Notes on Linear Algebra Done Right[1]

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[1] 4^{th} ed. Some exercises are omitted as trivial.

Contents

Chapter 1

Vector Spaces

1A $\mathbb{R}^n$ and $\mathbb{C}^n$

Defn 1.1 *Addition* and *multiplication* on $\mathbb{C}$ are defined by

$$
\begin{aligned}
(a+bi)+(c+di) &= (a+c)+(b+d)i \\
(a+bi)\cdot(c+di) &= (ac-bd)+(ad+bc)i
\end{aligned}
$$

where $a, b, c, d \in \mathbb{R}$. ◇

By properties of $\mathbb{R}$ and 1.1 we obtain properties of $\mathbb{C}$. By the existence of inverses we define $-\alpha$ and $\dfrac{1}{\alpha}$, and subtraction and division accordingly.

Note 1.2 Use $\mathbb{F}$ (i.e., fields) to denote either $\mathbb{R}$ or $\mathbb{C}$. Elements of $\mathbb{F}$ are *scalars*. Say that x_k is the k^{th} *coordinate* of the *list* $(x_1, \ldots, x_n)$ (n, the *length* of the list, has to be finite). Lists, when thought of as arrows, are *vectors*. Addition and scalar multiplication on lists are defined componentwise in the standard way. ◇

Exercises

5 Additive inverse of complex arithmetic. ◁

This is due to the additive inverse of real arithmetic.

6 Multiplicative inverse of complex arithmetic. ◁

For $\alpha = a+bi$, where $a \neq 0$, we have $\dfrac{a-bi}{a^2+b^2}$ as the multiplicative inverse of α. Thus one exists. Also all the multiplicative inverses of α equals it by real arithmetic.

8 Find two distinct square roots of i. $\triangleleft$

We solve the equation $(a+bi)^2 = i$ to obtain $\pm\left(\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i\right)$.

1B Definition of Vector Space

We need a space where things act like vectors, which is defined as follows.

Defn 1.3 A *vector space* over a field $\mathbb{F}$ is a set V equipped with two operations: vector addition $+ : V \times V \to V$, and scalar multiplication $\cdot : \mathbb{F} \times V \to V$, satisfying the following axioms:

1. $(V, +)$ is an abelian group, i.e.:
   (a) (Associativity) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$,
   (b) (Commutativity) $u + v = v + u$ for all $u, v \in V$,
   (c) (Identity) There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$,
   (d) (Inverses) For each $v \in V$, there exists an element $-v \in V$ such that $v + (-v) = 0$.
2. Scalar multiplication satisfies:
   (a) (Multiplicative identity) $1 \cdot v = v$ for all $v \in V$,
   (b) (Associativity) $a \cdot (b \cdot v) = (ab) \cdot v$ for all $a, b \in \mathbb{F}$, $v \in V$,
   (c) (Distributivity over vector addition) $a \cdot (u + v) = a \cdot u + a \cdot v$ for all $a \in \mathbb{F}$, $u, v \in V$,
   (d) (Distributivity over field addition) $(a + b) \cdot v = a \cdot v + b \cdot v$ for all $a, b \in \mathbb{F}$, $v \in V$. $\diamond$

The simplest vector space is $\{0\}$.

Defn 1.4 Elements of a vector space are called *vectors* or *points*. $\diamond$

Eg 1.5 For $f, g \in \mathbb{F}^S$ and $\lambda \in \mathbb{F}$, $f + g \in \mathbb{F}^S$ is defined by $\forall x \in S, (f + g)(x) = f(x) + g(x)$ and $\lambda f \in \mathbb{F}^S$ by $\forall x (\lambda f)(x) = \lambda f(x)$. If $S \neq \emptyset$, then $\mathbb{F}^S$ is a vector space over $\mathbb{F}$. The vector space $\mathbb{F}^n$ is a special case of $\mathbb{F}^S$, where $S = \{1, 2, \ldots, n\}$. $\diamond$

Thm 1.6 A vector space has a unique additive identity. $\diamond$

Proof. Suppose 0 and $0'$ are both additive identities for some vector space V. Then

$$0' = 0' + 0 = 0 + 0' = 0. \qquad \dashv$$

Thm 1.7 Every element in a vector space has a unique additive inverse. $\diamond$

Proof. Suppose $a \in V$ has two additive inverses b and c. Then

$$b = b + (a + c) = (b + a) + c = c. \qquad \dashv$$

Notations like $-v$ and $w - v$ make sense due to the uniqueness of additive inverses. From now on, V denotes a vector space over $\mathbb{F}$.

