability.estimate()
Percolation.random()
StdRandom.bernoulli()
    : n2 times
StdRandom.bernoulli()
return
Percolation.percolates()
    flow()
    return
return
: T times
Percolation.random()
StdRandom.bernoulli()
    : n2 times
StdRandom.bernoulli()
return
Percolation.percolates()
    flow()
    return
return
return
once for each point
PercolationProbability.estimate()
Percolation.random()
StdRandom.bernoulli()
    : n2 times
StdRandom.bernoulli()
return
Percolation.percolates()
    flow()
    return
return
: T times
Percolation.random()
StdRandom.bernoulli()
    : n2 times
StdRandom.bernoulli()
return
Percolation.percolates()
    flow()
    return
return
return
return

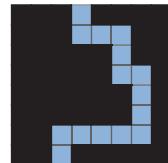
```

*Function-call trace for PercolationPlot*

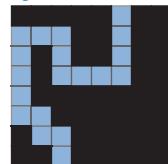
To model physical situations such as water flowing through porous substances, we need to use three-dimensional arrays. Is there a similar threshold in the three-dimensional problem? If so, what is its value? Depth-first search is effective for studying this question, though the addition of another dimension requires that we pay even more attention to the computational cost of determining whether a system percolates (see EXERCISE 2.4.18). Scientists also study more complex lattice structures that are not well modeled by multidimensional arrays—we will see how to model such structures in SECTION 4.5.

Percolation is interesting to study via *in silico* experimentation because no one has been able to derive the threshold value mathematically for several natural models. The only way that scientists know the value is by using simulations like Percolation. A scientist needs to do experiments to see whether the percolation model reflects what is observed in nature, perhaps through refining the model (for example, using a different lattice structure). Percolation is an example of an increasing number of problems where computer science of the kind described here is an essential part of the scientific process.

*percolates (path never goes up)*



*does not percolate*



*Directed percolation*

**Lessons** We might have approached the problem of studying percolation by sitting down to design and implement a single program, which probably would run to hundreds of lines, to produce the kind of plots that are drawn by PROGRAM 2.4.6. In the early days of computing, programmers had little choice but to work with such programs, and would spend enormous amounts of time isolating bugs and correcting design decisions. With modern programming tools like Java, we can do better, using the incremental modular style of programming presented in this chapter and keeping in mind some of the lessons that we have learned.

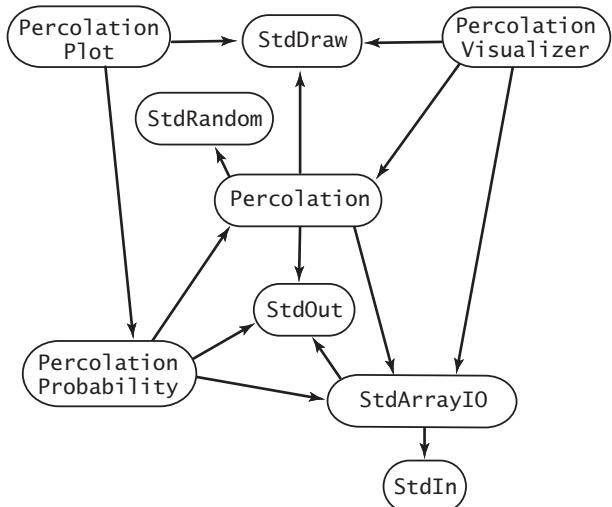
**Expect bugs.** Every interesting piece of code that you write is going to have at least one or two bugs, if not many more. By running small pieces of code on small test cases that you understand, you can more easily isolate any bugs and then more easily fix them when you find them. Once debugged, you can depend on using a library as a building block for any client.

**Keep modules small.** You can focus attention on at most a few dozen lines of code at a time, so you may as well break your code into small modules as you write it. Some classes that contain libraries of related methods may eventually grow to contain hundreds of lines of code; otherwise, we work with small files.

**Limit interactions.** In a well-designed modular program, most modules should depend on just a few others. In particular, a module that *calls* a large number of other modules needs to be divided into smaller pieces. Modules that *are called* by a large number of other modules (you should have only a few) need special attention, because if you do need to make changes in a module's API, you have to reflect those changes in all its clients.

**Develop code incrementally.** You should run and debug each small module as you implement it. That way, you are never working with more than a few dozen lines of unreliable code at any given time. If you put all your code in one big module, it is difficult to be confident that *any* of it is free from bugs. Running code early also forces you to think sooner rather than later about I/O formats, the nature of problem instances, and other issues. Experience gained when thinking about such issues and debugging related code makes the code that you develop later in the process more effective.

**Solve an easier problem.** Some working solution is better than no solution, so it is typical to begin by putting together the simplest code that you can craft that solves a given problem, as we did with vertical percolation. This implementation is the first step in a process of continual refinements and improvements as we develop a more complete understanding of the problem by examining a broader variety of test cases and developing support software such as our `PercolationVisualizer` and `PercolationProbability` classes.



Case study dependency graph (not including system calls)

*Consider a recursive solution.* Recursion is an indispensable tool in modern programming that you should learn to trust. If you are not already convinced of this fact by the simplicity and elegance of `Percolation` and `PercolationPlot`, you might wish to try to develop a nonrecursive program for testing whether a system percolates and then reconsider the issue.

*Build tools when appropriate.* Our visualization method `show()` and random boolean matrix generation method `random()` are certainly useful for many other applications, as is the adaptive plotting method of `PercolationPlot`. Incorporating these methods into appropriate libraries would be simple. It is no more difficult (indeed, perhaps easier) to implement general-purpose methods like these than it would be to implement special-purpose methods for percolation.

*Reuse software when possible.* Our `StdIn`, `StdRandom`, and `StdDraw` libraries all simplified the process of developing the code in this section, and we were also immediately able to reuse programs such as `PercolationVisualizer`, `PercolationProbability`, and `PercolationPlot` for percolation after developing them for vertical percolation. After you have written a few programs of this kind, you might find yourself developing versions of these programs that you can reuse for other Monte Carlo simulations or other experimental data analysis problems.

THE PRIMARY PURPOSE OF THIS CASE study is to convince you that modular programming will take you much further than you could get without it. Although no approach to programming is a panacea, the tools and approach that we have discussed in this section will allow you to attack complex programming tasks that might otherwise be far beyond your reach.

The success of modular programming is only a start. Modern programming systems have a vastly more flexible programming model than the class-as-a-library-of-static-methods model that we have been considering. In the next two chapters, we develop this model, along with many examples that illustrate its utility.

**Q&A**

**Q.** Editing `PercolationVisualizer` and `PercolationProbability` to rename `Percolation` to `PercolationVertical` or whatever method we want to study seems to be a bother. Is there a way to avoid doing so?

**A.** Yes, this is a key issue to be revisited in CHAPTER 3. In the meantime, you can keep the implementations in separate subdirectories, but that can get confusing. Advanced Java mechanisms (such as the *classpath*) are also helpful, but they also have their own problems.

**Q.** That recursive `flow()` method makes me nervous. How can I better understand what it's doing?

**A.** Run it for small examples of your own making, instrumented with instructions to print a function-call trace. After a few runs, you will gain confidence that it always marks as full the sites connected to the start site via a chain of neighboring open sites.

**Q.** Is there a simple nonrecursive approach to identifying the full sites?

**A.** There are several methods that perform the same basic computation. We will revisit the problem in SECTION 4.5, where we consider *breadth-first search*. In the meantime, working on developing a nonrecursive implementation of `flow()` is certain to be an instructive exercise, if you are interested.

**Q.** `PercolationPlot` (PROGRAM 2.4.6) seems to involve a huge amount of computation to produce a simple function graph. Is there some better way?

**A.** Well, the best would be a simple mathematical formula describing the function, but that has eluded scientists for decades. Until scientists discover such a formula, they must resort to computational experiments like the ones in this section.

## Exercises

**2.4.1** Write a program that takes a command-line argument  $n$  and creates an  $n$ -by- $n$  boolean matrix with the element in row  $i$  and column  $j$  set to `true` if  $i$  and  $j$  are relatively prime, then shows the matrix on the standard drawing (see EXERCISE 1.4.16). Then, write a similar program to draw the Hadamard matrix of order  $n$  (see EXERCISE 1.4.29). Finally, write a program to draw the boolean matrix such that the element in row  $n$  and column  $j$  is set to `true` if the coefficient of  $x^j$  in  $(1+x)^i$  (binomial coefficient) is odd (see EXERCISE 1.4.41). You may be surprised at the pattern formed by the third example.

**2.4.2** Implement a `print()` method for `Percolation` that prints 1 for blocked sites, 0 for open sites, and \* for full sites.

**2.4.3** Give the recursive calls for `flow()` in PROGRAM 2.4.5 given the following input:

```
3 3
1 0 1
0 0 0
1 1 0
```

**2.4.4** Write a client of `Percolation` like `PercolationVisualizer` that does a series of experiments for a value of  $n$  taken from the command line where the site vacancy probability  $p$  increases from 0 to 1 by a given increment (also taken from the command line).

**2.4.5** Describe the order in which the sites are marked when `Percolation` is used on a system with no blocked sites. Which is the last site marked? What is the depth of the recursion?

**2.4.6** Experiment with using `PercolationPlot` to plot various mathematical functions (by replacing the call `PercolationProbability.estimate()` with a different expression that evaluates a mathematical function). Try the function  $f(x) = \sin x + \cos 10x$  to see how the plot adapts to an oscillating curve, and come up with interesting plots for three or four functions of your own choosing.

**2.4.7** Modify `Percolation` to animate the flow computation, showing the sites filling one by one. Check your answer to the previous exercise.

**2.4.8** Modify `Percolation` to compute that maximum depth of the recursion used in the flow calculation. Plot the expected value of that quantity as a function of the site vacancy probability  $p$ . How does your answer change if the order of the recursive calls is reversed?

**2.4.9** Modify `PercolationProbability` to produce output like that produced by `Bernoulli` (PROGRAM 2.2.6). *Extra credit:* Use your program to validate the hypothesis that the data obeys a Gaussian distribution.

**2.4.10** Create a program `PercolationDirected` that tests for *directed* percolation (by leaving off the last recursive call in the recursive `flow()` method in PROGRAM 2.4.5, as described in the text), then use `PercolationPlot` to draw a plot of the directed percolation probability as a function of the site vacancy probability  $p$ .

**2.4.11** Write a client of `Percolation` and `PercolationDirected` that takes a site vacancy probability  $p$  from the command line and prints an estimate of the probability that a system percolates but does not percolate down. Use enough experiments to get an estimate that is accurate to three decimal places.

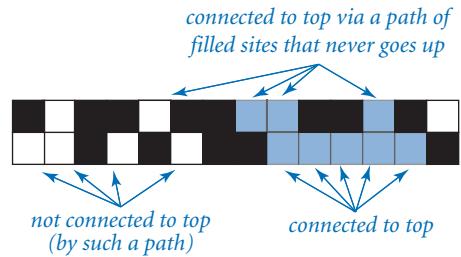
## Creative Exercises

**2.4.12** *Vertical percolation.* Show that a system with site vacancy probability  $p$  vertically percolates with probability  $1 - (1 - p)^n$ , and use `PercolationProbability` to validate your analysis for various values of  $n$ .

**2.4.13** *Rectangular percolation systems.* Modify the code in this section to allow you to study percolation in rectangular systems. Compare the percolation probability plots of systems whose ratio of width to height is 2 to 1 with those whose ratio is 1 to 2.

**2.4.14** *Adaptive plotting.* Modify `PercolationPlot` to take its control parameters (gap tolerance, error tolerance, and number of trials) as command-line arguments. Experiment with various values of the parameters to learn their effect on the quality of the curve and the cost of computing it. Briefly describe your findings.

**2.4.15** *Nonrecursive directed percolation.* Write a nonrecursive program that tests for directed percolation by moving from top to bottom as in our vertical percolation code. Base your solution on the following computation: if any site in a contiguous subrow of open sites in the current row is connected to some full site on the previous row, then all of the sites in the subrow become full.



*Directed percolation calculation*

**2.4.16** *Fast percolation test.* Modify the recursive `flow()` method in PROGRAM 2.4.5 so that it returns as soon as it finds a site on the bottom row (and fills no more sites). *Hint:* Use an argument `done` that is `true` if the bottom has been hit, `false` otherwise. Give a rough estimate of the performance improvement factor for this change when running `PercolationPlot`. Use values of  $n$  for which the programs run at least a few seconds but not more than a few minutes. Note that the improvement is ineffective unless the first recursive call in `flow()` is for the site *below* the current site.

**2.4.17 Bond percolation.** Write a modular program for studying percolation under the assumption that the edges of the grid provide connectivity. That is, an edge can be either empty or full, and a system percolates if there is a path consisting of full edges that goes from top to bottom. *Note:* This problem has been solved analytically, so your simulations should validate the hypothesis that the bond percolation threshold approaches  $1/2$  as  $n$  gets large.

**2.4.18 Percolation in three dimensions.** Implement a class `Percolation3D` and a class `BooleanMatrix3D` (for I/O and random generation) to study percolation in three-dimensional cubes, generalizing the two-dimensional case studied in this section. A percolation system is an  $n$ -by- $n$ -by- $n$  cube of sites that are unit cubes, each open with probability  $p$  and blocked with probability  $1-p$ . Paths can connect an open cube with any open cube that shares a common face (one of six neighbors, except on the boundary). The system percolates if there exists a path connecting any open site on the bottom plane to any open site on the top plane. Use a recursive version of `flow()` like PROGRAM 2.4.5, but with eight recursive calls instead of four. Plot the percolation probability versus site vacancy probability  $p$  for as large a value of  $n$  as you can. Be sure to develop your solution incrementally, as emphasized throughout this section.

**2.4.19 Bond percolation on a triangular grid.** Write a modular program for studying bond percolation on a triangular grid, where the system is composed of  $2n^2$  equilateral triangles packed together in an  $n$ -by- $n$  grid of rhombus shapes. Each interior point has six bonds; each point on the edge has four; and each corner point has two.

