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Quantum teleportation with qubits and qutrits

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1 Introduction

Quantum information technology and quantum computing are rapidly developing areas of quantum physics. Quantum computers as a whole have made huge leaps in the last couple of years, and are a very active field of research. Solutions to problems underivable by classical computers or even super computers are found using quantum computers in a much more efficient manner. However this does not imply that such solutions cannot be derived on classical computers, it would just take an absurd amount of time. The goal of this thesis is to derive a way to simulate quantum operations on a classical machine. We use this simulation to study the properties of quantum entanglement and quantum teleportation. It was derived entirely from one single circuit for quantum teleportation given by Qiskit. First, the derivation was performed for a qudit with $d = 5$, in order to be able to see all the properties of this generalised model. Here we only show the cases for $d = 2$ and $d = 3$ since they are physically realisable qudits. The first part consists of the space, states and operators that we will be using. The main goal is to define their method of construction (generation) as opposed to giving fundamental interpretations; the entire thesis follows this outline. We show that the operators can be generated in multiple ways. Computers do not understand interpretation, and so, the main focus is on deriving models that can be easily computed. The second part demonstrates that quantum entanglement as a phenomenon does in fact exist, and that it is a statistically measurable quantity of the particles, highlighting the fundamental implications of quantum mechanics. In the third part, we define what quantum teleportation is, and use both mathematical and computational procedures to derive it for both qubits and qutrits. A simulation and an experiment are achieved on Qiskit's platform for qubits. For qutrits, we write a simulator on Mathematica; however, instead of simulating quantum teleportation as functions acting in a way predefined by its purpose, we instead simulate it as if it was on an ideal quantum machine. The fourth and final part is a novel, creative take on quantum encryption based on the implications of the thesis itself. It shows that quantum teleportation can have many practical applications.

2 Spaces, states and operators

2.1 Schrodinger's equation

In Schrodinger's basis, the states of a system evolve over time, meaning, that for each interval of time, we move further along the wave function $\psi(t)$, but the operators acting on those states stay constant. The time evolution $U(t, t_0)$ has to be unitary, and so we have:

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle \text{, where } U(t, t_0) = e^{-i\hat{H}\cdot(t-t_0)/\hbar} \tag{2.1.1}$$

As you can easily see, the Hamiltonian, which is an operator, does in fact, stay constant throughout the entire process, it has no dependency on t, or put simply $\partial_t \hat{H} = 0$. From that we have the Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{2.1.2}$$

What kind of information does this equation give us ? Well if you read it, it says that for each passing moment of time t, the state evolves with a step, that would be equal to the Hamiltonian acting on that state at the exact moment of time. Note here, that we have excluded t_0, instead we always assume that at the initial moment we have $t_0 = 0$ and $|\psi(t_0)\rangle = |\psi(0)\rangle$ which is constant. This helps us not only in simplifying the notation, but also in interpreting the meaning behind Schrodinger's equation. For any system described by such an equation, would have a starting point, i.e. $|\psi(0)\rangle$, and any information about the system can be gained by evolving this state unitarily through time. It also allows us to pick a random state $\psi(t)$, at a random moment t, and evolve that state, to a point, where we return back to $|\psi(0)\rangle$ but with a global phase???. One may wonder whether this can be done, but the answer lies in the space itself in which the wave function propagates. For any system to be considered quantum mechanical, it needs to be Hermitian, meaning that the space itself, regardless of it's dimensions, is complete. Any system to be considered complete, means that after a number of unitary transformations, we will be able to map out the entire space, and come back to the state where we had originally started. A simple example to illustrate this point, say that we have state $k = |2\rangle$, in space four-dimensional space, and act on it with a unitary operator $\hat{A}$:

$$|k\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \hat{A} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies \hat{A}|k\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \hat{A}\hat{A}|k\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.1.3}$$

Again, the operator $\hat{A}$ stays constant, but the state of the system changes, and it does in fact go back to the initial state once it has reached the final possible achievable state of the space defined by the system. With just these few simple definitions, we are able to achieve quantum teleportation. Keep in mind, that the states, operators, etc. all have a classical character, and follow classical logic, in this case, matrix multiplication. There is nothing intrinsically quantum about these constructs, the quantum part comes from our interpretation on what value these states and operators give when we apply quantum logic on them. Well what is that quantum logic? Let's look again at our vector $|k\rangle$, it has a dimensionality of $N = 4$, and a rank of $r = 1$. The dimensionality of the vector is decided by the dimensions of the Hilbert space itself, while the rank is an intrinsic property of the vector itself, which tells us the number of linearly independent rows. Our operator on the other hand, has a dimensionality of $N = 4$, and a rank of $r = 4$, which is the maximum number of linearly independent rows in the space, meaning that the operator is of full-rank, it is invertible, and it can create any possible superposition from the states given an initial state. So just by defining one state, and one full-rank operator, we can map out the entire Hilbert space that we're working in.

2.2 d-dimensional Hilbert space

There are many different ways of interpreting the properties of Hilbert space, so let us start with the simplest most basic definition. A Hilbert space is an infinite dimensional complete inner product space; complete, meaning that every point of the space can be mapped out, and inner product space, meaning we have introduced a dot product that induces a norm on the space itself. But not only does it induce a norm on the space as a whole, it induces a specific norm on one of the bases of the space, depending on the basis states that are in the product. This allows us to introduce a variable for length and angle, which, when used properly, allows us to describe the time evolution of the system that preserves the inner product structure. We are mostly interested in the finite d-dimensional Hilbert space, which also satisfies the completeness theorem.

2.2.1 States in d-dimensional Hilbert space

A state in d-dimensional Hilbert space can be defined as the sum of linearly independent vectors $|k_i\rangle$ which form a basis where every element can be represented as a linear combination:

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |\psi_i\rangle = \begin{pmatrix} \alpha_0 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_1 \\ 0 \\ ... \\ 0 \end{pmatrix} + ... \begin{pmatrix} 0 \\ 0 \\ 0 \\ ... \\ \alpha_{d-1} \end{pmatrix} \tag{2.2.1}$$

where $|\psi\rangle$ is a basis state, d the dimension of the space itself, and α_i is a complex number whose modulus squared gives us the probability of the state being in that particular state. Of particular interest to quantum mechanics is when the basis states $|j\rangle$ and $|k\rangle$ are orthogonal to each other:

$$|j\rangle = \sum_{i=0}^{d-1} \alpha_i |j\rangle = \begin{pmatrix} \alpha_0 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_1 \\ 0 \\ ... \\ 0 \end{pmatrix} + ... \begin{pmatrix} 0 \\ 0 \\ 0 \\ ... \\ \alpha_{d-1} \end{pmatrix}; |k\rangle = \sum_{i=0}^{d-1} \beta_i |k_i\rangle = \begin{pmatrix} \beta_0 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ ... \\ 0 \end{pmatrix} + ... \begin{pmatrix} 0 \\ 0 \\ 0 \\ ... \\ \beta_{d-1} \end{pmatrix};$$

$$\langle j|k\rangle = \delta_{jk} \tag{2.2.2}$$

where $\langle j|$ is the conjugate transpose of $|j\rangle$. For the inner product we have two possible values:

$$\delta_{jk} = 0, j \neq k;$$
$$\delta_{jk} = 1, j = k$$

In other words, the basis is orthonormal. The sum of all probabilities is equal to one:

$$\sum_{i=0}^{d-1} |\alpha_i|^2 + |\beta_i|^2 = 1 \tag{2.2.3}$$

For the special case of $d = 2$, the states can be represented on the Bloch sphere using:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \cos\frac{\theta}{2} |0\rangle + e^{i\phi} \sin\frac{\theta}{2} \tag{2.2.4}$$

2.2.2 Operators in d-dimensional Hilbert space

In general, we begin by defining the quantum mechanical operators for position and momentum; however, since we will not be using them as much, it is better to instead define the group to which the operators belong for an arbitrary d. The operators need to be unitary, meaning they belong to the U(d) group, and also their determinant needs to be equal to one, which gives us the Special Unitary group of d - SU(d). We are mostly interested in the method of their construction, and subsequently their usage, instead of more profound interpretations.

