B.Sc. Semester-III Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 32114 **Course Code : SH/MTH/304/GE-3**

Course Title : Algebra

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

UNIT–I

1. Answer any **five** of the following questions:

$2\times5=10$

a) Prove that for any complex number z, $|z|\geq\frac{1}{\sqrt{2}}(|\mathrm{Re}\, z|+|Im z|)$.

b) Construct an equivalence relation on the set $A=\{1, 2, 3\}$.

c) Using principle of mathematical induction prove that $3^{2n}-8n-1$ is divisible by 64 where n is a positive integer.

d) Apply Descartes rule of signs to examine the nature of the roots of the equation $x^7+x^5-x^3=0$.

[Turn Over]

e) Let λ be an eigenvalue of an $n\times n$ matrix A. Show that λ^4 is an eigenvalue of the matrix A^4.

f) Show that the mapping $f : S \to \mathbb{R}$ defined by $f(x)=\dfrac{x}{1-|x|}$, where $S=(-1, 1)$, is bijective.

g) Prove that $S\times S$ is an equivalence in S.

h) Express the matrix A as a product of elementary matrices, where $A=\begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$.

UNIT–II

2. Answer any **four** of the following questions:

$5\times4=20$

a) Show that one of the values of $\left(1+i\sqrt{3}\right)^{\frac{3}{4}}+\left(1-i\sqrt{3}\right)^{\frac{3}{4}}$ is $\sqrt[4]{32}$.

b) Let $T : R^2 \to R^3$ defined by

$$T(x, y)=(x+y,\ x-2y,\ 3x+y).$$

Show that T is non-singular and find T^{-1}.

c) If α, β, γ are the roots of the equation $x^3+qx+r=0$, then find the equation whose roots are $\dfrac{\beta+\gamma}{\alpha^2}, \dfrac{\gamma+\alpha}{\beta^2}, \dfrac{\alpha+\beta}{\gamma^2}$.

d) A mapping $T : R^3 \to R^3$ is defined by

$T(x, y, z) = (x+2y+3z,\ 3x+2y+z,\ x+y+z),\ (x, y, z) \in R^3$.

Show that T is a linear mapping. Find Ker T and the dimension of Ker T.

e) Show that the solutions of the equation $(1+x)^{2n} + (1-x)^{2n} = 0$ are $x = \pm i \tan \frac{(2r-1)\pi}{4n}$, $r = 1, 2, \ldots, n$.

f) i) If d=gcd(a, m), then show that

$$ax \equiv ay \pmod{m} \leftrightarrow x \equiv y\left(\bmod \frac{m}{d}\right).$$

ii) Find the least positive residues in

3^{36} (mod 77). 3+2=5

UNIT–III

3. Answer any **one** of the following questions:

10×1=10

a) i) Find the sum of 99th powers of the roots of the equation $x^7 - 1 = 0$.

ii) Show that $3x^5 - 4x^2 + 6 = 0$ has at least two imaginary roots.

iii) Diagonalise the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}.$$

3+2+5=10

b) i) Find the linear mapping $T: R^3 \to R^3$ if $T(1, 0, 0) = (2, 3, 4)$, $T(0, 1, 0) = (1, 2, 3)$, $T(0, 0, 1) = (1, 1, 1)$. Find the matrix of T relative to the ordered basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

ii) Find the dimension of the subspace W of $\mathbb{R}^3$ where

$$W = \{(x, y, z) : x + 2y + z = 0,\ 2x + y + 3z = 0\}.$$

iii) If p is prime, greater than 3, show that 24 divides $(p^2 - 1)$.

(2+2)+3+3=10
