4. If λ be an eigen value of an $n \times n$ matrix A, prove that
(i) λ is also an eigen value of the matrix A^t;
(ii) $k\lambda$ is an eigen value of the matrix kA, k being a scalar;
(iii) λ^2 is an eigen value of the matrix A^2.

5. $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a non-singular matrix A of order n. Find the eigen values of the matrix (i) A^{-1}, (ii) adj A.

6. If λ be an eigen value of an $n \times n$ idempotent matrix A, prove that λ is either 1 or 0.

[**Hint.** $A^2 = A$. Let $AX = \lambda X$ for some $X \neq 0$. Then $\lambda X = AX = A^2X = \lambda^2 X$.]

7. Find the eigen values and the corresponding eigen vectors of the following real matrices.

(i) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, (ii) $\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$, (iii) $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

8. Find the eigen values and the corresponding eigen vectors of the following complex matrices.

(i) $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, (ii) $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, (iii) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

9. Find the algebraic and the geometric multiplicities of each eigen value of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$.

10. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are eigen vectors of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ a & 3 & 1 \\ 1 & b & c \end{pmatrix}$. Find a, b, c.

11. If λ be an eigen value of an $n \times n$ matrix A, prove that λ^m is an eigen value of the matrix A^m, where m is a positive integer.

[**Hint.** $AX = \lambda X$ implies $A^2X = \lambda(AX) = \lambda^2 X$ etc.]

12. If λ be an eigen value of an $n \times n$ non-singular matrix A, prove that λ^{-m} is an eigen value of the matrix A^{-m}, where m is a positive integer.

13. If A and B are square matrices of the same order and one of them is non-singular, prove that the matrices AB and BA have the same eigen values.

14. A and B are symmetric non-singular matrices such that $AB + BA = O$. Prove that each of the matrices A, B and AB have eigen values symmetric about the origin.

15. λ is an eigen value of a real skew symmetric matrix. Prove that $\left| \frac{1-\lambda}{1+\lambda} \right| = 1$.

16. If S be a real skew symmetric matrix of order n, prove that the matrices $I_n + S$ and $I_n - S$ are both non-singular.

17. If A be a real non-singular symmetric matrix, prove that the matrices A and A^{-1} have the same set of eigen vectors.

18. Let X be an eigen vector of an $n \times n$ matrix A associated with an eigen value λ. Prove that $P^{-1}X$ is an eigen vector of the matrix $P^{-1}AP$ associated with λ.

19. Show that the characteristic equation of an orthogonal matrix is a reciprocal equation.

[**Hint.** Let A be orthogonal and $\psi(x) = \det(A - xI_n)$. Then $\psi(x) = \pm x^n \psi(\frac{1}{x})$.]

20. P is a real orthogonal matrix with $\det P = -1$. Prove that -1 is an eigen value of P.

[**Hint.** $\det(P + I) = \det(P + PP^t) = \det P \det(I + P^t) = -\det(P + I)$.]

21. If S be a real skew symmetric matrix of order n, prove that

(i) the matrix $S - I_n$ is non-singular,

(ii) the matrix $(S - I_n)^{-1}(S + I_n)$ is orthogonal,

(iii) if X be an eigen vector of S with eigen value λ, then X is also an eigen vector of the matrix $(S - I_n)^{-1}(S + I_n)$ with eigen value $\frac{\lambda+1}{\lambda-1}$,

(iv) if $\bar{S} = (S - I_n)^{-1}(S + I_n)$, then $\bar{S} - I_n$ is also non-singular and $\bar{\bar{S}} = S$.

22. A is an $n \times n$ real symmetric matrix and $A^2 = I_n$ but $A \neq \pm I_n$. Prove that 1 and -1 are the only distinct eigen values of A.

23. A is a 3×3 real matrix having the eigen values $1, 2, 0$.
$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are the eigen vectors of A corresponding to the eigen values $1, 2, 0$ respectively. Find the matrix A.

24. A is a 3×3 real matrix having the eigen values $5, 2, 2$. The eigen vectors of A corresponding to the eigen values 5 and 2 are

$$c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, c \neq 0 \text{ and } c\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, (c, d) \neq (0, 0) \text{ respectively.}$$

Find the matrix A.

25. If every non-zero vector in $\mathbb{R}^n$ be an eigen vector of a real $n \times n$ matrix A corresponding to a real eigen value λ, prove that A is the scalar matrix λI_n.

[**Hint.** The solution space of the homogeneous system $(A - \lambda I_n)X = O$ is $\mathbb{R}^n$ of dimension n. Therefore the rank of $A - \lambda I_n$ is 0.]