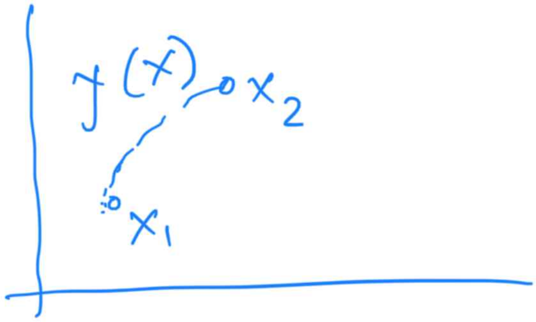
Action principle (Hamilton's principle)

minimum distance between two points on a plane



$$ds = \sqrt{dx^2 + dy^2}$$

$$S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{x_1}^{x_2} f(y, \dot{y})\, dx$$

$$y(x, \epsilon) = y(x) + \epsilon\, \eta(x)$$

$$\frac{dS}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \epsilon} + \underbrace{\frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \epsilon}}_{\text{integrating by parts}} \right) dx.$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\frac{\partial y}{\partial \epsilon}\,dx + \cancelto{0}{\frac{\partial f}{\partial \dot y}\frac{\partial y}{\partial \epsilon}\Big|_{x_1}^{x_2}} - \int \frac{d}{dx}\frac{\partial f}{\partial \dot y}\frac{\partial y}{\partial \epsilon}\,dx$$

$$= \int \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot y}\right)\right]\frac{\partial y}{\partial \epsilon}$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot y}\right)\right]\eta(x)$$

as this is true for any arbitrary $\eta(x)$

$$\cancel{\frac{\partial f}{\partial y}} \quad \frac{d}{dx}\left(\frac{\partial f}{\partial \dot y}\right) = \frac{\partial f}{\partial y}$$

For the above case, $f = \sqrt{1+\dot y^2}$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial \dot y} = \frac{\dot y}{\sqrt{1+\dot y^2}}$$

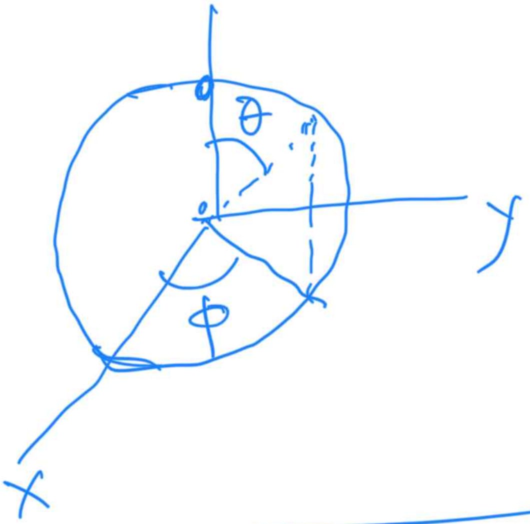
$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot y}\right) = 0 \Rightarrow \frac{\dot y}{\sqrt{1+\dot y^2}} = C$$

$$\Rightarrow \boxed{\dot y = K}$$

$$\Rightarrow \boxed{y = 1}$$

This is the equation of a straight line

Minimum distance on a sphere



$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

$r = 1$

$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$dx = \cos\theta \cos\phi \, d\theta - \sin\theta \sin\phi \, d\phi$

$dy = +\cos\theta \sin\phi \, d\theta + \sin\theta \cos\phi \, d\phi$

$dz = -\sin\theta \, d\theta$

$$\sqrt{dx^2 + dy^2 + dz^2} = \sqrt{d\theta + \sin^2\theta \, d\phi}$$

$$ds = \sqrt{d\theta^2 + \sin^2\theta \, d\phi^2}$$

$$S = \int_{\theta}^{\theta_2} \sqrt{1 + \sin^2\theta \, \dot{\phi}^2} \, d\theta$$

$$f = \sqrt{1 + \sin^2\theta \, \dot{\phi}^2}$$

$$\frac{\partial f}{\partial \phi} = 0 \qquad \frac{\partial f}{\partial \dot{\phi}} = \frac{\sin^2\theta \, \dot{\phi}}{\sqrt{1 + \sin^2\theta \, \dot{\phi}^2}}$$

$$\frac{\sin^2\theta \, \dot{\phi}}{\sqrt{1 + \sin^2\theta \, \dot{\phi}^2}} = \frac{1}{C}$$

$$\sin^4\theta \, \dot{\phi}^2 = \frac{1}{C^2}\left(1 + \sin^2\theta \, \dot{\phi}^2\right)$$

$$\left(C^2 \sin^4\theta - \sin^2\theta\right) \dot{\phi}^2 = 1$$

$$=$$

$$\dot{\phi}^2 = \frac{1}{\sin^2\theta\left(c^2\sin^2\theta - 1\right)}$$

$1/c^2 = a^2$

$$\dot{\phi}^2 = \frac{a^2}{\sin^2\theta\left(\sin^2\theta - a^2\right)}$$

$$\dot{\phi} = \frac{a}{\sin\theta\sqrt{\sin^2\theta - a^2}}$$

$$\phi = \phi_0$$

Action principle and Lagrangian formulation

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt.$$

$$q(t, \epsilon) = q(t) + \epsilon\, \eta(t)$$

$$\frac{dS}{d\epsilon} = \int^{t_2}\left(\frac{\partial L}{\partial q}\frac{\partial q}{\partial \epsilon} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \epsilon}\right) dt$$

