Lotka-Volterra (Prey-predator) mechanism:
universe : Fox (Y) and Rabbit (X) and grass (A) are the only components

Lotka-Volterra (Prey-predator) mechanism:
universe : Fox (Y) and Rabbit (X) and grass (A) are the only components

Various subprocesses are

- Rabbits consume grass to become more healthy and breed: $A + X \to 2X$,
- Foxes eat rabbits to be healthy and increase their population: $X + Y \to 2Y$
- Foxes die due to natural causes: $Y \to \Phi$

  (Null denoting nonexistence- death or decay)

Lotka-Volterra (Prey-predator) mechanism:
universe : Fox (Y) and Rabbit (X) and grass (A) are the only components

Various subprocesses are

- Rabbits consume grass to become more healthy and breed: $A + X \to 2X$,
- Foxes eat rabbits to be healthy and increase their population: $X + Y \to 2Y$
- Foxes die due to natural causes: $Y \to \Phi$

  (Null denoting nonexistence- death or decay)

assume : amount of grass (A) is non-varying, constant.
These three "equations" consitute the model.

A+X$\xrightarrow{k_a}$2X;

$$\frac{d[X]}{dt} = k_a[A][X] - k_b[X][Y]$$

$\mathrm{A}+\mathrm{X} \xrightarrow{k_a} 2\mathrm{X}$;

$$\frac{d[X]}{dt} = k_a[A][X] - k_b[X][Y]$$

$\mathrm{X}+\mathrm{Y} \xrightarrow{k_b} 2\mathrm{Y}$;

$$\frac{d[Y]}{dt} = k_b[X][Y] - k_c[Y]$$

$A+X \xrightarrow{k_a} 2X;$

$$\frac{d[X]}{dt} = k_a[A][X] - k_b[X][Y]$$

$X+Y \xrightarrow{k_b} 2Y;$

$$\frac{d[Y]}{dt} = k_b[X][Y] - k_c[Y]$$

$Y \xrightarrow{k_c} P$

A+X$\xrightarrow{k_a}$2X;

$\frac{d[X]}{dt} = k_a[A][X] - k_b[X][Y]$

X+Y$\xrightarrow{k_b}$2Y;

$\frac{d[Y]}{dt} = k_b[X][Y] - k_c[Y]$

Y$\xrightarrow{k_c}$P

use $[X] = \frac{k_c}{k_b}x$; $[Y] = \frac{k_c}{k_b}y$;

$[A] = \frac{k_c}{k_a}y$; $t = \frac{1}{k_c}\tau$

$A + X \xrightarrow{k_a} 2X;$

$$\frac{d[X]}{dt} = k_a[A][X] - k_b[X][Y]$$

$X + Y \xrightarrow{k_b} 2Y;$

$$\frac{d[Y]}{dt} = k_b[X][Y] - k_c[Y]$$

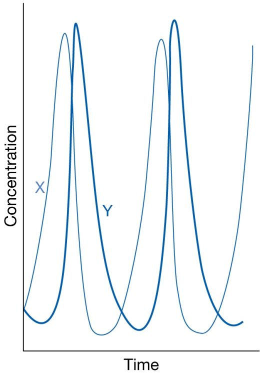
$Y \xrightarrow{k_c} P$

use $[X] = \frac{k_c}{k_b}x;\ [Y] = \frac{k_c}{k_b}y;$

$[A] = \frac{k_c}{k_a}y;\ t = \frac{1}{k_c}\tau$

$$\frac{dx}{d\tau} = ax - xy$$

$$\frac{dy}{d\tau} = xy - y$$



$$A + X \xrightarrow{k_a} 2X;$$

$$\frac{d[X]}{dt} = k_a[A][X] - k_b[X][Y]$$

$$X + Y \xrightarrow{k_b} 2Y;$$

$$\frac{d[Y]}{dt} = k_b[X][Y] - k_c[Y]$$

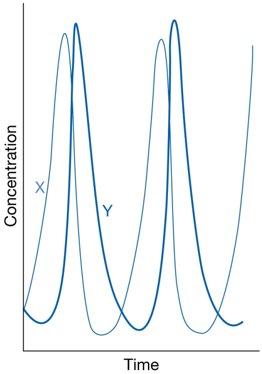
$$Y \xrightarrow{k_c} P$$

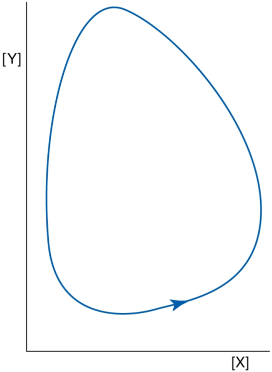
use $[X] = \frac{k_c}{k_b}x;\ [Y] = \frac{k_c}{k_b}y;$

$[A] = \frac{k_c}{k_a}y;\ t = \frac{1}{k_c}\tau$

$$\frac{dx}{d\tau} = ax - xy$$

$$\frac{dy}{d\tau} = xy - y$$





For a general dynamical system, the solution for an initial state $(X_o,\ Y_0)$ may be :
$X(t) = X_o + \Delta_x e^{\lambda t}$; and similar for Y
the behaviour depends on λ. Take $\Delta > 0$

For a general dynamical system, the solution for an initial state $(X_o,\ Y_0)$ may be :
$X(t) = X_o + \Delta_x e^{\lambda t}$; and similar for Y
the behaviour depends on λ. Take $\Delta > 0$

real $\lambda > 0 \implies X(t)$ monotonic increasing function of t

For a general dynamical system, the solution for an initial state $(X_o,\ Y_0)$ may be :
$X(t) = X_o + \Delta_x e^{\lambda t}$; and similar for Y
the behaviour depends on λ. Take $\Delta > 0$

real $\lambda > 0 \implies X(t)$ monotonic increasing function of t

real $\lambda < 0 \implies X(t)$ monotonic decreasing function of t

For a general dynamical system, the solution for an initial state $(X_o, \; Y_0)$ may be :
$X(t) = X_o + \Delta_x e^{\lambda t}$; and similar for Y
the behaviour depends on λ. Take $\Delta > 0$

real $\lambda > 0 \implies X(t)$ monotonic increasing function of t

real $\lambda < 0 \implies X(t)$ monotonic decreasing function of t

λ purely imaginary $\implies X(t)$ oscillatory function in t

For a general dynamical system, the solution for an initial state (X_o, Y_0) may be :
$X(t) = X_o + \Delta_x e^{\lambda t}$; and similar for Y
the behaviour depends on λ. Take $\Delta > 0$

real $\lambda > 0 \implies X(t)$ monotonic increasing function of t

real $\lambda < 0 \implies X(t)$ monotonic decreasing function of t

λ purely imaginary $\implies X(t)$ oscillatory function in t

$\lambda = p + iq$, where p and q are real $\implies$ interesting scenarios where $p > 0$ or, $p < 0$ while $q \neq 0$.

For a general dynamical system, the solution for an initial state $(X_o,\ Y_0)$ may be :
$X(t) = X_o + \Delta_x e^{\lambda t}$; and similar for Y
the behaviour depends on λ. Take $\Delta > 0$

real $\lambda > 0 \implies X(t)$ monotonic increasing function of t

real $\lambda < 0 \implies X(t)$ monotonic decreasing function of t

λ purely imaginary $\implies X(t)$ oscillatory function in t

$\lambda = p + iq$, where p and q are real $\implies$ interesting scenarios where $p > 0$ or, $p < 0$ while $q \neq 0$.

$p > 0 \implies$ "spiraling out" phase portrait

For a general dynamical system, the solution for an initial state $(X_o,\ Y_0)$ may be :
$X(t) = X_o + \Delta_x e^{\lambda t}$; and similar for Y
the behaviour depends on λ. Take $\Delta > 0$

real $\lambda > 0 \implies X(t)$ monotonic increasing function of t

real $\lambda < 0 \implies X(t)$ monotonic decreasing function of t

λ purely imaginary $\implies X(t)$ oscillatory function in t

$\lambda = p + iq$, where p and q are real $\implies$ interesting scenarios where $p > 0$ or, $p < 0$ while $q \neq 0$.

$p > 0 \implies$ "spiraling out" phase portrait

$p < 0$ "spiraling in" behaviour. They spiral about a "limit point" or a "limit cycle".

Finding steady states :

Finding steady states :

$$\frac{dx}{d\tau} = ax - xy = 0$$

$$\frac{dy}{d\tau} = xy - y = 0$$

Finding steady states :

$\frac{dx}{d\tau} = ax - xy = 0$

$\frac{dy}{d\tau} = xy - y = 0$

$\implies x = 1;\; y = a$ or, $x = y = 0$

Belousov-Zhabotinski reaction
($KBrO_3$, malonic acid, cerium (IV) salt in acidic solution):

Belousov-Zhabotinski reaction
($KBrO_3$, malonic acid, cerium (IV) salt in acidic solution):
$BrO_3^- + HBrO_2 + H_3O^+ \longrightarrow 2BrO_2\cdot + 2H_2O$

Belousov-Zhabotinski reaction
($KBrO_3$, malonic acid, cerium (IV) salt in acidic solution):
$BrO_3^- + HBrO_2 + H_3O^+ \longrightarrow 2BrO_2\cdot + 2H_2O$
$2BrO_2\cdot + 2Ce(III) + 2H_3O^+ \longrightarrow 2HBrO_2 + 2Ce(IV) + 2H_2O$

Belousov-Zhabotinski reaction
($KBrO_3$, malonic acid, cerium (IV) salt in acidic solution):
$BrO_3^- + HBrO_2 + H_3O^+ \longrightarrow 2BrO_2\cdot + 2H_2O$
$2BrO_2\cdot + 2Ce(III) + 2H_3O^+ \longrightarrow 2HBrO_2 + 2Ce(IV) + 2H_2O$
product $HBrO_2$is a reactant in first step and provides a feedback mechanism that enhances rate of formation of $HBrO_2$

