Linear Algebra

Row reduced form :-

A matrix $\underline{A}$ is said to be in row-reduced form if -

i) First non-zero element of each row is 1 which is called leading element.

ii) The all other elements in the column containing leading element are zero.

Example :-

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 2 & 7 & 9 \\ 1 & 0 & 0 & 3 & 4 & 6 & 1 \\ 0 & 1 & 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{row reduced but not echelon}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{pmatrix} \Rightarrow \text{row reduced but not echelon}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{row reduced echelon form}$$

Elementary row operation :-

To transform a given matrix into row-reduced form or row reduced echelon form, we perform the following three types of elementary row operations —

i) Multiplication of a row by a non-zero real number (cR_i)

ii) Adding $\underline{c}$ times $\underline{j}$th row to $\underline{i}$th row $(R_i + cR_j)$

iii) Interchange of two rows (R_{ij})

* Row reduce →

$$A = \begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow[R_4 - 3R_2]{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_4} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 - 2R_4} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= R$$

which is row reduced form

Here matrix $\underline{R}$ is called row equivalent to $\underline{A}$. Here number of non-zero rows of R is $\underline{3}$. So $\underline{\text{rank of}}$ $\underline{A}$ is $\underline{3}$.

* Solve the following system of linear equations :

$$3x_1 + 5x_2 - 4x_3 = 7$$
$$-3x_1 - 2x_2 + 4x_3 = -1$$
$$6x_1 + x_2 - 8x_3 = -4$$

→ Here the system of linear equations can be written as

$AX = B$

where $A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$

Here A_b (augmented) $= \begin{pmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix}$

$$\xrightarrow{\frac{1}{3}R_1} \begin{pmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{7}{3} \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix} \xrightarrow[R_3 - 6R_1]{R_2 + 3R_1} \begin{pmatrix} 1 & 5/3 & -4/3 & 7/3 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \begin{pmatrix} 1 & 5/3 & -4/3 & 7/3 \\ 0 & 1 & 0 & 2 \\ 0 & -9 & 0 & -18 \end{pmatrix} \xrightarrow[R_3 + 9R_2]{R_1 - \frac{5}{3}R_2} \begin{pmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here Rank A = 2 & Rank A_b = 2

∴ Rank of A = Rank of A_b = 2 < 3 (number of variables)

So the system of equations has infinitely many solutions.

So, the system of equations can be written as

$$1 \cdot x_1 + 0 \cdot x_2 - \frac{4}{3}x_3 = -1$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 = 2$$

$$\therefore x_1 - \frac{4}{3}x_3 = -1$$
$$x_2 = 2$$

Suppose, $x_1 = c$

$$\therefore \frac{4}{3}x_3 = c + 1$$
$$\Rightarrow x_3 = \frac{3c}{4} + \frac{1}{4}$$

Therefore the solutions are

$$\left(c, 2, \frac{3c}{4} + \frac{3}{4}\right), \quad c \in \mathbb{R}$$

$$= \left(0, 2, \frac{3}{4}\right) + c\left(1, 0, \frac{3}{4}\right)$$

* ii) If Rank of A = Rank of A_b = number of variables, then the system of equations has unique solution.

i) If Rank of A = Rank of A_b = $\rho < n$ (number of variables), then the system of equations has infinitely many solutions; where number of free variables is $n-\rho$.

iii) If Rank of $A \neq$ Rank of A_b, then the system of equations has no solution. In this case the system is called inconsistent.

Abstract and Linear Algebra (S.K. Mapa) Exercises - 10

7. i)

$$x + y + z = 1$$
$$2x + 3y - z = a + 1$$
$$2x + y + 5z = a^2 + 1$$

Here, $A_b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & a+1 \\ 2 & 1 & 5 & a^2+1 \end{pmatrix}$

$$\xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & a-1 \\ 0 & -1 & 3 & a^2-1 \end{pmatrix} \xrightarrow[R_3 + R_2]{R_1 - R_2} \begin{pmatrix} 1 & 0 & 4 & 2-a \\ 0 & 1 & -3 & a-1 \\ 0 & 0 & 0 & a^2+a-2 \end{pmatrix}$$

