3.6. Elementary operations.

An *elementary operation* on a matrix A over a field F is an operation of the following three types.

1. Interchange of two rows (or columns) of A.
2. Multiplication of a row (or column) by a non-zero scalar c in F.
3. Addition of a scalar multiple of one row (or column) to another row (or column).

When applied to rows, the elementary operations are said to be *elementary row operations*. And when applied to columns they are said to be *elementary column operations*.

The interchange of the ith and jth row is denoted by R_{ij}.

Multiplication of the ith row by a non-zero scalar c is denoted by cR_i [or $R_i(c)$].

Addition of c times the jth row to the ith row is denoted by $R_i + cR_j$ [or $R_{ij}(c)$].

In a similar manner, the *elementary column operations* C_{ij}, cC_i [or $C_i(c)$], $C_i + cC_j$ [or $C_{ij}(c)$] are defined.

If T be an elementary operation on the matrix A the transformed matrix is denoted as $T(A)$. If $B = T(A)$ the operation is expressed as

$$A \xrightarrow{T} B.$$

For example,

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 7 \\ 4 & 9 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{pmatrix} \xrightarrow{C_{23}} \begin{pmatrix} 2 & 0 & 4 \\ 4 & 5 & 9 \\ 1 & 7 & 3 \end{pmatrix};$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{2R_3} \begin{pmatrix} 2 & 1 & 3 \\ 4 & 5 & 0 \\ 6 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{2C_3} \begin{pmatrix} 2 & 1 & 6 \\ 4 & 5 & 0 \\ 3 & 2 & 2 \end{pmatrix};$$

$$\begin{pmatrix} 2 & 4 \\ 4 & 9 \\ 1 & 3 \end{pmatrix} \xrightarrow{R_2-2R_1} \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 4 & 9 \\ 1 & 3 \end{pmatrix} \xrightarrow{C_2-2C_1} \begin{pmatrix} 2 & 0 \\ 4 & 1 \\ 1 & 1 \end{pmatrix}.$$

If T be an elementary operation on A such that $T(A) = B$ and T_1 be an elementary operation on B such that $T_1(B) = C$ then $C = T_1\{T(A)\}$. C is obtained from A by applying two elementary operations T and T_1 successively.

If T be an elementary row (column) operation on a matrix A then T^{-1}, the inverse of T, is defined to be an elementary row (column) operation such that $T^{-1}(TA) = A$.

For example, if $T = R_{ij}$ then $T^{-1} = R_{ij}$;
if $T = R_i(c)$ then $T^{-1} = R_i(c^{-1})$;
if $T = R_{ij}(c)$ then $T^{-1} = R_{ij}(-c)$.

Clearly, the inverse of an elementary row (column) operation is an elementary row (column) operation of the same type.

Row equivalence. Column equivalence.

Let us consider the set S of all $m \times n$ matrices over a field F. A matrix B in S is said to be *row equivalent* (*column equivalent*) to a matrix A in S if B can be obtained by successive application of a finite number of elementary row operations (column operations) on A.

The relation of row equivalence (column equivalence) on the set S is an equivalence relation. Consequently, the set S is partitioned into classes of row equivalent (column equivalent) matrices.

We shall now discuss some properties of *row equivalent matrices*. Analogous properties hold in case of *column equivalent matrices*.

Row-reduced matrix. Row echelon matrix.

Definition. An $m \times n$ matrix A is said to be *row-reduced* if

(a) the first non-zero element in each non-zero row is 1 (called the leading 1); and

(b) in each column containing the leading 1 of some row, the leading 1 is the only non-zero element.

Examples of a row-reduced matrix are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Definition. An $m \times n$ matrix A is said to be a *row-reduced echelon matrix* (or a *row echelon* matrix) if

(a) A is row-reduced;

(b) there is an integer $r (0 \le r \le m)$ such that the first r rows of A are non-zero rows and the remaining rows (if there be any) are all zero rows; and

(c) if the leading element of the ith non-zero row occurs in the k_ith column of A, then $k_1 < k_2 < \cdots < k_r$.

Examples of a row echelon matrix are

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Worked Example.

1. Find a row-reduced matrix which is row equivalent to

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{pmatrix}.$$

Let us apply elementary row operations on the matrix.

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{pmatrix}$$

$$\xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 2 & 6 & 0 & 4 & 2 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Working procedure. The first row is a non-zero row and the first non-zero element is 2 (in the third column).

Step 1. Multiply the first row by $\frac{1}{2}$. The leading element in the row becomes 1.

Step 2. To reduce the other elements in the third column to zero, perform the operations $R_2 - 2R_1, R_3 - 2R_1$.

Observe that the first non-zero element in the 2nd row is 1 (in the first column).

Step 3. To reduce the other elements in the first column to zero, perform the operation $R_3 - 2R_2$.

Observe that the 3rd row becomes a zero row.

All the rows are exhausted and the process terminates.

Work...

2. Apply elementay row operations to reduce the following matrix to a row echelon matrix

$$\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_3-5R_1]{R_2-3R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_4-3R_2]{R_3-2R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_{34}} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R, \text{ say.}$$

R is a row echelon matrix.