2.2.3 SU(2)

First let us consider the case of $d = 2$, where the operators would be represented as 2x2 complex unitary matrices:

$$\hat{O}_2 = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \tag{2.2.5}$$

such that the matrix product between an operator and it's complex conjugate is:

$$\hat{O}_2 \hat{O}_2^\dagger = \hat{I}_2 \tag{2.2.6}$$

where $\hat{I}_2$ is the 2x2 identity matrix, and $\hat{O}_2$ is an operator belonging to the SU(2) group, or simply, they're unitary. In order to define these operators, we would need to define the algebra to which the group belongs to, namely the $su(2)$ algebra, whose generators are given by:

$$su_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; su_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; su_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \tag{2.2.7}$$

As we can see, the generators themselves are anti-Hermitian, have trace zero, dimensonality of $d = 2$, and they are of full-rank $r = 2$. Since the rank of these matrices are full-rank, any operator that would be generated by these matrices should also be of full-rank, giving the operator the ability to map out the entire basis on which it operates.

2.2.4 SU(3)

Now consider the case of $d = 3$, where the operators would be represented as 3x3 complex unitary matrices, which follow the same rules as the SU(2) operators. However, unlike the SU(2) operators. We will use the SU(2) group to construct them by imposing a trace condition along one of the main bases(axes) of expansion(evolution). For $d = 3$, we have a total of $d^2 - 1 = 8$ generators:

1. $\frac{d(d-1)}{2} = 3$ symmetric matrices:
   $\lambda_s = |j\rangle\langle k| + |k\rangle\langle j|$, where $0 \le j < k \le d-1$

2. $\frac{d(d-1)}{2} = 3$ anti-symmetric matrices:
   $\lambda_a = -i(|j\rangle\langle k|) + i(|k\rangle\langle j|)$, where $0 \le j < k \le d-1$

3. $d - 1 = 2$ diagonal matrices:
   $\lambda_d = -\sqrt{\frac{2}{l(l+1)}}\left(\sum_{j=1}^{l} |j\rangle\langle j| - l\,|l+1\rangle\langle l+1| \right)$, where $0 \le j < k \le d-1$

In fact, we still have one more generator, our norm(the generator of our basis?), or also known as the identity matrix which for SU(3) is:

$$\hat{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2.8}$$

However, for higher dimensions, Dirac notation becomes messy,and instead it would be useful to have a more visual way of constructing these generators. Begin with a very special case of the $su(2)$ generators:

$$\begin{aligned} su_0 &= -i\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \to \hat{\sigma}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \\ su_1 &= i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to \hat{\sigma}_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \\ su_2 &= -i\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \to \hat{\sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \end{aligned} \tag{2.2.9}$$

Expand each matrix by one row and one column of zeroes, giving us the first symmetric,anti-symmetric and diagonal matrices:

$$\begin{aligned} \lambda_1 &= |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\ \lambda_2 &= -i(|0\rangle\langle 1|) + i(|1\rangle\langle 0|) = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\ \lambda_3 &= |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \end{aligned} \tag{2.2.10}$$

If we move along the edges by one row/column, this would give us the the rest of the symmetric and anti-symmetric matrices:

$$\begin{aligned} \lambda_4 &= |0\rangle\langle 2| + |2\rangle\langle 0| = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \lambda_5 = -i(|0\rangle\langle 1|) + i(|1\rangle\langle 0|) = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \\ \lambda_6 &= |1\rangle\langle 2| + |2\rangle\langle 1| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \lambda_7 = -i(|1\rangle\langle 2|) + i(|2\rangle\langle 1|) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \end{aligned} \tag{2.2.11}$$

We have one last remaining matrix, λ_8, which is diagonal:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Impose the trace condition(standard normalization condition):

$$\mathrm{Tr}[\lambda_k \lambda_j] = 2\delta_{kj} \tag{2.2.12}$$

and solve the system:

$$\lambda_3\lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Applying the trace condition we get:

$$\mathrm{Tr}[\lambda_3\lambda_8] = a - b = 0 \to a = b$$

and since we already know that the matrices by themselves need to be traceless as well, we have:

$$\mathrm{Tr}[\lambda_8] = a + b + c = 0 \to \mathrm{Tr}\lambda_8 = a + a + c = 2a + c \to c = -2a \tag{2.2.13}$$

And one last time:

$$\mathrm{Tr}[\lambda_8\lambda_8] = \mathrm{Tr}\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 4a^2 \end{pmatrix} = 2$$

finally giving us $a^2 + a^2 + 4a^2 = 6a^2 = 2 \to a = \pm\frac{1}{\sqrt{3}}$ and λ_8 has the form:

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{2.2.14}$$

With that, we have all the matrices needed to generate any operator belonging to the SU(3) group. These matrices are known as the Gell-Mann matrices, and using them, we get the generators for the SU(3) group as:

$$G_i = \frac{\lambda_i}{2} \tag{2.2.15}$$

The visual method of moving the indices of the non-zero elements is much more intuitive, and can be easily done for $d > 3$. In the figure below you can see the general algorithm, where each λ_i are the two positions where our non-zero elements would be, for our non-diagonal matrices, and the diagonal matrices just have a different coefficient for the last element λ_{dd}. For $d = 4$ or higher, we also have elements that are inside the edges, denoted by the green lines.

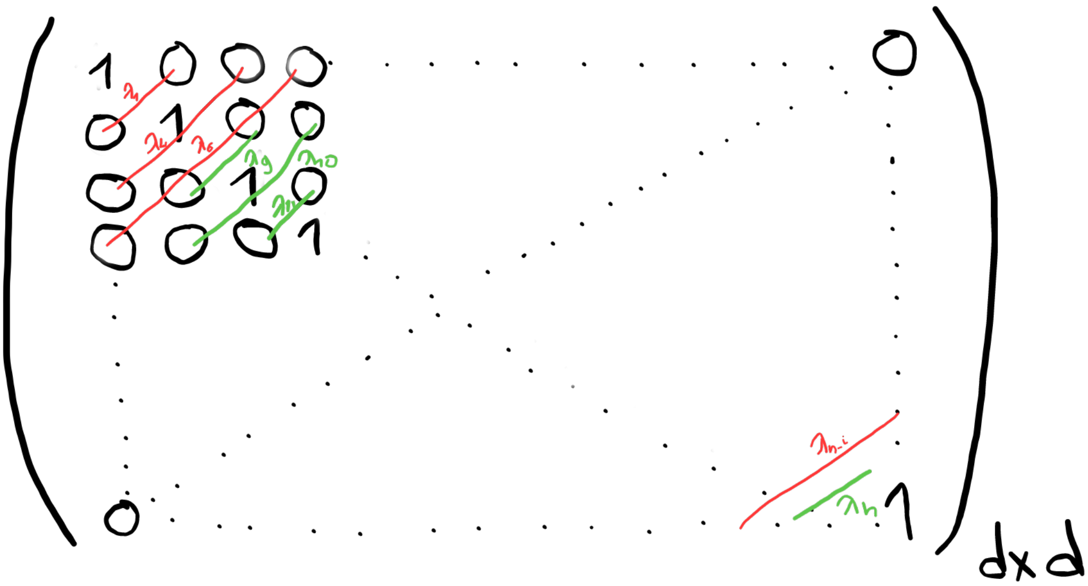

Fig. 1: Visual algorithm for generating Gell-Mann matrices

2.3 Representing real physical quantities

Like stated before, we will not be using the operators for position and momentum, however we will still be using an operator that represents a quantum mechanical property. That operator is the spin operator of the particle. In order for the operators to be able to act on the spin basis of particles, they need to be of a very special kind, a Pauli operator.

2.3.1 Pauli operators for $d = 2$

The Pauli operators for $d = 2$ are given by the special case of the $su(2)$ generators: $\hat{\sigma}_x$,$\hat{\sigma}_y$,$\hat{\sigma}_z$, where x, y, z are 0, 1, 2 respectively. Only in this configuration the generators and Pauli operators share the same matrices, for $d > 2$ they do not. Visually we can represent the $su(2)$ transformations as moving on the surface of a sphere. The Pauli transformations can be represented as 180 degree rotations around their specific basis on the Bloch sphere, as shown in the figure below.