t_1

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right] \eta(t) + \frac{\partial L}{\partial \dot{q}} \frac{\partial q}{\partial t} \Bigg|_{t_1}^{t_2}$$

$$\Rightarrow \quad \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

$$\frac{d}{dt} p = -\frac{\partial V}{\partial x} = + \frac{\partial}{\partial x}(T - V)$$

$$L = T - V \qquad \frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{x}}$$

$$L = T - V$$

Harmonic Oscillator

$$T = \frac{1}{2} m \dot{x}^2 \qquad V = \frac{1}{2} m \omega^2 x^2$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

$$\frac{\partial L}{\partial x} = -\cancel{\frac{1}{2}}\, m \omega^2 x$$

$$\frac{\partial L}{\partial \dot{x}} = m \dot{x} \qquad m\ddot{x} = -m\omega^2 x$$

Multiple dimension

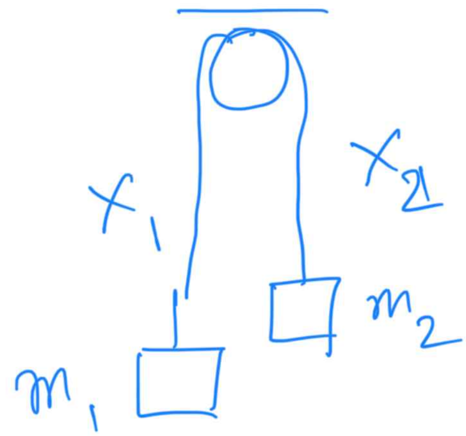
$$\delta S = \sum_i \left(\frac{\partial L}{\partial q_i} \partial q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

$$= \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i$$

$$\frac{\partial L}{} - \frac{d}{} \frac{\partial L}{} = 0$$

$\partial q_i \qquad dt \; \partial \dot{q}_i$

2.



Lagrangian

$$L = \frac{1}{2} m_1 \dot{x}_1^2 - m_1 g x_1 + \frac{1}{2} m_2 \dot{x}_2^2 - m_2 g x_2$$

$$x_1 + x_2 = l \qquad \dot{x}_1 = -\dot{x}_2$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2 - m_1 g x_1 - m_2 g (l - x_1)$$

$$= \frac{1}{2}(m_1+m_2)x_1 - (m_1-m_2)x_1$$
$$+ m_2 g l$$

$$\frac{\partial L}{\partial x_1} = -(m_1-m_2)g$$

$$\frac{\partial L}{\partial \dot{x}_1} = (m_1+m_2)\dot{x}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1}$$

$$(m_1+m_2)\ddot{x}_1 = -(m_1-m_2)g$$

$$\ddot{x}_1 = \frac{m_2-m_1}{m_2+m_1}g$$

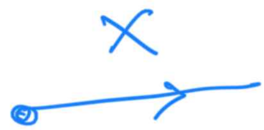
Lagrange Equation of motion

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Free particle moving on a straight line

Kinetic energy



$$T = \frac{1}{2} m \dot{x}^2$$

$$V = 0$$

$$L = \frac{1}{2} m \dot{x}^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \Rightarrow m\ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial x}\right)$$

With external force described by a potential $V(x)$

$$L = \frac{1}{2} m \dot{x}^2 - V(x)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$$

$$\Rightarrow m\ddot{x} = -\frac{\partial V}{\partial x}$$

Simple Harmonic Oscillator

$$L = \frac{1}{2} m \dot{x} - \frac{1}{2} K x^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{\partial L}{\partial x} = -Kx$$

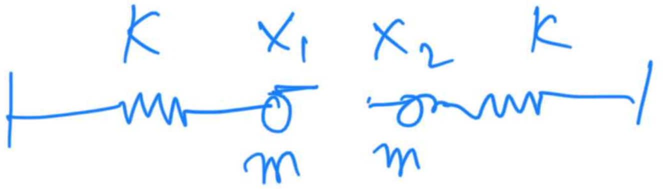
$$\Rightarrow m\dot{x} = -Kx$$

$\rightarrow$

Frequency $\omega = \sqrt{\frac{K}{m}}$

$$T = \frac{2\pi}{\omega}$$

Two independependent simple Harmonic oscillators



Kinetic energy

$$T = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right)$$

potential energy

$$V = \frac{1}{2} K x_1^2 + \frac{1}{2} K x_2^2$$

$$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right) - \frac{1}{2} K \left(x_1^2 + x_2^2 \right)$$

$$\frac{\partial L}{\partial \dot{x}_1} = m \dot{x}_1 \qquad \frac{\partial L}{\partial x_1} = -K x_1$$

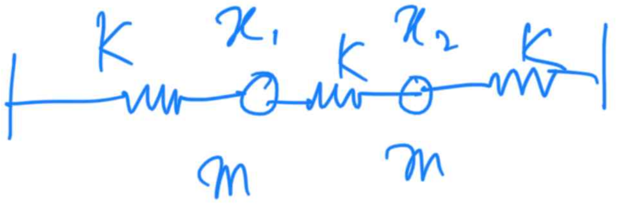
$\partial \dot{x}_1$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m\ddot{x}_1$$

$$m\ddot{x}_1 = -K x_1$$

Similarly $m\ddot{x}_2 = -K x_2$

Two interacting Harmonic Oscillators



$$KE = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2\right)$$

$$PE = \frac{1}{2} K x_1^2 + \frac{1}{2} K x_2^2 + \frac{1}{2} K (x_1 - x_2)^2$$

$$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} K \left(x_1^2 + x_2^2\right) - \frac{1}{2} K \left(x_1 - x_2\right)^2$$

$$\frac{\partial L}{} = m\dot{x}_1 \quad \frac{\partial L}{} = -K x_1 - K(x_1 - x_2)$$