Here rank of $A = 2$

The system will be consistent if rank of $A_b = 2$ which is possible only when

$$a^2 + a - 2 = 0$$
$$\Rightarrow a = 1 \text{ or } -2$$

Case 1: When $a = 1$

In this case the system of equations can be written as

$$x + 4z = 1$$
$$y - 3z = 0$$

Suppose, $z = c$

$\therefore$ $x = 1 - 4c$

$y = 3c$

$\therefore$ Solutions are $(1-4c, 3c, c)$, $c \in \mathbb{R}$

Case 2. When $a = -2$

In this case the system of equations can be written as

$$x + 4z = 4$$
$$y - 3z = -3$$

Suppose, $z = c$

$\therefore$ $x = 4 - 4c$

$y = 3c - 3$

$\therefore$ Solutions are $(4-4c, 3c-3, c)$ $c \in \mathbb{R}$

$= c(-4, 3, 1) + (4, -3, 0)$

H.W.

* Row reduce →

i) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix}$

$$\xrightarrow[R_3 - 6R_1]{R_2 - 4R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

$$\xrightarrow[R_3 + 5R_2]{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$, which is now reduced form

ii) $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix}$

$$\xrightarrow[R_3 - 5R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{pmatrix} \xrightarrow{-\frac{1}{4}R_2} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$\xrightarrow[R_3 + 8R_2]{R_1 - 3R_2} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{pmatrix} \xrightarrow{-\frac{1}{10}R_3} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 - 3R_3]{R_1 + 2R_3} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$, which is now reduced form

6. i)

$x+y+z=1$

$2x+y+2z=1$

$x+2y+3z=0$

Here the system of linear equations can be written as

$AX=B$

where $A=\begin{pmatrix}1&1&1\\2&1&2\\1&2&3\end{pmatrix}$, $X=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $B=\begin{pmatrix}1\\1\\0\end{pmatrix}$

Here $A_b=\begin{pmatrix}1&1&1&1\\2&1&2&1\\1&2&3&0\end{pmatrix}$

$$\xrightarrow[R_3-R_1]{R_2-2R_1}\begin{pmatrix}1&1&1&1\\0&-1&0&-1\\0&1&2&-1\end{pmatrix}\xrightarrow{-R_2}\begin{pmatrix}1&1&1&1\\0&1&0&1\\0&1&2&-1\end{pmatrix}$$

$$\xrightarrow[R_3-R_2]{R_1-R_2}\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&2&-2\end{pmatrix}\xrightarrow{\frac{1}{2}R_3}\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&-1\end{pmatrix}$$

$$\xrightarrow{R_1-R_3}\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\end{pmatrix}$$

Here rank of $A=3$ and rank of $A_b=3$

$\therefore$ Rank of A = Rank of A_b = 3 (number of variables)

So, the system of equations has unique solution

So, the system of equations can be written as

$x=1$

$y=1$

$z=-1$

Therefore the solutions are $(1,1,-1)$

ii)

$x+y+z=1$

$2x+y+2z=2$

$3x+2y+3z=5$

Here the system of linear equations can be written as

$AX=B$

where $A=\begin{pmatrix}1&1&1\\2&1&2\\3&2&3\end{pmatrix}$, $X=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $B=\begin{pmatrix}1\\2\\5\end{pmatrix}$

Here $A_b=\begin{pmatrix}1&1&1&1\\2&1&2&2\\3&2&3&5\end{pmatrix}$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1}\begin{pmatrix}1&1&1&1\\0&-1&0&0\\0&-1&0&2\end{pmatrix}\xrightarrow{-R_2}\begin{pmatrix}1&1&1&1\\0&1&0&0\\0&-1&0&2\end{pmatrix}$$

$$\xrightarrow[R_3+R_2]{R_1-R_2}\begin{pmatrix}1&0&1&1\\0&1&0&0\\0&0&0&2\end{pmatrix}\xrightarrow{\frac{1}{2}R_3}\begin{pmatrix}1&0&1&1\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

$$\xrightarrow{R_1-R_3}\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

Here rank of $A=2$ and rank of $A_b=3$

$\therefore$ Rank of $A\neq$ rank of A_b.

So, the system of equations has no solution.

