Q1.

$$L = e^{\lambda t}\left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2\right)$$

(1) $\dfrac{\partial L}{\partial \dot{x}} = e^{\lambda t} m \dot{x}$ $\quad \dfrac{\partial L}{\partial x} = +e^{\lambda t} m \omega^2 x$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = e^{\lambda t}\left(m \ddot{x} + \lambda m \dot{x}\right)$$

Equation of motion

$$m \ddot{x} + \lambda m \dot{x} = + m \omega^2 x$$

$$\ddot{x} + \lambda \dot{x} - \omega^2 x = 0$$

(2) momentum $p = \dfrac{\partial L}{\partial \dot{x}} = e^{\lambda t} m \dot{x}$

$$\Rightarrow \dot{x} = \frac{p}{m} e^{-\lambda t}$$

$$H = p \dot{x} - L$$

$$= \frac{p^2}{m} e^{-\lambda t} - e^{\lambda t}\left(\frac{1}{2} m \frac{p^2}{m^2} e^{-2\lambda t}\right.$$

$$+ \frac{1}{2} m \omega x)$$

$$= \frac{p^2}{2m} e^{-\lambda t} - \frac{1}{2} e^{\lambda t} m \omega^2 x^2$$

(3) Hamilton's equation of motion

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} e^{-\lambda t}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = + e^{\lambda t} m \omega^2 x$$

Q2. $a_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $a_2 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

(1)

$$[a_1, a_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} - \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} - \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(2) Eigenvalues and eigenvectors of a_1

$\lambda = \pm 1 \quad |\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\phi_2\rangle = \begin{pmatrix} 1 \end{pmatrix}$

Eigen values and eigen vectors of a_2

$$(\lambda - 1)^2 - 1 = 0 \qquad \lambda = 2 \quad \lambda = 0$$

$\lambda = 2$

$$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow -x_1 + i x_2 = 0$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

(3) The values of a_1 are $1, -1$

Chosen state is $|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ = 1st eigenstate of a_2

To calculate the probabilities the $|\psi_1\rangle$ is expanded in the eigen states of a_1

$$|\psi_1\rangle = c_1 |\phi_1\rangle + c_2 |\phi_2\rangle$$

$$\text{prob}(\lambda = 1) = |c_1|^2 = |\langle \phi_1 | \psi_1 \rangle|^2 = \frac{1}{2}$$

$$\text{prob}(\lambda = -1) = |c_2|^2 = |\langle \phi_2 | \psi_1 \rangle|^2 = \frac{1}{2}$$

(4) Average $\langle a_1 \rangle = \frac{1}{2}(+1) + \frac{1}{2}(-1) = 0$

$$\langle a_1^2 \rangle = \frac{1}{2}(+1)^2 + \frac{1}{2}(-1)^2 = 1$$

$$\Delta a_1 = 1.$$

Q3.

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

(1) $$\langle x \rangle = \int_{-\infty}^{+\infty} x \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{\hbar}x^2} dx$$

(Normal distribution)

$$= 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{\hbar}x^2} dx$$

$$= \frac{1}{2}\frac{\hbar}{m\omega}$$

(2) $\langle p \rangle = \langle \phi_0 | p | \phi_0 \rangle$

(2) $\langle p \rangle = \langle p ...$

$$p|\phi_0\rangle = -i\hbar \frac{\partial \phi_0}{\partial x} = -i\hbar \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\frac{m\omega}{2\hbar}\right)\cdot 2x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\langle\phi_0|p|\phi_0\rangle = i\int_{-\infty}^{+\infty} m\omega \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} x\, e^{-\frac{m\omega}{2\hbar}x^2}\, dx$$

$$= 0$$

Uncertainty in x: $\Delta x = \left[\langle x^2\rangle - \langle x\rangle^2\right]^{1/2}$

$$= \sqrt{\frac{1}{2}\frac{\hbar}{m\omega}}$$

(3) Momentum ~~w~~wavefunction

$$\phi_p = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} e^{-ip/\hbar\, x} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\, dx$$

$$\int_{-\infty}^{+\infty} \exp\left[-\frac{m\omega}{2\hbar}\left[x^2 + i\frac{2px}{m\omega}\right]\right] dx$$

$$= \int \exp\left[-\frac{m\omega}{2\hbar}\left[\left[x^2 + \frac{2ipx}{m\omega} + \left(\frac{ip}{m\omega}\right)^2\right] - \left(\frac{ip}{m\omega}\right)^2\right]\right]$$