Examples of oscillatory systems:
1. Demand and supply [Economics]
(2) process of sleep [Biology]
(3) Belousov-Zhabotinskii reaction ("oscillatory reaction") [Chemistry]

https://en.wikipedia.org/wiki/Brusselator

$A \rightarrow X$

$2X + Y \rightarrow 3X$

$B + X \rightarrow Y + D$

$X \rightarrow E$

https://en.wikipedia.org/wiki/Brusselator

$A \to X$

$2X + Y \to 3X$

$B + X \to Y + D$

$X \to E$

https://en.wikipedia.org/wiki/Oregonator

$A + Y \longrightarrow X + P$

$X + Y \longrightarrow 2\ P$

$A + X \longrightarrow 2\ X + 2\ Z$

$2\ X \longrightarrow A + P$

$B + Z \longrightarrow \frac{1}{2} f\ Y$

Monika Sharma and Praveen Kumar,
Chemical Oscillations - Basic Principles and Examples,
Resonance, vol.11, #2, Feb 2006 p. 43-50
Chemical Oscillations - Mathematical Modelling,
Resonance, vol.11, #7, July 2006 p. 61-69

Motion of a particle attached to a spring

Motion of a particle attached to a spring

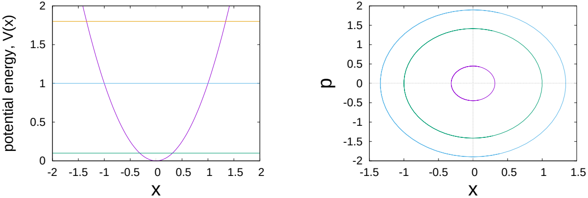
parabolic (simple harmonic) potential :
$m\ddot{x} = -k(x - x_0)$

Motion of a particle attached to a spring

parabolic (simple harmonic) potential :
$m\ddot{x} = -k(x - x_0)$

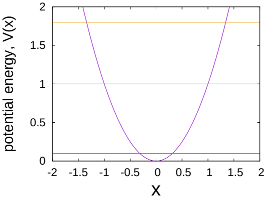
or, $\ddot{x} + \omega_o^2 x = 0$ $(x_0 = 0; \omega_o^2 = \frac{k}{m})$

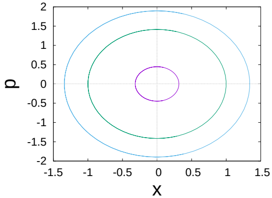
phase plane portrait for : $\dot{x} = y$; $\dot{y} = -x$
$y \equiv$ momentum p; and constants are taken unity;

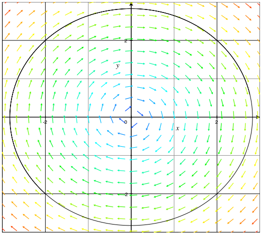


phase plane portrait for : $\dot{x} = y$; $\dot{y} = -x$

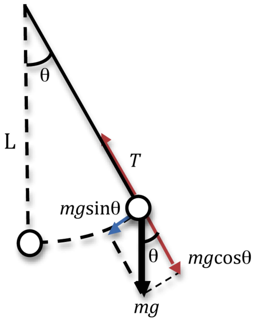
$y \equiv$ momentum p; and constants are taken unity;

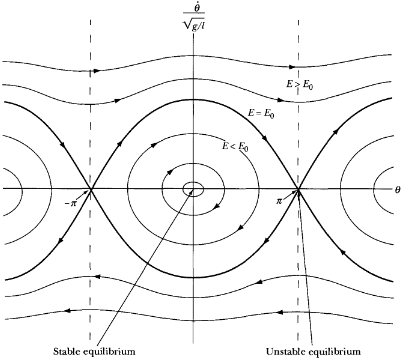




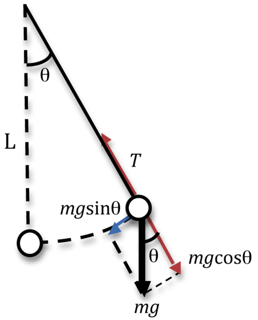


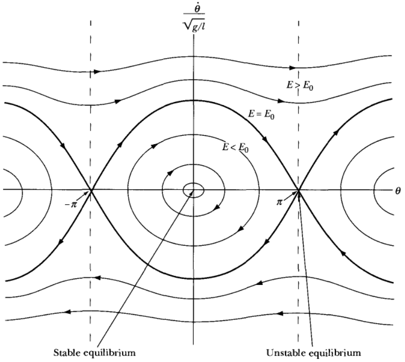
frictionless simple pendulum, $\ddot{\theta} = -\frac{g}{l}\sin\theta$, and its dynamics





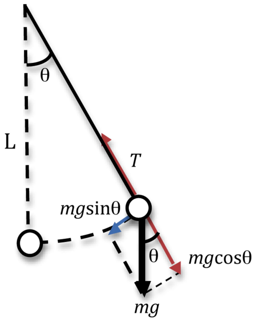
frictionless simple pendulum, $\ddot{\theta} = -\frac{g}{l}\sin\theta$, and its dynamics

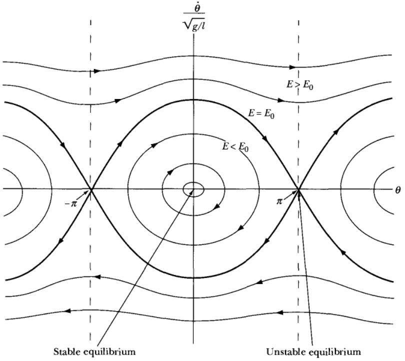




$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)$$

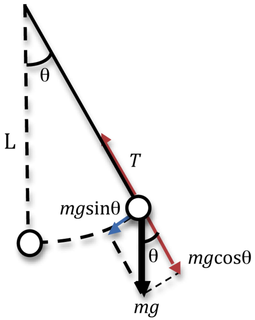
frictionless simple pendulum, $\ddot{\theta} = -\frac{g}{l}\sin\theta$, and its dynamics

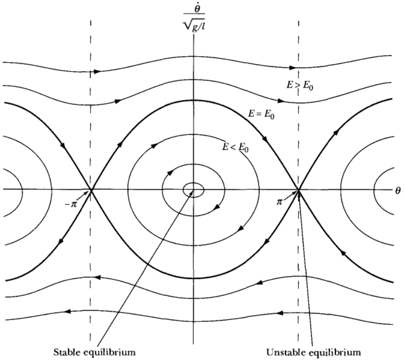




$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) \text{ or, } \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) + \omega^2\sin\theta = 0;\ \omega^2 = \frac{g}{l}$$

frictionless simple pendulum, $\ddot{\theta} = -\frac{g}{l}\sin\theta$, and its dynamics

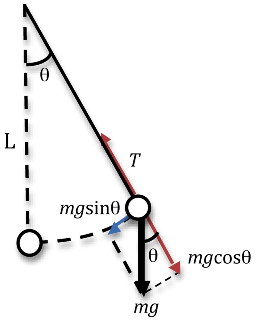


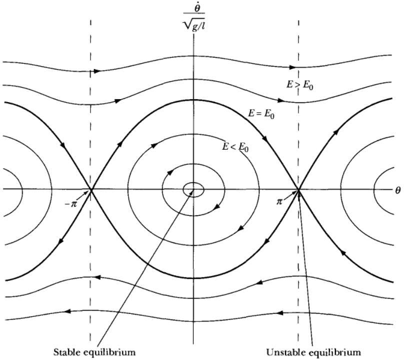


$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)$ or, $\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) + \omega^2\sin\theta = 0$; $\omega^2 = \frac{g}{l}$

integrating, $\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = C$

frictionless simple pendulum, $\ddot{\theta} = -\frac{g}{l}\sin\theta$, and its dynamics

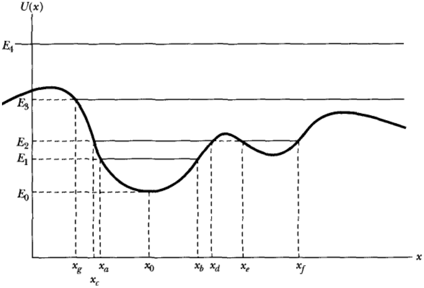




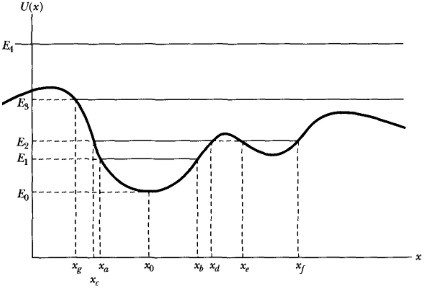
$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)$ or, $\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) + \omega^2\sin\theta = 0;\ \omega^2 = \frac{g}{l}$

integrating, $\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = C$ or, $\dot{\theta} = \pm\sqrt{2(C + \omega^2\cos\theta)}$

complex potentials



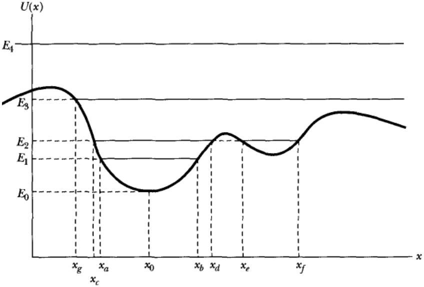
complex potentials



$\frac{1}{2}mv^2 \geq 0 \implies E \geq U$

motion bounded for energies E_1 and E_2

complex potentials

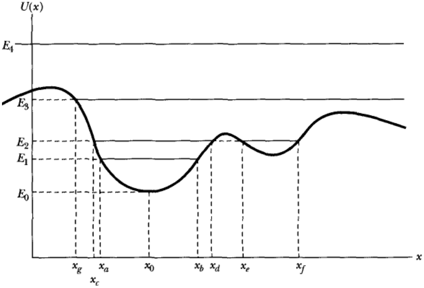


$\frac{1}{2}mv^2 \geq 0 \implies E \geq U$

motion bounded for energies E_1 and E_2

For E_1, motion **periodic** between **turning points** x_a and x_b

complex potentials



$\frac{1}{2}mv^2 \geq 0 \implies E \geq U$

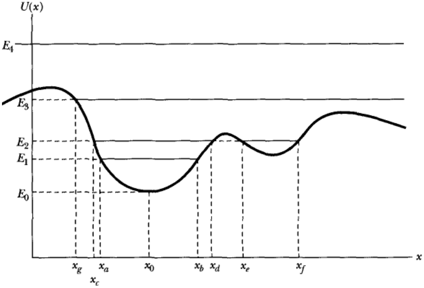
motion bounded for energies E_1 and E_2

For E_1, motion **periodic** between **turning points** x_a and x_b

For E_2, motion periodic
- two possible regions :