*percolates*



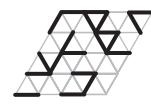
*does not*



*percolates*



*does not*

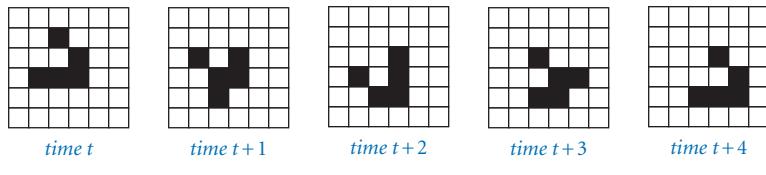




**2.4.20 Game of Life.** Implement a class `GameOfLife` that simulates Conway's *Game of Life*. Consider a boolean matrix corresponding to a system of cells that we refer to as being either live or dead. The game consists of checking and perhaps updating the value of each cell, depending on the values of its neighbors (the adjacent cells in every direction, including diagonals). Live cells remain live and dead cells remain dead, with the following exceptions:

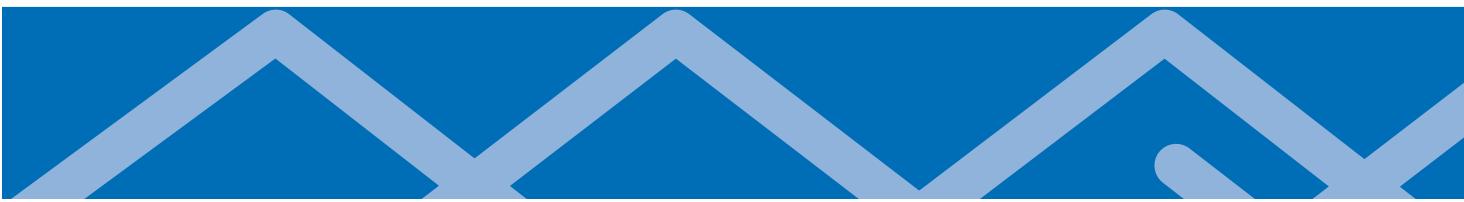
- A dead cell with exactly three live neighbors becomes live.
- A live cell with exactly one live neighbor becomes dead.
- A live cell with more than three live neighbors becomes dead.

Initialize with a random boolean matrix, or use one of the starting patterns on the booksite. This game has been heavily studied, and relates to foundations of computer science (see the booksite for more information).



Five generations of a glider

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# Index

## A

A-format instructions, 911  
Absolute value function, 199  
Absorption identity, 990  
Abstract machines, 737–738  
Abstract methods, 446  
Abstraction  
    color, 341–343  
    circuits, 1037–1039  
    data, 382  
    displays, 346  
    function-call, 590–591  
    libraries, 230, 429  
    object-oriented programming, 329  
    printing as, 76  
    recursion, 289  
    vs. representation, 69  
    standard audio, 155  
    standard drawing, 144  
    standard I/O, 129, 139–143  
Accept states  
    DFAs, 738–739  
    Turing machines, 766–767  
Access modifiers, 384  
Accessing references, 339  
Account information  
    dictionary lookup, 628–629  
    indexing, 634  
Accuracy  
    n-body simulation, 488  
    random web surfer, 185  
Adaptive plots, 314–318

Adders  
    binary, 771  
    combinational circuits, 1007  
    overview, 1028  
    ripple-carry, 1028–1030  
    sum-of-products, 1028  
**AddInts** program, 134  
Addition  
    complex numbers, 402–403  
    floating-point numbers, 24–26  
    integers, 22, 884  
    negative numbers, 887  
    spatial vectors, 442–443  
Address control lines, 1056  
Addresses  
    array elements, 94  
    memory, 909  
    symbolic names, 981  
Adelman, Leonard, 795  
Adjacency matrix, 692  
Adjacent vertices, 671  
Albers, Josef, 342  
**AlbersSquares** program, 341–342  
Alex, 380  
Algebra  
    boolean, 989–991  
    vectors, 442–443  
Algorithms, 493  
    computability, 787  
    decidability, 786–787  
    exponential-time, 826  
    overview, 786  
    performance. *See* Performance  
    polynomial-time, 825–826  
searching. *See* Searches  
sorting. *See* Sorts  
Aliasing  
    arrays, 516  
    bugs from, 439, 441  
    references, 363  
Allocating memory, 94, 367  
Alphabets  
    formal languages, 720–721  
    metasymbols, 725  
    regular expressions, 730  
    symbols, 718–719  
ALUs. *See* Arithmetic logic units (ALUs)  
Amortized analysis, 580–581  
Ampersands (&)  
    bitwise operations, 891–892  
    boolean type, 26–27, 991  
Analog circuits, 1013  
AND circuits in ALUs, 1031  
AND gates, 1014  
And operation  
    bitwise, 891–892  
    boolean type, 26–27, 987–989  
    TOY machine, 913  
Animations  
    **BouncingBall**, 152–153  
    double buffering, 151  
Annihilation identity, 990  
Antisymmetric property, 546  
Application programming  
    interfaces (APIs)  
    access modifiers, 384  
Body, 480  
built-in data types, 30–32

- C** Charge, 383  
**C**olor, 343  
**C**omparable, 545  
**C**omplex, 403  
**C**ounter, 436–437  
 data types, 388  
 designing, 233, 429–431  
**D**raw, 361  
**G**raph, 675–679  
**H**istogram, 392  
 implementing, 231  
**I**n, 354  
 libraries, 29, 230–232  
 modular programming, 432  
**O**ut, 355  
**P**athFinder, 683  
**P**icture, 347  
**Q**ueue, 592  
**S**ET, 652  
**S**ketch, 459  
 spatial vectors, 442–443  
**S**T, 625  
**S**tackOfStrings, 568  
**S**tandard, 237  
**S**tandardAudio, 159  
**S**tandardDraw, 149, 154  
**S**tandardIn, 132–133  
**S**tandardOut, 130  
**S**tandardRandom, 233  
**S**tandardStats, 244  
**S**tandardStockAccount, 410  
**S**topwatch, 390  
**S**tring, 332–333  
 symbol tables, 625–627  
**T**urtle, 394  
**U**niverse, 483  
**V**ector, 443  
 Approximation algorithms, 852  
 Arbitrary-size input streams, 137–138  
**args** argument, 7, 208
- A**rguments  
 arrays as, 207–210  
 command-line, 7–8, 11, 127  
 constructors, 333, 385  
 methods, 30  
 passing, 207–210, 364–365  
**p**rintf(), 130–132  
 static methods, 197  
**A**riane 5 rocket, 35  
**A**rithmetic  
 CPU instructions, 1079  
 floating point numbers, 890  
 integers, 884–885  
 operators, 22  
 TOY machine instructions, 912  
**A**rithmetic logic units (ALUs), 1031  
 bitwise operations, 1031  
 inputs, 1031  
 outputs, 1032  
 summary, 1032–1033  
 TOY machine, 910  
**A**rithmetic expression evaluation, 586–589  
**A**rithmetic shifts  
 bits, 891–892  
 purpose, 898–899  
**A**rrayIndexOutOfBoundsException, 95, 116, 466  
**A**rrays  
 aliasing, 516  
 as arguments, 207–210  
 assigning, 117  
 associative, 630  
 binary searches, 538–539  
 bitonic, 563  
 bounds checking, 95  
 comparing, 117  
 coupon collector problem, 101–103  
 decks of cards, 97–100
- declaring, 91, 116  
 default initialization, 93  
 exchanging values, 96  
 FIFO queues, 596  
 hash tables, 636  
 I/O libraries, 237–238  
 images, 346–347  
 immutable types, 439–440  
 iterable classes, 603  
 linked structures, 942–944  
 machine-language, 938–941  
 memory, 91, 94, 515–517  
 multidimensional, 111  
 overview, 90–92  
 parallel, 411  
 plotting, 246–248  
 precomputed values, 99–100  
 references, 365  
 resizing, 578–581, 635  
 as return values, 210  
 setting values, 95–96  
 shuffling, 97  
 side effects, 208–210  
 Sieve of Eratosthenes, 103–105  
 stacks, 568–570, 578–581  
 summary, 115  
 transposition, 120  
 two-dimensional.  
*See* Two-dimensional arrays  
**A**rrays.binarySearch(), 559  
**A**rrays.sort(), 559  
**A**rrayStackOfStrings program, 568–570, 603  
 Arrival rate in M/M/1 queues, 597–598  
*The Art of Computer Programming* book, 947  
 ASCII standard, 874, 894–895  
 Assemblers for TOY machine, 964  
 Assembly language  
 description, 930  
 symbolic names, 981

Assertions, 466–467  
Assignments  
    arrays, 117  
    chained, 43  
    compound, 60  
    description, 17  
    references, 363  
Associative arrays, 630  
Associative axiom, 990, 993  
Associativity, 17  
Asterisks (\*)  
    comments, 9  
    floating-point numbers, 24–26  
    integers, 22–23  
    regular expressions, 724  
Audio  
    plotting sound waves, 249  
    standard, 155–159  
    superposition, 211–215  
Autoboxing, 457, 585–586  
Automatic promotion, 33  
Average-case performance, 648  
Average magnitude, 164  
Average path lengths, 693  
Average power, 164  
Average program, 137–138  
Axioms in Boolean algebra, 990

## B

Backslashes (\)  
    escape sequences, 19  
    regular expressions, 731  
Backward compatibility, 976  
Bacon, Kevin, 684  
Balanced binary trees, 661  
Ball animation, 152–153  
Barnsley ferns, 240–243  
Base cases  
    binary search trees, 640  
    recursion, 264–265, 281

Base classes, 452–453  
Base64 encoding, 904  
Bases in positional notation, 875  
Basic scaffolding, 302–304  
Basic statistics, 244–246  
Beck exploit, 529  
Beckett, Samuel, 273  
Beckett program, 274–275  
Behavior of objects, 340  
Benford’s law, 224  
Bernoulli, Jacob, 398  
Bernoulli program, 249–250  
Best-case performance  
    binary search trees, 647  
    insertion sort, 544  
Big-O notation, 520–521  
**BigInteger** class, 827, 897–898  
Binary adders, 771  
Binary digits, 22  
Binary frequency count equality, 772–773  
Binary incrementers, 769–771  
Binary number system  
    conversions, 67–69  
    description, 38  
Binary operators, 17  
Binary program, 67–69  
Binary reflected Gray code, 274  
Binary representation  
    decimal conversions, 877  
    description, 875  
    examples, 878–879  
    hex conversions, 876–877  
    literals, 891  
Binary search trees (BSTs)  
    implementation, 645–646  
    insert process, 644–645  
    machine-language, 942–944  
    ordered operations, 651  
    overview, 640–643  
    performance, 647–648  
search process, 643–644  
symbol tables, 624–625  
traversing, 649–650  
Binary searches  
    binary representation, 536  
    correctness proof, 535  
    exception filters, 540  
    inverting functions, 536–538  
    overview, 533–534  
    random web surfer, 176  
    running time, 535  
    sorted arrays, 538–539  
    symbol tables, 635  
    weighing objects, 540–541  
Binary strings, 718–719  
Binary trees  
    balanced, 661  
    heap-ordered, 661  
    isomorphic, 661  
Binary16 format, 888  
BinarySearch program, 538–539  
Binomial coefficients, 125  
Binomial distributions, 125, 249  
Biology  
    computational, 732–734  
    DNA computers, 795  
    genomics application, 336–340  
    graphs, 672  
Bipartite graphs, 682  
Bisection searches, 537  
Bit-slice memory design, 1054–1056  
Bitmapmed images, 346  
Bitonic arrays, 563  
Bits  
    binary number system, 38, 875  
    bitwise operations, 891–892  
    computer dependence, 874  
    description, 22  
    logical instructions, 912–913  
    manipulating, 891–893  
    memory, 1056

- memory size, 513  
 register, 1051  
 shifting, 891–892
- Bitwise operations**  
 and, 913  
 arithmetic logic units, 1031  
 exclusive *or*, 39, 913  
 shift, 913
- Black–Scholes formula, 222, 565
- Blobs, 709
- Blocks**  
 statements, 50  
 variable scope, 200
- Bodies**  
 loops, 53  
 static methods, 196
- Body program**  
 memory, 514  
 N-body simulation, 479–482
- Bollobás–Chung graph model, 713
- Book indexes, 632–633
- Booksite, 2–3
- Boole, George, 986
- boolean data type**  
 conversion codes, 131–132  
 description, 14–15  
 input, 133  
 memory size, 513  
 overview, 26–27
- Boolean logic  
 cryptography application, 992–994  
 description, 27  
 expressions, 995–996  
 functions, 987–991, 994–997  
 overview, 986
- Boolean matrices, 302
- Boolean satisfiability, 832, 836  
 boolean equation satisfiability problem, 838  
**NP**-completeness, 844–846, 853–856
- Booting, 959–960, 968–969  
 Bootstrapping, 971
- BouncingBall** program, 152–153
- Bounding boxes for drawings, 146
- Bounds**  
 arrays, 95  
 exponential time, 826  
 polynomial time, 825
- Boxing, 457, 585–586
- Box–Muller formula, 47
- Breadth-first searches, 683, 687–688, 690, 692
- break statement**, 74
- Bridges, Brownian, 278–280
- Brin, Sergey, 184
- Brown, Robert, 400
- Brownian bridges, 278–280
- Brownian motion, 400–401
- Brownian program, 278–280
- Brute-force algorithm, 535–536
- BST program, 645–646
- BSTs.** *See* Binary search trees (BSTs)
- Buffer overflow  
 arrays, 95  
 attacks, 963
- Buffering drawings, 151
- Bugs  
 aliasing, 363, 439, 441  
 overview, 6  
 testing for, 318
- Built-in data types  
 boolean, 26–27  
 characters and strings, 19–21  
 comparisons, 27–29  
 conversions, 32–35  
 floating-point numbers, 24–26  
 integers, 22–24  
 library methods, 29–32  
 overview, 14–15  
 summary, 35–36  
 terminology, 15–18
- Built-in interfaces, 451
- Buses, 1034–1036  
 CPU connections, 1077  
 program counter connections, 1073–1074
- Buzzers, 1048
- byte** data type, 24
- Bytecode**  
 compiling, 589, 788  
 Java virtual machine, 965
- Bytes memory size, 513
- ## C
- C conversion specification, 131
- Caches**  
 and instruction time, 509  
 in top-down dynamic programming, 284
- Calculators, 908
- Callbacks in event-based programming, 451
- Calls, 193  
 chaining, 404  
 in machine language, 932–933  
 methods, 30, 197, 340  
 reverse Polish notation, 591
- Canvas, 151
- Card decks, arrays for, 97–100
- Carets (^)  
 bitwise operations, 891–892  
 regular expressions, 731
- Carroll, Lewis, 710
- Carry bits in adders, 1028
- Cartesian representation, 433
- Casts, 33–34
- Cat program, 356
- Cellular automata, 794
- Central processing units (CPUs)  
 bus connections, 1077  
 control lines, 1077–1078  
 execute phase, 1079

- fetch phase, 1078  
instructions, 1079–1080  
interfaces, 1076  
load address, 1080  
modules, 1076  
overview, 985  
TOY-8 machine, 1076–1080
- Centroids, 164  
Chained assignments, 43  
Chained comparisons, 43  
Chaining method calls, 404  
Characters and `char` data type  
    ASCII, 894  
    conversion to numbers, 880–881  
    description, 15  
    memory size, 513  
    representing, 894–895  
    Unicode, 894–895  
    working with, 19–21
- Charge program, 383–389, 515
- Checksums  
    description, 86  
    formula, 220
- Chords, 211
- Chromatic scale, 156
- Church, Alonso, 790
- Church–Turing thesis  
    extended, 823  
    overview, 790–791  
    Turing machine simulation, 798  
    virtual machines, 958
- Ciphers, Kamasutra, 377
- Ciphertext, 993
- Circuit models  
    building circuits, 1006–1008  
    connections, 1002–1004  
    controlled switches, 1005–1006  
    conventions, 1004  
    inputs, 1002–1004  
    logical design, 1008–1009  
    outputs, 1002–1004
- overview, 1002–1003  
    wires, 1002–1004
- Circuits  
    combinational.  