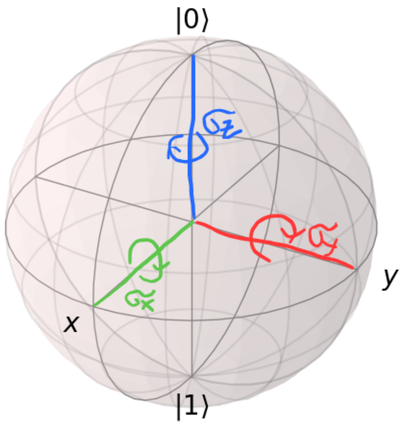

Fig. 2: Rotations on the Bloch sphere

As you can see, the direction of the rotation does not matter in this case since the rotations are always by 180 degrees. That is why, only in this case: $\sigma_x = \sigma_x^\dagger$, $\sigma_y = \sigma_y^\dagger$ and $\sigma_z = \sigma_z^\dagger$, or in other words, they are Hermitian.

2.3.2 Pauli operators for $d = 3$

Looking at the generators for the $su(3)$ algebra, we can see that only the diagonal ones are of full rank $r = 3$, the rest are of rank $r = 2$, meaning that they would not be able to map out their entire basis. That is the reason why we need, yet another group, one that is the generalization of the Pauli group, which, does not belong to the SU(3) but, the Pauli group itself of SU(2) is still a part of this new group. This new group is called the Generalized Pauli group, where the operators are:

$$\hat{\sigma}_{x_3} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$
$$\hat{\sigma}_{z_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{2.3.1}$$

where $\omega = e^{2i\pi/d}$ is the root of unity and it satisfies:

$$1 + \omega + \omega^2 + ... + \omega^{d-1} = 0$$

These new operators are generated by the Generalized Clifford algebra, where the operators $\hat{\sigma}_{x3}$ and $\hat{\sigma}_{z3}$ are generated by the shift and clock matrices respectively. The third and final operator of the generalized Pauli group is given by:

$$\hat{H}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix} \tag{2.3.2}$$

also known as the Hadamard operator, where $1/\sqrt{3}$ is a normalizing constant that follows from Von Neumann's generalization principle for spin operators of higher dimensions. However, it should still be possible to generate these matrices out of the SU(3) generators. Since we will only be using three of the generators, it is more convenient to use the following notation:

$$\lambda_1 \to \lambda_1;\ \lambda_4 \to \lambda_2;\ \lambda_6 \to \lambda_3.$$

For the operators we have:

$$\begin{aligned} \hat{\sigma}_{x3} &= \lambda_1\lambda_3 + \lambda_2\lambda_1 + \lambda_3\lambda_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ \hat{\sigma}_{z3} &= \frac{\omega - \omega^2 + 1}{2}\lambda_1\lambda_1 + \frac{\omega^2 - \omega + 1}{2}\lambda_2\lambda_2 + \frac{\omega^2 + \omega - 1}{2}\lambda_3\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \end{aligned} \tag{2.3.3}$$

In the same way we can get the Hadamard operator:

$$\begin{aligned} \sqrt{3}\hat{H}_3 = \frac{\omega - \omega^4 + 1}{2}\lambda_1\lambda_1 + \omega^2\lambda_1\lambda_2 + \lambda_1\lambda_3 + \omega^2\lambda_2\lambda_1 + \frac{\omega^4 - \omega + 1}{2}\lambda_2\lambda_2 + \lambda_2\lambda_3 \\ + \lambda_3\lambda_1 + \lambda_3\lambda_2 + \frac{\omega^4 + \omega - 1}{2}\lambda_3\lambda_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix} \end{aligned} \tag{2.3.4}$$

This is very easily scalable to much higher dimensions of d. If for $d = 2$ we needed a sphere, which is three dimensional, for $d = 3$ we would need a sphere that is four dimensional, or in other words a foursphere. However, such representations are tough to imagine and tough to show visually, so instead, we will be using a circle to represent only rotations around the X and Z axis, and we leave the Y axis separate, as it only moves us from the imaginary to real and back.

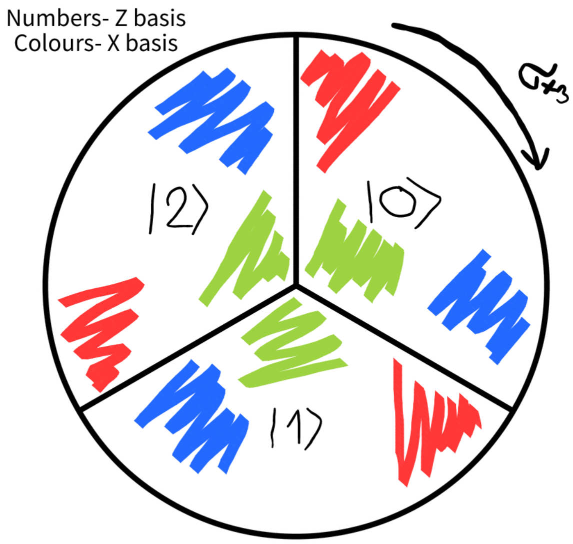

Fig. 3: Rotations on the XZ-circle

On the figure you can see that the rotations this time are 120 degrees about the X-axis and they are represented by rotations in the Z basis, namely, rotations between the states $|0\rangle$, $|1\rangle$ and $|2\rangle$. Meanwhile, rotations along the Z-axis are represented by rotations in the X-basis, and for clarity, we define them as the state being in a different colour(phase), namely red, blue and green. So every state can be in three colours, giving us a total of nine possible states. This can be very easily generalized for $d > 3$, where the rotations would be $360/d$ degrees, and we would have d different colours.

3 Quantum Entanglement

A very unintuitive, but important part of the Hilbert space is that it allows for correlations between particles, that cannot be described by any classical parameter. This correlation is what we call quantum entanglement, it is a quantum mechanical phenomenon which creates a "link" between said particles that only breaks once we have measured either particle.

3.1 EPR pairs

Such pairs of particles are more commonly known as the Einstein-Podolsky-Rosen pairs, where the name comes from their creators. The idea is simple, we have two particles in a known initial state, they interact with each other for a very brief moment of time, cease to interact and move away spatially. What quantum mechanics predicts is that, by measuring the first particle, we would be able to predict the outcome of the measurement of the second particle. It is wired in our classical brains to think that there must be some sort of a real physical link between these two particles, which is the medium that transfers our action on one of the particles onto the other, and allows for the validity of such predictions to be true. This was the main idea of the EPR paper, namely, the authors wanted to show that because such a correlation between particles is possible, there must exist a set of "hidden variables" which are the medium of transfer.

3.2 Hidden variable theory

Such variables, as the name implies, would be very hard to look for, requiring very precise measurements of the particle, something that was not available back when this idea was first conceived. It would take nearly four decades for a breakthrough in the matter, and it would come in the form of Bell's inequality. In order for any sort of experiment to be considered as representing quantum mechanical processes, we would have to repeat the experiment multiple times, with incredible consistency and get a statistical average of the results. With such a restriction in mind, it becomes increasingly difficult to realise such experiments in real life, not even the greatest painter could ever paint the same picture twice, and so, from the very start, we are forced into techniques where such restrictions are easily achievable.

3.2.1 Bell's Inequality

The original derivation of this inequality is due to Bell in 1964, however we will show a more modern approach to deriving it and experimentally proving it using Qiskit. Begin by having Alice and Bob measure the polarisations of an EPR pair(Φ^+) along three different angles φ_a, φ_b and φ_c off the Z-basis. Each measurement gives one of two values $|0_i\rangle$ and $|1_i\rangle$ where $i = a, b, c$ denotes the different angles of the polarisers. For example, the results of the measurements could be:

$$\begin{aligned} p(A) &= 0_a; \\ p(B) &= 0_b \end{aligned} \tag{3.2.1}$$

where Alice and Bob have measured a zero in the a-basis and b-basis given by the rotation φ_a and φ_b respectively. Since the bases of measurement are randomly generated we are allowed to switch from probabilities to measurement numbers. Using the fact that no matter how many measurements are carried out, the probabilities of these results are independent of the third measurement basis given by φ_c:

$$N(A = 0_a; B = 0_b) = N(A = 0_a; B = 0_b|c = 0) + N(A = 0_a; B = 0_b|c = 1) \tag{3.2.2}$$

We can rewrite by imposing the c condition onto Bob's measurement:

$$N(A = 0_a; B = 0_b|c = 0) \le N(A = 0_a|c = 0) = N(A = 0_a; B = 0_c); \tag{3.2.3}$$

and the c condition onto Alice's measurement:

$$N(A = 0_a; B = 0_b|c = 1) \le N(B = 0_b|c = 1) = N(A = 1_c; B = 0_b) \tag{3.2.4}$$

