B.Sc. 1st Semester (Honours) Examination, 2023-24

MATHEMATICS

Course ID : 12112 **Course Code : SH-MTH-102/C-2**

Course Title : Algebra

[New Syllabus-2022]

Time : 2 Hours Full Marks : 40

The figures in the right margin indicate marks.

Candidates are required to answer in their own words as far as practicable.

Notations and symbols have their usual meaning.

UNIT-I

1. Answer ***any five*** from the following questions: 5×2=10

a) Let $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$ and $c_1, c_2, \dots, c_n$ be all positive real numbers. Apply Cauchy-Schwarz inequality to prove that $(a_1b_1c_1 + a_2b_2c_2 + \cdots + a_nb_nc_n)^2 <$

$$({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)({b_1}^2 + {b_2}^2 + \cdots + b_n^2)\,({c_1}^2 + {c_2}^2 + \cdots + {c_n}^2)$$

b) Prove that for a complex number $z, |z| \geq \frac{1}{\sqrt{2}}(|Rez| + |Imz|)$.

c) If A and B are two invertible matrices, then show that AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

[Turn Over]

d) Prove that product of any three consecutive integers is divisible by 6.

e) Let W be the subspace of $\mathbb{R}^3$ defined by

$$W = \{(x, y, z) \in \mathbb{R}^3 : 2x+2y+z=0, 3x+3y-z=0, x+y-3z=0\}.$$

Find the dimension of W.

f) Solve for x and y, where $x, y \in Z$ and $35x + 40y = 50$.

g) The eigen values of the matrix A are 5 and 3. Find the eigen values of the matrix $(A^2 - 8A + 15I + 30A^{-1})$.

h) Let $X = \{a, b, c\}$ be a non-empty set and P(X) be the power set of X. Show that $(P(X), \leq)$ is a poset.

UNIT-II

2. Answer *any four* from the following questions: $4\times5=20$

a) i) Let $\rho = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b$ is a multiple of $6\}$. Show that ρ is an equivalence relation.

ii) Apply Descartes' rule of signs to examine the nature of the roots of the equation $x^6 - 3x^2 - 2x - 3 = 0$.

b) i) Use Principle of mathematical induction to prove that $2^{5n+1} + 3^{2n+1}$ is divisible by 23 for $n \in \mathbb{N}$.

ii) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two functions, given by $f(x) = |x| + x$ for all $x \in \mathbb{R}$ and $g(x) = |x| - x$ for all $x \in \mathbb{R}$. Find fog and gof.

c) Prove that the square of any integer is always in the form of either 3n or 3n+1.

d) If x, y, z be three unequal positive numbers, then show that $\left(\frac{y+z}{2}\right)^x \left(\frac{z+x}{2}\right)^y \left(\frac{x+y}{2}\right)^z < x^x\, y^y\, z^z$.

e) Solve the equation by Ferrari's Method $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

f) Solve, if possible the system of linear equations

$$x_1 + 2x_2 - x_3 = 10$$
$$-x_1 + x_2 + 2x_3 = 2$$
$$2x_1 + x_2 - 3x_3 = 8.$$

UNIT-III

3. Answer ***any one*** from the following questions: $1\times10=10$

a) i) Use Cayley-Hamilton theorem to find A^{100} where

$$A=\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}.$$

ii) Use the theory of Congruence to find the remainder when 3.4^{n+1} is divisible by 9 for all $n\in\mathbb{N}$.

iii) Let $f:S\to\mathbb{R}$ defined by $f(x)=\dfrac{x}{1-|x|}, x\in S$, where $S=\{x\in\mathbb{R}:-1<x<1\}$. Show that the mapping f is a bijection. Determine f^{-1}. 3+3+4

b) i) Using the Principle of MAthematical Induction, prove that $7^{2n}+16n-1$ is divisible by 64, for all $n\in N$..

ii) Solve the equation $x^4-2x^2+8x-3=0$, using Euler's method. 5+5