        *See* Combinational circuits  
    description, 1010  
    from gates, 1019–1021  
    memory, 1054–1057
- Circular linked lists, 622
- Circular queues, 620
- Circular shifts, 375
- .class extension, 3, 8, 228
- `ClassDefNotFoundError`, 160
- Classes, 4–5  
    accessing, 227–229  
    description, 226  
    implementing, 383–389  
    inner, 609  
    modules as, 228  
    variables, 284
- Classifying **NP**-complete problems, 851
- Client code  
    data types, 430  
    library methods, 230
- Clocks  
    CPU, 1077–1079  
    fetch and execute, 1059, 1061  
    overview, 1058–1059  
    run and halt inputs, 1060  
    write control, 1059–1060
- Closure operation in REs, 724
- Clouds, plasma, 280
- Clustering coefficients  
    global, 713  
    local, 693–694
- CMYK color format, 48–49, 371
- Code and coding  
    description, 2  
    encapsulating, 438  
    incremental development, 319, 701
- reuse, 226, 253, 701  
    static methods, 205–206
- Codebooks, 992
- Codons, genes, 336
- Coefficients for floating-point numbers, 889
- Coercion, 33
- Coin flip, 52–53
- Collatz function, 784
- Collatz problem, 296–297, 818
- Collatz sequence, 948
- Collections  
    description, 566  
    iterable, 601–605  
    objects, 582–583  
    queues. *See* Queues  
    stacks. *See* Stacks  
    symbol tables. *See* Symbol Tables
- Colons (:) *in* Turing machine tapes, 767  
    foreach statements, 601–602
- Color and Color data type  
    blobs, 709  
    compatibility, 344  
    conversion, 48–49  
    drawings, 150  
    grayscale, 344  
    luminance, 343  
    memory, 514  
    overview, 341–343
- Columns in 2D arrays, 106, 108
- Combinational circuits  
    adders, 1028–1030  
    ALUs, 1031–1033  
    buses, 1034–1036  
    decoders, 1021–1022  
    demultiplexers, 1022  
    description, 1007–1008  
    gates, 1013–1021  
    layers of abstraction, 1037–1039

- modules, 1034
- multiplexers, 1023
- overview, 1012
- sum-of-products, 1024–1027
- Comma-separated-value (.csv) files, 358, 360
- Command-line arguments, 7–8, 11, 127
- Commas (,)

  - arguments, 30
  - constructors, 333
  - lambda expressions, 450
  - methods, 30, 196
  - two-dimensional arrays, 108

- Comments, 5, 9
- Commercial data processing, 410–413
- Common sequences, longest, 285–288
- Commutative axiom, 990
- Compact trace format, 770
- Comparable interface, 451, 545
- Comparable keys

  - sorting, 546
  - symbol tables, 626–627

- CompareDocuments program, 462–463
- compareTo() method

  - description, 451
  - `String`, 332
  - user defined, 545–546

- Comparisons

  - arrays, 117
  - chained, 43
  - objects, 364, 545–546
  - operators, 27–29
  - performance, 508–509
  - sketches, 462–463

- Compatibility

  - backward, 976

- Color, 344
- Compile-time errors, 6
- Compilers

  - description, 3, 589
  - optimizing, 814
  - programs as data, 922–924
  - purpose, 788
  - TOY machine, 964–965

- Compiling

  - array values set at, 95–96, 108
  - classes in, 229
  - description, 2
  - programs, 3

- Complement operation

  - bitwise, 891
  - Boolean algebra, 990

- Complete small-world graphs, 694
- Complex program

  - chaining method calls, 404
  - encapsulation, 433–434
  - instance variables, 403–404
  - objects, 404
  - overview, 402–403
  - program, 405

- Complex numbers, 406–409
- Compound assignments, 60
- Compression, optimal, 814
- Computability

  - algorithms, 787
  - halting problem, 808–810
  - Hilbert’s program, 806–807
  - liar’s paradox, 807–808
  - overview, 806
  - reduction, 811–813
  - unsolvability proof, 810
  - unsolvable problems.

    - See* Unsolvable problems

- Computation: Finite and Infinite Machines, 780
- Computational biology, 732–734
- Computational models, 716
- Computer animations, 151
- Computer speed in performance, 507–508
- Computer systems, 1094–1095
- Computers and Intractability: A Guide to the Theory of NP-completeness* book, 859
- Computers Ltd.: What They Really Can’t Do* book, 780
- Computing devices

  - boolean logic. *See* Boolean logic
  - circuit models.

    - See* Circuit models

  - combinational circuits. *See* Combinational circuits
  - digital. *See* Digital devices
  - overview, 985
  - sequential circuits.

    - See* Sequential circuits

- Computing machines

  - machine-language programming. *See* Machine-language programming
  - overview, 873
  - representing information. *See* Representing information
  - TOY. *See* TOY machine

- Computing sketches, 459–460
- Concatenation

  - files, 356
  - strings, 19–20, 723–724

- Concert A, 155
- Concordances, 659
- Conditionals and loops, 50

  - applications, 64–73
  - break statement, 74
  - continue statement, 74
  - do-while loops, 75
  - examples, 61
  - for loops, 59–61
  - if statement, 50–53
  - infinite loops, 76

- miscellaneous, 74–75  
in modular programming, 227–228  
nesting, 62–64  
performance analysis, 500, 510  
static methods, 193–195  
summary, 77  
`switch` statement, 74–75  
TOY machine, 913, 918–921  
`while` loops, 53–59
- Connected components, 709
- Connecting programs, 141
- Connections
- buses, 1034
  - circuit models, 1002–1004
  - CPU, 1077
  - power source, 1003–1004
  - program counters, 1073–1075
- Constant order of growth, 503
- Constants, 16
- Constructors
- data types, 384–385
  - `String`, 333
- Containing symbol table keys, 624
- Context-free languages, 755
- Continue statements, 74
- Contracts
- APIs, 230–231
  - design by contract, 465–467
  - interface, 446–447
  - machine-language, 932
- Control characters, 894
- Control circuit
- CPU, 1078
  - execute signals, 1082–1083
  - fetch signals, 1080, 1082–1083
  - overview, 1080
- Control flow
- conditionals and loops.  
*See* Conditionals and loops
  - static method calls, 193–195
- Control lines
- CPU, 1077–1080
  - memory bits, 1056
  - multiplexers, 1019–1020
  - program counters, 1074–1075
  - register bits, 1051
- Controlled switches, 1002–1003, 1005–1006
- Conversion codes, 131–132
- Conversion specifications, 130–131
- Conversions
- casts, 33–34
  - color, 48–49
  - data types, 339
  - decimal to binary, 877
  - explicit, 34–35
  - hex and binary, 876–877
  - implicit, 33
  - numbers, 21, 67–69
  - overview, 32
  - strings, 21, 453, 880–881
- Convert program, 880–882
- Conway, John, 326, 794
- Cook, Stephen, 840, 845
- Cook–Levin theorem, 844–845, 847
- Cook reduction, 841
- Coordinates
- drawing, 144–146
  - images, 347
  - polar, 47
- Corner cases, 236
- Cosine similarity measure, 462
- Cost of immutable types, 440
- Coulomb's law, 383
- Counter circuits, 1008
- Counter machines, 794
- Counter program, 436–437
- Coupon collector problem, 101–103
- Coupon program, 206
- CouponCollector program, 101–103, 205
- CPUs. *See* Central processing units (CPUs)
- Craps game, 259
- Cray, Seymour, 971
- Crichton, Michael, 424
- Cross-coupled *NOR* gates, 1050
- Cross-coupled switches, 1049
- Cross products of vectors, 472
- Cryptographic keys, 992
- Cryptography application, 992–994
- Cryptosystems, 992–993
- <Ctrl-C> keys, 76
- <Ctrl-D> keys, 137
- <Ctrl-Z> keys, 137
- Cubic order of growth, 505–508
- Cumulative distribution function, 202–203
- Curly braces (`{}`)
- regular expressions, 724, 731
  - statements, 5, 78–79
  - static methods, 196
  - two-dimensional arrays, 108
- Curves
- Brownian bridges, 278–280
  - Dragon, 49, 424
  - Koch, 397
  - space-filling, 425
  - spirals, 398–399
- Cycles per second, 155
- D**
- Dantzig, George, 831
- Data abstraction, 329, 382
- Data as instructions, 922–924
- Data compression, 814
- Data-driven code, 141, 171, 184
- Data mining example, 458–459
- Data paths for buses, 1034

- Data structures, 493  
 arrays. *See* Arrays  
 binary search trees. *See* Binary search trees (BSTs)  
 linked lists, 571–578  
 queues. *See* Queues  
 resource allocation, 606–607  
 stacks. *See* Stacks  
 stock example, 411  
 summary, 608  
 symbol tables. *See* Symbol tables
- Data-type design  
 APIs, 429–431  
 data mining example, 458–464  
 design by contract, 465–467  
 encapsulation, 432–438  
 immutability, 439–446  
 subclassing, 452–457  
 subtyping, 446–451  
 overview, 428
- Data types  
 access modifiers, 384  
 APIs, 383  
**boolean**, 991  
 built-in. *See* Built-in data types  
 classes, 383  
**Color**, 341–345  
**Complex**, 402–405  
 constructors, 384–385  
 conversions, 34–35, 339  
 creating, 382  
 definitions, 331–335  
**DrunkenTurtle**, 400–401  
 elements summary, 383  
 generic, 583–585  
**Histogram**, 392–393  
 image processing, 346–352  
 immutable, 364, 439  
 input and output, 353–362  
 insertion sorts, 545–548  
 instance methods, 385–386
- instance variables, 384  
**Koch**, 397  
**Mandelbrot**, 406–409  
 output, 355  
 overview, 330  
 reference, 362–369  
**Spiral**, 398–399  
**StockAccount**, 410–413  
**Stopwatch**, 390–391  
**String**. *See* Strings and String  
 data type  
 summary, 368  
**TOY** machine, 907  
**Turtle**, 394–396  
 type safety, 18  
 variables within methods,  
 386–388
- Data visualization, 307–309  
 Davis, Martin, 816  
 Dead Sea Scrolls, 659
- Debugging  
 abstraction layers, 1037  
 assertions, 466–467  
 encapsulation for, 432  
 immutable types, 440  
 incremental, 317, 319  
 linked lists, 596  
 modular programming,  
 251–254  
 test client `main()` for, 235  
 unit testing, 246
- Decidability, 786–787
- Decimal number system  
 conversion to binary, 877  
 description, 38, 875  
 examples, 878–879
- Decision problems, **NP**, 835
- Decks of cards, 97–100
- Declaration statements, 15–16
- Declaring  
 arrays, 91, 116  
**String** variables, 333
- Decoders, 1021–1022  
 Decoding numbers, 889  
 Decrementers, binary, 770–771  
 Decryption devices, 992  
 Dedup operation  
 punched paper tape, 942–944  
 strings, 652–653
- DeDup** program, 652–653
- Default values  
 arrays, 93, 106–107  
 canvas size, 145  
 ink color, 150  
 instance variables, 415  
 Node objects, 572  
 pen radius, 146
- Defensive copies, 441
- Defining  
 functions, 192  
 interfaces, 446  
 static methods, 193, 196
- Definite integration, 816
- Degrees of separation  
 description, 670  
 shortest paths, 684–686
- DeMorgan’s laws, 989–991,  
 1014–1015
- Demultiplexers, 1022
- Denial-of-service attacks, 512
- Dependencies in subclasses, 453
- Dependency graphs, 252
- Deprecated methods, 469
- Depth-first searches  
 vs. breadth-first searches, 690  
 percolation case study, 312
- Deques, 618
- Derived classes, 452
- Descartes, René, 398
- Design  
 APIs, 233  
 by contract, 465–467  
 data types. *See* Data-type design

- Deterministic finite-state automata (DFAs)  
examples, 740–741  
implementation, 741–743  
Kleene’s theorem.  