$\partial \dot{x}_1 \qquad \partial x_1$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m\ddot{x}_1$$

$$m\ddot{x}_1 = -Kx_1 - K(x_1 - x_2) \rightarrow ①$$

$$\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 \qquad \frac{\partial L}{\partial x_2} = -Kx_2 - K(x_2 - x_1)$$

$$m\ddot{x}_2 = -Kx_2 - K(x_2 - x_1) \rightarrow ②$$

For variable $y_1 = x_1 + x_2$

Adding equations 1 and 2

$$(\ddot{x}_1 + \ddot{x}_2) = -\frac{K}{m}(x_1 + x_2)$$

$$\ddot{y}_1 = -\frac{K}{m} y_1$$

and $y_2 = x_1 - x_2$

$$(\ddot{x}_1 - \ddot{x}_2) = -\frac{3K}{m}(x_1 - x_2)$$

$$\ddot{y}_2 = -\frac{3K}{m} y_2$$

The equalion of molecusns are

$$m\ddot{x}_1 = -2Kx_1 + Kx_2$$

$$m\ddot{x}_2 = Kx_1 - 2Kx_2$$

which canbe written in matrix form

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2K/m & K/m \\ K/m & -2K/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The normal mode of freequencies can be calculated by the eigenvalue and eigen vectror of the matrix

$$\begin{vmatrix} -2K/m - \lambda & K/m \\ K/m & -2K/m - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\lambda + 2K/m\right)^2 - \left(K/m\right)^2 = 0$$

$$\left(\lambda + K/m\right)\left(\lambda + 3K/m\right) = 0$$

hence, $\lambda_1 = -K/m \quad \lambda_2 = -3K/m$

corresponding eigen ~~vectors~~ for λ_1

$$\begin{pmatrix} -K/m & K/m \\ K/m & -K/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \text{eigenvector } E_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -3K/m$

$$\begin{pmatrix} K/m & K/m \\ K/m & K/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow E_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

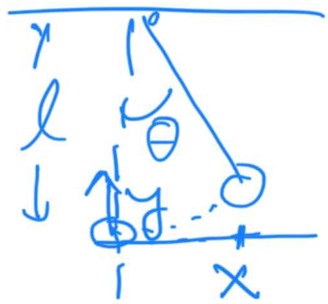
The new ~~valereables~~ coordinates

$$\begin{pmatrix} y_1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \end{pmatrix}$$

$$\begin{pmatrix} y_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \end{pmatrix}$$

$$y_1 = x_1 + x_2 \qquad y_2 = x_1 - x_2$$

Simple pendulum



$$y = l(1 - \cos\theta)$$

$$x = l \sin\theta$$

$$\dot{x} = + l \cos\theta \, \dot{\theta} \qquad \dot{y} = l \sin\theta \, \dot{\theta}$$

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = mgl(1 - \cos\theta)$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \dot\theta} = m l^2 \dot\theta \qquad \frac{\partial L}{\partial \theta} = - m g l \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = m l^2 \ddot\theta$$

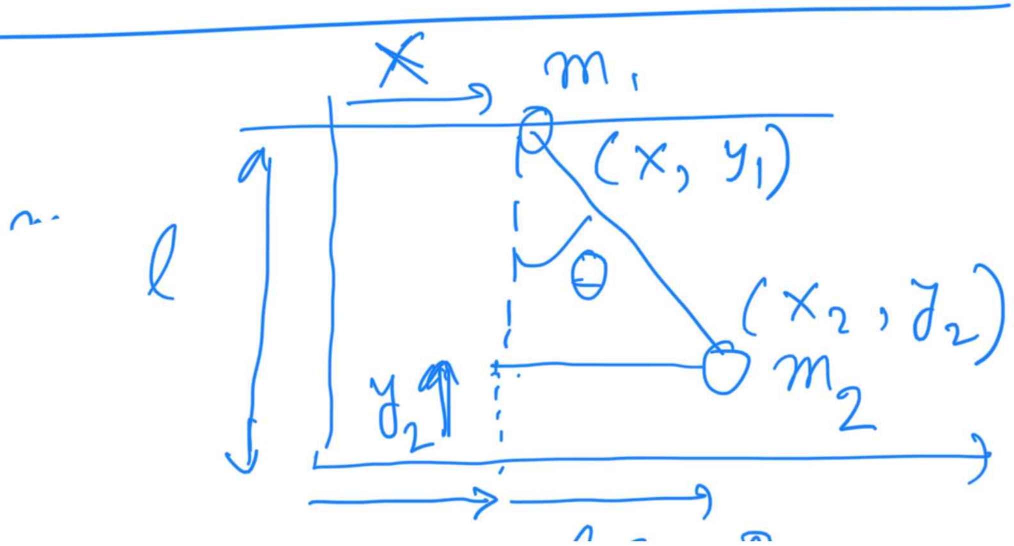
$$\Rightarrow \quad m l^2 \ddot\theta = - m g l \sin\theta$$