Substituting in (3.2.2):

$$N(A = 0_a; B = 0_b) \le N(A = 0_a; B = 0_c) + N(A = 0_b, B = 1_c) \tag{3.2.5}$$

where we have used $N(A = 0_b, B = 1_c) = N(A = 1_c, B = 0_b)$. This is one form of the Bell inequality, and if it holds, it means that hidden variables exist. However it is easily seen that this inequality does not hold always. For example, in the case of Φ^+ we have:

$$\begin{aligned} N(A = 0_a, B = 0_c) = N(A = 1_a, B = 1_c) = \cos^2(\varphi_a - \varphi_c); \\ N(A = 0_b, B = 1_c) = N(A = 1_b, B = 0_c) = \sin^2(\varphi_b - \varphi_c) \end{aligned} \tag{3.2.6}$$

Setting $\varphi_a = 0$, $\varphi_b = \vartheta$ and $\varphi_c = \pi/2 - \vartheta$:

$$\cos^2\vartheta \le \cos^2(\vartheta - \frac{\pi}{2}) + \sin^2(2\vartheta - \frac{\pi}{2}) \tag{3.2.7}$$

and the identities:

$$\begin{aligned} \cos(a + b) = \cos a \cos b - \sin a \sin b \\ \sin(a + b) = \sin a \cos b + \cos a \sin b \end{aligned}$$

we get:

$$\begin{aligned} \cos^2\vartheta &\le \big(\cos\vartheta\cos\frac{\pi}{2} - \sin\vartheta\sin\frac{\pi}{2}\big)^2 + \big(\sin\vartheta\cos\frac{\pi}{2} + \cos\vartheta\sin\frac{\pi}{2}\big)^2 \\ &\le sin^2\vartheta + \cos^2\vartheta \end{aligned} \tag{3.2.8}$$

which is only true for $\vartheta = 0/45$ degrees, proving that the inequality is violated.

3.2.2 CHSH Inequality

A more generalised version of Bell's inequality was derived by Clauser-Horne-Shimony-Holt a couple of years later. It expands the inequality so that it can be achieved on an experimental apparatus. However their method is quite outdated, so instead we use one which we can simulate in Qiskit. Again, we have Alice and Bob doing polarisation measurements, this time along two different basis a_1, a_2 and b_1, b_2 respectively. Each of the basis have eigenstates $\pm\left|1_{i/j}\right\rangle$ with eigenvalues ± 1. The experiment consists of averaging the number of simultaneous detections for both detectors in a fixed basis, given by:

$$E(i, j) = \frac{N_{00}^{i,j} - N_{01}^{i,j} - N_{10}^{i,j} + N_{11}^{i,j}}{N_{00}^{i,j} + N_{01}^{i,j} + N_{10}^{i,j} + N_{11}^{i,j}} \tag{3.2.9}$$

where $N_{00}^{i,j}$ is the number of simultaneous detections of 0 in Alice's and Bob's labs and so on. Since there are four total measurement basis, we have:

$$S = E(a_1, b_1) - E(a_1, b_2) + E(a_2, b_1) + E(a_2, b_2) \tag{3.2.10}$$

3.2.3 CHSH Simulation

To realise this idea on Qiskit's platform, we use a total of four qubits, where two of the qubits are in an arbitrary basis, while the other two qubits are entangled and rotated by $\pi/4$. Each of the qubits is measured and the number of detections is plotted. The circuit is shown on Fig.4 and is run for 512 shots. It is then iterated 100 times and the sum of the result is plotted on Fig.5.

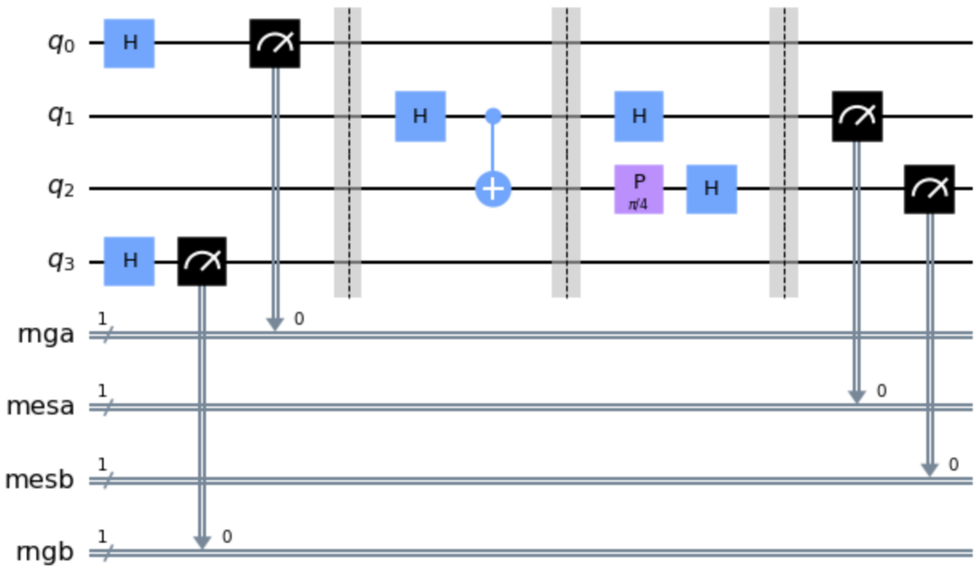

Fig. 4: Qiskit circuit for CHSH inequality

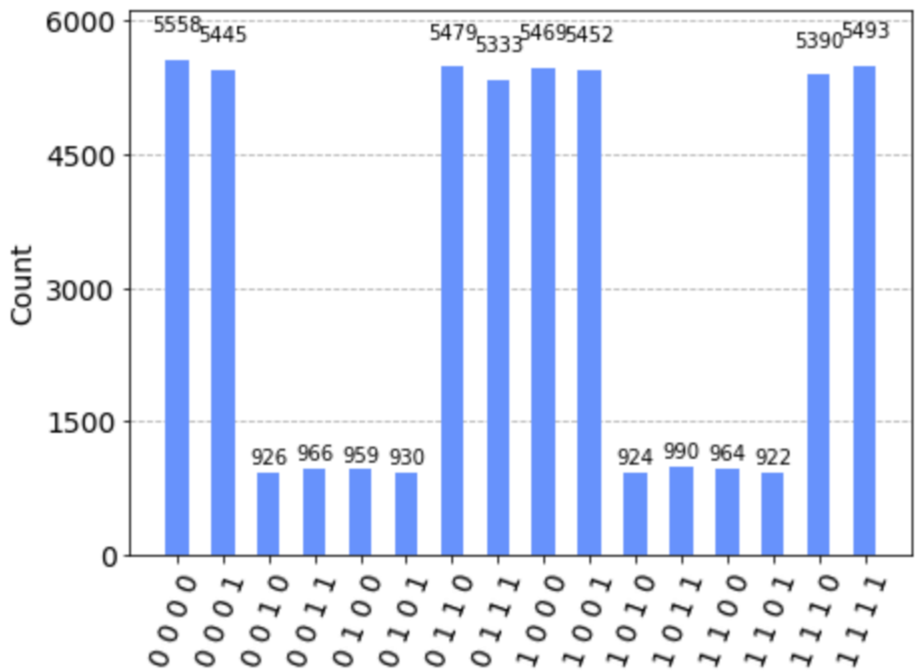

Fig. 5: Results from 100 iterations of the CHSH circuit

In order to interpret these results, we calculate S:

$$S = 0.708 - (-0.700) + 0.703 + 0.702 \approx 2.813 \leq 2\sqrt{2} \leq 2.828 \tag{3.2.11}$$

where we have used the averaged values:

$$
\begin{aligned}
E(a_1,b_1) &= \frac{\frac{5558}{100}-\frac{926}{100}-\frac{959}{100}+\frac{5479}{100}}{\frac{5558}{100}+\frac{926}{100}+\frac{959}{100}+\frac{5479}{100}} = \frac{91.52}{129.22} \approx 0.708;\\
E(a_2,b_1) &= \frac{-\frac{5445}{100}+\frac{966}{100}+\frac{930}{100}-\frac{5333}{100}}{\frac{5445}{100}+\frac{966}{100}+\frac{930}{100}+\frac{5479}{100}} = -\frac{88.83}{126.74} \approx -0.700;\\
E(a_1,b_2) &= \frac{\frac{5469}{100}-\frac{924}{100}-\frac{964}{100}+\frac{5390}{100}}{\frac{5469}{100}+\frac{924}{100}+\frac{964}{100}+\frac{5390}{100}} = \frac{89.71}{127.47} \approx 0.703;\\
E(a_2,b_2) &= \frac{\frac{5452}{100}+\frac{990}{100}+\frac{922}{100}+\frac{5493}{100}}{\frac{5452}{100}+\frac{990}{100}+\frac{922}{100}+\frac{5493}{100}} = \frac{90.33}{128.57} \approx 0.702;
\end{aligned}
$$

The indeces are switched because Qiskit enumerates the qubits by going from lowest to highest. Since $S = 2.813 > 2$, we have shown that there is a correlation between the qubits that violates the inequality, and so the correlation between these particles is bigger than any possible classical correlation.

