Matrix of a Linear Transformation

Exercises - 18

12. $T : \mathbb{R}^3 \to \mathbb{R}^3$

1st Part $T(x, y, z) = (y+z,\ z+x,\ x+y)$

The matrix of T relative to the ordered bases $\{(1,0,0), (0,1,0), (0,0,1)\}$ is A

$T(1,0,0) = (0,1,1) = 0(1,0,0) + 1(1,0,0) + 1(0,0,1)$

$T(0,1,0) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (1,1,0) = 1(1,0,0) + 1(0,1,0) + 0(0,0,1)$

$\therefore$ The matrix of $T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$*$ $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (y+z,\ z+x,\ x+y)$

$\therefore T(x, y, z) = AX$

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9. $T : \mathbb{R}^3 \to \mathbb{R}^3$

$\{(-1,1,1), (1,-1,1), (1,1,-1)\}$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$$

$T(-1,1,1) = 1(-1,1,1) + 2(1,-1,1) + 3(1,1,-1) = (4,2,0)$

$T(1,-1,1) = 2(-1,1,1) + 1(1,-1,1) + 3(1,1,-1) = (2,4,0)$

$T(1,1,-1) = 2(-1,1,1) + 3(1,-1,1) + 1(1,1,-1) = (\cancel{0}2,0,4)$

Let, $(x, y, z) = c_1(-1,1,1) + c_2(1,-1,1) + c_3(1,1,-1)$

$\therefore\quad c_2 + c_3 - c_1 = x$

$c_1 + c_3 - c_2 = y$

$c_1 + c_2 - c_3 = z$

$c_1 + c_2 + c_3 = x + y + z$

$2c_1 = y + z$

$c_1 = \frac{1}{2}(y+z)$

$c_2 = \frac{1}{2}(z+x)$

$c_3 = \frac{1}{2}(x+y)$

$$T(x,y,z) = c_1 T(-1,1,1) + c_2 T(1,-1,1) + c_3 T(1,1,-1)$$

$$= c_1(4,2,0) + c_2(2,4,0) + c_3(2,0,4)$$

$$= (4c_1 + 2c_2 + 2c_3,\ 2c_1 + 4c_2,\ 4c_3)$$

$$= (2x + 3y + 3z,\ 2x + y + 3z,\ 2x + 2y)$$

| H.W.

12. Remaining

$T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (y+z, x+z, x+y)$, $(x,y,z) \in \mathbb{R}^3$

The matrix of T relative to the ordered bases $\{(2,1,1), (1,2,1), (1,1,2)\}$ of $\mathbb{R}^3$ is B.

$T(2,1,1) = (2,3,3) = 0(2,1,1) + 1(1,2,1) + 1(1,1,2)$

$T(1,2,1) = (3,2,3) = 1(2,1,1) + 0(1,2,1) + 1(1,1,2)$

$T(1,1,2) = (3,3,2) = 1(2,1,1) + 1(1,2,1) + 0(1,1,2)$

$$\therefore \text{Matrix } B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$S: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$S(1,0,0) = (2,1,1) = 2(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$S(0,1,0) = (1,2,1) = 1(1,0,0) + 2(0,1,0) + 1(0,0,1)$

$S(0,0,1) = (1,1,2) = 1(1,0,0) + 1(0,1,0) + 2(0,0,1)$

The matrix of S relative to the ordered bases $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ is P.

$$\therefore P = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

13. $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (x+y+z, x+z, x+y)$

The matrix of T relative to the ordered bases $((1,0,0), (0,1,0), (0,0,1))$ of $\mathbb{R}^3$ is A

$T(1,0,0) = (1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$T(0,1,0) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (1,1,0) = 1(1,0,0) + 1(0,1,0) + 0(0,0,1)$

$$\therefore A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The matrix of T relative to the ordered bases $((0,1,1), (1,0,1), (1,1,0))$ of $\mathbb{R}^3$ is B.