$$\ddot\theta = -\frac{g}{l} \sin\theta$$

$$\ddot\theta = - g/l \, \theta \quad \text{for small } \theta$$

$$\omega^2 = \sqrt{g/l} \quad \text{frequency of oscillation}$$

Pendulum with two masses



$x \quad l\sin\theta$

degrees of freedom x, θ

$x_1 = x \quad y_1 = l$

$x_2 = x + l\sin\theta$

$y_2 = l(1-\cos\theta)$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_2 l^2 \theta$$

$$\dot{x}_2 = (\dot{x} + l\cos\theta\,\dot{\theta})$$

$$\dot{y}_2 = l\sin\theta\,\dot{\theta}$$

Kinetic energy

$$= \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x} + l\cos\theta\,\dot{\theta})^2$$

$$\frac{1}{2} m_2 l^2 \sin^2\theta\,\dot{\theta}$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x} + \frac{1}{2} m_2 l^2 \dot{\theta}^2$$

$$+ m_2 l \cos\theta \, \dot{x} \, \dot{\theta}$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\theta}^2$$

$$+ m_2 l \cos\theta \, \dot{x} \, \dot{\theta}$$

potential energy

$$V = m_2 g l (1 - \cos\theta)$$

The potential energy for m_1 is constant, so we can ignore it

$$L = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\theta}^2$$

$$+ m_2 l \cos\theta \, \dot{x} \dot{\theta} - m_2 g l (1 - \cos\theta)$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + \cancel{\tfrac{1}{2}}\, m_2 l \cos\theta\, \theta \quad \text{— (1)}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 l^2 \dot{\theta} + m_2 l \cos\theta\, \dot{x} \quad \text{— (2)}$$

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial \theta} = -m_2 \sin\theta\, \dot{x}\, \dot{\theta} - m_2 g l \sin\theta$$

From (1)

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1 + m_2)\ddot{x} + m_2 l \cos\theta\, \ddot{\theta} - m_2 l \sin\theta\, \dot{\theta}^2$$

$$\therefore \boxed{(m_1 + m_2)\ddot{x} + m_2 l \cos\theta\, \ddot{\theta} - m_2 l \sin\theta\, \dot{\theta}^2 = 0} \quad \text{— (3)}$$

From (2)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta} - m_2 l \cos\theta\, \ddot{x} - m_2 l \sin\theta\, \dot{\theta}\, \dot{x}$$

θ

So

$$m l^2 \ddot{\theta} - m_2 l \cos\theta \, \ddot{x} - m_2 l \sin\theta \, \dot{\theta} \dot{x}$$

$$= - m_2 l \sin\theta \, \dot{x} \dot{\theta} - m g l \sin\theta$$

$$\boxed{m l^2 \ddot{\theta} - m_2 l \cos\theta \, \ddot{x} = - m g l \sin\theta} \quad - (4)$$

For small θ

$$m_2 l^2 \ddot{\theta} - m_2 l \ddot{x} = - m_2 g l \theta$$

From (3)

$$(m_1 + m_2) \ddot{X} = - m_2 l \ddot{\theta} + m_2 l \theta \dot{\theta}^2$$

$$l \ddot{\theta} - \ddot{x} = - g \theta \qquad \boxed{\dot{\theta} \sim 1}$$

$$l \ddot{\theta} - \left(\frac{- m_2 l \ddot{\theta} + m_2 l \theta \dot{\theta}^2}{m_1 + m_2} \right) = - g \theta$$

$$l\left(1+\frac{m_2}{m_1+m_2}\right) = -\left[\frac{m_2 l}{m_1+m}+g\right]\theta$$

$$\frac{m_1+2m_2}{m_1+m_2}\ddot{\theta} = -\left(\frac{m_2}{m_1+m_2}+g/l\right)\theta$$

$$\ddot{\theta} = -\frac{m_1+m_2}{m_1+2m_2}\left(\frac{m_2}{m_1+m_2}+g/l\right)\theta$$

For $m_1 \gg m_2$ $\quad \frac{m_1}{m_2} \to \infty$

$$\ddot{\theta} = -g/l\ \theta$$

For $m_2 \gg m_1$

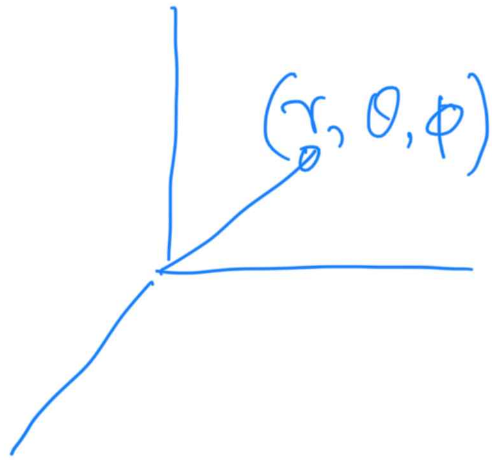
$$\ddot{\theta} = -\frac{1}{2}\left(1+g/l\right)\theta$$