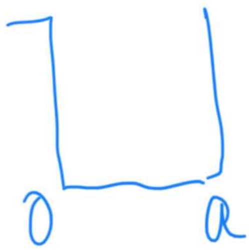
$$= \int \exp\left[-\frac{m\omega}{2\hbar}\frac{p^2}{...} - \frac{m\omega}{2\hbar}\left(x + \frac{ip}{m\omega}\right)^2\right]$$

$$\int \exp\left[\ \ldots\ 2\hbar\, m^2\omega^2\ \ldots\ 2\hbar\, (\ \ldots\ )\right]$$

$$= e^{-\frac{p^2}{2\hbar m\omega}} \frac{1}{\sqrt{2\pi \frac{2\hbar}{m\omega}}}$$

$$|\phi_p|^2 = \frac{e^{-p^2/\hbar m\omega}}{4\pi\hbar/m\omega}$$

Q4.



(1) $\psi = \sqrt{\frac{3}{5}}\,|\phi_1\rangle + \sqrt{\frac{2}{5}}\,|\phi_2\rangle$

$$E_1 = \frac{\hbar^2\pi^2}{2ma^2} \qquad E_2 = \frac{4\hbar^2\pi^2}{2ma^2}$$

prob of $E_1 = \frac{3}{5}$ prob of $E_2 = \frac{2}{5}$

(2) Average energy $\langle E\rangle$

$$\langle E\rangle = \frac{3}{5}\,\frac{\hbar^2\pi^2}{2ma^2} + \frac{2}{5}\,\frac{4\hbar^2\pi^2}{2ma^2}$$

$$= \frac{\hbar^2\pi^2}{10ma^2}(3+8) = \frac{11\hbar^2\pi^2}{10ma^2}$$

$$\langle E^2\rangle = \frac{3}{5}\left(\frac{\hbar^2\pi^2}{2ma^2}\right)^2 + \frac{2}{5}\left(\frac{4\hbar^2\pi^2}{2ma^2}\right)^2$$

$$= \frac{1}{5}\left(\frac{\hbar^2\pi^2}{2ma^2}\right)^2\left[3+32\right]$$

$$= 7\left(\frac{\hbar^2\pi^2}{2ma^2}\right)^2$$

$$(\Delta E)^2 = \langle E^2\rangle - \langle E\rangle^2$$

$$= 7\left(\frac{\hbar^2\pi^2}{2ma^2}\right)^2 - \left(\frac{11}{5}\right)^2\left(\frac{\hbar^2\pi^2}{2ma^2}\right)^2$$

$$= \left(7 - \left(\frac{11}{5}\right)^2\right)\left(\frac{\hbar^2\pi^2}{2ma^2}\right)^2$$

$$\Delta E = \left(7 - \left(\frac{11}{5}\right)^2\right)^{1/2}\frac{\hbar^2\pi^2}{2ma^2}$$

(3) probability of the electron to

... within $a/2$

$$p(x<a/2) = \int_0^{a/2} |\psi(x)|^2 \, dx$$

$$= \frac{2}{a}\left[\frac{3}{5}\int_0^{a/2} \sin^2\frac{\pi x}{a}\,dx + \frac{2}{5}\int \sin^2\frac{2\pi x}{a}\,dx + \frac{2\sqrt{6}}{5}\int \sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\,dx\right]$$

$$= \frac{1}{5}\frac{2}{a}\left(\frac{3a}{4} + \frac{2a}{4} + 2\sqrt{6}\,a/4\right)$$

$$= \frac{2}{5}\left(\frac{5}{4} + \frac{2\sqrt{6}}{4}\right) \approx 0.98$$

(4) $$\psi(x,t) = \sqrt{\frac{3}{5}}\, e^{-iE_1 t/\hbar}|\phi_1\rangle + \sqrt{\frac{2}{5}}\, e^{-iE_2 t/\hbar}|\phi_2\rangle$$

