Q1

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

I.

$$[\sigma_1, \sigma_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\,\sigma_3$$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= 2i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2i\,\sigma_1$$

$$[\sigma_3, \sigma_1] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= 2i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2i\,\sigma_2$$

II Eigenvalues and eigenvectors of σ_1

$\lambda = \pm 1$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\lambda = 1 \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = x_2$$

$$|a_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ Eigenvector 1

$$\lambda = -1 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = -x_2$$

$$|a_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ Eigenvector 2

Eigenvalues and eigenvectors of σ_2

$$\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm 1$$

$$\lambda = 1 \quad \begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = -i x_2$$

$$|b_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = +i x_2$$

$$|b_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Eigen values and eigen vectors of σ_3

$\lambda = \pm 1 \quad \lambda = 1 \Rightarrow |c_1\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -1 \Rightarrow |c_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

III <u>prob distribution of $\hat{\sigma}_1$ σ_3 in the eigen state $|b_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$</u>

For σ_1

$|b_1\rangle = \alpha_1 |a_1\rangle + \alpha_2 |a_2\rangle$

$\alpha_1 = \langle a_1 | b_1 \rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} (1 \quad 1) \begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1}{2}(1-i)$

$\alpha_2 = \langle a_2 | b_1 \rangle = \frac{1}{2} (1 \quad -1) \begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1}{2}(-i-1)$

For measuring σ_1

prob of getting value $1 = |\alpha_1|^2 = \frac{1}{2}$

prob of getting value $-1 = |\alpha_2|^2 = \frac{1}{2}$

For σ_3

$|b_1\rangle = \alpha_1 |c_1\rangle + \alpha_2 |c_2\rangle$

$\alpha_1 = \langle c_1 | b_1 \rangle = \frac{1}{\sqrt{2}} (1 \quad 0) \begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-i)$

$\langle \ldots \rangle = \frac{1}{\ldots} (0 \quad 1) \begin{pmatrix} -i \\ \ldots \end{pmatrix} = \frac{1}{\ldots}$

$\alpha_2 = \langle c_2 | \psi_2 \rangle = \frac{1}{\sqrt{2}} (0 \;\; 1)(1) \quad \sqrt{2}$

For measuring σ_3

prob of getting value $1 = |\alpha_1|^2 = \frac{1}{2}$

prob of getting value $-1 = |\alpha_2|^2 = \frac{1}{2}$

IV

$$\langle \sigma_1 \rangle = \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$$

$$\langle \sigma_1^2 \rangle = \frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2 = 1$$

$$\langle \Delta \sigma_1^2 \rangle = \langle \sigma_1^2 \rangle - \langle \sigma_1 \rangle^2 = 1$$

$$\langle \sigma_3 \rangle = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

$$\langle \sigma_3^2 \rangle = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$$

$$\Delta \sigma_3^2 = 1$$

$\Rightarrow \Delta \sigma_1 \cdot \Delta \sigma_3 = 1$ uncertainty relation

Q. 2

I. The first two lowest

energy states are

$$|\phi_1\rangle = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{L} \quad |\phi_1\rangle = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{L}$$

$$\psi = \frac{1}{\sqrt{2}} |\phi_1\rangle + \frac{1}{\sqrt{}} |\phi_2\rangle$$

Averge energy $\langle E\rangle = \frac{1}{2} E_1 + \frac{1}{2} E_2$

$$= \frac{1}{2}\left(\frac{\hbar^2}{2m}\frac{\pi^2}{L^2}\right) + \frac{1}{2}\left(\frac{\hbar^2}{2m}\frac{4\pi^2}{L^2}\right)$$