3.3 Bell basis and Bell states

The Bell basis is the basis which contains all the possible maximally entangled states of a given two-particle system, represented by their product states. We define them here for clarity and will later demonstrate them using quantum gates.

3.3.1 Qubits

For qubits we have $d^2 = 4$ Bell states:

$$
\begin{aligned}
\left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}}\big(\,|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\,\big);\\
\left|\Psi^-\right\rangle &= \frac{1}{\sqrt{2}}\big(\,|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle\,\big);\\
\left|\Psi^+\right\rangle &= \frac{1}{\sqrt{2}}\big(\,|0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle\,\big);\\
\left|\Phi^-\right\rangle &= \frac{1}{\sqrt{2}}\big(\,|0\rangle\otimes|0\rangle - |1\rangle\otimes|1\rangle\,\big)
\end{aligned}
\tag{3.3.1}
$$

where:

$$
\begin{aligned}
|00\rangle = |0\rangle\otimes|0\rangle &= \frac{1}{\sqrt{2}}\big(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\big);\\
|01\rangle = |0\rangle\otimes|1\rangle &= \frac{1}{\sqrt{2}}\big(\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle\big);\\
|10\rangle = |1\rangle\otimes|0\rangle &= \frac{1}{\sqrt{2}}\big(\left|\Psi^+\right\rangle - \left|\Psi^-\right\rangle\big);\\
|11\rangle = |1\rangle\otimes|1\rangle &= \frac{1}{\sqrt{2}}\big(\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle\big)
\end{aligned}
\tag{3.3.2}
$$

3.3.2 Qudits

For qudits we have $d^2 = 9$ Bell states:

$$\begin{aligned}
|\Phi_1\rangle &= \frac{1}{\sqrt{3}}\big(\,|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle\,\big);\\
|\Phi_2\rangle &= \frac{1}{\sqrt{3}}\big(\,|0\rangle \otimes |1\rangle + |1\rangle \otimes |2\rangle\,\big) + |2\rangle \otimes |0\rangle\,\big);\\
|\Phi_3\rangle &= \frac{1}{\sqrt{3}}\big(\,|0\rangle \otimes |2\rangle + |1\rangle \otimes |0\rangle + |2\rangle \otimes |1\rangle\,\big);\\
|\Phi_4\rangle &= \frac{1}{\sqrt{3}}\big(\,|0\rangle \otimes |0\rangle + \omega\,|1\rangle \otimes |1\rangle + \omega^2\,|2\rangle \otimes |2\rangle\,\big);\\
|\Phi_5\rangle &= \frac{1}{\sqrt{3}}\big(\,|0\rangle \otimes |1\rangle + \omega\,|1\rangle \otimes |2\rangle + \omega^2\,|2\rangle \otimes |0\rangle\,\big);\\
|\Phi_6\rangle &= \frac{1}{\sqrt{3}}\big(\,|0\rangle \otimes |2\rangle + \omega\,|1\rangle \otimes |0\rangle + \omega^2\,|2\rangle \otimes |1\rangle\,\big);\\
|\Phi_7\rangle &= \frac{1}{\sqrt{3}}\big(\,|0\rangle \otimes |0\rangle + \omega^2\,|1\rangle \otimes |1\rangle + \omega^4\,|2\rangle \otimes |2\rangle\,\big);\\
|\Phi_8\rangle &= \frac{1}{\sqrt{3}}\big(\,|0\rangle \otimes |1\rangle + \omega^2\,|1\rangle \otimes |2\rangle + \omega^4\,|2\rangle \otimes |0\rangle\,\big);\\
|\Phi_9\rangle &= \frac{1}{\sqrt{3}}\big(\,|0\rangle \otimes |2\rangle + \omega^2\,|1\rangle \otimes |0\rangle + \omega^4\,|2\rangle \otimes |1\rangle\,\big)
\end{aligned} \tag{3.3.3}$$

where:

$$\begin{aligned}
|00\rangle = |0\rangle \otimes |0\rangle &= \frac{1}{\sqrt{3}}\big(\,|\Phi_1\rangle + |\Phi_4\rangle + |\Phi_7\rangle\,\big);\\
|01\rangle = |0\rangle \otimes |1\rangle &= \frac{1}{\sqrt{3}}\big(\,|\Phi_2\rangle + |\Phi_5\rangle + |\Phi_8\rangle\,\big);\\
|02\rangle = |0\rangle \otimes |2\rangle &= \frac{1}{\sqrt{3}}\big(\,|\Phi_3\rangle + |\Phi_6\rangle + |\Phi_9\rangle\,\big);\\
|10\rangle = |1\rangle \otimes |0\rangle &= \frac{1}{\sqrt{3}}\big(\,|\Phi_3\rangle + \omega^2\,|\Phi_6\rangle + \omega^4\,|\Phi_9\rangle\,\big);\\
|11\rangle = |1\rangle \otimes |1\rangle &= \frac{1}{\sqrt{3}}\big(\,|\Phi_1\rangle + \omega^2\,|\Phi_4\rangle + \omega^4\,|\Phi_7\rangle\,\big);\\
|12\rangle = |1\rangle \otimes |2\rangle &= \frac{1}{\sqrt{3}}\big(\,|\Phi_2\rangle + \omega^2\,|\Phi_5\rangle + \omega^4\,|\Phi_8\rangle\,\big);\\
|20\rangle = |0\rangle \otimes |2\rangle &= \frac{1}{\sqrt{3}}\big(\,|\Phi_2\rangle + \omega^4\,|\Phi_5\rangle + \omega^2\,|\Phi_8\rangle\,\big);\\
|21\rangle = |2\rangle \otimes |1\rangle &= \frac{1}{\sqrt{3}}\big(\,|\Phi_3\rangle + \omega^4\,|\Phi_6\rangle + \omega^2\,|\Phi_9\rangle\,\big);\\
|22\rangle = |2\rangle \otimes |2\rangle &= \frac{1}{\sqrt{3}}\big(\,|\Phi_1\rangle + \omega^4\,|\Phi_4\rangle + \omega^2\,|\Phi_7\rangle\,\big)
\end{aligned} \tag{3.3.4}$$

4 Quantum teleportation

The culmination of the quantum entanglement saga is the realisation of an experiment where the information about two given particles is transmitted through quantum entanglement, for example, an experiment where we achieve quantum teleportation. The idea is simple, we entangle two particles, creating an EPR pair, move to the Bell basis and the system is now represented by a Bell state. Separate the pair, and send each particle to a different lab, making sure that along the way any interactions are prevented. Now, one of the particles is entangled again with another particle in an arbitrary state. In essence, this transfers the entanglement from the first EPR pair, to the second EPR pair, giving us in general one big entangled system. The second EPR pair is then measured, destroying the entanglement between all particles. Measurement results are then sent to the other lab, where they do operations based on them, and measure their particle, which should give the arbitrary state. Therefore, we have managed to teleport the information contained in our arbitrary particle, to the one in the other lab, without the two of them ever interacting with each other, which is a beautiful way to use quantum entanglement.

