Operators

$$\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\boxed{\text{Calculate } \langle \sigma_2 \rangle \text{ in the eigenstates of } \sigma_1}$

Eigenvalues and eigenvectors of σ_1

$$\lambda = \pm 1$$

For $\lambda = 1$

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-x_2 - i x_1 = 0 \implies x_2 = -i x_1$$

$$|+\rangle \qquad \begin{pmatrix} 1 \\ \end{pmatrix} \qquad \langle + | = \begin{pmatrix} 1 & i \end{pmatrix}$$

$|\phi_1\rangle = \begin{pmatrix} \\ -i \end{pmatrix} \qquad \langle \phi_1| = (\quad)$

$\langle \phi_1 | \phi_1 \rangle = 2 \Rightarrow |\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

For $\lambda = -1 \quad |\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\langle \phi_1 | \phi_2 \rangle = \frac{1}{2} \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 0$

The orthonormal basis for σ_1

$\lambda_1 = 1, \ |\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\lambda_1 = -1 \ |\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

For, the orthonormal basis for σ_2

$\lambda_1 = 1 \quad |\chi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_1 = -1 \quad |\chi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The orthonormal basis for σ_3

$\lambda_1 = 1 \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

$\lambda_1 = -1 \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

To calculate average of σ_1 in the eigenstates of σ_2

$|\chi_1\rangle$ and $|\chi_2\rangle$ should be written as a linear combination of $|\phi_1\rangle$ and $|\phi_2\rangle$

$|\chi_1\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$

$a_1 = \langle\phi_1|\chi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$= \frac{1}{\sqrt{2}}$

$a_2 = \langle\phi_2|\chi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$= -\frac{1}{\sqrt{2}} i$$

$$\Rightarrow |\psi_1\rangle = \frac{1}{\sqrt{2}} |\phi_1\rangle - \frac{1}{\sqrt{2}} i \, |\phi_2\rangle$$

The average of σ_1

$$= |a_1|^2 \lambda_1 + |a_2|^2 \lambda_2$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

One can also get the result by

$$\langle \phi_1 | \sigma_1 | \phi_1 \rangle$$

$$= \frac{1}{2} \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ +i \end{pmatrix} = 0$$

Time evolution and stationary states

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle$$

$$\Psi(0) = \sum a_n |\phi_n\rangle$$

$$|\Psi(t)\rangle = \sum_n a_n e^{-iE_n t} |\phi_n\rangle$$

$$\langle \phi_n | \Psi(t)\rangle = a_n e^{-iE_n t}$$

$$|\langle \phi_n | \Psi(t)\rangle|^2 = |a_n|^2$$

The probability distribution doesn't change with time

If $\Psi(0) = |\phi_n\rangle$

$$\langle \Psi(t) | A | \Psi(t)\rangle$$

$$= \left\langle \sum_n a_n^* e^{+iE_nt} \Psi(0) \middle| \sum_n a_n e^{iE_nt} \Psi(0) \right\rangle$$

If $\Psi(0) = |\phi_n\rangle$

$$\langle \Psi(t) | A | \Psi(t) \rangle = \langle \phi_n(0) | A \, \phi_n(0) \rangle$$

At stationary state all the observables expectation values remain the same.

Example

$$|\Psi(0)\rangle = a_1 |\phi_1\rangle + a_2 |\phi_2\rangle$$

$$|\Psi(t)\rangle = \left(a_1 e^{-iE_1t} |\phi_1\rangle + a_2 e^{-iE_1t} |\phi_2\rangle \right)$$

$$\langle \Psi(t) | x | \Psi(t) \rangle = a_1 e^{-iE_1t}$$

$$\langle x|\psi(t)\rangle = \left[a_1 e^{-iE_1 t}\phi_1(x) + a_2 e^{iE_1 t}\phi_2(x)\right]$$

$$= \left[a_1\phi_1(x) + a_2\phi_2(x)\right]\cos\omega t + i\left[a_1\phi_1(x) - a_2\phi_2(x)\right]\sin\omega t$$

$$|\psi(x,t)|^2 =$$

Free particle Hamiltonian

$$H = \frac{p^2}{2m} \Rightarrow H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x}$$