$x_c \leq x \leq x_d$ and $x_e \leq x \leq x_f$

complex potentials



$\frac{1}{2}mv^2 \geq 0 \implies E \geq U$

motion bounded for energies E_1 and E_2

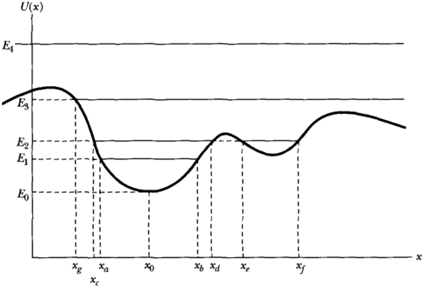
For E_1, motion **periodic** between **turning points** x_a and x_b

For E_2, motion periodic
- two possible regions :

$x_c \leq x \leq x_d$ and $x_e \leq x \leq x_f$

particle cannot "jump" from one "pocket" to the other

complex potentials



$\frac{1}{2}mv^2 \geq 0 \implies E \geq U$

motion bounded for energies E_1 and E_2

For E_1, motion **periodic** between **turning points** x_a and x_b

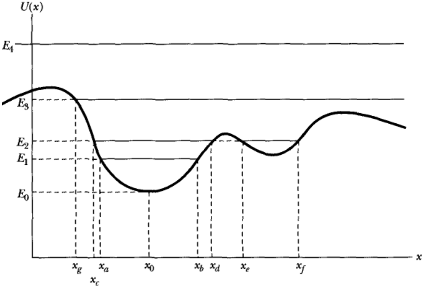
For E_2, motion periodic
- two possible regions :

$x_c \leq x \leq x_d$ and $x_e \leq x \leq x_f$

particle cannot "jump" from one "pocket" to the other

once in a pocket, it must remain there forever if energy$= E_2$

complex potentials



$\frac{1}{2}mv^2 \geq 0 \implies E \geq U$

motion bounded for energies E_1 and E_2

For E_1, motion **periodic** between **turning points** x_a and x_b

For E_2, motion periodic
- two possible regions :

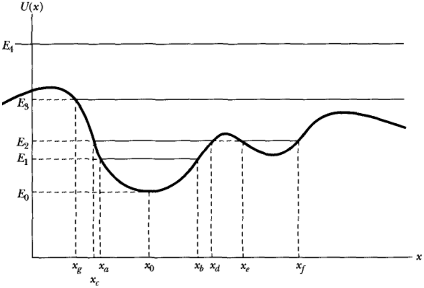
$x_c \leq x \leq x_d$ and $x_e \leq x \leq x_f$

particle cannot "jump" from one "pocket" to the other

once in a pocket, it must remain there forever if energy$= E_2$

For E_0, only one solution, $x = x_0$: particle at rest with $T = 0$

complex potentials



$\frac{1}{2}mv^2 \geq 0 \implies E \geq U$

motion bounded for energies E_1 and E_2

For E_1, motion **periodic** between **turning points** x_a and x_b

For E_2, motion periodic
- two possible regions :

$x_c \leq x \leq x_d$ and $x_e \leq x \leq x_f$

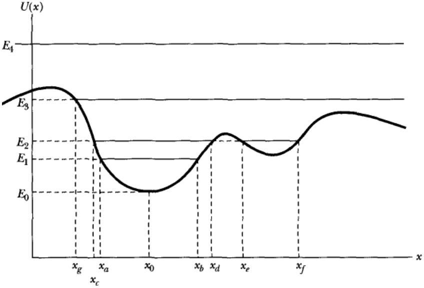
particle cannot "jump" from one "pocket" to the other

once in a pocket, it must remain there forever if energy$= E_2$

For E_0, only one solution, $x = x_0$: particle at rest with $T = 0$

For E_3: particle comes in from infinity, stops and turns at $x = x_g$, and returns to infinity— like a tennis ball bouncing against a wall

complex potentials



$\frac{1}{2}mv^2 \geq 0 \implies E \geq U$

motion bounded for energies E_1 and E_2

For E_1, motion **periodic** between **turning points** x_a and x_b

For E_2, motion periodic
- two possible regions :

$x_c \leq x \leq x_d$ and $x_e \leq x \leq x_f$

particle cannot "jump" from one "pocket" to the other

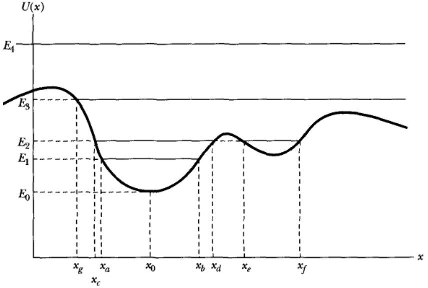
once in a pocket, it must remain there forever if energy$= E_2$

For E_0, only one solution, $x = x_0$: particle at rest with $T = 0$

For E_3: particle comes in from infinity, stops and turns at $x = x_g$, and returns to infinity— like a tennis ball bouncing against a wall

For E_4: motion unbounded and particle may be at any position
- speed varies as it depends on difference between E_4 and $U(x)$

complex potentials



$\frac{1}{2}mv^2 \geq 0 \implies E \geq U$

motion bounded for energies E_1 and E_2

For E_1, motion **periodic** between **turning points** x_a and x_b

For E_2, motion periodic
- two possible regions :

$x_c \leq x \leq x_d$ and $x_e \leq x \leq x_f$

particle cannot "jump" from one "pocket" to the other

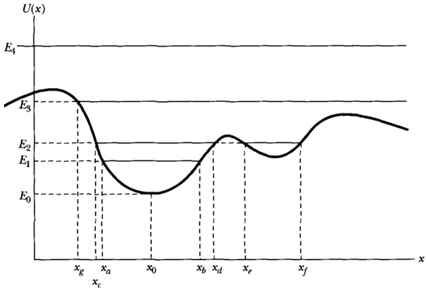
once in a pocket, it must remain there forever if energy$= E_2$

For E_0, only one solution, $x = x_0$: particle at rest with $T = 0$

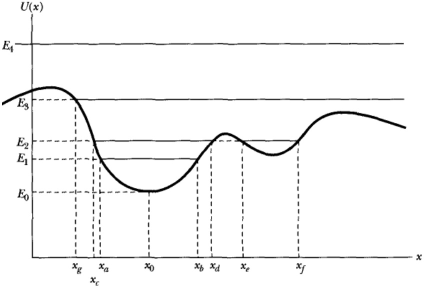
For E_3: particle comes in from infinity, stops and turns at $x = x_g$, and returns to infinity— like a tennis ball bouncing against a wall

For E_4: motion unbounded and particle may be at any position
- speed varies as it depends on difference between E_4 and $U(x)$

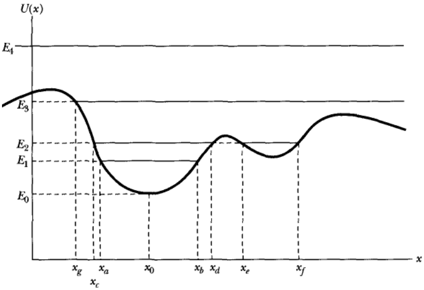
If moving to right, it speeds up and slows down and continues to infinity



U(x)
E4
E3
E2
E1
E0
x
xg
xa
xc
x0
xb
xd
xe
xf

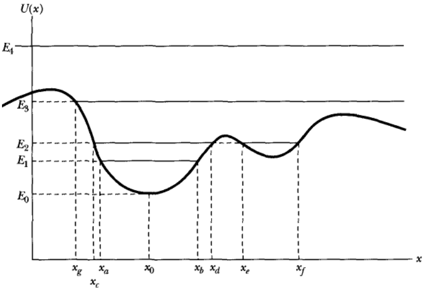


in region $x_a \leq x \leq x_b$
$U(x) = -k(x—x_0)^2$



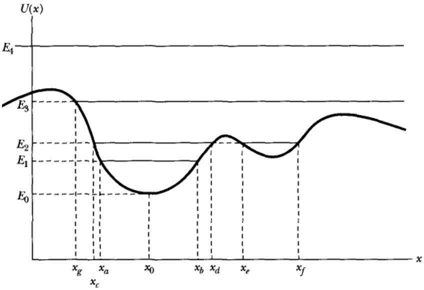
in region $x_a \leq x \leq x_b$
$U(x) = -k(x—x_0)^2$

particle with energy $\gtrsim E_0$ oscillates about equilibrium point $x = x_0$



in region $x_a \leq x \leq x_b$
$U(x) = -k(x—x_0)^2$

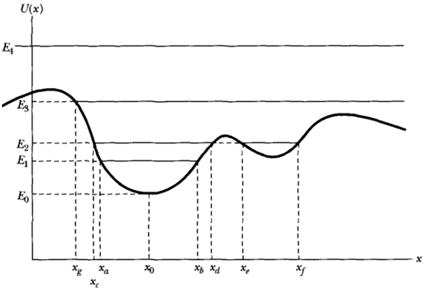
particle with energy $\gtrsim E_0$ oscillates about equilibrium point $x = x_0$
particle placed at x_0 remains there



in region $x_a \leq x \leq x_b$
$U(x) = -k(x—x_0)^2$

particle with energy $\gtrsim E_0$ oscillates about equilibrium point $x = x_0$
particle placed at x_0 remains there

