Vector Space

$\mathbb{R} = \{x : x \in \mathbb{R}\}$

$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R};\ y \in \mathbb{R}\}$

$\mathbb{R}^3 = \{(x, y, z) : x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}\}$.

$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R},\ i = 1 \text{ to } n \}$

in general $\mathbb{R}^n$ forms a vector space over $\mathbb{R}$ with respect to addition and sca~~l~~ler multiplication defined by

$$+ : (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)$$

$$\cdot : c(x_1, x_2, \ldots, x_n) \qquad c \in \mathbb{R}$$
$$= (cx_1, cx_2, \ldots, cx_n)$$

Example $\mathbb{R}^2(\mathbb{R}), \mathbb{R}^3(\mathbb{R}), \ldots$

form a vector space

Linear Combination

Suppose $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}^n$

A vector $\beta \in \mathbb{R}^n$ is said to be a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$ if there exists scalars $c_1, c_2, \ldots, c_n$ such that $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$

Let, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

Then the set of all linear combination of the vectors of S is called linear span of S and it is denoted by $L(S)$ and defined by $L(S) = \{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n : c_i \in \mathbb{R},\ \alpha_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}$

In $\mathbb{R}^3$ $\alpha_1 = (1,1,2)$

$\alpha_2 = \{2,1,3\}$

$L(S) = \{c_1\alpha_1 + c_2\alpha_2 : c_1, c_2 \in \mathbb{R}\}$

$= c_1(1,1,2) + c_2(2,1,3) : c_1 c_2 \in \mathbb{R}\}$

$= (c_1, c_1, 2c_1) + (2c_2, c_2, 3c_2) : c_1, c_2 \in \mathbb{R}\}$

$= \{c_1 + 2c_2, c_1 + c_2, 2c_1 + 3c_2 ; c_1, c_2 \in \mathbb{R}\}$

* Linear dependence & Independence :-

$\Rightarrow$ Let, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a subset of $\mathbb{R}^n$. $\alpha_i \in \mathbb{R}^n$ $i = 1, 2, \ldots, n$

S is said to be linearly dependent if there exists scalars $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \underbrace{(0, 0, \ldots, 0)}_{(n \text{ times})}$$

[null vector]

S is not Linearly dependent (L.D) then S is called Linearly Independent (L.I)

i.e if S is Linearly Independent

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = (0, 0, \ldots, 0)$$

if and only $c_1 = c_2 = \ldots = c_n = 0$

Q Let $\alpha = (1,1,2)$ and $\beta = (2,2,4) \in \mathbb{R}^3$, $S = \{\alpha, \beta\}$ is L.I / L.D ?

$\Rightarrow$ c, d be scalars from $\mathbb{R}$

$c\alpha + d\beta = (0,0,0)$

$c(1,1,2) + d(2,2,4) = (0,0,0)$

$\Rightarrow (c, c, 2c) + (2d, 2d, 4d) = (0,0,0)$

$\Rightarrow (c + 2d, c + 2d, 2c + 4d) = (0,0,0)$

$c + 2d = 0$ $\quad$ $2c + 4d = 0$

$c + 2d = 0$

$c = -2d$

$d \in \mathbb{R}$

$c \in \mathbb{R}$

e.g $d = 2$, $c = -4$

$-4(1,1,2) + 2(2,2,4) = (0,0,0)$

so, S is L.D

Q $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ L.D / L.I

$c_1, c_2, c_3 \in \mathbb{R}$

$c_1(1,0,0) + c_2(0,1,0) + c_2(0,0,1) = (0,0,0)$

if $c_1 = c_2 = c_3 = 0$ then S is L.I

c_1, c_2, c_3 not all zero then S is L.D

here $c_1, c_2 c_2 = 0, 0, 0)$ $\Rightarrow c_1 = c_2 = c_3 = 0$

$\therefore$ S is L.I

Q $S=\{(0,1,1),(1,0,1),(1,1,0)\}$

$c_1(0,1,1)+c_2(1,0,1)+c_3(1,1,0)=(0,0,0)$

~~if, $c_1 \neq c_2 \neq c_3 \neq 0$~~

~~$(c_1, c_2, c_3) = (0,0,0)$~~

~~$\therefore$ S is Linearly Independent~~

$0, c_1, c_1$
$c_2, 0, c_2$

$(0, c_1, c_1)+(c_2,0,c_2)+(c_3,c_3,0)=(0,0,0)$

$\Rightarrow (c_2+c_3,\ c_1+c_3,\ c_1+c_2)=(0,0,0)$

$c_2+c_3=0$
$\Rightarrow c_2=-c_3$

$c_1+c_3=0$,
$\Rightarrow c_1=-c_3$
$c_1+c_2=0$
$\Rightarrow c_1=-c_2$

$$\left.\begin{aligned} c_2+c_3&=0\\ c_1+c_3&=0\\ c_1+c_2&=0\end{aligned}\right\}\text{Homogeneous system}$$

$$\begin{vmatrix} 0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{vmatrix}$$

$= -1(0-1)+1(1-0)=1+1=2\neq 0$

So, the system has a unique solution ie $=(0,0,0)$

So, $c_1=c_2=c_3=0$ is the only soln

$\therefore$ S is Linearly Independent

Q $S=\{(2,3,1),(2,1,3),(1,1,1)\} \rightarrow$ LD/LI

$c_1(2,3,1)+c_2(2,1,3)+c_3(1,1,1)=(0,0,0)$

$\Rightarrow (2c_1, 3c_1, c_1)+(2c_2, c_2, 3c_2)+(c_3, c_3, c_3)=(0,0,0)$

$\Rightarrow (2c_1+2c_2+c_3),(3c_1+c_2+c_3),(c_1+3c_2+c_3)=(0,0,0)$

$$\left.\begin{aligned} 2c_1+2c_2+c_3&=0\\ 3c_1+c_2+c_3&=0\\ c_1+3c_2+c_3&=0\end{aligned}\right\}\text{Homogeneous system}$$

$$\begin{vmatrix} 2 & 2 & 1\\ 3 & 1 & 1\\ 1 & 3 & 1\end{vmatrix}$$

$= 2(1-3)-2(3-1)+1(9-1)$

$= 2\times -2 -4+8$

$= -4-4+8$

$= 0 = 0$

So, the system has ~~many solution~~ infinite solution other than $(0,0,0)$.

