Notes on A Mathematical Introduction to Logic

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July 9, 2025

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Chapter 0

Useful Facts about Sets

Lem 00A (0A) Assume that $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_m, \ldots, y_{m+k} \rangle$ Then

$$x_1 = \langle y_1, \ldots, y_{k+1} \rangle. \qquad \diamond$$

Proof Sketch. We use induction on m. $\dashv$

Suppose that A is a set such that no member of A is a finite sequence of other members, then if $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_n \rangle$, and each x_i and y_j is in A, the by the above lemma $m = n$. Whereupon we have $x_i = y_i$ as well.

Thm 00B (0B) Let A be a countable set. Then the set of all finite sequences of members of A is also countable. $\diamond$

Proof. The set S of all such finite sequences can be characterized by the equation

$$S = \bigcup_{n \in \mathbb{N}} A^{n+1}.$$

Since A is countable, we have a function f mapping A one-to-one into $\mathbb{N}$. The basic idea is to map S one-to-one into $\mathbb{N}$ by assigning to $\langle a_0, a_1, \ldots, a_m \rangle$ the number $2^{f(a_0)+1} 3^{f(a_1)+1} \cdots p_m^{f(a_m)+1}$, where p_m is the $(m+1)$st prime. This suffers from the defect that this assignment might not be well-defined. For conceivably there could be $\langle a_0, a_1, \ldots, a_m \rangle = \langle b_0, b_1, \ldots, b_n \rangle$, with a_i and b_j in A but with $m \neq n$. But this is not serious; just assign to each number of S the *smallest* number obtainable in the above fashion. This gives us a well-defined map; it is easy to see that it is one-to-one. $\dashv$

Chapter 1

Sentential Logic

1.0 Informal Remarks on Formal Languages

To describe a formal language, we specify:

1. The set of *symbols* (the alphabet).
2. The definition of *wff*s.
3. The meaning of the language, or its rule for translation into natual languages.

Item 3 is of less importance, especially considering that digital computers carry out caculation according to programming languages, which are formal languages, without knowledge of the meaning of them.

1.1 The Language of Sentential Logic

Symbols are described as follows.

1. Logical symbols.
   (a) Sentential connective symbols.
   (b) Parentheses.
2. Nonlogical symbols (parameters or proposition or sentence symbols).

Note that symbols can be themselves formulas in another language, say a first-order language. Note also that we assume no symbols is a finite sequence of other symbols, to assure that finite sequences of symbols are uniquely decomposable (see 00A, and the subsequent remarks).

An *expression* is a finite sequence of symbols.

We introduce *formula-building operation* $\mathcal{E}$ before the definition of *wffs*.

$$\begin{aligned}\mathcal{E}_\neg(\alpha) =& (\neg\alpha)\\ \mathcal{E}_\wedge(\alpha,\beta) =& (\alpha\wedge\beta)\\ \mathcal{E}_\vee(\alpha,\beta) =& (\alpha\vee\beta)\\ \mathcal{E}_\to(\alpha,\beta) =& (\alpha\to\beta)\\ \mathcal{E}_\leftrightarrow(\alpha,\beta) =& (\alpha\leftrightarrow\beta)\end{aligned}$$

We want our definition of *wffs* as a formal version of the following descrptions.

1. Every sentence symbol is a wff.
2. If α and β are wffs, then so are $\mathcal{E}_\neg(\alpha), \mathcal{E}_\wedge(\alpha,\beta), \mathcal{E}_\vee(\alpha,\beta), \mathcal{E}_\to(\alpha,\beta)$ and $\mathcal{E}_\leftrightarrow(\alpha,\beta)$.
3. No expression is a wff, unless it is compelled to be one by Item 1 and 2.

The following principle qualifies.

Thm 11A Induction Principle If S is a set of wffs containing all the sentence symbols and closed under $\mathcal{E}$, then S is the set of all wffs. $\diamond$

Proof Sketch. For arbitrary wff α, consider some construction sequence $\langle\varepsilon_1,\dots,\varepsilon_n\rangle$. We use strong induction on $i \leq n$ with the hypothesis claiming $\forall j < i\ \varepsilon_j \in S$. $\dashv$

Intuitively, an arbitrary wff α is built up by finitely applying $\mathcal{E}$, and thus belongs to S. Also, note that S contains only wffs, by starting upon some of them.

Eg 11B Any expression with more left parentheses than right parentheses is not a wff. $\diamond$

Proof Sketch. We prove that the set of "balanced" wffs contains all sentence symbols and is closed under $\mathcal{E}$ and apply 11A. $\dashv$

Exercises

1 Question to fill in. $\triangleleft$

Omitted as I do not like this exercise.

2 Question to fill in. $\triangleleft$

For the first part, we show first that any wff, in whose construction $\mathcal{E}$ is used at most 2 times, is not of length 2, 3, or 6. Then we show that the length of any other wff is at least 7. For the second part, We use induction on the length of a wff.

3 Question to fill in. $\triangleleft$

We use induction on the number of times $\mathcal{E}$ is used in the construction of a wff.

4 Question to fill in. $\triangleleft$

* We can show that the last expression in the sequence containing A_4 is not used by any other expression in the sequence, and neither is any expression containing A_4. Note that an expression in the sequence may not be used in its successors.

5 Question to fill in. $\triangleleft$

1. We use induction on the number of times $\mathcal{E}$ is used in the construction of a wff.
2. We use again induction to show that the length is of the form $4k+1$.

1.2 Truth Assignments

A *truth assignment* v for a set $\mathcal{S}$ of sentence symbols is a function

$$v : \mathcal{S} \to \{F, T\}$$

Let $\mathcal{S}$ be the set of wffs that can be built up from $\mathcal{S}$ by $\mathcal{E}$. An extention $\overline{v}$ of v

$$\overline{v} : \overline{\mathcal{S}} \to \{F, T\}$$

is intuitively defined on $\overline{\mathcal{S}}$.

Thm 12A (12A) For any truth assignment v for a set $\mathcal{S}$ these is a unique extension $\overline{v}$ on $\overline{\mathcal{S}}$. $\diamond$

Proof. See 1.3. $\dashv$

A truth assignment v *satisfies* φ iff $\overline{v}(\varphi) = T$.

Defn 12B A set Σ of wffs *tautologically implies* a wff τ (written $\Sigma \vDash \tau$) iff every truth assignment for the sentence symbols in Σ and τ that satisfies every member of Σ also satisfies τ. Say that τ is a *tautology* (written $\vDash \tau$) when $\varnothing \vDash \tau$. If Σ is singleton $\{\sigma\}$, then we write "$\sigma \vDash \tau$" instead. If both $\sigma \vDash \tau$ and $\tau \vDash \sigma$, then they are said to be *tautologically equivalent* (written $\sigma \vDash\dashv \tau$). $\diamond$

17B is a nontrivial fact.

Applying the *truth-table* method to a wff with n sentence symbols requires 2^n lines. By the way, the famous "*P versus NP*" problem can be formulated as, might there be some general method that, given any wff α with n sentence symbols, will determine whether or not α is a tautology in polynomial time?

Eg 12C It is hardly evident that

$$((((P \wedge Q) \to R) \to S) \to ((P \to R) \to S))$$

is a tautology, unless we observe

$$\begin{aligned} (P \wedge Q) &\models P, \\ (P \to R) &\models ((P \wedge Q) \to R), \\ (((P \wedge Q) \to R) \to S) &\models ((P \to R) \to S). \end{aligned} \qquad \diamond$$

Exercises

1 Question to fill in. ◁

Omitted as trivial. One will find it highly nontrivial if they neglect 'neither', as I did.

2 Question to fill in. ◁

Omitted as trivial. We can consider this exercise intuitively by observing that $P \to Q$ is as helpful as a tautology to the truthfulness of P.

3 Question to fill in. ◁

Omitted as trivial.

4 Show that

(a) $\Sigma; \alpha \models \beta$ iff $\Sigma \models (\alpha \to \beta)$.

(b) $\alpha \models\!\!\!\!\!\!\;\dashv \beta$ iff $\models (\alpha \to \beta)$. ◁

(a) Omitted for brevity. A key step should be to state that "for each truth assignment $\overline{v}$ that satisfies $\Sigma; \alpha$, it also satisfies β" iff "for each truth assignment $\overline{v}$ that satisfies Σ, if $\overline{v}(\alpha) = T$ then $\overline{v}(\beta) = T$".

(b) "For each truth assignment $\overline{v}$ that satisfies α, it also satisfies β" iff "for each truth assignment $\overline{v}$, if $\overline{v}(\alpha) = T$ then $\overline{v}(\beta) = T$." The rest is omitted for brevity.

5 Prove or refute each of the following assertions:

(a) If either $\Sigma \models \alpha$ or $\Sigma \models \beta$, then $\Sigma \models (\alpha \vee \beta)$.

(b) If $\Sigma \models (\alpha \vee \beta)$, then either $\Sigma \models \alpha$ or $\Sigma \models \beta$. ◁

* (a) Obviously true. (b) $\vDash (\alpha \vee \neg\alpha)$, but in general neither $\vDash \alpha$ nor $\vDash (\neg\alpha)$ (except one of them is tautology.) Actually (b) is true iff Σ is complete (26X).

6 Question to fill in. $\triangleleft$

(a) Let B be the set of all sentence symbols in α, C the set generated from B by $\mathcal{E}$ and the set on which $\bar{v}_1$ and $\bar{v}_2$ agree S. We can show that $\alpha \in C$, $B \subseteq S$ and S is inductive.

(b) Let $\mathcal{S}' \subseteq \mathcal{S}$ be the set of sentence symbols used in $\Sigma; \tau$. Every truth assignment for $\mathcal{S}'$ that satisfies every member of Σ agrees with some truth assignment for S that satisfies every member of Σ, and thus satisfies τ.

7 Question to fill in. $\triangleleft$

Which fork leads to the capital, if you belong to the people who always tell falsehoods?

8 (*Substitution*) Consider a sequence $\alpha_1, \alpha_2, \dots$ of wffs. For each wff φ let φ^* be the result of replacing the sentence symbol $\mathrm{A_n}$ by α_n for each n.

(a) Let v be a truth assignment for the set of all sentence symbols; define u to be the truth assignment for which $u(A_n) = \bar{v}(\alpha_n)$. Show that $\bar{u}(\varphi) = \bar{v}(\varphi^*)$.

(b) Show that if φ is a tautology, then so is φ^*. $\triangleleft$

(a) Say S is the set of wffs such that satisfies what substitution requires. We can show by induction that S include all sentence symbols and is closed under $\mathcal{E}$.

(b) It follows trivially from (a).

9 (*Duality*) Let α be a wff whose only connective symbols are $\wedge, \vee$ and $\neg$. Let α^* be the result of interchanging $\wedge$ and $\vee$ and replacing each sentence symbol by its negation. Show that α^* is tautologically equivalent to $(\neg\alpha)$. $\triangleleft$

Say D is the set of wffs such that $\overline{v}(\neg\alpha) = \overline{v}(\alpha^*)$, where α is a wff and $\overline{v}$ is any truth assignment for α. We can show that D contains all sentence symbols and is closed under $\mathcal{E}_\neg$, $\mathcal{E}_\wedge$ and $\mathcal{E}_\vee$. We can apply substitution (Ex.1.2.8) in the process.

10 Say that a set Σ_1 of wffs is *equivalent* to a set Σ_2 of wffs iff for any wff α, we have $\Sigma_1 \vDash \alpha$ iff $\Sigma_2 \vDash \alpha$. A set Σ is *independent* iff no member of Σ is tautologically implied by the remaining members in Σ. Show that the following hold.

(a) A finite set of wffs has an independent equivalent subset.

(b) An infinite set need not have an independent equivalent subset.

(c) Let $\Sigma = \{\sigma_0, \sigma_1, \dots\}$, show that there is an independent equivalent set Σ'. $\triangleleft$

(a) We do the following to any finite set Σ of wffs until we obtain a independent subset of it: remove a wff $\alpha \in \Sigma$ such that $\Sigma - \alpha \vDash \alpha$. We claim that equivalence is kept during this procedure. A key part of the proof of this claim should be: a truth assignment v satisfies Σ iff it satisfies $\Sigma - \alpha$.

(b) Note that we are supposed to provide a counterexample by giving a set that is infinite and not independent. One could be $\Gamma = \{A_1, (A_1 \wedge A_2), \ldots, (A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_n), \ldots\}$. * The set of independent subsets of Γ is $\{\emptyset, \{A_1\}, \{A_1 \wedge A_2\}, \ldots, \{A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_n\}, \ldots\}$, in which none is equivalent to Γ.

(c) The leading motivation during our set construction should be to achieve "independence", which means that we should make sure that for each formula σ_i some truth assignment $\overline{v}$ exists such that $\overline{v}(\sigma_i) = F$ and $\overline{v}(\sigma) = T$, where σ is any formula other than σ_i. That is why we may consider

$$\Sigma' = \{\sigma_0, (\sigma_0 \to \sigma_1), ((\sigma_0 \wedge \sigma_1) \to \sigma_2), \ldots,$$
$$((\sigma_0 \wedge \sigma_1 \wedge \cdots \wedge \sigma_n) \to \sigma_{n+1}), \ldots\} \setminus \{\sigma :\vDash \sigma\}.$$

It is equivalent to Σ, for we can verify that $\Sigma' \vDash \Sigma$ using induction and $\Sigma \vDash \Sigma'$ is trivial. We can also easily verify that it is independent.

11 Question to fill in. ◁

This is trivial considering that $\leftrightarrow$ is associative and commutative.

12 Question to fill in. ◁

Omitted for brevity.

13 Question to fill in. ◁

Watching Tennis. The formalization is omitted for brevity.

14 Question to fill in. ◁

Omitted for it is a trivial specification of Ex.1.2.6(a).

15 Question to fill in. ◁

We could use a truth table, but the results are straightforward if we consider to simplify (b) and (c).

1.3 A Parsing Algorithm

A parsing algorithm for wffs is crucial (explained in 1.4) in that it guarantees the uniqueness of the construction trees of wffs, and thus proves the existence of the extension $\overline{v}$ of a truth assignment v. That is to say, For any wff φ there is a unique tree constructing it, so we can unambiguously arrive at a value for $\overline{v}(\varphi)$.

We must show that with our current notation (with the parentheses) ambiguity does not arise. There is one sense in which this fact is unimportant: If it failed, we would simply change notation until it was true.

This paragraph introduces some technical details and is not of vital importance. To describe our algorithm, we might first prove that every wff has the same number of left as right parentheses (see 11B) and that any proper initial segment of a wff contains an excess of left parentheses (which can be proved similarly). The essence of the algorithm is that we know when to stop when we recognize a vertex according to the count of parentheses, scanning a expression from the left until first observing a balance between left and right parentheses. Also, the procedure is finite and the given expression is not a wff if it is rejected.

Polish notation avoids both ambiguity and parentheses.

Conventions on notations are similar to 2.1.

Exercises

1 Question to fill in. $\triangleleft$

Omitted as trivial.

2 Question to fill in. $\triangleleft$

$\alpha =(\gamma_0 \wedge \theta_0),$
$\beta =\beta_0,$
$\gamma =(\gamma_0,$
$\delta =\theta_0) \wedge \beta_0.$

3 Question to fill in. $\triangleleft$

Omitted as trivial.

4 Question to fill in. $\triangleleft$

A working intuition can be to put back the right parentheses. We do this by examining vertices. Specifically, we mark all sentence symbols as vertices, and then proceed iteratively with the following procedure:

1. If any connective symbol is on the proper position of vertices, a right parenthesis is added accordingly.

2. A newly added right parenthesis is then paired to the closest left parenthesis on its left, forming a new vertex.

A tree can be therefore constructed. We should then examine various properties of this tree, especially that it remains identical with or without right parentheses.

An other observation, as suggested, states that any wff

1. has the same number of parentheses as connective symbols and,

2. is a sentence symbol *or* ends with a connective symbol followed by a sentence symbol.

We can state the algorithm accordingly.

5 The English language has a tendency to use two-part connectives: "both ... and ..." "either ... or ..." "if ..., then ...." How does this affect unique readability in English? ◁

I personally, due to limited proficiency in English, do not feel constrained or ambiguity when expressing using these connectives. We might though note that "if" often means iff in daily usage. For example, "you can pass the exam if you score more than 60 percent of the points".

6 We have given an algorithm for analyzing a wff by constructing its tree from the top down. There are also ways of constructing the tree from the bottom up. This can be done by looking through the formula for innermost pairs of parentheses. Give a complete description of an algorithm of this sort. ◁

Omitted for brevity. This can be done very similarly to what we did in Ex.1.3.4. The key is to recognize the vertices in a bottom up manner.

7 Question to fill in. ◁

Yes. The corresponding algorithm is the same as that in Exercise 6 (which is, unfortunately, omitted). This can be obvious considering that the algorithm never directly worked with parentheses but with lower vertices on our parsing tree. Also we may note that any sff has twice as many parentheses as connective symbols and give accordingly an parsing algorithm. Remember in both ways we need to carefully reject expressions that are not wffs.

1.4 Induction and Recursion

Induction

We may occationally want to seek a smallest subset of a set U whose members can be built up from some initial elements by applying some operations some finite number of times.

Defn 14A Let U be the set of expressions, $B \subseteq U$ an initial set and a class $\mathcal{F}$ of functions, for the sake of simplicity, containing two members f and g, where

$$f : U \times U \to U \text{ and } g : U \to U.$$

Say C is the set we wish to construct, namely the set *generated* from B by $\mathcal{F}$. We give two definitions ($C^\star$ and $C_\star$) of C and verify their equivalence.

1. Say a subset S of U is *inductive* iff $B \subseteq S$ and S is closed under f and g. Let $C^\star$ be the intersection of all the inductive subsets of U. Note that $C^\star$ is the smallest (by which we mean, being a subset of any set in comparison) inductive set.

2. Temporarily define a *construction sequence* to be a finite sequence $\langle x_1, \dots, x_n \rangle$ of elements of U such that for each $i \leq n$ we have at least one of

$$\begin{aligned} &x_i \in B, \\ &x_i = f(x_j, x_k) \text{ for some } j < i, k < i, \\ &x_i = g(x_j) \text{ for some } j < i. \end{aligned}$$

   Let $C_\star$ be the set of points x such that some construction sequence ends with x. To better describe this, let C_n be the set of points x such that some construction sequence of length n ends with x. Then

$$B = C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots \text{ and } C_\star = \bigcup_n C_n. \qquad \diamond$$

Now we verify their equivalence.

1. To demonstrate that $C^\star \subseteq C_\star$ it suffices to show that $C_\star$ is inductive, which is obvious.

2. For a construction sequence $\langle x_0, \dots, x_n \rangle$, we use strong induction (which is a little bit similar to what we did in 11A) on i to see that $x_i \in C^\star, i \leq n$, thus $x_n \in C^\star$ and hence $C_\star \subseteq C^\star$.

We have consequently an extended version of 11A:

Thm 14B Induction Principle If S is a subset of C that is inductive then $S = C$. $\diamond$

Proof. It suffices to show $C \subseteq S$, which is given by definition $C^\star$. $\dashv$

Eg 14C Inductively constructed sets.

1. Let U be the set of all real numbers, $B = \{0\}$ and $\mathcal{F} = \{S\}$. Then C is the set of natural numbers.

2. Let U contain all functions whose domain and range are each sets of real numbers, B contain the identity function and all constant functions and $\mathcal{F}$ contain the operations of addition, multiplication, division and root extraction. Then C is the set of algebraic functions.

3. The set of wffs, as a generated set, is of special interest in that each consitituent in the family tree of a wff is a proper segment of the end product.

4. Let U be a set of propositions that can be indexed by natual numbers like P_i, $B = \{P_0\}$, $\mathcal{F} = \{f : P_i \to P_{S(i)}\}$, C the set generated from B by $\mathcal{F}$ (we have a trivial fact that $C = U$), and T the largest subset of U whose members are true. If we can show that $B \subseteq T$ and T is closed under f (in other words, T is inductive), then it follows that $T = C = U$, according to 14B. $\diamond$

Recursion

The problem we now want to consider is that of defining a function on C recursively, which means, when given $h(x)$ on B, we would like to extend it to $\overline{h}(x)$ on C with regard to given rules for computing $\overline{h}(f(x, y))$ and $\overline{h}(g(x))$.

