Orthonormal basis for eigenstates of Hamiltonean

$$|\phi_n\rangle = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

$$E_n = \frac{\hbar^2}{2m} \frac{n^2\pi^2}{a^2}$$

$$\langle \phi_n | \phi_m \rangle = \delta_{nm}$$

The eigenstates in possition basis

$$|\phi_n\rangle = \int dx \; a_n \, |x\rangle$$

$$a_n = \langle x | \phi_n \rangle = \phi_n(x)$$

The averge of $\hat{x}$

$$\langle \hat{x} \rangle = \int_0^a x \, |\phi_n(x)|^2 \, dx$$

$$= \frac{2}{a} \int_0^a x \sin^2 \frac{n\pi x}{a} dx = \frac{a}{2}$$

The variance in x

$$\langle x^2 \rangle = \frac{2}{a}\int_0^a x^2 \sin^2 \frac{n\pi x}{a} dx$$

$$\langle x^2 \rangle = a^2 \left[\frac{1}{3} - \frac{1}{2n^2\pi^2}\right]$$

The eigenstate in momentum basis

$$|\phi_n\rangle = \int dp \; a_p \; |p\rangle$$

$$a_p = \langle p | \phi_n \rangle \text{ where } |p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{+ip/\hbar x}$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\int_0^a e^{ip/\hbar x} \sin \frac{n\pi x}{a} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\int_0^a e^{ip/\hbar x} \frac{e^{\frac{in\pi x}{a}} - e^{-\frac{in\pi x}{a}}}{2} dx$$

$$= \frac{2\sqrt{\pi a/\hbar}}{\frac{p^2 a^2}{\hbar^2} - \pi^2} \cos\left(\frac{pa}{2\hbar}\right)$$

$\propto$

$$\langle p \rangle = \int_{-\infty}^{+\infty} p\, |a_p|^2\, dp = 0$$

$$\langle p^2 \rangle = \int_{-\infty}^{+\infty} p^2\, |a_p|^2\, dp = \left(\frac{n\pi\hbar}{a}\right)^2$$

Average of $\hat{p}$ and $\hat{p}^2$ can also be calculated by

$$\langle \hat{p} \rangle = \langle \phi_n | \hat{p} | \phi_n \rangle$$

$$\langle p^2 \rangle = \langle \phi_n | p^2 | \phi_n \rangle$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \qquad \hat{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$\langle x \rangle = \frac{a}{2} \qquad \langle p \rangle = 0$$

$$\langle x^2 \rangle = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$

$$\langle p \rangle = 0$$

$$\langle p^2 \rangle = \left(\frac{n\pi\hbar}{a}\right)^2$$

$$\langle p \rangle = (\quad a)$$

$$\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2 = a^2\left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)$$

$$\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2 = \left(\frac{n\pi\hbar}{a}\right)^2$$

The uncertainty relation

$$\Delta x^2 \Delta p^2 = \frac{\hbar^2}{4}\left(\frac{n^2\pi^2}{3} - 2\right) > \frac{\hbar^2}{4}$$

$$\Rightarrow \Delta x \Delta p > \frac{\hbar}{2}$$

Time evolution:

If the particle is in an eigenstates of ~~H~~ Hamiltonian, it will remain stationary. So the probability distribution of any observable will not change with time.

If, $|\Psi(0)\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$

$\phi_1 \rightarrow$ ground state ϕ_2 : first excited state

$$|\Psi(t)\rangle = a_1 e^{-iE_1 t/\hbar}|\phi_1\rangle + a_2 e^{-iE_2 t/\hbar}|\phi_2\rangle$$

