Selected Homework of Mathematical Logic

Hao Su

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Contents

Homework 1

1 Question to fill in. ◁

to do

2 Verify the validity of the deduction step in the following statement:

I can doubt that the physical world exists. I can even doubt whether my body really exists. I cannot doubt that I myself exist. So I am not my body.

— Descartes

◁

A formalization could be:

Premises:

1. $can\_doubt(I, physical\_world)$,
2. $can\_doubt(I, my\_body)$,
3. $cannot\_doubt(I, I)$.

Conclusion:

$\neg = (I, my\_body)$.

It is straightforward that the deduction step is invalid.

Note that *cannot_doubt* instead of $\neg can\_doubt$ are used to capture the subjective nature of "doubt". *be* instead of $=$ might be used in conclusion, in repect of that relations between pairs of expressions in natual languages may not qualify as "equality" that we refer to in formal languages.

Homework 6

1 Question to fill in. $\triangleleft$

to do

Homework 7

1 [Enderton, ex. 1, 2, 5, p. 17] Translate between English and the specified first-order language.

(a) Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting. ($\forall$, for all things; N, is a number; I, interesting; $<$, is less than; 0, a constant symbol intended to denote zero.)

(b) There is no number such that no number is less than it. (The same language as in Item a.)

(c) $\forall x(Nx \to Ix \to \neg\forall y(Ny \to Iy \to \neg x < y))$. (The same language as in Item a. There exists a relatively concise translation.)

(d) (i) You can fool some of the people all of the time. (ii) You can fool all of the people some of the time. (iii) You can't fool all of the people all of the time. ($\forall$, for all things; P, is a person; T, is a time; $F\ x\ y$, you can fool x at y. One or more of the above may be ambiguous, in which case you will need more than one translation.) $\triangleleft$

(a) $\forall x(Nx \wedge \neg Ix \wedge \forall y(Ny \wedge y < x \to Iy) \to Ix)$.

(b) $\neg\exists x(Nx \wedge \neg\exists y(Ny \wedge y < x))$.

(c) Any interesting number is less than some interesting number.

(d) (i) Here ambiguity is in that it either says that there are some (fixed) people you can fool all time, or says that at every moment there are (some, not fixed) people you can fool, i.e. either $\exists x(Px \wedge \forall y(Ty \to Fxy))$ or $\forall y(Ty \to \exists x(Px \wedge Fxy))$. (ii) Here ambiguity is in that it either says that you can fool each person at some time (times can be different for different people), or says that at some (fixed) time you can fool everyone (at that specific time): $\forall x(Px \to \exists y(Ty \wedge Fxy))$ or $\exists y(Ty \wedge \forall x(Px \to Fxy))$. (iii) $\neg\forall x(Px \to \forall y(Ty \to Fxy))$.

2 [Enderton, ex. 1, p. 129] For a term u, let u^x_t be the expression obtained from u by replacing the variable x by the term t. Restate this definition without using any form of the word "replace" or its synonyms. $\triangleleft$

For term t and variable x, we define $\sigma_{x\mapsto t} : V \to T$, where V is the set of variables, T the set of terms and $\sigma_{x\mapsto t}$ identity except that it maps x to t and then recursively define the extension $\overline{\sigma_{x\mapsto t}} : T \to T$:

1. For each variable v, $\overline{\sigma_{x\mapsto t}}(v) = h(v)$.
2. For each constant symbol c, $\overline{\sigma_{x\mapsto t}}(c) = c$.
3. For terms $t_1, \ldots, t_n$ and n-place function symbol f,
$$\overline{\sigma_{x\mapsto t}}(f(t_1, \ldots, t_n)) = f(\overline{\sigma_{x\mapsto t}}(t_1), \ldots, \overline{\sigma_{x\mapsto t}}(t_n)).$$

3 [Enderton, ex. 9, p. 130]

(a) Show by two examples that $(\varphi^x_y)^y_x$ is not in general equal to φ, where the first shows that x may occur in $(\varphi^x_y)^y_x$ at a place where it does not occur in φ and the second shows that x may occur in a φ at a place where it does not occur in $(\varphi^x_y)^y_x$.