Such $\overline{h}$ may not exist, in that if any member of C can be constructed according to two different trees, we can easily design the rules to generate contradiction. We are basically intended to construct $\overline{h}$ as a homomorphism from C into ran $\overline{h}$, which is not likely to exist, except that C is *freely* generated, which is described as follows:

1. f_C and g_C are one-to-one.

2. Ran f_C, ran g_C and B are pairwise disjoint.

Thm 14D Recursion Theorem Assume that the subset C of U is freely generated from B by f and g, where

$$\begin{aligned} f : U \times U \to& U, \\ g : U \to& U. \end{aligned}$$

Further assume that V is a set and F, G and h are functions such that

$$\begin{aligned} h : B \to& V, \\ F : V \times V \to& V, \\ G : V \to& V. \end{aligned}$$

Then there is a unique function

$$\overline{h} : C \to V$$

such that

1. For x in B, $\overline{h}(x) = h(x)$;
2. For x, y in C,

$$\begin{aligned} \overline{h}(f(x,y)) =& F(\overline{h}(x), \overline{h}(y)), \\ \overline{h}(g(x)) =& G(\overline{h}(x)). \end{aligned} \qquad \diamond$$

Proof Sketch. The idea is to construct $\overline{h}$ as the union of many approximating functions. Temporarily call a function v acceptable iff dom v is a subset of C, ran v a subset of V, and for any x and y in C:

1. If x belongs to B and to dom v, then $v(x) = h(x)$.
2. If $f(x, y)$ belongs to the domain of v, then so do x and y, and $v(f(x, y)) = F(v(x), v(y))$. If $g(x)$ belongs to the domain of v, then so does x, and $v(g(x)) = G(v(x))$.

Let K be the collecion of all acceptable functions, and let $\overline{h} = \bigcup K$, thus $\langle x, z\rangle \in \overline{h}$ iff $v(x) = z$ for some acceptable v.

We claim that $\overline{h}$ meets our requirements, which we can prove in four steps:

1. We claim that $\overline{h}$ is a function. Let

$$S = \{x \in C | \text{ for at most one } z, \langle x, z\rangle \in \overline{h}\}$$

   We can then show that $S = C$ by verifying that S is inductive by using what we require from v. A tricky thing to think of is that S is not the domain of $\overline{h}$.

2. We claim that $\overline{h}$ itself is an acceptable function.

3. We claim that $\overline{h}$ is defined throughout C. It suffices to show that the domain of $\overline{h}$ is inductive. It is here that the assumption of freeness is used. For example is the following: Suppose that x is in the domain of $\overline{h}$. Then $\overline{h}; \langle g(x), G(\overline{h}(x))\rangle$ is acceptable. Consequently, $g(x)$ is in the domain of $\overline{h}$.

4. We claim that $\overline{h}$ is unique, which follows from step 1 and step 2. $\dashv$

The detailed proof in the book (page 42) was done in a respected manner and is worth reading (except for the 4th step).

This theorem says that any map h of B into V can be extended to a homomorphism $\overline{h}$ from C (with operations f and g) into V (with operations F and G), if C is freely generated.

Note that the set of wffs is freely generated from the set of sentence symbols by $\mathcal{E}$ (because of 1.3).

Exercises

1 Question to fill in. $\triangleleft$

Omitted as trivial.

2 Question to fill in. $\triangleleft$

We can show that $\to\wedge$ is not a proper segment of any wff by considering a inductive subset of the set of wffs, any of whose members does not have $\to\wedge$ as a proper segment.

3 Question to fill in. $\triangleleft$

Omitted as trivial. Note that that $\mathcal{F}$ is restricted did not play a definite part in the original proof.

1.5 Sentential Connectives

Suppose that α is a wff whose sentence symbols are at most $A_1, \ldots, A_n$. We say that an n-place Boolean function B^n_α is *realized* by α, if

$$\begin{aligned} B^n_\alpha(X_1, \ldots, X_n) = {} & \text{the truth value given to } \alpha \text{ when} \\ & A_1, \ldots, A_n \text{ are given the values } X_1, \ldots, X_n. \end{aligned}$$

Thm 15A (15A) Let α and β be wffs whose sentence symbols are among $A_1, \ldots, A_n$. Then

1. $\alpha \vDash \beta$ iff for all $X \in \{F, T\}^n$, $B_\alpha(X) \leq B_\beta(X)$.
2. $\alpha \vDash\dashv \beta$ iff $B_\alpha = B_\beta$.
3. $\vDash \alpha$ iff B_α is the constant function with the value T. $\diamond$

Proof of 1.

$$\begin{aligned}
\alpha \vDash \beta \text{ iff for all } 2^n \text{ assignments } v, &\qquad \overline{v}(\alpha) = T \Rightarrow \overline{v}(\beta) = T,\\
\text{iff for all } 2^n\ n\text{-tuples } X, &\qquad B^n_\alpha(X) = T \Rightarrow B^n_\beta(X) = T,\\
\text{iff for all } 2^n\ n\text{-tuples } X, &\qquad B^n_\alpha(X) \leq B^n_\beta(X),
\end{aligned}$$

where $F < T$. $\dashv$

Thm 15B (15B) Let G be an n-place Boolean function, $n \geq 1$. We can find a wff α such that $G = B^n_\alpha$, i.e., such that α realizes G. $\diamond$

Proof. Case I: G is the constant function with value F. Let $\alpha = A_1 \wedge \neg A_1$.

Case II: Otherwise there are k points at which G has the value T, $k > 0$. List these:

$$\begin{aligned}
X_1 &= \langle X_{11}, X_{12}, \ldots, X_{1n} \rangle,\\
X_2 &= \langle X_{21}, X_{22}, \ldots, X_{2n} \rangle,\\
&\cdots\\
X_k &= \langle X_{k1}, X_{k2}, \ldots, X_{kn} \rangle.
\end{aligned}$$

Let

$$\begin{aligned}
\beta_{ij} &= \begin{cases} A_j & \text{iff } X_{ij} = T,\\ (\neg A_j) & \text{iff } X_{ij} = F, \end{cases}\\
\gamma_i &= \beta_{i1} \wedge \cdots \wedge \beta_{in},\\
\alpha &= \gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_k.
\end{aligned}$$

We claim that $G = B^n_\alpha$, and the proof is omitted as trivial. $\dashv$

Say a wff α is in *disjunctive normal form* (abbr. DNF), if α is a disjunction

$$\alpha = \gamma_1 \vee \cdots \vee \gamma_k,$$

where each γ_i is a conjunction

$$\gamma_i = \beta_{i1} \wedge \cdots \wedge \beta_{in_i}$$

and each β_{ij} is a sentence symbol or the negation of one. DNFs are desirable in that they explicitly list the truth assignments that satisfy them. CNF is similarly defined and is desirable in that they explicitly list the truth assignments that fial to satisfy them.

Cor 15C (15C) For any wff φ, we can find a tautologically equivalent wff α in disjunctive normal form. $\diamond$

A set B of Boolean functions is *complete*, if every boolean function $G : \{F, T\}^n \to \{F, T\}$ for $n \geq 1$ can be "realized" by B.

It follows from Ex.1.2.9 and 15C that both $\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are complete.

Eg 15D $\{\wedge, \to\}$ is not complete. $\diamond$

Proof. We can show by induction that for any wff α using only these connectives and having A as its only sentence symbol, we have $A \models \alpha$. Thus, none of them are tautologically equivalent to $\neg A$. (This is usually how we prove that a certain set of connectives is not complete.) $\dashv$

There are two 0-place Boolean functions, F and T. For the corresponding connective symbols we take $\bot$ and $\top$. Note that $\bot$ is a wff by itself, which is always assigned F. Also we have $A \to \bot \models \neg A$.

Eg 15E $\{\downarrow\}$, $\{|\}$, $\{\neg, \to\}$ and $\{\bot, \to\}$ are complete. $\diamond$

Exercises

1 Question to fill in. $\triangleleft$

(a) Omitted as trivial.

(b) A valid one should be $((A_1 \vee A_2) \to (\neg(A_3 \vee (A_1 \wedge A_2))))$. The idea is to construct in form $\alpha \to \beta$.

2 Question to fill in. $\triangleleft$

$<$, $>$ and $+$ suffer the same limit that they fail to map two Fs to a T.

1.6 Switching Circuits

We may simulate circuits using wffs (or equivalently, Boolean functions).

Given a circuit (or its wff), it is of particular interest to find an equivalent circuit (or a tautologically equivalent wff) for which the cost (for example, the number of connective symbols used) is a minimum, subject to constraints such as a maximum allowable delay (the depth of the construction tree of the wff). This is easily a highly nontrivial problem, and the book features a briliant example (page 58) which is approached like a dimensionality reduction task.

Relay switch (page 57) is also of particular interest in that it is very easy to realize using circuits and switches.

(For exercise 2) Define a *literal* to be a wff which is either a sentence symbol or the negation of a sentence symbol. An *implicant* of φ is a conjunction α of literals (using distinct sentence symbols) such that $\alpha \vDash \varphi$. For example, in 15C we showed that any satisfiable wff φ is tautologically equivalent to a disjunction $\alpha_1 \vee \cdots \vee \alpha_n$ where each α_i is an implicant of φ. An implicant α of φ is *prime* iff it ceases to be an implicant upon the deletion of any of its literals. The motivation for us to find prime implicants could be to find a wff α in DNF such that $\alpha \vDash\Dashv \varphi$ and that α is of minimum length.

Exercises

1 Question to fill in. ◁

Omitted for brevity. I suggest to use enumeration and to utilize symmetry and no better solutions occurred me.

2 Question to fill in. ◁

1. $((\neg A) \wedge C)$, $(A \wedge B)$ and * $(B \wedge C)$. To see this, we verify the following matrix, which states the assignments satisfying the wff:

$$\begin{array}{ccc} A & B & C \\ T & T & T \\ T & T & F \\ F & F & T \\ F & T & T \end{array}$$

   For example, we may consider two implicants $(A \wedge B \wedge C)$ and $(A \wedge B \wedge \neg C)$, thus the assignment of C doesn't really matter, and hence obtain $(A \wedge B)$.

2. $(((\neg A) \wedge C) \vee (A \wedge B))$ and * $(((\neg A) \wedge C) \vee (A \wedge B) \vee (B \wedge C))$. We would like our disjunctions to span all rows of the matrix.

3 Question to fill in. ◁

The structure of wff did not play a definite part in exercise 2, so this exercise is omitted for brevity.

1.7 Compactness and Effectiveness

Compactness

Call a set Σ of wffs *satisfiable* iff there is a truth assignment that satisfies every member of Σ.

Rmk 17A If a set Σ of wffs is satisfiable, so is at least one of $\Sigma;\alpha$ and $\Sigma;\neg\alpha$, where α is a wff. $\diamond$

Proof. Suppose otherwise for the sake of contradiction.

$$\begin{aligned}
&\Sigma;\alpha \text{ is unsatisfiable}\\
\text{iff }&\text{for any truth assignment } \overline{v} \text{ such that } \overline{v}(\Sigma)=T, \overline{v}(\alpha)=F\\
\text{iff }&\Sigma \vDash \neg\alpha,
\end{aligned}$$

which contradicts that $\Sigma;\neg\alpha$ is unsatisfiable. $\dashv$

Thm 17B Compactness Theorem A set of wffs is satisfiable iff every finite subset is satisfiable. $\diamond$

It is called "compactness theorem", for it does assert the compactness of a certain topological space, and thus can be proved by using Tychonoff's theorem on product spaces.

We temporarily say that Σ is *finitely satisfiable* iff every finite subset of Σ is satisfiable, which coincides with satisfiability according to the compactness theorem.

Proof. The proof consists of two distinct parts. We first take our given finitely satisfiable set Σ and extend it to a maximal such set Δ (such that (1) $\Sigma \subseteq \Delta$, that (2) for any wff α either $\alpha \in \Delta$ or $\neg\alpha \in \Delta$, and that (3) Δ is finitely satisfiable). We then utilize Δ to make a truth assignment that satisfies Σ.

For the first part, let $\alpha_1, \alpha_2, \ldots$ be a fixed enumeration of the wffs. (This is possible since the set of sentence symbols, and hence the set of expressions, is countable; see 00B.) Define by recursion (on the natural numbers)

$$\begin{aligned}
\Delta_0 =& \Sigma,\\
\Delta_{n+1} =& \begin{cases} \Delta_n;\alpha_{n+1} & \text{if this is finitely satisfiable,}\\ \Delta_n;\neg\alpha_{n+1} & \text{otherwise.}\end{cases}
\end{aligned}$$

We show that this definition is valid by asserting that if Σ is finitely satisfiable, then so is at least one of the sets $\Sigma;\alpha$ and $\Sigma;\neg\alpha$. If not, then $\Sigma_1;\alpha$ and $\Sigma_2;\neg\alpha$ are unsatisfiable for some finite $\Sigma_1 \subseteq \Sigma$ and $\Sigma_2 \subseteq \Sigma$, and hence $\Sigma_1 \cup \Sigma_2; \alpha$ and $\Sigma_1 \cup \Sigma_2; \neg\alpha$ are both unsatisfiable, which contradicts 17A. Let $\Sigma = \bigcup_n \Delta_n$, the limit of the Δ_n's. Note that Δ is finitely satisfiable, for any finite subset is already a finite subset of some Δ_n and hence is satisfiable. Thus we now have a set Δ having properties (1)-(3). This can also be done employing Zorn's lemma, which I am not familiar with for now.

For the second part we define a truth assignment v for the set of all sentence symbols:

$$v(A) = T \text{ iff } A \in \Delta$$

for any sentence symbol A. Consider a set $C = \{\alpha | \overline{v}(\alpha) = T \text{ iff } \alpha \in \Delta\}$, where α is a wff and $\overline{v}$ the extended v, we show by induction as follows that $C = S$, where S is the set of all wffs.

1. B of all sentence symbols is a subset of C according to the definition of v.
2. We show as follows that C is closed under $\mathcal{E}_\neg$ and $\mathcal{E}_\wedge$, since $\{\neg, \wedge\}$ is complete. For $\alpha \in C$ and $\beta \in C$, (1) $\overline{v}(\neg\alpha) = T$ iff $\overline{v}(\alpha) = F$ iff $\alpha \notin \Delta$ iff $\neg\alpha \in \Delta$, (2) $\overline{v}(\alpha \wedge \beta) = T$ iff $\overline{v}(\alpha) = T$ and $\overline{v}(\beta) = T$ iff $\alpha \in \Delta$ and $\beta \in \Delta$ iff $\neg(\alpha \wedge \beta) \notin \Delta$ (otherwise we have $\{\alpha, \beta, \neg(\alpha \wedge \beta)\} \subseteq \Delta$ contradicting that Δ is finitely satisfiable) iff $(\alpha \wedge \beta) \in \Delta$.

Therefore $\overline{v}$ satisfies Δ and also its subset Σ. ⊣

Cor 17C (17A) If $\Sigma \vDash \tau$ then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash \tau$. ⋄

Proof. We contrapositively state the 17B: for a set Σ of wffs, Σ is unsatisfiable iff there is a finite $\Sigma_0 \subseteq \Sigma$ that Σ_0 is unsatisfiable.

$$\begin{aligned} &\Sigma \vDash \tau \\ \text{iff } &\Sigma; \neg\tau \text{ is unsatisfiable} \\ \text{iff } &\text{there is a finite } \Sigma_0 \subseteq \Sigma; \neg\tau \text{ that is unsatisfiable} \end{aligned}$$

If $\neg\tau \in \Sigma_0$ then $\Sigma_0 - \neg\tau; \neg\tau$ is unsatisfiable, thus $\Sigma_0 - \neg\tau \vDash \tau$ and $\Sigma_0 - \neg\tau \subseteq \Sigma$. Else we have that $\Sigma_0 \vDash \tau$. ⊣

Note that this proof can be done in the opposite direction, like what is done on page 60 of the book.

Second Proof. We construct Σ_0 from $\Sigma; \tau$. Let $V = \{v_1, v_2, \ldots, v_k\}$ denote the set of truth assignments that fail to satisfy τ. Apparently $k \leq 2^n$, where n is the number of distinct variables in τ. For each v_i select $\alpha_i \in \Sigma$ such that $\overline{v_i}(\alpha_i) = F$. Let $\Sigma_0 = \{\alpha_i | 1 \leq i \leq k\}$. Thus Σ_0 is finite. We claim that $\Sigma_0 \vDash \tau$. This construction mainly takes advantage of the finiteness of the truth table of τ. ⊣

Effectiveness and Computability[1]

Defn 17D A procedure is *effective*, if it is an algorithm that can be mechanically implemented and takes finite time to execute. A set Σ of expressions is *decidable* iff there exists an effective procedure that, given an expression α, will decide whether or not $\alpha \in \Sigma$. $\diamond$

There are $2^{\aleph_0}$ set of expressions and $\aleph_0$ effective procedures (for there are $\aleph_0$ finite sequences of letters), therefore some sets are not decidable. (Note that procedures and sets of expressions correspond one to one.)

The truth-table is an effective procedure that decides whether or not $\Sigma \vDash \tau$, where $\Sigma; \tau$ is a finite set of wffs. Therefore we have that the set of tautological consequences of Σ is decidable. In particular, the set of tautologies is decidable.

Defn 17E A set A of expressions is *effectively enumerable* iff there exists an effective procedure that lists, in some order, the members of A (or, for any specified member of A, it must eventually appear on the list). Say that A is *semidecidable* iff there exists an effective procedure that, given any expression ε, produces the answer "yes" iff $\varepsilon \in A$. $\diamond$

Similarly, say that a function f is *effectively computable* (or simply *computable*) iff there exists an effective procedure that, given an input x, will produce the output $f(x)$.

***Thm 17F** (17E) A set is effectively enumerable iff it is semidecidable. $\diamond$

Proof. We consider the nontrivial direction, that is, to create a listing of A given a procedure that makes A semidecidable. Note that we can effectively enumerate all expressions:

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$$

Then any scheme that spends infinite effort testing each ε_i qualifies. For example, one step on the testing of ε_1; two steps on the testing of ε_1 and ε_2 each; three steps on the testing of ε_1, ε_2 and ε_3 each, and so forth. Print ε_j if "yes" is produced upon it. $\dashv$

Comment. The following scheme is not very desirable, for (1) time is not well defined (nothing is well defined currently, but *step* is actually better than *time*: it indicates that the procedure *actually* proceeds) and (2) when dividing up time (or even count of steps), it is not obvious that *all* procedures *will* proceed. Suppose T is a fixed period of time. Spend $\frac{T}{2^i}$ testing ε_i in each iteration that takes T. If "yes" is produced upon ε_j, output ε_j on the list and wait the rest of $\frac{T}{2^j}$. In the following iterations, skip ε_j

[1]Discussions in this subsection are carried out informally.

and spend $\frac{T}{2^{(i-1)}}$ testing ε_i for $i > j$. The list's order is fixed due to the determinism of our procedure.

***Thm 17G** (17F) A set of expressions is decidable iff both it and its complement (in a larger decidable set) are effectively enumerable. ◇

Proof. Suppose T is a fixed period of time, Σ a set of expressions and α the expression we would like to decide. We are essentially given two effective procedures. To make an effective procedure that decides whether or not $\alpha \in \Sigma$, we excute both procedures alternately. ⊣

***Thm 17H** (17G) If Σ is a semidecidable set of wffs, then the set of tautological consequences of Σ is effectively enumerable. ◇

Proof. We can effectively enumerate all wffs in Σ:

$$\alpha_1, \alpha_2, \alpha_3, \ldots$$

and the finite subsets of Σ:

$$\emptyset, \{\alpha_1\}, \{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_2, \alpha_3\}, \ldots$$

and the set of tautological consequences of each of the finite subsets. Therefore we can easily design a scheme according to which we can effectively enumerate the tautological consequences of all finite subsets of Σ, which is equivalent to the set of tautological consequences of Σ according to 17C. ⊣

Exercises

1 Question to fill in. ◁

See 17B.