Equilibrium may be stable, unstable, or neutral

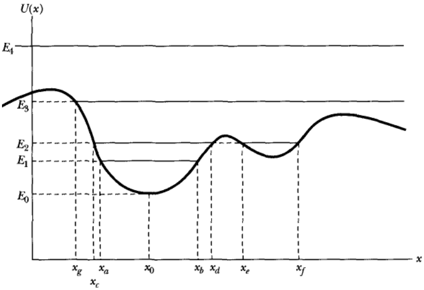


in region $x_a \leq x \leq x_b$
$U(x) = -k(x—x_0)^2$

particle with energy $\gtrsim E_0$ oscillates about equilibrium point $x = x_0$
particle placed at x_0 remains there

Equilibrium may be stable, unstable, or neutral

equilibrium at x_0 is stable because if particle is placed on either side of it, it would eventually return there



in region $x_a \leq x \leq x_b$
$U(x) = -k(x—x_0)^2$

particle with energy $\gtrsim E_0$ oscillates about equilibrium point $x = x_0$
particle placed at x_0 remains there

Equilibrium may be stable, unstable, or neutral

equilibrium at x_0 is stable because if particle is placed on either side of it, it would eventually return there

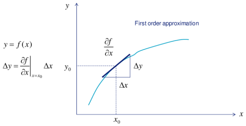
point at maximum between x_d and x_e is unstable equilibrium

Taylor series : a series expansion of a function $f(x)$ about a point $x = x_0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{unitary method}} + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

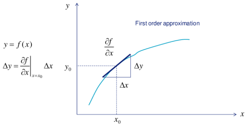
Taylor series : a series expansion of a function $f(x)$ about a point $x = x_0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{unitary method}} + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$



Taylor series : a series expansion of a function $f(x)$ about a point $x = x_0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{unitary method}} + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

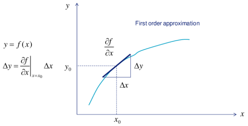


for arbitrary x_0,

$$f(x + a) = f(x) + f'(x)a + \frac{f''(x)}{2!}a^2 + \frac{f'''(x)}{3!}a^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}a^n + \cdots$$

Taylor series : a series expansion of a function $f(x)$ about a point $x = x_0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{unitary method}} + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$



for arbitrary x_0,

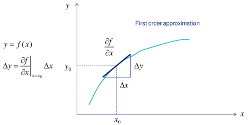
$$f(x + a) = f(x) + f'(x)a + \frac{f''(x)}{2!}a^2 + \frac{f'''(x)}{3!}a^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}a^n + \cdots$$

when $x_0 = 0$, it is called a Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Taylor series : a series expansion of a function $f(x)$ about a point $x = x_0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{unitary method}} + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$



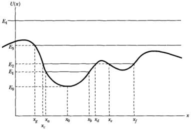
for arbitrary x_0,

$$f(x + a) = f(x) + f'(x)a + \frac{f''(x)}{2!}a^2 + \frac{f'''(x)}{3!}a^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}a^n + \cdots$$

when $x_0 = 0$, it is called a Maclaurin series

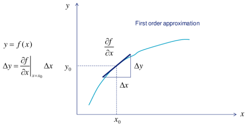
$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

express potential $U(x)$ in a Taylor series about equilibrium point



Taylor series : a series expansion of a function $f(x)$ about a point $x = x_0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{unitary method}} + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$



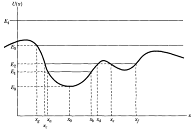
for arbitrary x_0,

$$f(x+a) = f(x) + f'(x)a + \frac{f''(x)}{2!}a^2 + \frac{f'''(x)}{3!}a^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}a^n + \cdots$$

when $x_0 = 0$, it is called a Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

express potential $U(x)$ in a Taylor series about equilibrium point

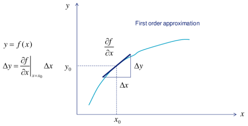


$$U(x) = U(x_0) + (x - x_0)\frac{dU}{dx}\bigg|_{x_0} + \frac{(x-x_0)^2}{2!}\frac{d^2U}{dx^2}\bigg|_{x_0} + \frac{(x-x_0)^3}{3!}\frac{d^3U}{dx^3}\bigg|_{x_0} + \cdots$$

first order or linear term

Taylor series : a series expansion of a function $f(x)$ about a point $x = x_0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{unitary method}} + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$



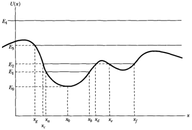
for arbitrary x_0,

$$f(x + a) = f(x) + f'(x)a + \frac{f''(x)}{2!}a^2 + \frac{f'''(x)}{3!}a^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}a^n + \cdots$$

when $x_0 = 0$, it is called a Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

express potential $U(x)$ in a Taylor series about equilibrium point



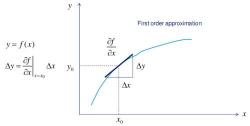
$$U(x) = U(x_0) + (x - x_0)\left.\frac{dU}{dx}\right|_{x_0} + \frac{(x - x_0)^2}{2!}\left.\frac{d^2U}{dx^2}\right|_{x_0} + \frac{(x - x_0)^3}{3!}\left.\frac{d^3U}{dx^3}\right|_{x_0} + \cdots$$

first order or linear term

at equilibrium, $\left.\frac{dU}{dx}\right|_{x_0} = 0$, $\quad \therefore U(x) = U(x_0) + \frac{(x - x_0)^2}{2!}\left.\frac{d^2U}{dx^2}\right|_{x_0} + \cdots$

Taylor series : a series expansion of a function $f(x)$ about a point $x = x_0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{unitary method}} + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$



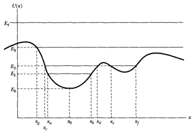
for arbitrary x_0,

$$f(x + a) = f(x) + f'(x)a + \frac{f''(x)}{2!}a^2 + \frac{f'''(x)}{3!}a^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}a^n + \cdots$$

when $x_0 = 0$, it is called a Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

express potential $U(x)$ in a Taylor series about equilibrium point



$$U(x) = U(x_0) + (x - x_0)\left.\frac{dU}{dx}\right|_{x_0} + \frac{(x - x_0)^2}{2!}\left.\frac{d^2U}{dx^2}\right|_{x_0} + \frac{(x - x_0)^3}{3!}\left.\frac{d^3U}{dx^3}\right|_{x_0} + \cdots$$

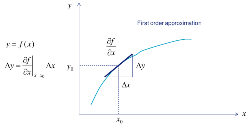
first order or linear term

at equilibrium, $\left.\frac{dU}{dx}\right|_{x_0} = 0$, $\quad \therefore U(x) = U(x_0) + \frac{(x - x_0)^2}{2!}\left.\frac{d^2U}{dx^2}\right|_{x_0} + \cdots$

$\left.\frac{d^2U}{dx^2}\right|_{x_0} > 0$: stable equilibrium;

Taylor series : a series expansion of a function $f(x)$ about a point $x = x_0$

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{unitary method}} + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$



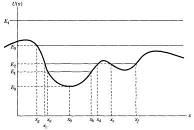
for arbitrary x_0,

$$f(x+a) = f(x) + f'(x)a + \frac{f''(x)}{2!}a^2 + \frac{f'''(x)}{3!}a^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}a^n + \cdots$$

when $x_0 = 0$, it is called a Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

express potential $U(x)$ in a Taylor series about equilibrium point



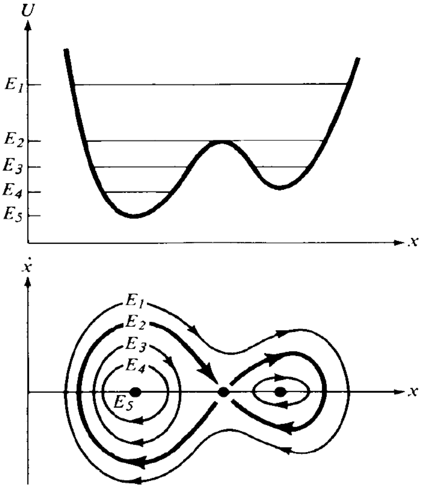
$$U(x) = U(x_0) + (x - x_0)\frac{dU}{dx}\Big|_{x_0} + \frac{(x-x_0)^2}{2!}\frac{d^2U}{dx^2}\Big|_{x_0} + \frac{(x-x_0)^3}{3!}\frac{d^3U}{dx^3}\Big|_{x_0} + \cdots$$

first order or linear term

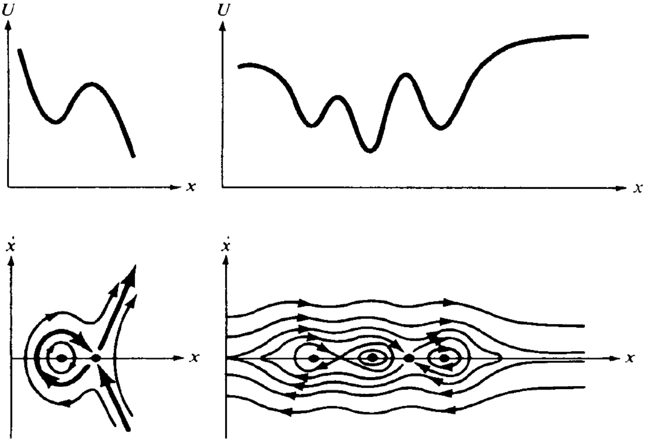
at equilibrium, $\frac{dU}{dx}\Big|_{x_0} = 0$, $\quad \therefore U(x) = U(x_0) + \frac{(x-x_0)^2}{2!}\frac{d^2U}{dx^2}\Big|_{x_0} + \cdots$