$$p(x<a/2) = \int_0^a |\psi(x,t)|^2 \, dx$$

$$= \int \frac{2}{a}\left|\sqrt{\frac{3}{5}}\sin\frac{\pi x}{a} + \sqrt{\frac{2}{5}}\sin\frac{2\pi x}{a}\, e^{-i\omega t}\right|^2 dx$$

$$= \int \frac{2}{a}\left[\frac{3}{5}\sin^2\frac{\pi x}{a} + \frac{2}{5}\sin^2\frac{2\pi x}{a}\right.$$

$$\pi x \quad 2\pi x]$$

$$+ \frac{2\sqrt{6}}{5} \cos\omega t \sin\frac{\pi x}{a} \sin\frac{2\pi x}{a} \Big] dx$$

$$= \frac{2}{5}\left(\frac{5}{4} + \cos\omega t \frac{2\sqrt{6}}{4}\right)$$

$$\omega = \frac{E_2 - E_1}{\hbar} = \frac{\hbar}{2m}\frac{3\pi^2}{a^2} \qquad \boxed{\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{a^2}}$$

minimum value $= \frac{2}{5}\left(5/4 - \frac{2\sqrt{6}}{4}\right) \simeq 0.01$

when $t = \frac{\pi}{\omega}$

Q5 (1) Lagrangian in polar coordinate

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{e^2}{r}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Hamiltonian $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{e^2}{r}$

(2) Hamilton's equation of motion

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{e^2}{r^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \qquad p_\theta = L$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

(3) For $E<0$ and $L=0$

Maximum radius r_{max} is given by

$$E = -\frac{e^2}{r_{max}} \qquad r_{max} = -\frac{e^2}{E}$$

(4) For ground state eigenstate

$$E = -\frac{e^2}{2a_0} \Rightarrow r_{max} = 2a_0$$

The probability that the electron remains within r_{max}

$$p(r<r_{max}) = \int_0^{r_{max}} 4\pi r^2 |\psi_0(r)|^2 dr$$

$$= 4\pi \int_0^{r_{max}} r^2 \frac{1}{\pi a_0^3} e^{-2r/a_0} dr$$

$-2r/a_0$ (r ... r) $\Big|^{r_{max}}$

$$= \frac{4}{a_0^3}\left[-\frac{a_0}{2} e^{\cdots}\left(\frac{2a_0}{4} + \frac{2a_0\cancel{x}+r}{2}\right)\right|_0$$

$$\boxed{r_{max} = 2a_0}$$

$$= \frac{4}{2a_0^2}\left[-e^{-4}\left(\frac{a_0}{2} + 2a_0^2 + 4a_0^2\right) + \frac{a_0^2}{2}\right]$$

$$= 2\left(\frac{1}{2} - \frac{13}{2e^4}\right) = \left(1 - \frac{13}{e^4}\right) \simeq 0.75$$

Q6 The schrodenger equaten

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$

$$E = 0$$

$$\psi(x) = a\exp\left(-x^2/L^2\right)$$

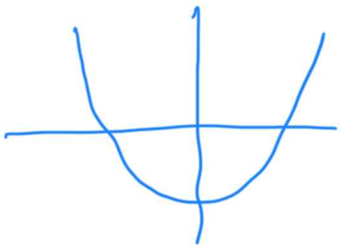
$$V(x) = \frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2\psi}{\partial x^2}$$

$$\frac{\hbar^2}{\ } \quad 1 \; e^{x^2/L^2} d\left(-2ax\right) e^{-x^2/L^2}$$

$$= + \frac{\hbar^2}{2m}\frac{1}{a} e^{x^2/L^2} \frac{d}{dx}\left(-\frac{2ax}{L^2}\right) e^{-x^2/L^2}$$

$$= + \frac{\hbar^2}{2ma} e^{x^2/L^2}\left[-\frac{2a}{L^2}e^{-x^2/L^2} + \frac{4ax^2}{L^4}e^{-x^2/L^2}\right]$$

$$= + \frac{\hbar^2}{mL^2}\left[\frac{2x^2}{L^2} - 1\right]$$



(2) It's a harmonic oscillator shifted by energy $-\frac{\hbar^2}{2mL^2}$

The ground state energy

$$E_0 = \frac{1}{2}\hbar\omega - \frac{\hbar^2}{2mL^2}$$

$$\omega^2 = \left(\frac{2\hbar^2}{m^2L^4}\right)$$