2 Question to fill in. ◁

See 17B.

3 Prove 17B from 17C. ◁

For a non-empty set Σ of wffs and $\alpha \in \Sigma$, we have that

$$\begin{aligned} &\Sigma - \alpha; \alpha \text{ is unsatisfiable} \\ \text{iff } &\Sigma - \alpha \vDash \neg\alpha \\ \text{iff } &\text{there is a finite } \Sigma_0 \subseteq (\Sigma - \alpha) \text{ such that } \Sigma_0 \vDash \neg\alpha \\ \text{iff } &(\Sigma_0; \alpha) \subseteq \Sigma \text{ is unsatisfiable.} \end{aligned}$$

The choice of α is arbitrary, thus we have 17B stated contrapositively.

5 Where Σ is a set of wffs, define a *deduction* from Σ to be a finite sequence $\langle \alpha_0, \ldots, \alpha_n \rangle$ of wffs such that for each $k \leq n$, either (a) α_k is a tautology, (b) $\alpha_k \in \Sigma$, or (c) for some i and j less than k, α_i is $(\alpha_j \to \alpha_k)$ (*modus ponens*). Give a decution from the set $\{\neg \mathrm{S} \vee \mathrm{R}, \mathrm{R} \to \mathrm{P}, \mathrm{S}\}$, the last component of which is P. ◁

$\langle \mathrm{S}, \neg\mathrm{S} \wedge \mathrm{R}, (\neg\mathrm{S} \wedge \mathrm{R}) \to (\mathrm{S} \to \mathrm{R}), \mathrm{S} \to \mathrm{R}, \mathrm{R}, \mathrm{R} \to \mathrm{P}, \mathrm{P} \rangle$.

6 (*Soundness*) Let $\langle \alpha_0, \ldots, \alpha_n \rangle$ be a deduction from Σ. Show that $\Sigma \vDash \alpha_k$ for each $k \leq n$. ◁

We use strong induction and we mainly consider case (c), where for some i and j less than k, $\alpha_i = \alpha_j \to \alpha_k$. For every truth assignment v that satisfies every member of Σ, we have $\bar{v}(\alpha_j) = \bar{v}(\alpha_j \to \alpha_k) = T$ considering the induction hypothesis, then according to the defintion of $\bar{v}$ we have $\bar{v}(\alpha_k)$.

7 (*Completeness*) Show that whenever $\Sigma \vDash \tau$, then there exists a deduction from Σ, the last component of which is τ. ◁

Whenever $\Sigma \vDash \tau$, it follows from 17C that there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash \tau$. Let $\Sigma_0 = \{\alpha_1, \ldots, \alpha_k\}$, we have that $\vDash \alpha_1 \to \alpha_2 \to \cdots \to \alpha_k \to \tau$. A deduction the last component of which is τ is then immediate.

8 Question to fill in. ◁

See 17G.

11 Question to fill in. ◁

We are essentially trying to express the relation between sets by operations upon procedures. The details are ommitted for brevity.

Chapter 2

First-Order Logic

2.0 Preliminary Remarks

The model of sentential logic may be inadequate to capture the subtlety of some deduction, and to help with that we introduce first-order logic, the essence of which is well respected in the following remark: when the "working mathematician" finds a proof, almost invariably what is meant is a proof that can be mirrored in first-order logic.

2.1 First-Order Languages

Defn 21A *Symbols* are arranged as follows:

1. Logical symbols
   - (a) Parenthesis: $(,)$.
   - (b) Sentential connective symbols: $\rightarrow, \neg$.
   - (c) Variables: $v_1, v_2, \ldots$.
   - (d) Equality symbol (optional): $=$.
2. Parameters
   - (a) Quantifier symbol: $\forall$.
   - (b) Predicate symbols (possibly empty).
   - (c) Function symbols (possibly empty).

(d) Constant symbols (0-place function symbols). ◇

As before, we assume that the symbols are distinct and that no symbol is a finite sequence of other symbols.

We specify our first-order language by saying (1) whether or not the equality symbol is present and (2) what the parameters are. An example is as follows.

Eg 21B *Language of set theory*: Equality: Yes (usually). Predicate symbols: $\in$. Function symbols: None (or a constant symbol $\emptyset$). ◇

It is generally agreed that, by and large, mathematics can be embedded into set theory. By this is meant that

1. Statements in mathematics (such as the fundamental theorem of calculus) can be expressed in the language of set theory; and
2. The theorems of mathematics follow logically from the axioms of set theory.

Formulas

An *expression* is any finite sequence of symbols.

Defn 21C We define for each n-place function symbol f, an n-place term-building operation $\mathcal{F}_f$ on expressions:

$$\mathcal{F}_f(\varepsilon_1, \ldots, \varepsilon_n) = f\varepsilon_1 \cdots \varepsilon_n$$

and say that the set of *terms* is the set generated (14A) from B by the $\mathcal{F}_f$ operations, where B is the set of constant symbols and variables. ◇

An *atomic formula* (*atomic*) an expression of the form

$$P\ t_1 \cdots t_n,$$

where P is an n-place predicate symbol and $t_1 \cdots t_n$ are terms.

We define the formula-building operations on expressions:

$$\begin{aligned} \mathcal{E}_\neg(\gamma) &= (\neg\gamma), \\ \mathcal{E}_\to(\gamma, \delta) &= (\gamma \to \delta), \\ \mathcal{Q}_i(\gamma) &= \forall v_i \gamma. \end{aligned}$$

The set of *well-formed fomulas* (*wffs*, or just *formulas*) the set generated from B by the formula-building operations, where B is the set of atomics.

The terms are the expressions that are translated as names of objects (noun phrases), in contrast to the wffs which are translated as assertions about objects.

Free Variables

We define function h on atomics:

$$h(\alpha) = \text{ the set of all variables, if any, in the atomic formula } \alpha.$$

And we want to extend h to a function $\overline{h}$ defined on all wffs in such a way that

$$\begin{aligned} \bar{h}(\mathcal{E}_{\neg}(\alpha)) &= \bar{h}(\alpha), \\ \bar{h}(\mathcal{E}_{\rightarrow}(\alpha, \beta)) &= \bar{h}(\alpha) \cup \bar{h}(\beta), \\ \bar{h}(\mathcal{Q}_i(\alpha)) &= \bar{h}(\alpha) \text{ after removing } v_i, \text{ if present.} \end{aligned}$$

Then we say that x occurs *free* in α (or that x is a *free variable* of α) iff $x \in \bar{h}(\alpha)$. The existence of a unique such $\bar{h}$ (and hence the meaningfulness of our definition) follows from 14D and from the fact that each wff has a unique decomposition (2.3).

If no variable occurs free in the wff α (i.e., if $\bar{h}(\alpha) = \emptyset$), then α is a *sentence*.

On Notation

1. Outermost parentheses may be dropped.
2. $\neg, \forall$, and $\exists$ apply to as little as possible.
3. $\wedge$ and $\vee$ apply to as little as possible, subject to item 2.
4. When one connective is used repeatedly, the expression is grouped to the right.

Exercises

1 Translate from English to the first-order language specified as follows. ($\forall$, for all things; N, is a number; I, interesting; $<$, is less than; 0, a constant symbol intended to denote zero.)

(a) Zero is less than any number.

(b) If any number is interesting, then zero is interesting.

(c) No number is less than zero.

(d) Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting.

(e) There is no number such that all numbers are less than it.

(f) There is no number such that no number is less than it. $\triangleleft$

1. $\forall x(Nx \to 0 < x)$.
2. $\forall x(\neg Nx \vee \neg Ix) \to \neg I0$.
3. $\forall x(x < 0 \to \neg Nx)$.
4. $\forall x(Nx \wedge \neg Ix \wedge \forall y(Ny \wedge y < x \to Iy) \to Ix)$. Note that this is a translation task. Sentences can be contradictory.
5. $\forall x(Nx \to \neg\forall y(Ny \to y < x))$.
6. $\neg\exists x(Nx \wedge \neg\exists y(Ny \wedge y < x))$.

2 Translate with the language specified in Ex.2.1.1 $\forall x(Nx \to Ix \to \neg\forall y(Ny \to Iy \to \neg x < y))$. ◁

Any interesting number has the property that that it is not less than any insteresting number is not true (can be rephrased as: Any interesting number is less than some interesting number.)

5 Translate from English to the first-order language specified as follows. ($\forall$, for all things; P, is a person; T, is a time; Fxy, you can fool x at y. One or more of the above may be ambiguous, in which case you will need more than one translation.) (a) You can fool some of the people all of the time. (b) You can fool all of the people some of the time. (c) You can't fool all of the people all of the time. ◁

(a) Here ambiguity is in that it either says that there are some (fixed) people you can fool all time, or says that at every moment there are (some, not fixed) people you can fool, i.e. either $\exists x(Px \wedge \forall y(Ty \to Fxy))$ or $\forall y(Ty \to \exists x(Px \wedge Fxy))$. (b) Here ambiguity is in that it either says that you can fool each person at some time (times can be different for different people), or says that at some (fixed) time you can fool everyone (at that specific time): $\forall x(Px \to \exists y(Ty \wedge Fxy))$ or $\exists y(Ty \wedge \forall x(Px \to Fxy))$. (c) $\neg\forall x(Px \to \forall y(Ty \to Fxy))$.

9 Question to fill in. ◁

This can be trivially done by recursion as an extension to the definition of free variables, considering the structures of wffs. I, however, personally believe we can comfortably say that x occurs free as the ith symbol in α if x occurs free in α and is the ith symbol in α.

2.2 Truth and Models

A *structure* $\mathfrak{A}$ for a given first-order language is a function whose domain is the set of parameters satisfying

1. $\forall^{\mathfrak{A}} = |\mathfrak{A}|$, where $\forall$ is the quantifier symbol.
2. $P^{\mathfrak{A}} \in |\mathfrak{A}|^n$, where P is an n-place predicate symbol.
3. $f^{\mathfrak{A}} : |\mathfrak{A}|^n \to |\mathfrak{A}|$, where f is an n-place function symbol.
4. $c^{\mathfrak{A}} \in |\mathfrak{A}|$, where c is a constant symbol, as a special case for item 3.

Note that we require $|\mathfrak{A}|$ to be nonempty. A structure essentially explains to us the universal quantifier symbol and other parameters of a first-order language.

Defn 22A Let ψ be a wff of our language, $\mathfrak{A}$ a structure for the language, $s : V \to |\mathfrak{A}|$, a function from the set V of all variables (21A) into $|\mathfrak{A}|$. We as follows formally define what it means for $\mathfrak{A}$ to *satisfy* φ *with* s, written $\models_{\mathfrak{A}} \varphi[s]$.

We first recursively define the extension $\overline{s} : T \to |\mathfrak{A}|$, a function from the set T of all terms (21C) into $|\mathfrak{A}|$.

1. For each variable x, $\overline{s}(x) = s(x)$.
2. For each c, $\overline{s}(c) = c^{\mathfrak{A}}$.
3. For terms $t_1, \ldots, t_n$ and n-place function symbol f,

$$\overline{s}(f(t_1, \ldots, t_n)) = f^{\mathfrak{A}}(\overline{s}(t_1), \ldots, \overline{s}(t_n)).$$

The idea is that $\bar{s}(t)$ should be the member of the universe $|\mathfrak{A}|$ that is named by the term t. The existence of a unique such extension $\bar{s}$ of s follows from 14D, by using the fact that the terms have unique decompositions (2.3).

Then we define satisfaction of atomics. (We should see that 1 and 2 are essentially the same.)

1. $\models_{\mathfrak{A}} = t_1 t_2[s]$ iff $\bar{s}(t_1) = \bar{s}(t_2)$.
2. For an n-place predicate parameter P,

$$\models_{\mathfrak{A}} P\ t_1 \cdots t_n[s] \text{ iff } \langle \bar{s}(t_1), \ldots, \bar{s}(t_n) \rangle \in P^{\mathfrak{A}}.$$

Finally we define recursively satisfaction of other wffs. Note that we essentially define a truth assignment, or a homomorphism, that preserves formula-building operations ($\mathcal{E}_\neg$, $\mathcal{E}_\to$ and $\mathcal{Q}_i$) from the set of wffs to (informally) $\{\models, \not\models\}$, which again follows from 14D and the fact that wffs have unique decompositions (2.3).

1. For atomic formulas, the definition is above.
2. $\models_{\mathfrak{A}} \neg\varphi[s]$ iff $\not\models_{\mathfrak{A}} \varphi[s]$.

3. $\models_{\mathfrak{A}} (\varphi \to \psi)[s]$ iff either $\not\models_{\mathfrak{A}} \varphi[s]$ or $\models_{\mathfrak{A}} \psi[s]$ or both.
4. $\models_{\mathfrak{A}} \forall x\ \varphi[s]$ iff for every $d \in |\mathfrak{A}|$, we have $\models_{\mathfrak{A}} \varphi[s(x|d)]$,

where

$$s(x|d)(y) = \begin{cases} s(y) & y \neq x, \\ d & y = x. \end{cases} \qquad \diamond$$

Defn 22B To give an alternative definition of *satisfaction*, we first define a function $h : A \to \mathcal{P}(|\mathfrak{A}|^V)$, where A is the set of atomics: for an n-place predicate parameter P (we include $=$ as a 2-place predicate if it exists),

$$h(P\ t_1 \cdots t_n) = \{s : V \to |\mathfrak{A}| | \langle \bar{s}(t_1), \ldots, \bar{s}(t_n)\rangle \in P^{\mathfrak{A}}.\}$$

Then we extend h to $\bar{h}$ with the set of wffs as its domain.

1. $h(\varphi) \subseteq \bar{h}(\varphi)$.
2. $\bar{h}(\neg\varphi) = \{s : V \to |\mathfrak{A}| | s \notin \bar{h}(\varphi)\}$.
3. $\bar{h}(\varphi \to \psi) = \bar{h}(\varphi) \cup \bar{h}(\psi)$.
4. $\bar{h}(\forall x\ \varphi) = \{s : V \to |\mathfrak{A}| |$ for every $d \in |\mathfrak{A}|, s(x|d) \in \bar{h}(\varphi)\}$.

We at last define

$$\models_{\mathfrak{A}} \varphi[s] \text{ iff } s \in \bar{h}(\varphi). \qquad \diamond$$

Thm 22C (22A) Assume that s_1 and s_2 are functions from V into $|\mathfrak{A}|$ which agree at all free variables (if any) of the wff φ. Then

$$\models_{\mathfrak{A}} \varphi[s_1] \text{ iff } \models_{\mathfrak{A}} \varphi[s_2] \qquad \diamond$$

Proof Sketch. We use 14B. ⊣

The proof amounts to seeing what infomation given by s was actually used. An analogous fact regarding structures is if $\mathfrak{A}$ and $\mathfrak{B}$ agree at all the parameters that occur in φ, then $\models_{\mathfrak{A}} \varphi[s]$ iff $\models_{\mathfrak{B}} \varphi[s]$.

Suppose that φ is a formula such that all variables occuring free in φ are included among $v_1, \ldots, v_k$. Then for elements $a_1, \ldots, a_k$ of $|\mathfrak{A}|$,

$$\models_{\mathfrak{A}} \varphi[\![a_1, \ldots, a_k]\!]$$

means that $\mathfrak{A}$ satisfies φ with some (and hence with any) function $s : V \to |\mathfrak{A}|$ for which $s(v_i) = a_i, 1 \leq i \leq k$.

Cor 22D (22B) For a sentence σ, either

(a) $\mathfrak{A}$ satisfies σ with every function $s : V \to |\mathfrak{A}|$, or

(b) $\mathfrak{A}$ does not satisfy σ with any such function. $\diamond$

If alternative (a) holds, then we say that σ is *true* in $\mathfrak{A}$ (written $\models_{\mathfrak{A}} \sigma$) or that $\mathfrak{A}$ is a *model* of σ. And if alternative (b) holds, then σ is *false* in $\mathfrak{A}$. (They cannot both hold since $|\mathfrak{A}|$ is nonempty.) $\mathfrak{A}$ is a *model* of a set Σ of sentences iff it is a model of every member of Σ.

Eg 22E The sentence $\exists x(x \cdot x = 1 + 1)$ is true in $(\mathbb{R}; 0, 1, +, \times)$ and false in $(\mathbb{Q}; 0, 1, +, \times)$. We state again here that structures determine the interpretation of the parameters in a formula while choices of $s : V \to |\mathfrak{A}|$ determine that of the free variables in it. $\diamond$

Logical Implication

Defn 22F Let Γ be a set of wffs, φ a wff. Then Γ *logically implies* φ, written $\Gamma \models \varphi$, iff for every structure $\mathfrak{A}$ for the language and every function $s : V \to |\mathfrak{A}|$ such that $\mathfrak{A}$ satisfies every member of Γ with s, $\mathfrak{A}$ also satisfies φ with s. $\diamond$

Defn 22G We write "$\gamma \models \varphi$" in place of "$\{\gamma\} \models \varphi$." Say that φ and ψ are *logically equivalent* ($\varphi \models\!\!\!\dashv \psi$) iff $\varphi \models \psi$ and $\psi \models \varphi$. The first-order analogue of the concept of a tautology is the concept of a valid formula: A wff φ is *valid* iff $\emptyset \models \varphi$ (written simply "$\models \varphi$"). $\diamond$

Cor 22H (22C) For a set $\Sigma; \tau$ of sentences, $\Sigma \models \tau$ iff every model of Σ is also a model of τ. A sentence τ is valid iff it is true in every structure. $\diamond$

Definability in a Structure

Consider a structure $\mathfrak{A}$ and a formula φ whose free variables are among $v_1, \ldots, v_k$. Then we can construct the k-ary relation on $|\mathfrak{A}|$:

$$\{\langle a_1, \ldots, a_k\rangle \mid \models_{\mathfrak{A}} \varphi[\![a_1, \ldots, a_k]\!]\}.$$

Call this the k-ary relation φ *defines* in $\mathfrak{A}$. In general, a k-ary relation on $|\mathfrak{A}|$ is said to be *definable* in $\mathfrak{A}$ iff there is a formula (whose free variables are among $v_1, \ldots, v_k$) that defines it there.

Eg 22I Consider $\mathfrak{N} = (\mathbb{N}; 0, S, +, \cdot)$.

1. Some relations on $\mathbb{N}$ are not definable, for there are uncountably many relations on $\mathbb{N}$ but only countably many possible defining formulas. There is an inherent difficulty in giving a specific example. After all, if something is undefinable, then it is hard to say exactly what it is!

2. We say that 2 is a *definable element* in $\mathfrak{N}$, for $\{2\}$ is defined by $v_1 = SS0$.

3. The set of primes is defined by

$$1 < v_1 \wedge \forall v_2 \forall v_3 (v_1 = v_2 \cdot v_3 \to v_2 = 1 \vee v_3 = 1),$$

where parameters 1 and $<$ are definable.

4. Exponentiation, $\{\langle m, n, p\rangle | p = m^n\}$ is definable but by no means obviously. $\diamond$

In fact, we will argue later that any decidable relation on $\mathbb{N}$ is definable in $\mathfrak{N}$, as is any effectively enumerable relation and a great many others. To some extent the complexity of a definable relation can be measured by the complexity of the simplest defining formula.

Definability of a Class of Structures

Consider some concepts we come upon in mathematics, say *graphs*, *groups*, *vector spaces*, and so forth. In each case, the objects of study are *structures* for a suitable language. They are required to satisfy a certain set Σ of sentences (referred to as "axioms"). The relevant theory then studies the models of the set Σ of axioms (or at least some of them).

For a set Σ of sentences, let Mod Σ be the class of all models of Σ (write Mod τ in place of Mod $\{\tau\}$). Note that "class" instead of "set" is used here.