$T(0,1,1) = (2,1,1) = 0(0,1,1) + 1(1,0,1) + 1(1,1,0)$

$T(1,0,1) = (2,2,1) = \frac{1}{2}(0,1,1) + \frac{1}{2}(1,0,1) + \frac{3}{2}(1,1,0)$

$T(1,1,0) = (2,1,2) = \frac{1}{2}(0,1,1) + \frac{3}{2}(1,0,1) + \frac{1}{2}(1,1,0)$

$$\therefore B = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

14. $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (x+2y+z,\ 2x+y+z,\ x+y+z)$

The matrix of T relative to the ordered bases $((1,0,0),$ $(0,1,0), (0,0,1))$ of $\mathbb{R}^3$ is A.

$T(1,0,0) = (1,2,1) = 1(1,0,0) + 2(0,1,0) + 1(0,0,1)$

$T(0,1,0) = (2,1,1) = 2(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$$\therefore A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The matrix of T relative to the ordered bases $((0,1,1),$ $(1,0,1), (1,1,0))$ of $\mathbb{R}^3$ is B

$T(0,1,1) = (3,2,2) = \frac{1}{2}(0,1,1) + \frac{3}{2}(1,0,1) + \frac{3}{2}(1,1,0)$

$T(1,0,1) = (2,3,2) = \frac{3}{2}(0,1,1) + \frac{1}{2}(1,0,1) + \frac{3}{2}(1,1,0)$

$T(1,1,0) = (3,3,2) = 1(0,1,1) + 1(1,0,1) + 2(1,1,0)$

$$\therefore B = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ \frac{3}{2} & \frac{1}{2} & 1 \\ \frac{3}{2} & \frac{3}{2} & 2 \end{pmatrix}$$

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9. 2nd Part i)

We have $T(x,y,z) = (2x+3y+3z,\ 2x+y+3z,\ 2x+2y)$

$T(1,0,0) = (2,2,2) = 2(1,0,0) + 2(0,1,0) + 2(0,0,1)$

$T(0,1,0) = (3,1,2) = 3(1,0,0) + 1(0,1,0) + 2(0,0,1)$

$T(0,0,1) = (3,3,0) = 3(1,0,0) + 3(0,1,0) + 0(0,0,1)$

$\therefore$ The matrix of T relative to the ordered bases $((1,0,0),$

$$(0,1,0), (0,0,1)) = \begin{pmatrix} 2 & 3 & 3 \\ 2 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$$

ii) $T(0,1,1) = (6,4,2) = 0(0,1,1) + 2(1,0,1) + 4(1,1,0)$

$T(1,0,1) = (5,5,2) = 1(0,1,1) + 1(1,0,1) + 4(1,1,0)$

$T(1,1,0) = (5,3,4) = 1(0,1,1) + 3(1,0,1) + 2(1,1,0)$

So, the matrix of T relative to the ordered bases ((0,1,1), (1,0,1), (1,1,0)) is = $\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \\ 4 & 4 & 2 \end{pmatrix}$.

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* Let, A be a square matrix of order n.

Then $\det(A - xI_n)$ is called <u>characteristic polynomial</u> of A and the equation $\det(A - xI_n) = 0$ is called <u>characteristic equation</u> of A.

The roots of the characteristic equation of A are called <u>eigen value</u> of A.

* <u>Cayley : Hamilton theorem</u> :- Every square matrix satisfies its own characteristic equation.

<u>Example</u> :-

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

$$A - xI_2 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

$$= \begin{pmatrix} 2-x & 1 \\ 0 & 3-x \end{pmatrix}$$

$$\det(A - xI_2) = \begin{vmatrix} 2-x & 1 \\ 0 & 3-x \end{vmatrix} = x^2 - 5x + 6$$ → Characteristic Polynomial of A

$\det(A - xI_2) = 0$

i.e. $x^2 - 5x + 6 = 0 \longrightarrow$ characteristic equation of A

$$A^2 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix}$$

$A^2 - 5A + 6I_2$

$$= \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix} - 5\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} 10 & 5 \\ 0 & 15 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore\ A^2 - 5A + 6I_2 = O$

