## Course Title : Algebra (Old)

### Unit–I

1. **Answer any *five* question :** 2×5=10

(a) Find the remainder when $11^{40}$ is divided by 8.

(b) Let $a, b, c$ be all positive real numbers. Then prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \geq a+b+c.$$

(c) Let $f: \mathbb{Z} \to E_+^0$ be defined by $f(x)=|x|+x \ \forall x \in \mathbb{Z}$ and $E_+^0$ be the set of all nonnegative even integers. Let $g: E_+^0 \to \mathbb{Z}$ be defined by $g(x)=\frac{x}{2} \ \forall x \in E_+^0$.

Check whether $g$ is inverse of $f$.

(d) Check whether $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(n)=4n-5$ $\forall n \in \mathbb{Z}$ is bijective.

(e) If $a_1, a_2, \ldots, b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_n$ are all positive real numbers then prove that $(a_1 b_1 c_1 + a_2 b_2 c_2 + \cdots + a_n b_n c_n)^2 < \left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2\right)\left(c_1^2+c_2^2+\cdots+c_n^2\right)$.

(f) Solve the equation $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

(g) Apply Descartes' rule of signs to examine the nature of the roots of the equation $x^4 + 2x^2 + 3x - 1 = 0$.

(h) Use Cayley–Hamilton theorem to compute $A^{-1}$ where

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}.$$

## Unit–II

2. Answer **any *four*** questions : 5×4=20

(a) (i) Let $f : A \to B$ and $g : B \to C$ be both bijections. Prove that $g \circ f : A \to C$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. 3

(ii) Suppose $f$ and $g$ are two functions from $\mathbb{R}$ into $\mathbb{R}$ such that $f \circ g = g \circ f$. Does it necessarily imply that $f = g$? Justify your answer. 2

(b) (i) Find gcd of 615 and 1080 and find integers $s$ and $t$ such that gcd $(615, 1080) = 615\ s + 1080\ t$. 2

(ii) What is the remainder when $1! + 2! + 3! + \cdots + 99! + 100!$ is divided by 18? 3

(c) A relation $\rho$ is defined on $\mathbb{Z}$ by "$x\ \rho\ y$ if and only if $x^2 - y^2$ is divisible by 5" for $x, y, \in \mathbb{Z}$. Prove that $\rho$ is an equivalence relation on $\mathbb{Z}$. Show that there are three distinct equivalence classes. 5

(d) (i) Find the principal value of $(1 + i)^i$. 2

(ii) Find the general solution of $\cos z = 2$. 3

(e) Find the equation whose roots are the roots of $x^4 - 8x^2 + 8x + 6 = 0$, each diminished by 2. 5

(f) Show that eigen values of a real symmetric matrix are all real. 5

## Unit–III

3. Answer **any *one*** question : 10×1=10

(a) (i) If $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$, show that $A^2 - 10A + 16I_3 = 0$.

Hence obtain $A^{-1}$. 3

(ii) Find all real $x$ for which the rank of the matrix is less than 4. 2

$$\begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix}$$

(iii) Determine the conditions for which the system

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

admits of (i) only one solution, (ii) no solution, (iii) infinitely many solution. 5

(b) (i) A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3,\ x^2 + 4x_3,\ x_1 - x_2 + 3x_3),$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$.

Find the matrix to $T$ relative to the ordered basis ((0,0,1), (1,0,0), (0,1,0)) of $\mathbb{R}^3$. 3

(ii) Check whether the relations $p$ on $\mathbb{Z}$ defined by "$x\ p\ y$ *if f* $x \mid y$" is antisymmetric. 2

(iii) Find whether the following is bijective; if so, find their inverse.

Let $S = \{x \in \mathbb{R} : -1 < x < 1\}$ and $f : \mathbb{R} \to S$ defined by

$$f(x) = \frac{x}{1 + |x|}.$$ 5

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