Linear Transformation

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a linear transformation if

i) $T(\alpha+\beta) = T(\alpha) + T(\beta) \quad \forall\, \alpha, \beta \in \mathbb{R}^n$

ii) $T(c\alpha) = cT(\alpha) \quad \forall\, \alpha \in \mathbb{R}^n,\ c \in \mathbb{R}$

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* Show that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x+y, x+2y, y)$ is a linear transformation.

→ Let, $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$

$$T(\alpha) = (x_1+y_1,\ x_1+2y_1,\ y_1)$$

$$T(\beta) = (x_2+y_2,\ x_2+2y_2,\ y_2)$$

$$\alpha+\beta = (x_1+x_2,\ y_1+y_2)$$

$$T(\alpha+\beta) = (x_1+x_2+y_1+y_2,\ x_1+x_2+2y_1+2y_2,\ y_1+y_2)$$

$$= (x_1+y_1,\ x_1+2y_1,\ y_1) + (x_2+y_2,\ x_2+2y_2,\ y_2)$$

$$= T(\alpha) + T(\beta)$$

$$c\alpha = (cx_1, cy_1)$$

$$T(c\alpha) = (cx_1+cy_1,\ cx_1+2cy_1,\ cy_1)$$

$$= c(x_1+y_1,\ x_1+2y_1,\ y_1)$$

$$= c \cdot T(\alpha)$$

$\therefore$ T is a linear transformation.

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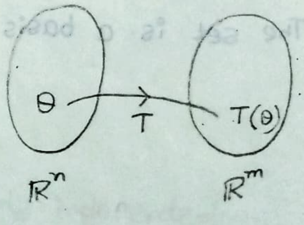
* If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $T(\theta) = \theta'$ where θ and θ' are null vectors of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

→ Now,

$$\theta + \theta = \theta$$



$$\Rightarrow T(\theta+\theta) = T(\theta)$$

$$\Rightarrow T(\theta) + T(\theta) = T(\theta)$$

$$\Rightarrow T(\theta) + T(\theta) - T(\theta) = T(\theta) - T(\theta)$$

$\Rightarrow T(\theta) + \theta' = \theta'$

$\Rightarrow T(\theta) = \theta'$

Exercises - 16

3. $T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(0,1,1) = (2,1,1)$

$T(1,0,1) = (1,2,1)$

$T(1,1,0) = (1,1,2)$

Let, $(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$

$c_2 + c_3 = \cancel{0}x$

$c_1 + c_3 = y$

$c_1 + c_2 = z$

$2(c_1+c_2+c_3) = x+y+z$

$\Rightarrow c_1+c_2+c_3 = \frac{1}{2}(x+y+z)$

$c_1 = \frac{1}{2}(y+z-x)$

$c_2 = \frac{1}{2}(x-y+z)$

$c_3 = \frac{1}{2}(x+y-z)$

$\therefore T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$

$= c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2)$

$= (2c_1+c_2+c_3,\ c_1+2c_2+c_3,\ c_1+c_2+2c_3)$

$= (y+z,\ x+z,\ x+y)$

H.W.

Exercises 16

1.i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+y, x-y)$.

Let, $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$

$T(\alpha) = (x_1+y_1,\ x_1-y_1)$

$T(\beta) = (x_2+y_2,\ x_2-y_2)$

$\alpha+\beta = \cancel{(x_1+y_1, x_2} (x_1+x_2,\ y_1+y_2)$

$T(\alpha+\beta) = (x_1+x_2+y_1+y_2,\ x_1+x_2-y_1-y_2)$

$= (x_1+y_1,\ x_1-y_1) + (x_2+y_2,\ x_2-y_2)$

$= T(\alpha) + T(\beta)$

$c\alpha = (cx_1, cy_1)$

$T(c\alpha) = (cx_1 + cy_1 , cx_1 - cy_1)$

$= c(x_1 + y_1 , x_1 - y_1)$

$= c \cdot T(\alpha)$

$\therefore$ T is a linear mapping.