    *See* Kleene’s theorem  
language recognized, 739–740  
NFA equivalence, 749–750  
nondeterminism, 744–748  
operations, 739  
overview, 738  
power limitations, 753–755  
summary, 756  
universal virtual, 788–789  
DFA program, 742–743  
Diameters of graphs, 711  
Diamond operators ( $\langle\rangle$ ), 585  
Dice  
    Sicherman, 259  
    simulation, 121  
Dictionary lookup, 624, 628–632  
Difficult problems  
    intractability, 828–829  
    search problems, 837–838  
Digital circuits, 1013  
Digital devices, 1070  
    control, 1080–1082  
    CPU, 1076–1080  
    program counters, 1073–1075  
Digital image processing  
    digital images, 346–347  
    fade effect, 351–352  
    grayscale, 347–349  
    overview, 346  
    scaling, 349–350  
Digital signal processing, 155, 158  
Dijkstra, Edsger  
    Dutch-national-flag problem, 564  
    goto statements, 926  
    two-stack algorithm, 587  
Dijkstra’s algorithm, 692  
Diophantine, 816  
Directed graphs, 711  
Directed percolation, 317  
Discrete distributions, 172  
Disjunctive normal forms, 996–997  
Distances of graph paths, 683,  
    687–688  
Distributive axiom, 990  
Divide-and-conquer approach  
    linearithmic order of growth, 504  
    mergesort, 550–551, 554  
Division  
    floating-point numbers, 24–26  
    integers, 22–23  
    polar representation, 433  
DivisorPattern program, 62–64  
DNA computers, 795  
DNS (domain name system), 629  
do-while loops, 75  
Documents, searching for, 464  
Dollar signs (\$) in REs, 731  
Domain name system (DNS), 629  
Domains, function, 192  
Dot products  
    function implementation, 209  
    vectors, 92, 442–443  
Double.parseDouble() method  
    calls to, 30–31  
    type conversion, 21, 34  
Double buffering drawings, 151  
double data type  
    conversion codes, 132  
    description, 14–15  
    input, 133  
    memory size, 513  
    overview, 24–26  
Double negation identity, 990  
Double negatives in gates,  
    1015–1016  
Double quotes ("")  
    escape sequences, 19  
    text, 5, 10  
Doublet game, 710  
Doubling hypotheses, 496, 498–499  
DoublingTest program, 496,  
    498–499  
Downscaling in image processing,  
    349  
Dragon curves, 49, 424  
Dragon program, 163  
Draw library, 361  
Drawings  
    recursive graphics, 276–277  
    standard. *See* Standard drawing  
DrunkenTurtle program, 400  
DrunkenTurtles program, 401  
Dumping virtual machines,  
    960–961  
Dutch-national-flag problem, 564  
Dynamic dictionaries, 628  
Dynamic dispatch, 448  
Dynamic programming  
    bottom-up, 285  
    longest common subsequence,  
        285–288  
    overview, 284  
    summary, 289  
    top-down, 284

## E

- Easy problems  
    intractability, 829  
    search, 837  
Eavesdroppers, 992–993  
Eccentricity in vertices, 711  
Eckert, J. Presper, 924–925  
Edges  
    graphs, 671, 674  
    self-loops and parallel, 676  
EDVAC computer, 924–925

Efficiency  
 n-body simulation, 488  
 random web surfer, 185  
 Turing machines, 772

Efficient algorithms, 532

Einstein, Albert, 400

Election voting machine errors, 436

Electric charge, 383–389

Element distinctness problem, 554

Elements in arrays, 90

`else` clauses, 51–52

Empirical analyses, 496–497

Empty strings with REs, 724

Emulators, 965

Encapsulation  
 code clarity, 438  
 error prevention, 436–437  
 example, 433–434  
 modular programming, 432  
 overview, 432  
 planning for future, 435  
 private access modifier, 433

Encoding numbers, 889

Encryption devices, 992

End-of-file sequence, 137

Enhancements for Turing machines, 792–793

ENIAC computer, 924–925

Enigma code, 717

Entropy  
 Shannon, 378  
 text corpus, 667–668

Equals signs (=)  
 assignment statements, 17  
 assignment vs. boolean, 42, 78  
 comparisons, 27–29, 364  
 compound assignments, 60  
 vs. `equals()`, 369–370

Equality of objects, 364, 454–456

`equals()` method  
`Color`, 343  
 vs. equals signs, 369–370  
`Object`, 453–455  
`String`, 332

Equilateral triangles, 144–145

Equivalence problem for REs, 728

Equivalent models for Turing machines, 792–793

Erdős, Paul, 686

Erdős–Renyi model, 695, 712

Errors  
 aliasing, 363  
 debugging. *See* Debugging  
 encapsulation for, 436–437  
 overview, 6  
 syntax, 10–11  
 testing for, 318

Escape characters, 730, 757

Escape sequences, 19

Euclidean distance  
 sketch comparisons, 462–463  
 vectors, 118

Euclid’s algorithm  
 description, 85  
 machine-language, 931  
 recursion, 267–268  
 TOY machine, 918–921

Euler, Leonhard, 89

Euler’s constant, 222

Euler’s sum-of-powers conjecture, 89

Euler’s totient function, 222

Evaluate program, 588–589

Evaluating expressions, 17, 586–589

Event-based programming, 451

`Exception` class, 465

Exception filters, 540

Exceptions, 465–467

Exchanging values  
 arrays, 96  
 function implementation, 209

Exclamation points (!)  
 not operator, 26–27, 991  
 comparisons, 27–29

Exclusive or operation  
 bitwise, 891–892, 913  
 boolean, 987–989  
 sum-of-products, 1024–1025

Execute phase in CPU, 1079

Explicit casts, 33–34

Exponential distributions, 597

Exponential order of growth, 505  
 difficult problems, 828–829  
 intractability, 826  
 overview, 272–273, 506  
 playing card possibilities, 823  
 running time, 507–508  
 SAT problem, 856  
 usefulness, 858

Expressions  
 arithmetic evaluation, 586–589  
 boolean, 995–996  
 description, 17  
 Lambda, 450  
 method calls, 30  
 regular. *See* Regular expressions

Extended Church–Turing thesis, 823

Extensible libraries, 452

`ExtractFloat` program, 893

Extracting data, 358, 360

## F

*Factor* problem, 838, 859

Factorials, 264–265

Factoring, 72–73, 827, 838

`Factors` program, 72–73

Fade effect, 351–352

`Fade` program, 351–352

- Fair coin flip, 52–53  
Falsifiable hypotheses, 495  
Fecundity parameter, 89  
Feedback circuits, 1048–1049  
Fermat’s Last Theorem, 89, 722  
Ferns, Barnsley, 240–243  
Fetch-increment-execute cycle, 910–911  
Fibonacci numbers  
  formulas, 82  
  machine language, 935–936  
  recursion, 282–283  
FIFO queues. *See* First-in first-out (FIFO) queues  
Files  
  concatenating and filtering, 356  
  format, 237  
  in I/O, 126  
  n-body simulation, 483  
  redirection, 139–141  
  splitting, 360  
  stock example, 411  
  symbol tables, 629  
Filled shapes, 149  
Filters  
  exception, 540  
  files, 356  
  image processing, 379  
  piping, 142–143  
  standard drawing data, 146–147  
  standard input, 140  
**final** keyword  
  description, 384  
  immutable types, 440  
  instance variables, 404  
Financial systems, graphs for, 673  
Finite-state transducers, 762  
Finite sums, 64–65  
First-in first-out (FIFO) queues  
  applications overview, 597  
  array implementation, 596  
linked-list implementation, 593  
*M/M/1*, 597–600  
  overview, 566, 592–593  
Flexibility, 702  
**Flip** program, 52–53  
Flip-flops, 1049–1050  
**float** data type, 26, 513  
Floating-point numbers  
  conversion codes, 131–132  
  exponents, 889  
  overview, 24–26  
  precision, 40  
  representing, 888–890  
  storing, 40  
Flow of control  
  conditionals and loops. *See* Conditionals and loops  
  static method calls, 193–195  
Flowcharts, 51–52  
**for** loops  
  **continue** statement, 74  
  examples, 61  
  nesting, 62–64  
  working with, 59–61  
Foreach statements, 601–602  
Formal languages  
  abstract machines, 737–738  
  alphabets, 720–721  
  binary strings, 718–719  
  definitions, 718–723  
  DFAs. *See* Deterministic finite-state automata (DFAs)  
  recognition problem, 722  
  regular, 723–729  
  regular expressions. *See* Regular expressions (REs)  
  specification problem, 722  
Format, files, 237  
Format strings, 130–131  
Formatted input, 135  
Formatted printing, 130–132  
Forth language, 590  
Fortran language, 1094  
Fourier series, 211  
Fractal dimensions, 280  
Fractals, 278–280  
Fractional Brownian motion, 278  
Fractions, 889–890  
Fragile base class problem, 453  
Freeing memory, 367  
Frequencies  
  counting, 555  
  sorting, 556  
  Zipf’s law, 556  
**FrequencyCount** program, 555–557  
Fully parenthesized arithmetic expressions, 587  
Function calls  
  abstraction, 590–591  
  static methods, 197  
  traces, 195  
  trees, 269, 271  
Function graphs, 148, 248  
Functional interfaces, 450  
Functional programming, 449  
Functional property of programs, 812–813  
Functions  
  boolean, 987–991, 994–997  
  computing with, 449  
  defining, 192  
  inverting, 536–538  
  iterated function systems, 239–243  
  libraries. *See* Libraries  
  machine language, 931–933  
  mathematical, 202–204  
  modules. *See* Modules  
  overview, 191  
  recursive. *See* Recursion  
  static methods, 193–201  
  tables of, 907–908

**G**

Gambler program, 70–71  
 Gambler’s ruin simulation, 69–71  
 Game of Life, 326, 794  
 Garbage collection, 367, 516  
 Gardner, Martin, 424  
 Garey, Michael R., 859  
 Gates  
     abstraction layers, 1037  
      $AND$ , 1014  
     circuits from, 1019–1021  
     multiway, 1015–1017  
      $NOR$ , 1014  
      $NOT$ , 1013–1014  
      $OR$ , 1014  
     overview, 1013  
     sum-of-products, 1026–1027  
     summary, 1018–1019  
     universal sets of, 1045  
 Gaussian distribution functions  
     API, 231  
     cumulative, 202–203  
     probability density, 202–203  
 Gaussian elimination, 830  
 Gaussian program, 203  
 Gaussian random numbers, 47  
 General purpose computers, 790  
 Generalized multiway gates,  
     1016–1017  
 Generalized regular expressions,  
     730–732  
 Generic types, 583–585  
 Genomics  
     application, 336–340  
     indexing, 634  
     regular expressions, 727,  
         732–734  
     symbol tables, 629  
 Geometric mean, 162

**Geometry**

abstraction layers, 1037–1039  
 gates, 1015–1016  
 German Enigma code, 717  
 Get operations  
     hash tables, 639  
     symbol tables, 624  
 Gilbert–Shannon–Reeds model,  
     125  
 Glass filters, 379  
 Global clustering coefficients, 713  
 Global variables, 284  
 Glossary of terms, 1097–1101  
 Gödel, Kurt, 807, 840  
 Golden ratio, 83  
 Goldstine, Herman, 925  
 Gore, Al, 436  
 Gosper, R., 805  
 Goto statements, 926  
 Graph data type, 675–679  
 Graph program, 676–679  
 Graphics  
     recursive, 276–277, 397  
     turtle, 394–396  
 Graphs  
     bipartite, 682  
     client example, 679–682  
     connected components, 709  
     dependency, 252  
     description, 671  
     DFAs, 738  
     diameters, 711  
     directed, 711  
     examples, 695  
     function, 148, 248  
     generators, 700  
     Graph data type, 675–679  
     grid, 708  
     isomorphism problem, 859  
     lessons, 700–702  
     matching, 713

overview, 670–671

random web surfer, 170  
 small-world, 693–699  
 systems examples, 671–674  
 vertex cover, 828, 834, 842

**Gravity**

Gray codes, 273–275

**Grayscale**

Color, 344  
 image processing, 347–349  
 Grayscale program, 347–349  
 Greater than signs ( $>$ )  
     bitwise operations, 891–892  
     comparisons, 27–29  
     lambda expressions, 450  
      redirection, 139–140  
 Greatest common divisor (gcd)  
     machine language, 931  
     recursive algorithm, 267–268  
     TOY machine, 918–921

**Grep program**

grep tool  
     filters, 142–143  
     regular expressions, 734–736  
 Grid graphs, 708  
 Guarantees  
     NP-complete problems, 852  
     performance, 512, 627  
     worst-case analysis, 825

**H**

H-trees of order  $n$ , 276–277  
 Hadamard matrices, 122  
 Halt instructions  
     CPU, 1079  
     TOY machine, 912  
 Halting problem, 808–810  
 Hamilton, William, 424  
 Hamming distances, 295  
 Handles for pointers, 371

Hardy, G. H., 86  
Harel, David, 780  
Harmonic mean, 162  
Harmonic numbers  
  finite sums, 64–65  
  function implementation, 199  
Harmonic program, 193–195  
HarmonicNumber program, 64–65  
Harmonics and chords, 211  
Hash codes and hashing operation  
  object equality, 454–455  
  sketches, 460  
  strings, 515  
  symbol tables, 624  
Hash functions, 636  
Hash tables, 636–639  
Hash values, 636  
Hashable keys, 626  
hashCode() method  
  Object, 453, 455–456  
  String, 332  
HashMap class, 655  
HashST program, 637–638  
Heap memory, 516  
Heap-ordered binary trees, 661  
Height in binary search trees, 640  
HelloWorld program, 4–6  
Hertz, 155  
Hexadecimal (hex) notation  
  conversions with binary, 876–877  
  description, 875–876  
  examples, 878–879  
  literals, 891  
  memory, 909  
Hilbert, David, 425, 806, 816  
Hilbert curves, 425  
Hilbert’s 10th problem, 816  
Hilbert’s program, 806–807  
Histogram program, 392–393  
Histograms, 177

Hoare, C. A. R., 518  
Hollywood numbers, 711  
Horner, William, 957  
Horner’s method, 223, 882,  
  956–957  
Htree program, 276–277  
Humanists, 716–717  
Hurst exponent, 280  
Hyperbolic functions, 256  
Hyperlinks, 170  
Hypotenuse of right triangles, 199  
Hypotheses  
  doubling, 496, 498–499  
  falsifiable, 495  
  mathematical analysis, 498,  
    500–502  
  overview, 496

**I**

I/O. *See* Input; Output  
Identifiers, 15–16  
Identities  
  Boolean algebra, 989–990  
  exclusive or function, 993  
  objects, 338, 340  
IEEE 754 standard, 40, 888–889  
if statements  
  nesting, 62  
  working with, 50–53  
IFS program, 241, 251  
IllegalFormatException, 131  
ILP problem (integer linear programming problem), 831  
NP-completeness, 846  
  vertex cover problem, 842  
Immutable types, 364, 439  
  advantages, 440  
  arrays and strings, 439–440  
  cost, 440  
  example, 442–445

final modifier, 440  
references, 441  
symbol table keys, 625, 655  
Implementation  
  API methods, 231  
  interfaces, 447  
Implements clause, 447  
Implicit type conversions, 33  
In data type, 354–356  
Incremental development, 319, 701  
Incrementers, binary, 769–771  
Index program, 632–634  
IndexGraph program, 680–682  
Indexing  
  arrays, 90, 116  
  String, 332  
  symbol tables, 624, 632–634  
  zero-based, 92  
Induced subgraphs, 705  
Induction  
  mathematical, 262, 266  
  recursion step, 266  
Infinite loops, 76, 808–812  
Infinite tape for Turing machines,  
  769, 774  
Infinity value, 26, 40  
Information content of strings, 378  
Information representation. *See*  
  Representing information  
Inheritance  
  multiple, 470  
  subclassing, 452–457  
  subtyping, 446–451  
Initialization  
  array, 93  
  inline, 18  
  instance variables, 415  
  two-dimensional array, 106–107  
Inline variable initialization, 18  
Inner classes, 609

- Inner loops
  - description, 62
  - performance, 500, 510
- Inorder tree traversal, 649
- Input**
  - arithmetic logic units, 1031
  - array libraries, 237–238
  - circuit models, 1002–1004
  - clocks, 1060
  - command-line arguments, 7
  - data types, 353
  - demultiplexers, 1022
  - file concatenation, 356
  - gates, 1013
  - insertion sorts, 548–549
  - machine-language, 936–938
  - multiplexers, 1019–1020
  - overview, 126–129
  - in performance, 510
  - program counters, 1073–1075
  - random web surfer, 171
  - screen scraping, 357–359
  - standard, 132–138
  - stream data type, 354–355
  - virtual machines, 969–970
- Input/off switches, 1005
- InputMismatchException**, 135
- Inserting**
  - BST nodes, 644–645
  - linked list nodes, 573–574
- Insertion** program, 546–547
- Insertion sorts**
  - data types, 545–548
  - input sensitivity, 548–549
  - overview, 543–544
  - performance, 544–545
- InsertionDoublingTest** program, 548–549
- Instance methods
  - data types, 385–386
  - invoking, 334
  - vs. static, 340
- Instance variables
  - Complex program, 403–404
  - data types, 384
  - initial values, 415
- Instances of objects, 333
- Instruction register (IR), 910
- Instructions
  - components, 911
  - CPU, 1079–1080
  - as data, 922–924
  - execution time, 509
  - instruction sets, 911–913
  - parsing, 966–967
  - TOY machine, 909
  - TOY-8 machine, 1070–1071
- Integer linear inequality
  - satisfiability, 831, 838, 845
- Integer linear programming, 831
  - NP**-completeness, 846
  - vertex cover problem, 842
- Integer.parseInt()** method
  - calls to, 30–31
  - type conversion, 21, 23, 34
  - strings, 880–882
- Integers and `int` data type
  - arithmetic, 884–885
  - bitwise operations, 891–892
  - conversion codes, 131–132
  - description, 14–15
  - input, 133–134
  - overview, 22–24
- Integrals, approximating, 449
- Integrated development environments (IDEs), 3
- Integration, definite, 816
- Interactions between modules, 319
- Interactive user input, 135–136
- Interface construct, 446
- Interfaces
  - APIs, 430
  - built-in, 451
  - circuit models, 1003
  - CPU, 1076
  - defining, 446
  - functional, 450
  - gates, 1016–1017
  - implementing, 447
  - memory, 1054
  - multiplexers, 1020
  - program counters, 1073
  - using, 447–448
- Internet DNS, 629–630
- Internet Protocol (IP), 435
- Interpolation in fade effect, 351
- Interpreters
  - Evaluate program, 589
  - TOY machine, 964
- IntOps** program, 23
- Intractability
  - difficult problems, 828–829
  - easy problems, 829
  - exponential-time algorithms, 826
  - main question, 840–841
  - NP**-completeness.