$\frac{d^2U}{dx^2}\Big|_{x_0} > 0$: stable equilibrium; $\quad \frac{d^2U}{dx^2}\Big|_{x_0} < 0$: unstable equilibrium

Phase space description :



curve separatring the two distinct dynamical behaviours is called **Separatrix**



Pot. energy

Phase curves

Nonlinear Oscillations and Chaos

Nonlinear Oscillations and Chaos

Laplace : if we know position and velocities of all particles in universe then we know future for all time $\Leftarrow$ determinism

Nonlinear Oscillations and Chaos

Laplace : if we know position and velocities of all particles in universe then we know future for all time $\Leftarrow$ determinism

knowing laws of nature is not enough. Much of nature seems to be chaotic

Nonlinear Oscillations and Chaos

Laplace : if we know position and velocities of all particles in universe
then we know future for all time $\Leftarrow$ determinism

knowing laws of nature is not enough. Much of nature seems to be chaotic

deterministic chaos (as opposed to randomness) : motion of a system whose time evolution has a sensitive dependence on initial conditions

Nonlinear Oscillations and Chaos

Laplace : if we know position and velocities of all particles in universe then we know future for all time $\Leftarrow$ determinism

knowing laws of nature is not enough. Much of nature seems to be chaotic

deterministic chaos (as opposed to randomness) : motion of a system whose time evolution has a sensitive dependence on initial conditions

determinism : system develops from one moment to next, where present depends on past in a well-determined way through physical laws

Nonlinear Oscillations and Chaos

Laplace : if we know position and velocities of all particles in universe then we know future for all time $\Leftarrow$ determinism

knowing laws of nature is not enough. Much of nature seems to be chaotic

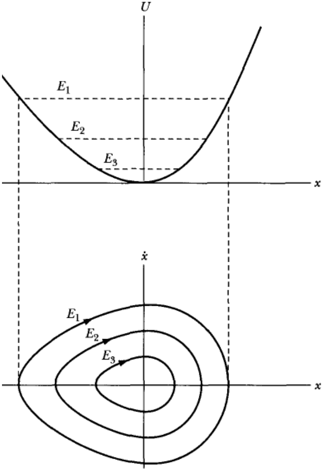
deterministic chaos (as opposed to randomness) : motion of a system whose time evolution has a sensitive dependence on initial conditions

determinism : system develops from one moment to next, where present depends on past in a well-determined way through physical laws

random process : present has no causal connection to past (e.g., flipping of coin)

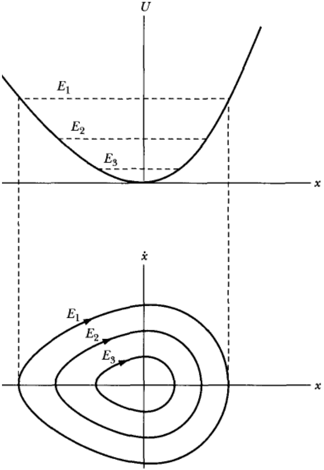
Phase Diagrams for Nonlinear Systems : $\dot{x} \propto \sqrt{E - U(x)}$
ex. : asymm. pot. - soft for $x < 0$ and hard for $x > 0$: no damping

Phase Diagrams for Nonlinear Systems : $\dot{x} \propto \sqrt{E - U(x)}$
ex. : asymm. pot. - soft for $x < 0$ and hard for $x > 0$: no damping



Phase Diagrams for Nonlinear Systems : $\dot{x} \propto \sqrt{E - U(x)}$

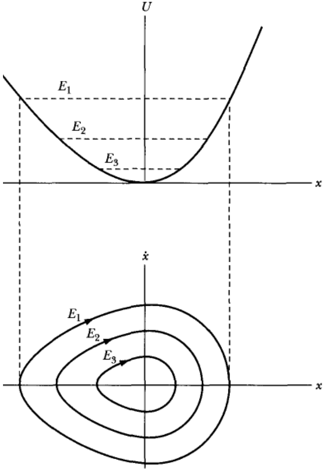
ex. : asymm. pot. - soft for $x < 0$ and hard for $x > 0$: no damping



attractor : set of points (or point) in phase space toward which a system is 'attracted' when damping is present

Phase Diagrams for Nonlinear Systems : $\dot{x} \propto \sqrt{E - U(x)}$

ex. : asymm. pot. - soft for $x < 0$ and hard for $x > 0$: no damping

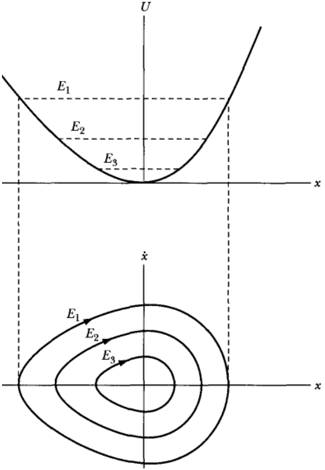


attractor : set of points (or point) in phase space toward which a system is 'attracted' when damping is present

here, point $x = 0$ is an attractor

Phase Diagrams for Nonlinear Systems : $\dot{x} \propto \sqrt{E - U(x)}$

ex. : asymm. pot. - soft for $x < 0$ and hard for $x > 0$: no damping

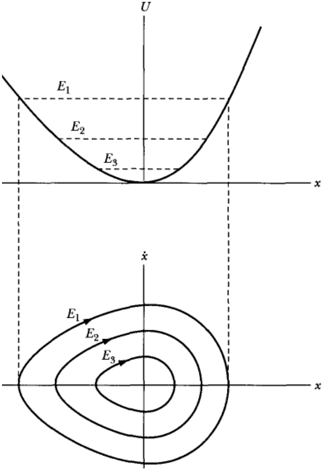


attractor : set of points (or point) in phase space toward which a system is 'attracted' when damping is present here, point $x = 0$ is an attractor

if total energy $E <$ height to which potential rises on either side of $x = 0$, then particle is 'trapped' in the potential well ($x_a < x < x_b$)

Phase Diagrams for Nonlinear Systems : $\dot{x} \propto \sqrt{E - U(x)}$

ex. : asymm. pot. - soft for $x < 0$ and hard for $x > 0$: no damping



attractor : set of points (or point) in phase space toward which a system is 'attracted' when damping is present

here, point $x = 0$ is an attractor

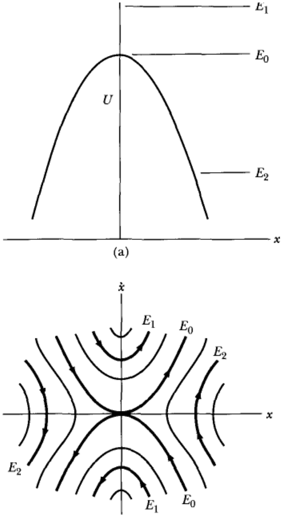
if total energy $E <$ height to which potential rises on either side of $x = 0$, then particle is 'trapped' in the potential well ($x_a < x < x_b$)

$x = 0$: stable equilibrium, $\frac{d^2U}{dx^2} > 0$

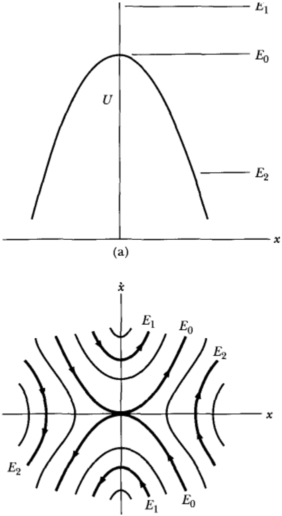
small disturbance results in locally bounded motion

Inverted asymmetric potential

Inverted asymmetric potential

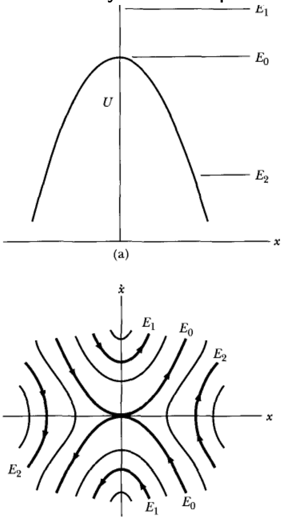


Inverted asymmetric potential



$x = 0$ is unstable equilibrium : $\frac{d^2U}{dx^2} < 0$

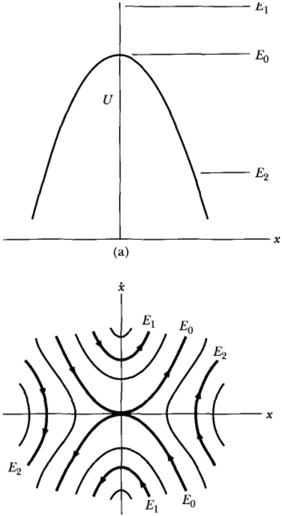
Inverted asymmetric potential



$x=0$ is unstable equilibrium : $\frac{d^2U}{dx^2} < 0$

If potential parabolic, $U(x) = -\frac{1}{2}kx^2$, then phase paths for energy E_0 are straight lines

Inverted asymmetric potential

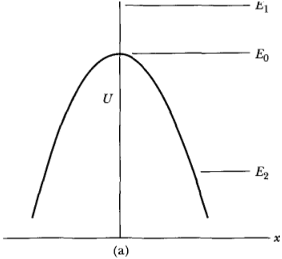


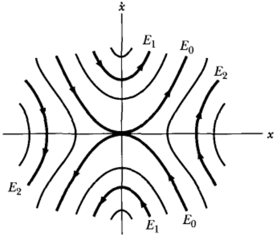
$x=0$ is unstable equilibrium : $\frac{d^2U}{dx^2} < 0$

