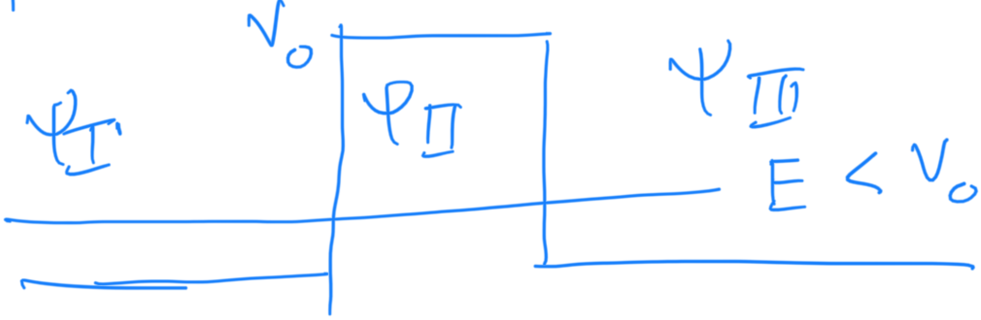
Quantum tunnelling through a potential barrier



The equations for three regions

$$\psi_I(x) = A_1 e^{iK_1 x} + B_1 e^{-iK_1 x}$$

$$\psi_{II}(x) = A_2 e^{K_2 x} + B_2 e^{-K_2 x}$$

$$\psi_{III}(x) = A_3 e^{iK_1 x}$$

$$K_1 = \frac{2mE}{\hbar^2} \qquad K_2 = \frac{2m(V_0 - E)}{\hbar^2}$$

The boundary conditions at 0

$$\psi_I(0) = \psi_{II}(0)$$

$$\Rightarrow A_1 + B_1 = A_2 + B_2 \quad — (1)$$

$$\psi_I'(0) = \psi_{II}'(0)$$

$$\Rightarrow \quad iK_1(A_1 - B_1) = K_2(A_2 - B_2) \quad -(2)$$

The boundary conditions at $x = a$

$$\psi_{II}(a) = \psi_{III}(a)$$

$$\Rightarrow \quad A_2 e^{K_2 a} + B_2 e^{-K_2 a} = C e^{+iK_1 a} \quad -(3)$$

$$\psi_{II}'(a) = \psi_{III}'(a)$$

$$K_2\left(A_2 e^{K_2 a} - B_2 e^{-K_2 a}\right) = iK_1 C e^{+iK_1 a} \quad -(4)$$

Solving (1) → (4) one can get the values of the B_1 A_2 B_2 and A_3 parameters as functions of A_1

Transmission probability

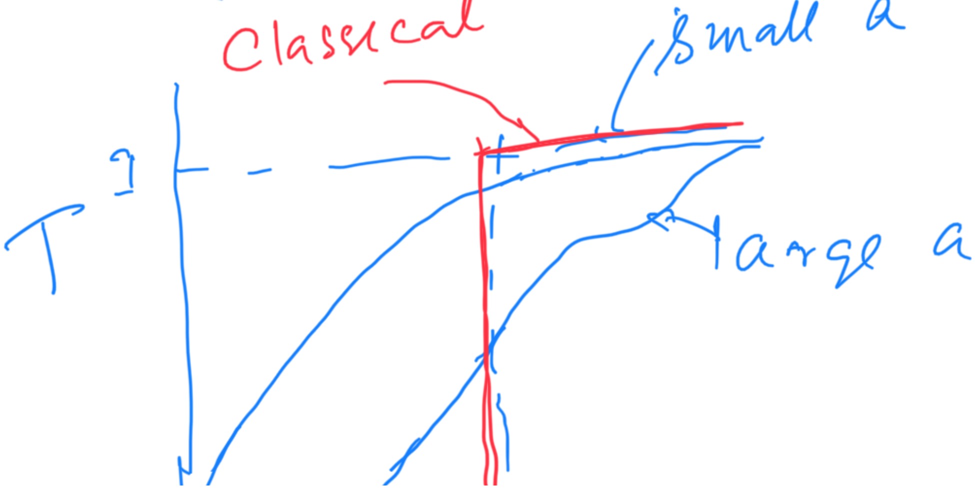
$$T = \left|\frac{A_3}{A_1}\right|^2$$ $|A_3|^2$ is the amplitude in incoming wave at I