    - See **NP**-completeness
  - numbers, 827
  - overview, 822–824
  - path problems, 829
  - polynomial-time algorithms, 825–826
  - polynomial-time reductions, 841–843
  - problem size, 824
  - satisfiability, 830–832
  - search problems, 833–840
  - subset sum problem, 827–828
  - vertex cover, 828
  - worst case, 825

*Introduction to the Theory of Computation* book, 780  
 Invariants in assertions, 467  
 Inverse permutations, 122  
 Inverters, 1013–1014  
 Inverting functions, 536–538  
 Invoking instance methods, 334  
 IP (Internet Protocol), 435  
 IPv4  
     vs. IPv6, 435  
     number of addresses, 900, 904  
 IPv6  
     vs. IPv4, 435  
     number of addresses, 901  
 IR (instruction register), 910  
 IR write control line, 1082  
 ISBN (International Standard Book Number), 86  
 Isolated vertices in graphs, 703  
 Isomorphic binary trees, 661  
 Isomorphism in graphs, 859  
 Items in collections, 566  
**Iterable** interface, 451, 602  
 Iterable collections, 601–605  
     arrays, 603  
     linked lists, 604–605  
     Queue, 604–605  
     SET, 652  
     Stack, 603  
 Iterated function systems, 239–243  
 Iterations in BSTs, 650  
**Iterator** interface, 451, 602–605

**J**

Java command, 3, 134  
 .java extension, 3, 6, 8, 197, 383  
 Java language  
     benefits, 9  
     libraries, 1094  
     overview, 1–8  
 Java platform, 2

Java Virtual Machine (JVM)  
     description, 3  
     overview, 965–966  
     as program, 788  
 Java virtual machines, 429  
 Johnson, David S., 859  
 Josephus problem, 619  
 Julia sets, 427  
 Jump and link instruction, 931  
 Jump register instruction, 931

**K**

K-ring graphs, 694–695  
 K-way multiplexers, 1019–1020  
 Kamasutra ciphers, 377  
 Karp, Richard, 845–848  
 Karp's reductions  
**NP**-completeness, 845–848  
     polynomial-time, 841  
 Kevin Bacon game, 684–686  
 Key-sorted tree traversal, 649  
 Keys  
     BSTs, 640–642, 650  
     cryptographic, 992  
     immutable, 625  
     Kamasutra ciphers, 377  
     symbol tables, 624–626, 655  
 Key–value pairs, 624–626  
 Kleene, Stephen, 748  
 Kleene's theorem  
     applications, 753–756  
     DFA, NFA, and RE equivalence, 749–752  
     overview, 748  
     power limitations, 753–756  
     proof strategy, 748  
     RE recognition, 753  
 Knuth, Donald  
     MIX machine, 947  
     optimization, 518  
     running time, 496, 501  
     SAT solvers, 832

Koch program, 397

**L**

Ladders, word, 710  
 Ladner, R., 859  
 Lambda calculus, 790, 794  
 Lambda expressions, 450  
 Languages. *See* Formal languages; Programming languages  
 Last-in first-out (LIFO), 566–567  
 Lattices in random walks, 112–115  
 Layers of abstraction, 1037–1039  
 LCS (longest common subsequence), 285–288  
 Leading zeros, 883  
 Leaf nodes in BSTs, 640  
 Leaks, memory, 367, 581  
 LeapYear program, 28–29  
 Left associativity, 17  
 Left shift operations  
     bitwise, 891–892  
     TOY machine, 913  
 Left subtrees, 640  
 Length  
     arrays, 91–92  
     graphs paths, 674, 683  
     strings, 332  
 Less than signs (<)  
     bitwise operations, 891–892  
     comparisons, 27–29  
     redirection, 140–141  
 Let's Make a Deal simulation, 88  
 Levin, Leonid, 845  
 Liar's paradox, 807–808  
 Libraries  
     APIs, 230–232  
     array I/O, 237–238  
     clients, 230  
     extensible, 452  
     Java, 1094  
     methods, 29–32

- modifying, 255  
in modular programming, 227–228, 251–254  
modules, 191  
overview, 226, 230  
random numbers, 232–236  
statistics, 244–250  
stress testing, 236  
unit testing, 235
- LIFO (last-in first-out), 566–567  
Lights for TOY machine, 916  
Lindenmayer systems, 803  
Linear algebra for vectors, 442–443  
Linear equation satisfiability problem, 830, 839  
Linear feedback shift registers (LFSRs), 1000–1001  
Linear inequality satisfiability problem, 831, 839  
Linear interpolation, 351  
Linear order of growth, 504–505, 507–508  
Linear programming problem, 831  
Linearithmic order of growth, 504–505, 507–508  
Linked lists  
  circular, 622  
  FIFO queues, 593, 596  
  hash tables, 636  
  iterable classes, 604–605  
  overview, 571–574  
  stacks, 574–576  
  summary, 578  
  symbol tables, 635  
  traversal, 574, 577
- Linked structures. *See* Binary search trees (BSTs)
- LinkedStackOfStrings** program, 574–576
- Links in BSTs, 640–642  
Lipton, R. J., 856
- Lissajous, Jules A., 168  
Lissajous patterns, 168  
Lists, linked. *See* Linked lists
- Literals  
  array elements, 116  
  binary and hex, 891  
  booleans, 26  
  characters, 18–19  
  description, 15  
  floating-point numbers, 24  
  integers, 22  
  strings, 19, 334
- Little’s law, 598
- Load address instruction, 1080  
Load instructions, 938, 1080  
**LoadBalance** program, 606–607  
Local clustering, 693–694  
Local variables  
  vs. instance variables, 384  
  static methods, 196
- Logarithmic order of growth, 503  
Logarithmic spirals, 398–399  
Logical design, 1008–1009  
Logical instructions, 912–913  
Logical shifts, 891–892  
Logical switches  
  bus muxes, 1036  
  demultiplexers, 1022  
  multiplexers, 1020
- Logo language, 400  
Loitering condition, 581  
Long data type, 24, 513  
Long path problems, 829  
Longest common subsequence (LCS), 285–288  
Longest path problem, 838  
**LongestCommonSubsequence** program, 286–288  
Lookup program, 630–632  
Loops. *See* Conditionals and loops  
Lost letter, 840
- Lower bounds, 826  
Luminance, 343–345  
**Luminance** program, 344–345
- ## M
- M/M/1* queues, 597–600  
MAC addresses, 877  
Machine-language programming  
  arrays, 938–941  
  benefits, 945  
  description, 907  
  functions, 931–933  
  overview, 930  
  standard input, 936–938  
  standard output, 934–936  
  summary, 945–946  
  TOY machine, 914
- Magnitude  
  complex numbers, 402–403  
  spatial vectors, 442–443
- Magritte, René, 363  
**main()** methods, 4–5  
  multiple, 229  
  transfer of control, 193–194
- Majority function  
  adder circuits, 1028–1030  
  sum-of-products circuits, 1027  
  truth tables for, 1025
- Mandelbrot, Benoît, 297, 406  
**Mandelbrot** program, 406–409  
Maps, Mercator projections, 48  
Markov, Andrey, 176  
Markov chains  
  impact, 184  
  mixing, 179–184  
  overview, 176  
  power method, 180–181  
  squaring, 179–180
- Markov model paradigm, 460  
**Markov** program, 180–182

- Markov systems, 802–803  
Markovian queues, 597  
Marsaglia’s method, 85, 259  
Masking bitwise operations, 892–893  
**Matcher** class for REs, 763  
Matching graphs, 713  
**Math** library, 192  
    accessing, 228  
    methods, 30–32, 193, 198  
Mathematical analysis, 498–502  
Mathematical functions, 202–204  
Mathematical induction, 262, 266  
Mathematical models, 716  
Matiyasevich, Yuri, 816  
Matlab language, 1094  
Matrices  
    boolean, 302  
    Hadamard, 122  
    images, 346–347  
    matrix multiplication, 109  
    sparse, 666  
    transition, 172–173  
    two-dimensional arrays, 106, 109–110  
        vector multiplication, 110, 180  
Mauchly, John, 924–925  
Maximum values in arrays, 209  
Maximum keys in BSTs, 651  
Maxwell–Boltzmann distributions, 257  
McCarthy’s 91 function, 298  
Mechanical systems, graphs for, 673  
Memoization, 284  
Memory  
    arrays, 91, 94, 515–517  
    **ArrayStackOfStrings**, 569–570  
    available, 520  
    bit-slice design, 1054–1056  
    circuits, 1054–1057  
    feedback loops as, 1048  
    flip-flops, 1049–1050  
    interfaces, 1054  
    leaks, 367, 581  
    linked lists, 571  
    memory bits, 1056  
    objects, 338, 514  
    performance, 513–517  
    recursion, 282  
    references, 367  
    safe pointers, 366  
    strings, 515  
    TOY machine, 908–909  
    two-dimensional arrays, 107  
        virtual, 972, 975–976  
Memory dumps, 909  
Memory instructions  
    address instructions, 912  
    TOY machine, 913  
Memory writes for CPU, 1079  
Memoryless queues, 597  
Mercator projections, 48  
**Merge** program, 550–552  
Mergesort  
    divide-and-conquer, 554  
    overview, 550–552  
    performance, 553  
Metacharacters, 724, 730–731  
Method references, 470  
Methods  
    abstract, 446  
    call chaining, 404  
    deprecated, 469  
    instance, 334, 385–386  
    instance vs. static, 340  
    library, 29–32  
    **main()**, 4–5  
    overriding, 452  
    static. *See Functions; Static methods*  
    stub, 303  
    variables within, 386–388  
MIDI Tuning Standard, 161  
Midpoint displacement method, 278, 280  
Milgram, Stanley, 670  
Minimum keys in BSTs, 651  
Minsky, Marvin, 780, 794  
Minus signs (-)  
    compound assignments, 60  
    floating-point numbers, 24–26  
    integers, 22  
    lambda expressions, 450  
MIX machine, 947  
Mixed-type operators, 27–29  
Mixing Markov chains, 176, 179–184  
**MM1Queue** program, 598–600  
Models  
    circuit. *See Circuit models*  
    computational, 716  
    mathematical, 716  
    universal, 794–797  
Modular programming, 191  
    classes in, 227–229  
    code reuse, 226, 253  
    debugging, 253  
    encapsulation, 432  
    flow of control in, 227–228  
    libraries in, 251–254  
    machine language, 932  
    maintenance, 253  
    program size, 252–253  
Modules  
    abstraction layers, 1037  
    as classes, 228  
    CPU, 1076  
    description, 1034  
    interactions, 319  
    overview, 191  
    program counters, 1073  
    size, 319  
    summary, 254

- Monochrome luminance, 343–344  
 Monte Carlo simulation, 300, 307–308  
 Moore’s Law  
     coping with, 971  
     description, 507–508  
 Move-to-front strategy, 620  
 Movie–performer graph, 680  
 Multidimensional arrays, 111  
 Multiple arguments, 197  
 Multiple inheritance, 470  
 Multiple `main()` methods, 229  
 Multiple `return` statements, 198  
 Multiple I/O streams, 143  
 Multiplexers  
     bus switching, 1035  
     description, 1023  
     selection, 1019–1020  
 Multiplication  
     complex numbers, 402–403  
     floating-point numbers, 24–26  
     integers, 22–23, 885  
     matrices, 109–110  
     P search problems, 839  
     polar representation, 433  
 Multiway gates, 1015–1017, 1023  
 Music, 155–159  
 Mutable types, 364, 439
- N**
- N-body simulation  
     Body data type, 479–480  
     file format, 483  
     force, 480–482  
     overview, 478–479  
     summary, 488  
     Universe data type, 483–487  
 Names  
     arrays, 91  
     methods, 5, 30, 196  
     objects, 362
- variables, 16  
 vertices, 675  
 NaN value, 26, 40  
**NAND** function, 989–991  
 Nash, John, 840  
 Natural numbers, 875  
 Natural recursion, 262  
 Negation axiom, 990  
 Negative numbers  
     array indexes, 116  
     representing, 38, 886–888  
 Neighbor vertices, 671  
 Nested classes  
     iterators, 574  
     linked lists, 603–605  
 Nesting conditionals and loops, 62–64  
 new keyword  
     constructors, 385  
     Node objects, 609  
     String objects, 333  
 Newcomb, Simon, 224  
 Newline characters (`\n`)  
     compiler considerations, 10  
     escape sequences, 19  
 Newton, Isaac  
     dice question, 88  
     motion simulation, 478–479  
     square root method, 65  
 Newton’s law of gravitation, 481  
 Newton’s method, 65–67  
 Newton’s second law of motion, 480–481  
 NFAs. *See* Nondeterministic finite-state automata (NFAs)  
 90–10 rule, 170, 176  
 Nodes  
     BSTs, 640–642, 942  
     linked lists, 571–573  
     new keyword, 609
- Nondeterministic finite-state automata (NFAs)  
     DFA equivalence, 749–750  
     Kleene’s theorem.  
