Selected Homework of Mathematical Logic

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Homework 1

1 Question to fill in. ◁

to do

2 Verify the validity of the deduction step in the following statement:

I can doubt that the physical world exists. I can even doubt whether my body really exists. I cannot doubt that I myself exist. So I am not my body.

— Descartes

◁

A formalization could be:

Premises:

1. $can\_doubt(I, physical\_world)$,
2. $can\_doubt(I, my\_body)$,
3. $cannot\_doubt(I, I)$.

Conclusion:

$\neg = (I, my\_body)$.

It is straightforward that the deduction step is invalid.

Note that *cannot_doubt* instead of $\neg$*can_doubt* are used to capture the subjective nature of "doubt". *be* instead of $=$ might be used in conclusion, in repect of that relations between pairs of expressions in natual languages may not qualify as "equality" that we refer to in formal languages.

Homework 6

1 Question to fill in. ◁

to do

Homework 7

1 [Enderton, ex. 1, 2, 5, p. 17] Translate between English and the specified first-order language.

(a) Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting. ($\forall$, for all things; N, is a number; I, interesting; $<$, is less than; 0, a constant symbol intended to denote zero.)

(b) There is no number such that no number is less than it. (The same language as in Item a.)

(c) $\forall x(Nx \to Ix \to \neg\forall y(Ny \to Iy \to \neg x < y))$. (The same language as in Item a. There exists a relatively concise translation.)

(d) (i) You can fool some of the people all of the time. (ii) You can fool all of the people some of the time. (iii) You can't fool all of the people all of the time. ($\forall$, for all things; P, is a person; T , is a time; $F\ x\ y$, you can fool x at y. One or more of the above may be ambiguous, in which case you will need more than one translation.) $\lhd$

(a) $\forall x(Nx \land \neg Ix \land \forall y(Ny \land y < x \to Iy) \to Ix)$.

(b) $\neg\exists x(Nx \land \neg\exists y(Ny \land y < x))$.

(c) Any interesting number is less than some interesting number.

(d) (i) Here ambiguity is in that it either says that there are some (fixed) people you can fool all time, or says that at every moment there are (some, not fixed) people you can fool, i.e. either $\exists x(Px \land \forall y(Ty \to Fxy))$ or $\forall y(Ty \to \exists x(Px \land Fxy))$. (ii) Here ambiguity is in that it either says that you can fool each person at some time (times can be different for different people), or says that at some (fixed) time you can fool everyone (at that specific time): $\forall x(Px \to \exists y(Ty \land Fxy))$ or $\exists y(Ty \land \forall x(Px \to Fxy))$. (iii) $\neg\forall x(Px \to \forall y(Ty \to Fxy))$.

2 [Enderton, ex. 1, p. 129] For a term u, let u^x_t be the expression obtained from u by replacing the variable x by the term t. Restate this definition without using any form of the word "replace" or its synonyms. $\lhd$

For term t and variable x, we define $\sigma_{x\mapsto t} : V \to T$, where V is the set of variables, T the set of terms and $\sigma_{x\mapsto t}$ identity except that it maps x to t and then recursively define the extension $\overline{\sigma_{x\mapsto t}} : T \to T$:

1. For each variable v, $\overline{\sigma_{x\mapsto t}}(v) = h(v)$.
2. For each constant symbol c, $\overline{\sigma_{x\mapsto t}}(c) = c$.
3. For terms $t_1, \ldots, t_n$ and n-place function symbol f,
$$\overline{\sigma_{x\mapsto t}}(f(t_1, \ldots, t_n)) = f(\overline{\sigma_{x\mapsto t}}(t_1), \ldots, \overline{\sigma_{x\mapsto t}}(t_n)).$$

3 [Enderton, ex. 9, p. 130]

(a) Show by two examples that $(\varphi^x_y)^y_x$ is not in general equal to φ, where the first shows that x may occur in $(\varphi^x_y)^y_x$ at a place where it does not occur in φ and the second shows that x may occur in a φ at a place where it does not occur in $(\varphi^x_y)^y_x$.

(b) Prove *Re-replacement lemma*: If y does not occur in φ, then x is substitutable for y in φ_y^x and $(\varphi_y^x)_x^y = \varphi$. $\triangleleft$

(a) $\varphi = P\ y$ (y occurs free in φ) and $\forall y\ P\ x$ (not substitutable).

(b) We use induction on φ.

Case 1: For atomic $\varphi = P\ t_1, \ldots, t_n$, we have that x is substitutable for y in φ_y^x and that

$$(\varphi_y^x)_x^y = ((P\ t_1, \ldots, t_n)_y^x)_x^y = P\ ((t_1)_y^x)_x^y, \ldots, ((t_n)_y^x)_x^y = \varphi.$$

Case 2: Given the inductive hypothesis, the inductive step holds by definition for formula building operations $\mathcal{E}_\neg$, $\mathcal{E}_\to$ and $\mathcal{Q}_i$, where $v_i \neq x$ and $v_i \neq y$ (since y does not occur in φ).
Case 3: $\varphi = \forall x\ \psi$. Then $(\forall x\ \psi)_y^x = \forall x\ \psi$, in which y does not occur (free, and thus x is substitutable.) Therefore $(\varphi_x^y)_y^x = ((\forall x\ \psi)_y^x)_x^y = (\forall x\ \psi)_y^x = \forall x\ \psi = \varphi$.