ii) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x+2y, 2x+y, x+y)$

Let, $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$

$T(\alpha) = (x_1 + 2y_1 , 2x_1 + y_1 , x_1 + y_1)$

$T(\beta) = (x_2 + 2y_2 , 2x_2 + y_2 , x_2 + y_2)$

$\alpha + \beta = (x_1 + x_2 , y_1 + y_2)$

$\therefore T(\alpha+\beta) = (x_1 + x_2 + 2y_1 + 2y_2 , 2x_1 + 2x_2 + y_1 + y_2 , x_1 + x_2 + y_1 + y_2)$

$= (x_1 + 2y_1 , 2x_1 + y_1 , x_1 + y_1) + (x_2 + 2y_2 , 2x_2 + y_2 , x_2 + y_2)$

$= T(\alpha) + T(\beta)$

$c\alpha = (cx_1, cy_1)$

$T(c\alpha) = (cx_1 + 2cy_1 , 2cx_1 + cy_1 , cx_1 + cy_1)$

$= c(x_1 + 2y_1 , 2x_1 + y_1 , x_1 + y_1)$

$= c\,T(\alpha)$

$\therefore$ T is a linear mapping.

iii) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (yz, zx, xy)$

Let, $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$\therefore T(\alpha) = (y_1 z_1 , z_1 x_1 , x_1 y_1)$

$T(\beta) = (y_2 z_2 , z_2 x_2 , x_2 y_2)$

$\alpha + \beta = (x_1 + x_2 , y_1 + y_2 , z_1 + z_2)$

$T(\alpha+\beta) = ((y_1 + y_2)(z_1 + z_2) , (z_1 + z_2)(x_1 + x_2) , (x_1 + x_2)(y_1 + y_2))$

$= (y_1 z_1 + y_1 z_2 + y_2 z_1 + y_2 z_2 , z_1 x_1 + z_1 x_2 + z_2 x_1 + x_2 z_2 ,$
$x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2)$

$T(\alpha+\beta) \neq T(\alpha) + T(\beta)$

$\therefore$ T is not a linear mapping.

iv) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+2y+3z,\ 3x+2y+z,\ x+y+z)$

Let, $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$T(\alpha) = (x_1 + 2y_1 + 3z_1,\ 3x_1 + 2y_1 + z_1,\ x_1 + y_1 + z_1)$

$T(\beta) = (x_2 + 2y_2 + 3z_2,\ 3x_2 + 2y_2 + z_2,\ x_2 + y_2 + z_2)$

$\alpha + \beta = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

$T(\alpha+\beta) = (x_1 + x_2 + 2y_1 + 2y_2 + 3z_1 + 3z_2,\ 3x_1 + 3x_2 + 2y_1 + 2y_2 + z_1 + z_2,$
$x_1 + x_2 + y_1 + y_2 + z_1 + z_2)$

$= (x_1 + 2y_1 + 3z_1,\ 3x_1 + 2y_1 + z_1,\ x_1 + y_1 + z_1) +$
$(x_2 + 2y_2 + 3z_2,\ 3x_2 + 2y_2 + z_2,\ x_2 + y_2 + z_2)$

$= T(\alpha) + T(\beta)$

$c\alpha = (cx_1, cy_1, cz_1)$

$T(c\alpha) = (cx_1 + 2cy_1 + 3cz_1,\ 3cx_1 + 2cy_1 + cz_1,\ cx_1 + cy_1 + cz_1)$

$= c(x_1 + 2y_1 + 3z_1,\ 3x_1 + 2y_1 + z_1,\ x_1 + y_1 + z_1)$

$= cT(\alpha)$

$\therefore$ T is a linear mapping.

v) $T: \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T(x,y,z) = (-x+y+z,\ x-y+z,\ x+y-z,$
$x+y+z)$

Let, $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$T(\alpha) = (-x_1 + y_1 + z_1,\ x_1 - y_1 + z_1,\ x_1 + y_1 - z_1,\ x_1 + y_1 + z_1)$

$T(\beta) = (-x_2 + y_2 + z_2,\ x_2 - y_2 + z_2,\ x_2 + y_2 - z_2,\ x_2 + y_2 + z_2)$