So, the system has non trivial soln that is c_1, c_2, c_3 not all zero

Hence S is L.D

Q) $\{(1,2,3),(2,3,1),(3,1,2)\}$ L.D/L.I

$\Rightarrow c_1(1,2,3)+c_2(2,3,1)+c_3(3,1,2)=0$

$\Rightarrow (c_1, 2c_1, 3c_1)+(2c_2, 3c_2, c_2)+(3c_3, c_3, 2c_3)=0$

$\Rightarrow (c_1+2c_2+3c_3,\ 2c_1+3c_2+c_3,\ 3c_1+c_2+2c_3)=(0,0,0)$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$= 1(6-1)-2(4-3)+3(2-9)$

$= 1\times5-2-21$

$= 5-2-21 \neq 0$

$= -18 \neq 0$

S is L.I

Q $\{(1,1,1,0),(1,0,1,1)\ (1,2,1,2)\ (1,1,1,1)\}$

L.D/L.I

$\Rightarrow c_1(1,1,1,0)+c_2(1,0,1,1)+c_3(1,2,1,2)+c_4(1,1,1,1)$
$=0,0,0,0$

$\Rightarrow (c_1+c_2+c_3+c_4,\ c_1+2c_3+c_4,\ c_1+c_2+c_3+c_4,$
$c_2+2c_3+c_4)=(0,0,0,0)$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

$R_2' = R_2' - R_3'$

$R_3' = R_3 - R_1'$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

$R_2' = R_2 \times -1$
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

$R_1' = R_1 - R_4$

$R_4' = R_4 - R_1$
$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{vmatrix}$$

$c_1+c_2+c_3+c_4=0$

$c_1+2c_3+c_4=0$

$c_1+c_2+c_3+c_4=0$

$c_2+2c_3+c_4=0.$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

Here $R_1 = R_3$

So, Rank of A < 4 = no. of unknowns

so the system has infinite solution

c_1, c_2, c_3, c_4 has not all zero solution

$\therefore$ S is L.D

Basis - Let V be a vector space over $\mathbb{R}$. A subset $S = \{\alpha_1, \alpha_2 \ldots, \alpha_n\}$ of V is said to be a basis of V if

(i) S is L.I

(ii) $L(S) = V$

No. of element in the basis of a vector space is called its dimension it is denoted by $\dim V$.

$\mathbb{R}^3$ $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

(i) $S \to$ L.I

(ii) $L(S) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$
$= (c_1, c_2, c_3) \quad c_1, c_2, c_3 \in \mathbb{R}$
$= \mathbb{R}^3$

S is a basis of $\mathbb{R}^3$

so, $\dim \mathbb{R}^3 = 3$

In a V.S $V(\mathbb{R})$. dim V is unique but basis is not.

$S = \{(1,1,0), (1,0,1), (0,1,1)\} \to$ L.I.

$L(S) = \mathbb{R}^3$

Let $\dim V = n$. then any L.I subset of V containing n vectors is a basis itself

Exercise-7

i) $S = \{(2,3,1), (2,1,3), (1,1,1)\}$

$c_1(2,3,1) + c_2(2,1,3) + c_3(1,1,1) = (0,0,0)$

$\Rightarrow (2c_1, 3c_1, c_1) + (2c_2, c_2, 3c_2) + (c_3, c_3, c_3) = (0,0,0)$

$\Rightarrow$ ~~$(2c_1+2c_2+c_3)$~~

$\Rightarrow (2c_1+2c_2+c_3,\ 3c_1+c_2+c_3;\ c_1+3c_2+c_3) = (0,0,0)$

$$\left.\begin{aligned} 2c_1+2c_2+c_3 &= 0 \\ 3c_1+c_2+c_3 &= 0 \\ c_1+3c_2+c_3 &= 0 \end{aligned}\right\} \text{Homogeneous system}$$

$$\begin{vmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$

$\Rightarrow 2(1-3) - 2(3-1) + 1(9-1)$

$\Rightarrow -4 - 4 + 8 = 0$

So, the system has infinite solution other than $(0,0,0)$

So, the system has non-trivial solution that mean c_1, c_2, c_3 not all zero

Hence S is L.D

ii) ~~$S = \{$~~

2) ~~i) $S = \{(1,1,1,0), (1,0,1,1), (1,2,1,2)$~~

2) ii) $S = \{(2,3,1,4), (3,2,4,1), (1,1,1,1)\}$

$c_1(2,3,1,4) + c_2(3,2,4,1) + c_3(1,1,1,1) + c_4(0,0,0,0)$
$= (0,0,0,0)$

$\Rightarrow (2c_1 + 3c_2 + c_3, 3c_1 + 2c_2 + c_3, c_1 + 4c_2 + c_3, 4c_1 + c_2 + c_3) = (0,0,0,0)$

$$\left.\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \\ c_1 + 4c_2 + c_3 &= 0 \\ 4c_1 + c_2 + c_3 &= 0 \end{aligned}\right\} \text{Homogeneous solution}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 4 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

$R_1' = \frac{R_1}{2}$
$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 3 & 2 & 1 \\ 1 & 4 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

$2 - 3\times 4$
$= 2 - 12$
$= -10$
$4 - \frac{3}{2}$
$= \frac{5}{2}$
$1 - \frac{3}{2}$
$1 - 4\times 1$

$R_2' = R_2 - 3R_3$
$R_3' = R_3 - R_1$
$R_4' = R_4 - 4R_3$
$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -10 & -2 \\ 0 & \frac{5}{2} & \frac{1}{2} \\ 0 & -15 & -3 \end{bmatrix}$$

$R_2' = -\frac{R}{10}$
$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{5} \\ 0 & \frac{5}{2} & \frac{1}{2} \\ 0 & -15 & -3 \end{bmatrix}$$

$\frac{1}{2} - \frac{3}{10}$
$= \frac{5-3}{10}$
$=$
$\frac{1}{2} - \frac{5}{2}\times\frac{1}{5}$
$-3 + \frac{15}{5}$

$R_1' = R_1 - \frac{3}{2}R_2$
$R_3' = R_3 - \frac{5}{2}R_2$
$R_4' = R_4 + 15R_2$
$$\begin{bmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of $A <$ no. of unknowns $= 3$
So, the system has infinite solution. c_1, c_2, c_3 has not all zero solution

$\therefore$ S is linearly dependent.

