B.Sc. Semester-III Examination, 2022-23

MATHEMATICS [Programme]

Course ID : 32118 **Course Code : SP/MTH/301/C-1C**

Course Title : Algebra

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks. Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meanings.

UNIT–I

1. Answer any **five** of the following questions: $2\times5=10$

a) Solve the equation $x^4+x^2-2x+6=0$, where $1+i$ is one root of the given equation.

b) If n is a positive integer, prove that $\dfrac{1.3.7...(2^n-1)}{2.4.8...2^n}<\dfrac{2^n}{2^{n+1}-1}$.

c) Find the product of all the values of $(1+i)^{\frac{4}{5}}$.

d) Give an example of an infinite set S and a mapping $F:S\to S$ such that F is surjective but not injective.

e) Show that the relation ρ defined on $\mathbb{R}$ by the rule "$x\rho y$ iff $x-y$ is irrational" is not an equivalence relation.

f) Let $f: A \to B$ and $g: B \to C$ be two invertible functions. Then show that $g \circ f$ is invertible.

g) Determine the rank of the matrix $A^3 + A^2 + A$, where

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & 3 \end{bmatrix}.$$

h) Prove that $n^2 - 2$ is not divisible by 4, $n \in Z$.

UNIT–II

2. Answer any **four** of the following questions:

$5\times4=20$

a) Use De Moiver's theorem to prove that

$$tan\, 5\theta = \frac{5\, tan\theta - 10\, tan^3\theta + tan^5\theta}{1 - 10\, tan^2\theta + 5\, tan^4\theta}.$$

b) Solve the equation : $x^4 + x^3 - 2x^2 + 4x - 24 = 0$.

c) Suppose A is a 2×2 real matrix with trace 5 and determinant 6. Find the eigenvalues of the matrix $B = A^2 - 2A + I_2$.

d) Find a linear operator T on $\mathbb{R}^3$ such that $Ker\ T$ is the subspace $U = \{(x, y, z) \in R^3 : x + y + z = 0\}$ of $\mathbb{R}^3$.

e) i) Prove that the rank of a real skew-symmetric matrix cannot be 1.

ii) If $\gcd(m, n)=1$, prove that $\gcd(mn, m+n)=1$, where m and n are positive integers. 3+2

f) For what values of k the following system of equations has a non-trivial solution? Solve in any one case.

$$x + 2y + 3z = kx\ ;\ 2x + y + 3z = ky;\ 2x + 3y + z = kz$$

UNIT–III

3. Answer any **one** of the following questions:

10×1=10

a) i) Prove that two eigenvectors of a square matrix A over the field $\mathbb{R}$ corresponding to two distinct eigen values of A are linearly independent.

ii) Find the sum of 99^{th} powers of the roots of the equation $x^7 - 1 = 0$.

iii) Find the least positive residue of 3^{36} $(mod\ 77)$. 4+3+3=10

b) i) Show that $S = \{(2, -5, 3)\}$ is not a subspace of $V_3(\mathbb{R})$ generated by the vectors $(1, -3, 2)$, $(2, -4, 1)$, $(1, -5, 7)$.

ii) Show that the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ is not diagonalizable.

iii) Solve the equation:

$$x^6 - x^5 + x^4 - 2x^3 + x^2 - x + 1 = 0.$$

3+3+4=10