Lagrangian in 3D for a single particle under central force

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$



$$\dot{x} = \dot{r} \sin\theta \cos\phi + r \cos\theta \cos\phi \, \dot{\theta} - r \sin\theta \sin\phi \, \dot{\phi}$$

$$\dot{y} = \dot{r} \sin\theta \sin\phi + r \cos\theta \sin\phi \, \dot{\theta} + r \sin\theta \cos\phi \, \dot{\phi}$$

$$\dot{z} = \dot{r} \cos\theta - r \sin\theta \, \dot{\theta}$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) - V(r)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial \dot{\theta}} = m\dot{\theta}$$

$$\boxed{\frac{\partial L}{\partial r} = -\frac{\partial V}{\partial r} + m r \dot{\theta}^2 + m r \sin^2\theta \, \dot{\phi}^2}$$

$$\frac{\partial L}{\partial \theta} = 2 r^2 \sin\theta \cos\theta \, \dot{\phi}^2$$

$$\frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi} \qquad \frac{\partial L}{\partial \phi} = 0$$

$m r^2 \sin\theta$

$$\frac{d}{dt}\left(m r^2 \sin^2\theta \, \dot{\phi}\right) = 0$$

$m r^2 \sin^2\theta \, \dot{\phi} \rightarrow$ angular momentum component along z axis

$\therefore \; L_z = C$

Since, potential depends on only r, thus spherically symmetric

$L_z = C$ for any arbitrary

z axes

choice of z axis

Thus, total angular momentum must be constant

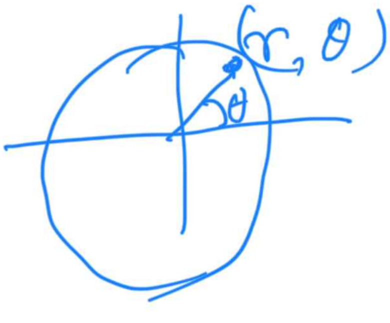
as we know, $\vec{L} = \vec{r} \times \vec{p}$

$\vec{r}$ is perpendicular to $\vec{L}$

The motion must happen on a plane perpendicular to the direction of $\vec{L}$

Choose $\vec{L}$ along Z axis

Two degrees of freedom r, θ



$x = r\cos\theta \qquad y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial r} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0$$

$\Rightarrow$ Equation for θ

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \Rightarrow mr^2\dot{\theta} = A$$

$$\Rightarrow \dot{\theta}^2 = \frac{A}{m^2r^4}$$

The equation for r

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\partial V \qquad mr\dot{\theta}^2$$

$$m\ddot{r} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2$$

$$= -\frac{\partial V}{\partial r} + mr\frac{A}{m^2r^4}$$

$$= -\frac{\partial V}{\partial r} + \frac{A}{mr^3}$$

$$\ddot{r} = -\frac{1}{m}\frac{\partial V}{\partial r} + \frac{A}{m^2r^3} \qquad V = -\frac{K}{r}$$

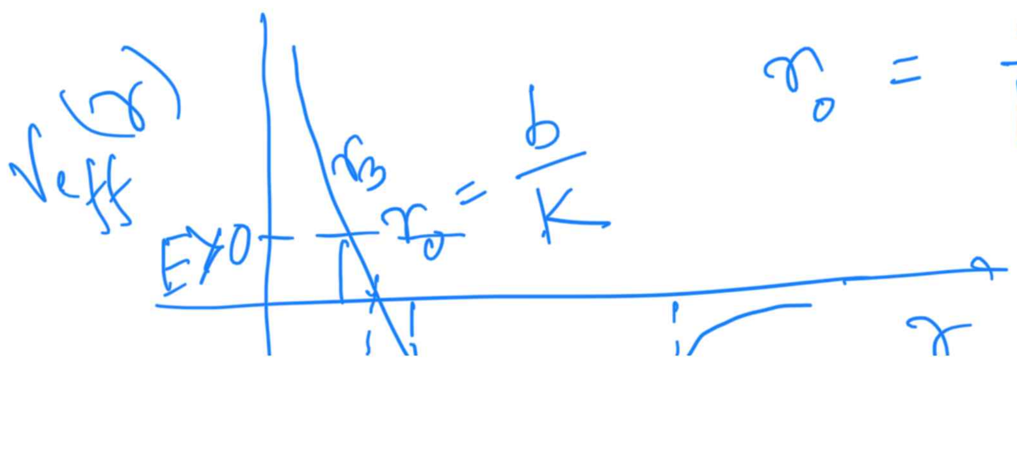
For Kepler's problem (Gravity)

$$\ddot{r} = -\frac{K}{r^2} + \frac{b}{r^3} \qquad b = \frac{A}{m^2}$$

$$\ddot{r} = -\frac{\partial}{\partial r}\left(V + \frac{b}{r^2}\right)$$

$$= -\frac{\partial}{\partial r}\left(-\frac{K}{r} + \frac{b}{r^2}\right) = -\frac{\partial}{\partial r}V_{eff}(r)$$

$$r_0 = \frac{b}{K}$$

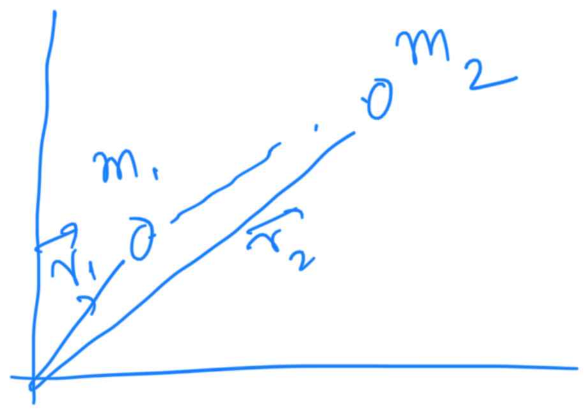


For $E < 0$ the orbit will be bound between r_1 and r_2

$$V_{eff}(r_1) = -E \qquad V_{eff}(r_2) = E$$

$E > 0$ the orbit will not be bound but minimum distance from the centre $V_{eff}(r_3) = E$