$c_1, c_2, c_3 \neq 0$ many $= 0$

3) i) $S = \{(1,2,1), (k,3,1), (2,k,0)\}$; $k \in \mathbb{R}$

$c_1(1,2,1) + c_2(k,3,1) + c_3(2,k,0) = (0,0,0)$

$\Rightarrow$ ~~$c_1 + c_2$~~ $(c_1 + kc_2 + 2c_3, 2c_1 + 3c_2 + k, c_1 + c_2 + 0) = (0,0,0)$

$$\left.\begin{aligned} c_1 + kc_2 + 2c_3 &= 0 \\ 2c_1 + 3c_2 + k &= 0 \\ c_1 + c_2 + 0 &= 0 \end{aligned}\right\} \text{Homogeneous system}$$

if
$$\begin{vmatrix} 1 & k & 2 \\ 2 & 3 & k \\ 1 & 1 & 0 \end{vmatrix} = 0$$
then the system ~~has many~~ infinite solution and c_1, c_2, c_3 not all zero solutions.

$\therefore 1(-k) - k(-k) + 2(2-3) = 0$
$\Rightarrow -k + k^2 - 2 = 0$
$\Rightarrow k^2 - k - 2 = 0$
$\Rightarrow k^2 - 2k + k - 2 = 0$
$\Rightarrow k(k-2) + 1(k-2) = 0$

$(2k-k)$

$\Rightarrow$ $\cancel{(K+2)}$ $(K-2)(K+1) = 0$

$\therefore$ $K = +2$; $K = -1$

$\therefore$ for the value of $K = +2$ or $K = -1$

the S is Linearly Dependent.

9) i) $S = \{(K,1,1), (1,K,1), (1,1,K)\}$

$c_1(K,1,1) + c_2(1,K,1) + c_3(1,1,K) = (0,0,0)$

to form the basis of $\mathbb{R}^3$ S must be

(i) S is L.I

(ii) $L(S) \in \mathbb{R}^3$

so,

$(Kc_1 + c_2 + c_3, c_1 + Kc_2 + c_3, c_1 + c_2 + Kc_3) = (0,0,0)$

and $Kc_1 + c_2 + c_3, c_1 + Kc_2 + c_3, c_1 + c_2 + Kc_3 \in \mathbb{R}^3$

$$\left.\begin{aligned} Kc_1 + c_2 + c_3 &= 0 \\ c_1 + Kc_2 + c_3 &= 0 \\ c_1 + c_2 + Kc_3 &= 0 \end{aligned}\right\} \text{Homogeneous system}$$

~~to be~~ S be L.I

$$\begin{vmatrix} K & 1 & 1 \\ 1 & K & 1 \\ 1 & 1 & K \end{vmatrix} \neq 0$$

$K(K^2-1) - 1(K-1) + 1(1-K) \neq 0$

$\Rightarrow K^3 - K - K + 1 + 1 - K \neq 0$

$\Rightarrow K^3 - 3K + 2 \neq 0$

$\Rightarrow K^3 - K^2 + K^2 - K - 2K + 2 \neq 0$

$\Rightarrow K^2(K-1) + K(K-1) - 2(K-1) \neq 0$

$\Rightarrow (K^2 + K - 2)(K-1) \neq 0$

$\therefore$ $(K-1) \neq 0$

$\therefore K \neq 1$

or,

$K^2 + 2K - K - 2 \neq 0$

$\therefore K(K+2) - 1(K+2) \neq 0$

$\Rightarrow (K+2)(K-1) \neq 0$

$\therefore K \neq 1$ $K \neq -2$

$\therefore$ So if $K \neq 1$, $K \neq -2$ the set S form a basis of $\mathbb{R}^3$

<u>Basis</u>

Let V be a vector space over $\mathbb{R}$. A subset S of V is said to be a basis of V if

(i) S is linearly independent

(ii) $L(S) = V$

Now for a vector space $V(\mathbb{R})$ and S subset of V $(S \subseteq V)$ then it always true that $L(S) \subseteq V$

To show S is a basis of V we have only to check

(i) S is linearly independent.

(ii) $V \subseteq L(S)$

$\mathbb{R}^2(\mathbb{R})$

$S = \{(1,0)\ (0,1)\}$

(i) S is L.I

(ii) We know that $L(S) \subseteq \mathbb{R}^2$

To proof $L(S) = \mathbb{R}^2$

we have only to show $\mathbb{R}^2 \subseteq L(S)$

Let $(a,b) \in \mathbb{R}^2$, $a, b \in \mathbb{R}$

$(a,b) = c_1(1,0) + c_2(0,1) = (c_1, c_2)$

let $c_1, c_2 \in \mathbb{R}$

$\therefore$ $c_1 = a$, $c_2 = b$

$(a,b) \in L(S)$

So, $\mathbb{R}^2 = L(S)$

Note:- This basis is called standard basis of $\mathbb{R}^2$

$\mathbb{R}^3(\mathbb{R})$

$S = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$ is called standard basis over $\mathbb{R}^3(\mathbb{R})$

(i) $S \to$ L.I

(ii) $L(S) \subseteq \mathbb{R}^3$

$\mathbb{R}^3 \subseteq L(S)$

In a vector space basis is not unique but the no. of element in the ~~basis~~ all basis are same. This number is called dimension of that vector space.