So the probability of finding the particle at x and $x+\Delta x$

$$p(x,t) = |\langle x|\Psi(t)\rangle|^2 \Delta x$$

$$\langle x|\Psi(t)\rangle = \sqrt{\frac{2}{a}}\left(a_1 e^{-iE_1 t/\hbar}\sin\frac{n\pi x}{a} + a_2 e^{-\frac{iE_2 t}{\hbar}}\sin\frac{2n\pi x}{a}\right)$$

$$= \sqrt{\frac{2}{a}}\, e^{-iE_1 t}\left[a_1 \sin\frac{n\pi x}{a} + a_2\cos\omega t\sin\frac{2n\pi x}{a} + i a_2 \sin\omega t\sin\frac{2n\pi x}{a}\right]^2$$

$$|\langle x | \Psi(t)\rangle|^2 = \left(a_1 \sin\frac{n\pi x}{a} + a_2 \cos\omega t \sin\frac{2n\pi x}{a}\right)^2$$

$$+ \; a_2^2 \sin^2\omega t \sin^2\frac{2n\pi x}{a}$$

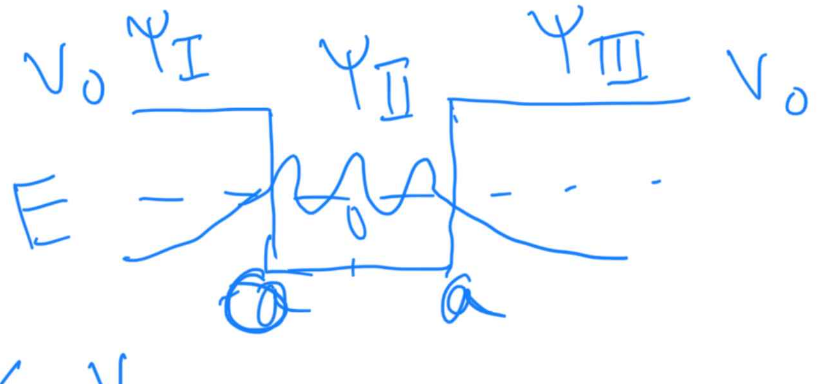
$$= a_1^2 \sin^2\frac{n\pi x}{a} + a_2^2 \sin^2\frac{2n\pi x}{a}$$

$$+ \; 2a_1 a_2 \cos\omega t \sin\frac{n\pi x}{a} \sin\frac{2n\pi x}{a}$$

which depends on time

so the position operator is time dependent

Finite potential well



$E < V_0$

solutions at the

The sol...

three regions

$$\psi_{I} = A_1 e^{K_1 x}$$

$$\psi_{II} = A_2 \sin K_2 x + B_2 \cos K_2 x$$

$$\psi_{III} = A_3 e^{-K_1 x}$$

$$K_1^2 = \frac{2m}{\hbar^2}(V_0 - E) \qquad K_2^2 = \frac{2m}{\hbar^2} E$$

Boundary conditions at $x = 0$

$$\psi_{I}(0) = \psi_{II}(0)$$

$$\psi_{I}'(0) = \psi_{II}(0) \quad \text{yeild}$$

$$A_1 = B_2$$

$$K_1 A_1 = K_2 A_2$$

$$\Rightarrow \psi_{II} = A_1\left[\frac{K_1}{K_2} \sin K_2 x + \cos K_2 x\right]$$

Boundary condition at $x = a$

$$\psi_{II}(a) = \psi_{III}(a)$$

$$\psi'_{II}(a) = \psi'_{III}(a)$$

$$A_1\left(K_1 \sin K_2 a + K_2 \cos K_2 a\right) = K_2 A_3 e^{-K_1 a}$$

$$A_1\left(K_1 \cos K_2 a - K_2 \sin K_2 a\right) = -K_1 A_3 e^{-K_1 a}$$

$$A_1 \sin(K_2 a + \theta) = K_2 A_3 e^{-K_1 a}$$

$$-A_1 \cos(K_2 a + \theta) = K_1 A_3 e^{-K_1 a}$$

$$\Rightarrow \tan(K_2 a + \theta) = \frac{K_2}{K_1} \qquad \theta = \tan^{-1}\left(\frac{K_2}{K_1}\right)$$

This equation is the condition for

These quan...

- discrete energy states

$$\tan\left(\sqrt{\frac{2m}{\hbar^2}E\,a^2} + \theta\right) = \sqrt{\frac{E}{V_0 - E}}$$

Solutions of the equation are the energy eigen values.