Two body problem and center of mass coordinates



$$\frac{1}{2} m_1 (\dot{\vec{r}}_1)^2 + \frac{1}{2} m_2 (\dot{\vec{r}}_2)^2 - V(\vec{r}_1, \vec{r}_2)$$

$$L = \frac{1}{2} m_1 (\dot{\vec{r}}_1)^2 + \frac{1}{2} m_2 (\dot{\vec{r}}_2)^2 \ldots$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r})}{m_1 + m_2}$$

$$(m_1 + m_2)\vec{R} = (m_1 + m_2) r_1 - m_2 \vec{r}$$

$$\vec{r}_1 = R + \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r}_2 = \vec{r}_1 - \vec{r} = R - \frac{m_1}{m_1 + m_2} \vec{r}$$

$$L = \frac{1}{2} m_1 \left(\dot{\vec{R}} + \frac{m_2}{m_1 + m_2} \dot{\vec{r}} \right)^2 + \frac{1}{2} m_2 \left(\dot{\vec{R}} - \frac{m_1}{m_1 + m_2} \dot{\vec{r}} \right)^2 - V(\vec{R}, \vec{r})$$

$$= \frac{1}{2}(m_1 + m_2) \dot{\vec{R}}^2 + \frac{m_1 m_2}{m_1 + m_2} \dot{\vec{r}}^2$$

$$- V(\vec{R}, \vec{r})$$

$- V(R, r)$

For central force $\quad M = m_1 + m_2 \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \mu \dot{r}^2 - V(r)$$

Hamiltons' equation

Generalized momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad p_i \neq m\dot{q}_i$$

$$H = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t)$$

$$\frac{dH}{dt} = \sum_i \dot{p}_i \dot{q}_i + p_i \ddot{q}_i - \frac{\partial L}{\partial q_i} \dot{q}_i - \frac{\partial L}{\partial \dot{q}} \ddot{q}_i - \frac{\partial L}{\partial t}$$

0 from Lagrange Equation

$$= \sum \underbrace{\left(\dot{p}_i - \frac{\partial L}{\partial q_i}\right)}_{0} \dot{q}_i + \sum \underbrace{\left(p_i - \frac{\partial L}{\partial \dot{q}_i}\right)}_{0} \ddot{q}_i$$

$$= \sum \left(\cdots \right)$$

as $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$= -\frac{\partial L}{\partial t}$$

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} \partial q_i$$

$$dH = \sum_i p_i \, d\dot{q}_i + dp_i \, \dot{q}_i - \frac{\partial L}{\partial q_i} \delta q_i$$

$$- \frac{\partial L}{\partial \dot{q}_i} \partial \dot{q}_i$$

$$= \sum_i \dot{q}_i \, dp_i - \dot{p}_i \, \delta q_i$$

$$\dot{q}_i \, \dot{p}_i - \dot{p} \, \dot{q}$$

Hamilton's equation

$$H = \sum_i p_i \dot{q}_i - L(\dot{q}_i, q, t)$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$\frac{dH}{dt} = \sum_i \left[\dot{p}_i \dot{q}_i + p_i \ddot{q}_i - \frac{\partial L}{\partial q_i} \dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right]$$

$$= \sum_i \overbrace{\left(p_i - \frac{\partial L}{\partial \dot{q}_i}\right)}^{0} \ddot{q}_i + \overbrace{\left(\dot{p}_i - \frac{\partial L}{\partial q_i}\right)}^{=0} \dot{q}_i - \frac{\partial L}{\partial t}$$

$$\frac{dH}{dt} = \frac{\partial L}{\partial t}$$

$$dH = \sum_i p_i \, d\dot{q}_i + \dot{q}_i \, dp_i - \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i$$

$$- \frac{\partial L}{\partial q_i} \delta q_i$$

$$= \dot{q}_i \, dp_i - \dot{p}_i \, \delta q_i$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \frac{\partial H}{\partial q_i}$$

Hamiltonian in one dimension

$$H(x, p, \cancel{t}) = \frac{p^2}{2m} + V(x)$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$$

Hamiltonian is the total energy

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - T + V$$

$$x \frac{\partial f}{\partial x} = nf$$ ~~for~~ homogeneous equation

If T is a homonegeneous function of $\dot{q}$ of degree 2 $\quad \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2T$

$$\mathcal{H} = T + V$$

Simple Harmonic Oscillator

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$$

Generalized momentum for x

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$$

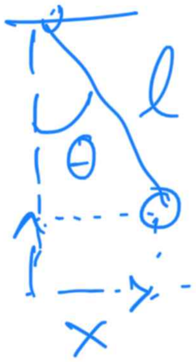
Then $\mathcal{H} = \frac{p_x^2}{2m} + \frac{1}{2} k x^2$

Equation of motion

$$\dot{x} = \frac{\partial H}{\partial p_x} \Rightarrow \boxed{\dot{x} = \frac{p_x}{m}}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} \Rightarrow \boxed{\dot{p}_x = -Kx}$$