Now, $x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 3x - 2x + 6 = 0$

$\Rightarrow x(x-3) - 2(x-3) = 0$

$\therefore\ x = 3,\ x = 2 \longrightarrow$ eigen value of A

Exercises - 13

1.ii) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$

$$A - xI_3 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} - \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$$

$$= \begin{pmatrix} 1-x & 2 & 1 \\ 1 & -1-x & 1 \\ 2 & 3 & -1-x \end{pmatrix}$$

$$\det(A - xI_3) = \begin{vmatrix} 1-x & 2 & 1 \\ 1 & -1-x & 1 \\ 2 & 3 & -1-x \end{vmatrix}$$

$$= (1-x)\{(1+x)^2 - 3\} - 2(-1-x-2) + 1(3+2+2x)$$

$$= (1-x)(2x+x^2-2) - 2(-3-x) + 1(2x+5)$$

$$= 2x + x^2 - 2 - 2x^2 - x^3 + 2x + 6 + 2x + 2x + 5$$

$$= -x^3 - x^2 + 8x + 9$$

$$\Rightarrow x^3 + x^2 - 8x - 9$$

$\therefore$ Characteristic equation is $x^3 + x^2 - 8x - 9 = 0$

Now, $A^2 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 2 \\ 2 & 6 & -1 \\ 3 & -2 & 6 \end{pmatrix}$

$A^3 = \begin{pmatrix} 5 & 3 & 2 \\ 2 & 6 & -1 \\ 3 & -2 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 12 & 13 & 6 \\ 6 & -5 & 9 \\ 13 & 26 & -5 \end{pmatrix}$

$\therefore \quad A^3 + A^2 - 8A - 9I_3$

$$= \begin{pmatrix} 12 & 13 & 6 \\ 6 & -5 & 9 \\ 13 & 26 & -5 \end{pmatrix} + \begin{pmatrix} 5 & 3 & 2 \\ 2 & 6 & -1 \\ 3 & -2 & 6 \end{pmatrix} - \begin{pmatrix} 8 & 16 & 8 \\ 8 & -8 & 8 \\ 16 & 24 & -8 \end{pmatrix} - \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, A ~~st~~ satisfies its own characteristic equation, hence Cayley-Hamilton theorem verified.

Here $\det A = 9 \neq 0$

$\therefore A^{-1}$ exists

Now, $A^3 + A^2 - 8A - 9I_3 = O$

$$\Rightarrow 9I = A^3 + A^2 - 8A$$

$$\Rightarrow 9A^{-1} = A^{-1}(A^3 + A^2 - 8A)$$

$$= A^2 + A - 8I_3$$

$$\Rightarrow A^{-1} = \frac{1}{9}(A^2 + A - 8I_3)$$

$$= \frac{1}{9}\begin{pmatrix} 5 & 3 & 2 \\ 2 & 6 & -1 \\ 3 & -2 & 6 \end{pmatrix} + \frac{1}{9}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} - \frac{1}{9}\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{pmatrix}$$

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* If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be a square matrix of order n.

and let, $\det(A - xI_n) = \begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix}$

$$= c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n$$

Then $c_0 = (-1)^n$

$c_1 = (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})$

$= (-1)^{n-1} \cdot$ trace of A

$c_n = \det A$

* If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigen values of A, then

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = (-1)\frac{c_1}{c_0}$$

$$= \frac{(-1)(-1)^{n-1}\text{ trace } A}{(-1)^n} = \text{trace } A$$

$$\lambda_1\lambda_2 \cdots \lambda_n = (-1)^n \frac{c_n}{c_0} = c_n = \det A$$

2. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Characteristic equation is

$$\det(A - xI_3) = 0$$

$$\Rightarrow \begin{vmatrix} 1-x & 0 & 0 \\ 1 & -x & 1 \\ 0 & 1 & -x \end{vmatrix} = 0$$