Thm 1.8 $\forall v \in V, 0v = 0.$ $\diamond$

Proof. We have for any $v \in V$

$$0v = (0 + 0)v = 0v + 0v.$$

Adding the additive inverse of $0v$ to both sides of the equation above gives $0 = 0v$. $\dashv$

Comment. We *have* to use distributivity for that's where vector addition and scalar multiplication are connected in 1.3. The first equation holds because $0 \in \mathbb{F}$.

Thm 1.9 $\forall a \in \mathbb{F}, a0 = 0.$ $\diamond$

Proof. We have for any $a \in \mathbb{F}$

$$a0 = a(0 + 0) = a0 + a0.$$

Adding the additive inverse of $a0$ to both sides of the equation above gives $0 = a0$. $\dashv$

Similarly, $0 = (1 + (-1))v$ gives us

Thm 1.10 $\forall v \in V, (-1)v = -v.$ $\diamond$

Exercises

1 Prove that $\forall v \in V, -(-v) = v$. $\triangleleft$

$0 = (-v) + (-(-v)) = v + (-v)$.

2 Suppose $a \in \mathbb{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$. $\triangleleft$

If $a \neq 0$ we have that $v = (a \cdot \frac{1}{a})v = \frac{1}{a} \cdot (av) = 0$, which is equivalent to what we are asked to prove.

5 Show that (d) of item 1 in 1.3 can be replaced with 1.8. $\triangleleft$

We are to show the existence of additive inverse from 1.8 and the rest of 1.3.

$$0 = 0v = (1-1)v = v + (-1)v.$$

Thus the additive inverse of v exists, namely $(-1)v$.

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$: the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$? Explain. $\triangleleft$

Consider $(\mathbb{R} \cup \{\infty, -\infty\}, +)$. Commutativity and existence of identity and inverses holds by definition. However $(u, v, w) = (3, \infty, -\infty)$ violates commutativity, hence $(\mathbb{R} \cup \{\infty, -\infty\}, +)$ is not an abelian group, thus $\mathbb{R} \cup \{\infty, -\infty\}$ not a vector space.

8 Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.

- Addition on $V_{\mathbb{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all $u_1, v_1, u_2, v_2 \in V$.

- Complex scalar multiplication on $V_{\mathbb{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbb{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbb{C}}$ is a complex vector space. $\triangleleft$

We verify each of the requirements specified by 1.3 by properties of V as an vector space, very much like verifying the properties of $\mathbb{C}$ by those of $\mathbb{R}$.

Comment. Think of V as a subset of $V_{\mathbb{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbb{C}}$ can then be thought of as generalizing the construction of $\mathbb{C}^n$ from $\mathbb{R}^n$ (thought of as a subset of $\mathbb{C}^n$.)

1C Subspaces

Defn 1.11 A subset U of V is called a *subspace* of V if U is also a vector space with the same additive identity, addition, and scalar multiplication as on V. $\diamond$

Thm 1.12 $U \subseteq V$ is a subspace of V iff it (1) has the additive identity of V (or is nonempty, because we can take $u \in U$ then $0u \in U$.), (2) is closed under addtion, and (3) is closed under scalar multiplication. $\diamond$

Proof. Both directions hold by definition. In paticular, closure ensures that addition and multiplication are reasonably defined ($U \times U \to U$ and $\mathbb{F} \times U \to U$) and properties such as associativity hold because they hold on V and $U \subseteq V$. $\dashv$

Eg 1.13 The set of differentiable real-valued functions f on the interval (0,3) such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ iff $b = 0$ for closure under scalar multiplication, which shows the linear structure underlying parts of calculus. The subspaces of $\mathbb{R}^2$ are precisely $\{0\}$ all lines in $\mathbb{R}^2$ containing the origin and $\mathbb{R}$, which intuitively justifies the word "linear". $\diamond$

Defn 1.14 Suppose $V_1, \ldots, V_m$ are subspaces of V. The *sum* of them is

$$V_1 + \cdots + V_m = \{v_1 + \cdots + v_m \mid \bigwedge v_i \in V_i\} \qquad \diamond$$

Thm 1.15 Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is the smallest subspace of V containing $V_1, \ldots, V_m$. $\diamond$