7. i)

$x-y+z=1$

$x+2y+4z=a$

$x+4y+6z=a^2$

Here $A_b=\begin{pmatrix}1&-1&1&1\\1&2&4&a\\1&4&6&a^2\end{pmatrix}$

$$\xrightarrow[R_3-R_1]{R_2-R_1}\begin{pmatrix}1&-1&1&1\\0&3&3&a-1\\0&5&5&a^2-1\end{pmatrix}\xrightarrow{\frac{1}{3}R_2}\begin{pmatrix}1&-1&1&1\\0&1&1&\frac{a-1}{3}\\0&5&5&a^2-1\end{pmatrix}$$

$$\xrightarrow[R_3-5R_2]{R_1+R_2}\begin{pmatrix}1&0&2&1+\frac{a-1}{3}\\0&1&1&(a-1)/3\\0&0&0&a^2-1-\left(\frac{5a-5}{3}\right)\end{pmatrix}$$

Here rank of $A=2$

The system will be consistent if rank of $A_b=2$

which is possible only when

$$a^2-1-\left(\frac{5a-5}{3}\right)=0$$

$\Rightarrow 3a^2-3a-5a+5=0$

$\Rightarrow 3a(a-1)-5(a-1)=0$

$\Rightarrow a=1$ or, $a=\frac{5}{3}$

$\Rightarrow 3a^2 - 3 - 5a + 5 = 0$

$\Rightarrow 3a^2 - 5a + 2 = 0$

$\Rightarrow 3a^2 - 3a - 2a + 2 = 0$

$\Rightarrow 3a(a-1) - 2(a-1) = 0$

$\Rightarrow a = \frac{2}{3}$ or $a = 1$

<u>Case 1</u>: When $a = \frac{2}{3}$

In this case the system of equations can be written as

$$x + 2z = \cancel{0}\ 8/9$$
$$y + z = -1/9$$

Suppose, $z = c$

$\therefore x = -2c + 8/9$

$y = -\frac{1}{9} - c$

$\therefore$ Solutions are $\left(-2c + \frac{8}{9}, -\frac{1}{9} - c, c\right)$, $c \in \mathbb{R}$

$$= c(-2,-1,1) + \left(\frac{8}{9}, -\frac{1}{9}, 0\right)$$

<u>Case 2</u>: When $a = 1$

In this case the system of equations can be written as

$$x + 2z = 1$$
$$y + z = 0$$

Suppose, $z = c$

$\therefore x = 1 - 2c$

$y = -c$

$\therefore$ Solutions are $(1-2c, -c, c)$, $c \in \mathbb{R}$

$$= c(-2,-1,1) + (1,0,0)$$

9.i) $x + 2y + z = 1$

$2x + y + 3z = b$

$x + ay + 3z = b+1$

Here the system of linear equations can be written as $AX = B$

where $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & a & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ b \\ b+1 \end{pmatrix}$

Here $A_b = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & b \\ 1 & a & 3 & b+1 \end{pmatrix}$

$$\xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & 1 & b-2 \\ 0 & a-2 & 2 & b \end{pmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} & \frac{2-b}{3} \\ 0 & a-2 & 2 & b \end{pmatrix}$$

$$\xrightarrow[R_3 - (a-2)R_2]{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 5/3 & \frac{2b-1}{3} \\ 0 & 1 & -1/3 & \frac{2-b}{3} \\ 0 & 0 & \frac{a+4}{3} & b - \frac{(a-2)(2-b)}{3} \end{pmatrix}$$

a) The system of equations has only one solution when

$\frac{a+4}{3} \neq 0 \Rightarrow a + 4 \neq 0 \Rightarrow a \neq -4$

b) The system of equations has no solution if

$\frac{a+4}{3} = 0$ and $b - \frac{(a-2)(2-b)}{3} \neq 0$

$\Rightarrow a = -4$ $\qquad \Rightarrow \frac{3b - 2a + ab + 4 - 2b}{3} \neq 0$

$\Rightarrow 3b - \underset{2(-4)}{\cancel{2a}} + (-4)\cdot b + 4 - 2b \neq 0$

$\Rightarrow -3b + 12 \neq 0$

$\Rightarrow b \neq 4$

$\therefore a = -4, b \neq 4$

c) The system of equations has many solution when

$\frac{a+4}{3} = 0$ and $b - \frac{(a-2)(2-b)}{3} = 0$

$\Rightarrow a = -4, b = 4$

ii) $x + y + z = b$

$2x + y + 3z = b+1$

$5x + 2y + az = b^2$

Here the system of linear equations can be written as $AX = B$

where $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 5 & 2 & a \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} b \\ b+1 \\ b^2 \end{pmatrix}$