If potential parabolic, $U(x) = -\frac{1}{2}kx^2$, then phase paths for energy E_0 are straight lines

those corresponding to energies E_1 and E_2 are hyperbolas

Inverted asymmetric potential





$x=0$ is unstable equilibrium : $\frac{d^2U}{dx^2} < 0$

If potential parabolic, $U(x) = -\frac{1}{2}kx^2$, then phase paths for energy E_0 are straight lines

those corresponding to energies E_1 and E_2 are hyperbolas

This is the limit to which phase paths approach if nonlinear term for force decreases in magnitude

van der Pol Equation : nonlinear oscillations in vacuum tube circuits

van der Pol Equation : nonlinear oscillations in vacuum tube circuits

$$\ddot{x} + \mu \left(x^2 - a^2\right) \dot{x} + \omega_o^2 x = 0$$

van der Pol Equation : nonlinear oscillations in vacuum tube circuits

$$\ddot{x} + \mu \left(x^2 - a^2\right) \dot{x} + \omega_o^2 x = 0$$

For $a = 1$ and $\omega_o = 1$, we have $\ddot{x} + \mu \left(x^2 - 1\right) \dot{x} + x = 0$

van der Pol Equation : nonlinear oscillations in vacuum tube circuits

$$\ddot{x} + \mu \left(x^2 - a^2\right) \dot{x} + \omega_o^2 x = 0$$

For $a = 1$ and $\omega_o = 1$, we have $\ddot{x} + \mu \left(x^2 - 1\right) \dot{x} + x = 0$

If $|x| > |a|$, then coefficient of $\dot{x}$ is $+$ve and system is damped,

i.e., max. amplitude decreases in time

van der Pol Equation : nonlinear oscillations in vacuum tube circuits

$$\ddot{x} + \mu\left(x^2 - a^2\right)\dot{x} + \omega_o^2 x = 0$$

For $a = 1$ and $\omega_o = 1$, we have $\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0$

If $|x| > |a|$, then coefficient of $\dot{x}$ is $+$ve and system is damped,
i.e., max. amplitude decreases in time

if $|x| < |a|$, then negative damping occurs, i.e., max. amplitude increases in time

van der Pol Equation : nonlinear oscillations in vacuum tube circuits

$$\ddot{x} + \mu\left(x^2 - a^2\right)\dot{x} + \omega_o^2 x = 0$$

For $a = 1$ and $\omega_o = 1$, we have $\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0$

If $|x| > |a|$, then coefficient of $\dot{x}$ is +ve and system is damped,

i.e., max. amplitude decreases in time

if $|x| < |a|$, then negative damping occurs, i.e., max. amplitude increases in time

$\therefore \exists$ soln. for which max. amplitude neither increases nor decreases with time

van der Pol Equation : nonlinear oscillations in vacuum tube circuits

$$\ddot{x} + \mu \left(x^2 - a^2\right) \dot{x} + \omega_o^2 x = 0$$

For $a = 1$ and $\omega_o = 1$, we have $\ddot{x} + \mu \left(x^2 - 1\right) \dot{x} + x = 0$

If $|x| > |a|$, then coefficient of $\dot{x}$ is $+$ve and system is damped,

i.e., max. amplitude decreases in time

if $|x| < |a|$, then negative damping occurs, i.e., max. amplitude increases in time

$\therefore \exists$ soln. for which max. amplitude neither increases nor decreases with time

Such a curve in phase plane is called a limit cycle and is an attractor for the system

van der Pol Equation : nonlinear oscillations in vacuum tube circuits

$$\ddot{x} + \mu\left(x^2 - a^2\right)\dot{x} + \omega_o^2 x = 0$$

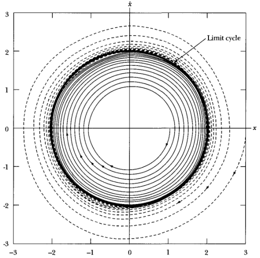
For $a = 1$ and $\omega_o = 1$, we have $\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0$

If $|x| > |a|$, then coefficient of $\dot{x}$ is +ve and system is damped, i.e., max. amplitude decreases in time

if $|x| < |a|$, then negative damping occurs, i.e., max. amplitude increases in time

$\therefore \exists$ soln. for which max. amplitude neither increases nor decreases with time

Such a curve in phase plane is called a limit cycle and is an attractor for the system



damping parameter $\mu = 0.05$

solution slowly approaches limit cycle

van der Pol Equation : nonlinear oscillations in vacuum tube circuits

$$\ddot{x} + \mu \left(x^2 - a^2\right) \dot{x} + \omega_o^2 x = 0$$

For $a = 1$ and $\omega_o = 1$, we have $\ddot{x} + \mu \left(x^2 - 1\right) \dot{x} + x = 0$

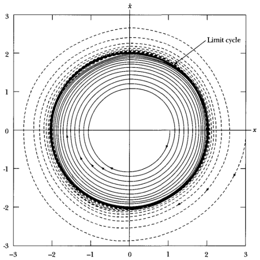
If $|x| > |a|$, then coefficient of $\dot{x}$ is +ve and system is damped,

i.e., max. amplitude decreases in time

if $|x| < |a|$, then negative damping occurs, i.e., max. amplitude increases in time

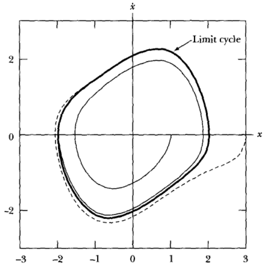
$\therefore\ \exists$ soln. for which max. amplitude neither increases nor decreases with time

Such a curve in phase plane is called a limit cycle and is an attractor for the system



damping parameter $\mu = 0.05$

solution slowly approaches limit cycle



damping parameter $\mu = 0.5$

solution approaches limit cycle faster

Phase paths outside the limit cycle spiral inward

Phase paths outside the limit cycle spiral inward
and those inside the limit cycle spiral outward

Phase paths outside the limit cycle spiral inward
and those inside the limit cycle spiral outward

limit cycle defines locally bounded stable motion

Phase paths outside the limit cycle spiral inward
and those inside the limit cycle spiral outward

limit cycle defines locally bounded stable motion

A system described by van der Pol's equation is **self-limiting**

Phase paths outside the limit cycle spiral inward
and those inside the limit cycle spiral outward

limit cycle defines locally bounded stable motion

A system described by van der Pol's equation is **self-limiting**

i.e., once set into motion under conditions that lead to an increasing amplitude, it is automatically prevented from growing without bound

Phase paths outside the limit cycle spiral inward
and those inside the limit cycle spiral outward

limit cycle defines locally bounded stable motion

A system described by van der Pol's equation is **self-limiting**

i.e., once set into motion under conditions that lead to an increasing amplitude, it is automatically prevented from growing without bound
system has this property whether the initial amplitude is greater or smaller than critical (limiting) amplitude

Phase paths outside the limit cycle spiral inward
and those inside the limit cycle spiral outward

limit cycle defines locally bounded stable motion

A system described by van der Pol's equation is **self-limiting**

i.e., once set into motion under conditions that lead to an increasing amplitude, it is automatically prevented from growing without bound
system has this property whether the initial amplitude is greater or smaller than critical (limiting) amplitude

For a small value of $\mu(0.05)$, x and $\dot{x}$ are sinusoidal with time

Phase paths outside the limit cycle spiral inward
and those inside the limit cycle spiral outward

limit cycle defines locally bounded stable motion

A system described by van der Pol's equation is **self-limiting**

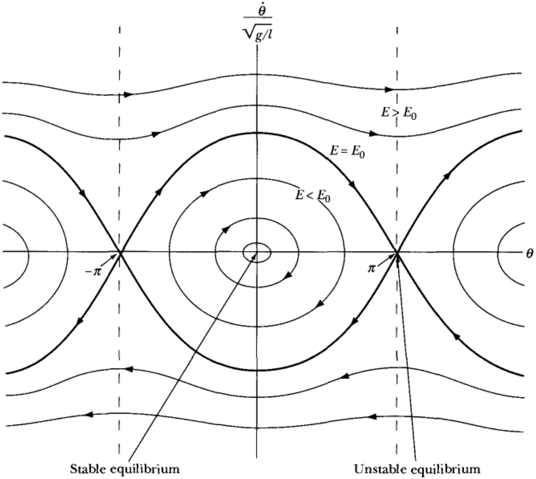
i.e., once set into motion under conditions that lead to an increasing amplitude, it is automatically prevented from growing without bound
system has this property whether the initial amplitude is greater or smaller than critical (limiting) amplitude

For a small value of $\mu(0.05)$, x and $\dot{x}$ are sinusoidal with time

for higher values of $\mu(0.5)$, sinusoidal shapes become skewed

Plane pendulum : phase diagram

Plane pendulum : phase diagram



map : use n to denote time sequence of a system
and x to denote a physical observable

describe progression of nonlinear system at a particular moment by investigating how the $(n+1)$th state (or iterate) depends on the nth state

map : use n to denote time sequence of a system
and x to denote a physical observable

describe progression of nonlinear system at a particular moment by investigating how the $(n+1)$th state (or iterate) depends on the nth state

e.g., $x_{n+1} = (2x_n + 3)^2$

map : use n to denote time sequence of a system
and x to denote a physical observable

describe progression of nonlinear system at a particular moment by investigating how the $(n+1)$th state (or iterate) depends on the nth state

e.g., $x_{n+1} = (2x_n + 3)^2$

The relationship, $x_{n+1} = f(x_n)$, is called a map

Logistic equation : $f(a,x) = ax(1-x)$

Logistic equation : $f(a,x) = ax(1-x)$

Logistic map : $x_{n+1} = ax_n(1-x_n)$

Logistic equation : $f(a,x) = ax(1-x)$

Logistic map : $x_{n+1} = ax_n(1-x_n)$

biological application : population growth of fish in a pond
iterations, or n, represent fish population,
$x_1 = $ #fish in the pond at the beginning
If x_1 is small, population grows rapidly in early years because of available resources