(b) Prove *Re-replacement lemma*: if y does not occur in φ, then x is substitutable for y in φ^x_y and $(\varphi^x_y)^y_x = \varphi$. $\lhd$

(a) $\varphi = P\ y$ (y occurs free in φ) and $\forall y\ P\ x$ (not substitutable).

(b) We use induction on φ.

Case 1: For atomic $\varphi = P\ t_1, \ldots, t_n$, we have that x is substitutable for y in φ^x_y and that

$$(\varphi^x_y)^y_x = ((P\ t_1, \ldots, t_n)^x_y)^y_x = P\ ((t_1)^x_y)^y_x, \ldots, ((t_n)^x_y)^y_x = \varphi.$$

Case 2: Given the inductive hypothesis, the inductive step holds by definition for formula building operations $\mathcal{E}_\neg$, $\mathcal{E}_\to$ and $\mathcal{Q}_i$, where $v_i \neq x$ and $v_i \neq y$ (since y does not occur in φ).

Case 3: $\varphi = \forall x\ \psi$. Then $(\forall x\ \psi)^x_y = \forall x\ \psi$, in which y does not occur (free, and thus x is substitutable.) Therefore $(\varphi^y_x)^x_y = ((\forall x\ \psi)^x_y)^y_x = (\forall x\ \psi)^x_y = \forall x\ \psi = \varphi$.

Homework 8

1 [Enderton, ex. 4, 7, 10, p. 130] Show by deduction that

1. $\vdash \forall x\ \varphi \to \exists x\ \varphi$;
2. $\vdash \exists x(Px \to \forall x\ Px)$;
3. $\{Qx, \forall y(Qy \to \forall z\ Pz)\} \vdash \forall x\ Px$.
4. $\forall x \forall y\ Pxy \vdash \forall y \forall x\ Pyx$. ◁

1. $\vdash \forall x\ \varphi \to \exists x\ \varphi \Leftarrow \forall x\ \varphi \vdash \exists x\ \varphi$ by deduction theorem,
   $\Leftarrow \forall x\ \varphi \vdash \neg\forall x\ \neg\varphi$ by rewriting,
   $\Leftarrow \forall x\ \neg\varphi \vdash \neg\forall x\ \varphi$ by contraposition and rule T,
   $\Leftarrow \neg\varphi \vdash \neg\forall x\ \varphi$ by Ax.2 and MP,
   $\Leftarrow \forall x\ \varphi \vdash \varphi$ by contraposition and rule T,
   $\Leftarrow \vdash \forall x\ \varphi \to \varphi$ by MP, which is Ax.2.

2. $\vdash \exists x(Px \to \forall x Px)$
   $\Leftarrow \{\forall x \neg(Px \to \forall x Px)\}$ is inconsistent by RAA,
   $\Leftarrow \forall x \neg(Px \to \forall x Px) \vdash \forall x Px$
   $\wedge\ \forall x \neg(Px \to \forall x Px) \vdash \neg\forall x Px$,
   where
   $\forall x \neg(Px \to \forall x Px) \vdash \forall x Px$
   $\Leftarrow \vdash \forall x \neg(Px \to \forall x Px) \to \forall x Px$ by MP,
   $\Leftarrow \vdash \forall x(\neg(Px \to \forall x Px) \to Px)$ by Ax.3 and MP,
   $\Leftarrow \neg(Px \to \forall x Px) \to Px$, by generalization theorem and MP,
   which is Ax.1, and
   $\forall x \neg(Px \to \forall x Px) \vdash \neg\forall x Px$,
   $\Leftarrow \neg(Px \to \forall x Px) \to Px$ by Ax.2, which is Ax.1.

3. $\{Qx, \forall y(Qy \to \forall z\ Pz)\} \vdash \forall x\ Px$
   $\Leftarrow \{Qx, \forall y(Qy \to \forall z\ Pz)\} \vdash \forall w\ Pw$ by EAV (cf. 2),
   $\Leftarrow \{Qx, \forall y(Qy \to \forall z\ Pz)\} \vdash Pw$ by generalization theorem,
   which we show directly:
   1. $\vdash \forall y(Qy \to \forall z Pz) \to Qx \to \forall z Pz$ Ax.2.
   2. $\{Qx, \forall y(Qy \to \forall z Pz)\} \vdash \forall z Pz$ 1; ded.
   3. $\vdash \forall z Pz \to Pw$ Ax.2.
   4. $\{Qx, \forall y(Qy \to \forall z Pz)\} \vdash Pw$ 2; 3; MP.