Simple pendulum



$$x = l\sin\theta \qquad y = l(1-\cos\theta)$$

$$L = \frac{1}{2}m\dot{x}^2 - mgl(1-\cos\theta)$$

$$\dot{x} = l\cos\theta\,\dot{\theta} \qquad \dot{y} = l\sin\theta\,\dot{\theta}$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

One degree of freedom θ

Generalized moment $p_\theta = \frac{\partial L}{\partial \dot{\theta}}$

$$p_\theta = ml^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{ml^2}$$

$$\therefore \mathcal{H} = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)$$

Equations of motion

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \Rightarrow \boxed{\dot{\theta} = \frac{p_\theta}{ml^2}}$$

$$\theta = \frac{\partial H}{\partial p_\theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \Rightarrow \boxed{\dot{p}_\theta = -mgl\sin\theta}$$

$$\Rightarrow ml\ddot{\theta} = -mgl\sin\theta$$

Hamiltonian in 3D

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

The generalized coordinates for x, y, z

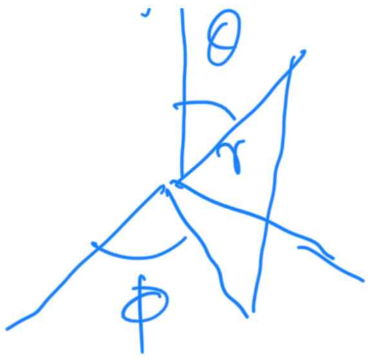
$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \qquad \dot{y} = \frac{p_y}{m}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad \dot{z} = \frac{p_z}{m}$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(x, y, z)$$

In spherical polar coordinates (r, θ, ϕ)



$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x,y,z)$$

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta}$$
$$- r\sin\theta\sin\phi\,\dot{\phi}$$

$$\dot{y} = \dot{r}\sin\theta\sin\phi - r\cos\theta\cos\phi\,\dot{\theta}$$
$$+ r\sin\theta\cos\phi\,\dot{\phi}$$

$$\dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\Rightarrow \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - V(r,\theta,\phi)$$

The generalized momenta p_r, p_θ, p_ϕ

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$$

The Hamiltonian (by substituting in L)

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r,\theta,\phi)$$

Central force problem

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r)$$

Equations of motion

$$\dot{r} = \frac{\partial H}{\partial p_r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{\phi} = \frac{p_\phi}{}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} \Rightarrow \dot{\phi} = \frac{\quad}{mr^2\sin^2\theta}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} \Rightarrow \dot{p}_r = -\frac{\partial V}{\partial r} - \frac{p_\theta^2}{mr^3} - \frac{p_\phi^2}{mr^3\sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \Rightarrow \dot{p}_\theta = -\frac{2p_\phi^2\cos\theta}{2mr^2\sin^3\theta}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} \Rightarrow \dot{p}_\phi = 0$$

$$\Downarrow$$

$$\Rightarrow \frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0$$

$\therefore$ angular momentum along z axis is constant.

Hence, motion is happing on a plane perpendular to the angular momentum

Equations of motion for central force

$$\dot{r} = \frac{\partial H}{\partial p_r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} \Rightarrow \dot{\phi} = \frac{p_\phi}{mr^2 \sin^2\theta}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} \Rightarrow \dot{p}_r = -\frac{\partial V}{\partial r} - \frac{p_\theta^2}{mr^3} - \frac{p_\phi^2}{mr^3 \sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \Rightarrow \dot{p}_\theta = -\frac{2 p_\phi^2 \cos\theta}{2mr^2 \sin^3\theta}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} \Rightarrow \dot{p}_\phi = 0$$

$$\Downarrow$$

Angular momentum

$$\Rightarrow \frac{d}{dt}\left(mr^2 \dot{\phi}\right) = 0 \Rightarrow p_\phi = l$$

$$\frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r) = 0$$

$$\frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + V(r) = E$$

$$\frac{1}{2} m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{K}{r} = E$$

$$\dot{\phi} = \frac{l^2}{2mr^2}$$

$$\dot{r}^2 = \sqrt{\frac{2\left(E + \frac{K}{r} - \frac{l^2}{2mr^2}\right)}{m}}$$

Dividing $\dot{\phi}$ by $\dot{r}$

$$\frac{d\phi}{dr} = \frac{l^2}{2mr^2}\sqrt{\frac{m}{2\left(E + K/r - l^2/2mr^2\right)}}$$

$$d\phi = \frac{l^2}{2} \frac{1/r^2 \, dr.}{\sqrt{2m\left(E + \frac{K}{r} - \frac{l^2}{2mr^2}\right)}}$$

$$\boxed{\text{Putting } \frac{1}{r} = u} = \frac{l^2}{2} \frac{du}{\sqrt{2m\left(E + Ku - \frac{l^2}{2m}u^2\right)}}$$