Logistic equation : $f(a, x) = ax(1 - x)$

Logistic map : $x_{n+1} = ax_n(1 - x_n)$

biological application : population growth of fish in a pond
iterations, or n, represent fish population,
$x_1 = \#$fish in the pond at the beginning
If x_1 is small, population grows rapidly in early years because of available resources

overpopulation may eventually deplete number of fish

Logistic equation : $f(a, x) = ax(1-x)$

Logistic map : $x_{n+1} = ax_n(1-x_n)$

biological application : population growth of fish in a pond
iterations, or n, represent fish population,
$x_1 = \#$fish in the pond at the beginning
If x_1 is small, population grows rapidly in early years because of available resources

overpopulation may eventually deplete number of fish

population x_n is scaled so that its value fits in the interval $(0, 1)$

Logistic equation : $f(a, x) = ax(1 - x)$

Logistic map : $x_{n+1} = ax_n(1 - x_n)$

biological application : population growth of fish in a pond
iterations, or n, represent fish population,
$x_1 = \#$fish in the pond at the beginning
If x_1 is small, population grows rapidly in early years because of available resources

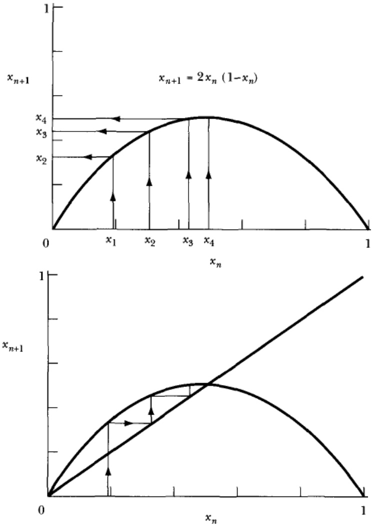
overpopulation may eventually deplete number of fish

population x_n is scaled so that its value fits in the interval $(0, 1)$

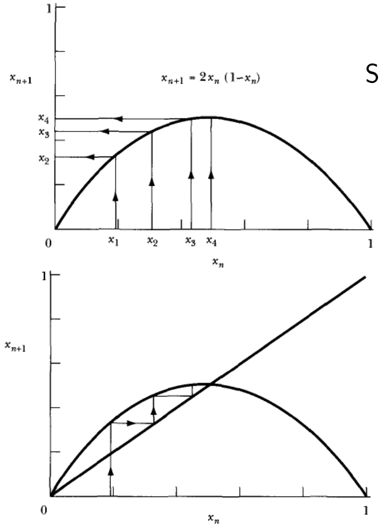
α is a model-dependent parameter representing average effects of environment

$\alpha = 2$

$\alpha = 2$

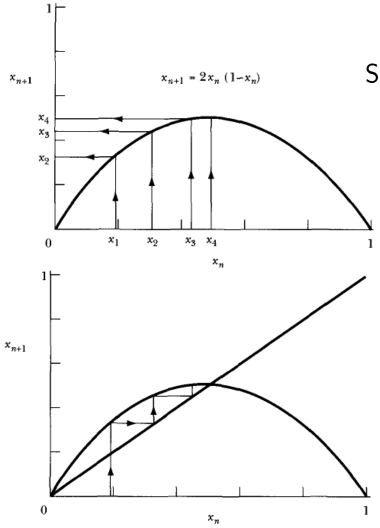


$\alpha = 2$



Start with initial value x_1 on horizontal axis, move up until we intersect with the curve $x_{n+1} = 2x_n(1 - x_n)$

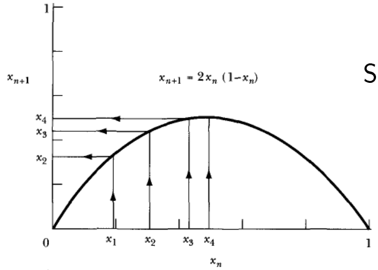
$\alpha = 2$

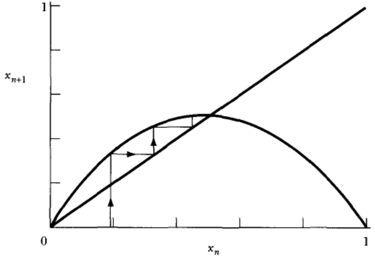


Start with initial value x_1 on horizontal axis, move up until we intersect with the curve $x_{n+1} = 2x_n(1 - x_n)$

then move left where we find x_2 on the vertical axis

$\alpha = 2$



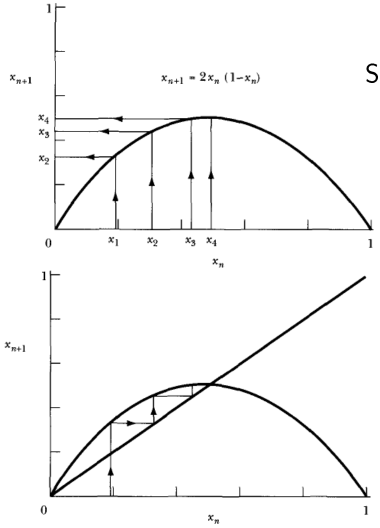


Start with initial value x_1 on horizontal axis, move up until we intersect with the curve $x_{n+1} = 2x_n(1 - x_n)$

then move left where we find x_2 on the vertical axis

then start with x_2 on horizontal axis and repeat the process to find x_3 on vertical axis

$\alpha = 2$



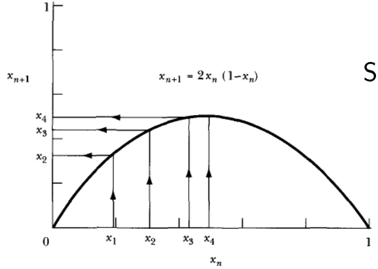
Start with initial value x_1 on horizontal axis, move up until we intersect with the curve $x_{n+1} = 2x_n(1 - x_n)$

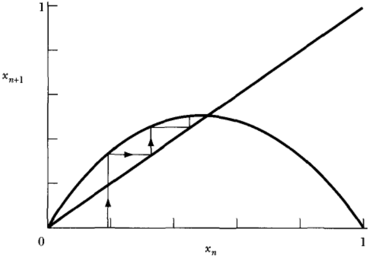
then move left where we find x_2 on the vertical axis

then start with x_2 on horizontal axis and repeat the process to find x_3 on vertical axis

after a few iterations, converge on $x = 0.5$

$\alpha = 2$





Start with initial value x_1 on horizontal axis, move up until we intersect with the curve $x_{n+1} = 2x_n(1 - x_n)$

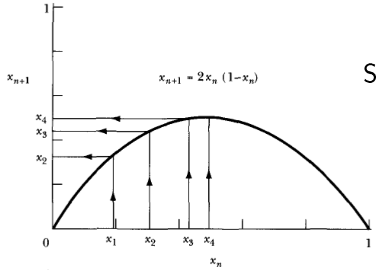
then move left where we find x_2 on the vertical axis

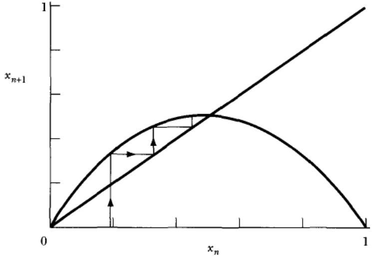
then start with x_2 on horizontal axis and repeat the process to find x_3 on vertical axis

after a few iterations, converge on $x = 0.5$

fish population stabilizes at half its maximum

$\alpha = 2$





Start with initial value x_1 on horizontal axis, move up until we intersect with the curve $x_{n+1} = 2x_n(1 - x_n)$

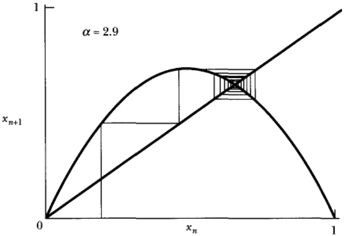
then move left where we find x_2 on the vertical axis

then start with x_2 on horizontal axis and repeat the process to find x_3 on vertical axis

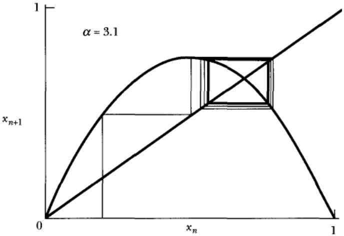
after a few iterations, converge on $x = 0.5$

fish population stabilizes at half its maximum

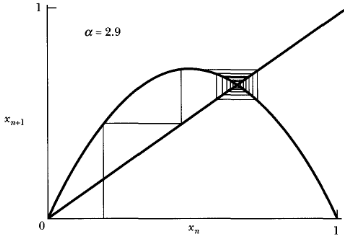
result is independent of initial value as long as it is not 0 or 1



(a)

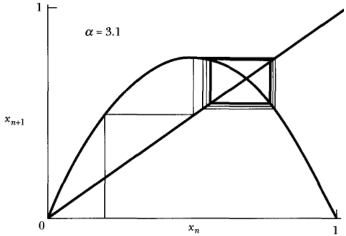


(b)

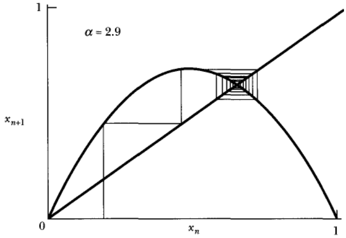


(a)