$$H\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2} = -k^2\psi$$

$$\left(e^{+ikx} \qquad e^{-ikx}\right)$$

$$\psi(x) = \left(A e \quad + \quad B e \quad\right)$$

$$\langle e^{ikx} | e^{ik'x} \rangle = \delta(k - k')$$

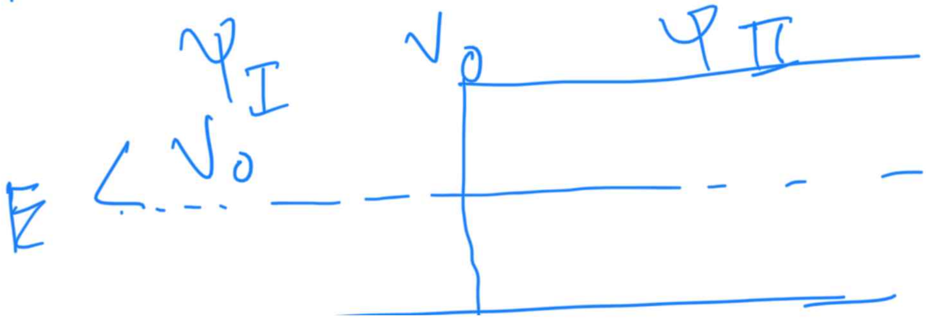
With a potential

$$H = \frac{p^2}{2m} + V_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V_0 \psi = E\psi$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi \qquad k^2 = \frac{2m}{\hbar^2}(E - V_0)$$

Energy eigenstates for a potential barrier.



The Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x)$$

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases}$$

For region I

$$H \psi_I = E \psi_I$$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_I}{\partial x^2} = E \psi_I$$

$$\Rightarrow \psi_I(x) = A e^{ik_1 x} + B e^{-ik_1 x}$$

$$k_1^2 = \frac{2mE}{\hbar^2}$$

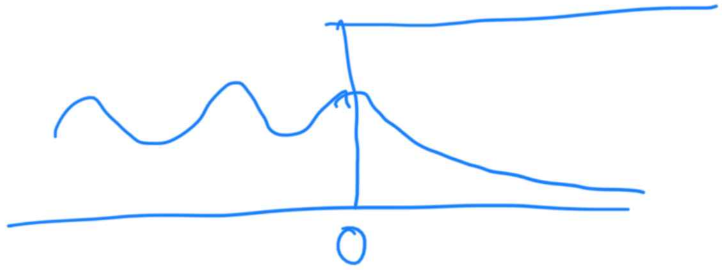
Similarly

$-k_1 x$

$$\psi_{II}(x) = C e$$

$$K_2^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

The general soluten



At $x=0$ the boundary conditions are.

$$\psi_I(0) = \psi_{II}(0)$$

$$\psi_I'(0) = \psi_{II}'(0)$$

$$\Rightarrow \quad A + B = C$$

$$iK_1(A - B) = -K_2 C$$

The incoming wave toward

the barrier will get reflected. The wave will penetrate through the barrier to some distance due to quantum effect

when $E > V_0$ from the same equation

$$\psi_I(x) = Ae^{ik_1 x} + Be^{-ik_1 x}$$

$$\psi_{II}(x) = Ce^{ik_2 x} \qquad k_2^2 = \frac{2m(E-V_0)}{\hbar^2}$$

Boundary conditions

$$\psi_I(0) = \psi_{II}(0)$$

$$\psi_I'(0) = \psi_{II}'(0)$$

$$\Rightarrow \quad A + B = C$$

$$ik_1 (A - B) = ik_2 C$$

Solving for A, B

$$A = \frac{C}{2}\left(1 + \frac{k_2}{k_1}\right) \qquad B = \frac{C}{2}\left(1 - \frac{k_2}{k_1}\right)$$

The reflection probability

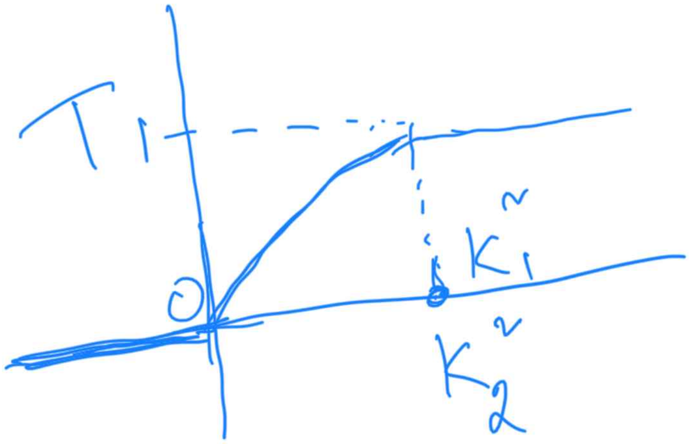
$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

Transmission probability

$$T = 1 - R = \frac{4 k_1 k_2}{(k_1 + k_2)^2}$$

$\Rightarrow$ as k_2 ~~increases~~ T increases

k_2 increases when V_0 reduces



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