         *See* Kleene’s theorem  
     overview, 744  
     RE equivalence, 750–751  
     recognition problem, 744–745  
     trace example, 747  
 Nondominant inner loops, 510  
**NOR** function, 989–991  
**NOR** gates  
     cross-coupled, 1050  
     description, 1014  
 Normal distribution functions  
     cumulative, 202–203  
     probability density, 202–203  
**NOT** gates, 1013–1014  
 Not operation, 26–27, 987–989  
**NP**-completeness  
     addressing problems, 852  
     boolean satisfiability, 853–856  
     classifying problems, 851  
     Cook–Levin theorem, 844–847  
     coping, 850–857  
     Karp’s reductions, 845–848  
     overview, 843–844  
     proving, 844–849  
**NP**-hard problems, 858  
**NP** search problems  
     difficult, 837  
     easy, 837  
     main question, 840–841  
     nondeterminism, 835  
     overview, 833  
     solutions, 835  
     subset sum, 834  
     TSP problem, 862  
     vertex cover problem, 834, 842  
     0/1 ILP problem, 835  
 Null calls, 312

- Null keys in symbol tables, 626  
Null links in BSTs, 640  
Null nodes in linked lists, 571–572  
`null` keyword, 415  
Null transitions in NFAs, 744–746  
Null values in symbol tables, 626  
`NullPointerException`, 370  
Numbers  
    conversions, 21, 67–69, 880–881  
    intractability, 827  
    negative, 886–888  
    real, 888–890  
Numerical integration, 449  
Nyquist frequency, 161
- O**
- `Object` class, 453–455  
Object-oriented programming  
    data types. *See* Data types  
    description, 254  
    overview, 329  
Objects  
    arrays, 365  
    collections, 582–583  
    comparing, 364, 545–546  
    Complex, 404  
    equality, 454–456  
    memory, 514  
    names, 362  
    orphaned, 366  
    references, 338–339  
    `String`, 333–334  
    type conversions, 339  
    uninitialized variables, 339  
    working with, 338–339  
Observations, 495–496  
Occam’s Razor, 814  
Octal representation, 898  
Odd parity function  
    adder circuits, 1028–1030  
    sum-of-products circuits, 1026  
    truth tables for, 1026
- Off-by-one errors, 92  
Offscreen canvas, 151  
Offset binary representation, 889  
*On computable numbers, with an application to the Entscheidungsproblem* article, 717  
On/off switches, 1005  
One-dimensional arrays, 90  
One-hot *OR* gates, 1023  
Onscreen canvas, 151  
Opcodes, 911  
Operands, 17  
Operators and operations  
    boolean, 26–27, 989–991  
    comparisons, 27–29, 364  
    compound assignments, 60  
    data types, 14, 331  
    description, 15  
    expressions, 17, 587  
    floating-point numbers, 24  
    integers, 22, 891  
    lambda, 450  
    overloading, 416  
    precedence, 17  
    reverse Polish notation, 590  
    stacks, 590  
    strings, 19, 21, 334, 453  
    TOY machine, 906  
Optimal data compression, 814  
Optimization  
    *NP* problems, 835  
    premature, 518  
Optimizing compilers, 814  
*OR* function, 987–989  
*OR* gates, 1014, 1023  
Or operation  
    bitwise, 891–892  
    `boolean` type, 26–27  
    TOY machine, 913  
Order in BSTs, 640, 642–643  
Order statistics, 651  
Order-of-growth classifications  
constant, 503  
cubic, 505–508  
exponential, 505–508  
linear, 504–505, 507–508  
linearithmic, 504–505, 507–508  
logarithmic, 503  
overview, 503  
performance analysis, 500–501  
quadratic, 504–505, 507–508  
Ordered operations  
    binary search trees, 651  
    symbol tables, 624  
Orphaned objects, 366  
Orphaned stack items, 581  
Out library, 355–356  
Outer loops, 62  
Outline shapes, 149  
Output  
    arithmetic logic units, 1032  
    array libraries, 237–238  
    circuit models, 1002–1004  
    clocks, 1059–1060  
    data types, 353  
    file concatenation, 356  
    gates, 1013  
    machine language, 934–936  
    print statements, 8  
    `printf()` method, 126–129  
    standard, 127, 129–132  
    standard audio, 155–159  
    standard drawing.  
        *See* Standard drawing  
    stream data types, 355  
    two-dimensional arrays, 107  
    virtual machines, 969–970  
Overflow  
    arithmetic, 885  
    arrays, 95  
    attacks, 963  
    guarding against, 898  
    integers, 23  
    negative numbers, 38

Overhead for objects, 514

Overloading  
operators, 416  
static methods, 198

Overriding methods, 452

## P

*The P=NP Question and Gödel's Lost Letter* book, 856

P search problems, 837  
examples, 839  
main question, 840–841

Padding object memory, 514

Page, Lawrence, 184

Page ranks, 176–177

Palindromes  
description, 719  
Watson–Crick, 374

Paper size, 294

Paper tape, 934–938

Papert, Seymour, 400

Parallel arrays, 411

Parallel edges, 676

Parameter variables

lambda expressions, 450  
static methods, 196–197, 207

Parameterized data types, 582–586

Parameters

in performance, 511  
TOY-8 machine, 1070

Parentheses ()

casts, 33  
constructors, 333, 385  
expressions, 17, 27  
functions, 24, 197  
lambda expressions, 450  
methods, 30, 196  
operator precedence, 17  
regular expressions, 724  
stacks, 587, 590  
static methods, 196

vectors, 442

Parity in ripple-carry adders, 1028

Parsing  
instructions, 966–967  
strings, 880–882

Pascal's triangle, 125

Passing arguments  
references by value, 364–365  
static methods, 207–210

PathFinder program, 683–686,  
690–692

Paths

graphs, 674, 683–692  
intractability problems, 829  
shortest. *See* Shortest paths  
simple, 710

Pattern class for REs, 763

PCs. *See* Program counters (PCs)

PDA (pushdown automata),  
755–756

PDP-8 computers, 906

Peaks in terrain analysis, 167

Pell's equation, 869

Pens

color, 150  
drawings, 146

Pepys, Samuel, 88

Pepys problem, 88

Percent signs (%)

conversion codes, 131–132  
remainder operation, 22–23

Percolation case study

adaptive plots, 314–318  
lessons, 318–320  
overview, 300–301  
**Percolation**, 303–304  
**PercolationPlot**, 315–317  
**PercolationProbability**,  
310–311  
**PercolationVisualizer**,  
308–309

probability estimates, 310–311

recursive solution, 312–314

scaffolding, 302–304

testing, 305–308

vertical percolation, 305–306

Performance

binary search trees, 647–648

binary searches, 535

caveats, 509–511

comparing, 508–509

guarantees, 512, 627

hypotheses, 496–502

importance, 702

insertion sorts, 544–545

memory use, 513–517

mergesort, 553

multiple parameters, 511

order of growth, 503–506

overview, 494–495

perspective, 518

prediction, 507–509

scientific method, 495–502

shortest paths, 690

wrapper types, 369

Performer program, 697–699

Periods (.)

classes, 227

regular expressions, 724

Permutations

inverse, 122

sampling, 97–99

Phase transitions, 317

Phone books, 628

Photographs, 346

Physical systems, graphs for, 672

Pi constant, 31–32

Picture library, 346–347

Piecewise approximation, 148

Pigeonhole principle, 754–755

- Piping  
  connecting programs, 141  
  filters, 142–143
- Pixels in image processing, 346
- Plasma clouds, 280
- Playing card possibilities, 823
- PlayThatTune program, 157–158
- PlayThatTuneDeluxe program, 213–215
- PlotFilter program, 146–147
- Plotting  
  array values, 246–248  
  experimental results, 249–250  
  function graphs, 148, 248  
  percolation case study, 314–318  
  sound waves, 249
- Plus signs (+)  
  compound assignments, 60  
  floating-point numbers, 24–26  
  integers, 22  
  regular expressions, 731  
  string concatenation, 19–20
- Pointers  
  array elements, 94  
  handles, 371  
  object references, 338  
  safe, 366
- Poisson processes, 597
- Polar coordinates, 47
- Polar representation, 433–434
- Polling, statistical, 167
- Polymorphism, 448
- Polynomial time, 823
- Polynomial-time algorithms  
  intractability, 825–826  
   $P$  search problems, 837, 839  
  usefulness, 858
- Polynomial-time reductions, 841–843
- Pop operation  
  reverse Polish notation, 590–591
- in stacks, 567–568
- Positional notation, 875
- Post, Emil, 813–814
- Post correspondence problem, 813–814
- Postconditions in assertions, 467
- Postfix notation, 590
- Postorder tree traversal, 649
- PostScript language, 400, 590
- PotentialGene program, 336–337
- Pound signs (#), 769
- Power method, 180–181
- Power source, 1003–1004
- PowersOfTwo program, 56–58
- Precedence  
  arithmetic operators, 17  
  regular expressions, 724
- Precision  
  floating-point numbers, 25, 40  
  printf(), 130–131  
  standard output, 129–130
- Precomputed array values, 99–100
- Preconditions in assertions, 467
- Prediction, performance, 507–509
- Preferred attachment process, 713
- Prefix-free strings, 564
- Premature optimization, 518
- Preorder tree traversal, 649
- Primality-testing function, 198–199
- Prime numbers  
  in factoring, 72–73  
  Sieve of Eratosthenes, 103–105
- PrimeSieve program, 103–105
- Primitive data types, 14  
  memory size, 513  
  overflow checking, 39  
  performance, 369  
  wrappers, 457
- Principle of superposition, 483
- print() method, 31  
  arrays, 237–238  
  impurity, 32  
  Out, 355  
  vs. println(), 8  
  standard output, 129–130
- Print statements, 5
- printf() method, 129–132, 355
- Printing, formatted, 130–132
- println() method, 31  
  description, 5  
  impurity, 32  
  Out, 355  
  vs. print(), 8  
  standard output, 129–130  
  string concatenation, 20
- private keyword  
  access modifier, 384  
  encapsulation, 433
- Probabilities, 308, 310–311
- Probability density function, 202–203
- Problem reduction  
  overview, 811  
  program equivalence, 812  
  Rice’s theorem, 812–813  
  totality problem, 811–812
- Problem size in intractability, 824
- Procedural programming style, 329
- Program counters (PCs)  
  bus connections, 1073–1074  
  connections and timing, 1075  
  control lines, 1074–1075  
  interfaces, 1073  
  modules, 1073  
  overview, 1073  
  TOY machine, 910
- Program equivalence problem, 812
- Program size, 252–253
- Programming environments, 1094

- Programming languages  
 indexing, 634  
 stack-based, 590  
 symbol tables, 629
- Programming overview, 1  
*HelloWorld* example, 4–6  
 input and output, 7–8  
 process, 2–3
- Programs  
 connecting, 141  
 processing programs, 788–790, 964–966
- Proof by contradiction, 754
- Pseudo-code, 911
- public** keyword  
 access modifiers, 384  
 description, 228  
 static methods, 196
- Pulses, clock, 1058
- Punched cards, 940
- Punched paper tape, 934–938
- Pure functions, 201
- Pure methods, 32
- Push operation  
 reverse Polish notation, 590–591  
 stacks, 567–568
- Pushbuttons for TOY machine, 916
- Pushdown automata, 755–756
- Pushdown stacks, 567–568
- Put operations  
 hash tables, 639  
 symbol tables, 624
- Putnam, Hilary, 816
- Q**
- Quad play, 273
- Quadratic Koch island fractal, 803
- Quadratic order of growth, 504–505, 507–508
- Quadratic** program, 25–26
- Quadrature integration, 449
- Quaternions, 424
- Question marks (?) in REs, 731
- Questions** program, 533–535
- Queue** program, 592–596, 604–605
- Queues  
 circular, 620  
 deques, 618  
 FIFO. *See* First-in first-out (FIFO) queues  
 overview, 566  
 random, 596  
 summary, 608
- Queuing theory, 597–600
- Quotes ("") in text, 5
- R**
- Race conditions in flip-flops, 1050
- Ragged arrays, 111
- Ramanujan, Srinivasa, 86
- Ramanujan's taxi, 86
- Random graphs, 695
- Random numbers  
 fair coin flips, 52–53  
 function implementation, 199  
 Gaussian, 47  
 impurity, 32  
 libraries, 232–236  
 random sequences, 127–128  
 Sierpinski triangles, 239–240  
 simulations, 72–73  
`Math.random()`, 30–31
- Random queues, 596
- Random shortcuts, 699
- Random walks  
 Brownian bridges, 278  
 self-avoiding, 112–115  
 two-dimensional, 86  
 undirected graphs, 712
- Random web surfer case study  
 histograms, 177  
 input format, 171
- lessons, 184–185
- Markov chains, 176, 179–184
- overview, 170–171
- page ranks, 176–177
- simulation, 174–178
- transition matrices, 172–173
- RandomInt** program, 33–34
- RandomSeq** program, 127–128
- RandomSurfer** program, 175–177
- RangeFilter** program, 140–143
- Ranges  
 binary search trees, 651  
 functions, 192