(a) $a \lessapprox 3$ stable populations : solns. follow spiral path to final value

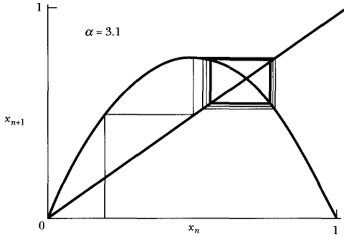


(b)



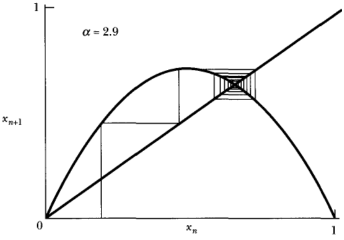
(a)

(a) $a \lessapprox 3$ stable populations : solns. follow spiral path to final value



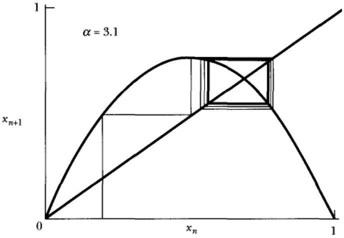
(b)

(b) $a \gtrapprox 3$ multiple possible solutions depending on initial condition : solns. follow path converging to two points at which the square intersects the iteration line



(a)

(a) $a \lesssim 3$ stable populations : solns. follow spiral path to final value

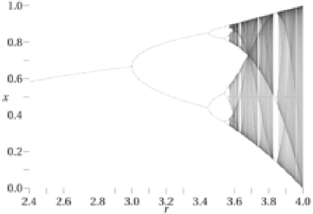


(b)

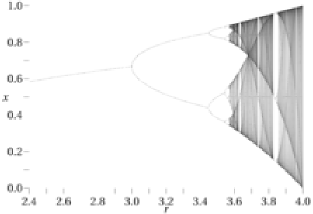
(b) $a \gtrsim 3$ multiple possible solutions depending on initial condition : solns. follow path converging to two points at which the square intersects the iteration line

Such a change in number of solutions to an equation,
when a parameter such as α is varied, is called a **bifurcation**

bifurcation diagram :

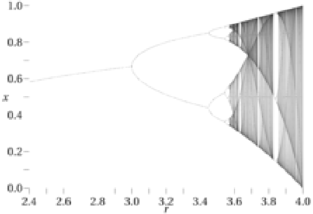


bifurcation diagram :



At $\alpha = 3.45$, two-cycle bifurcation evolves into a four cycle

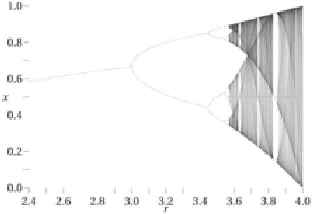
bifurcation diagram :



At $\alpha = 3.45$, two-cycle bifurcation evolves into a four cycle

bifurcation and period doubling continue up to an infinite number of cycles near $\alpha = 3.57$

bifurcation diagram :

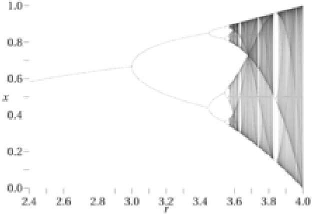


At $\alpha = 3.45$, two-cycle bifurcation evolves into a four cycle

bifurcation and period doubling continue up to an infinite number of cycles near $\alpha = 3.57$

Chaos occurs for many values of *a* between 3.57 and 4.0, but there are still windows of periodic motion, with an especially wide window around 3.84

bifurcation diagram :



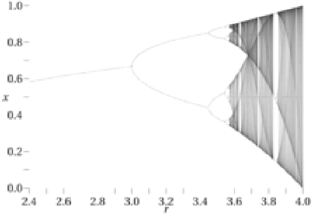
At $\alpha = 3.45$, two-cycle bifurcation evolves into a four cycle

bifurcation and period doubling continue up to an infinite number of cycles near $\alpha = 3.57$

Chaos occurs for many values of a between 3.57 and 4.0, but there are still windows of periodic motion, with an especially wide window around 3.84

interesting behavior occurs for $\alpha = 3.82831 \implies$ apparent periodic cycle of 3 units seems to occur for several periods, then it suddenly violently changes for some time, and then returns again to the 3-cycle

bifurcation diagram :



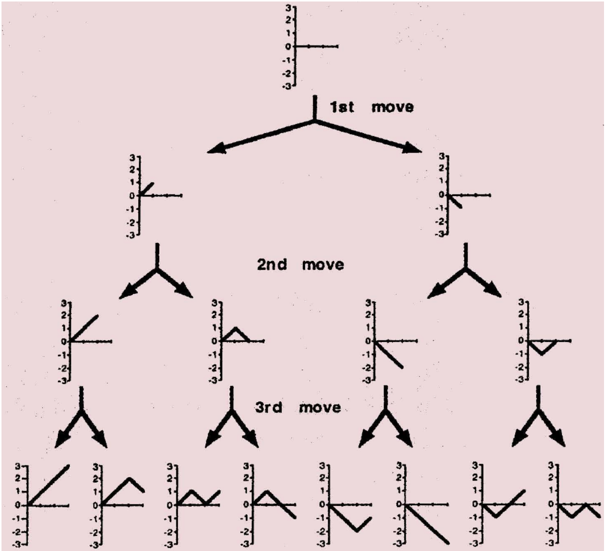
At $\alpha = 3.45$, two-cycle bifurcation evolves into a four cycle

bifurcation and period doubling continue up to an infinite number of cycles near $\alpha = 3.57$

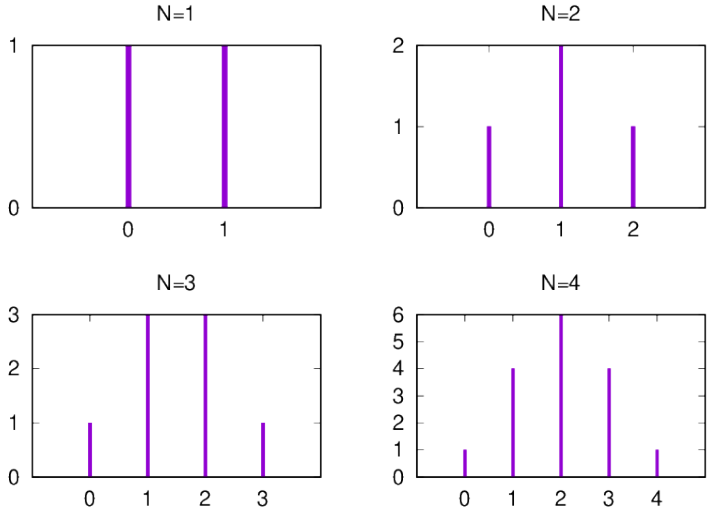
Chaos occurs for many values of a between 3.57 and 4.0, but there are still windows of periodic motion, with an especially wide window around 3.84

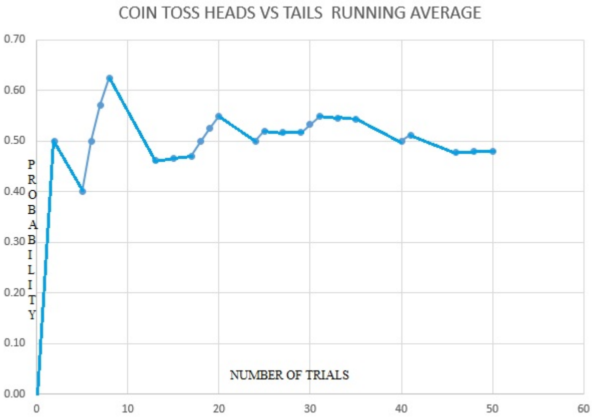
interesting behavior occurs for $\alpha = 3.82831 \implies$ apparent periodic cycle of 3 units seems to occur for several periods, then it suddenly violently changes for some time, and then returns again to the 3-cycle

This intermittent behavior could prove devastating to a biological study operating over several years that suddenly turns chaotic without apparent reason

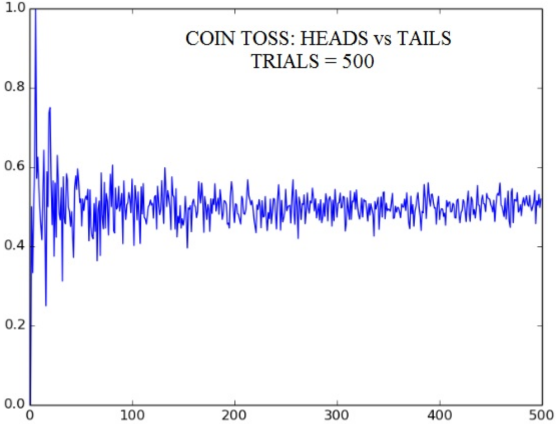


1st move
2nd move
3rd move

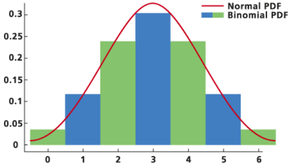

N=1
N=2
N=3
N=4
0
1
2
3
4
5
6



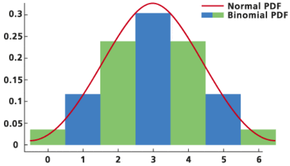
COIN TOSS HEADS VS TAILS RUNNING AVERAGE
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PROBABILITY
NUMBER OF TRIALS
0
10
20
30
40
50
60


COIN TOSS: HEADS vs TAILS
TRIALS = 500
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500

In the limit of large numbers :
Binomial distribution $\overset{N\to\infty}{\longrightarrow}$ Gaussian



In the limit of large numbers :
Binomial distribution $\xrightarrow{N \to \infty}$ Gaussian



where one deals with very many identical particles $\mathcal{O}\left(10^{23}\right)$, statistical arguments become particularly effective