and $|A_1|^2$ is the amplitude of outgoing wave in region III

$$T = \frac{4k_1^2 k_2^2}{(k_1^2 + k_2^2)\sinh^2 a k_2 + 4k_1^2 k_2^2}$$

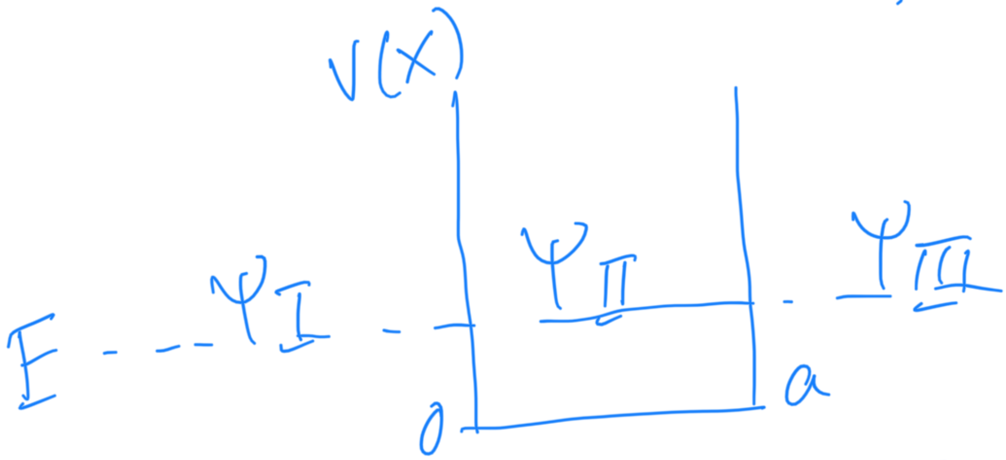
When $E > V_0$

$$T = \frac{4k_1^2 k_2^2}{(k_1^2 - k_2^2)\sin^2 a k_2 + 4k_1^2 k_2^2}$$



$E < V_0 \quad E = V_0 \quad E > V_0 \quad E/V_0$

Particle inside a box (infinite potential well)



$$V(x) = \begin{cases} \infty & x > a \quad x < 0 \\ 0 & x < a \quad x > 0 \end{cases}$$

Since $V(x) = \infty$ in region I and III $\Psi_I = \Psi_{III} = 0$

The energy eigen states are

$$\Psi_{II}(x) = A_1 e^{iKx} + Be^{-iKx}$$

$$\Psi_{II}(0) = 0 \qquad \Psi_{II}(a) = 0$$

$$\Rightarrow A = -B$$

$\Rightarrow$ ''

$$\psi_{II}(x) = A \sin Kx$$

$$\psi_{II}(a) = 0 \Rightarrow \sin Ka = 0$$

$$Ka = n\pi$$

The energy eigenvalues and corresponding eigenstates are

$$E_n = \frac{\hbar^2}{2m} \frac{\pi^2 n^2}{a^2}$$

$$|\phi_n\rangle = A \sin \frac{n\pi x}{a}$$

For normalization $\langle \phi_n | \phi_n \rangle = 1$

$$\int_0^a |A|^2 \sin^2 \frac{n\pi x}{a} dx = 1$$

$$|A|^2 \int \frac{1}{2}\left(1 - \cos \frac{2n\pi x}{a}\right) = 1$$

$$\Rightarrow A = \sqrt{\frac{2}{a}}$$

$\therefore \quad |\phi_n\rangle = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$ are the orthonormal basis of energy eigenstates for different values of n