- Ranks  
 binary search trees, 651  
 random web surfer, 176–177
- Raphson, Joseph, 65
- Raster images, 346
- Real numbers, 888–890
- Receivers in cryptography, 992
- Recognition problem  
 formal languages, 722  
 NFAs, 744–745  
 REs, 728–729, 735, 753
- Recomputation, 282–283
- Rectangle rule, 449
- Recurrence relations, 272
- Recursion, 191  
 base cases, 281  
 BSTs, 640–641, 644, 649  
 binary searches, 533  
 Brownian bridges, 278–280  
 considering, 320  
 convergence issues, 281–282  
 dynamic programming, 284–289
- Euclid's algorithm, 267–268
- exponential time, 272–273
- factorial example, 264–265
- function-call trees, 269, 271
- graphics, 276–277, 397

- Gray codes, 273–275  
linked lists, 571  
mathematical induction, 266  
memory requirements, 282  
mergesort, 550  
overview, 262–263  
percolation case study, 312–314  
perspective, 289  
pitfalls, 281–283  
recomputation issues, 282–283  
towers of Hanoi, 268–272
- Red-black trees, 648  
Redirection, 139  
  piping, 142–143  
  standard input, 140–141  
  standard output, 139–140
- Reduced instruction set computing (RISC), 974
- Reductio ab absurdum*, 808
- Reduction  
  binary search trees, 640  
  mergesort, 554  
  polynomial-time, 841–843  
  problem, 811–813  
  recursion, 264–265
- References  
  accessing, 339  
  aliasing, 363  
  arrays, 365  
  equality, 454–455  
  garbage collection, 367  
  immutable types, 364, 441  
  linked lists, 572  
  memory, 367  
  method, 470  
  object-oriented programming, 330  
  objects, 338–339  
  orphaned objects, 366  
  passing, 207, 210, 364–365  
  performance, 369
- properties, 362–363  
safe pointers, 366
- Reflexive property, 454
- Registers  
  implementing, 1052  
  machine language, 931  
  overview, 1051–1052  
  TOY machine, 909, 911  
  writing to, 1052–1053
- Regular expressions (REs)  
  applications, 732–736  
  computational biology, 732–734  
  generalized, 730–732  
  NFA equivalence, 750–752  
  overview, 724–725  
  recognition problem, 728–729, 735, 753  
  regular languages, 725–727  
  searches, 734–736  
  shorthand notations, 730–731  
  validity checking, 732
- Regular languages, 723  
  basic operations, 723–724  
  regular expressions. *See* Regular expressions (REs)
- Reject states  
  DFAs, 738–739  
  Turing machines, 766–767
- Relative entropy, 667–668
- Relays in circuit models, 1006
- Remainder operation, 22–23
- Removing  
  array items, 569  
  collection items, 566, 602–603  
  linked list items, 573–574  
  NFA nodes, 751  
  queue items, 592, 596  
  set keys, 652  
  stack items, 567–569  
  symbol table keys, 624–627
- Rendell, Paul, 805
- Repetitive code, simplifying, 100  
Representation in APIs, 431  
Representing information  
  binary and hex, 875–880  
  bit manipulation, 891–893  
  characters, 894–895  
  integer arithmetic, 884–885  
  negative numbers, 886–888  
  overview, 874  
  real numbers, 888–890  
  strings, 880–883  
  summary, 896
- Reproducible experiments, 495
- Reserved words, 16
- Resetting flip-flops, 1050
- Resizing arrays, 578–581, 635
- ResizingArrayStackOf-Strings** program, 578–581
- Resource allocation  
  graphs for, 673  
  overview, 606–607
- Resource-sharing systems, 606–607
- Return addresses, 931
- return** statements, 194, 196, 198
- Return values  
  arrays as, 210  
  methods, 30, 196, 200, 207–210  
  reverse Polish notation, 591
- Reuse, code, 226, 253, 701
- Reverse Polish notation, 590
- RGB color format, 48–49, 341, 371
- Rice, Henry, 812
- Rice’s theorem, 812–813
- Riemann integral, 449
- Riffle shuffles, 125
- Right shift operations  
  bitwise, 891–892  
  TOY machine, 913
- Right subtrees, 640
- Right triangles, 199
- Ring buffers, 620

Ring graphs, 694–695, 699  
 Ripple-carry adders, 1028–1030  
 RISC (reduced instruction set computing), 974  
 Robinson, Julia, 816  
 Roots in binary search trees, 640  
 Rotation filters, 379  
 Roulette-wheel selection, 174  
 Round-robin policies, 606  
 Rows in 2D arrays, 106, 108  
 RR-format instructions, 911  
**Ruler** program, 19–20  
 Run-time errors, 6  
 Running time. *See* Performance  
 Running virtual machines, 969  
**RuntimeException**, 466

## S

Safe pointers, 366  
**Sample** program, 98–99  
 Sample standard deviation, 246  
 Sample variance, 244  
 Sampling  
     audio, 156–157  
     function graphs, 148  
     scaling, 349–350  
     without replacement, 97–99  
**SAT** problem, 832  
     nondeterministic TM, 836  
     **NP**-completeness, 844–846, 853–856  
**SAT** program, 855–856  
 Satisfiability, 830  
     boolean, 832, 836  
     integer linear inequality, 831  
     linear equation, 830  
     linear inequality, 831  
     **NP**-completeness, 844–846, 853–856  
 Saving audio files, 157  
 Scaffolding, 302–304

**Scale** program, 349–350  
 Scaling  
     drawings, 146  
     image processing, 349–350  
     spatial vectors, 442–443  
 Scientific computing, 1094  
 Scientific method, 494–495  
     hypotheses, 496–502  
     observations, 495–496  
 Scientific notation  
     conversion codes, 131–132  
     real numbers, 888–889  
 Scope of variables, 60, 200  
 Screen scraping, 357–359  
 Search problems  
     difficult, 837–838  
     easy, 837  
     nondeterminism, 836  
     overview, 833  
     solutions, 835  
     subset sum, 834  
     TSP problem, 862  
     vertex cover problem, 834, 842  
     0/1 ILP problem, 835, 842  
 Searches  
     binary. *See* Binary searches  
     binary search trees. *See* Binary search trees (BSTs)  
     bisection, 537  
     breadth-first, 683, 687–688, 690, 692  
     data mining example, 458–464  
     depth-first, 312, 690  
     indexing, 634  
     overview, 532  
     regular expressions, 734–736  
     for similar documents, 464  
 Secret messages, 992  
 Seeds for random numbers, 475  
 Select control lines, 1056  
 Self-avoiding walks, 112–115, 710  
 Self-loops for edges, 676  
 Self-modifying code, 922–924  
**SelfAvoidingWalk** program, 112–115  
 Semantics, 52  
 Semicolons ( ; )  
     **for** loops, 59  
     statements, 5  
 Sequential circuits  
     clocks, 1058–1061  
     description, 1008  
     feedback circuits, 1048–1049  
     flip-flops, 1049–1050  
     memory, 1054–1057  
     overview, 1048  
     registers, 1051–1053  
     summary, 1062–1063  
 Sequential searches, 535–536  
 Server farms, 976  
 Servers, 606  
 Service rate, 597–598  
**SET** library, 652–653  
 Sets  
     elementary functions, 1001  
     gates, 1045  
     graphs, 676  
     Julia, 427  
     Mandelbrot, 406–409  
     overview, 652–653  
     of values, 14  
 Setting flip-flops, 1050  
 Shadow variables, 419  
 Shannon, Claude, 1013, 1041  
 Shannon entropy, 378  
 Shapes, outline and filled, 149  
 Shifts  
     bits, 891–892  
     circular, 375  
     linear feedback shift registers (LFSRs), 1000–1001  
     purpose, 898–899  
 TOY machine, 913

- s**hort data type, 24  
Shortcuts in ring graphs, 699  
Shortest paths  
    adjacency-matrix, 692  
    breadth-first searches, 690  
    degrees of separation, 684–686  
    distances, 687–688  
    graphs, 674, 683  
    implementation, 691  
    *P* search problems, 829, 839  
    performance, 690  
    single-source clients, 684  
    trees, 688–689  
Shuffling arrays, 97  
Sicherman dice, 259  
Side effects  
    arrays, 208–210  
    assertions, 467  
    importance, 217  
    methods, 32, 126, 201  
Sierpinski triangles, 239–240  
Sieve of Eratosthenes, 103–105  
Sign-and-magnitude, 886  
Sign extension convention, 899  
Signatures  
    constructors, 385  
    methods, 30, 196  
    overloading, 198  
Similarity measures, 462  
Simple paths, 710  
Simplex method, 831  
Simulations  
    coupon collector, 174–178  
    dice, 121  
    gambler’s ruin, 69–71  
    Let’s Make a Deal, 88–89  
    load balancing, 606–607  
    *M/M/1* queues, 598–600  
    Monte Carlo, 300, 307–308  
    n-body. *See* N-body simulation  
    random web surfer, 174–178
- Single-line comments, 5  
Singles quotes ('), 19  
Singly linked lists, 571  
Sipser, Michael, 780  
Six degrees of separation, 670  
Size  
    arrays, 578–581, 635  
    binary search trees, 651  
    modules, 319  
    paper, 294  
    problems, 495, 824  
    program, 252–253  
    symbol tables, 624  
    words, 874, 897
- Sketch program, 459–462  
Sketches  
    comparing, 462–463  
    computing, 459–460  
    hashing, 460  
    overview, 458–459
- Slashes (/)  
    comments, 5  
    floating-point numbers, 24–26  
    integers, 22–23
- Slide rules, 907–908
- Small-world case study. *See* Graphs
- Small-world phenomenon, 670, 693
- SmallWorld program, 696
- Smith–Waterman algorithm, 286
- Social network graphs, 672
- Sorts  
    `Arrays.sort()`, 559  
    frequency counts, 555–557  
    insertion, 543–549  
    lessons, 558  
    mergesort, 550–555  
    overview, 532  
    *P* search problems, 839
- Sound. *See* Standard audio
- Sound waves  
    plotting, 249  
    superposition of, 211–215
- Source vertices, 683
- Space-filling curves, 425
- Spaces, 10
- Space–time tradeoff, 99–100
- Sparse matrices, 666
- Sparse small-world graphs, 693
- Sparse vectors, 666
- Spatial vectors, 442–445
- Specification problem  
    APIs, 430  
    formal languages, 722  
    programs, 596
- Speed  
    clocks, 1058  
    in performance, 507–508
- Spider traps, 176
- Spira mirabilis, 398
- Spiral program, 398–399
- Spirographs, 167
- Split program, 358, 360
- Spreadsheets, 108
- Sqrt program, 65–67
- Square brackets ([])  
    arrays, 91, 106  
    regular expressions, 731
- Square roots  
    computing, 65–67  
    double value, 25
- Squares, Albers, 341–342
- Squaring Markov chains, 179–180
- SR flip-flops, 1050
- ST library, 625–627
- Stable circuits with feedback, 1049
- Stack program, 583–585
- StackOfStrings program, 568
- StackOverflowError, 282

- Stacks  
 arithmetic expression  
     evaluation, 586–589  
 arrays, 568–570, 578–581  
 function calls, 590–591  
 linked lists, 574–576  
 overview, 566  
 parameterized types, 582–586  
 pushdown, 567–568  
 stack-based languages, 590  
 summary, 608
- Standard audio  
 concert A, 155  
 description, 126, 128–129  
 music example, 157–158  
 notes, 156  
 overview, 155  
 sampling, 156–157  
 saving files, 157  
 summary, 159
- Standard deviation, 246
- Standard drawing  
 control commands, 145–146  
 description, 126, 128–129  
 double buffering, 151  
 filtering data to, 146–147  
 function graphs, 148  
 outline and filled shapes, 149  
 overview, 144–145  
 summary, 159  
 text and color, 150
- Standard input  
 arbitrary size, 137–138  
 description, 126, 128–129  
 formatted, 135  
 interactive, 135–136  
 machine language, 936–938  
 multiple streams, 143  
 overview, 132–133  
 redirecting, 140–141  
 summary, 159
- typing, 134  
 virtual machines, 969–970
- Standard output  
 description, 127  
 formatted, 130–132  
 machine language, 934–936  
 multiple streams, 143  
 overview, 129–130  
 piping, 141–143  
 redirecting, 139–140  
 summary, 159  
 virtual machines, 969–970
- Standard statistics, 244–250
- Standards, API, 429
- Start codons, 336
- Statements  
 assignment, 17  
 blocks, 50  
 declaration, 15–16  
 methods, 5
- States  
 DFAs, 738–739  
 NFAs, 744–746  
 objects, 340  
 Turing machines, 766–772  
 virtual machines, 968
- Static methods, 191–192  
 accessing, 227–229  
 arguments, 197  
 for code organization, 205–206  
 control flow, 193–195  
 defining, 193, 196  
 function-call traces, 195  
 function calls, 197  
 implementation examples, 199  
 vs. instance, 340  
 libraries. *See Libraries*  
 overloading, 198  
 passing arguments, 207–210  
 returning values, 207–210
- Side effects, 201  
 summary, 215  
 superposition example, 211–215  
 terminology, 195–196  
 variable scope, 200
- Static variables, 284
- Statistical polling, 167
- Statistics, 244–250
- StdArrayIO library, 237–238
- StdAudio library, 128–129, 155
- StdDraw library, 128–129,  
 144–145, 150, 154