$\alpha + \beta = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

$T(\alpha+\beta) = (-x_1 - x_2 + y_1 + y_2 + z_1 + z_2,\ x_1 + x_2 - y_1 - y_2 + z_1 + z_2,$
$x_1 + x_2 + y_1 + y_2 - z_1 - z_2,\ x_1 + x_2 + y_1 + y_2 + z_1 + z_2)$

$= (-x_1 + y_1 + z_1,\ x_1 - y_1 + z_1,\ x_1 + y_1 - z_1,\ x_1 + y_1 + z_1) +$
$(-x_2 + y_2 + z_2,\ x_2 - y_2 + z_2,\ x_2 + y_2 - z_2,\ x_2 + y_2 + z_2)$

$= T(\alpha) + T(\beta)$

$c\alpha = (cx_1, cy_1, cz_1)$

$T(c\alpha) = (-cx_1 + cy_1 + cz_1, cx_1 - cy_1 + cz_1, cx_1 + cy_1 - cz_1, cx_1 + cy_1 + cz_1)$

$= c(-x_1 + y_1 + z_1, x_1 - y_1 + z_1, x_1 + y_1 - z_1, x_1 + y_1 + z_1)$

$= c\,T(\alpha)$

$\therefore$ T is a linear mapping.

vi) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x,y,z) = x+y+z$

Let, $\alpha = (x_1, y_1, z_1)$; $\beta = (x_2, y_2, z_2)$

$T(\alpha) = x_1 + y_1 + z_1$; $T(\beta) = x_2 + y_2 + z_2$

$\alpha + \beta = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$T(\alpha+\beta) = x_1 + x_2 + y_1 + y_2 + z_1 + z_2$

$= x_1 + y_1 + z_1 + x_2 + y_2 + z_2$

$= T(\alpha) + T(\beta)$

$c\alpha = (cx_1, cy_1, cz_1)$

$= cx_1 + cy_1 + cz_1$

$= c(x_1 + y_1 + z_1)$

$= c\,T(\alpha)$

$\therefore$ T is a linear mapping.

vii) $T : \mathbb{R}_{2\times 2} \to \mathbb{R}_{2\times 2}$ defined by $T(A) = \frac{1}{2}(A + A^t)$

Let, $\alpha = A_1$, $\beta = A_2$

$T(\alpha) = \frac{1}{2}(A_1 + A_1^t)$

$T(\beta) = \frac{1}{2}(A_2 + A_2^t)$

$\alpha + \beta = A_1 + A_2$

$T(\alpha+\beta) = \frac{1}{2}\left\{(A_1 + A_2) + (A_1 + A_2)^t\right\}$

$= \frac{1}{2}(A_1 + A_1^t) + \frac{1}{2}(A_2 + A_2^t)$

$= T(\alpha) + T(\beta)$

$c\alpha = cA_1$

$T(c\alpha) = \frac{1}{2}(cA_1 + cA_1^t)$

$= c\{\frac{1}{2}(A_1 + A_1^t)\}$

$= cT(\alpha)$

$\therefore$ T is a linear mapping.

2. $T(1,0,0) = (0,1,0)$

$T(0,1,0) = (0,0,1)$ $\quad T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(0,0,1) = (1,0,0)$

Let, $(x,y,z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$\therefore c_1 = x, \quad c_2 = y, \quad c_3 = z$

$\therefore T(x,y,z) = c_1 T(1,0,0) + c_2 T(0,1,0) + c_3 T(0,0,1)$

$= c_1(0,1,0) + c_2(0,0,1) + c_3(1,0,0)$

$= (c_3, c_1, c_2)$

$= (z, x, y)$

4. $T(2,1,1) = (1,1,1)$

$T(1,2,1) = (1,1,1)$ $\quad T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(1,1,2) = (1,1,1)$