Working procedure. The element in the $(1, 1)$ position is 2.

Step 1. Multiply the 1st row by $\frac{1}{2}$. The leading element in the first row becomes 1 in the $(1, 1)$ position.

Step 2. To reduce all other elements in the first column to zero, perform the operations $R_2 - 3R_1, R_3 - 5R_1$.

Consider the submatrix obtained by deleting the first row and the first column. Observe that the element in the (1, 1) position of the submatrix, i.e., the element in the (2, 2) position of the original matrix is 2.

Step 3. Multiply the second row by $\frac{1}{2}$. The leading element in the second row becomes 1 in the (2, 2) position .

Step 4. To reduce all other elements in the second column to zero, perform the operations $R_3 - 2R_2, R_4 - 3R_2$.

Observe that the third row becomes a zero row.

Step 5. Perform R_{34} to bring the zero row to the last.

Consider the submatrix obtained by deleting first two rows and first two columns. Observe that the element in the (1, 1) position of the submatrix is 2.

Step 6. Multiply the third row by $\frac{1}{2}$. The leading element in the third row becomes 1 in the (3, 3) position.

Step 7. To reduce all other elements in the third column to zero, perform the operation $R_1 - 2R_3$.

The rows are exhausted and the process terminates.

3. Find a row echelon matrix which is row equivalent to

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}.$$

Let us apply elementary row operations on the matrix.

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

$$\xrightarrow[R_4 - 3R_1]{R_3 - 2R_1} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow[\substack{R_3+2R_2\\R_4+5R_2}]{R_1-2R_2}\begin{pmatrix}1&3&0&2&1\\0&0&1&1&0\\0&0&0&0&0\\0&0&0&3&3\end{pmatrix}\xrightarrow{R_{34}}\begin{pmatrix}1&3&0&2&1\\0&0&1&1&0\\0&0&0&3&3\\0&0&0&0&0\end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3}\begin{pmatrix}1&3&0&2&1\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&0\end{pmatrix}\xrightarrow[R_2-R_3]{R-2R_3}\begin{pmatrix}1&3&0&0&-1\\0&0&1&0&-1\\0&0&0&1&1\\0&0&0&0&0\end{pmatrix}.$$

Working procedure. The first column is a non-zero column. The element in the (1,1) position is zero.

Step 1. Perform R_{12} to bring a non-zero element to (1, 1) position. The leading 1 in the first row occurs in the first column.

Step 2. To reduce the other elements in the first column to zero, perform the operations $R_3 - 2R_1, R_4 - 3R_1$.

Consider the submatrix obtained by deleting the first row and the first column. Observe that the first column of the submatrix is a zero column, the second column is the next non-zero column. The element in the (2, 3) position is a non-zero element 2.

Step 3. Multiply the second row by $\frac{1}{2}$. The leading element in the row becomes 1.

Step 4. Reduce all other elememts in the 3rd column to 0 by performing the operations $R_1 - 2R_2, R_3 + 2R_2, R_4 + 5R_2$.

Observe that the third row becomes a zero row.

Step 5. Perform R_{34} to bring the zero row to the last.

Step 6. Multiply the 3rd row by $\frac{1}{3}$. The leading element in the row becomes 1 in the 4th column.

Step 7. Reduce all other elements in the 4th column to zero by performing the operations $R_1 - 2R_3, R_2 - R_3$.

The rows are exhausted and the process terminates.

Fully reduced normal form.

An $m \times n$ matrix B is said to be *equivalent* to an $m \times n$ matrix A over the same field F, if B can be obtained from A by a finite number of elementary row and column operations. Thus row equivalence and column equivalence are particular cases of equivalence of matrices.

For a given $m \times n$ matrix A, a row-reduced echelon matrix B can be obtained by applying on A a finite number of elementary row operations. Now by applying suitable column operations on B, a column-reduced echelon matrix C can be obtained. C has the following properties–

(i) No zero row is followed by a non-zero row;
(ii) no zero column is followed by a non-zero column;
(iii) the leading 1 in each non-zero row is the only non-zero element;
(iv) the leading 1 in each non-zero column is the only non-zero element;

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(v) the leading 1 in the kth row is the leading 1 in the kth column.

Thus C takes the form $\begin{pmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{pmatrix}$,

where I_r is the identity matrix of order r and O_{pq} is a zero matrix of order $p \times q$.

C is said to be the *fully reduced normal form* of the matrix A.

Note. In particular, if A be an $n \times n$ matrix of rank r, then C is a diagonal matrix of order n whose first r diagonal elements are 1 and the remaining diagonal elements (if $r < n$) are all 0.

Example 5. Find the fully reduced normal form of the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix}.$$

Let us apply elementary operations on the matrix.

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix}$$

$$\xrightarrow[R_4-3R_1]{R_3-2R_1} \begin{pmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow[\substack{R_3-2R_2 \\ R_4-R_2}]{R_1-R_2} \begin{pmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2-2R_3]{R_1+2R_3} \begin{pmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[C_5-2C_1]{C_2-3C_1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_5-C_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{C_{23}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_{34}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R, \text{ say.}$$

R is the fully reduced normal form.