Here $A_b = \begin{pmatrix} 1 & 1 & 1 & b \\ 2 & 1 & 3 & b+1 \\ 5 & 2 & a & b^2 \end{pmatrix}$

$$\xrightarrow[R_3 - 5R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & b \\ 0 & -1 & 1 & 1-b \\ 0 & -3 & a-5 & b^2-5b \end{pmatrix} \xrightarrow{-R_2} \begin{pmatrix} 1 & 1 & 1 & b \\ 0 & 1 & -1 & b-1 \\ 0 & -3 & a-5 & b^2-5b \end{pmatrix}$$

$$\xrightarrow[R_3 + 3R_2]{R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & b-1 \\ 0 & 0 & a-8 & b^2-2b-3 \end{pmatrix}$$

a) The system of equations has only one solution when

$a - 8 \neq 0$

$\Rightarrow a \neq 8$

b) The system of equations has no solution when

$a - 8 = 0$ and $b^2 - 2b - 3 \neq 0$

$\Rightarrow a = 8$ $\quad\Rightarrow b^2 - 3b + b - 3 \neq 0$

$\Rightarrow b(b-3) + 1(b-3) \neq 0$

$\Rightarrow (b-3)(b+1) \neq 0$

$\therefore a = 8, \ (b-3)(b+1) \neq 0$

c) The system of equations has many solutions when

$a - 8 = 0$ and $b^2 - 2b - 3 = 0$

$\Rightarrow a = 8$ $\quad\Rightarrow (b-3)(b+1) = 0$

$\Rightarrow b = 3$ or $b = -1$

$\therefore a = 8, b = 3$ or $a = 8, b = -1$

* Elementary matrix :-

$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Elementary matrix $\quad$ (A)

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Identity matrix $\quad$ Elementary matrix

$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ $\xleftarrow{R_2 - R_1}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Elementary matrix $\quad$ (A)

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Identity matrix $\quad$ Elementary matrix

$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$ $\xleftarrow{2R_3}$

* $R_i + cR_j \Longrightarrow E_{ij}(c) \qquad E_{ij}^{-1}(c) = E_{ij}(-c)$

$cR_i \Longrightarrow E_i(c) \qquad E_i^{-1}(c) = E_i(\frac{1}{c})$

$R_{ij} \Longrightarrow E_{ij} \qquad E_{ij}^{-1} = E_{ij}$

* Write the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ as a product of elementary matrices

$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_{13}(-1)E_{32}(-1)E_{31}(-1)E_{21}(-1)A = I_3$$

$$\Rightarrow A = \left\{E_{13}(-1)E_{32}(-1)E_{31}(-1)E_{21}(-1)\right\}^{-1}$$

$$= E_{21}^{-1}(-1)E_{31}^{-1}(-1)E_{32}^{-1}(-1)E_{13}^{-1}(-1) \quad \left[\because (ABC)^{-1} = C^{-1}B^{-1}A^{-1}\right]$$

$$\Rightarrow A = E_{21}(1)E_{31}(1)E_{32}(1)E_{13}(1)$$

Exercises 4

10. iii) The problem is done above.

* Inverse determination

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Exercises 10

8. ii)

$x + 2y + 3z = kx$ or, $(1-k)x + 2y + 3z = 0$

$2x + y + 3z = ky$ or, $2x + (1-k)y + 3z = 0$

$2x + 3y + z = kz$ or, $2x + 3y + (1-k)z = 0$

Here the system of equations can be written as $AX = B$

where $A = \begin{pmatrix} 1-k & 2 & 3 \\ 2 & 1-k & 3 \\ 2 & 3 & 1-k \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Here the system of equations is homogeneous.

So, $(0,0,0)$ is a solution which is a trivial solution.

