Assignment 1: solns.

1. Estimation is an important aspect of modeling in sciences. Do the following exercises to test your ability to estimate quantities :

   (a) The distance between the classroom (H-205) and the institute main gate $\approx n \times 100$ m. Estimate n as an integer.
   Ans. 1 mark for $n = 4$ or 5; $\frac{1}{2}$mark for $n = 3$ or 6; zero otherwise.

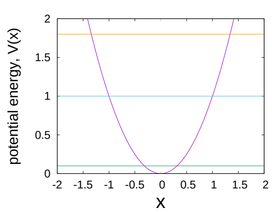
   (b) The time required to walk from the classroom to the main gate at an average pace $\approx n$ s. Estimate n as an integer.
   Ans. 1 mark for $n = 200 - 300$; $\frac{1}{2}$mark for $n = 150 - 199$ or $301 - 350$; zero otherwise.

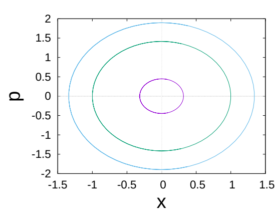
   (c) If a thin conducting cable (say, copper or aluminum, used for fittings in homes) was laid from the classroom to the main gate on the regular path that you use for walking, and typical current (as in wires in home fittings) made to flow through it, what would be the time taken by an individual electron to go from one to the other end on average? Assume each atom in the cable to contribute one electron for conduction.
   Ans. Typically, in home fittings, current ≈ 10 A,
   From density of metals and atomic mass, we can find the number of atoms per m^3. The vale is $\approx 10^{29}$
   current=charge flowing per unit time
   assume one free electron coming from each atom and cross-section of wire$\approx$ 1 mm$^3 = 10^{-6}$m^3.
   $\therefore$current=charge for each electron$\times$area of cross-section$\times$number of free electrons m$^3\times$ speed of an electron
   $\therefore$ speed$\approx \frac{10}{1.6\times10^{-19}\times10^{-6}\times10^{29}} = \frac{1}{1600}$m/s
   time taken$\approx \frac{500}{\frac{1}{1600}} = 8 \times 10^5$s

   (d) If the electron in the ground state of hydrogen atom were to go around the hydrogen nucleus following Bohr model, approximately how many rounds will it make in the time estimated in the question above (c).
   Ans. It is convenient to do this in atomic units,
   atomic unit of time=time taken by an electron to move a distance equal to the atomic unit of length (Bohr radius= $a_0 = 1$)$\approx 2.4 \times 10^{-17}$.
   per round, distance traveled= $2\pi.a_0 = 2\pi$
   $\therefore$ no. of rounds $\approx \frac{1}{2\pi}.\frac{8\times10^5}{2.4\times10^{-17}} \approx 5 \times 10^{21}$

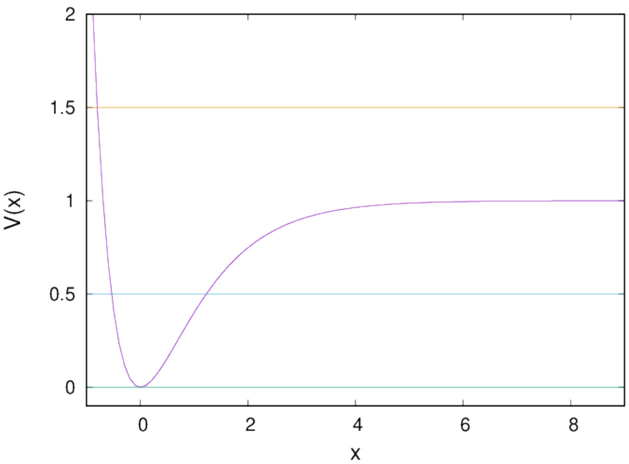
   (e) Estimate the power consumed for the flow of current for the time estimated above (c).
   Typically, the voltage in home fittings is 220V.
   For a DC current of 10 A to flow we need power= $10 \times 220 = 2200$W. For AC, the current needs to be averaged over the cycles.
   Energy consumed= $2200 \times 8 \times 10^5$J= 1.76×10^9J or Ws. It is usually expressed in kWH.
   1.76×10^9Ws= $\frac{1.76}{3.6} \times 10^3$kWH or $\approx$ 500kWH.