4. 1. $\forall x \forall y Pxy \vdash \forall a \forall b Pab$ EAV (cf. 2).
   2. $\forall a \forall b Pab \vdash \forall y \forall x Pyx$ EAV (cf. 2).
   3. $\forall x \forall y Pxy \vdash \forall y \forall x Pyx$ 1; 2; ded; MP.

2 Give a complete proof of The Existence of Alphabetic Variants. ◁

Existence of Alphabetic Variants Let φ be a formula, t a term, and x a variable. Then we can find a formula φ' (which differs from φ only in the choice of quantified variables) such that (a) $\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi$ and (b) t is substitutable for x in φ'.

Proof Sketch. We consider fixed t and x and construct φ' by recursion on φ. For atomic φ we take $\varphi' = \varphi$, and then $(\neg\varphi)' = (\neg\varphi')$,$(\varphi \to \psi)' = (\varphi' \to \psi')$. We define $(\forall y\ \varphi)' = \forall z(\varphi')^y_z$, where z is a variable that does not occur in φ' or t or x. By inductive hypothesis we see that (b) holds. For (a) we only prove that $\forall y\ \varphi \vdash \forall z(\varphi')^y_z$. Consider a sequence $\langle \forall y\ \varphi, \forall y\ \varphi', (\varphi')^y_z, \forall z(\varphi')^y_z \rangle$, where each formula (except the first one) can be obtained from former ones. To see the converse, consider $\langle \forall z(\varphi')^y_z, ((\varphi')^y_z)^z_y, \varphi', \varphi, \forall y\ \varphi \rangle$. □

3 [Enderton, ex. 15, p. 131] Prove **Rule EI**: Assume that the constant symbol c does not occur in φ, ψ or Γ, and that $\Gamma; \varphi^x_c \vdash \psi$. Then $\Gamma; \exists x\ \varphi \vdash \psi$ and there is a deduction of ψ from $\Gamma; \exists x\ \varphi$ in which c does not occur. ("EI" stands for "existential instantiation".) Then use it to show that deductions from $\emptyset$ of the following formulas exist:

1. $\exists x\ \alpha \lor \exists x\ \beta \leftrightarrow \exists x(\alpha \lor \beta)$;
2. $\forall x\ \alpha \lor \forall x\ \beta \to \forall x(\alpha \lor \beta)$. ◁

Proof. By contraposition we have $\Gamma; \neg\psi \vdash \neg\varphi^x_c$. By generalization on constants we have a deduction (without c) from $\Gamma; \neg\psi$ of $\forall y((\neg\varphi^x_c)^c_y)$, where y does not occur in $\neg\varphi^x_c$. Since c does not occur in $\neg\varphi$, we have that $(\neg\varphi^x_c)^c_y = \neg\varphi^x_y$. Also, by 3 we have that x is substitutable for y in $\neg\varphi^x_y$ and $(\neg\varphi^x_y)^y_x = \neg\varphi$, thus $(\forall y \neg\varphi^x_y) \to \neg\varphi$ is Ax.2, by which we obtain $\forall y \neg\varphi^x_y \vdash \forall x \neg\varphi$. So $\Gamma; \neg\psi \vdash \forall x \neg\varphi$. Apply contrapostion again and we are done. □

1. a. $\vdash \exists x\ \alpha \lor \exists x\ \beta \to \exists x(\alpha \lor \beta)$