- StdIn library, 128–129, 132–133
- StdOut library, 129–131
- StdRandom program, 232–236
- StdStats program, 244–247
- StockAccount program, 410–413
- StockQuote program, 358–359
- Stop codons, 336
- Stopwatch program, 390–391
- Store instruction, 938, 1080
- Stored-program computers,  
 922–924
- Streams  
 input, 354–355  
 output, 355  
 screen scraping, 357–359
- Stress testing, 236
- Strings and `String` data type  
 alphabet symbols, 718  
 API, 332–333  
 binary, 718–719  
 circular shifts, 375  
 concatenation, 19–20, 723–724  
 conversion codes, 131–132  
 conversions, 21, 453  
 description, 14–15  
 genomics application, 336–340  
 immutable types, 439–440  
 input, 133  
 internal storage, 37

- invoking instance methods, 334  
memory, 515  
objects, 333–334  
overview, 331  
parsing, 880–882  
prefix-free, 564  
representation, 882–883  
as sequence of characters, 19  
shortcuts, 334–335  
string replacement systems, 795  
unions, 723  
variables, 333  
vertices, 675  
working with, 19–21
- Strogatz, Stephen, 670, 693, 713
- Structured programming, 926
- Stub methods, 303
- Subclassing inheritance, 452–457
- Subgraphs, induced, 705
- Subset sum problem  
intractability, 827–828  
**NP**, 834, 838
- Subtraction  
floating-point numbers, 24–26  
integers, 22  
negative numbers, 887
- Subtrees, 640, 651
- Subtyping inheritance, 446–451
- Sum-of-powers conjecture, 89
- Sum-of-products  
adders, 1028  
boolean representation, 996–997  
circuits, 1024–1027
- Sums, finite, 64–65
- Superclasses, 452
- Superposition  
force vectors, 483  
sound waves, 211–215
- Swirl filters, 379
- Switch control lines, 1005
- Switch statements, 74–75
- Switches  
bus muxes, 1036  
circuit models, 1002, 1005–1006  
demultiplexers, 1022  
gates, 1013  
multiplexers, 1020  
TOY machine, 916–917
- Switching circuit analysis, 1007
- Switching time of gates, 1013
- Symbol tables  
APIs, 625–627  
BSTs. *See* Binary search trees  
dictionary lookup, 628–632  
graphs, 676  
hash tables, 636–639  
implementations, 635–636  
indexing, 632–634  
machine language, 944  
overview, 624–625  
perspective, 654  
sets, 652–653
- Symbolic names in assembly, 981
- Symbols  
definition, 757  
description, 718–719  
DFA, 738  
NFA, 744  
regular expressions, 724  
Turing machines, 766–767
- Symmetric order in BSTs, 640
- Symmetric property, 454
- Syntax errors, 10–11
- T**
- Tables  
of functions, 907–908  
hash, 636–639  
symbol. *See* Symbol tables
- Tabs  
compiler considerations, 10  
escape sequences, 19
- Tape and tape readers  
DFAs, 738–739  
Turing machines, 766–769, 774–776
- Tape program, 776
- Taylor series approximations, 204
- Templates, 50
- TenHello program, 54–55, 60
- Terminal windows, 127
- Terms, glossary for, 1097–1101
- Terrain analysis, 167
- Testing  
for bugs, 318  
importance, 701  
percolation case study, 305–308
- Text. *See also* Strings and String data type  
drawings, 150  
printing, 5, 10
- Text editors, 3
- Theory of computing, 715–717
- this keyword, 445
- Thompson, Ken, 735
- 3n+1 problem, 296–297
- ThreeSum program, 497–502
- Throwing exceptions, 465–466
- Thue word problem, 819
- Ticks, clock, 1058
- Tilde notation, 500
- Tildes (~)  
bitwise operations, 891  
boolean type, 991  
frequency analysis, 500
- Time  
exponential, 272–273, 823  
performance. *See* Performance  
polynomial, 823  
Stopwatch timers, 390–391
- TimePrimitives program, 519
- Timesharing, 965
- Tools, building, 320

- Top-level domains, 375  
**toString()** method  
  **Charge**, 383, 387  
  **Color**, 343  
  **Complex**, 403, 405  
  **Convert**, 881–882  
  **Counter**, 436–437  
  description, 339  
  **Graph**, 678–679  
  linked lists, 574, 577  
  **Object**, 453  
  **Sketch**, 459  
  **Tape**, 776  
  **Vector**, 443  
**Total orderings**, 546  
**Totality problem**, 811–812  
**Towers of Hanoi problem**, 268–272  
**TOY machine**  
  arithmetic logic unit, 910  
  conditionals and loops, 918–921  
  family of computers, 972–977  
  fetch-increment-execute cycle, 910–911  
  first program, 914–915  
  historical note, 907–908  
  instruction register, 910  
  instructions, 909, 911–913  
  in Java, 966–972  
  machine-language programming. *See* Machine-language programming  
  memory, 908–909  
  operating, 916–917  
  overview, 906–907  
  program counter, 910  
  registers, 909  
  stored-program computer, 922–924  
  virtual. *See* Virtual machines  
  von Neumann machines, 924–925  
**TOY program**, 967  
**TOY-8 machine**, 974–975  
  basic parameters, 1070  
  control circuit, 1080–1082  
  CPU, 1076–1080  
  instruction set, 1070–1071  
  perspective, 1084–1087  
  **sum.toy** program, 1071–1072, 1082–1083  
**TOY-64 machine**, 973–974  
**Tracing**  
  function-call, 195  
  programs with **random()**, 103  
  variable values, 18, 56–57  
**Transfer of control**, 193–195  
**Transistors**, 1006  
**Transition matrices**, 172–173  
**Transition** program, 172–173  
**Transitions**  
  DFAs, 738–739  
  NFAs, 744–746  
  Turing machines, 766–767  
**Transitive property**  
  comparisons, 546  
  equivalence, 454  
  polynomial-time reduction, 843  
**Transposition of arrays**, 120  
**Traveling salesperson problem**, 862  
**Traversal**  
  binary search trees, 649–650  
  linked lists, 574, 577  
**TreeMap library**, 655  
**Tree nodes**, 269  
**Trees**  
  BSTs. *See* Binary search trees  
  function-call, 269, 271  
  H-trees, 276–277  
  shortest paths, 688–689  
**Triangles**  
  drawing, 144–145  
  right, 199  
  Sierpinski, 239–240  
**Trigonometric functions**, 256  
**Truth tables**, 26–27, 988–989  
**TSP problem**, 862  
**Turing**, Alan, 766  
  bio, 410–411, 717  
  code breaking, 907  
  von Neumann influenced by, 924–925  
**Turing-complete models**, 794  
**Turing machines**  
  binary adders, 771  
  binary incrementers, 769–771  
  compact trace format, 770  
  constant factor, 824  
  efficiency, 772  
  frequency count, 772–773  
  model, 766–769  
  overview, 766  
  related machines, 770–771  
  restrictions, 792–793  
  **SAT** problem, 836  
  universal, 789–790  
  universal virtual, 774–779  
  universal virtual DFAs, 789  
  universality. *See* Universality  
  variations, 791–794  
**TuringMachine** program, 777–778  
**Turtle** program, 394–396  
**Twenty questions game**, 135–136, 533–535  
**TwentyQuestions** program, 135–136  
**Two-dimensional arrays**  
  description, 90  
  initialization, 106–107  
  matrices, 109–110  
  memory, 107, 516  
  output, 107  
  overview, 106  
  ragged, 111

- self-avoiding walks, 112–115
  - setting values, 108
  - spreadsheets, 108
  - Two’s complement, 38, 886–888
  - Type arguments, 585, 611
  - Type conversions, 34–35
  - Type parameters, 585
  - Type safety, 18
  - Types. *See* Data types
- U**
- Unboxing, 457, 585–586
  - Undirected graphs, 675
  - Unicode characters
    - description, 19
    - overview, 894–895
    - strings, 37
  - Uniform random numbers, 199
  - Uninitialized variables, 94, 339
  - Union operation in REs, 723
  - Unit testing, 235
  - Universal models, 794–797
  - Universal sets
    - elementary functions, 1001
    - gates, 1045
  - Universal Turing machines (UTMs), 789–790
  - Universal virtual DFAs, 741–743
  - Universal virtual TMs, 774–779
  - Universality
    - algorithms, 786–787
    - Church–Turing thesis, 790–791
    - overview, 786
    - programs processing programs, 788–790
    - Turing machine variations, 791–794
    - universal models, 794–797
    - virtual DFA/NFA, 788–789
  - Universe program, 483–487
  - Unreachable code error, 216
- Unsigned integers, 884
  - Unsolvability proof, 810
  - Unsolvable problems, 430
    - blank tape halting problem, 820
    - definite integration, 816
    - description, 806
    - examples, 815
    - halting problem, 808–810
    - Hilbert’s 10th problem, 816
    - implications, 816–817
    - liar’s paradox, 807–808
    - optimal data compression, 814
    - optimizing compilers, 814
    - Post correspondence, 813–814
    - program equivalence, 812
    - totality, 811–812
  - Upper bounds, 825
  - Upscaling in image processing, 349
  - UseArgument program, 7–8
  - User-defined libraries, 230
  - UTF-8 encoding, 895
  - UTMs, 789–790
- V**
- Validate program, 729
  - Validity checking, 732
  - Values
    - array, 95–96
    - data types, 14, 331
    - passing arguments by, 207, 210, 364–365
    - precomputed, 99–100
    - symbol tables, 624–626
  - Variables
    - assignment statements, 17
    - boolean, 987, 994–997
    - compound assignments, 60
    - constants, 16
    - description, 15–16
    - initial values, 415
    - inline initialization, 18
- instance, 384
  - within methods, 196, 386–388
  - names, 16
  - scope, 60, 200
  - shadow, 419
  - static, 284
  - string, 333
  - tracing values, 18
  - uninitialized, 339
  - Vector images, 346
  - Vector program, 443–445, 515
  - Vectors
    - arrays, 92
    - cross products, 472
    - dot products, 92, 442–443
    - matrix–vector multiplication, 110
    - n-body simulation, 479–480
    - sparse, 666
    - spatial, 442–445
    - vector–matrix multiplication, 110, 180
  - Vertex cover problem
    - intractability, 828
  - NP**-completeness, 846–847
  - NP** search problems, 834, 842
  - Vertical bars (|)
  - bitwise operations, 891–892
  - boolean type, 26–27, 991
  - piping, 141
  - regular expressions, 724
  - Vertical *OR* gates, 1023
  - Vertical percolation, 305–306
  - Vertices
    - bipartite graphs, 682
    - creating, 676
    - eccentricity, 711
    - graphs, 671, 674
    - isolated, 703
    - names, 675
    - PathFinder, 683
    - String, 675

- Virtual machines  
 booting, 959–960, 968–969  
 cautions, 961–963  
 and cloud computing, 924  
 description, 965  
 dumping, 960–961  
 instructions, 966–967  
 Moore’s law, 971  
 overview, 958–959  
 program development, 970–971  
 programs that process  
   programs, 964–966  
 running, 969  
 standard input, 969–970  
 standard output, 969–970  
 states, 968  
 TOY machine family, 972–977  
 universal virtual DFAs, 742  
   universal virtual TM, 774–779
- Viruses, 963
- Viterbi algorithm, 286
- void** keyword, 201, 216
- Volatility  
 Black–Scholes formula, 565  
 Brownian bridges, 278, 280
- Von Neumann, John, 906  
 ballistics tables, 907  
 ENIAC improvements, 924–925  
 Gödel letter, 840  
 mergesort, 554
- Von Neumann architecture, 790,  
 906, 924–925
- Voting machine errors, 436
- W**
- Walks  
 random. *See* Random walks  
 self-avoiding, 112–115, 710
- Watson–Crick palindrome, 374
- Watts, Duncan, 670, 693, 713
- Watts–Strogatz graph model, 713
- .wav format, 157
- Wave filters, 379
- Web graphs, 695
- Web pages, 170  
  indexes searches, 634  
  preferential attachment, 713
- Weighing objects, 540–541
- Weighted averages, 120
- Weighted superposition, 212
- while** loops, 53–59  
  examples, 61  
  nesting, 62
- Whitelists, binary searches for, 540
- Whitespace characters  
  compiler considerations, 10  
  input, 135
- Wide interfaces  
  APIs, 430  
  examples, 610–611
- Wildcard operation in REs, 724
- Wiles, Andrew, 722
- Wind chill, 47
- Wires  
  circuit models, 1002–1004  
  gates, 1013
- Word ladders, 710
- Words  
  binary representation, 875  
  computer, 874  
  memory size, 513  
  size, 897
- Worst-case performance  
  big-*O* notation, 520–521  
  binary search trees, 648  
  description, 512  
  insertion sort, 544  
  intractability, 825  
  **NP**-completeness, 852
- Wrapper types  
  autoboxing, 585–586  
  references, 369, 457
- Write control lines  
  CPU, 1079–1080  
  memory bits, 1056  
  register bits, 1051
- X**
- XOR** circuits  
  in arithmetic logic units, 1031  
  sum-of-products, 1024–1025
- xor (exclusive *or*) operation, 891–892, 913
- Y**
- Y2K problem, 435, 976
- Young tableaux, 530
- Z**
- Zero-based indexing, 92
- Zero crossings, 164
- Zero extension convention, 899
- 0/1 ILP problem, 831, 835
- NP**-completeness, 845–846  
  vertex cover problem, 842
- Zeros, leading, 883
- ZIP codes, 435
- Zipf’s law, 556