Defn 22J A class $\mathcal{K}$ of structures for our language is an *elementary class* (EC) iff $\mathcal{K} =$ Mod τ for some sentence τ. $\mathcal{K}$ is an *elementary class in the wider sense* (EC_Δ) iff $\mathcal{K} = \text{Mod } \Sigma$ for some set Σ of sentences. (The adjective "elementary" is employed as a synonym for "first-order".) $\diamond$

Eg 22K

1. Assuming equality and $\forall$ and and a two-place predicate symbol E, then a *graph* is a structure for this language $\mathfrak{A} = (V; E^{\mathfrak{A}})$, with the axiom stating that the edge relation is symmetric and irreflexive can be translated by the sentence

$$\forall x(\neg xEx \wedge \forall y(xEy \to yEx)).$$

So the class of all graphs is an elementary class. But the class of all finite graphs is not one.

2. Assuming equality, $\forall$ and a two-place predicate symbol P, the class of *nonempty*

ordered sets is an elementary class Mod τ, where τ is the conjunction of

$$\begin{aligned}
&\forall x\forall y\forall z(xPy \to yPz \to xPz);\\
&\forall x\forall y(xPy \lor x = y \lor yPx);\\
&\forall x\forall y(xPy \to \neg yPx).
\end{aligned}$$

3. Assuming equality, $\forall$ and a two-place function symbol $\circ$, the class of all *groups* is an elementary class Mod τ, where τ is the conjunction of *the group axioms*:

$$\begin{aligned}
&\forall x\forall y\forall z(x \circ y) \circ z = x \circ (y \circ z);\\
&\forall x\forall y\exists z\ x \circ z = y;\\
&\forall x\forall y\exists z\ z \circ x = y.
\end{aligned}$$

   Let

$$\begin{aligned}
&\lambda_2 = \exists x\exists y\ x \neq y,\\
&\lambda_3 = \exists x\exists y\exists z(x \neq y \land x \neq z \land y \neq z),\\
&\dots
\end{aligned}$$

   Thus λ_n translates, "There are at least n things." Then the group axioms together with $\{\lambda_2, \lambda_3, \dots\}$ form a set Σ for which Mod Σ is the class of infinite groups, thus EC_Δ. It is not EC though.

4. Assuming equality and the parameters $\forall, 0, 1, +, \cdot$. *Fields* can be reagarded as structures for this language. The class of all fields is an elementary class. The class of fields of characteristic zero is EC_Δ. It is not EC. $\diamond$

Homomorphisms

Defn 22L Let $\mathfrak{A}, \mathfrak{B}$ be structure for the language. A *homomorphism* h of $\mathfrak{A}$ into $\mathfrak{B}$ is a function $h : |\mathfrak{A}| \to |\mathfrak{B}|$ with the properties:

(a) For each n-place predicate parameter P and each n-tuple $\langle a_1, \dots, a_n\rangle$ of elements of $|\mathfrak{A}|$,

$$\langle a_1, \dots, a_n\rangle \in P^{\mathfrak{A}} \text{ iff } \langle h(a_1), \dots, h(a_n)\rangle \in P^{\mathfrak{B}}.$$

(b) For each n-place function symbol f and each such n-tuple,

$$h(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(h(a_1), \dots, h(a_n)).$$

In the case of a constant symbol c this becomes

$$h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

Say that h *preserves* the relation and functions. If, in addition, h is one-to-one, it is then called an *isomorphism* (or an *isomorphic embedding*) of $\mathfrak{A}$ into $\mathfrak{B}$. If there is an isomorphism of $\mathfrak{A}$ *onto* $\mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *isomorphic* (written $\mathfrak{A} \cong \mathfrak{B}$). $\diamond$

Consider two sturctures $\mathfrak{A}$ and $\mathfrak{B}$ for the language such that $|\mathfrak{A}| \subseteq |\mathfrak{B}|$. Then the identity map from $|\mathfrak{A}|$ into $|\mathfrak{B}|$ is an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ iff

(a) $P^{\mathfrak{A}}$ is the restriction of $P^{\mathfrak{B}}$ to $|\mathfrak{A}|$, for each predicate symbol P;

(b) $f^{\mathfrak{A}}$ is the restriction of $f^{\mathfrak{B}}$ to $|\mathfrak{A}|$, for each function symbol f, and $c^{\mathfrak{A}} = c^{\mathfrak{B}}$ for each constant symbol c.

Say that $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, and $\mathfrak{B}$ is an *extension* of $\mathfrak{A}$.

Eg 22M Let $\mathbb{P}$ be the set of positive integers, $<_P$ the usual ordering relation on $\mathbb{P}$ and so be $<_N$. Then the identity map $Id : \mathbb{P} \to \mathbb{N}$ is an isomorphism of $(\mathbb{P}; <_P)$ into $(\mathbb{N}; <_N)$. Thus $(\mathbb{P}; <_P)$ is a substructure of $(\mathbb{N}; <_N)$. Similarly we have that $(\mathbb{Q}; +_Q)$ is a substructure of $(\mathbb{C}; +_C)$. $\diamond$

Thm 22N Homomorphism Theorm Let h be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, and let s map the set of variables into $|\mathfrak{A}|$.

(a) For any term t, we have $h(\bar{s}(t)) = \overline{h \circ s}(t)$, where $\bar{s}(t)$ is computed in $\mathfrak{A}$ and $\overline{h \circ s}(t)$ is computed in $\mathfrak{B}$.

(b) For any quantifier-free formula α not containing the equality symbol,

$$\models_{\mathfrak{A}} \alpha[s] \text{ iff } \models_{\mathfrak{B}} \alpha[h \circ s].$$

(c) "Not containing the equality symbol" in (b) is not necessary if h is one-to-one.

(d) "Quantifier-free" in (b) is not necessary if h is a homomorphism of $\mathfrak{A}$ *onto* $\mathfrak{B}$. $\diamond$

Proof. (a) We use induction on t.

(b) For an atomic formula *such as* $P\ t$, we have

$$\begin{aligned} \models_{\mathfrak{A}} P\ t[s] &\Leftrightarrow \bar{s}(t) \in P^{\mathfrak{A}} \\ &\Leftrightarrow h(\bar{s}(t)) \in P^{\mathfrak{B}} \\ &\Leftrightarrow \overline{h \circ s}(t) \in P^{\mathfrak{B}} \\ &\Leftrightarrow \models_{\mathfrak{B}} P\ t[h \circ s]. \end{aligned}$$

Then we use induction on wffs.

(c) Note that the second $\Leftrightarrow$ in (b) holds for the special predicate symbol "$=$" iff h is one-to-one.

(d) For any element a of $|\mathfrak{A}|$,

$$\begin{aligned}\models_{\mathfrak{B}} \forall x\ \varphi[h \circ s] &\Rightarrow \models_{\mathfrak{B}} \varphi[(h \circ s)(x|h(a))]\\ &\Leftrightarrow \models_{\mathfrak{B}} \varphi[h \circ (s(x|a))]\\ &\Leftrightarrow \models_{\mathfrak{A}} \varphi[s(x|a)]\end{aligned}$$

Thus $\models_{\mathfrak{B}} \forall x\ \varphi[h \circ s] \Rightarrow \models_{\mathfrak{A}} \forall x\ \varphi[s]$. If h maps $|\mathfrak{A}|$ *onto* $|\mathfrak{B}|$, the counterpart of the above argument is immediate. $\dashv$

The above theorem essentially depicts how homomorphic structures "preserve" satisfaction.

Defn 22O Two structures $\mathfrak{A}$ and $\mathfrak{B}$ for the language are said to be *elementarily equivalent* (written $\mathfrak{A} \equiv \mathfrak{B}$) iff for any sentence σ,

$$\models_{\mathfrak{A}} \sigma \Leftrightarrow \models_{\mathfrak{B}} \sigma. \qquad \diamond$$

Cor 22P (22D) $\mathfrak{A} \cong \mathfrak{B} \Rightarrow \mathfrak{A} \equiv \mathfrak{B}$. $\diamond$

Actually more is true. Isomorphic structures are alike in every "structural" way; not only do they satisfy the same first-order sentences, they also satisfy the same second-order (and higher) sentences, i.e., they are secondarily equivalent and more.

Eg 22Q

1. $(\mathbb{R}; <_R) \equiv (\mathbb{Q}; <_Q)$, but they are not isomorphic. (See the comment for 26AF.)
2. In 22M, Id and $h : n \mapsto n - 1$ are both isomorphisms. The latter, in paticular, is *onto* $(\mathbb{N}; <_N)$. Therefore $(\mathbb{P}; <_P)$ and $(\mathbb{N}; <_N)$ are indistinguishable by first-order sentences. But we may tell a difference, by applying wffs that contain quantifiers, in case we use Id as the isomorphism. $\diamond$

An *automorphism* of the structure $\mathfrak{A}$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}$. The identity function on $|\mathfrak{A}|$ is trivially an automorphism of $\mathfrak{A}$. Say that $\mathfrak{A}$ is *rigid* if the identity function is its only automorphism.

Cor 22R (22E) Let h be an automorphism of the structure $\mathfrak{A}$, and R an n-ary relation on $|\mathfrak{A}|$ definable in $\mathfrak{A}$. Then for any $a_1, \ldots, a_n$ in $|\mathfrak{A}|$,

$$\langle a_1, \ldots, a_n\rangle \in R \Leftrightarrow \langle h(a_1), \ldots, h(a_n)\rangle \in R. \qquad \diamond$$

Eg 22S

1. Consider sturcture $(\mathbb{R}; <)$. An automorphism of it is simply a function h from $\mathbb{R}$ onto $\mathbb{R}$ that is strictly increasing:

$$a < b \Leftrightarrow h(a) < h(b).$$

Therefore $h : a \mapsto a^3$ qualifies. Since this function maps points outside of $\mathbb{N}$ into $\mathbb{N}$, the set $\mathbb{N}$ is not definable in $(\mathbb{R}; <)$.

2. Consider structure $(E; +, f_r)_{r\in\mathbb{R}}$, where E is a plane, $+$ a binary function symbol of vector addition and f_r a unary function of scalar multiplication by r. $h : \mathbf{x} \mapsto 2\mathbf{x}$ is an automorphism of it, but h does not preserve the set of unit vectors,

$$\{\mathbf{x}|\mathbf{x} \in E \text{ and } \mathbf{x} \text{ has length } 1\}.$$

So this set, and therefore lengths of the vectors in the plane, is not definable in $(E; +, f_r)$. (Incidentally, the homomorphisms of vector spaces are called *linear transformations.*) $\diamond$

Exercises

1 Show that (a) $\Gamma; \alpha \vDash \varphi$ iff $\Gamma \vDash (\alpha \to \varphi)$; and (b) $\varphi \vDash\!\!\!\!\Dashv \psi$ iff $\vDash (\varphi \leftrightarrow \psi)$. $\triangleleft$

(a)
$$\begin{aligned}
\Gamma; \alpha \vDash \varphi &\Leftrightarrow (\forall\tau\ \tau \in \Gamma; \alpha \to \vDash_{\mathfrak{A}} \tau[s]) \to \vDash_{\mathfrak{A}} \varphi[s] \\
&\Leftrightarrow (\forall\tau\ \tau \in \Gamma \to \vDash_{\mathfrak{A}} \tau[s]) \land \vDash_{\mathfrak{A}} \alpha[s] \to \vDash_{\mathfrak{A}} \varphi[s] \\
&\Leftrightarrow (\forall\tau\ \tau \in \Gamma \to \vDash_{\mathfrak{A}} \tau[s]) \to \vDash_{\mathfrak{A}} (\alpha \to \varphi)[s] \\
&\Leftrightarrow \Gamma \vDash (\alpha \to \varphi).
\end{aligned}$$

(b) This should be a specification of (a), and is omitted for brevity.

2 Question to fill in. $\triangleleft$

Observe that P is a transitive relation in (a), an antisymmetric one in (b) and one with a right-absorbing element in the universe in (c).

1. $(\{a, b\}, \{\langle a, b\rangle, \langle b, b\rangle, \langle b, a\rangle\})$.

2. $(\{a, b\}, \{\langle a, b\rangle, \langle b, b\rangle, \langle b, a\rangle, \langle a, a\rangle\})$.

3. $(\mathbb{Z}_{>0}, |)$, where $|$ is the "divides" relation on positive integers.

3 Show that

$$\{\forall x(\alpha \to \beta), \forall x\alpha\} \vDash \forall x\beta. \quad \triangleleft$$

Consider a fixed $\mathfrak{A}$ and s, for every $d \in |\mathfrak{A}|$,

$$
\begin{aligned}
&(\vDash_{\mathfrak{A}} \forall x(\alpha \to \beta)[s]) \wedge (\vDash_{\mathfrak{A}} \forall x\ \alpha[s]) \\
\Leftrightarrow&(\vDash_{\mathfrak{A}} (\alpha \to \beta)[s(x|d)]) \wedge (\vDash_{\mathfrak{A}} \alpha[s(x|d)]) \\
\Leftrightarrow&(\vDash_{\mathfrak{A}} \alpha[s(x|d)] \to \vDash_{\mathfrak{A}} \beta[s(x|d)]) \wedge (\vDash_{\mathfrak{A}} \alpha[s(x|d)]) \\
\Rightarrow& \vDash_{\mathfrak{A}} \beta[s(x|d)] \\
\Leftrightarrow& \vDash_{\mathfrak{A}} \forall x\ \beta[s].
\end{aligned}
$$

Thus $\{\forall x(\alpha \to \beta), \forall x\ \alpha\} \vDash \forall x\ \beta$. One should note that the $\wedge$ and $\Leftrightarrow$ and $\Rightarrow$ used in this proof are only simplifications of meta-reasoning in English. The same works for many exercises (for example, Ex.2.2.1).

4 Show that if x does not occur free in α, then $\alpha \vDash \forall x\alpha$. ◁

Consider a fixed $\mathfrak{A}$ and s. For every $d \in |\mathfrak{A}|$, we have that s and $s(x|d)$ agree at all free variables of α, then by 22C $\vDash_{\mathfrak{A}} \alpha[s] \Leftrightarrow \vDash_{\mathfrak{A}} \alpha[s(x|d)] \Leftrightarrow \vDash_{\mathfrak{A}} \forall x\ \alpha[s]$.

5 Question to fill in. ◁

By Ex.2.2.1 it suffices to show that $\{= xy, Pzfx\} \vDash Pzfy$. Consider a fixed $\mathfrak{A}$ and s and arbitrary $d \in |\mathfrak{A}|$,

$$
\begin{aligned}
(\vDash_{\mathfrak{A}} = xy[s]) \wedge (\vDash_{\mathfrak{A}} Pzfx[s]) \Leftrightarrow&(\bar{s}(x) = \bar{s}(y)) \wedge (\langle \bar{s}(z), f\bar{s}(x)\rangle \in P^{\mathfrak{A}}) \\
\Rightarrow&\langle \bar{s}(z), f\bar{s}(y)\rangle \in P^{\mathfrak{A}} \\
\Leftrightarrow& \vDash_{\mathfrak{A}} Pzfy[s].
\end{aligned}
$$

Thus $\{= xy, Pzfx\} \vDash Pzfy$.

6 Show that a formula θ is valid iff $\forall x\theta$ is valid. ◁

By Ex.2.2.4 it suffices to show that if a wff φ has free variables then it is not valid. To show that we first prove that for any wff φ there exist $\mathfrak{A}$ and s such that $\vDash_{\mathfrak{A}} \varphi[s]$ and then show that if φ is not a sentence, there exist $\mathfrak{A}'$ and s' such that $\nvDash_{\mathfrak{A}'} \varphi[s']$. I suppose the above is a working proof sketch but we might as well instead prove this directly for the sake of brevity: Consider a fixed $\mathfrak{A}$ and s, $\vDash_{\mathfrak{A}} \varphi[s(x|d)]$ holds for every $d \in |\mathfrak{A}|$ since φ is *valid*, and that is exactly $\vDash_{\mathfrak{A}} \forall x\ \varphi[s]$. For the other direction we take $d = s(x)$. Thus $\varphi \Leftrightarrow \forall x\ \varphi$.

7 Restate the definition of "$\mathfrak{A}$ satisfies φ with s" by defining recursively a function $\overline{h}$ such that $\mathfrak{A}$ satisfies φ with s iff $s \in \overline{h}(\varphi)$. ◁

See 22B.

9 Assume that the language has equality and a two-place predicate symbol P. For each of the following conditions, find a sentence σ such that the structure $\mathfrak{A}$ is a model of σ iff the condition is met.

1. $|\mathfrak{A}|$ has exactly two members.
2. $P^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|$ into $|\mathfrak{A}|$.
3. $P^{\mathfrak{A}}$ is a permutation of $|\mathfrak{A}|$. ◁

1. $\exists a \exists b \forall c(\neg a = b \wedge (c = a \vee c = b))$.
2. $\forall x \exists y \forall z(P\ xy \wedge (P\ xz \to y = z))$.
3. $\forall x \exists y \forall z \exists p \forall q \forall r(P\ xy \wedge (P\ xz \to y = z) \wedge P\ pq \wedge (P\ rq \to p = r))$.

10 Show that

$$\models_{\mathfrak{A}} \forall v_2\, Qv_1v_2[[c^{\mathfrak{A}}]] \quad \text{iff} \quad \models_{\mathfrak{A}} \forall v_2\, Qcv_2.$$

Here Q is a two-place predicate symbol and c is a constant symbol. ◁

See 25A.

11 For each of the following relations, give a formula which defines it in $(\mathbb{N}; +, \cdot)$. (The language is assumed to have equality and the parameters $\forall$, $+$, and $\cdot$).

1. $\{0\}$.
2. $\{1\}$.
3. $\{\langle m, n\rangle \mid n \text{ is the successor of } m \text{ in } \mathbb{N}\}$.
4. $\{\langle m, n\rangle \mid m < n \text{ in } \mathbb{N}\}$. ◁

1. $\forall x\ x + a = x$.
2. $\forall x\ x \cdot a = x$.
3. $\exists y \forall x(x \cdot y = x \wedge n = m + y)$.
4. $\exists y \forall x \exists k(x + y = x \wedge \neg k = y \wedge n = m + k)$.

13 Question to fill in. ◁

Omitted as trivial.

16 Give a sentence having models of size $2n$ for every positive integer n, but no finite models of odd size. (The language should have equality and whatever parameters you choose.) ◁

Idea: One method is to make a sentence that says, "Everything is either red or blue, and f is a color-reversing permutation." So we may write $\forall x(f(f(x)) = x \wedge \neg f(x) = x)$. (We are not exactly following the method, but instead making sure that things occur in pairs.) Alternatively, assuming equality and a binary predicate symbole R, consider the conjunction of:

(a) $\forall x \exists y(\neg x = y \wedge Ryx)$;

(b) $\forall x \forall y \forall z(Rxy \wedge Rxz \to y = z)$;

(c) $\forall x \forall y(Ryx \to Rzx)$.

17 (a) Consider a language with equality whose only parameter (aside from $\forall$) is a two-place predicate symbol P. Show that if $\mathfrak{A}$ is finite and $\mathfrak{A} \equiv \mathfrak{B}$ then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. (b) Show that the result of part (a) holds regardless of what the parameters the language contains. $\triangleleft$

(a) Let $V_c = \{v_1, \ldots, v_n\}$, where $n = \text{card}|\mathfrak{A}|$. Consider fixed $s_{\mathfrak{A}} : V \to |\mathfrak{A}|$ that is one-to-one when restricted to V_c. Make a sentence σ of the form $\exists v_1 \cdots \exists v_n(\bigwedge \Sigma)$, where and Σ contains the following formulas: (i) $\bigwedge_{i \neq j} \neg = v_i v_j$; (ii) $\forall x\ x \in \{v_1, \ldots, v_n\}$; (iii) one saying that Pv_iv_j iff $\models_{\mathfrak{A}} Pv_iv_j[s_{\mathfrak{A}}]$. Then we have $\models_{\mathfrak{A}} \sigma$. Now make $s_{\mathfrak{B}} : V \to |\mathfrak{B}|$ such that (i) is one-to-one when restricted to V_c and (ii) if $\models_{\mathfrak{A}} Pv_iv_j[s_{\mathfrak{A}}]$, then $\models_{\mathfrak{B}} Pv_iv_j[s_{\mathfrak{B}}]$. This is valid due to that $\models_{\mathfrak{B}} \sigma$. Then we have $(s_{\mathfrak{B}}|_{V_c}) \circ (s_{\mathfrak{A}}|_{V_c})^{-1}$ as an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. (b) This is a trivial extension of part (a). We only need to make sure that there is a sentence σ that "describes" $\mathfrak{A}$ to preserve the parameters, which is feasible when $\mathfrak{A}$ is finite.