$\Leftarrow \exists x\ \alpha \lor \exists x\ \beta \vdash \exists x(\alpha \lor \beta)$	by deduction theorem,
$\Leftarrow \forall x \neg(\alpha \lor \beta) \vdash \forall x \neg\alpha \land \forall x \neg\beta$	by contraposition and Ax.1,
which we show directly:	
1. $\vdash \neg(\alpha \lor \beta)^x_c \to \neg\alpha^x_c$	Ax.1. c does not occur in α or β.
2. $\vdash \forall x \neg(\alpha \lor \beta) \to \neg(\alpha \lor \beta)^x_c$	Ax.2.
3. $\vdash \forall x \neg(\alpha \lor \beta) \to \neg\alpha^x_c$	1; 2; MP.
4. $\alpha^x_c \vdash \neg\forall x \neg(\alpha \lor \beta)$	3; ded; contraposition.
5. $\exists x\alpha \vdash \neg\forall x \neg(\alpha \lor \beta)$	4; EI.
6. $\forall x \neg(\alpha \lor \beta) \vdash \forall x \neg\alpha$	5; contraposition.
7. $\forall x \neg(\alpha \lor \beta) \vdash \forall x \neg\beta$	same as how 6 is deduced.
8. $\forall x \neg(\alpha \lor \beta) \vdash \forall x \neg\alpha \land \forall x \neg\beta$	7; 8; rule T.

b. $\vdash \exists x(\alpha \vee \beta) \rightarrow \exists x\alpha \vee \exists x\beta$

$\Leftarrow (\alpha \vee \beta)^x_c \vdash \exists x\alpha \vee \exists x\beta$ by ded and EI (c does not occur in α or β),

which we show directly:

1.$\forall x \neg\alpha \vdash \neg\alpha^x_c$ Ax.2; ded.

2.$\alpha^x_c \vdash \exists x\alpha$ 1; contraposition.

3.$\alpha^x_c \vdash \exists x\alpha \vee \exists x\beta$ 2; Ax.1; rule T.

4.$\neg(\exists x\alpha \vee \exists x\beta) \vdash \neg\alpha^x_c$ 3; contraposition; Ax.1.

5.$\neg(\exists x\alpha \vee \exists x\beta) \vdash \neg\beta^x_c$ same as how 4 is deduced.

6.$\neg(\exists x\alpha \vee \exists x\beta) \vdash (\neg\alpha \wedge \neg\beta)^x_c$ 4; 5; rule T.

7.$(\alpha \vee \beta)^x_c \vdash \exists x\alpha \vee \exists\beta$ 6; contraposition.

c. $\vdash \exists x\alpha \vee \exists x\beta \leftrightarrow \exists x(\alpha \vee \beta)$ a; b; rule T.

2. $\vdash \forall x\alpha \vee \forall x\beta \rightarrow \forall x(\alpha \vee \beta)$

$\Leftarrow \forall x\alpha \vee \forall x\beta \vdash \forall x(\alpha \vee \beta)$ by deduction theorem,

$\Leftarrow \exists x\neg(\alpha \vee \beta) \vdash \exists x\neg\alpha \wedge \exists x\neg\beta$ by contraposition and Ax.1,

$\Leftarrow \neg(\alpha \vee \beta)^x_c \vdash \exists x\neg\alpha \wedge \exists x\neg\beta$ by EI, where c does not occur in α or β,

which we show directly:

1. $\vdash \forall x\alpha \rightarrow \alpha^x_c$ Ax.2.

2.$\forall x\alpha \vdash \alpha^x_c$ 1; ded.

3. $\vdash \alpha^x_c \rightarrow (\alpha \vee \beta)^x_c$ Ax.1.

4.$\forall\alpha \vdash (\alpha \vee \beta)^x_c$ 2; 3; MP.

5.$\neg(\alpha \vee \beta)^x_c \vdash \exists x\neg\alpha$ 4; contraposition.

6.$\neg(\alpha \vee \beta)^x_c \vdash \exists x\neg\beta$ same as how 5 is deduced.

7.$\neg(\alpha \vee \beta)^x_c \vdash \exists x\neg\alpha \wedge \exists x\neg\beta$ 5; 6; rule T.