Let, $(x,y,z) = c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2)$

$\therefore \quad 2c_1 + c_2 + c_3 = x$

$c_1 + 2c_2 + c_3 = y$

$c_1 + c_2 + 2c_3 = z$

$\therefore \quad 4(c_1 + c_2 + c_3) = x + y + z$

$\Rightarrow c_1 + c_2 + c_3 = \frac{1}{4}(x+y+z)$

$\therefore T(x,y,z) = c_1 T(2,1,1) + c_2 T(1,2,1) + c_3 T(1,1,2)$

$= c_1(1,1,1) + c_2(1,1,1) + c_3(1,1,1)$

$= (c_1 + c_2 + c_3, \ c_1 + c_2 + c_3, \ c_1 + c_2 + c_3)$

$= \left(\frac{x+y+z}{4}, \frac{x+y+z}{4}, \frac{x+y+z}{4}\right)$

5. $T: \mathbb{R}^3 \to \mathbb{R}^4$

$T(0,1,1) = (0,1,1,1)$

$T(1,0,1) = (1,0,1,1)$

$T(1,1,0) = (1,1,0,1)$

Let, $(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$

$\therefore\ c_2 + c_3 = x$

$c_1 + c_3 = y$

$c_1 + c_2 = z$

$\Rightarrow 2(c_1+c_2+c_3) = x+y+z$

$\Rightarrow c_1+c_2+c_3 = \frac{1}{2}(x+y+z)$

$c_1 = \frac{1}{2}(y+z-x)$

$c_2 = \frac{1}{2}(x+z-y)$

$c_3 = \frac{1}{2}(x+y-z)$

$T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$

$= c_1(0,1,1,1) + c_2(1,0,1,1) + c_3(1,1,0,1)$

$= (c_2+c_3,\ c_1+c_3,\ c_1+c_2,\ c_1+c_2+c_3)$

$= \left(x, y, z, \frac{x+y+z}{2}\right)$

* **Kernel of T :-**

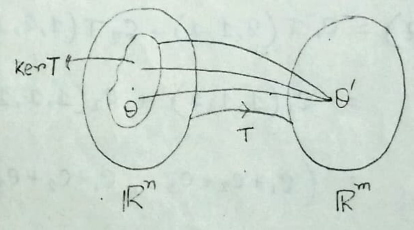
Let, $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Then kernel of T, denoted by $\ker T$

where $\ker T = \{\alpha \in \mathbb{R}^n : T(\alpha) = \theta'\}$

* Show that $\ker T$ is a subspace of $\mathbb{R}^n$.

→ Since, $T(\theta) = \theta'$, where θ, θ' are null vectors of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.



So, $\theta \in \ker T$

Let, $\alpha, \beta \in \ker T$ and $c \in \mathbb{R}$

then $T(\alpha) = \theta'$, $T(\beta) = \theta'$

$\therefore T(\alpha+\beta) = T(\alpha) + T(\beta) = \theta' + \theta' = \theta'$

and $T(c\alpha) = cT(\alpha) = c\theta' = \theta'$

$\therefore \alpha+\beta, c\alpha \in \ker T$

Therefore $\ker T$ is a subspace of $\mathbb{R}^n$.

* The dimension of $\ker T$, denoted by $\dim \ker T$ and which is called <u>nullity of T</u>.

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* A mapping $f : A \to B$ is said to be <u>injective</u> if

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

<u>or</u>

$x_1 \neq x_2$ in $A \Rightarrow f(x_1) \neq f(x_2)$ in B.

* f is said to be <u>surjective</u> if for each element $y \in B$ there exists at least one element $x \in A$ such that $f(x) = y$

* A mapping $f : A \to B$ is said to be <u>bijective</u> if it is both injective as well as surjective.

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* i) Let, $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Then T is injective iff $\ker T = \{\theta\}$

ii) Let, $T : \mathbb{R}^n \to \mathbb{R}^m$ be an injective linear transformation.

If $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ are linearly independent set of vectors of $\mathbb{R}^n$, then $\{T(\alpha_1), T(\alpha_2), \dots, T(\alpha_k)\}$ are linearly independent set of vectors in $\mathbb{R}^m$.