101 Consider a language having equality and a binary predicate symbol R. Give a sentence σ such that *finite* members of Mod σ are unions of disjoint directed cycles (by disjoint we mean that no vertices or edges are shared), where R is thought of as the edge relation. Now give an infinite model for σ. $\triangleleft$

σ could be the conjunction of:

(a) $\forall x \exists y \exists z(Ryx \wedge Rxz)$,

(b) $\forall x \forall y \forall z(Rxy \wedge Rxz \to y = z)$,

(c) $\forall x \forall y \forall z(Ryx \wedge Rzx \to y = z)$.

To give an infinite model $\mathfrak{A}$ for σ, let $|\mathfrak{A}| = \mathbb{Z}$, and

$$R^{\mathfrak{A}} = \{\ldots, \langle -2, -1\rangle, \langle -1, 0\rangle, \langle 0, 1\rangle, \langle 1, 2\rangle, \ldots\}.$$

2.3 A Parsing Algorithm

We *must* show that we can decompose formulas (and terms) in a unique way to justify our definitions by recursion. Omitting technical details (which, like 1.3, brilliantly utilized some traits of the notations we use), we state that the set of terms *is* freely generated from the set of variables and constant symbols by the $\mathcal{F}_f$ operations, and that the set of wffs *is* freely generated from the set of atomic formulas by the operations $\mathcal{E}_\neg$, $\mathcal{E}_\to$ and $\mathcal{Q}_i(i = 1, 2, \ldots)$. The solutions to the exercises are omitted as trivial.

2.4 A Deductive Calculus

Suppose $\Sigma \vDash \tau$ and we want to demonstrate it. We can achieve this goal in the context of sentential logic by definition (using the truth table method) (or we may use deduction introduced in Ex.1.7.5, but we might not bother to, for that the truth table method is already effective). But to do this by definition (22F) in the context of first-order logic is significantly more difficult. So we consider *formal proofs* and ask them to be finitely long and decidable (for otherwise there is no point in introducing them).

Then we argue that such proofs exist. The finiteness demands 25H and the effectiveness demands 25I. (The set of proofs from $\emptyset$ should be decidable, then we can effectively enumerate all strings and sort out non-proofs, thus the validities should be effectively enumerable.) These two theorems are sufficient. For there exists by 25H a finite set $\{\sigma_0, \ldots, \sigma_n\} \subseteq \Sigma$ that logically implies τ. Then $\sigma_0 \to \cdots \to \sigma_n \to \tau$ is valid, and by 25I we can find it after a finite number of enumerations. The record of enumeration procedure *is* a proof. It *is* finite and decidable.

According to the above motivation we aim to derive one type of proofs, or *deduction*, that is, in the context of first-order logic, adequate (by proving the two theorem) and correct (by describing a correct procedure that enumerates validities).

Formal Deductions

To present a deductive calculus for first-order logic is to choose a set Λ of formulas that are called *logical axioms* and a set of *rules* of inference. In our case we will have a infinite Λ (with detailed discussion to follow) and only one rule of inference, traditionally known as *modus ponens* and usually stated: From the formulas α and $\alpha \to \beta$ we may infer β:

$$\frac{\alpha, \alpha \to \beta}{\beta}.$$

Then we can define that a *deduction of* φ *from* Γ is a finite sequence $\langle \alpha_0, \ldots, \alpha_n \rangle$ of formulas such that α_n is φ and for each $k \leq n$, either (a) α_k is in $\Gamma \cup \Lambda$, or (b) for some i and j less than k, α_j is $\alpha_i \to \alpha_k$. If such a deduction exists, we say that φ is *deducible* from Γ, or that φ is a *theorem* of Γ, for which we write $\Gamma \vdash \varphi$.

We may adopt a viewpoint that is similar to 14A. This would differ from the generation of wffs in that the set of theorems is *not* freely generated from $\Gamma \cup \Lambda$ by modus ponens, and that the domain is "partial", in the form $\langle \alpha, \alpha \to \beta \rangle$, instead of arbitrary wffs. Say that a set S of formulas is *closed* under modus ponens if whenever both $\alpha \in S$ and $(\alpha \to \beta) \in S$ then also $\beta \in S$.

Thm 24A Induction Principle Suppose that S is a set of wffs that includes $\Gamma \cup \Lambda$ and is closed under modus ponens. Then S contains every theorem of Γ. $\diamond$

Defn 24B Say that a wff φ is a *generalization* of wff ψ iff for some $n \geq 0$ and some variables $x_1, \ldots, x_n$,

$$\varphi = \forall x_1 \cdots \forall x_n \psi.$$

The set Λ we give contains all generalizations of wffs of the following forms, where x and y are variables and α and β are wffs:

1. Tautologies;
2. $\forall x\ \alpha \to \alpha^x_t$, where t is substitutable for x in α;
3. $\forall x(\alpha \to \beta) \to (\forall x\ \alpha \to \forall x\ \beta)$;
4. $\alpha \to \forall x\ \alpha$, where x does not occur free in α.

And if the language includes equality, we add

5. $x = x$;
6. $x = y \to (\alpha \to \alpha')$, where α is atomic and α' is obtained from α by replacing x in zero or more places by y. $\diamond$

For the origins of the axioms see 2.4.

Substitution

Defn 24C For term t and variable x, we define $\sigma_{x \mapsto t} : V \to T$, where V is the set of variables, T the set of terms and $\sigma_{x \mapsto t}$ identity except that it maps x to t and then recursively define the extension $\overline{\sigma_{x \mapsto t}} : T \to T$:

1. For each variable v, $\overline{\sigma_{x \mapsto t}}(v) = \sigma_{x \mapsto t}(v)$.
2. For each constant symbol c, $\overline{\sigma_{x \mapsto t}}(c) = c$.
3. For terms $t_1, \ldots, t_n$ and n-place function symbol f,

$$\overline{\sigma_{x \mapsto t}}(f(t_1, \ldots, t_n)) = f(\overline{\sigma_{x \mapsto t}}(t_1), \ldots, \overline{\sigma_{x \mapsto t}}(t_n)). \qquad \diamond$$

The fact that ran $\overline{\sigma_{x\mapsto t}} \subseteq T$, though trivial, follows from 14B, where the two sets of interest are ran $\overline{\sigma_{x\mapsto t}}$ and the subset of ran $\overline{\sigma_{x\mapsto t}}$ containing only terms.

Then we define α^x_t recursively:

1. For atomic $\alpha = P\ t_1, \ldots, t_n$, $\alpha^x_t = P\ \overline{\sigma_{x\mapsto t}}(t_1), \ldots, \overline{\sigma_{x\mapsto t}}(t_n)$.
2. $(\neg\alpha)^x_t = (\neg\alpha^x_t)$.
3. $(\alpha \to \beta)^x_t = (\alpha^x_t \to \beta^x_t)$.
4. $(\forall y\ \alpha)^x_t = \begin{cases} \forall y\ \alpha & x = y \\ \forall y(\alpha^x_t) & x \neq y \end{cases}$

Therefore α^x_t is the expression (also formula, trivially) obtained from α by replacing x, wherever it occurs free in α, by t.

Defn 24D The phrase "t is *substitutable* for x in α" is recursively defined as follows:

1. For atomic α, t is always substitutable for x in α.
2. t is substitutable for x in $(\neg\alpha)$ iff it is substitutable for x in α. t is substitutable for x in $(\alpha \to \beta)$ iff it is substitutable for x in both α and β.
3. t is substitutable for x in $\forall y\ \alpha$ iff either
   (a) x does not occur free in $\forall y\ \alpha$, or
   (b) y does not occur in t and t is substitutable for x in α. ◇

Tautologies

We devide the set of wffs into two groups:

1. The *prime* formulas are the atomic formulas and those of the form $\forall x\ \alpha$.
2. The nonprime formulas are the others.

Thus any formula is generated from the set of prime formulas by operations $\mathcal{E}_\neg$ and $\mathcal{E}_\to$. Say that any tautology (12B) of sentential logic is a *tautology* in axiom group 1, where the sentence symbols are prime formulas.

Note that we are employing an extension of Chapter 1 to the case of an uncountable set of sentence symbols. Note also that we could use less tautologies and obtain others by use of modus ponens. The whole set of tautologies is a nice decidable set, but not one polynomial-time decidable.

Now that first-order formulas are also wffs of sentential logic, we can apply concepts from both Chapters 1 and 2 to them.

Defn 24E Define on the set of prime formulas a truth assignment

$$v(\alpha) = T \text{ iff } \models_{\mathfrak{A}} \alpha[s],$$

where $\mathfrak{A}$ is a sturcture and $s : V \to |\mathfrak{A}|$. $\diamond$

We want to show that this definition, when extended to $\bar{v}$, is closed under $\mathcal{E}_\neg$ and $\mathcal{E}_\to$, and this is trivial by definition. Thus we have

$\quad\Gamma$ tautologically implies φ

$\Leftrightarrow$ every truth assignment for prime formulas in $\Gamma; \varphi$ that satisfies every number of Γ also satisfies φ

$\Rightarrow$ for every structure $\mathfrak{A}$ and every function $s : V \to |\mathfrak{A}|$ such that $\mathfrak{A}$ satisfies every number of Γ with s, $\mathfrak{A}$ also satisfies φ with s

$\Leftrightarrow \Gamma \models \varphi$.

The converse of $\Rightarrow$ does not hold, because cases exist where a logical implication is not a tautological one. An example could be $\forall x\ Ax \models Ab$.

Thm 24F (24B) $\Gamma \vdash \varphi$ iff $\Gamma \cup \Lambda$ tautologically implies φ. $\diamond$

Proof Sketch. For ($\Rightarrow$) we use induction on φ. For ($\Leftarrow$) we use 17C. $\dashv$

The above proof is related to Ex.1.7.6 and Ex.1.7.7. We are using sentential compactness for a possibly uncountable language.

Deductions and Metatheorems

Thm 24G Generalization Theorem If $\Gamma \vdash \varphi$ and x do not occur free in any formula in Γ, then $\Gamma \vdash \forall x\ \varphi$. $\diamond$

Proof. It suffices (by the 24A) to show that the set $\{\varphi | \Gamma \vdash \forall x\ \varphi\}$ includes $\Gamma \cup \Lambda$ and is closed under modus ponens.

Case 1: $\varphi \in \Lambda$. We have $\Gamma \vdash \forall x\ \varphi$. Thus $\Lambda \subseteq \{\varphi | \Gamma \vdash \forall x\ \varphi\}$.
Case 2: $\varphi \in \Gamma$. We have by axiom group 4 (24B) that $\Gamma \vdash \forall x\ \varphi$. Thus $\Gamma \subseteq \{\varphi | \Gamma \vdash \forall x\ \varphi\}$.
Case 3: φ is obtained by modus ponens from ψ and $\psi \to \varphi$. We have by inductive hypothesis and axiom group 3 (24B) that $\Gamma \vdash \forall x\ \varphi$. Thus $\{\varphi | \Gamma \vdash \forall x\ \varphi\}$ is closed under modus ponens. $\dashv$

Eg 24H $\forall x \forall y\ \alpha \vdash \forall y \forall x\ \alpha$. ◇

Proof Sketch. We use axiom group 2 (24B) and then 24G. Note that a variable is always substitutable for itself. ⊣

Lem 24I Rule T (24C) If $\Gamma \vdash \alpha_1, \dots, \Gamma \vdash \alpha_n$ and $\{\alpha_1, \dots, \alpha_n\}$ tautologically implies β, then $\Gamma \vdash \beta$. ◇

Proof. We have a tautology $\alpha_1 \to \cdots \to \alpha_n \to \beta$, and by axiom group 1 (24B) and modus ponens (applied n times) we have $\Gamma \vdash \beta$. ⊣

Thm 24J Deduction Theorem If $\Gamma; \gamma \vdash \varphi$, then $\Gamma \vdash (\gamma \to \varphi)$. ◇

Proof. By 24F and Ex.1.2.4(a) we have that

$$\begin{aligned} \Gamma; \gamma \vdash \varphi &\Leftrightarrow \Gamma \cup \Lambda; \gamma \text{ tautologically implies } \varphi \\ &\Leftrightarrow \Gamma \cup \Lambda \text{ tautologically implies } (\gamma \to \varphi) \\ &\Leftrightarrow \Gamma \vdash (\gamma \to \varphi). \end{aligned}$$ ⊣

Second Proof. Alternatively, to avoid usage of sentential compactness, we can show by induction that for every theorem φ of $\Gamma; \gamma$ the formula $(\gamma \to \varphi)$ is a theorem of Γ.

Case 1: $\varphi = \gamma$. We have by axiom group 1 (24B) $\Gamma \vdash (\gamma \to \varphi)$.
Case 2: $\varphi \in \Gamma \cup \Lambda$. Thus $\Gamma \vdash \varphi$. And φ tautologically implies $(\gamma \to \varphi)$, whence by 24I we have $\Gamma \vdash (\gamma \to \varphi)$.
Case 3: φ is obtained by modus ponens from ψ and $\psi \to \varphi$. By the inductive hypothesis, $\Gamma \vdash (\gamma \to \psi)$ and $\Gamma \vdash (\gamma \to (\psi \to \varphi))$. And the set $\{\gamma \to \psi, \gamma \to (\psi \to \varphi)\}$ tautologically implies $\gamma \to \varphi$. Thus by 24I, $\Gamma \vdash (\gamma \to \varphi)$. ⊣

When proving by induction, to decide the inductive hypothesis is to find a trait that is described by the theorem to prove.

Cor 24K Contraposition (24D) $\Gamma; \varphi \vdash \neg\psi \Leftrightarrow \Gamma; \psi \vdash \neg\varphi$. ◇

Defn 24L Say that a set of formulas is *inconsistent* iff for some β, both β and $\neg\beta$ are the theorems of the set. (In this event, any formula α is a theorem of the set, since $\beta \to \neg\beta \to \alpha$ is a tautology.) Say that a set of formulas is *consistent* if it is not inconsistent. ◇

Cor 24M Reductio ad Absurdum, RAA (24E) If $\Gamma; \varphi$ is inconsistent, then $\Gamma \vdash \neg\varphi$. ◇

Proof. Say that $\Gamma; \varphi \vdash \beta$ and $\Gamma; \varphi \vdash \neg\beta$, then we have by 24J that $\Gamma \vdash (\varphi \to \beta)$ and $\Gamma \vdash (\varphi \to \neg\beta)$. Thus we have by 24I $\Gamma \vdash \neg\varphi$. ⊣

Reductio ad Absurdum essentially states that to show that φ is deducible from Γ, it suffices to show that $\Gamma; \neg\varphi$ is inconsistent.

Strategy

Thm 24N Generalization on Constants (24F) Assume that $\Gamma \vdash \varphi$ and that c is a constant symbol that does not occur in Γ. Then there is a variable y (which does not occur in φ) such that $\Gamma \vdash \forall y\, \varphi^c_y$. Furthermore, there is a deduction of $\forall y\, \varphi^c_y$ from Γ in which c does not occur. ◇

Proof. Let $\langle \alpha_0, \ldots, \alpha_n \rangle$ be a deduction of φ from Γ. (Thus $\alpha_n = \varphi$.) Let y be the first variable that does not occur in any of the a_i's. We claim that

$$\langle (\alpha_0)^c_y, \ldots, (\alpha_n)^c_y \rangle \tag{*}$$

is a deduction from Γ of φ^c_y. So we must check that each $(\alpha_k)^c_y$ is in $\Gamma \cup \Lambda$ or is obtained from earlier formulas by modus ponens.

Case 1: $\alpha_k \in \Gamma$. Then c does not occur in α_k. So $(\alpha_k)^c_y = \alpha_k$, which is in Γ.
Case 2: $\alpha_k \in \Lambda$. Then $(\alpha_k)^c_y \in \Lambda$ (trivially).
Case 3: α_k is obtained by MP from α_i and $\alpha_j = (\alpha_i \to \alpha_k)$ for i, j less than k. Then $(\alpha_j)^c_y = ((\alpha_i)^c_y \to (\alpha_k)^c_y)$ by 2.4. Thus we have $(\alpha_k)^c_y$ by MP.

Let Φ be the finite subset of Γ that is used in (*). Thus (*) is a deduction of φ^c_y from Φ, and y does not occur in Φ. So by 24G $\Phi \vdash \forall y\, \varphi^c_y$. This is *the* deduction of $\forall y\, \varphi^c_y$ in which c does not occur. ⊣

Lem 24O Re-replacement Lemma If y does not occur in φ, then x is substitutable (24D) for y in φ^x_y and $(\varphi^x_y)^y_x = \varphi$. ◇

Proof. We use induction on φ.

Case 1: For atomic $\varphi = P\ t_1, \ldots, t_n$, by 2.4 we have that x is substitutable for y in φ^x_y and that

$$(\varphi^x_y)^y_x = ((P\ t_1, \ldots, t_n)^x_y)^y_x = P\ ((t_1)^x_y)^y_x, \ldots, ((t_n)^x_y)^y_x = \varphi.$$

Case 2: Given the inductive hypothesis, the inductive step holds by definition for formula building operations $\mathcal{E}_\neg$, $\mathcal{E}_\to$ and $\mathcal{Q}_i$, where $v_i \neq x$ and $v_i \neq y$ (since y does not occur in φ).

Case 3: $\varphi = \forall x\ \psi$. Then $(\forall x\ \psi)^x_y = \forall x\ \psi$, in which y does not occur (free, and thus x is substitutable.) Therefore $(\varphi^y_x)^x_y = ((\forall x\ \psi)^x_y)^y_x = (\forall x\ \psi)^x_y = \forall x\ \psi = \varphi$. $\dashv$

Cor 24P (24G) Assume that $\Gamma \vdash \varphi^x_c$, where the constant symbol c does not occur in Γ or in φ. Then $\Gamma \vdash \forall x\ \varphi$, and there is a deduction of $\forall x\ \varphi$ from Γ in which c does not occur. $\diamond$

Proof. By 24N we have a deduction (without c) from Γ of $\forall y((\varphi^x_c)^c_y) = \forall y\ \varphi^x_y$ (for c does not occur in φ), where y does not occur in φ^x_c. By axiom group 2 (24B) and 24O we have $\vdash \forall y\ \varphi^x_y \to \varphi$. We then have $\vdash \forall y\ \varphi^x_y \to \forall x\ \varphi$ by 24J and 24G. Thus $\Gamma \vdash \forall x\ \varphi$. (We used the fact that if $\Theta \subseteq \Gamma$ and $\Theta \vdash \varphi$, $\Gamma \vdash \varphi$.) $\dashv$

Cor 24Q Rule EI (24H) Assume that the constant symbol c does not occur in φ, ψ or Γ, and that $\Gamma; \varphi^x_c \vdash \psi$. Then $\Gamma; \exists x\ \varphi \vdash \psi$ and there is a deduction of ψ from $\Gamma; \exists x\ \varphi$ in which c does not occur. ("EI" stands for "existential instantiation".) $\diamond$

Proof. By contraposition we have $\Gamma; \neg\psi \vdash \neg\varphi^x_c$. By the preceding corollary we obtain $\Gamma; \neg\psi \vdash \forall x\ \neg\varphi$. We use contraposition again. $\dashv$

Alphabetic Variants

Thm 24R Existence of Alphabetic Variants (24I) Let φ be a formula, t a term, and x a variable. Then we can find a formula φ' (which differs from φ only in the choice of quantified variables) such that (a) $\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi$ and (b) t is substitutable for x in φ'. $\diamond$

Proof. We consider fixed t and x and construct φ' by recursion on φ. For atomic φ we take $\varphi' = \varphi$, and then $(\neg\varphi)' = (\neg\varphi')$,$(\varphi \to \psi)' = (\varphi' \to \psi')$. We define $(\forall y\ \varphi)' = \forall z(\varphi')^y_z$, where z is a variable that does not occur in φ' or t or x. By inductive hypothesis we see that (b) holds. To see that $\forall y\ \varphi \vdash \forall z(\varphi')^y_z$, consider a sequence $\langle \forall y\ \varphi, \forall y\ \varphi', (\varphi')^y_z, \forall z(\varphi')^y_z \rangle$, where each formula (except the first one) can be obtained from former ones. To see the converse, consider $\langle \forall z(\varphi')^y_z, ((\varphi')^y_z)^z_y, \varphi', \varphi, \forall y\ \varphi \rangle$. This is very similar to part of the proof of 24P. $\dashv$

This theorem essentially states that we can *in effect always* perform substitution.