Homework 9

1 [Enderton, ex. 1, p. 99] Show that (a) $\Gamma; \alpha \vDash \varphi$ iff $\Gamma \vDash (\alpha \to \varphi)$; and (b) $\varphi \vDash\dashv \psi$ iff $\vDash (\varphi \leftrightarrow \psi)$. $\triangleleft$

(a) $$\begin{aligned} \Gamma; \alpha \vDash \varphi &\Leftrightarrow (\forall \tau\ \tau \in \Gamma; \alpha \to \vDash_{\mathfrak{A}} \tau[s]) \to \vDash_{\mathfrak{A}} \varphi[s] \\ &\Leftrightarrow (\forall \tau\ \tau \in \Gamma \to \vDash_{\mathfrak{A}} \tau[s]) \wedge \vDash_{\mathfrak{A}} \alpha[s] \to \vDash_{\mathfrak{A}} \varphi[s] \\ &\Leftrightarrow (\forall \tau\ \tau \in \Gamma \to \vDash_{\mathfrak{A}} \tau[s]) \to \vDash_{\mathfrak{A}} (\alpha \to \varphi)[s] \\ &\Leftrightarrow \Gamma \vDash (\alpha \to \varphi). \end{aligned}$$

(b) Let $\Gamma = \emptyset$ in (a) to have that $\varphi \vDash \psi$ iff $\varphi \to \psi$ and that $\psi \to \varphi$ iff $\psi \to \varphi$, thus we are done.

Note that the $\wedge$ and $\Leftrightarrow$ and $\Rightarrow$ used in this proof are only simplifications of meta-reasoning in English.

2 Show that if x does not occur free in α, then $\alpha \vDash \forall x\alpha$. $\triangleleft$

Consider a fixed $\mathfrak{A}$ and s. For every $d \in |\mathfrak{A}|$, we have that s and $s(x|d)$ agree at all free variables of α, then by theorem 2.17 in slides (which says that s only matters when it comes to free variables) $\vDash_{\mathfrak{A}} \alpha[s] \Leftrightarrow \vDash_{\mathfrak{A}} \alpha[s(x|d)] \Leftrightarrow \vDash_{\mathfrak{A}} \forall x\ \alpha[s]$.

3 Show that a formula θ is valid iff $\forall x\theta$ is valid. $\triangleleft$

Consider a fixed $\mathfrak{A}$ and s, $\vDash_{\mathfrak{A}} \varphi[s(x|d)]$ holds for every $d \in |\mathfrak{A}|$ since φ is *valid*, and that is exactly $\vDash_{\mathfrak{A}} \forall x\ \varphi[s]$. For the other direction we take $d = s(x)$ to ensure that $s(x|d) = s$. Thus $\varphi \Leftrightarrow \forall x\ \varphi$.

4 Restate the definition of "$\mathfrak{A}$ satisfies φ with s" by defining recursively a function $\bar{h}$ such that $\mathfrak{A}$ satisfies φ with s iff $s \in \bar{h}(\varphi)$. $\triangleleft$

To give an alternative definition of *satisfaction*, we first define a function $h : A \to \mathcal{P}(|\mathfrak{A}|^V)$, where A is the set of atomics: for an n-place predicate parameter P (we include $=$ as a 2-place predicate if it exists),

$$h(P\ t_1 \cdots t_n) = \{s : V \to |\mathfrak{A}| | \langle \bar{s}(t_1), \ldots, \bar{s}(t_n)\rangle \in P^{\mathfrak{A}}.\}$$

Then we extend h to $\bar{h}$ with the set of wffs as its domain.

1. $h(\varphi) \subseteq \bar{h}(\varphi)$.
2. $\bar{h}(\neg\varphi) = \{s : V \to |\mathfrak{A}| | s \notin \bar{h}(\varphi)\}$.
3. $\bar{h}(\varphi \to \psi) = \bar{h}(\varphi) \cup \bar{h}(\psi)$.
4. $\bar{h}(\forall x\ \varphi) = \{s : V \to |\mathfrak{A}| |$ for every $d \in |\mathfrak{A}|, s(x|d) \in \bar{h}(\varphi)\}$.

We at last define
$$\vDash_{\mathfrak{A}} \varphi[s] \text{ iff } s \in \bar{h}(\varphi).$$

5 [Enderton, ex. 9, p. 100] Assume that the language has equality and a two-place predicate symbol P. For each of the following conditions, find a sentence σ such that the structure $\mathfrak{A}$ is a model of σ iff the condition is met.

(a) $|\mathfrak{A}|$ has exactly two members.

(b) $P^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|$ into $|\mathfrak{A}|$. (A *function* is a single-valued relation, as in Chapter 0. For f to be a function from A into B, the domain of f must be all of A; the range of f is a subset, not necessarily proper, of B.)