Ex→ Dimension of $\mathbb{R}^2 = 2$

" " $\mathbb{R}^3 = 3$

Let V be a vector space over $\mathbb{R}$. $\dim V = n \in \mathbb{N}$ then any linearly independed subset $S(V)$ containing n no. of element is a basis of V

Subspace:-

Subspace:→ A subset S of $\mathbb{R}^n$ is said to be a Subspace of $\mathbb{R}^n(\mathbb{R})$; if ———

(i) The null vector $(0,0,\ldots,0)$ of $\mathbb{R}^n$ belongs to S

(ii) if $\alpha, \beta \in S$ then $\alpha+\beta \in S$

(iii) if $\alpha \in S$ the $c\alpha \in S\ \forall c \in \mathbb{R}$

Ex-6

2. (i) $S = \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$; check whether S is a Subspace of $\mathbb{R}^3$ or not)

$\Rightarrow \quad S = \{(0,y,z) : y, z \in \mathbb{R}\}$

(i) choose $y = z = 0 \in \mathbb{R}$
we get $(0,0,0) \in S$

(ii) choose $\alpha = (0, y_1, z_1)$; $\beta = (0, y_2, z_2)$ from S where $y_1, z_1 \cdot y_1 y_2, z_1 z_2 \in \mathbb{R}$

Now,

$\alpha+\beta = (0, y_1, z_1) + (0, y_2, z_2)$
$= (0, y_1+y_2, z_1+z_2)$

Since y_1+y_2 and z_1+z_2 are real number

$\therefore \alpha+\beta \in S$

(iii) Let $\alpha = (0, y_1, z_1)$
$c \in \mathbb{R}$

$c\alpha = c(0, y_1, z_1) = (0, cy_1, cz_1)$

Since $cy_1, cz_1 \in \mathbb{R}$; $c\alpha \in S$

therefore S is a subspace of $\mathbb{R}^3(\mathbb{R})$

(iii) $S = \{(x,y,z) \in \mathbb{R}^3 : xy = z\}$;

i) choose $x = y = z = 0 \in \mathbb{R}$
we get $(0,0,0) \in S$

ii) choose $\alpha = (x_1, y_1, x_1 y_1)$,
$\beta = (x_2, y_2, x_2 y_2)$
from S where $x_1, x_2, y_1, y_2, x_1 y_1, x_2 y_2 \in \mathbb{R}$

$\alpha+\beta = (x_1+x_2, y_1+y_2, x_1 y_1 + x_2 y_2)$

since x_1+x_2 and y_1+y_2, $x_1 y_1 + x_2 y_2$ are real number

$\therefore \alpha+\beta \in S$

iii) Let $\alpha = (x_1, y_1, \cancel{x_1 y_1})$

$c \in R$

$c\alpha = (cx_1, cy_2, cx_1 y_1)$

$cx_1 \in R$, $cy_2 \in R$, $cx_1 y_1 \in R$

$c\alpha \in S$

$\therefore$ therefore S is a subspace of $\mathbb{R}^3(\mathbb{R})$

Q $S = \{(x, y, z) : xy = z\}$

i) $(0,0,0) \in S$ as $0 \cdot 0 = 0$

ii) $\alpha = (x_1, y_1, z_1) \in S \Rightarrow x_1 y_1 = z_1$

$\beta = (x_2, y_2, z_2) \in S \Rightarrow x_2 y_2 = z_2$

$\alpha + \beta = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$(x_1 + x_2)(y_1 + y_2) = x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2$

$= z_1 + z_2 + x_1 y_2 + x_2 y_1$

$\neq z_1 + z_2$

$\therefore \alpha + \beta \notin S$

Hence S is not subspace of $\mathbb{R}^3$

vi) $S = \{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0,\ 2x - y + z = 0\}$

i) $(0,0,0) \in S$

as $(0,0,0)$ satisfies the two conditions $x + 2y - z = 0$ & $2x - y + z = 0$

(ii) $\alpha = (x_1, y_1, z_1)$ & $\beta = (x_2, y_2, z_2) \in S$

$x_1 + 2y_1 - z_1 = 0$ $\quad$ $x_2 + 2y_2 - z_2 = 0$

$2x_1 - y_1 + z_1 = 0$ $\quad$ $2x_2 - y_2 + z_2 = 0$

$\alpha + \beta = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$x_1 + x_2 + 2(y_1 + y_2) - (z_1 + z_2) = (x_1 + 2y_1 - z_1) + (x_2 + 2y_2 - z_2)$

$= 0 + 0$

$= 0$

$2(x_1 + x_2) - (y_1 + y_2) - (z_1 + z_2)$

$= (2x_1 - y_1 + z_1) + (2x_2 - y_2 + z_2) = 0 + 0 = 0$

So, $\alpha + \beta \in S$

(iii) ~~$c\alpha = (cx_1, cy_1, cz_1)$~~

$c\alpha = c(x_1, y_1, z_1)$

$= (cx_1, cy_1, cz_1)$ $\quad$ $c \in \mathbb{R}$

$cx_1 + 2cy_1 - cz_1$

$= c(x_1 + 2y_1 - z_1)$

$= c \times 0$

$= 0$

$2cx_1 - cy_1 + cz_1$

$= c(2x_1 - y_1 + z_1)$

$= c \times 0$

$= 0$

$c\alpha \in S$

Hence S is a subspace of $\mathbb{R}^3$

iv) $S = \{(x, y, z) \in \mathbb{R}^3 : x+y+z = 0\}$

(i) $(0,0,0) \in S$

as it $(0,0,0)$ satisfies the equation $x+y+z=0$

(ii) $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$x_1+y_1+z_1 = 0$, $x_2+y_2+z_2 = 0$

$\alpha+\beta = (x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

$\Rightarrow$

$x_1+x_2+y_1+y_2+z_1+z_2 =$

$(x_1+y_1+z_1)+(x_2+y_2+z_2) =$

$= 0+0$

$= 0$

so $\alpha+\beta \in S$

(iii) $c\alpha = c(x_1+y_1+z_1)$ $c \in \mathbb{R}$

$\Rightarrow$

$cx_1+cy_2+cz_2$

$= c(x_1+y_2+z_2)$

$= c.0$

$= 0$

$c \in S$

Hence S is a subspace of $\mathbb{R}^3$

9) ii) $c_1(k,1,1,1)+c_2(1,k,1,1)+c_3(1,1,k,1)+c_4(1,1,1,k)$
$= (0,0,0,0)$

$kc_1+c_2+c_3+c_4 = 0$ — (1)
$c_1+kc_2+c_3+c_4 = 0$ — (2)
$c_1+c_2+kc_3+c_4 = 0$ — (3)
$c_1+c_2+c_3+kc_4 = 0$ — (4)