Equality

Assuming $=$ in our language, we want to show that the relation defined by $v_1 = v_2$ is reflexive, symmetric, and transitive (i.e., is an equivalence relation) and that equality is compatible with the predicate and function symbols. The relevant facts are listed as follows. (P and f are both 2-place. Similarly for n-place predicate symbols and function symbols.)

Rmk 24S Facts about Equality

$$\vdash \forall x\ x = x; \tag{Eq.1}$$
$$\vdash \forall x \forall y (x = y \to y = x); \tag{Eq.2}$$
$$\vdash \forall x \forall y \forall z (x = y \to y = z \to x = z); \tag{Eq.3}$$
$$\vdash \forall x_1 \forall x_2 \forall y_1 \forall y_2 (x_1 = y_1 \to x_2 = y_2 \to P x_1 x_2 \to P y_1 y_2); \tag{Eq.4}$$
$$\vdash \forall x_1 \forall x_2 \forall y_1 \forall y_2 (x_1 = y_1 \to x_2 = y_2 \to f x_1 x_2 \to f y_1 y_2). \tag{Eq.5}$$

Proofs can be found on page 122, 127 and 128 of the book. ◇

Final Comments

Here we discuss roughly the reasons for our choices of axioms (24B). Axiom goup 1 is included to handel sentential connective symbols and axiom group 2 reflects the intended meaning of the quantifier symbol. And in order to be able to prove the 24G we added axiom groups 3 and 4 and arranged for generalizations of axioms to be axioms. Axiom groups 5 and 6 are included to prove the properties we want from equality.

By 25B every logical axiom is a valid formula. *All* valid formulas are not used as logical axioms, for that (a) we need a class Λ with a finitary, *syntatical* definition (instead of a *semantical* one) to prove certain things and that (b) we perfer a decidable Λ, which is not the case for the set of validities. Cf. 25I.

The author intend to present in this chapter not just a deductive calculus, but also facts of its development. The discussion is carried out at a meta level, which can be employed in any correct mathematical reasoning. I personally believe the use of "If __, then__." instead of $\to$ is to separate the levels of reasoning. We are actually operating a machine (in this case, the object language or a description of some first-order language) according to our reasoning.

Exercises

1 For a term u, let u^x_t be the expression obtained from u by replacing the variable x by the term t. Restate this definition without using any form of the word "replace" or its synonyms. ◁

See 24C.

3 Question to fill in. ◁

See 24E.

4 Show by Hilbert style deduction that $\vdash \forall x\varphi \to \exists x\varphi$. ◁

$$\begin{aligned}
\vdash \forall x\ \varphi \to \exists x\ \varphi \Leftarrow & \forall x\ \varphi \vdash \exists x\ \varphi && \text{by deduction theorem,}\\
\Leftarrow & \forall x\ \varphi \vdash \neg\forall x\ \neg\varphi && \text{by rewriting,}\\
\Leftarrow & \forall x\ \neg\varphi \vdash \neg\forall x\ \varphi && \text{by contraposition and rule T,}\\
\Leftarrow & \neg\varphi \vdash \neg\forall x\ \varphi && \text{by Ax.2 and MP,}\\
\Leftarrow & \forall x\ \varphi \vdash \varphi && \text{by contraposition and rule T,}\\
\Leftarrow & \vdash \forall x\ \varphi \to \varphi && \text{by MP, which is Ax.2.}
\end{aligned}$$

7 Show that

(a) $\vdash \exists x(Px \to \forall x\ Px)$;

(b) $\{Qx, \forall y(Qy \to \forall z\ Pz)\} \vdash \forall x\ Px$. ◁

(a)

$$\begin{aligned}
& \vdash \exists x(Px \to \forall x Px)\\
\Leftarrow & \{\forall x\neg(Px \to \forall x Px)\} \text{ is inconsistent} && \text{by RAA,}\\
\Leftarrow & \forall x\neg(Px \to \forall x Px) \vdash \forall x Px\\
& \wedge \forall x\neg(Px \to \forall x Px) \vdash \neg\forall x Px,\\
& \text{where}\\
& \forall x\neg(Px \to \forall x Px) \vdash \forall x Px\\
\Leftarrow & \vdash \forall x\neg(Px \to \forall x Px) \to \forall x Px && \text{by MP,}\\
\Leftarrow & \vdash \forall x(\neg(Px \to \forall x Px) \to Px) && \text{by Ax.3 and MP,}\\
\Leftarrow & \neg(Px \to \forall x Px) \to Px, && \text{by generalization theorem and MP,}\\
& \text{which is Ax.1, and}\\
& \forall x\neg(Px \to \forall x Px) \vdash \neg\forall x Px,\\
\Leftarrow & \neg(Px \to \forall x Px) \to \neg\forall x Px && \text{by Ax.2, which is Ax.1.}
\end{aligned}$$

(b)

$$\begin{aligned}
& 1. \vdash \forall y(Qy \to \forall z Pz) \to Qx \to \forall z Pz && \text{Ax.2.}\\
& 2. \{Qx, \forall y(Qy \to \forall z Pz)\} \vdash \forall z Pz && \text{1; ded.}\\
& 3. \{Qx, \forall y(Qy \to \forall z\ Pz)\} \vdash \forall x Px && \text{EAV.}
\end{aligned}$$

9 (*Re-replacement*)

(a) Show by two examples that $(\varphi^x_y)^y_x$ is not in general equal to φ, where the first shows that x may occur in $(\varphi^x_y)^y_x$ at a place where it does not occur in φ and the second shows that x may occur in a φ at a place where it does not occur in $(\varphi^x_y)^y_x$.

(b) Prove 24O. ◁

(a) $\varphi = P\ y$ (y occurs free in φ) and $\forall y\ P\ x$ (not substitutable).

(b) See 24O.

10 Show that $\forall x \forall y\ Pxy \vdash \forall y \forall x\ Pyx$. ◁

This is immediate by 24R.

15 Show that

(a) $\exists x\ \alpha \vee \exists x\ \beta \leftrightarrow \exists x(\alpha \vee \beta)$;

(b) $\forall x\ \alpha \vee \forall x\ \beta \to \forall x(\alpha \vee \beta)$. ◁

(a) a. $\vdash \exists x\ \alpha \vee \exists x\ \beta \to \exists x(\alpha \vee \beta)$

$\Leftarrow \exists x\ \alpha \vee \exists x\ \beta \vdash \exists x(\alpha \vee \beta)$ by deduction theorem,

$\Leftarrow \forall x \neg(\alpha \vee \beta) \vdash \forall x \neg\alpha \wedge \forall x \neg\beta$ by contraposition and Ax.1,

which we show directly:

1. $\vdash \neg(\alpha \vee \beta)^x_c \to \neg\alpha^x_c$ Ax.1. c does not occur in α or β.
2. $\vdash \forall x \neg(\alpha \vee \beta) \to \neg(\alpha \vee \beta)^x_c$ Ax.2.
3. $\vdash \forall x \neg(\alpha \vee \beta) \to \neg\alpha^x_c$ 1; 2; MP.
4. $\alpha^x_c \vdash \neg\forall x \neg(\alpha \vee \beta)$ 3; ded; contraposition.
5. $\exists x\alpha \vdash \neg\forall x \neg(\alpha \vee \beta)$ 4; EI.
6. $\forall x \neg(\alpha \vee \beta) \vdash \forall x \neg\alpha$ 5; contraposition.
7. $\forall x \neg(\alpha \vee \beta) \vdash \forall x \neg\beta$ same as how 6 is deduced.
8. $\forall x \neg(\alpha \vee \beta) \vdash \forall x \neg\alpha \wedge \forall x \neg\beta$ 7; 8; rule T.

b. $\vdash \exists x(\alpha \vee \beta) \to \exists x\alpha \vee \exists x\beta$

$\Leftarrow (\alpha \vee \beta)^x_c \vdash \exists x\alpha \vee \exists x\beta$ by ded and EI (c does not occur in α or β),

which we show directly:

1. $\forall x \neg\alpha \vdash \neg\alpha^x_c$ Ax.2; ded.
2. $\alpha^x_c \vdash \exists x\alpha$ 1; contraposition.
3. $\alpha^x_c \vdash \exists x\alpha \vee \exists x\beta$ 2; Ax.1; rule T.
4. $\neg(\exists x\alpha \vee \exists x\beta) \vdash \neg\alpha^x_c$ 3; contraposition; Ax.1.
5. $\neg(\exists x\alpha \vee \exists x\beta) \vdash \neg\beta^x_c$ same as how 4 is deduced.
6. $\neg(\exists x\alpha \vee \exists x\beta) \vdash (\neg\alpha \wedge \neg\beta)^x_c$ 4; 5; rule T.
7. $(\alpha \vee \beta)^x_c \vdash \exists x\alpha \vee \exists\beta$ 6; contraposition.

c. $\vdash \exists x\alpha \vee \exists x\beta \leftrightarrow \exists x(\alpha \vee \beta)$ a; b; rule T.

(b) $\vdash \forall x\alpha \vee \forall x\beta \to \forall x(\alpha \vee \beta)$

$\Leftarrow \forall x\alpha \vee \forall x\beta \vdash \forall x(\alpha \vee \beta)$ by deduction theorem,

$\Leftarrow \exists x\neg(\alpha \vee \beta) \vdash \exists x\neg\alpha \wedge \exists x\neg\beta$ by contraposition and Ax.1,

$\Leftarrow \neg(\alpha \vee \beta)^x_c \vdash \exists x\neg\alpha \wedge \exists x\neg\beta$ by EI, where c does not occur in α or β,

which we show directly:

1. $\vdash \forall x\alpha \to \alpha^x_c$ Ax.2.
2. $\forall x\alpha \vdash \alpha^x_c$ 1; ded.
3. $\vdash \alpha^x_c \to (\alpha \vee \beta)^x_c$ Ax.1.
4. $\forall\alpha \vdash (\alpha \vee \beta)^x_c$ 2; 3; MP.
5. $\neg(\alpha \vee \beta)^x_c \vdash \exists x\neg\alpha$ 4; contraposition.
6. $\neg(\alpha \vee \beta)^x_c \vdash \exists x\neg\beta$ same as how 5 is deduced.
7. $\neg(\alpha \vee \beta)^x_c \vdash \exists x\neg\alpha \wedge \exists x\neg\beta$ 5; 6; rule T.

2.5 Soundness and Completeness Theorems

A desirable and significant fact is that *some* deductive calculus is sound and complete. In this section we state that our chosen one qualifies.

The proof of 25C proceeds via 25B, whose proof, in turn, depends on 25A.

Lem 25A Substitution Lemma If the term t is substitutable for the variable x in the wff φ then $\models_{\mathfrak{A}} \varphi^x_t[s] \Leftrightarrow \models_{\mathfrak{A}} \varphi[s(x|\overline{s}(t))]$. $\diamond$

Proof. We first state the fact that for any term $u, \overline{s}(u^x_t) = \overline{s(x|\overline{s}(t))}(u)$. (This can be proved by induction on u.) Then we use induction on φ. Consider fixed $\mathfrak{A}$ and s.

Case 1: $\varphi = P\ t_1, \ldots, t_n$. We have that

$$\begin{aligned} \models_{\mathfrak{A}} \varphi^x_t[s] &\Leftrightarrow \langle \overline{s}((t_1)^x_t), \ldots, \overline{s}((t_n)^x_t)\rangle \in P^{\mathfrak{A}} \\ &\Leftrightarrow \langle \overline{s(x|\overline{s}(t))}(t_1), \ldots, \overline{s(x|\overline{s}(t))}(t_n)\rangle \in P^{\mathfrak{A}} \\ &\Leftrightarrow\ \models_{\mathfrak{A}} \varphi[s(x|\overline{s}(t))]. \end{aligned}$$

Case 2: $\varphi = \neg\psi$ or $\varphi = \psi \to \theta$. The inductive step follows from the induction hypotheses for ψ and θ.
Case 3: $\varphi = \forall y\ \psi$, and x does not occur free in φ. Then s and $s(x|\overline{s}(t))$ agree on all variables that occur free in φ. Also $\varphi^x_t = \varphi$. By 22C the conclusion is immediate.(I tried to state something similar but way more cumbersome when trying to prove this case.)
Case 4: $\varphi = \forall y\ \psi$, where $y \neq x$ and y does not occur in t. For every $d \in |\mathfrak{A}|$ we have

$$\models_{\mathfrak{A}} (\forall y\ \psi)^x_t[s] \Leftrightarrow\ \models_{\mathfrak{A}} \psi^x_t[s(y|d)] \Leftrightarrow\ \models_{\mathfrak{A}} \psi[s(y|d)(x|\overline{s(y|d)}(t))] \Leftrightarrow\ \models_{\mathfrak{A}} \forall y\ \psi[s(x|\overline{s}(t))].$$

This holds exactly for that $y \neq x$ and y does not occur in t. ⊣

Lem 25B (25A) Every logical axiom (24B) is valid. ◇

Proof. By Ex.2.2.6 any generalization of a valid formula is valid. Therefore it suffices to examine various axiom groups.

Axiom group 1: From 24E we have that if $\emptyset$ tautologically implies $\alpha, \emptyset \vDash \alpha$.
Axiom group 2: Immediate given 25A.
Axiom group 3: See Ex.2.2.3.
Axiom group 4: See Ex.2.2.4.
Axiom group 5: Trivial.
Axiom group 6: For an example see Ex.2.2.5. We can use induction on terms and rest of the proof is trivial. ⊣

Thm 25C Soundness Theorem $\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$. ◇

Proof. We use induction on φ. By Ex.1.7.6 this is trivial. ⊣

Cor 25D (25C) If $\vdash (\varphi \leftrightarrow \psi)$, then φ and ψ are logically equivalent. ◇

Cor 25E (25D) If φ' is an alphabetic variant of φ by 24R, then φ and φ' are logically equivalent. ◇

Define Γ to be *satisfiable* iff there is some $\mathfrak{A}$ and s such that $\mathfrak{A}$ satisfies every member of Γ with s.

Cor 25F (25E) If Γ is satisfiable then it is consistent (24L). ◇

Second Proof of 25C.

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma; \neg\varphi \text{ is inconsistent} \Rightarrow \Gamma; \neg\varphi \text{ is not satisfiable} \Leftrightarrow \Gamma \vDash \varphi.$$ ⊣

This proof states that 25F is equivalent to 25C. Note that "unsatisfiable", like "inconsistent", is indeed a very strong assertion.

Thm 25G Completeness Theorem (a) If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$. (b) Any consistent set of formulas is satisfiable. ◇

Proof Sketch. This is a proof for a countable language with equality symbol. By Ex.2.5.2 it suffices to prove part (b). Similar to the proof of 17B, we begin with a consistent set Γ and extend it to a set Δ of formulas for which (i) $\Gamma \subseteq \Delta$. (ii) Δ is consistent and is maximal in the sense that for any formula α, either $\alpha \in \Delta$

or $(\neg\alpha) \in \Delta$. (iii) For any formula φ and variable x, there is a constant c such that $(\neg\forall x\ \varphi \to \neg\varphi^x_c) \in \Delta$. Then we form a structure $\mathfrak{A}$ in which members of Γ not containing equality symbol can be satisfied. Finally, we change $\mathfrak{A}$ to accomodate formulas containing the equality symbol.

Let Γ be a consistent set of wffs in a countable language.

Step 1: Expand the language by adding a countably infinite set of new constant symbols. Then Γ remains consistent as a set of wffs in the new language. By contradiction and 24N this is valid.

Step 2: For each wff φ in the new language and each variable x, we add to Γ the wff

$$\neg\forall x\ \varphi \to \neg\varphi^x_c,$$

where c is one of the new constant symbols. The idea is that c volunteers to name a counterexample to φ, if there is any. We can do this in such a way that Γ together with the set Θ of all the added wffs is still consistent. This feels very valid, because the newly added constant symbols do not seem to carry inconsistency. For a proof see page 136.

Step 3: Now we extend $\Gamma\cup\Theta$ to a consistent set Δ which is maximal in the sense that for any wff φ either $\varphi \in \Delta$ or $(\neg\varphi) \in \Delta$. Let Λ be the set of logical axioms for the expanded language. Since $\Gamma\cup\Theta$ is consistent, by 24F there is no formula β such that $\Gamma\cup\Theta\cup\Lambda$ tautologically implies both β and $\neg\beta$. Hence there is a truth assignment v for the set of all prime formulas that satisfies $\Gamma\cup\Theta\cup\Lambda$. Let $\Delta = \{\varphi|\bar{v}(\varphi) = T\}$. Clearly Δ qualifies. Also we have that Δ is deductively closed, that is, $\Delta \vdash \varphi \Rightarrow \varphi \in \Delta$ (by 24F or maximality and consistency). Cf. 17B.

Step 4: We make structure $\mathfrak{A}$, replacing the equality symbol temporarily with a new two-place predicate symbol E.

(a) $|\mathfrak{A}|$ is the set of all terms of the new language.

(b) $\langle u, t\rangle \in E^{\mathfrak{A}}$ iff $= ut \in \Delta$.

(c) For each n-place predicate symbol P, $\langle t_1, \ldots, t_n\rangle \in P^{\mathfrak{A}}$ iff $Pt_1 \cdots t_n \in \Delta$.

(d) For each n-place function symbol f, $f^{\mathfrak{A}}(t_1, \ldots, t_n) = ft_1 \cdots t_n$.

The constant symbols are treated as 0-place functions. Define also $s : V \to |\mathfrak{A}|$ identity on V. It then follows that for any term t, $\bar{s}(t) = t$. For any wff φ, let φ^* be the result of replacing the equality symbol in φ by E. Then $\models_{\mathfrak{A}} \varphi^*[s]$ iff $\varphi \in \Delta$.

We can prove this by induction, where the quantification case is not immediately trivial: To show that $\models_{\mathfrak{A}} \forall x\ \varphi^*[s] \Leftrightarrow (\forall x\ \varphi) \in \Delta$, first show that $(\Rightarrow)$ holds. By

25A and IH we have that $\varphi^x_c \in \Delta$ We have in Δ that $\neg\forall x\ \varphi \to \neg\varphi^x_c$, which gives us $(\forall x\ \varphi) \in \Delta$ contrapositively. To intuitively understand this, c was chosen to be a counterexample to φ, but now φ^x_c holds, lest φ. The converse can be proven very much the same, except that we have to deal with the case where t is not substitutible for x in φ. By 25E this is repairable.

Step 5: We need to deal with the equality symbol in the language. For example, if Γ contains sentence $c = d$, then we need a structure $\mathfrak{B}$ in which $c^{\mathfrak{B}} = d^{\mathfrak{B}}$. We obtain $\mathfrak{B}$ as the quotient structure $\mathfrak{A}/E$ of $\mathfrak{A}$ modulo $E^{\mathfrak{A}}$. For the full definition see page 140. Let $h : |\mathfrak{A}| \to |\mathfrak{A}/E|$ be the natural map $h(t) = [t]$. Then we have for any φ: $\varphi \in \Delta \Leftrightarrow \vDash_{\mathfrak{A}} \varphi^*[s] \Leftrightarrow \vDash_{\mathfrak{A}/E} \varphi^*[h \circ s] \Leftrightarrow \vDash_{\mathfrak{A}/E} \varphi[h \circ s]$, because $E^{\mathfrak{A}/E}$ is the equality relation on $|\mathfrak{A}/E|$. That is, $\mathfrak{A}/E$ satisfies every member of Δ with $h \circ s$. The validity of this step is that by 24S $E^{\mathfrak{A}}$ is a *congruence relation* for $\mathfrak{A}$.