4.1 No cloning theorem

As nice as it would be to be able to teleport any arbitrary state, unfortunately it is not possible. We can teleport an arbitrary state, but the state must be arbitrary in a pre-determined basis, a basis in which all of the particles involved in the experiment are part of. This can be proven easily. Begin by defining a cloning operator which clones the state $|j\rangle$ onto $|k\rangle$:

$$\hat{K}(|j\rangle \otimes |k\rangle) = |j\rangle \otimes |j\rangle \tag{4.1.1}$$

where $\hat{K}$ is unitary. Taking the scalar product we have:

$$\left\langle \hat{K}(|j\rangle \otimes |k\rangle) \middle| \hat{K}(|l\rangle \otimes |k\rangle) \right\rangle = \langle j \otimes j | l \otimes l \rangle \tag{4.1.2}$$

However, since the operator is unitary, it also satisfies:

$$\left\langle \hat{K}(|j\rangle \otimes |k\rangle) \middle| \hat{K}(|l\rangle \otimes |k\rangle) \right\rangle = \langle k \otimes j | \hat{K}^\dagger \hat{K} | l \otimes k \rangle = \langle k \otimes j | l \otimes k \rangle \tag{4.1.3}$$

Evaluate both equations:

$$\langle j \otimes j | l \otimes l \rangle = \langle k \otimes j | l \otimes k \rangle \tag{4.1.4}$$

simplify:

$$\begin{aligned} \langle j|l\rangle \,.\, \langle j|l\rangle &= \langle j|l\rangle \,.\, \langle k|k\rangle \\ \implies (\langle j|l\rangle)^2 &= \langle j|l\rangle \end{aligned} \tag{4.1.5}$$

where we have used $\langle k|k\rangle = 1$. The equation above can only be true if $\langle j|l\rangle = 1$ or $\langle j|l\rangle = 0$, where the former one is the trivial solution since it implies that $|j\rangle = |l\rangle$, while the latter is true only if the vectors are orthogonal. And with that, we have proven that we can only entangle particles that are either in the same basis, or in a mutually orthogonal bases, which is also known as the no-cloning theorem.

4.2 Qubits

The smallest element in computer memory, is a bit, with a value of 0 or 1 and 8 bits make a byte. A byte is any symbol you see on the computer screen, whether it's a number, a punctuation sign, a letter from the alphabet, every single one of them has a byte with a certain configuration assigned to it. Our aim is to construct quantum logic operators using quantum bits. A quantum bit is a representation of a quantum mechanical state that gives us certain information about the system. In general, we take the quantum bit, also known as qubit to be two dimensional, meaning it has two basis:

$$\begin{aligned} |a\rangle &= |0\rangle \\ |b\rangle &= |1\rangle \end{aligned} \tag{4.2.1}$$

If the system is taken to be representing spin, it means that we are allowed to act on the states with the Pauli operators, also known as Pauli matrices:

$$\begin{aligned} \hat{\sigma_x} &= |0\rangle\langle 1| + |1\rangle\langle 0| \\ \hat{\sigma_z} &= |0\rangle\langle 0| - |1\rangle\langle 1| \\ \hat{\sigma_y} &= -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0| \end{aligned} \tag{4.2.2}$$

This seems a little strange however, at least it is not the way that we are used to seeing Pauli matrices. We are used to them being in their widely known matrix form:

$$\hat{\sigma_x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma_z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{\sigma_y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.2.3}$$

The former one is in Dirac notation, while the latter is in matrix notation, and the states are:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.2.4}$$

Or represented visually, on the Bloch sphere:

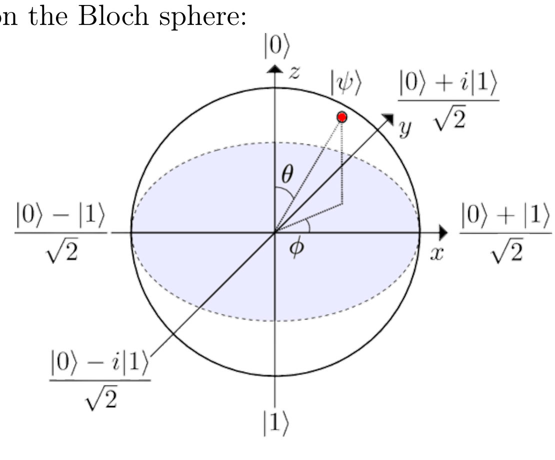

Fig. 6: Qubit

These two notations denote the same idea, but how do we establish a connection between them? The physicist would say, well we can substitute for $|0\rangle$ and $|1\rangle$ in the equations, and then find their Hermitian conjugate and solve the equation, which would lead us to the matrices we see below them. And yes, one could get very good at these calculations and be able to do it fast. However, the programmer would argue that the states themselves, are actually the indices of the ones and zeroes in the matrix notation. How so? When defining a matrix in any programming language, we use two loops, for example, take any element of a matrix to be given by a variable

a, with indices i and j. a_{ij} is the element of the matrix that is in the i-th row, and the j-th column, where the rows are represented by the outer loop, and the columns are represented by the inner loop. In reality, this would enable us to make an infinitely large matrices, with as many rows, and as many columns as we want, but for now we restrict ourselves on $N = 2$, $i, j = 0, 1$ so as to be part of the SU(2) algebra:

$$\hat{\sigma}_2 = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \tag{4.2.5}$$

Now if we look back to the Dirac notation, we can easily see, that we can get all the Pauli matrices, simply by replacing the element where the indices match the ket-bra product. This all may seem very obvious at a first glance, but this little "trick" will allow us to more easily expand this logic and algebra to higher dimensions.

4.2.1 Gates for qubits

Operators, or more commonly known as gates in quantum information, are one of the most basic interactions we have with qubits. So far we have used them, but not clearly defined them, so let us do that. Gates are unitary matrices, which when acting upon a certain state, transform the state into a unitarily equivalent one. Their general form is given by:

$$U\big(\theta, \phi, \lambda\big) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\phi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix} \tag{4.2.6}$$

where $e^{i\lambda} = \cos\lambda + i\sin\lambda$ we get:

$$\begin{aligned}
\hat{H} &= U\big(\frac{\pi}{2}, 0, \pi\big) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\big(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\big) \\
\hat{X} &= U\big(\pi, 0, \pi\big) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| \\
\hat{Z} &= U\big(0, 0, \pi\big) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left|0\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right| \\
\hat{Y} &= U\big(\pi, \frac{\pi}{2}, \frac{\pi}{2}\big) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\left|0\right\rangle\left\langle 1\right| + i\left|1\right\rangle\left\langle 0\right| \\
\hat{P} &= U\big(0, 0, \lambda\big) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix} = \left|0\right\rangle\left\langle 0\right| + e^{i\lambda}\left|1\right\rangle\left\langle 1\right| \\
\hat{S} &= U\big(0, 0, \frac{\pi}{2}\big) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \left|0\right\rangle\left\langle 1\right| + i\left|1\right\rangle\left\langle 0\right| \\
\hat{I} &= U\big(0, 0, \frac{\pi}{4}\big) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|
\end{aligned}$$

For the most part we will be using the $\hat{X}$, $\hat{Z}$ and $\hat{Y}$, as they represent the Pauli-operators and lead to actual physical states. The Hadamard gate is also of great interest, it contains all the possible superpositions of states, allowing us to move between the X and Z basis:

$$\begin{aligned}
\hat{H}\hat{Z}\hat{H} &= \frac{1}{\sqrt{2}}\big(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\big)\big(\left|0\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\big)\frac{1}{\sqrt{2}}\big(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\big) \\
&= \frac{1}{2}\big(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\big)\big(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| - \left|1\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\big) \\
&= \frac{1}{2}\big(2(\left|0\right\rangle\left\langle 1\right|) + 2(\left|1\right\rangle\left\langle 0\right|)\big) = \hat{X}; \\
\hat{H}\hat{X}\hat{H} &= \frac{1}{\sqrt{2}}\big(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\big)\big(\left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right|\big)\frac{1}{\sqrt{2}}\big(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\big) \\
&= \frac{1}{2}\big(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\big)\big(\left|0\right\rangle\left\langle 0\right| - \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\big) \\
&= \frac{1}{2}\big(2(\left|0\right\rangle\left\langle 0\right|) - 2(\left|1\right\rangle\left\langle 1\right|)\big) = \hat{Z}; \qquad (4.2.7)
\end{aligned}$$

So far we have only considered, single qubit gates, but there are also multi-qubit gates which we will use, most notably the Control gate, given by:

$$\hat{C} = \begin{pmatrix} \hat{I} & 0 \\ 0 & \hat{U} \end{pmatrix} = \left|0\right\rangle\left\langle 0\right| \otimes \hat{I} + \left|1\right\rangle\left\langle 1\right| \otimes \hat{U} \qquad (4.2.8)$$

of particular interest is the case where $\hat{U} = \hat{X}$, famously known as the control-NOT (CNOT) gate:

$$\hat{C} = \begin{pmatrix} \hat{I} & 0 \\ 0 & \hat{X} \end{pmatrix} = \left|0\right\rangle\left\langle 0\right| \otimes \hat{I} + \left|1\right\rangle\left\langle 1\right| \otimes \hat{X} = \left|00\right\rangle\left\langle 00\right| + \left|01\right\rangle\left\langle 01\right| + \left|10\right\rangle\left\langle 11\right| + \left|11\right\rangle\left\langle 10\right| \qquad (4.2.9)$$