(1) − (2): $kc_1+c_2-c_1-kc_2 = 0$
$c_1(k-1)+c_2(1-k) = 0$

(3) − (4): $c_3(k-1)+c_4(1-k) = 0$

$c_1(k-1)+c_2(1-k) = 0$ — (5)
$c_3(k-1)+c_4(1-k) = 0$ — (6)

18) i) $S = \{(x, y, z) \in \mathbb{R}^3 \,;\, 2x + y - z = 0\}$

$2x + y - z = 0$, let $x = c$, $y = d$

$\boxed{\text{No of Free variable} = \text{No. of unknown} - \text{No. of equation}}$

$c, d \in \mathbb{R}$

$z = 2c + d$

$S = \{(c, d, 2c+d) : c, d \in \mathbb{R}\}$

$= \{c(1, 0, 2) + d(0, 1, 1) : c, d \in \mathbb{R}\}$

$= L\{(1,0,2), (0,1,1)\}$ $\quad \left[\begin{array}{l} S' = \{\alpha, \beta\} \\ \alpha = (1,0,2) \\ \beta = (0,1,1) \end{array} \right.$

$= L(S')$

Now $S' = \{(1,0,2), (0,1,1)\}$

Let $c_1, c_2 \in \mathbb{R}$

$c_1(1,0,2) + c_2(0,1,1) = (0,0,0)$

$\left. \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array} \right]$ $\quad 2c_1 + c_2 = 0$, $c_2 = -2c_1$ $\quad c_1 = c_2 = 0$ is the only solution

$\therefore$ So, S' is Linearly Independent.

S' forms a basis of S

$\therefore$ So $\dim S = 2$

19) $S = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2y - z = 0,\ 2x + y + w = 0\}$.

No of free variable $= 2$

Let $x = c$, $y = d$, $c, d \in \mathbb{R}$.

$z = c + 2d$, $w = -2c - d$

$S = \{(c, d, c+2d, -2c-d) : c, d \in \mathbb{R}\}$

$= \{c(1,0,1,-2) + d(0,1,2,-1) : c, d \in \mathbb{R}\}$

$= L(S')$ $\quad$ [Where $S' = \{(1,0,1,-2), (0,1,2,-1)\}$

S' generates S

$c_1(1,0,1,-2) + c_2(0,1,2,-1)$

$= (c_1, 0, c_1, -2c_1) + (0, c_2, 2c_2, -1c_2)$ $= (0,0,0,0)$

$c_1 = 0$

$c_2 = 0$

$c_1 + 2c_2 = 0$

$-2c_1 - 1c_2 = 0$

S' is L.I

S' form a basis of S

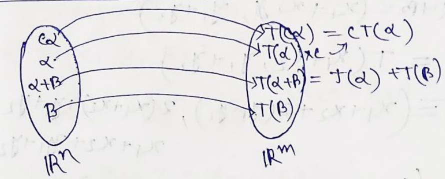
$\therefore$ $\dim S = 2$

Linear Transformation

A mapping ~~IR~~ $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be a linear transformation if —

(i) $T(\alpha+\beta) = T(\alpha)+T(\beta)$; $\forall \alpha, \beta \in \mathbb{R}^n$

(ii) $T(c\alpha) = cT(\alpha)$; $\forall \alpha \in \mathbb{R}^n$, $c \in \mathbb{R}^n$



Q Show that if $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by —

$T(x,y) = (x+y, x-y)$, $(x,y) \in \mathbb{R}^2$

Let, $\alpha = (x_1, y_1)$ $\beta = (x_2, y_2)$ are two arbitory elements of $\mathbb{R}^2$ and $c \in \mathbb{R}$

$\alpha + \beta = (x_1+x_2, y_1+y_2)$

(i) $T(\alpha+\beta) = T(x_1+x_2, y_1+y_2)$

$= (x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2)$

$= (x_1+y_1, x_1-y_1) + (x_2+y_2, x_2-y_1)$

$= T(\alpha) + T(\beta)$

(ii) $T(c\alpha) = T(cx_1, cy_1)$

$= cx_1 + cy_1, cx_1 - cy_1$

$= c(x_1+y_1), x_1-y_1)$

$= cT(\alpha)$

So, from (i) and (ii) we can say that T is a linear mapping.

iii) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(x,y) = (x+2y,\ 2x+y,\ x+y)$, $(x,y) \in \mathbb{R}^2$

Let $\alpha = (x_1, y_1)$, $\beta(x_2, y_2)$ are two arbitory elements of $\mathbb{R}^3$

$\alpha + \beta = (x_1 + x_2, y_1 + y_2)$

i) $T(\alpha+\beta) = T(x_1+x_2, y_1+y_2)$

$= (x_1+x_2+2(y_1+y_2),\ 2(x_1+x_2)+y_1+y_2,\ x_1+x_2+y_1+y_2)$

$= ((x_1+2y_1, 2x_1+y_1, x_1+y_1) + (x_2+2y_2, 2x_2+y_2, x_2+y_2))$

$= T(\alpha) + T(\beta)$

ii) $T(c\alpha) = T(cx_1, cy_1)$

$= (cx_1 + 2cy_1,\ 2cx_1 + cy_1,\ cx_1 + cy_1)$

$= c(x_1+2y_1,\ 2x_1+y_1,\ x_1+y_1)$

$= cT(\alpha)$

$\therefore$ From (i) and (ii) we can say that T is a linear mapping.

iv) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x,y,z) = (x+2y+3z,\ 3x+2y+z,\ x+y+z)$, $(x,y,z) \in \mathbb{R}^3$

Let $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$ are two arbitory elements of $\mathbb{R}^3$