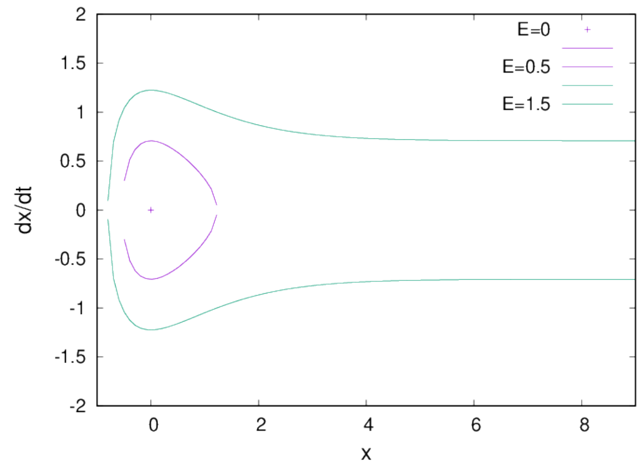
2. Draw phase plane trajectories with three different values of total energies for a dynamical (mechanical) system
   (i) exhibiting SHM



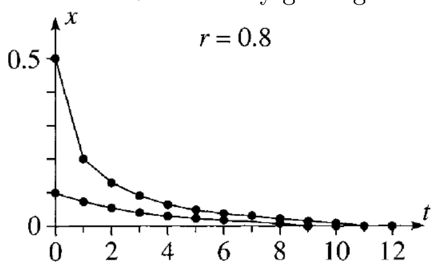


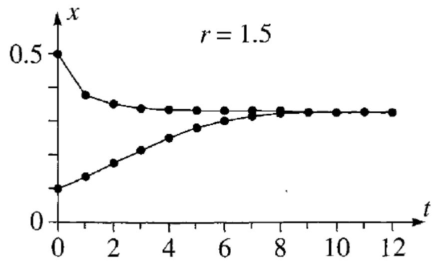
   Ans.
   (ii) subject to an anharmonic potential with the energy in one case higher than the dissociation limit.
   The simplest case is to take one value of energy as the minimum of the potential energy (say, 0, when the phase plot is just a point) and two other cases, one when the system is confined in the well (E= 0.5 in the plot) and when it goes beyond the dissociation limit. (E= 1.5 in the plot)

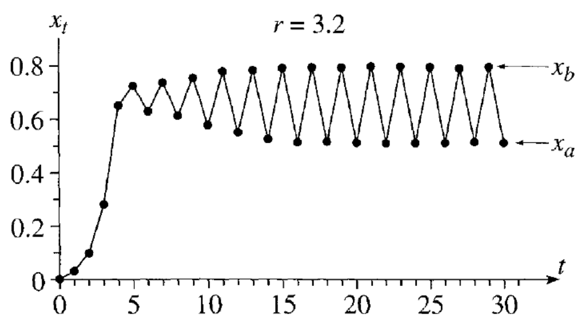




3. The logistic map $x_{n+1} = \alpha x_n(1 - x_n)$ is usually studied for the growth parameter, α, in the range $0 \le \alpha \le 4$. Why? Write a small code for the logistic map and compare the results graphically for $\alpha = 0.8$ and $\alpha = 1.5$ for initial conditions $x_0 = 0.1$ and 0.5. Take another case with $\alpha = 3.2$ and $x_0 \gtrsim 0$ and plot the results.
Ans. For $\alpha > 4$ we may get negative or unrealistic solutions like $x_n > 1$.







4. Show that in the limit of large N, the binomial distribution of n out of N objects becomes a Gaussian distribution
Ans. $P(N|n) = \begin{pmatrix} N \\ n \end{pmatrix} = \frac{N!}{n!(N-n)!}$
For large N and n, $\ln P \approx N \ln N - N - n \ln n + n - (N - n)\ln(N - n) + (N - n) = N \ln N - n \ln n - (N - n)\ln(N - n)$
$\frac{d \ln P}{dn} = -\ln n - 1 + \ln(N - n) + 1 = \ln \frac{N-n}{n}$
P reaches maximum when $\frac{N-n}{n} = 1 \implies n = n^* = \frac{N}{2}$
Expand $P(N|n)$ around $n = n^*$using Taylor's expansion :
$\ln P = \ln P^* + \left.\frac{d(\ln P)}{dn}\right|_{n=\frac{N}{2}} \left(n - \frac{N}{2}\right) + \frac{1}{2} \left.\frac{d^2(\ln P)}{dn^2}\right|_{n=\frac{N}{2}} \left(n - \frac{N}{2}\right)^2 + \cdots$
or, taking first derivative to be zero, $\ln P \approx \ln P^* + \frac{1}{2} \left.\frac{d^2(\ln P)}{dn^2}\right|_{n=\frac{N}{2}} \left(n - \frac{N}{2}\right)^2$
or, $P = P^*.e^{-\alpha\left(n-\frac{N}{2}\right)^2}$, where $\alpha = -\frac{1}{2} \left.\frac{d^2(\ln P)}{dn^2}\right|_{n=\frac{N}{2}} = -\frac{1}{2} \left.\frac{d(\ln \frac{N-n}{n})}{dn}\right|_{n=\frac{N}{2}} = \frac{1}{2} \left.\frac{N}{n(N-n)}\right|_{n=\frac{N}{2}} = \frac{1}{2} \frac{N}{\frac{N}{2}.\frac{N}{2}} = \frac{2}{N}$