Proof. That $V_1 + \cdots + V_m$ is a subspace and contains $V_1, \ldots, V_m$ is trivial. Suppose that V' contains $V_1, \ldots, V_m$ and is a subspace. By 1.14 and closure under addition we have that $V_1 + \cdots + V_m \subseteq V'$, thus the minimality. $\dashv$

Defn 1.16 Suppose $V_1, \ldots, V_m$ are subspaces of V. The sum $V_1 + \cdots + V_m$ is called a *direct sum* if each element of $V_1 + \cdots + V_m$ can be written in only one way as a sum $v_1 + \cdots + v_m$ where each $v_k \in V_k$, denoted $V_1 \oplus \cdots \oplus V_m$. $\diamond$

The definition of direct sum requires every vector in the sum to have a unique representation as an appropriate sum.

Thm 1.17 Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum iff the only way to write 0 as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$, is by taking each v_k equal to 0. $\diamond$

Proof. ($\Rightarrow$ is trivial. Consider $\Leftarrow$.) Suppose for sake of contradiction that $V_1 + \cdots + V_m$ is *not* a direct sum. Then there exists $v \in V_1 + \cdots + V_m$ such that $v = v_1' + \cdots + v_m' = v_1'' + \cdots + v_m''$, where $v_k' \in V_k$ and $v_k'' \in V_k$ for each k and $(v_1', \ldots, v_m') \neq (v_1'', \ldots, v_m'')$. Then we have $0 = (v_1' - v_1'') + \cdots + (v_m' - v_m'')$ and at least a j such that $v_j' - v_j'' \neq 0$. Hence contradiction. $\dashv$

Thm 1.18 Suppose U and W are subspaces of V. Then

$$U + W \text{ is a direct sum} \Leftrightarrow U \cap W = \{0\}. \qquad \diamond$$

Proof. $\Rightarrow$: Say that $v \in U \cap W$ and $v \neq 0$, then $0 = v + (-v)$, where $v \in U$ and $-v \in W$, hence $U + W$ is not a direct sum, and contradiction.
$\Leftarrow$: Say that $0 = a + b$, where $a \in U$, $b \in W$, and $a \neq 0$. Then $b = -a \in U$. Hence $b \in U \cap W$ and $b \neq 0$, and contradiction. $\dashv$

Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets.

Exercises

6 Is $\{(a,b,c) \in \mathbb{R}^3 \mid a^3 = b^3\}$ a subspace of $\mathbb{R}^3$? How about $\mathbb{C}^3$? $\triangleleft$

By $a^3 = b^3$ we have that $(a-b)(a^2+ab+b^2) = 0$. That is, if $b \neq 0$ then $\frac{a}{b} \in \left\{1, \frac{-1 \pm \sqrt{3}i}{2}\right\}$. We need a unique $\frac{a}{b}$ so that it is closed under addition, which holds in $\mathbb{R}$ but not in $\mathbb{C}$.

7 Prove or give a counterexample: If U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and contains additive inverses, then U is a subspace of $\mathbb{R}^2$. $\triangleleft$

A counterexample is $\mathbb{Z}^2$.

8 Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}$. $\triangleleft$

C.f. Ex.1C.6. Easily we have that $\{(a,b) \mid a = 2b \vee a = b\}$ is not closed under addition. Note that $\{(a,b) \mid a = 2b\} + \{(a,b) \mid a = b\}$ is a direct sum, yet their union is not a subspace.

12 Prove that the union of two subspaces of V is a subspace of V iff one of the subspaces is a subset of another. $\triangleleft$

Suppose otherwise that U and W are subspaces of V, $u \in U \setminus W$, and $w \in W \setminus U$. Then $u + w \notin U \cup W$, for that otherwise contradiction will arise. Hence $U \cup W$ is not a subspace of V.

13 Prove that the union of three subspaces of V is a subspace of V iff one of the subspaces contains the other two. $\triangleleft$

to do

18 Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses? $\triangleleft$

$\{0\}$ is *the* additive identity of addition on subspaces of V. Also only $\{0\}$ has an additive inverse, namely itself. Suppose otherwise for sake of contradiction that $W \subseteq V$ has an additive inverse $-W$, $w \in W$, and $w \neq 0$. Then $w = w + 0 \in W + (-W) = \{0\}$, hence contradiction.