$T(\alpha) = T(x_1, y_1, z_1)$

$= (x_1+2y_1+3z_1,\ 3x_1+2y_1+z_1,\ x_1+y_1+z_1)$

$T(\beta) = T(x_2, y_2, z_2)$

$= (x_2+2y_2+3z_2,\ 3x_2+2y_2+z_2,\ x_2+y_2+z_2)$

$\alpha+\beta = (x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

$T(\alpha+\beta) = T(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

$= (x_1+x_2+2(y_1+y_2)+3(z_1+z_2),\ 3(x_1+x_2)+2(y_1+y_2)+z_1+z_2,\ x_1+x_2+y_1+y_2+z_1+z_2)$

$= (x_1+x_2+2y_1+2y_2+3z_1+3z_2,\ 3x_1+3x_2+2y_1+2y_2+z_1+z_2,\ x_1+x_2+y_1+y_2+z_1+z_2)$

$= (x_1+2y_1+3z_1,\ 3x_1+2y_1+z_1,\ x_1+y_1+z_1) + (x_2+2y_2+3z_2,\ 3x_2+2y_2+z_2,\ x_2+y_2+z_2)$

$= T(\alpha) + T(\beta)$

$T(c\alpha) = T(cx_1, cy_1, cz_1)$

$= (cx_1+2cy_1+3cz_1,\ 3cx_1+2cy_1+cz_1,\ cx_1+cy_1+cz_1)$

$= c(x_1+2y_1+3z_1,\ 3x_1+2y_1+z_1,\ x_1+y_1+z_1)$

$= cT(\alpha)$

$\therefore$ From (i) and (ii) we can say that T is linear mapping.

Kernel of T :→

Let, $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation Then, Kernel of T is denoted by KerT and defined by —

$\ker T \{ \alpha \in \mathbb{R}^n : T(\alpha) = \theta \}$

where, θ is the null vector of $\mathbb{R}^m$

N.B KerT forms a subspace of $\mathbb{R}^n$

Image of T →

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation then Image of T is denoted by ImT and defined by

Range of T / $\operatorname{Im} T = \{ T(\alpha) : \alpha \in \mathbb{R}^n \}$

N.B. ImT forms a subspace of $\mathbb{R}^m$

---

1) T is linear

$\ker T = \{ \alpha \in \mathbb{R}^2 : T(\alpha) = (0,0) \}$

$= \{ (x,y) \in \mathbb{R}^2 : (x+y, x-y) = (0,0) \}$

[Let $\alpha = (x,y)$ be an arbitary element of $\mathbb{R}^2$]

$= \{ (x,y) \in \mathbb{R}^2 : x+y=0, \; x-y=0$

$= \{ (0,0) \}$

$\operatorname{Im} T = \{ T(x,y) : (x,y) \in \mathbb{R}^2 \}$

$= \{ (x+y, x-y) : x,y \in \mathbb{R}^2 \}$

$= \{ x(1,1) + y(1,-1) : x,y \in \mathbb{R} \}$

$= L\{(1,1),(1,-1)\}$

$S = \{(1,1),(1,-1)\}$ generats ImT

Now, $c_1(1,1) + c_2(1,-1) = (0,0)$

$c_1 + c_2 = 0$

$c_1 - c_2 = 0$

$c_1 = c_2 = 0$

∴ so S is linearly independent

dimension of ImT = 2

this is called Rank of T

dimension of KerT = 0 ; this is called Nullity of T

$\boxed{\text{Rank of T + Nullity of T = Dimension of the domain vectorspace}}$

Matrix representation of a linear transformation

12). $T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x, y, z) = (y+z, x+z, x+y)$

Basis : $((1,0,0), (0,1,0), (0,0,1))$

$T(1,0,0) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$T(0,1,0) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (1,1,0) = 1(1,0,0) + 1(0,1,0) + 0(0,0,1)$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

A is the matrix representation of the given linear transformation

Basis : $((2,1,1); (1,2,1)\ (1,1,2))$

$T(2,1,1) = (2,3,3) = 0(2,1,1) + 1(1,2,1) + 1(1,1,2)$

$T(1,2,1) = (3,2,3) = 1(2,1,1)\ \ 0(1,2,1)\ \ 1(1,1,2)$

$T(1,1,2) = (3,3,2) = 1(2,1,1) + 1(1,2,1) + 0(1,1,2)$

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

B is the matrix representation of the given linear transformation

13) $T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x,y,z) = (x+y+z,\ x+z,\ x+y)$

Basis : $((1,0,0),(0,1,0),(0,0,1))$

$T(1,0,0) = (1,1,1) = 1(1,0,0)+1(0,1,0)+1(0,0,1)$

$T(0,1,0) = (1,0,1) = 1(1,0,0)+0(0,1,0)+1(0,0,1)$

$T(0,0,1) = (1,1,0) = 1(1,0,0)+1(0,1,0)+0(0,0,1)$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

A is the matrix representation of ~~linear~~ given linear transformation

Characteristic equation

⇒ Let A be a square matrix of order n, then $\det(A - \lambda I_n)$ is called characteristic polynomial of A and $\det(A-\lambda I_n)=0$ is called characteristic equation of A. Solving the equation we get value of λ. These values are called eigen values of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$\det(A - \lambda I_2)$

$$= \det\left\{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

~~$\det(A-\lambda I_2)$~~ Characteristic equation of A $\begin{vmatrix} 2-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$(2-\lambda)(1-\lambda) = 0 \Rightarrow 2 - 3\lambda + \lambda^2 = 0$

$\lambda = 2;\ \lambda = 1$

Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic equation

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

char. equation : $\lambda^2 - 3\lambda + 2I = 0$

by cayley Hamilton theorem

$A^2 - 3A + 2I_2 = 0$

$\det A = 2 \neq 0$

$\therefore A^{-1}$ exists

$A^2 - 3A + 2I_2 = 0$

$2I_2 = 3A - A^2$

$2A^{-1} = 3A \cdot A^{-1} - A^2 A^{-1}$

$\therefore 2A^{-1} = 3I_2 - A$

$$A^{-1} = \frac{1}{2}[3I_2 - A] = \frac{1}{2}\left\{\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$$

Exercise-13

1) i) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$

Char of equation $= \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ 2 & 3 & 2-\lambda \end{bmatrix} = 0$