$$\Rightarrow (1-x)(x^2-1) = 0$$

$$\Rightarrow x^2 - 1 - x^3 + x = 0$$

$$\Rightarrow x^3 - x^2 - x + 1 = 0$$

So by Cayley-Hamilton theorem,

$$A^3 - A^2 - A + I_3 = O$$

Now,

$$A - I_3 = A - I_3$$

$$A^2 - A = A^2 - A$$

$$A^3 - A^2 = A - I_3$$

$$A^4 - A^3 = A^2 - A$$

$$A^5 - A^4 = A^3 - A^2 = A - I_3$$

$$A^6 - A^5 = A^2 - A$$

$$A^7 - A^6 = A^3 - A^2 = A - I_3$$

$$\vdots$$

$$A^{100} - A^{99} = A^2 - A$$

$$A^{100} - I_3 = 50(A - I_3) + 50(A^2 - A)$$

$$\Rightarrow A^{100} - I_3 = 50A^2 - 50I_3$$

$$\Rightarrow A^{100} = 50A^2 - 49I_3$$

Now, $A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$\therefore A^{100} = 50\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - 49\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 50 & 0 & 0 \\ 50 & 50 & 0 \\ 50 & 0 & 50 \end{pmatrix} - \begin{pmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{pmatrix}$$

7.iii) $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

1st Part

Characteristic equation is

$$\det(A - xI_3) = 0$$

$$\Rightarrow \begin{vmatrix} 2-x & 2 & 1 \\ 1 & 3-x & 1 \\ 1 & 2 & 2-x \end{vmatrix} = 0$$

$$\Rightarrow (2-x)\{(3-x)(2-x) - 2\} - 2(2-x-1) + 1(2-3+x) = 0$$

$$\Rightarrow (2-x)(6-3x-2x+x^2-2) - 2(1-x) + 1(x-1) = 0$$

$$\Rightarrow (2-x)(x^2-5x+4) - 2 + 2x + x - 1 = 0$$

$$\Rightarrow 2x^2 - 10x + 8 - x^3 + 5x^2 - 4x - 2 + 3x - 1 = 0$$

$$\Rightarrow -x^3 + 7x^2 - 11x + 5 = 0$$

$$\Rightarrow x^3 - 7x^2 + 11x - 5 = 0$$

$$\Rightarrow x^2(x-1) - 6x(x-1) + 5(x-1) = 0$$

$$\Rightarrow (x-1)(x^2-6x+5) = 0$$

$$\Rightarrow (x-1)(x-5)(x-1) = 0 \quad \Rightarrow x = 1, 1, 5$$

H.W.

Exercises - 13

1.i) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$

Characteristic equation is

$$\det(A - xI_3) = 0$$

$$\Rightarrow \begin{vmatrix} 1-x & 0 & 0 \\ 1 & 2-x & 1 \\ 2 & 3 & 2-x \end{vmatrix} = 0$$

$$\Rightarrow (1-x)\{(2-x)^2 - 3\} = 0$$

$$\Rightarrow (1-x)(4 - 4x + x^2 - 3) = 0$$

$$\Rightarrow (1-x)(x^2 - 4x + 1) = 0$$

$$\Rightarrow x^2 - 4x + 1 - x^3 + 4x^2 - x = 0$$

$$\Rightarrow -x^3 + 5x^2 - 5x + 1 = 0$$

$$\Rightarrow x^3 - 5x^2 + 5x - 1 = 0$$

Now, $A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{pmatrix}$

$A^3 = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 20 & 26 & 15 \\ 35 & \cancel{35}\ 45 & 26 \end{pmatrix}$

$\therefore \quad A^3 - 5A^2 + 5A - I_3$

$= \begin{pmatrix} 1 & 0 & 0 \\ 20 & 26 & 15 \\ 35 & \cancel{35}\ 45 & 26 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 5 & 10 & 5 \\ 10 & 15 & 10 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 25 & 35 & 20 \\ 45 & 60 & 35 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$

So, A satisfies its own characteristic equation.
Hence Cayley-Hamilton theorem verified.