(c) $P^{\mathfrak{A}}$ is a permutation of $|\mathfrak{A}|$; i.e., $P^{\mathfrak{A}}$ is a one-to-one function with domain and range equal to $|\mathfrak{A}|$. ◁

1. $\exists a \exists b \forall c(\neg a = b \wedge (c = a \vee c = b))$.
2. $\forall x \exists y \forall z(P\ xy \wedge (P\ xz \to y = z))$.
3. $\forall x \exists y \forall z \exists p \forall q \forall r(P\ xy \wedge (P\ xz \to y = z) \wedge P\ pq \wedge (P\ rq \to p = r))$.

Homework 10

1 Axiom group 3. ◁

Consider a fixed $\mathfrak{A}$ and s, for every $d \in |\mathfrak{A}|$,

$$\begin{aligned}
&(\models_{\mathfrak{A}} \forall x(\alpha \to \beta)[s]) \wedge (\models_{\mathfrak{A}} \forall x\ \alpha[s]) \\
\Leftrightarrow &(\models_{\mathfrak{A}} (\alpha \to \beta)[s(x|d)]) \wedge (\models_{\mathfrak{A}} \alpha[s(x|d)]) \\
\Leftrightarrow &(\models_{\mathfrak{A}} \alpha[s(x|d)] \to \models_{\mathfrak{A}} \beta[s(x|d)]) \wedge (\models_{\mathfrak{A}} \alpha[s(x|d)]) \\
\Rightarrow &\models_{\mathfrak{A}} \beta[s(x|d)] \\
\Leftrightarrow &\models_{\mathfrak{A}} \forall x\ \beta[s].
\end{aligned}$$

Thus $\{\forall x(\alpha \to \beta), \forall x\ \alpha\} \models \forall x\ \beta$. And that is equivalent to axiom group 3.

2 Axiom group 4. ◁

This is the same question as 2.

3 Axiom group 5. ◁

Fix $\mathfrak{A}$ and s. We have that $\models_{\mathfrak{A}} x = x[s]$ iff $s(x) = s(x)$, and that is always true.

4 Axiom group 6. ◁

Assume that α is atomic and α' is obtained from α by replacing x at some places by y. It suffices to show that

$$\{x = y, \alpha\} \models \alpha'.$$

So take any $\mathfrak{A}, s$ such that

$$\models_{\mathfrak{A}} x = y[s], \quad \text{i.e., } s(x) = s(y).$$

Then any term t has the property that if t' is obtained from t by replacing x at some places by y, then $\bar{s}(t) = \bar{s}(t')$. This is obvious; a full proof would use induction on t.

If α is $t_1 = t_2$, then α' must be $t_1' = t_2'$, where t_i' is obtained from t_i as described.

$$\begin{aligned}
\models_{\mathfrak{A}} \alpha[s] &\iff \bar{s}(t_1) = \bar{s}(t_2) \\
&\iff \bar{s}(t_1') = \bar{s}(t_2') \\
&\iff \models_{\mathfrak{A}} \alpha'[s].
\end{aligned}$$

Similarly, if α is $Pt_1 \cdots t_n$, then α' is $Pt_1' \cdots t_n'$ and an analogous argument applies.

Homework 11

1 [Enderton, ex. 4, p. 146] Let $\Gamma = \{\neg\forall v_1 Pv_1, Pv_2, Pv_3, \dots\}$. Is Γ consistent? Is Γ satisfiable? $\triangleleft$

It is consistent and satisfiable. Define $P^{\mathfrak{A}} = \{v_2, v_3, \dots\}$ and $s : V \to |\mathfrak{A}|$ as identity, it follows that for all $\gamma \in \Gamma$, $\vDash_{\mathfrak{A}} \gamma[s]$.

2 [Enderton, ex. 7, p. 146] For each of the following sentences, either show there is a deduction or give a counter-model (i.e., a structure in which it is false.)