$\Rightarrow (1-\lambda)\{(2-\lambda)^2-3\} = 0$

$\Rightarrow (1-\lambda)\{4-2\cdot2\cdot\lambda+\lambda^2-3\} = 0$

$\Rightarrow (1-\lambda)\{\lambda^2-4\lambda+1\} = 0$

$\cancel{\lambda=1}$, $\cancel{\lambda^2-4\lambda+1=0}$

$\cancel{\lambda^2}$

$\Rightarrow \lambda^2-4\lambda+1-\lambda^3+4\lambda^2-\lambda = 0$

$\Rightarrow -\lambda^3+5\lambda^2-5\lambda+1 = 0$

$\Rightarrow \lambda^3-5\lambda^2+5\lambda-1 = 0$

By cayley Hamilton theorem

$A^3-5A^2+5A-I_3 = 0$

$I_3 = A^3-5A^2+5A$

$A^{-1} = A^3 A^{-1} - 5A^2A^{-1}+5AA^{-1}$

$A^{-1} = A^2-5A+5I_3$

L.H.S $= A\times A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{bmatrix}$

$2+3+4$

$\therefore A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$

$=$

-- Then prove $A^3-5A^2+5A-I_3 = 0$, then calculate A^{-1}

2) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

char of equation $= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(\lambda^2-1) = 0$

$\Rightarrow \lambda^2-1-\lambda^3+\lambda = 0$

$\Rightarrow \lambda^3-\lambda^2-\lambda+1 = 0$

By cayley Hamilton theorem we get

$A^3-A^2-A+I_3 = 0_{3\times3}$

$A-I_3 = A-I_3$

$A^2-A = A^2-A$

$\Rightarrow A^3-A^2 = A-I_3$

$\Rightarrow$

$\Rightarrow A^4-A^3 = A^2-A$

$\Rightarrow A^5-A^4 = A^3-A^2 = A-I_3$

$\Rightarrow A^6-A^5 = A^2-A$

$\Rightarrow A^7-A^6 = A-I_3$

$\Rightarrow A^{99}-A^{98} = A-I_3$

$\Rightarrow A^{100}-A^{99} = A^2-A$

$\Rightarrow A^{100}-I_3 = 50(A^2-A)+50(A-I_3)$

$A^{100}-I_3 = 50(A^2-\cancel{A}+\cancel{A}-I_3)$

$A^{100}-I_3 = 50A^2-50I_3$

$A^{100} = 50A^2-50I_3+I_3$

$A^{100} = 50A^2-49I_3$

$A^2 = A\times A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1+0+0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$A^{100} = 50\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & 0 & 0 \\ 50 & 50 & 0 \\ 50 & 0 & 50 \end{bmatrix} - \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{bmatrix}$$

- Eigen vector :- A be a square matrix of order n. A vector $X \in \mathbb{R}^n$ is said to be an eigen vector ~~of~~ corresponding ~~the matrix A if~~ the eigen value λ of the matrix A if

$$\boxed{AX = \lambda X}$$

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7) ii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$

char. of equation $\begin{vmatrix} 1-\lambda & -1 & 2 \\ 2 & -2-\lambda & 4 \\ 3 & -3 & 6-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\{(\lambda-6)(\lambda+2)+12\} - 2(\lambda-6+6)$
$+3(-4+4+2\lambda) = 0$

$\Rightarrow (1-\lambda)\{\lambda^2-4\lambda-12+12\} - 2\lambda + 6\lambda = 0$

$\Rightarrow \lambda^2 - 4\lambda - \lambda^3 + 4\lambda^2 + 4\lambda = 0$

$\Rightarrow -\lambda^3 + 5\lambda^2 = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 = 0$

$\Rightarrow \lambda^2(\lambda-5) = 0$

$\lambda^2 = 0$ $\quad\quad \lambda = 5$

$\lambda = 0, 0$

~~Let AX be the~~ Let, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen
~~Let AX be~~

vector corresponding to eigen value $\lambda_1 = 0$

So, $AX = \lambda_1 X$

$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

or,

$\Rightarrow x_1 - x_2 + 2x_3 = 0$
$2x_1 - 2x_2 + 4x_3 = 0$
$3x_1 - 3x_2 + 6x_3 = 0$

$x_1 - x_2 + 2x_3 = 0$

Let $x_2 = c$, $x_3 = d$ $\quad [c, d \in \mathbb{R}]$

$x_1 = c - 2d$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c-2d \\ c \\ d \end{bmatrix} = c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Let X be an eigen vector corresponding to the eigen value $\lambda_2 = 5$

$AX = \lambda_2 X$

• Algebraic Multiplicity — If λ is a root of the characteristic equation of A of multiplicity n. then n is called algebraic multiplicity of λ

Here in the above problem Algebraic multiplicity of zero (0) is 2. Here the eigen vector corresponding $\lambda = 0$ is $c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}-2\\0\\1\end{bmatrix}$

$c, d \in \mathbb{R}$

Let $\alpha = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\beta = \begin{bmatrix}-2\\0\\1\end{bmatrix}$

then $S = \{c\alpha + d\beta : c, d \in \mathbb{R}\}$ forms a subspace of $\mathbb{R}^3$

the dimension of the subspace is called Geometric multiplicity of the eigen value $\lambda = 0$; Here dimension of $S = 2$

So, Geometric multiplicity of $\lambda = 0$ is 2.