As you can easily see, for this kind of operation, we would require two qubits, where one of them is the control(the gate acts with the identity matrix), and the other one is the target(the gate acts with the $\hat{X}$ gate). In other words, we have two types of CNOT gates, one that acts on the first state, and one that acts on the second one:

$$\begin{aligned}
CNOT_a \left|a\right\rangle\left|b\right\rangle &= \left|a\right\rangle\left|b + a \mod d\right\rangle \\
CNOT_b \left|a\right\rangle\left|b\right\rangle &= \left|a + b \mod d\right\rangle\left|b\right\rangle \qquad (4.2.10)
\end{aligned}$$

For qubits:

$$\begin{aligned}
\mathrm{CNOT}_a &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \\
\mathrm{CNOT}_b &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \qquad (4.2.11)
\end{aligned}$$

It is worth mentioning that the CNOT gate is of course, unitary, but also non-unique, meaning there are multiple possible CNOTs for a given d. We are mostly interested in the first $CNOT_a$, which can be raised to higher dimensions quite easily.

4.2.2 General Procedure

The protocol for quantum teleportation of qubits will be done on Qiskit. We will be using three qubits and three classical registers.

1. Create a quantum circuit with three qubits $q_0 = |0\rangle$, $q_1 = |0\rangle$, $q_2 = |0\rangle$ and three classical registers;
2. Begin by creating a Bell pair between q_1 and q_2 and move to the Bell basis;
3. Transfer q_2 to Bob;
4. Create a bell pair between q_0 and q_1 and return to the computational basis;
5. Measure q_0 and q_1 in the computational basis and send the results to Bob;
6. Bob applies gates to the third qubit depending on the results;
7. Bob measures $q_2 = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$; which is the same as q_0.

4.2.3 Mathematical procedure

1. We have three states: $q_0 = \alpha|0\rangle + \beta|1\rangle$, $q_1 = |0\rangle$, $q_2 = |0\rangle$;
2. Apply Hadamard gate on q_1, then Apply $CNOT_a$ on q_1 and q_2 :

$$\hat{H}|q_1\rangle = \hat{H}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + |1\rangle \implies \mathrm{CNOT}_a|q_1\rangle|q_2\rangle = \mathrm{CNOT}_a\frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)|0\rangle$$
$$= \frac{1}{\sqrt{2}}\Bigg(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|\Bigg)\big(|00\rangle + |10\rangle\big) = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big);$$

3. Apply $CNOT_a$ on q_0 and q_1:

$$\mathrm{CNOT}_a\big(\alpha|0\rangle + \beta|1\rangle\big)\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) = \frac{1}{\sqrt{2}}\Bigg(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|\Bigg)$$
$$\big(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\big) = \frac{1}{\sqrt{2}}\big(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\big);$$

4. Apply Hadamard gate on q_0, meaning we map the Bell states into the computational states of the two qubits($|\Phi^+\rangle \to |00\rangle$; $|\Psi^-\rangle \to |10\rangle$; $|\Psi^+\rangle \to |01\rangle$; $|\Phi^-\rangle \to |11\rangle$):

$$\hat{H}\frac{1}{\sqrt{2}}\big(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\big)$$
$$= \frac{1}{2}\Bigg(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\Bigg)\big(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\big)$$
$$= \frac{1}{2}(\alpha|000\rangle + \alpha|011\rangle + \alpha|100\rangle + \alpha|111\rangle + \beta|010\rangle + \beta|001\rangle - \beta|110\rangle - \beta|101\rangle)$$

5. Since Alice measures q_0 and q_1, we make groups (or in other words, we group by the Bell states):

$$|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |01\rangle \otimes (\beta|0\rangle + \alpha|1\rangle) + |10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) + |11\rangle \otimes (-\beta|0\rangle + \alpha|1\rangle)$$

or in the Bell basis:

$$\left|\Phi^+\right\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + \left|\Psi^+\right\rangle \otimes (\beta|0\rangle + \alpha|1\rangle) + \left|\Phi^-\right\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) + \left|\Psi^-\right\rangle \otimes (-\beta|0\rangle + \alpha|1\rangle)$$

6. Alice measures q_0 and q_1 and send results to Bob;

7. Bob performs operations on q_2 based on the results he got from Alice:
   $|00\rangle$ - Do nothing;
   $|01\rangle$ - Apply X gate;
   $|10\rangle$ - Apply Z gate;
   $|11\rangle$ - Apply ZX gate

8. Bob measures $q_2 = \alpha|0\rangle + \beta|1\rangle$ which should give us the state q_0.

4.2.4 Teleportation protocol

On Fig.7 you can see the algorithm that will be used to simulate quantum teleportation with qubits. It is done on the Qiskit platform, and is run for 512 shots. The figures summarise the results of a simulation and demonstration on a quantum machine accessible through Qiskit. As you can see, we have added one more Hadamard gate at each side since we want to show that even a superposition of states can be teleported, not just a single state.