Step 6: Restrict the structure $\mathfrak{A}/E$ to the original language. This restriction of $\mathfrak{A}/E$ satisfies every member of Γ with $h \circ s$. ⊣

Thm 25H Compactness Theorem (a) If $\Gamma \vDash \varphi$ then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \vDash \varphi$. (b) If every finite subset Γ_0 of Γ is satisfiable, then Γ is satisfiable. ⋄

Proof. These are trivial, given 25G and 25C. (a) and (b) are indeed equivalent. Cf. Ex.1.7.3. ⊣

***Thm 25I Enumerability Theorem** For a reasonable language, the set of valid wffs can be effectively enumerated (17E). ⋄

By a reasonable language we mean one whose set of parameters can be effectively enumerated and such that the sets of predicate symbols and function symbols are decidable. It has to be countable. We actually ask it to be "communicatable", indicating a finite alphabet. Actually, when we want to analysis an expression or string ε, we are *already* assuming it to be finite and from a countable language, for there are only countably many things eligible to be given by one person to another.

Proof. We have that Λ is decidable. By 17H and 24F we are done. ⊣

Second Proof. We can actually enumerate all validities by enumerating all finite sequences of wffs and check if each is a deduction. ⊣

***Cor 25J** (25F) Let Γ be a decidable set of formulas in a reasonable language. The set of theorems (or $\{\varphi|\Gamma \vDash \varphi\}$) of Γ is effectively enumerable. ⋄

This is indeed powerful. We are stating that the theorems of a system with decidable axioms is effectively enumerable.

***Cor 25K** (25G) Let Γ be a decidable set of formulas in a reasonable language, and for any sentence σ either $\Gamma \vDash \sigma$ or $\Gamma \vDash \neg\sigma$. Then the set of sentences implied by Γ is decidable. $\diamond$

Proof Idea. If Γ is inconsistent, then the set of all sentences is decidable. Otherwise cf. 17G. $\dashv$

The set of sentences implied by Γ is decidable, yet we do not have a fixed procedure for it.

Exercises

2 Prove the equivalence of parts (a) and (b) of the 25G. $\triangleleft$

Proof. (a)$\Rightarrow$(b): We prove this contrapositively.

$$\Gamma; \varphi \text{ is unsatisfiable} \Leftrightarrow \Gamma \vDash \neg\varphi \Rightarrow \Gamma \vdash \neg\varphi \Leftrightarrow \Gamma; \varphi \text{ is inconsistent.}$$

(b)$\Rightarrow$(a): By 24M we have that

$$\Gamma \vDash \varphi \Leftrightarrow \Gamma; \neg\varphi \text{ is unsatisfiable} \Rightarrow \Gamma; \neg\varphi \text{ is not consistent} \Leftrightarrow \Gamma \vdash \varphi. \qquad \dashv$$

4 Let $\Gamma = \{\neg\forall v_1 P v_1, P v_2, P v_3, \dots\}$. Is Γ consistent? Is Γ satisfiable? $\triangleleft$

It is consistent and satisfiable. Define $P^{\mathfrak{A}} = \{v_2, v_3, \dots\}$ and $s : V \to |\mathfrak{A}|$ as identity, it follows that for all $\gamma \in \Gamma$, $\vDash_{\mathfrak{A}} \gamma[s]$.

5 Show that an infinite map (of countries) can be colored with four colors iff every finite submap of it can be. $\triangleleft$

Proof. We prove only ($\Leftarrow$). Let $\mathcal{C}$ denote the set of countries on the map. Consider a first-order language L with no equality, no function symbols except the countable infinite set of constants $C = \{c_1, \dots, c_n, \dots\}$ and five 1-place predicate symbols C_1, C_2, C_3, C_4, V. Use arbitrary injective $f : \mathcal{C} \to C$. $C_1 x$ denotes "if x is a country on the map, then it is colored $\mathbf{C_1}$" and so forth, where $\mathbf{C_1}, \mathbf{C_2}, \mathbf{C_3}, \mathbf{C_4}$ are distinct colors. Px denotes "if x is a country on the map, then its coloring is *valid*". Let $\alpha = \forall x((C_1 x \vee C_2 x \vee C_3 x \vee C_4 x) \wedge (\bigwedge_{P \neq Q} \neg(Px \wedge Qx)))$, where $P, Q \in \{C_1, C_2, C_3, C_4\}$, and $\beta = \forall x\ Vx$. Consider a *coloring* $\Sigma = \{C_j c_i | j \in \{1,2,3,4\}, i \in \mathbb{Z}_{>0}\}$. We have that for every finite subset Σ_0 of Σ, $\Sigma_0 \cup \{\alpha, \beta\}$ is satisfiable. It follows that every finite subset Γ_0 of $\Gamma = \Sigma \cup \{\alpha, \beta\}$ is satisfiable. By 25H we are done. $\dashv$

Comment. In fact α is not necessary if one finds it reasonable to color one country with two colors and have the validness of the coloring of a country defined accordingly. We are essentially trying to abstract the problem to an extent where it is in its simplest form that can be solved taking advantage of 25H. By common sense, given a map (a set of countries and some topological relations on it, for example, a set of vetices and a set of edges) and a coloring (a set of formulas like Σ), we can easily (through an effective procedure) find a structure $\mathfrak{A}$ that safisfies every member of the coloring and tell if it is valid w.r.t. some country (that is, to define $V^{\mathfrak{A}}$), and predicate V is an abstraction of that procedure. This actually requires the set of vertices and the set of edges to be decidable, for otherwise our predicate V is not well defined.

6 Let Σ_1 and Σ_2 be sets of sentences such that nothing is a model of both Σ_1 and Σ_2. Show that there is a sentence τ such that

$$\text{Mod } \Sigma_1 \subseteq \text{Mod } \tau \quad \text{and} \quad \text{Mod } \Sigma_2 \subseteq \text{Mod } \neg\tau. \qquad \triangleleft$$

We may suppose Σ_1 and Σ_2 are satisfiable (the other cases are ommitted as trivial). $\Sigma_1 \cup \Sigma_2$ is not satisfiable, thus not finitely satisfiable. Say that a finite subset Σ_0 is inconsistent. Let α be the conjunction of $\Sigma_0 \cap \Sigma_1$. Clearly $\Sigma_1 \vdash \alpha$ and $\Sigma_2 \vdash \neg\alpha$, and we are done.

Comment. This can be stated: Disjoint EC_Δ classes can be separated by an EC class.

7 For each of the following sentences, either show there is a deduction or give a counter-model (i.e., a structure in which it is false.)

(a) $\forall x(Qx \to \forall y\, Qy)$

(b) $(\exists x\, Px \to \forall y\, Qy) \to \forall z(Pz \to Qz)$

(c) $\forall z(Pz \to Qz) \to (\exists x\, Px \to \forall y\, Qy)$

(d) $\neg\exists y\, \forall x(Pxy \leftrightarrow \neg Pxx)$ $\triangleleft$

(a) Not valid. Let $|\mathfrak{A}| = \{0, 1\}$ and $Q^{\mathfrak{A}} = \{1\}$. Then $\mathfrak{A}$ is a counter-model.

(b) $\quad \vdash (\exists x Px \to \forall y Qy) \to \forall z(Pz \to Qz)$

$\Leftarrow \exists x Px \to \forall y Qy \vdash \forall z(Pz \to Qz)$ by ded,

$\Leftarrow \{\exists x Px \to \forall y Qy, Pz\} \vdash Qz$ by gen and ded,

which we show directly:

1. $\forall x \neg Px \vdash \neg Pz$ Ax.2; ded.
2. $Pz \vdash \neg\forall x \neg Px$ 1; contraposition.
3. $\{\exists x Px \to \forall y Qy, Pz\} \vdash \forall y Qy$ 2; MP.
4. $\vdash \forall y Qy \to Qz$ Ax.2.
5. $\{\exists x Px \to \forall y Qy, Pz\} \vdash Qz$ 3; 4; MP.

(c) Not valid. Let $|\mathfrak{A}| = \{0, 1\}$ and $P^{\mathfrak{A}} = Q^{\mathfrak{A}} = \{1\}$. Then $\mathfrak{A}$ is a counter-model.

(d) $\quad \vdash \neg\exists y \forall x(Pxy \leftrightarrow \neg Pxx)$

$\Leftarrow \vdash \forall y \neg\forall x(Pxy \leftrightarrow \neg Pxx)$ by Ax.1 and MP,

$\Leftarrow \vdash \neg\forall x(Pxy \leftrightarrow \neg Pxx)$ by gen,

which we show directly:

1. $\vdash \neg(Pxy \leftrightarrow \neg Pxx)^x_y$ Ax.1.
2. $\vdash \forall x(Pxy \leftrightarrow \neg Pxx) \to (Pxy \leftrightarrow \neg Pxx)^x_y$ Ax.2.
3. $\vdash \neg(Pxy \leftrightarrow \neg Pxx)^x_y \to \neg\forall x(Pxy \leftrightarrow \neg Pxx)$ 2; Ax.1; MP.
4. $\vdash \neg\forall x(Pxy \leftrightarrow \neg Pxx)$ 1; 3; MP.

8 Assume the language (with equality) has just the parameters $\forall$ and P, where P is a two-place predicate symbol. Let $\mathfrak{A}$ be the structure with $|\mathfrak{A}| = \mathbb{Z}$, and with $\langle a, b\rangle \in P^{\mathfrak{A}}$ iff $|a - b| = 1$. Thus $\mathfrak{A}$ looks like an infinite graph:

$$\cdots \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \bullet \longleftrightarrow \cdots$$

Show that there is an elementarily equivalent structure $\mathfrak{B}$ that is not connected. (Being *connected* means that for every two members of $|\mathfrak{B}|$, there is a path between them. A *path* — of length n — from a to b is a sequence $\langle p_0, p_1, \ldots, p_n\rangle$ with $a = p_0$ and $b = p_n$ and $\langle p_i, p_{i+1}\rangle \in P^{\mathfrak{B}}$ for each i.) *Suggestion*: Add constant symbols c and d. Write down sentences saying c and d are far apart. $\triangleleft$

Cf. 26P. Expand the language by adding two new constant symbols c and d. For each integer $k \geq 0$, we can find a sentence λ_k that translates, "The distance between c and d is *not* k." For example,

$$\lambda_0 = \neg c = d$$
$$\lambda_1 = \forall p_1(Pcp_1 \to \neg p_1 = d),$$
$$\lambda_2 = \forall p_1 \forall p_2(Pcp_1 \to Pp_1p_2 \to \neg p_2 = d).$$

Let $\Sigma = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$. Consider a finite subset of $\Sigma \cup \mathrm{Th}\mathfrak{A}$. That subset is true in $\mathfrak{A}_k$ such that $|c^{\mathfrak{A}_k} - d^{\mathfrak{A}_k}| > k$ for some large k. So by 25H $\Sigma \cup \mathrm{Th}\mathfrak{A}$ has a model

$$\mathfrak{B} = (|\mathfrak{B}|; \mathrm{P}^{\mathfrak{B}}, =^{\mathfrak{B}}, c^{\mathfrak{B}}, d^{\mathfrak{B}})$$

Let $\mathfrak{B}_0$ be the restriction of $\mathfrak{B}$ to the original language: $\mathfrak{B}_0 = (|\mathfrak{B}|, \mathrm{P}^{\mathfrak{B}}, =^{\mathfrak{B}})$. By 26O $\mathfrak{B}_0 \equiv \mathfrak{A}$. Note that $c^{\mathfrak{B}} \in |\mathfrak{B}|$ and $d^{\mathfrak{B}} \in |\mathfrak{B}|$, but there is no path between them.

Comment. One might ask: every member of the universe should be connected to two unique nodes, then to which nodes is $c^{\mathfrak{B}}$ connected? Well, consider not c and d are located on the single infinite graph that $\mathfrak{A}$ indicates, but that c and d and $\mathbb{Z}$ are on 3 seperate infinite graphs, which together consititute our construction of $|\mathfrak{B}|$. That should make a better (possible) interpretation of what we have been effectively doing. One may feel that c and d are far apart but connected, but that is not the case in $\mathfrak{B}$. All we have is that every finite piece of $\mathfrak{B}_0$ looks like a finite segment of $\mathfrak{A}$.

2.6 Models of Theories

Finite Models

A sentence is *finitely valid* if it is true in every finite structure. For example, the negation of one saying that $<$ is an ordering with no largest element.

Thm 26A (26A) If a set Σ of sentences has arbitrarily large finite models, then it has an infinite model. $\diamond$

This straightforward fact can be proved as follows.

Proof. For each integer $k \geq 2$ we can find a sentence λ_k that translates, "There are at least k things." For example $\lambda_2 = \exists v_1 \exists v_2 v_1 \neq v_2$. Consider the set $\Sigma \cup \{\lambda_2, \lambda_3, \dots\}$. By hypothesis any finite subset of it has a model. So by compactness the entire set has a model, which clearly must be infinite. $\dashv$

For example, it is a priori conceivable that there might be some very subtle equation of group theory that was true in every finite group but false in every infinite group. But by the above theorem, no such equation exists.

Cor 26B (26B) The class of all finite structures (for a fixed language) is not EC_Δ (22J). The class of all infinite structures is not EC. $\diamond$

Proof. The first sentence follows immediately from 26A. If the class of all infinite structures is Mod τ, then the class of all finite structures is Mod $\neg\tau$. But this class is not even EC_Δ, much less EC. $\dashv$

The class of infinite structures is EC_Δ, being $\mathrm{Mod}\,\{\lambda_2, \lambda_3, \dots\}$.

Defn 26C For any structure $\mathfrak{A}$, define the *theory* of $\mathfrak{A}$ (written $\mathrm{Th}\mathfrak{A}$) to be the set of all sentences true in $\mathfrak{A}$. $\diamond$

Rmk 26D Any finite structure $\mathfrak{A}$ is isomorphic to a sturcture with the universe $\{1, 2, \dots, n\}$ where $n = \mathrm{card}|\mathfrak{A}|$. The idea here is, where $|\mathfrak{A}| = \{a_1, \dots, a_n\}$, to replace a_i by i. $\diamond$

Rmk 26E A finite structure of the sort in 26D can, for a finite language, be specified by a finite string of symbols. So it can be *communicated*. $\diamond$

***Rmk 26F** Given a finite structure $\mathfrak{A}$ for a finite language, with universe $\{1, \dots, n\}$, a wff φ and an assignment s_φ of numbers in this universe to the variables free in φ, we can effectively decide whether or not $\models_{\mathfrak{A}} \varphi[s_\varphi]$. The scheme is to just enumerate. $\diamond$

***Thm 26G** (26C) For a finite sturcture $\mathfrak{A}$ in a finite language, $\mathrm{Th}\mathfrak{A}$ is decidable. $\diamond$

Proof. By 26D and 26F this is immediate. $\dashv$

Second Proof. There exists a sentence $\delta_{\mathfrak{A}}$ that specifies $\mathfrak{A}$ up to isomorphism (cf. Ex.2.2.17.) It follows that $\mathrm{Th}\mathfrak{A} = \{\sigma | \delta_{\mathfrak{A}} \models \sigma\}$. (If σ is true in $\mathfrak{A}$, then it is true in all isomorphic copies, and hence all models of $\delta_{\mathfrak{A}}$. So $\delta_{\mathfrak{A}} \models \sigma$. The other direction is trivial.) Apply 25K, noting that for each σ, either $\models_{\mathfrak{A}}$ or $\models_{\mathfrak{A}} \neg\sigma$. $\dashv$

***Rmk 26H** The binary relation $\{\langle\sigma, n\rangle | \sigma$ has a model of size $n\}$ is decidable, where σ is a sentence and n is a positive integer. (There are finitely many structures to check, and we check them according to 26G.) $\diamond$

The *spectrum* of a sentence σ is $\{n | \sigma$ has a model of size $n\}$. Cf. Ex.2.2.16.

***Rmk 26I** The spectrum of any sentence is a decidable set of positive integers. $\diamond$

***Thm 26J** (26D) For a finite language, $\{\sigma | \sigma$ has a finite model$\}$ is effectively enumerable. $\diamond$

Proof. By 26H we check size $1, 2, \dots$. If a model of size n satisfies the given σ we are done. $\dashv$

***Cor 26K** (26E) Assume the language is finite, and let Φ be the set of sentences true in every finite structure. Then its complement, $\overline{\Phi}$, is effectively enumerable. $\diamond$

Proof. By 26H we just enumerate structures of size $i \in \mathbb{Z}_{>0}$ till σ does not have a model of size n. $\dashv$

***Thm 26L Trakhtenbrot's Theorem** Φ is not in general decidable or effectively enumerable. $\diamond$

Thus the analogue of 25I for finite structures only is false.

Size of Models

Thm 26M Löwenheim-Skolem Theorem (a) Let Γ be a satisfiable set of formulas in a countable language. Then Γ is satisfiable in some countable structure. (b) Let Σ be a set of sentences in a countable language. If Σ has any model, then it has a countable model. $\diamond$

Proof. Γ is consistent by 25F. In the proof of 25G we actually formed a countable structure $\mathfrak{A}/E$ from a consistent set, and that completes the proof. $\dashv$

Eg 26N Skolem's paradox: Let A_{ST} be a (hopefully consistent) set of axioms for set theory. By 26M A_{ST} has a countable model $\mathfrak{S}$. $\mathfrak{S}$ is would also be a model of the sentence σ : "There are uncountably many sets." The puzzling part here is that in $|\mathfrak{S}|$ there is no member (set) that can be interpreted as a bijection between natural numbers (defined by $\mathfrak{S}$) and $|\mathfrak{S}|$ (for this is what σ says) but there *exists* one (between $\mathbb{N}$ and $|\mathfrak{S}|$) outside of $|\mathfrak{S}|$, for that $\mathfrak{S}$ is countable. There *is* countably many sets inside $|\mathfrak{S}|$ yet we cannot "count" from inside $|\mathfrak{S}|$, so surprisingly, no contradiction arises. $\diamond$

Cor 26O For any structure $\mathfrak{A}$ for a countable language, there is a countable model $\mathfrak{B}$ of $\mathrm{Th}\mathfrak{A}$ and if so then $\mathfrak{A} \equiv \mathfrak{B}$ (22O). $\diamond$

Proof. $(\Rightarrow)$ $\models_{\mathfrak{A}} \sigma \Rightarrow \sigma \in \mathrm{Th}\mathfrak{A} \Rightarrow \models_{\mathfrak{B}} \sigma$
$(\Leftarrow)$ $\not\models_{\mathfrak{A}} \sigma \Rightarrow \models_{\mathfrak{A}} \neg\sigma \Rightarrow (\neg\sigma) \in \mathrm{Th}\mathfrak{A} \Rightarrow \models_{\mathfrak{B}} \neg\sigma \Rightarrow \not\models_{\mathfrak{B}} \sigma$. $\dashv$

The real field $(\mathbb{R}; 0, 1, +, \cdot)$ is an uncountable structure for a finite language. And Tarski showed the field of algebraic real numbers is a countable structure satisfying exactly the same sentences.