19 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that $V_1 + U = V_2 + U$ then $V_1 = V_2$. $\triangleleft$

$V_1 + U = V_2 + U$ should always hold, as long as $V_1 \subseteq U$ and $V_2 \subseteq U$, considering 1.15.

20 Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 \mid x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$. ◁

It is easy to verify that $\{(x, -x, y, -y) \in \mathbb{F}^4 \mid x, y \in \mathbb{F}\}$ suffices.

21 Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$. ◁

to do

22 Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that

$$\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3.$$ ◁

to do

23 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that $V = V_1 \oplus U = V_2 \oplus U$ then $V_1 = V_2$. ◁

A counterexample is that $V_1 = \{(x, 0) \mid x \in \mathbb{R}\}, V_2 = \{(x, x) \mid x \in \mathbb{R}\}, U = \{(0, y) \mid y \in \mathbb{R}\}$, and $V = \mathbb{R}^2$. The intuition is well illustrated by the counterexample: there are different sets of vectors that span a plane, which may share same components.

24 Let V_e denote the set of real-valued even functions on $\mathbb{R}$ and V_o the set of real-valued odd functions. Show that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$. ◁

Trivial considering that $f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$.

Chapter 2

Finite-Dimensional Vector Spaces

2A Span and Linear Independence

We will usually write lists of vectors without surrounding parentheses.

Defn 2.1 A *linear combination* of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \ldots, a_m \in \mathbb{F}$. ◇

Defn 2.2 The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$, denoted $\operatorname{span}(v_1, \ldots, v_m)$. The span of the empty list $()$ is defined to be $\{0\}$. ◇

Thm 2.3 The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list. ◇

The proof is omitted as trivial.

Defn 2.4 If $\operatorname{span}(v_1, \ldots, v_m)$ equals V, say that the list $v_1, \ldots, v_m$ *spans* V. ◇

Defn 2.5 A vector space is called *finite-dimensional* if some list of vectors in it spans the space. A vector space is called *infinite-dimensional* if it is not finite-dimensional. ◇

Defn 2.6 A function $p : \mathbb{F} \to \mathbb{F}$ is called a *polynomial* with coefficients in $\mathbb{F}$ if there exist $a_0, \ldots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$. $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$. $\diamond$

If a polynomial is represented by two set of coefficients, then subtracting one representation of the polynomial from the other produces a polynomial that is identically zero as a function on $\mathbb{F}$ and hence has all zero coefficients (which is obvious). Thus the coefficients of a polynomial are uniquely determined by the polynomial.

Defn 2.7 A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have *degree* m if there exist scalars $a_0, a_1, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that for every $z \in \mathbb{F}$, we have

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m.$$

The polynomial that is identically 0 is said to have degree $-\infty$. The degree of a polynomial p is denoted by $\deg p$. For m a nonnegative integer, $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m. $\diamond$

$\mathcal{P}_m(\mathbb{F}) = \operatorname{span}(1, z, \ldots, z^m)$ and thus a finite-dimensional vector space for each $m \in \mathbb{N}$. Consider any list of elements of $\mathcal{P}(\mathbb{F})$. Let m denote the highest degree of the polynomials in this list. Then every polynomial in the span of this list has degree at most m. Thus z^{m+1} is not in the span of our list. Hence no list spans $\mathcal{P}(\mathbb{F})$. Thus $\mathcal{P}(\mathbb{F})$ is infinite-dimensional.

Defn 2.8 A list $v_1, \ldots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that makes

$$a_1 v_1 + \cdots + a_m v_m = 0$$

is $a_1 = \cdots = a_m = 0$. The empty list () is declared to be linearly independent. $\diamond$

$\operatorname{span}(v_1, \ldots, v_m)$ is by definition $\operatorname{span}(v_1) + \cdots + \operatorname{span}(v_m)$ and is a direct sum iff $v_1, \ldots, v_m$ is linearly independent. C.f. 1.17.

If some vectors are removed from a linearly independent list, the remaining list is also linearly independent.

Defn 2.9 A list of vectors in V is called *linearly dependent* if it is not linearly independent. $\diamond$

If some vector in a list of vectors in V is a linear combination of the other vectors, then the list is linearly dependent.

Thm 2.10 Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, \ldots, m\}$ such that

$$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}).$$

Furthermore, if k satisfies the condition above and the k^{th} term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$. $\diamond$

Proof. $\dashv$