The system has non-trivial solution iff rank $A < 3$

i.e. $\det(A) = 0$

Now, $\det(A) = \begin{vmatrix} 1-k & 2 & 3 \\ 2 & 1-k & 3 \\ 2 & 3 & 1-k \end{vmatrix} = 0$

$$\Rightarrow (1-k)\{(1-k)^2 - 9\} - 2(2-2k-6) + 3(6-2+2k) = 0$$

$$\Rightarrow (1-k)(1-2k+k^2-9) - 2(-2k-4) + 3(2k+4) = 0$$

$$\Rightarrow (1-k)(k^2-2k-8) + 2(2k+4) + 3(2k+4) = 0$$

$$\Rightarrow k^2 - 2k - 8 - k^3 + 2k^2 + 8k + 4k + 8 + 6k + 12 = 0$$

$$\Rightarrow -k^3 + 3k^2 + 16k + 12 = 0$$

$$\Rightarrow k^3 - 3k^2 - 16k - 12 = 0$$

$$\Rightarrow k^2(k+1) - \cancel{2}4k(k+1) - 12(k+1) = 0$$

$$\Rightarrow (k+1)(k^2-4k-12) = 0$$

$$\Rightarrow (k+1)\{k(k+2) - 6(k+2)\} = 0$$

$$\Rightarrow (k+1)(k+2)(k-6) = 0$$

$$\Rightarrow k = \underline{-1} \text{ or } \underline{-2} \text{ or } \underline{6}$$

Case 1:

When $k = 6$.

$$A = \begin{pmatrix} -5 & 2 & 3 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{pmatrix}$$

$$\xrightarrow{R_1 + R_2 + R_3} \begin{pmatrix} -1 & 0 & 1 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{pmatrix} \xrightarrow{-R_1} \begin{pmatrix} 1 & 0 & -1 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{pmatrix}$$

$$\xrightarrow[R_3-2R_1]{R_2-2R_1}\begin{pmatrix}1&0&-1\\0&-5&5\\0&3&-3\end{pmatrix}\xrightarrow{-\frac{1}{5}R_2}\begin{pmatrix}1&0&-1\\0&1&-1\\0&3&-3\end{pmatrix}$$

$$\xrightarrow{R_3-3R_2}\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}$$

The system of equations can be written as

$$x-z=0 \quad \Rightarrow x=z$$
$$y-z=0 \quad \Rightarrow y=z$$
$$\therefore x=y=z$$

Suppose, $z=c$

$\therefore$ Solutions are $(c,c,c)=\underline{c(1,1,1)}$, $c\in\mathbb{R}$

Case 2: When $k=-1$

$$A=\begin{pmatrix}2&2&3\\2&2&3\\2&3&2\end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1}\begin{pmatrix}1&1&3/2\\2&2&3\\2&3&2\end{pmatrix}$$

$$\xrightarrow[R_3-2R_1]{R_2-2R_1}\begin{pmatrix}1&1&3/2\\0&0&0\\0&1&-1\end{pmatrix}\xrightarrow{R_1-R_3}\begin{pmatrix}1&0&5/2\\0&0&0\\0&1&-1\end{pmatrix}$$

The system of equations can be written as

$$x+5/2\,z=0$$
$$y-z=0$$

Suppose, $z=c \quad \therefore x=-\frac{5}{2}c,\ y=c$

$\therefore$ Solutions are $\left(-\frac{5}{2}c,\ c,\ c\right)$

$$= \cancel{c2\ -5c(}$$

$$= -\frac{1}{2}c\,(5,-2,-2)$$

$$= \underline{c_1(5,-2,-2)},\quad c_1\in\mathbb{R}$$

Case 3: When $k=-2$,

$$A=\begin{pmatrix}3&2&3\\2&3&3\\2&3&3\end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_1} \begin{pmatrix} 1 & 2/3 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 2/3 & 1 \\ 0 & 5/3 & 1 \\ 0 & 5/3 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{3}{5}R_2} \begin{pmatrix} 1 & 2/3 & 1 \\ 0 & 1 & 3/5 \\ 0 & 5/3 & 1 \end{pmatrix} \xrightarrow[R_3 - \frac{5}{3}R_2]{R_1 - \frac{2}{3}R_2} \begin{pmatrix} 1 & 0 & 3/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{pmatrix}$$

The system of equations can be written as

$$x + \frac{3}{5}z = 0$$

$$y + \frac{3}{5}z = 0$$

Suppose, $z = c$ $\quad \therefore x = -\frac{3}{5}c$, $y = -\frac{3}{5}c$

$\therefore$ Solutions are $\left(-\frac{3}{5}c, -\frac{3}{5}c, c\right)$

$= -\frac{1c}{5}(3, 3, -5)$

$= \underline{c_2\,(3, 3, -5)}$ $\quad c_2 \in \mathbb{R}$