$8 + \frac{1}{2}$

$$= \frac{5}{2}\frac{\hbar^2\pi^2}{2mL^2}$$

$$\langle E^2\rangle = \frac{1}{2}\left(\frac{\hbar^2}{2m}\frac{\pi^2}{L^2}\right)^2 + \frac{1}{2}\left(\frac{\hbar^2}{2m}\frac{4\pi^2}{L^2}\right)^2$$

$$= \frac{17}{2}\left(\frac{\hbar^2\pi^2}{2mL^2}\right)^2$$

$$\langle \Delta E^2\rangle = \frac{17}{2}\left(\frac{\hbar^2\pi^2}{2mL^2}\right)^2 - \frac{25}{4}\left(\frac{\hbar^2\pi^2}{2mL^2}\right)^2$$

$$\langle \Delta E^2\rangle = \frac{9}{4}\left(\frac{\hbar^2\pi^2}{2mL^2}\right)$$

later time t iE/t

At a later ...

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-iE_1/\hbar t} |\phi_1\rangle + \frac{1}{\sqrt{2}} e^{-iE_2t/\hbar} |\phi_1\rangle$$

$$\langle E \rangle^2 = \frac{1}{2} E_1 + \frac{1}{2} E_2 = \frac{25}{4}\left(\frac{\hbar^2 \pi^2}{2mL^2}\right)$$

$$\langle E^2 \rangle = \frac{1}{2} E_1^2 + \frac{1}{2} E_2^2 = \frac{17}{2}\left(\frac{\hbar^2 \pi^2}{2mL^2}\right)^2$$

$$\langle \Delta E^2 \rangle = \frac{9}{4}\left(\frac{\hbar^2 \pi^2}{2mL^2}\right)$$

II probability density find the particle at $x \to |\Psi(x)|^2$

$$a_x = \langle x | \Psi \rangle = \frac{1}{\sqrt{2}} \langle x | \phi_1 \rangle + \frac{1}{\sqrt{2}} \langle x | \phi_2 \rangle$$

$$|\Psi(x)|^2 = \left| \frac{1}{\sqrt{2}} \phi_1(x) + \frac{1}{\sqrt{2}} \phi_2(x) \right|^2$$

$$= \frac{1}{2} |\phi_1(x)|^2 + \frac{1}{2} |\phi_2(x)|^2 + \phi_1(x)\phi_2(x)$$

$\because$ $\phi_1(x)$ and $\phi_2(x)$ are real in these case

$$|\Psi(x)|^2 \; \frac{1}{L}\left[\sin^2 \pi \quad + \sin^2 \pi \right.$$

$$|\Psi(t=?)|^2 = \frac{1}{L}\Big(\sin\frac{?}{2} \ldots + \sin\frac{\pi}{2}\sin\pi \Big)$$

$$= \frac{1}{L}$$

III. For any value of x

probability density $= |\Psi(x)|^2$

$$\langle x\rangle = \int_0^a x\,|\Psi(x)|^2\,dx$$

$$= \frac{2}{L}\Big[\int_0^a \frac{1}{2}x\sin^2\frac{\pi x}{L}\,dx + \int_0^a \frac{1}{2}x\sin^2\frac{2\pi x}{L}\,dx + \int_0^a x\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\,dx\Big]$$

$$= \frac{L}{2} - \frac{16L}{9\pi^2} = 0.32L$$

$$\langle x^2\rangle = \int_0^a x^2\,|\Psi(x)|^2$$

$$= \frac{1}{L}\left[\int_0^a x^2 \sin^2\frac{\pi x}{L}\,dx + \int_0^a x^2 \sin^2\frac{2\pi x}{L} + \int_0^a 2x^2 \sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\,dx\right]$$

$$= \frac{1}{2}L^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right) + \frac{1}{2}L^2\left(\frac{1}{3} - \frac{1}{8\pi^2}\right) - \frac{8L^2}{9\pi^2}$$

$$= \frac{L^2}{3} - \frac{1}{2}\frac{L^2}{\pi^2}\left[\frac{1}{2} + \frac{1}{8} + \frac{16}{9}\right]$$