Here $\det A = 1 \neq 0$

$\therefore A^{-1}$ exists

Now, $A^3 - 5A^2 + 5A - I_3 = O$

$$\Rightarrow I_3 = A^3 - 5A^2 + 5A$$

$$\Rightarrow A^{-1} = A^{-1}(A^3 - 5A^2 + 5A)$$

$$= A^2 - 5A + 5I_3$$

$$= \cancel{\begin{pmatrix} 1 & 0 & 0 \\ 20 & 26 & 15 \\ 35 & 45 & 26 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 5 & 10 & 5 \\ 10 & 15 & 10 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\cancel{= \begin{pmatrix} 1 & 0 & 0 \\ 15 & 16 & 10 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 5 & 10 & 5 \\ 10 & 15 & 10 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$$

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7. i) 1st Part

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Characteristic equation is

$$\det(A - xI_3) = 0$$

$$\Rightarrow \begin{vmatrix} 2-x & 0 & 0 \\ 0 & 3-x & 0 \\ 0 & 0 & 5-x \end{vmatrix} = 0$$

$$\Rightarrow (2-x)\{(3-x)(5-x) - 0\} = 0$$

$$\Rightarrow (2-x)(3-x)(5-x) = 0$$

$$\Rightarrow x = 2, 3, 5$$

ii)

1st Part

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$$

Characteristic equation is

$$\det(A - xI_3) = 0$$

$$\Rightarrow \begin{vmatrix} 1-x & -1 & 2 \\ 2 & -2-x & 4 \\ 3 & -3 & 6-x \end{vmatrix} = 0$$

$$\Rightarrow (1-x)\{(x+2)(x-6)+12\} + 1(12-2x-12) + 2(-6+6+3x) = 0$$

$$\Rightarrow (1-x)(x^2-4x-\cancel{12}+\cancel{12}) + 1(-2x) + 2(3x) = 0$$

$$\Rightarrow x^2 - 4x - x^3 + 4x^2 - 2x + 6x = 0$$

$$\Rightarrow -x^3 + 5x^2 = 0$$

$$\Rightarrow x \cdot x \cdot (-x+5) = 0$$

$$\Rightarrow x = 0, 0, 5$$

9.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$$

Characteristic equation is

~~det (x-I₃)~~

$$\det(A - xI_3) = 0$$

$$\Rightarrow \begin{vmatrix} 1-x & -1 & 0 \\ 1 & 2-x & -1 \\ 3 & 2 & -2-x \end{vmatrix} = 0$$

$$\Rightarrow (1-x)\{(x-2)(x+2)+2\} + 1(-2-x+3) = 0$$

$$\Rightarrow (1-x)(x^2-4+2) + (-x+1) = 0$$

$$\Rightarrow (1-x)(x^2-2) - x + 1 = 0$$

$$\Rightarrow x^2 - 2 - x^3 + 2x - x + 1 = 0$$

$$\Rightarrow -x^3 + x^2 + x - 1 = 0$$

$$\Rightarrow -x^2(x-1) + 1(x-1) = 0$$

$$\Rightarrow (x-1)(1-x^2) = 0$$

$$\Rightarrow (x-1)(1+x)(1-x) = 0$$

$$\Rightarrow x = 1, 1, -1$$

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* We shall show by an example that $\operatorname{adj}(A - xI_n)$ is a matrix polynomial of degree $n-1$.

Suppose, $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad A - xI_3 = \begin{bmatrix} 1-x & 0 & 1 \\ 0 & 1-x & 2 \\ 1 & 0 & -x \end{bmatrix}$

$$\operatorname{adj}(A - xI_3) = \begin{bmatrix} x^2 - x & 2 & x-1 \\ 0 & x^2 - x - 1 & 0 \\ x-1 & 2x-2 & (x-1)^2 \end{bmatrix}^T$$

$$= \begin{bmatrix} x^2 - x & 0 & x-1 \\ 2 & x^2 - x - 1 & 2x-2 \\ x-1 & 0 & x^2 - 2x + 1 \end{bmatrix}$$

$$= x^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

So, we see that here for this 3×3 order matrix A, $\operatorname{adj}(A - xI_3)$ is a matrix polynomial of degree 2.

