Q1.

1) $$KE = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$PE = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_1 - x_2)^2$$

2) $$\frac{d}{dt}\left(\frac{\partial L}{\partial x_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial L}{\partial x_1} = -k_1 x_1 - k_2 (x_1 - x_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_2}\right) = m_2 \ddot{x}_2$$

$$\frac{\partial L}{\partial x_2} = k_2 (x_1 - x_2)$$

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2 (x_1 - x_2)$$

$$m_2 \ddot{x}_2 = k_2(x_1 - x_2)$$

$$\Rightarrow \quad m_1 \ddot{x}_1 = -(k_1+k_2)x_1 + k_2 x_2$$

$$m_2 \ddot{x}_2 = k_2 x_1 - k_2 x_2$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{pmatrix}$$

3) For eigen frequencies

$$\begin{vmatrix} -\frac{k_1+k_2}{m_1} - \lambda & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} - \lambda \end{vmatrix} = 0$$

$$\left(\lambda + \frac{k_1+k_2}{m_1}\right)\left(\lambda + \frac{k_2}{m_2}\right) - \frac{k_2^2}{m_1 m_2} = 0$$

$$\lambda^2 + \left(\frac{K_1 + K_2}{m_1} + \frac{K_2}{m_2}\right)\lambda + \frac{K_1 K_2}{m_1 m_2} = 0$$

$$\lambda^2 + \frac{K_1 m_2 + K_2(m_1+m_2)}{m_1 m_2} - \frac{K_1 K_2}{m_1 m_2} = 0$$

$$\lambda_{12} = -\frac{K_1 m_1 + K_2(m_1+m_2)}{2 m_1 m_2} \pm \sqrt{\left[\frac{K_1 m_1 + K_2(m_1+m_2)}{2 m_1 m_2}\right]^2 + \frac{K_1 K_2}{m_1 m_2}}$$

Q2.

1) $$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgl(1-\cos\theta)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

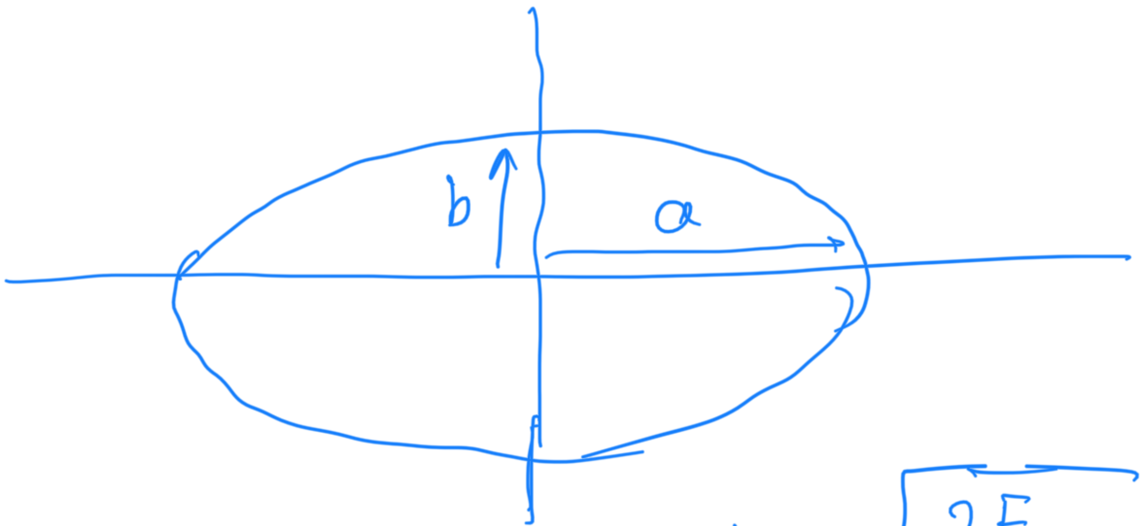
2) $$H(p_\theta, \theta) = \frac{p_\theta^2}{2mr^2} + mgl(1-\cos\theta)$$

3) For small θ $\qquad \cos\theta \simeq 1 - \frac{\theta^2}{2}$

p^2 $\qquad \theta^2$

$$\mathcal{H}(p_\theta, \theta) = \frac{p_\theta^2}{2mr^2} + mgl\,\frac{\theta^2}{2} = E$$

Equation of phase trajectory for constant energy E is an ellipse



$$a = \sqrt{2mr^2 E} \qquad b = \sqrt{\frac{2E}{mgl}}$$

Q3. 1) The Lagrangian

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\alpha}{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

2) The Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{\alpha}{r}$$

Equation of motion

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \qquad p_\theta = A$$

Total energy

$$E = \frac{p_r^2}{2m} + \frac{A}{2mr^2} - \alpha/r$$

$$= \frac{p_r^2}{2m} + V_{eff}(r)$$

For bound orbit

$$E < 0 \Rightarrow \frac{p_r^2}{2m} + \frac{A}{2mr} - \alpha/r < 0$$

2) For circular orbit

$$\frac{\partial V_{eff}}{} =$$

$$\frac{[illegible]}{2r} = 0$$

$$\Rightarrow \quad -\frac{A}{m r_0^3} + \frac{\alpha}{r_0^2} = 0$$

$$\Rightarrow \quad r_0 = \frac{A}{m\alpha}$$

For circular orbit $p_r = 0$

$$E = \frac{A}{2 m r_0^2} - \frac{\alpha}{r_0}$$

$$= \frac{A}{2m\left(A/m\alpha\right)^2} - \alpha \, \frac{m\alpha}{A}$$

$$= \frac{A}{2m} \, \frac{m^2\alpha^2}{A^2} - \frac{m\alpha^2}{A}$$

$$= \frac{1}{2} \, \frac{m\alpha^2}{A} - \frac{m\alpha^2}{A} = -\frac{m\alpha^2}{2A}$$

3) For an arbitrary energy $-E$

$$\frac{A}{2mr^2} - \alpha/r = -E$$

$$\frac{A}{2m} - \alpha r + Er^2 = 0$$

$$r^2 - \frac{\alpha}{E} r + \frac{A}{2mE} = 0$$

$$r = \frac{\alpha/E \pm \sqrt{(\alpha/E)^2 - \frac{2A}{mE}}}{2}$$

$$r_{min} = \frac{\alpha}{2E} - \frac{1}{2}\left[(\alpha/E)^2 - \frac{2A}{mE}\right]^{1/2}$$

$$r_{max} = \frac{\alpha}{2E} + \frac{1}{2}\left[(\alpha/E)^2 - \frac{2A}{mE}\right]^{1/2}$$