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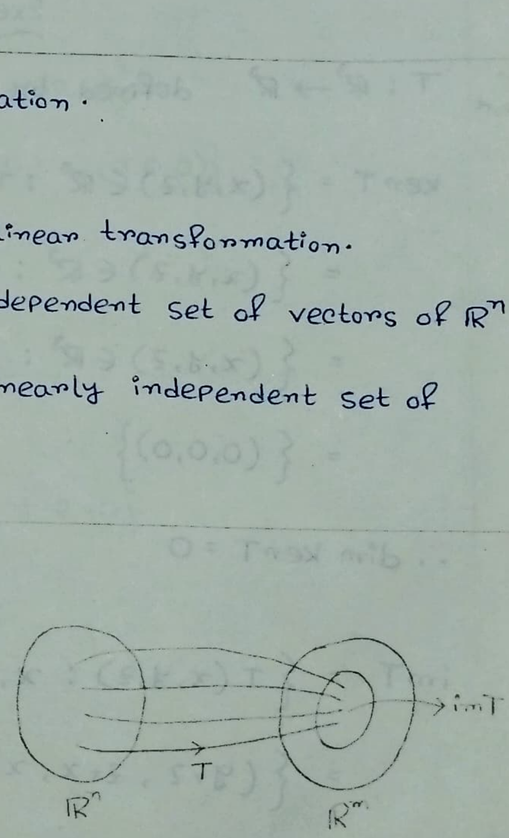
* <u>Image of T</u> :-

Let, $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Then image of T, denoted by $\operatorname{im} T$,

where $\operatorname{im} T = \{T(\alpha) : \alpha \in \mathbb{R}^n\}$



* Im T is a subspace of $\mathbb{R}^m$.

→ Since, $\theta \in \mathbb{R}^n$, $T(\theta) \in \mathbb{R}^m$

i.e. $\theta' \in \mathbb{R}^m$

Let, $\alpha', \beta' \in$ im T, $c \in \mathbb{R}$

then there exist $\alpha, \beta \in \mathbb{R}^n$ such that $T(\alpha) = \alpha'$ and $T(\beta) = \beta'$

So, $T(\alpha+\beta) = T(\alpha) + T(\beta) = \alpha' + \beta'$

$T(c\alpha) = cT(\alpha) = c\alpha'$

So, $\alpha' + \beta'$, $c\alpha' \in$ im T

So, im T is a subspace of $\mathbb{R}^m$.

* Now dimension of im T, denoted by dim im T which is called rank of T.

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* If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$\dim \ker T + \dim \operatorname{im} T = n$$

or, Nullity of T + rank of T $= n$

Exercises 16

3. (2nd Part) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (y+z, z+x, x+y)$.

$$\ker T = \{(x,y,z) \in \mathbb{R}^3 : T(x,y,z) = (0,0,0)\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (y+z, z+x, x+y) = (0,0,0)\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 : y+z=0,\ z+x=0,\ x+y=0\}$$

$$= \{(0,0,0)\}$$

$y+z=0$

$z+x=0$

$x+y=0$

$\therefore x+y+z=0$

$\therefore \dim \ker T = 0$

$$\operatorname{im} T = \{T(x,y,z) : x,y,z \in \mathbb{R}\}$$

$$= \{(y+z, z+x, x+y) : x,y,z \in \mathbb{R}\}$$

$$= \{x(0,1,1) + y(1,0,1) + z(1,1,0) : x,y,z \in \mathbb{R}\}$$

Suppose,

$$c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0) = (0,0,0)$$

$$\Rightarrow c_2 + c_3 = 0$$

$$c_1 + c_3 = 0$$

$$c_1 + c_2 = 0$$

$$\therefore 2(c_1 + c_2 + c_3) = 0$$

$$\Rightarrow c_1 + c_2 + c_3 = 0$$

$$\therefore c_1 = c_2 = c_3 = 0$$

$\therefore (0,1,1), (1,0,1), (1,1,0)$ is linearly independent

$\therefore (0,1,1), (1,0,1), (1,1,0)$ is a basis of imT

$\therefore$ dim imT $= 3$

dim KerT + dim ImT $= 0 + 3 = 3$

---

7. $T: \mathbb{R}^3 \to \mathbb{R}^3$

$\ker T = U = \{(x,y,z) \in \mathbb{R}^3 : x - y - z = 0\}$

$T(x,y,z) = (x-y-z, 0, 0)$

$$\ker T = \{(x,y,z) \in \mathbb{R}^3 : T(x,y,z) = (0,0,0)\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (x-y-z, 0, 0) = (0,0,0)\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 : x - y - z = 0\}$$