Alway $1 \leq$ Geometric Multiplicity $\leq$ Algebraic Multiplicity

if for a eigen value λ alg. mul. = Geo. mul. then λ is called regular

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vi) Let $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$ are two elements of $\mathbb{R}^3$ and $c \in \mathbb{R}$

(a) $\alpha + \beta = (x_1+x_2, y_1+y_2, z_1+z_2)$ $T(\alpha) = x_1+y_1+z_1$, $T(\beta) = x_2+y_2+z_2$

$= x_1+x_2+y_1+y_2+z_1+z_2$

$= x_1+y_1+z_1 + x_2+y_2+z_2$

$= T(\alpha) + T(\beta)$

(b) $T(c\alpha) = T(cx_1 + cy_1 + cz_1)$

$= c(x_1+y_1+z_1)$

$= cT(\alpha)$

KerT

T is linear

$\ker T = \{(x_1, y_1, z_1) \in \mathbb{R}^3 : T(x_1, y_1, z_1) = 0\}$

$= \{(x_1, y_1, z_1) \in \mathbb{R}^3 : x_1+y_1+z_1 = 0\}$

let $y_1 = c$; $z_1 = d$, $x_1 = -c-d$

$= \{(-c-d, c, d) : c, d \in \mathbb{R}\}$

$= \{c(-1,1,0) + d(-1,0,1) : c, d \in \mathbb{R}\}$

$= L\{(-1,1,0), (-1,0,1)\}$

$c_1(-1,1,0) + c_2(-1,0,1) = (0,0,0)$

$\Rightarrow \left.\begin{array}{l} -c_1 - c_2 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{array}\right\}$ $c_1 = c_2 = 0$ only solution

So, $\{(-1,1,0), (-1,0,1)\}$ is linearly Independent

Therefore $\dim \ker T$ = Nullity of $T = 2$

P.T.O

Im T

$$\text{Im}T = \{T(x,y,z) : (x,y,z) \in \mathbb{R}^3\}$$
$$= \{x+y+z : x+y+z \in \mathbb{R}\}$$
$$= \{\mathbb{R}\} ; \quad \dim \text{Im}T = \text{Rank of } T = 1$$

$$\dim \ker T + \dim \text{Im}T = 2+1 = 3 = \dim \text{ of } \mathbb{R}^3$$

2) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T(1,0,0) = (0,1,0)$$
$$T(0,1,0) = (0,0,1)$$
$$T(0,0,1) = (1,0,0)$$

let $(x,y,z) \in \mathbb{R}^3$

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$
$$= x(0,1,0) + y(0,0,1) + z(1,0,0)$$
$$\cancel{=} (x,y,z)$$

3) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T(0,1,1) = (2,1,1)$$
$$T(1,0,1) = (1,2,1)$$
$$T(1,1,0) = (1,1,2)$$

let $(x,y,z) \in \mathbb{R}^3$

$$(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_2(1,1,0)$$
$$= (c_2+c_3,\ c_1+c_3,\ c_1+c_2)$$

$c_2+c_3 = x$ ---- (i)

$c_1+c_3 = y$ ---- (ii)

$c_1+c_2 = z$ ---- (iii)

i + ii + iii

$$2(c_1+c_2+c_3) = x+y+z$$
$$c_1+c_2+c_3 = \frac{x+y+z}{2} \text{ --- (iv)}$$

$$\text{(iv)} - \text{(i)} = c_1 = \frac{x+y+z}{2} - x = \frac{y+z-x}{2}$$
$$\text{(iv)} - \text{(ii)} = c_2 = \frac{x+y+z}{2} - y = \frac{x-y+z}{2}$$
$$\text{(iv)} - \text{(iii)} = c_3 = \frac{x+y+z}{2} - z = \frac{x+y-z}{2}$$

$$T(x,y,z) = \frac{y+z-x}{2}(0,1,1) + \frac{x-y+z}{2}(1,0,1) + \frac{x+y-z}{2}(1,1,0)$$

$$T(x,y,z) = \frac{y+z-x}{2}(2,1,1) + \frac{x-y+z}{2}(1,2,1) + \frac{x+y-z}{2}(1,1,2)$$
$$= \left(y+z-x + \frac{x-y+z}{2} + \frac{x+y-z}{2},\ \frac{y+z-x}{2} + x-y+z + \frac{x+y-z}{2},\ \frac{y+z-x}{2} + \frac{x-y+z}{2} + x+y-z\right)$$

$$= (y+z,\ x+z,\ x+y)$$

$$\ker T = \{(x,y,z) \in \mathbb{R}^3 : T(x,y,z) = (0,0,0)\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 : (y+z, z+x, x+y) = (0,0,0)\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 : y+z=0,\ z+x=0,\ x+y=0\}$$
$$= \{(0,0,0) \in \mathbb{R}^3\}$$

dimension of ker T = Nullity of T = 0

$\mathrm{Im}\,T = \{T(x,y,z) : (x,y,z) \in \mathbb{R}^3\}$

$= \{(y+z,\ z+x,\ x+y) : (x,y,z) \in \mathbb{R}^3\}$

$= \{x(0,1,1) + y(1,0,1) + z(1,1,0) : (x,y,z) \in \mathbb{R}^3\}$

$\mathrm{Im}\,T = L\{(0,1,1),(1,0,1),(1,1,0)\}$

To show $(0,1,1),(1,0,1)(1,1,0)$ is Linealy independent.

dim of Im T = Rank of T = 3

dim ker T + dim Im T = 3 = Rank of T [Proved]

iii) $3^{4n+2} + 5^{2n+1}$ is divisible by 14 for all $n \in \mathbb{N}$

$\Rightarrow$ for $P(n) = 3^{4n+2} + 5^{2n+1}$

for $n = 1$

$3^{4\times1+2} + 5^{2\times1+1}$ is divisible by 14

for $n = k$

$\therefore P(k) = 3^{4k+2} + 5^{2k+1} = 14m$ [m is integer]

for $n = k+1$

$P(k+1) = 3^{4(k+1)+2} + 5^{2(k+1)+1}$

$= 3^{4k+4+2} + 5^{2k+2+1}$

$= 3^{4k+2+4} + 5^{2k+1+2}$

$= 3^{4k+2} \cdot 3^4 + 5^{2k+1} \cdot 5^2$

$= (14m - 5^{2k+1})3^4 + 5^{2k+1} \cdot 5^2$

$= 81 \cdot 14m - 5^{2k+1} \cdot 81 + 5^{2k+1} \cdot 5^2$

$= 81 \cdot 14m - 5^{2k+1}(81-25)$

$= 81 \cdot 14m - 5^{2k+1} \cdot 56$

$= 14(81m - 4 \cdot 5^{2k+1})$

is divisible by 14 as $(81m - 4 \cdot 5^{2k+1})$ is an integer, therefore by 1st principle of M.I

$3^{4n+2} + 5^{2n+1}$ is divisible by 14 for all $n \in \mathbb{N}$