UNIT 2

Wednesday, June 14, 2023 4:46 PM

Integers

$\mathbb{N} = \{1, 2, 3, \dots\} \to$ set of Natural numbers.

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\} \to$ set of Integers.

$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\} \to$ " " rational numbers

$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ $\quad \mathbb{R} \to$ Real numbers

set of irrational numbers $\sqrt{2}, \sqrt{3}, \dots, \pi, e, \dots$

$\mathbb{C} = \{a + ib ; a, b \in \mathbb{R}, i = \sqrt{-1}\} \to$ Complex numbers.

Empty set - ϕ has no element. $\{\ \}$

⊛ Well Ordering Property :→ Every non-empty subset of natural numbers has a least element.

Ex :→ $\mathbb{N} = \{1, 2, 3, \dots\}$ $\quad S \subseteq \mathbb{N}$

$S = \{3, 5, 7, 8, 11\}$
l.e

$S = \{9, 8, 7, 6, 4, 1, 2\}$
l.e

$S \subseteq \mathbb{R}$ $\quad S = (a, b) = (1, 2)$

$S \subseteq \mathbb{N}$ $\quad S = (1, 8) = \{2, 3, 4, \dots, 7\}$

⊛ Principle of Mathematical Induction :→

A. First Principle :→

Let $P(n)$ be a statement such that

(i) $P(n)$ is true for $n = 1$

(ii) We assume $P(n)$ is true for $n = k \Rightarrow P(n)$ is true for $n = k+1$.

Then by 1st principle of mathematical induction $P(n)$ is true for all natural numbers n.

1. Show that by 1st principle of mathematical Induction :-

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}.$$

$\Rightarrow$ Let $P(n) = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$

Now, (i) for $n = 1$ $\quad$ L.H.S $= 1$ $\quad$ R.H.S $= \frac{1(1+1)}{2} = \frac{2}{2} = 1$

L.H.S = R.H.S

So, $P(n)$ is true for $n = 1$

$L.H.S = R.H.S$

So, $P(n)$ is true for $n = 1$

(ii) Let, $P(n)$ is true for $n = k$.

So, $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$

for $n = k+1$

To show: $1 + 2 + 3 + \cdots + k+1 = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$

$$L.H.S = 1 + 2 + 3 + \cdots + k+1$$
$$= \underline{1 + 2 + 3 + \cdots + k} + (k+1)$$
$$= \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$
$$= R.H.S \text{ (proved)}$$

So, $P(n)$ is true for $n = k+1$.

Therefore by principle of mathematical Induction

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad \forall n \in N.$$

2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \in N$

3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 \quad \forall n \in N$

Exercise - 3A

1.(i) $P(n) = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1 \quad \forall n \in N.$

(ii) $P(n) = 3^{2n-1} + 2^{n+1}$ is divisible 7 $\forall n \in N$.

for, $n = 1$ $\quad P(1) = 3^{2 \cdot 1 - 1} + 2^{1+1} = 3^1 + 2^2 = 3 + 4 = 7$, is div. by 7.

i.e $P(n)$ is true for $n = 1$.

$\boxed{63 = 7 \times 9}$

let, $P(n)$ is true for $n = k$

i.e $P(k) = 3^{2k-1} + 2^{k+1}$ is div. by 7

so, $3^{2k-1} + 2^{k+1} = 7 \times m \quad m \in Z$

Now, for $n = k+1$

$P(k+1) = 3^{2(k+1)-1} + 2^{(k+1)+1}$

Now, for $n = k+1$

$$P(k+1) = 3^{2(k+1)-1} + 2^{(k+1)+1}$$
$$= 3^{2k+1} + 2^{k+2}$$
$$= 3^{2k-1} \cdot 3^2 + 2^{k+2}$$
$$= (7m - 2^{k+1})\, 9 + 2^{k+2}$$
$$= 63m - 9 \cdot 2^{k+1} + 2^{k+1} \cdot 2$$
$$= 63m - 2^{k+1}(9-2)$$
$$= 7 \cdot 9m - 7 \cdot 2^{k+1};$$
$$= 7(9m - 2^{k+1})$$ is div. by 7.

as $9m - 2^{k+1}$ is an integer.

Therefore by 1st .. ..

$3^{2k-1} + 2^{k+1}$ is div by 7.

⊛ Second Principle of mathematical Induction :→

⇒ Let $P(n)$ be a statement s.t

(i) $P(n)$ is true for $n = 1$

(ii) $P(n)$ is true for $n \le k \Rightarrow P(k+1)$ is true.

Then by second principle of M.I $P(n)$ is true for $\forall n \in N$

2(ii)

$$P(n) = z^n + \frac{1}{z^n} = 2\cos n\theta \quad \forall n \in N.$$

for $n = 1$ $\quad z^1 + \frac{1}{z^1} = z + \frac{1}{z} = 2\cos\theta = 2\cos 1 \cdot \theta$

Suppose the result is true for $n \le k$

Now, for $n = k+1$

$$L.H.S = z^{k+1} + \frac{1}{z^{k+1}}$$
$$= \left(z^k + \frac{1}{z^k}\right)\left(z + \frac{1}{z}\right) - \left(z^{k-1} + \frac{1}{z^{k-1}}\right)$$
$$= 2\cos k\theta \cdot 2\cos\theta - 2\cos(k-1)\theta$$
$$= 4\cos k\theta\cos\theta - 2\cos(k-1)\theta$$
$$= 4\cos k\theta\cos\theta - 2\cos(k\theta - \theta)$$
$$= 4\cos k\theta\cos\theta - 2\left[\cos k\theta\cos\theta + \sin k\theta\sin\theta\right]$$
$$= 2\cos k\theta\cos\theta - 2\sin k\theta\sin\theta$$

$$= 4\cos k\theta \cos\theta - 2\left[\cos k\theta \cos\theta + \text{[illegible]}\right]$$

$$= 2\cos k\theta \cos\theta - 2\sin k\theta \sin\theta$$

$$= 2\left[\cos k\theta \cos\theta - \sin k\theta \sin\theta\right]$$

$$= 2\cos(k\theta + \theta) = 2\cos(k+1)\theta = R.H.S$$

Therefore by 2nd principle of M.I $P(n)$ is true $\forall\, n \in \mathbb{N}$.