Mechanics: Assignment 4

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Question 1: Coupled Oscillators

This question analyzes a system of two masses and two springs.

1. Kinetic Energy, Potential Energy, and Lagrangian

- **Kinetic Energy (T):** The sum of the kinetic energies of the two masses.

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

- **Potential Energy (V):** The sum of the potential energies stored in the two springs. The extension of the second spring is $(x_2 - x_1)$.

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2$$

- **Lagrangian (L):** The Lagrangian is $L = T - V$.

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_2 - x_1)^2$$

2. Lagrange Equations

The Lagrange equation for a coordinate q_i is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$.

- **For x_1:**

$$\frac{d}{dt}(m_1\dot{x}_1) - (-k_1x_1 - k_2(x_2 - x_1)(-1)) = 0$$
$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

- **For x_2:**

$$\frac{d}{dt}(m_2\dot{x}_2) - (-k_2(x_2 - x_1)) = 0$$
$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

3. Normal Mode Frequencies

We seek oscillatory solutions of the form $x_j(t) = A_je^{i\omega t}$. Substituting these into the equations of motion yields a matrix equation. For a non-trivial solution, the determinant of the coefficient matrix must be zero.

$$\begin{vmatrix} (k_1 + k_2 - m_1\omega^2) & -k_2 \\ -k_2 & (k_2 - m_2\omega^2) \end{vmatrix} = 0$$

Expanding the determinant gives a quadratic equation for ω^2:

$$m_1m_2\omega^4 - (m_1k_2 + m_2(k_1 + k_2))\omega^2 + k_1k_2 = 0$$

Solving this equation provides the two normal mode frequencies.

Question 2: Simple Pendulum

This question deals with a simple pendulum of mass m and length l.

1. Lagrangian and Generalized Momentum

- **Lagrangian (L):**
$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$
- **Generalized Momentum (p_θ):**
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

2. Hamiltonian $H(p_\theta, \theta)$

The Hamiltonian is defined by the Legendre transformation $H = p_\theta\dot{\theta} - L$. First, we express $\dot{\theta}$ in terms of p_θ: $\dot{\theta} = p_\theta/(ml^2)$.
$$H = p_\theta\left(\frac{p_\theta}{ml^2}\right) - \left[\frac{1}{2}ml^2\left(\frac{p_\theta}{ml^2}\right)^2 + mgl\cos\theta\right]$$
$$H(p_\theta, \theta) = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

3. Phase Space Trajectories for small θ

For small angles, we use the approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$. The Hamiltonian becomes:
$$H \approx \frac{p_\theta^2}{2ml^2} + \frac{1}{2}mgl\theta^2 - mgl$$
The conserved energy is $E = H + mgl$. Rearranging the equation for E gives the equation of an ellipse in the phase space (p_θ vs. θ):
$$\frac{p_\theta^2}{2ml^2E} + \frac{\theta^2}{2E/mgl} = 1$$
The trajectories are concentric ellipses centered at the origin.

Question 3: Inverse Square Potential

The motion is on a plane with potential $V(r) = -a/r$ and constant angular momentum A.

1. Generalized Momenta and Bound Orbit Condition

The Lagrangian in polar coordinates is $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + a/r$.

- **Generalized Momenta:** $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$. The momentum p_θ is the conserved angular momentum, A.
- **Bound Orbit Condition:** For a bound orbit, the total energy E must be less than the potential energy at infinity. Since $V(r \to \infty) = 0$, the condition is $\mathbf{E < 0}$.

2. Energy and Radius for a Circular Orbit

A circular orbit occurs at a constant radius, which corresponds to the minimum of the effective potential $V_{\text{eff}}(r) = \frac{A^2}{2mr^2} - \frac{a}{r}$. We find this by setting $\frac{dV_{\text{eff}}}{dr} = 0$.
$$\frac{d}{dr}\left(\frac{A^2}{2mr^2} - \frac{a}{r}\right) = -\frac{A^2}{mr^3} + \frac{a}{r^2} = 0$$
The radius of the circular orbit is $r_c = \frac{A^2}{ma}$. The energy of this orbit is $E_c = V_{\text{eff}}(r_c) = -\frac{ma^2}{2A^2}$.

3. Minimum and Maximum Radius

The minimum (r_{min}) and maximum (r_{max}) radii for a given energy E are the turning points where the radial velocity is zero. This occurs when $E = V_{\text{eff}}(r)$.

$$E = \frac{A^2}{2mr^2} - \frac{a}{r}$$

This is a quadratic equation for $1/r$. The two solutions give the inverse of the minimum and maximum radii.

$$\frac{1}{r} = \frac{ma}{A^2}\left(1 \pm \sqrt{1 + \frac{2EA^2}{ma^2}}\right)$$