Homework 8

1 [Enderton, ex. 4, 7, 10, p. 130] Show by deduction that

1. $\vdash \forall x\ \varphi \to \exists x\ \varphi$;
2. $\vdash \exists x(Px \to \forall x\ Px)$;
3. $\{Qx, \forall y(Qy \to \forall z\ Pz)\} \vdash \forall x\ Px$.
4. $\forall x \forall y\ Pxy \vdash \forall y \forall x\ Pyx$. ◁

to do

2 Give a complete proof of The Existence of Alphabetic Variants. ◁

to do

3 [Enderton, ex. 15, p. 131] Prove **Rule EI**: Assume that the constant symbol c does not occur in φ, ψ or Γ, and that $\Gamma; \varphi^x_c \vdash \psi$. Then $\Gamma; \exists x\ \varphi \vdash \psi$ and there is a deduction of ψ from $\Gamma; \exists x\ \varphi$ in which c does not occur. ("EI" stands for "existential instantiation".) Then use it to show that deductions from $\emptyset$ of the following formulas exist:

1. $\exists x\ \alpha \vee \exists x\ \beta \leftrightarrow \exists x(\alpha \vee \beta)$;
2. $\forall x\ \alpha \vee \forall x\ \beta \to \forall x(\alpha \vee \beta)$. ◁

to do

Homework 9

1 [Enderton, ex. 1, p. 99] Show that (a) $\Gamma; \alpha \vDash \varphi$ iff $\Gamma \vDash (\alpha \to \varphi)$; and (b) $\varphi \vDash\Dashv \psi$ iff $\vDash (\varphi \leftrightarrow \psi)$. ◁

to do

2 Show that if x does not occur free in α, then $\alpha \vDash \forall x \alpha$. ◁

to do

3 Show that a formula θ is valid iff $\forall\theta$ is valid. ◁

to do

4 Restate the definition of "$\mathfrak{A}$ satisfies φ with s" by defining recursively a function $\overline{h}$ such that $\mathfrak{A}$ satisfies φ with s iff $s \in \overline{h}(\varphi)$. ◁

to do

5 [Enderton, ex. 9, p. 100] Assume that the language has equality and a two-place predicate symbol P. For each of the following conditions, find a sentence σ such that the structure $\mathfrak{A}$ is a model of σ iff the condition is met.

(a) $|\mathfrak{A}|$ has exactly two members.

(b) $P^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|$ into $|\mathfrak{A}|$. (A *function* is a single-valued relation, as in Chapter 0. For f to be a function from A into B, the domain of f must be all of A; the range of f is a subset, not necessarily proper, of B.)

(c) $P^{\mathfrak{A}}$ is a permutation of $|\mathfrak{A}|$; i.e., $P^{\mathfrak{A}}$ is a one-to-one function with domain and range equal to $|\mathfrak{A}|$. ◁

to do

Homework 10

Homework 11

1 [Enderton, ex. 4, p. 146] Let $\Gamma = \{\neg\forall v_1 Pv_1, Pv_2, Pv_3, \dots\}$. Is Γ consistent? Is Γ satisfiable? $\triangleleft$

to do

2 [Enderton, ex. 7, p. 146] For each of the following sentences, either show there is a deduction or give a counter-model (i.e., a structure in which it is false.)

(a) $\forall x(Qx \to \forall y\, Qy)$

(b) $(\exists x\, Px \to \forall y\, Qy) \to \forall z(Pz \to Qz)$

(c) $\forall z(Pz \to Qz) \to (\exists x\, Px \to \forall y\, Qy)$

(d) $\neg\exists y\, \forall x(Pxy \leftrightarrow \neg Pxx)$ $\triangleleft$

to do

3 [Enderton, ex. 8, p. 146] Assume the language (with equality) has just the parameters $\forall$ and P, where P is a two-place predicate symbol. Let $\mathfrak{A}$ be the structure with $|\mathfrak{A}| = \mathbb{Z}$, the set of integers (positive, negative, and zero), and with $\langle a, b\rangle \in P^{\mathfrak{A}}$ iff $|a - b| = 1$. Thus $\mathfrak{A}$ looks like an infinite graph:

$$\cdots \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \cdots$$

Show that there is an elementarily equivalent structure $\mathfrak{B}$ that is not connected. (Being *connected* means that for every two members of $|\mathfrak{B}|$, there is a path between them. A *path* — of length n — from a to b is a sequence $\langle p_0, p_1, \dots, p_n\rangle$ with $a = p_0$ and $b = p_n$ and $\langle p_i, p_{i+1}\rangle \in P^{\mathfrak{B}}$ for each i.) *Suggestion*: Add constant symbols c and d. Write down sentences saying c and d are far apart. Apply compactness. $\triangleleft$

to do

4 [Enderton, ex. 11, p. 100] For each of the following relations, give a formula which defines it in $(\mathbb{N}; +, \cdot)$. (The language is assumed to have equality and the parameters $\forall$, $+$, and $\cdot$).

1. $\{0\}$.
2. $\{1\}$.
3. $\{\langle m, n\rangle \mid n \text{ is the successor of } m \text{ in } \mathbb{N}\}$.
4. $\{\langle m, n\rangle \mid m < n \text{ in } \mathbb{N}\}$. $\triangleleft$

Assuming Definability

5 [Enderton, ex. 6, p. 146] Let Σ_1 and Σ_2 be sets of sentences such that nothing is a model of both Σ_1 and Σ_2. Show that there is a sentence τ such that

$$\text{Mod } \Sigma_1 \subseteq \text{Mod } \tau \quad \text{and} \quad \text{Mod } \Sigma_2 \subseteq \text{Mod } \neg\tau.$$

(This can be stated: Disjoint EC_Δ classes can be separated by an EC class.) *Suggestion*: $\Sigma_1 \cup \Sigma_2$ is unsatisfiable; apply compactness. $\triangleleft$

Assuming Mod