$$\simeq 0.21\,L^2$$

$$\langle \Delta x^2 \rangle = 0.2L^2 - 0.11L^2 = 0.1L^2$$

Theme time evolution of $|\Psi(t)\rangle$

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-\frac{iE_1 t}{\hbar}}|\phi_1\rangle + \frac{1}{\sqrt{2}}e^{-\frac{iE_2 t}{\hbar}}|\phi_2\rangle$$

$$\langle x|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_1 t/\hbar}\left[\phi_1(x)\right.$$

$$+\ e^{-i\frac{(E_1-E_2)}{\hbar}t}\,\phi_2(x)\Big]$$

$$\left|\langle x(\psi(t))\rangle\right|^2 = \frac{1}{2}\Big[\phi_1(x) + \cos\omega t\,\phi_2(x)\Big]^2$$

$$+\frac{1}{2}\,\phi_2^2(x) \qquad \boxed{\omega = \left(\frac{E_1-E_2}{\hbar}\right)}$$

$$= \frac{1}{2}\Big[|\phi_1^2(x)|^2 + |\phi_1(x)|^2 + 2\cos\omega t\,\phi_1(x)\phi_2(x)\Big]$$

$$\langle x(t)\rangle = \frac{1}{L}\Big[\int_0^a x\sin^2\frac{\pi x}{L}dx + \int_0^a x\sin^2\frac{2\pi x}{L}dx$$

$$+\int_0^a 2\cos\omega t\ x\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}dx\Big]$$

$$= \left(L - \frac{16L}{9\pi^2}\cos\omega t\right)$$

$$= L\,(1 - 0.18\cos\omega t)$$

$$\langle x^2(t)\rangle = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right) + L^2\left(\frac{1}{3} - \frac{1}{8\pi^2}\right)$$

$\langle x(t)\rangle$

$$- \cos\omega t \, \frac{16L}{9\pi^2}$$

$$\simeq \left(0.6 - 0.18\cos\omega t\right) L^2$$

$$\langle \Delta x^2(t)\rangle \simeq \left(0.6 - 0.18\cos\omega t\right) L^2$$

$$- L^2\left(1 - 0.18\cos\omega t\right)^2$$

IV The average momentum

$$\langle p\rangle = \langle \psi | \hat{p} | \psi\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$$

$$\langle p\rangle = \frac{1}{2}\langle \phi_1 | p | \phi_1\rangle + \frac{1}{2}\langle \phi_2 | p | \phi_2\rangle$$

$$+ \frac{1}{2}\langle \phi_1 | p | \phi_2\rangle + \frac{1}{2}\langle \phi_2 | p | \phi_1\rangle$$

$$p = -i\hbar \frac{\partial}{\partial x}$$

$$\langle \phi_1 | p | \phi_1\rangle = 0 \qquad \langle \phi_2 | p | \phi_2\rangle = 0$$

$$\langle \phi_1 | p | \phi_2 \rangle = \frac{2}{a}\int_0^a \sin\frac{\pi x}{a}\cos\frac{2\pi x}{a}dx$$

$$= -i\hbar\frac{2}{a}\frac{2\pi}{a}\left(-\frac{2a}{3\pi}\right) = i\frac{8\hbar}{3a};$$

Similarly,

$$\langle \phi_2 | p | \phi_1 \rangle = -i\hbar\frac{\pi}{a}\frac{2}{a}\int_0^a \sin\frac{2\pi x}{a}\cos\frac{\pi x}{a}dx$$

$$= -i\hbar\frac{2}{a}\frac{\pi}{a}\times\frac{4a}{3\pi} = -i\frac{8\hbar}{3a}$$

$$\Rightarrow \langle \phi_2 | p | \phi_1 \rangle + \langle \phi_1 | p | \phi_2 \rangle = 0$$

$$\Rightarrow \langle \psi | p | \psi \rangle = 0$$

Averge momentum is zero.