Eg 26P Consider the structure $\mathfrak{N} = (\mathbb{N}; 0, S, <, +, \cdot)$. We claim that there is a countable structure $\mathfrak{M}_0$ such that $\mathfrak{M}_0 \equiv \mathfrak{N}$ but $\mathfrak{M}_0 \not\cong \mathfrak{N}$ (22L). $\diamond$

Proof. Expand the language by adding a new constant symbol c. Let

$$\Sigma = \{0 < c, \mathrm{S}0 < c, \mathrm{SS}0 < c, \dots\}.$$

Consider a finite subset of $\Sigma \cup \mathrm{Th}\mathfrak{N}$. That subset is true in $\mathfrak{N}_k = (\mathbb{N}; 0, S, <, +, \cdot, k)$ (where $k = c^{\mathfrak{N}_k}$) for some large k. So by 25H $\Sigma \cup \mathrm{Th}\mathfrak{N}$ has a model. By 26M $\Sigma \cup \mathrm{Th}\mathfrak{N}$ has a countable model

$$\mathfrak{M} = (|\mathfrak{M}|; 0^{\mathfrak{M}}, \mathrm{S}^{\mathfrak{M}}, <^{\mathfrak{M}}, +^{\mathfrak{M}}, \cdot^{\mathfrak{M}}, c^{\mathfrak{M}}).$$

Let $\mathfrak{M}_0$ be the restriction of $\mathfrak{M}$ to the original language:

$$\mathfrak{M}_0 = (|\mathfrak{M}|; 0^{\mathfrak{M}}, \mathrm{S}^{\mathfrak{M}}, <^{\mathfrak{M}}, +^{\mathfrak{M}}, \cdot^{\mathfrak{M}}).$$

By 26O $\mathfrak{M}_0 \equiv \mathfrak{N}$. Also note that $c^{\mathfrak{M}} \in |\mathfrak{M}|$. Suppose that there is a homomorphism h of $\mathfrak{M}_0$ into $\mathfrak{N}$, then $\forall n \in \mathbb{N}\ \langle n, h(c^{\mathfrak{M}})\rangle \in <^{\mathfrak{N}}$, but $h(c^{\mathfrak{M}}) \in \mathbb{N}$, hence contradiction. $\dashv$

Comment. The construction does *not* add a single new point "∞"; it produces an infinite tail of elements greater than every standard numeral. A key observation is that *no first-order formula* can define the set of standard naturals inside the resulting model. (If $\psi(x)$ picked out precisely the standard elements, then $\mathfrak{N} \models \forall x\, \psi(x)$ would force $\mathfrak{M} \models \forall x\, \psi(x)$, contradicting the assumption.) On the other hand, a *single* non-standard element *can* be parameter-free definable; this does not violate $\mathfrak{M}_0 \equiv \mathfrak{N}$ because the same formula will simply pick out a different singleton in $\mathfrak{N}$. (Note that a finite formula can *not* say "x is bigger than every standard natural".)

Thm 26Q LST Theorem Let Γ be a set of formulas in a language of cardinality λ and assume that Γ is satisfiable in some infinite structure. Then for every cardinal $\kappa \geq \lambda$, there is a structure of cardinality κ in which Γ is satisfiable. $\diamond$

Cor 26R (26F) (a) Let Σ be a set of sentences in a countable language. If Σ has some infinite model, then Σ has models of every infinite cardinality. (b) Let $\mathfrak{A}$ be an infinite structure for a countable language. Then for any infinite cardinal λ, there is a structure $\mathfrak{B}$ of cardinality λ such that $\mathfrak{B} \equiv \mathfrak{A}$. $\diamond$

Defn 26S Call a set Σ of sentences *categorical* iff any two models of Σ are isomorphic. $\diamond$

26R implies that if Σ has any infinite models, then it is not categorical. This is indicative of a limitation in the expressiveness of first-order languages.

Theories

Defn 26T T is a *theory* iff T is a set of sentences such that for any sentence σ of the language, $T \vDash \sigma \Rightarrow \sigma \in T$. $\diamond$

There is always a smallest theory, consisting of the valid sentences of the language. There is also the theory consisting of all the sentences of the language; it is the only unsatisfiable theory.

Defn 26U For a class $\mathcal{K}$ of structures (for the language), define the *theory* of $\mathcal{K}$ (written Th$\mathcal{K}$) by the equation

$$\mathrm{Th}\mathcal{K} = \{\sigma | \sigma \text{ is true in every member of } \mathcal{K}\}. \qquad \diamond$$

26C is a special case where $\mathcal{K} = \{\mathfrak{A}\}$.

Thm 26V (26G) Th$\mathcal{K}$ is indeed a theory. $\diamond$

We have that $\mathcal{K} \subseteq \mathrm{Mod\ Th}\mathcal{K}$. Consider $(\mathbb{R}; <_R) \in \mathrm{Mod\ Th}(\mathbb{Q}; <_Q)$ (22Q). Actually the examples need not be elementarily equivalent. Consider $\models_{\mathfrak{A}} \alpha \wedge \beta \wedge \neg\gamma$, $\models_{\mathfrak{B}} \alpha \wedge \neg\beta \wedge \gamma$ and $\models_{\mathfrak{C}} \alpha \wedge \neg\beta \wedge \neg\gamma$. They are not pairwise elementarily equivalent, but $\mathfrak{C} \in \mathrm{Mod\ Th}\{\mathfrak{A}, \mathfrak{B}\}$.

Defn 26W The set of *consequences* of Σ is $\mathrm{Cn}\Sigma = \{\sigma | \Sigma \models \sigma\} = \mathrm{Th\ Mod}\Sigma$. $\diamond$

For example, set theory is the set of consequences of a certain set of sentences known as axioms for set theory. A set T of sentences is a theory iff $T = \mathrm{Cn}T$.

Defn 26X A theory T is *complete* iff for every sentence σ, either $\sigma \in T$ or $(\neg\sigma) \in T$. $\diamond$

For example, Th$\{\mathfrak{A}\}$ is a complete theory.

Rmk 26Y A theory T is complete iff any two models of T are elementarily equivalent. $\diamond$

So the theory of fields is not complete, since the sentences $1+1 = 0$ and $\exists x\ x \cdot x = 1+1$ are true in some fields but false in others. Cf. 26AE.

Defn 26Z A theory T is *axiomatizable* iff there is a decidable set Σ of sentences such that $T = \mathrm{Cn}\Sigma$ and *finitely axiomatizable* iff Σ is finite. In the latter case we have $T = \mathrm{Cn}\{\sigma\}$ (or $\mathrm{Cn}\sigma$) where σ is the conjunction of the finitely many members of Σ. $\diamond$

For example, the theory of fields is finitely axiomatizable. For the class $\mathcal{F}$ of fields is ModΦ, where Φ is the finite set of field axioms (item 4 of 22K). And the theory of fields is CnΦ.

Thm 26AA If CnΣ is finitely axiomatizable, then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\mathrm{Cn}\Sigma_0 = \mathrm{Cn}\Sigma$. $\diamond$

Proof. We have $\mathrm{Cn}\Sigma = \mathrm{Cn}\tau$ for some sentence τ and $\Sigma \models \tau$. By 25H there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \tau$ and $\mathrm{Cn}\tau \subseteq \mathrm{Cn}\Sigma_0 \subseteq \mathrm{Cn}\Sigma$, whence equality holds. $\dashv$

The theory of fields of characteristic 0 is axiomatizable, being $\mathrm{Cn}\Phi_0$, where Φ_0 consists of the (finitely many) field axioms together with the infinitely many sentences:

$$\begin{aligned} 1+1 &\neq 0, \\ 1+1+1 &\neq 0, \\ &\dots \end{aligned}$$

Suppose that this theory is finitely axiomatizable. By 26AA it is $\mathrm{Cn}\Phi_0'$, where Φ_0' is some finite subset of Φ_0. But it would be true in some field of a *large* characteristic, hence contradiction.

By 25J and 25K we have:

***Cor 26AB** (26I) In a reasonable language, (a) an axiomatizable theory is effectively enumerable and (b) a complete axiomatizable theory is decidable. ⋄

Defn 26AC Say that a theory T is *κ-categorical* iff all models of T having cardinality κ are isomorphic. ⋄

Thm 26AD Łoś–Vaught Test (1954) Let T be a theory in a countable language. Assume that T has no finite models. If T is κ-categorical for some infinite cardinal κ, then T is complete. ⋄

Proof. Consider any two models $\mathfrak{A}$ and $\mathfrak{B}$ of T. By 26R there exists structures $\mathfrak{A}' \equiv \mathfrak{A}$ and $\mathfrak{B}' \equiv \mathfrak{B}$ having cardinality κ. And we have that $\mathfrak{A}' \cong \mathfrak{B}'$, thus $\mathfrak{A} \equiv \mathfrak{B}$. By 26Y we are done. ⊣

Comment. If T is a theory in a language of cardinality λ, then we must demand that $\lambda \leq \kappa$.

Thm 26AE (26J) (a) The theory of algebraically closed fields of characteristic 0 is complete. (b) The theory of the complex field $\mathfrak{C} = (\mathbb{C}; 0, 1, +, \cdot)$ is decidable. ⋄

Proof. (a) Let $\mathcal{A}$ be the class of algebraically closed fields of characteristic 0. Then $\mathcal{A} = \mathrm{Mod}(\Phi_0 \cup \Gamma)$, where Φ_0 consists as before of the axioms for fields of characteristic 0, and Γ consists of the sentences

$$\begin{aligned} &\forall a \forall b \forall c (a \neq 0 \to \exists x \; a \cdot x \cdot x + b \cdot x + c = 0), \\ &\forall a \forall b \forall c \forall d (a \neq 0 \to \exists x \; a \cdot x \cdot x \cdot x + b \cdot x \cdot x + c \cdot x + d = 0), \\ &\dots \end{aligned}$$

We have that $\mathrm{Mod}\,\mathrm{Th}\mathcal{A} = \mathrm{Mod}\,\mathrm{Cn}(\Phi_0 \cup \Gamma) = \mathrm{Mod}(\Phi_0 \cup \Gamma) = \mathcal{A}$, which are all infinite. By 26AD it suffices to show that $\mathrm{Th}\mathcal{A}$ is categorical in any uncountable cardinality (this is actually more than what is needed to show), which is a known result of algebra. (b) The set $\Phi_0 \cup \Gamma$ is decidable and $\mathrm{Th}\mathcal{A} = \mathrm{Cn}(\Phi_0 \cup \Gamma)$, so this theory is axiomatizable. By part (b) of 26AB it is decidable. We have that $\mathfrak{C} \in \mathcal{A}$, whence $\mathrm{Th}\mathcal{A} \subseteq \mathrm{Th}\mathfrak{C}$. By (a) $\mathrm{Th}\mathcal{A}$ is complete. By Ex.2.6.2 we are done. $\dashv$

The theory of the real field $(\mathbb{R}; 0, 1, +, \cdot)$ is also decidable (due to Tarski), but it is not categorical in any infinite cardinality, so 26AD cannot be applied.

Consider a language with equality and parameters $\forall$ and $<$. Let δ be the conjunction of the following sentences:

1. Ordering axioms (trichotomy and transitivity):

$$\begin{aligned}
&\forall x \forall y (x < y \vee x = y \vee y < x),\\
&\forall x \forall y (x < y \rightarrow y \not< x),\\
&\forall x \forall y \forall z (x < y \rightarrow y < z \rightarrow x < z).
\end{aligned}$$

2. Density: $\forall x \forall y (x < y \rightarrow \exists z (x < z < y))$.
3. No endpoints: $\forall x \exists y \exists z (y < x < z)$.

The dense linear orderings without endpoints are, by definition, the structures for this language that are models of δ.

Thm 26AF Cantor Any countable model of δ is isomorphic to $(\mathbb{Q}, <_Q)$. $\diamond$

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be such structures. Fix enumerations $|\mathfrak{A}| = \{a_0, a_1, \dots\}$ and $|\mathfrak{B}| = \{b_0, b_1, \dots\}$. We construct an isomorphism h in stages. Let $A_i = B_i = h_i = \emptyset$ for $i \leq 0$ (we will use notation $h_i \subseteq |\mathfrak{A}| \times |\mathfrak{B}|$). At stage $2n$ let $h_{2n} = h_{2n-1} \cup \{\langle a_n, b_j \rangle\}$, where j is the least index such that h_{2n} preserves $<$ when restricted to A_n (by density and no endpoints this is valid.) Also let $A_n = A_{n-1} \cup \{a_n\}$ and $B_n = B_{n-1} \cup \{b_j\}$. At stage $2n + 1$ do the counterpart for h_{2n+1} and A_n. Let $h = \bigcup_0^\infty h_i$. Then h is indeed a one-to-one homomorphism, so $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, and we are done. $\dashv$

Comment. This is the classical Cantor back-and-forth proof. Note that we used no endpoints implicitly, for that without it it would be possible that an endpoint is mapped to a non-endpoint, whence we may fail to find an "extended" mapping that preserves $<$. By this theorem we have that $\mathrm{Cn}\delta$ is $\aleph_0$-categorical, thus by 26AD $\mathrm{Cn}\delta$ is complete. Hence by 26Y any two models of δ are elementarily equivalent; in paricular, $(\mathbb{Q}; <_Q) \equiv (\mathbb{R}; <_R)$. By 26AB we can also conclude these sturctures have decidable theories.

Prenex Normal Form

Defn 26AG Define a *prenex* formula to be one of the form (for some $n \geq 0$) $Q_1 x_1 \cdots Q_n x_n \alpha$ where Q_i is $\forall$ or $\exists$ and α is quantifier-free. $\diamond$

Thm 26AH Prenex Normal Form Theorem For any formula, we can find a logically equivalent prenex formula. $\diamond$

This seems very trivial. For proof see page 160 of the book.

Retrospectus

The interest symbolic logic holds for mathematicians is largely due to the accuracy with which it mirrors mathematical deductions. First-order logic, in particular, is well suited for formalizing mathematics, though it is less applicable to everyday discourse. Certain fragments of first-order logic, such as Horn clauses, are of particular interest to computer scientists, as they can express computation and even form the basis of Turing-complete models. (This paragraph has been kindly revised by `GPT-4o`.)

Exercises

2 Let T_1 and T_2 be theories (in the same language) such that (i) $T_1 \subseteq T_2$, (ii) T_1 is complete, and (iii) T_2 is satisfiable. Show that $T_1 = T_2$. $\triangleleft$

Say that T_2 is true in $\mathfrak{A}$. Obviously both T_1 and T_2 are $\mathrm{Th}\mathfrak{A}$, so we are done.

4 Prove 26AF. $\triangleleft$

See 26AF.

6 Prove the converse to part (a) of 26AB. $\triangleleft$

to do

Chapter 3

Undecidability

3.0 Number Theory

The language of number theory is a first-order language with equality and parameters $\forall, 0, \mathrm{S}, <, +, \cdot$ and E, where E is a two-place function symbol. The intended structure for it is $\mathfrak{N} = (\mathbb{N}; 0, S, <, +, \cdot, E)$, where E is exponentiation on $\mathbb{N}$. Thus *number theory* is $\mathrm{Th}\mathfrak{N}$.

We can assign to each formula α of the language of number theory an integer $\sharp\alpha$, called the Gödel number of α. Any way of assigning should suffice, as long as we can effectively find $\sharp\alpha$ from α and vice versa. Similarly, to each finite sequence D of formulas we assign an integer $\mathcal{G}(D)$.

Self-reference

Thm 30A Let $A \subseteq \mathrm{Th}\mathfrak{N}$, and assume that $\{\sharp\alpha | \alpha \in A\}$ is a set definable in $\mathfrak{N}$. Then we can find a sentence σ such that $\sigma \in \mathrm{Th}\mathfrak{N}$ but σ is not deducible from A. $\diamond$

Proof. We construct σ to express that σ itself is not a theorem of A. Thus if $A \vdash \sigma$, then what σ says is false, contradicting the fact that A consists of true sentences. And so $A \nvdash \sigma$, whence σ is true.

To begin with, consider ternary relation R: $\langle a, b, c\rangle \in R$ iff a is the Gödel number of some formula α and c is the value of $\mathcal{G}$ at some deduction from A of $\alpha(\mathrm{S}^b 0)$. We use here the notation: $\varphi(t) = \varphi^{v_1}_t, \varphi(t_1, t_2) = (\varphi^{v_1}_{t_1})^{v_2}_{t_2}$, and so forth.

Then because $\{\sharp\alpha | \alpha \in A\}$ is definable in $\mathfrak{N}$, it follows that R is definable also. Let ρ

be a formula that defines R in $\mathfrak{N}$. Let q be the Gödel number of

$$\forall v_3 \neg\rho(v_1, v_1, v_3).$$

Then let σ be

$$\forall v_3 \neg\rho(\mathrm{S}^q 0, \mathrm{S}^q 0, v_3).$$

Thus σ says that no number is the value of $\mathcal{G}$ at a deduction from A of the result of replacing, in formula number q, the variable v_1 by the numeral for q; i.e., no number is the value of $\mathcal{G}$ at a deduction of σ.

Suppose that there is a deduction of σ from A. Let k be the value of $\mathcal{G}$ at a deduction. Then $\langle q, q, k\rangle \in R$ and hence

$$\models_{\mathfrak{N}} \rho(\mathrm{S}^q 0, \mathrm{S}^q 0, \mathrm{S}^k 0).$$

And by

$$\sigma \vdash \neg\rho(\mathrm{S}^q 0, \mathrm{S}^q 0, \mathrm{S}^k 0)$$

we have that $\not\models_{\mathfrak{N}} \sigma$. But $A \vdash \sigma$ and the members of A are true in $\mathfrak{N}$, so we have a contradiction.

Hence there is no deduction of σ from A. And so for every k, we have that $\langle q, q, k\rangle \notin R$. i.e., $\models_{\mathfrak{N}} \sigma$. ⊣

Cor 30B The set $\{\sharp\tau | \models_{\mathfrak{N}} \tau\}$ of Gödel numbers of sentences true in $\mathfrak{N}$ is a set that is not definable in $\mathfrak{N}$. ◇

Proof. Say that $\mathrm{Th}\mathfrak{N}$ is definable. Then take $A = \mathrm{Th}\mathfrak{N}$ in 30A, we will have that $A \vdash \sigma$, obtaining a contradiction. ⊣

Diagonalization

An approach that feels less of magic. Define a binary relation P on the natural numbers: $\langle a, b\rangle \in P$ iff a is the Gödel number of a formula $\alpha(v_1)$ (with just v_1 free) and $\models_{\mathfrak{N}} \alpha(\mathrm{S}^b 0)$. ("$\alpha$ is true of b.") Then any set of natural numbers that is definable in $\mathfrak{N}$ equals, for some a, the "vertical section" $P_a = \{b | \langle a, b\rangle \in P\}$ of P. So any definable set of natural numbers is somewhere on the list $P_1, P_2, \ldots$.

Now we "diagonalize out" of the list. Define the set $H = \{b | \langle b, b\rangle \notin P\}$. ("$b$ is not true of b.") Then H is nowhere on the list. ($H \neq P_k$ because $k \in H \Leftrightarrow k \notin P_k$. i.e., H differs from P_k upon k.) Therefore H is not definable in $\mathfrak{N}$.

Yet it should be, defined by

$$\neg[(b \text{ is the Gödel number of a formula } \alpha(v_1)) \wedge \models_{\mathfrak{N}} \alpha(\mathrm{S}^b 0)].$$

Then why? An answer is proposed by 30C. Thus, we cannot assert a sentence in the language.

Thm 30C

(a) $\{\sharp\tau \mid \models_{\mathfrak{N}} \tau\}$ is not definable in $\mathfrak{N}$.

(b) *Th$\mathfrak{N}$ is undecidable.

(c) *Th$\mathfrak{N}$ is not axiomatizable (26Z). $\diamond$

Proof. (a): If to the contrary Th$\mathfrak{N}$ is definable, then H should be definable. For that say φ defines Th$\mathfrak{N}$, then we can easily say that $\models_{\mathfrak{N}} \alpha(\mathrm{S}^b 0)$ by the sentence $\varphi(\sharp\alpha(\mathrm{S}^b 0))$. Cf. 30B.
(b): It follows from the argument that every decidable set of natural numbers must be definable in $\mathfrak{N}$.
(c): By 26AB and (b) and that Th$\mathfrak{N}$ is complete. $\dashv$

Computability

Thm 30D For any decidable (or even effectively enumerable) set A of axioms, Cn$A \neq$ Th$\mathfrak{N}$. Because the CnA is effectively enumerable that Th$\mathfrak{N}$ is not. $\diamond$

3.1 Natual Numbers with Successor

Chapter 4

Second-Order Logic

4.1 Second-Order Languages