Cayley-Hamilton theorem :-

Every square matrix satisfies its own characteristic equation.

Proof :-

Let, A be a square matrix of order n and

$$\det(A-xI_n) = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n$$

We shall prove that $c_0A^n + c_1A^{n-1} + c_2A^{n-2} + \cdots + c_{n-1}A + c_nI_n = O$

We know,

$$(A-xI_n)\,\mathrm{adj}(A-xI_n) = \det(A-xI_n)\,I_n$$

Since, $\mathrm{adj}(A-xI_n)$ is a matrix polynomial of degree $n-1$.

Let, $\mathrm{adj}(A-xI_n) = B_0x^{n-1} + B_1x^{n-2} + B_2x^{n-3} + \cdots + B_{n-2}x + B_{n-1}$

So,

$$(A-xI_n)(B_0x^{n-1} + B_1x^{n-2} + B_2x^{n-3} + \cdots + B_{n-2}x + B_{n-1})$$

$$= (c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n)I_n$$

Equating coefficients of $x^n, x^{n-1}, x^{n-2}, \cdots, x$ and constant terms from both sides we get,

$$-B_0 = c_0I_n$$

$$AB_0 - B_1 = c_1I_n$$

$$AB_1 - B_2 = c_2I_n$$

$$\cdots \quad \cdots \quad \cdots$$

$$AB_{n-2} - B_{n-1} = c_{n-1}I_n$$

$$AB_{n-1} = c_nI_n$$

Multiplying the equations by $A^n, A^{n-1}, A^{n-2}, \cdots, A$ and 1 respectively we get,

$$-A_n B_0 = C_0 A^n$$

$$A^n B_0 - A^{n-1} B_1 = C_1 A^{n-1}$$

$$A^{n-1} B_1 - A^{n-2} B_2 = C_2 A^{n-2}$$

$$\vdots$$

$$A^2 B_{n-2} - A B_{n-1} = C_{n-1} A$$

$$A B_{n-1} = C_n I_n$$

Adding we get,

$$C_0 A^n + C_1 A^{n-1} + C_2 A^{n-2} + \cdots + C_{n-1} A + C_n I_n = O$$

7.iii) 2nd Part

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \qquad \lambda = 1, 1, 5$$

$\lambda = 5$

Suppose $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 5$

$\therefore \; AX = 5X$

$\Rightarrow (A - 5I_3)X = 0$

$$\Rightarrow \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore \; -3x + 2y + z = 0$ ——— (i)

$x - 2y + z = 0$ ——— (ii)

$x + 2y - 3z = 0$ ——— (iii)

(i) + (ii) $\Rightarrow$ $-2x + 2z = 0$

(ii) + (iii) $\Rightarrow$ $2x - 2z = 0$

$\Rightarrow x = z$

$\therefore y = z$

$\therefore \; x = y = z$

So, the solution of the system of equations are

Let, $x = c$

$$\therefore \begin{pmatrix} c \\ c \\ c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad c \in \mathbb{R}$$

So, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = 5$

$\underline{\underline{\lambda = 1}}$

Suppose $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 1$

$\therefore AX = X$

$\Rightarrow (A - I_3)X = 0$

$$\therefore \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore x + 2y + z = 0$

So, the solution of the system of equ^n are

$x = c, \; y = d, \; z = -c - 2d \quad , \quad c, d \in \mathbb{R}$

$$\therefore \begin{pmatrix} c \\ d \\ -c-2d \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

So, ~~the solution of the system of eq~~

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ are two linearly independent eigen vector corresponding to $\lambda = 1$

---

* In this problem the set of all eigen vectors corresponding to $\lambda = 5$ including zero vector forms a subspace of $\mathbb{R}^3$ and the subspace is $S = \{c\alpha : c \in \mathbb{R}\}$, $\alpha = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

this subspace is called <u>eigen space</u> or <u>characteristic space</u> corresponding to $\lambda = 5$.

and the dimension of the subspace is called geometric multiplicity of λ.