(a) $\forall x(Qx \to \forall y\, Qy)$

(b) $(\exists x\, Px \to \forall y\, Qy) \to \forall z(Pz \to Qz)$

(c) $\forall z(Pz \to Qz) \to (\exists x\, Px \to \forall y\, Qy)$

(d) $\neg\exists y\, \forall x(Pxy \leftrightarrow \neg Pxx)$ $\triangleleft$

(a) Not valid. Let $|\mathfrak{A}| = \{0, 1\}$ and $Q^{\mathfrak{A}} = \{1\}$. Then $\mathfrak{A}$ is a counter-model.

(b)
$$\begin{aligned}
&\vdash (\exists x Px \to \forall y Qy) \to \forall z(Pz \to Qz) && \\
\Leftarrow\; &\exists x Px \to \forall y Qy \vdash \forall z(Pz \to Qz) && \text{by ded,} \\
\Leftarrow\; &\{\exists x Px \to \forall y Qy, Pz\} \vdash Qz && \text{by gen and ded,}
\end{aligned}$$
which we show directly:
$$\begin{aligned}
&1. \forall x \neg Px \vdash \neg Pz && \text{Ax.2; ded.} \\
&2. Pz \vdash \neg\forall x \neg Px && \text{1; contraposition.} \\
&3. \{\exists x Px \to \forall y Qy, Pz\} \vdash \forall y Qy && \text{2; MP.} \\
&4. \vdash \forall y Qy \to Qz && \text{Ax.2.} \\
&5. \{\exists x Px \to \forall y Qy, Pz\} \vdash Qz && \text{3; 4; MP.}
\end{aligned}$$

(c) Not valid. Let $|\mathfrak{A}| = \{0, 1\}$ and $P^{\mathfrak{A}} = Q^{\mathfrak{A}} = \{1\}$. Then $\mathfrak{A}$ is a counter-model.

(d)
$$\begin{aligned}
&\vdash \neg\exists y \forall x(Pxy \leftrightarrow \neg Pxx) && \\
\Leftarrow\; &\vdash \forall y \neg\forall x(Pxy \leftrightarrow \neg Pxx) && \text{by Ax.1 and MP,} \\
\Leftarrow\; &\vdash \neg\forall x(Pxy \leftrightarrow \neg Pxx) && \text{by gen,}
\end{aligned}$$
which we show directly:
$$\begin{aligned}
&1. \vdash \neg(Pxy \leftrightarrow \neg Pxx)^x_y && \text{Ax.1.} \\
&2. \vdash \forall x(Pxy \leftrightarrow \neg Pxx) \to (Pxy \leftrightarrow \neg Pxx)^x_y && \text{Ax.2.} \\
&3. \vdash \neg(Pxy \leftrightarrow \neg Pxx)^x_y \to \neg\forall x(Pxy \leftrightarrow \neg Pxx) && \text{2; Ax.1; MP.} \\
&4. \vdash \neg\forall x(Pxy \leftrightarrow \neg Pxx) && \text{1; 3; MP.}
\end{aligned}$$

3 [Enderton, ex. 8, p. 146] Assume the language (with equality) has just the parameters $\forall$ and P, where P is a two-place predicate symbol. Let $\mathfrak{A}$ be the structure with $|\mathfrak{A}| = \mathbb{Z}$, the set of integers (positive, negative, and zero), and with $\langle a, b\rangle \in P^{\mathfrak{A}}$ iff $|a - b| = 1$. Thus $\mathfrak{A}$ looks like an infinite graph:

$$\cdots \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \cdots$$

Show that there is an elementarily equivalent structure $\mathfrak{B}$ that is not connected. (Being *connected* means that for every two members of $|\mathfrak{B}|$, there is a path between them. A *path* — of length n — from a to b is a sequence $\langle p_0, p_1, \ldots, p_n\rangle$ with $a = p_0$ and $b = p_n$ and $\langle p_i, p_{i+1}\rangle \in P^{\mathfrak{B}}$ for each i.) *Suggestion*: Add constant symbols c and d. Write down sentences saying c and d are far apart. Apply compactness. ◁

Expand the language by adding two new constant symbols c and d. For each integer $k \geq 0$, we can find a sentence λ_k that translates, "The distance between c and d is *not* k." For example,

$$\begin{aligned}
&\lambda_0 = \neg c = d\\
&\lambda_1 = \forall p_1(Pcp_1 \to \neg p_1 = d),\\
&\lambda_2 = \forall p_1 \forall p_2(Pcp_1 \to Pp_1p_2 \to \neg p_2 = d).
\end{aligned}$$