$$\frac{d}{r} \quad \sqrt{2m\left(E + ku - \frac{l^2}{2m}u^2\right)}$$

Integrating

$$\frac{1}{r} = c(1 + e\cos\phi)$$

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}$$

$E = 0$ parabola

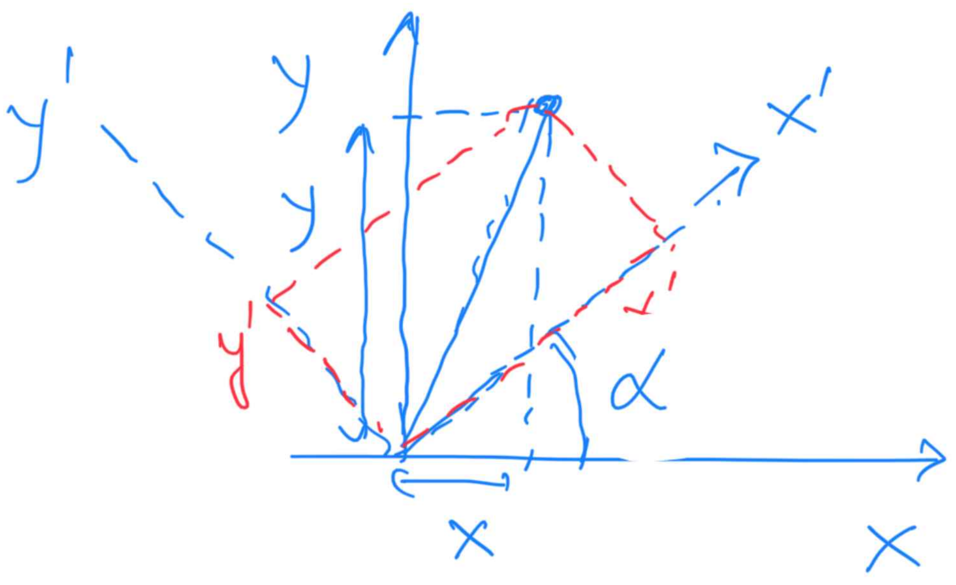
$E < 0$ ellipse

$E > 0$ Hyperbola

$e = 0$ circle

Symmetry, invariance and conservation

Rotation of axes by angle θ



Rotation matrix

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$x' = x\cos\alpha - y\sin\alpha$$

$$y' = x\sin\alpha + y\cos\alpha$$

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

For rotation invariance of L with respect to α $\frac{\partial L}{\partial \alpha} = 0$

$$\frac{\partial L}{\partial \alpha} = \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha}$$

$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right)$$

$$= \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \alpha}\right) = 0$$

$$\sum_i p_i \frac{\partial q_i}{\partial \alpha}\bigg|_{\alpha=0} = C$$

$$\sum_i p_i \frac{\partial q_i}{\partial \alpha}\bigg|_{\alpha=0} = C.$$

For rotation invariance

$$p_x \frac{\partial x'}{\partial \alpha}\bigg|_{\alpha=0} + p_y \frac{\partial y'}{\partial \alpha}\bigg|_{\alpha=0} = p_x y - p_y x = L_z$$

$\therefore L_z$ is conserved

Hamilton's formulation
poisson bracket and conservation

$$\frac{dA}{dt} = \sum \frac{\partial A}{\partial q_i}\frac{\partial q_i}{dt} + \frac{\partial A}{\partial p_i}\frac{\partial p_i}{dt}$$

$\frac{d}{dt}$ ... $\frac{\partial}{\partial q_i} \frac{\partial}{\partial t}$ ...

$$= \left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

$$= \{A, H\}$$ poisson bracket

Runge - Lenz vector $\vec{A} = \vec{p} \times \vec{L} - K\frac{\vec{r}}{r}$

Conservation of angular momentum for central force

$$L_z = x p_y - y p_x$$

$$\frac{\partial L_z}{\partial x} = p_y \quad \frac{\partial L_z}{\partial y} = -p_x \quad \frac{\partial L_z}{\partial z} = 0$$

$$\frac{\partial L_z}{\partial p_x} = -y \quad \frac{\partial L_z}{\partial p_y} = x \quad \frac{\partial L_z}{\partial p_z} = 0$$

$$V(x, y) = -\frac{K}{\sqrt{x^2+y^2}}$$

∂H ... $K x$

$$\frac{\partial H}{\partial x} = \frac{k x}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial H}{\partial y} = \frac{k y}{(x^2+y^2)^{3/2}} \qquad \frac{\partial H}{\partial z} = \frac{k z}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_y} = \frac{p_y}{m} \qquad \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$\{L_z, H\} = \frac{\partial L_z}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial L_z}{\partial p_x}\frac{\partial H}{\partial x}$$

$$= p_y \frac{p_x}{m} + y \frac{k x}{\sqrt{x^2+y^2+z^2}}$$

$$+ \frac{\partial L_z}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial L_z}{\partial p_y}\frac{\partial H}{\partial y}$$

$$- p_x \frac{p_y}{m} - x \frac{k y}{(x^2+y^2+z^2)^{3/2}}$$

$$+ \cancel{\frac{\partial L_z}{\partial z}}^{0}\frac{\partial H}{\partial p_z} - \cancel{\frac{\partial L_z}{\partial p_z}}^{0}\frac{\partial H}{\partial z} = 0$$

Linear momentum p_x

$$\{p_x, H\} = \frac{\partial p_x}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial p_x}{\partial p_x}\frac{\partial H}{\partial x}$$

$$+ \frac{\partial p_x}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial p_x}{\partial p_y}\frac{\partial H}{\partial y}$$

$$+ \frac{\partial p_x}{\partial z}\frac{\partial H}{\partial p_z} - \frac{\partial p_x}{\partial p_z}\frac{\partial H}{\partial z}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x}$$

$$= \frac{[illegible]}{m} \sqrt{2/m\left(E - \frac{M}{m} u^2 + \alpha u\right)}$$