Here geometric multiplicity of 5 is 1.

Also the eigen space corresponding to $\lambda = 1$ is

$$S = \{ c\alpha + d\beta : c, d \in \mathbb{R} \} \quad \text{where} \quad \alpha = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \; \beta = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

So, $c\alpha + d\beta$ is a basis of S.

$\therefore \dim S = 2$.

$\therefore$ Therefore geometric multiplicity of 1 is 2.

* **Algebraic multiplicity :-**

If λ is a root of the characteristic equation of A of multiplicity r, then λ is called r-fold eigen value and r is called algebraic multiplicity of λ.

Always, $1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity

Here algebraic multiplicity of 1 is 2 and 5 is 1.

* If for a eigen value λ, algebraic multiplicity = geometric multiplicity, then λ is called regular.

Here both 1 and 5 are regular eigen value.

* **Eigen Vector :-**

Let, A be a square matrix of order n.

A vector $X \in \mathbb{R}^n$ is said to be an eigen vector corresponding to a real number λ if $\boxed{AX = \lambda X}$

* The eigen values of a diagonal matrix are the diagonal elements of the matrix.

* The eigen values of a upper-triangular and lower-triangular matrix are diagonal elements of the matrix.

* If λ is an eigen value of A, then λ^m is an eigen value of A^m, where m is a +ve integer.

* 0 is an eigen value of A iff A is singular i.e. $\det A = 0$

* If λ is an eigen value of a non-singular matrix A, then λ^{-1} is an eigen value of A^{-1}.

* If λ is an eigen value of A, then $m\lambda$ is an eigen value of mA.

---

* If λ is an eigen value of A, then λ^2 is an eigen value of A^2.

→ Since, λ is an eigen value of A,

$$\det(A-\lambda I_n) = 0$$

So, the system of homogeneous equation $(A-\lambda I_n)X = 0$ has a non-zero solution X_1 (say)

So, $(A-\lambda I_n)X_1 = 0$

$\Rightarrow AX_1 = \lambda X_1$

Now, $A^2X_1 = A(AX_1)$

$= A\lambda X_1$

$= \lambda A X_1$

$= \lambda \cdot \lambda X_1$

$= \boxed{\lambda^2 X_1}$

$\Rightarrow (A^2 - \lambda^2 I_n)X_1 = 0$

So, the system of homogeneous equation $(A^2-\lambda^2 I_n)X = 0$ has a non-zero solution X_1

$\Longrightarrow \det(A^2-\lambda^2 I_n) = 0$

So, λ^2 is an eigen value of A^2.

|H.W.

7.1) 2nd Part

$\lambda = 2$

Suppose $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 2$

$\therefore AX = 2X$

$\Rightarrow (A - 2I_3)X = 0$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore y = 0$

$3z = 0 \Rightarrow z = 0$

Let, $x = c$

So, the solutions of the system of equation are

$$\begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c \in \mathbb{R}$$

So, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = 2$

$\lambda = 3$

Suppose, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 3$

$\therefore AX = 3X$

$\Rightarrow (A - 3I_3)X = 0$

$$\Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore -x = 0 \Rightarrow x = 0$

$2z = 0 \Rightarrow z = 0$

Let, $y = c$

So, the solutions of the system of equation are $\begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix} = c \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $c \in \mathbb{R}$

So, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = 3$

$\underline{\underline{\lambda = 5}}$

Suppose, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 5$

$\therefore \quad AX = 5X$

$\Rightarrow (A - 5I_3)X = O$

$\Rightarrow \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\therefore \quad -3x = 0 \Rightarrow x = 0$