Let $\Sigma = \{\lambda_0, \lambda_1, \lambda_2, \ldots\}$. Consider a finite subset of $\Sigma \cup \mathrm{Th}\mathfrak{A}$. That subset is true in $\mathfrak{A}_k$ such that $|c^{\mathfrak{A}_k} - d^{\mathfrak{A}_k}| > k$ for some large k. So by compactness $\Sigma \cup \mathrm{Th}\mathfrak{A}$ has a model

$$\mathfrak{B} = (|\mathfrak{B}|; \mathrm{P}^{\mathfrak{B}}, =^{\mathfrak{B}}, c^{\mathfrak{B}}, d^{\mathfrak{B}})$$

Let $\mathfrak{B}_0$ be the restriction of $\mathfrak{B}$ to the original language: $\mathfrak{B}_0 = (|\mathfrak{B}|, \mathrm{P}^{\mathfrak{B}}, =^{\mathfrak{B}})$. We have that $\mathfrak{B}_0$ is a model of $\mathrm{Th}\mathfrak{A}$, so $\mathfrak{B}_0 \equiv \mathfrak{A}$ (you may try to prove this). Note that $c^{\mathfrak{B}} \in |\mathfrak{B}|$ and $d^{\mathfrak{B}} \in |\mathfrak{B}|$, but there is no path between them.

Comment. One might ask: every member of the universe should be connected to two unique nodes, then to which nodes is $c^{\mathfrak{B}}$ connected? Well, consider not c and d are located on the single infinite graph that $\mathfrak{A}$ indicates, but that c and d and $\mathbb{Z}$ are on 3 seperate infinite graphs, which together consititute our construction of $|\mathfrak{B}|$. That should make a better (possible) interpretation of what we have been effectively doing. One may feel that c and d are far apart but connected, but that is not the case in $\mathfrak{B}$. All we have is that every finite piece of $\mathfrak{B}_0$ looks like a finite segment of $\mathfrak{A}$.

4 [Enderton, ex. 11, p. 100] For each of the following relations, give a formula which defines it in $(\mathbb{N}; +, \cdot)$. (The language is assumed to have equality and the parameters $\forall$, $+$, and $\cdot$).

1. $\{0\}$.
2. $\{1\}$.
3. $\{\langle m, n\rangle \mid n \text{ is the successor of } m \text{ in } \mathbb{N}\}$.
4. $\{\langle m, n\rangle \mid m < n \text{ in } \mathbb{N}\}$. ◁

1. $\forall x\ x + a = x$.
2. $\forall x\ x \cdot a = x$.
3. $\exists y \forall x(x \cdot y = x \land n = m + y)$.
4. $\exists y \forall x \exists k(x + y = x \land \neg k = y \land n = m + k)$.

5 [Enderton, ex. 6, p. 146] Let Σ_1 and Σ_2 be sets of sentences such that nothing is a model of both Σ_1 and Σ_2. Show that there is a sentence τ such that

$$\mathrm{Mod}\ \Sigma_1 \subseteq \mathrm{Mod}\ \tau \quad \text{and} \quad \mathrm{Mod}\ \Sigma_2 \subseteq \mathrm{Mod}\ \neg\tau.$$

(This can be stated: Disjoint EC_Δ classes can be separated by an EC class.) *Suggestion*: $\Sigma_1 \cup \Sigma_2$ is unsatisfiable; apply compactness. ◁

We may suppose Σ_1 and Σ_2 are satisfiable (the other cases are ommitted as trivial). $\Sigma_1 \cup \Sigma_2$ is not satisfiable, thus not finitely satisfiable. Say that a finite subset Σ_0 is inconsistent. Let α be the conjunction of members of $\Sigma_0 \cap \Sigma_1$. Clearly $\Sigma_1 \vdash \alpha$. We also have that $\Sigma_2 \vdash \neg\alpha$, for that if $\Sigma_2; \alpha$ is satisfiable so would be Σ_0 and thus contradiction, then we are done.