$$= U$$

---

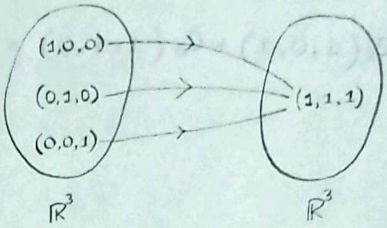
9. $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$\operatorname{Im} T = U = \{(x,y,z) \in \mathbb{R}^3 : x - y = 0, y - z = 0\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 : x = y = z\}$$

$$= \{(x,x,x) : x \in \mathbb{R}\}$$

$$= \{x(1,1,1) : x \in \mathbb{R}\}$$



$T(1,0,0) = (1,1,1)$

$T(0,1,0) = (1,1,1)$

$T(0,0,1) = (1,1,1)$

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$\Rightarrow T(x,y,z) = x(1,1,1) + y(1,1,1) + z(1,1,1)$

$= (x+y+z,\ x+y+z,\ x+y+z)$

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1. vii) 2nd Part

$T : \mathbb{R}_{2\times 2} \rightarrow \mathbb{R}_{2\times 2}$

$T(A) = \frac{1}{2}(A + A^T)$

$\therefore \ker T = \{A \in \mathbb{R}_{2\times 2} : T(A) = 0\}$

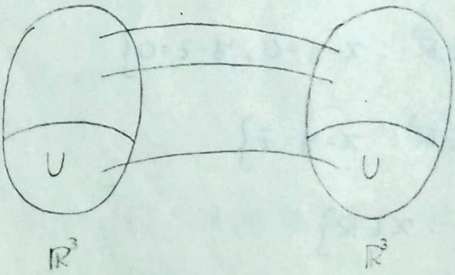
$= \{A \in \mathbb{R}_{2\times 2} : \frac{1}{2}(A + A^T) = 0\}$

$= \{A \in \mathbb{R}_{2\times 2} : A^T = -A\}$

$\operatorname{Im} T = \{T(A) : A \in \mathbb{R}_{2\times 2}\}$

$= \{\frac{1}{2}(A + A^T) : A \in \mathbb{R}_{2\times 2}\}$

10.



$U = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 0\}$

Let, $x = c,\ y = d$

$z = -c-d$

$= \{(c,d,-c-d) : c,d \in \mathbb{R}\}$

$= \{c(1,0,-1) + d(0,1,-1) : c,d \in \mathbb{R}\}$

Now, $c_1(1,0,0) + c_2(1,0,-1) + c_3(0,1,-1) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 &= 0 \\ c_3 &= 0 \\ -c_2 - c_3 &= 0 \end{aligned}\right\} \Rightarrow c_1 = c_2 = c_3 = 0$$

$\therefore$ $\{(1,0,0), (1,0,-1), (0,1,-1)\}$ is linearly independent

$\therefore$ $\{(1,0,0), (1,0,-1), (0,1,-1)\}$ is a basis of U

$\because$ $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(1,0,0) = (1,0,0)$

$T(1,0,-1) = (0,1,-1)$

$T(0,1,-1) = (1,0,-1)$

Let, $(x,y,z) = c_1(1,0,0) + c_2(1,0,-1) + c_3(0,1,-1)$

$\therefore$ $c_1 + c_2 = x$

$c_3 = y$

$-c_2 - c_3 = z$

$\Rightarrow c_2 = -z - y$

$\therefore c_1 = x + y + z$

$T(x,y,z) = c_1 T(1,0,0) + c_2 T(1,0,-1) + c_3 T(0,1,-1)$

$= c_1(1,0,0) + c_2(0,1,-1) + c_3(1,0,-1)$

$= (c_1 + c_3, c_2, -c_2 - c_3)$

$= (x + 2y + z, -z - y, z)$