$-2y = 0 \Rightarrow y = 0$

Let, $z = c$

So, the solutions of the system of equations are

$\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c \in \mathbb{R}$

So, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigen ~~value~~ vector corresponding to $\lambda = 5$

ii) $\underline{\underline{\lambda = 0}}$

Suppose, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen value corresponding to $\lambda = 0$

$\therefore \quad AX = OX$

$\Rightarrow AX = O$

$\Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\therefore \quad x - y + 2z = 0$

Let, $x = c, \quad z = d \quad \Rightarrow y = c + 2d$

So, the solutions of the system of equations are

$\begin{pmatrix} c \\ c+2d \\ d \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad c, d \in \mathbb{R}$

$\lambda = 5$

Suppose, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 5$

$\therefore \quad AX = 5X$

$\Rightarrow (A - 5I_3)X = 0$

$$\Rightarrow \begin{pmatrix} -4 & -1 & 2 \\ 2 & -7 & 4 \\ 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore \quad -4x - y + 2z = 0$ ——— (i)

$2x - 7y + 4z = 0$ ——— (ii)

$3x - 3y + z = 0$ ——— (iii)

(i) + 2(ii) $\Rightarrow \quad -15y + 10z = 0 \quad \Rightarrow y = \frac{2}{3}z$

~~3(i) + 4(iv) ⟹ 15y + 10~~

(i) − 2(iii) $\Rightarrow \quad -10x + 5y = 0$

$\Rightarrow 15y - 30x = 0$

$\Rightarrow y = 2x$

$\therefore$ (i) $\Rightarrow \quad -6x + 2z = 0$

(iii) $\Rightarrow \quad 3x - z = 0 \quad \Rightarrow z = 3x$

Let, $x = c \quad \Rightarrow y = 2c, \; z = 3c$

So, the solutions of the system of equations are

$$\begin{pmatrix} c \\ 2c \\ 3c \end{pmatrix} = c \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad c \in \mathbb{R}$$

So, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = 5$

9.

Remaining

$\lambda = 1$

Let, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 1$

$\therefore \quad AX = X$

$\Rightarrow (A - I_3)X = 0$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & -1 \\ 3 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore \quad y = 0$

$x - z = 0 \Rightarrow x = z$

Let, $x = c \Rightarrow z = c$

So, the system of equation's solutions are

$$\begin{pmatrix} c \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad c \in \mathbb{R}$$

Dimension of this eigen space $= 1$

$\therefore$ Geometric multiplicity of 1 is 1

Also algebraic multiplicity of 1 is 2.

$\lambda = -1$

Suppose, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to $\lambda = -1$

$\therefore \quad AX = -X$

$\Rightarrow (A + I_3)X = 0$

$$\Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore \quad 2x - y = 0 \Rightarrow y = 2x$

$x + 6x - z = 0 \Rightarrow z = 7x$

$3x + 4x - 7x = 0$ ✓

Let, $x = c$

$\therefore y = 2c,\ z = 7c$

So, the solutions of the system of equations are

$$\begin{pmatrix} c \\ 2c \\ 7c \end{pmatrix} = c\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}, \quad c \in \mathbb{R}$$

Dimension of this eigen space = 1

$\therefore$ Geometric multiplicity of -1 is 1

Also algebraic multiplicity of -1 is 1

6. A is an $n \times n$ idempotent matrix, then $A^2 = A$

Let, X be an eigen vector corresponding to λ.

Then $AX = \lambda X$

$\Rightarrow A^2X = \lambda \cdot AX$

$\Rightarrow \lambda X = \lambda \cdot \lambda X$

$\Rightarrow \lambda X = \lambda^2 X$

$\Rightarrow (\lambda - \lambda^2)X = 0$

$\Rightarrow \lambda - \lambda^2 = 0$ [$\because X$ is a non null vector]

$\Rightarrow \lambda(1-\lambda) = 0$

$\Rightarrow \lambda = 0$ or $\lambda = 1$

Proofs

* The eigen values of a diagonal matrix are the diagonal elements of the matrix.

$\rightarrow$ Let, $A = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$ be a diagonal matrix ~~order~~ of order n.