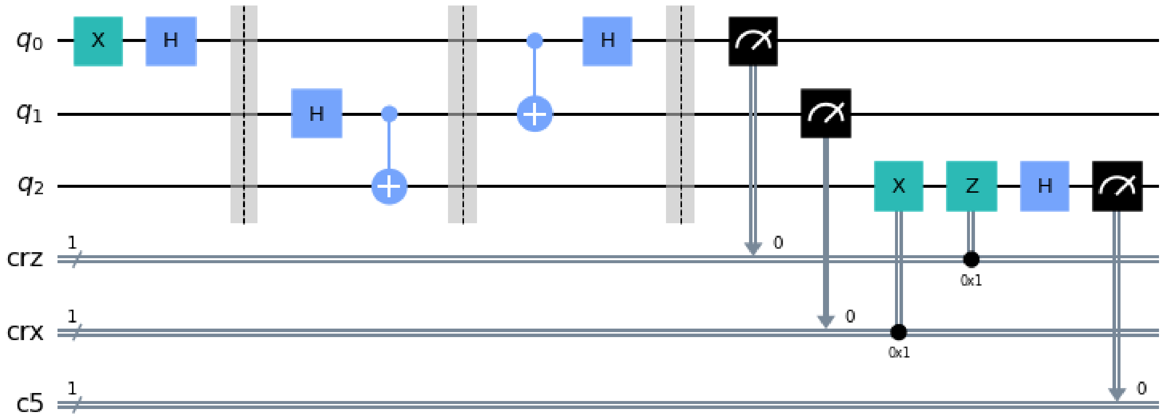

Fig. 7: Qiskit circuit for teleporting $q_0 = |1\rangle$

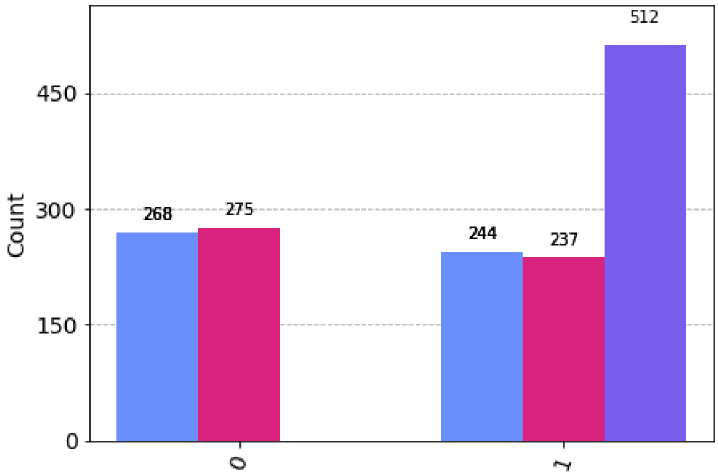

Fig. 8: Results from teleporting $q_0 = |1\rangle$ on a simulator

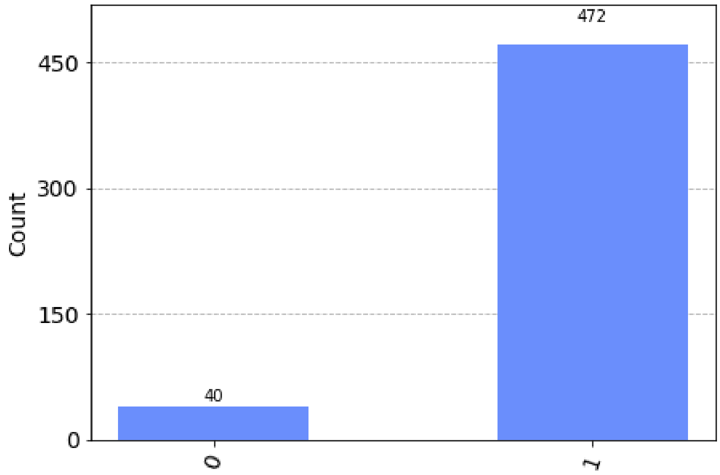

Fig. 9: Results from teleporting $q_0 = |1\rangle$ on a quantum machine

4.3 Qudits

A qudit is a quantum mechanical construct, which contains d bits of information, where $d > 2$. So far we have been using gates from the Pauli group and states from the two dimensional Hilbert space. However, $n = 2$ is a very ideal case, so in order to generalize it to higher levels we need a new basis and new gates.

4.3.1 Gates for qudits

Like stated before, the Pauli matrices generalize into the generalized Clifford algebra matrices, also known as shift and clock matrices:

1. $V = |d-1\rangle\langle 0| + \sum_{j=0,k=1}^{d-2} + |j\rangle\langle k+1|$, where $0 \leq j < k \leq d-2$;
2. $U = \sum_{j=k=0}^{d-1}\left(\omega^j |j\rangle\langle k|\right)$, where $0 \leq j, k \leq d-1$;
3. $W = \sum_{i=k=0}^{d-1} \omega^{jk} |i\rangle\langle k|$, where $0 \leq j, k, i \leq d-1$;

From this, we can easily get the generalised Pauli gates for qutrits:

$$\hat{X}_3 = |0\rangle\langle 2| + |1\rangle\langle 0| + |2\rangle\langle 1| \, ; \hat{Z}_3 = |0\rangle\langle 0| + \omega |1\rangle\langle 1| + \omega^2 |2\rangle\langle 2| \, ; \tag{4.3.1}$$
$$\hat{H}_3 = |0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 2| + |1\rangle\langle 0| + \omega |1\rangle\langle 1| + \omega^2 |1\rangle\langle 2| + |2\rangle\langle 0| + \omega^2 |2\rangle\langle 1| + \omega^4 |2\rangle\langle 2| \, ;$$

The control gates are defined in the same way as the ones for qubits, but instead we use the 3x3 identity matrix $\hat{I}_3$. The $CNOT_3$ is generated by:

$$CNOT_3 = \hat{I}_3 + |jk\rangle\langle j(3 + (k-j) \mod 3)| \, , \text{ where } 0 \leq k < j \leq d-1 \tag{4.3.2}$$

which gives us:

$$CNOT_3 = |00\rangle\langle 00| + |01\rangle\langle 01| + |02\rangle\langle 02| + |10\rangle\langle 12| + |11\rangle\langle 10| + |12\rangle\langle 11| + |20\rangle\langle 21| + |21\rangle |22\rangle + |22\rangle\langle 20| \tag{4.3.3}$$

or in matrix form:

$$CNOT_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.3.4}$$

And with that, we have all the nescessary parts for doing quantum teleportation with qutrits.

4.3.2 General Procedure

The protocol for quantum teleportation of qudits will be done on Mathematica. We will be using three qutrits.

1. Create a quantum circuit with three qutrits: $q_0 = \alpha\left|0\right\rangle + \beta\left|1\right\rangle + \gamma\left|2\right\rangle, q_1 = \left|0\right\rangle + q_2 = \left|0\right\rangle$;
2. Begin by generating the $\hat{X}_3$,$\hat{Z}_3$ and $\hat{H}_3$ gates from the clock and shift matrices;
3. Generate $CNOT_3$;
4. Create a Bell pair between q_1 and q_2 and move to the Bell basis;
5. Transfer q_2 to Bob;
6. Create a Bell pair between q_0 and q_1 and return to the computational basis;
7. Measure q_0 and q_1 in the computational basis and send the results to Bob;
8. Bob applies gates based on the result;
9. End result should be a superposition of $q_2 = \alpha\left|0\right\rangle + \beta\left|1\right\rangle + \gamma\left|2\right\rangle$, which is the same as q_0.

4.3.3 Mathematical Procedure

1. We have three states: $q_0 = \alpha\left|0\right\rangle + \beta\left|1\right\rangle + \gamma\left|2\right\rangle$, $q_1 = \left|0\right\rangle$, $q_2 = \left|0\right\rangle$;
2. Apply Hadamard gate on q_1, then Apply $CNOT_a$ on q_1 and q_2 :

$$\hat{H}\left|q_1\right\rangle = \hat{H}\left|0\right\rangle = \frac{1}{\sqrt{3}}(\left|0\right\rangle + \left|1\right\rangle + \left|2\right\rangle) \implies CNOT_3\left|q_1\right\rangle\left|q_2\right\rangle = CNOT_3\frac{1}{\sqrt{3}}\big(\left|0\right\rangle + \left|1\right\rangle + \left|2\right\rangle\big)\left|0\right\rangle$$
$$= \frac{1}{\sqrt{3}}\Bigg(\left|00\right\rangle\left\langle 00\right| + \left|01\right\rangle\left\langle 01\right| + \left|02\right\rangle\left\langle 02\right| + \left|10\right\rangle\left\langle 12\right| + \left|11\right\rangle\left\langle 10\right| + \left|12\right\rangle\left\langle 11\right| + \left|20\right\rangle\left\langle 21\right| + \left|21\right\rangle\left|22\right\rangle$$
$$+ \left|22\right\rangle\left\langle 20\right|\Bigg)\big(\left|00\right\rangle + \left|10\right\rangle + \left|20\right\rangle\big) = \frac{1}{\sqrt{3}}\big(\left|00\right\rangle + \left|11\right\rangle + \left|22\right\rangle\big);$$

3. Apply $CNOT_3$ on q_0 and q_1:

$$CNOT_3\big(\alpha\,|0\rangle+\beta\,|1\rangle+\gamma\,|2\rangle\,\big)\frac{1}{\sqrt{3}}\big(\,|00\rangle+|11\rangle+|22\rangle\,\big)$$
$$=\frac{1}{\sqrt{3}}\Big(\,|00\rangle\langle 00|+|01\rangle\langle 01|+|02\rangle\langle 02|+|10\rangle\langle 12|+|11\rangle\langle 10|+|12\rangle\langle 11|+|20\rangle\langle 21|+|21\rangle\,|22\rangle$$
$$+\,|22\rangle\langle 20|\,\Big)\big(\alpha\,|000\rangle+\alpha\,|011\rangle+\alpha\,|022\rangle+\beta\,|100\rangle+\beta\,|111\rangle+\beta\,|122\rangle+\gamma\,|200\rangle+\gamma\,|211\rangle+\gamma\,|222\rangle\,\big)$$
$$=\frac{1}{\sqrt{3}}\big(\alpha\,|000\rangle+\alpha\,|011\rangle+\alpha\,|022\rangle+\beta\,|110\rangle+\beta\,|121\rangle+\beta\,|102\rangle+\gamma\,|220\rangle+\gamma\,|201\rangle+\gamma\,|212\rangle\,\big);$$

4. Apply $CNOT_3$ again on q_0 and q_1:

$$CNOT_3\frac{1}{\sqrt{3}}\big(\alpha\,|000\rangle+\alpha\,|011\rangle+\alpha\,|022\rangle+\beta\,|110\rangle+\beta\,|121\rangle+\beta\,|102\rangle+\gamma\,|220\rangle+\gamma\,|201\rangle$$
$$+\,\gamma\,|212\rangle\,\big)=\frac{1}{\sqrt{3}}\Big(\,|00\rangle\langle 00|+|01\rangle\langle 01|+|02\rangle\langle 02|+|10\rangle\langle 12|+|11\rangle\langle 10|+|12\rangle\langle 11|+|20\rangle\langle 21|$$
$$+\,|21\rangle\langle 22|+|22\rangle\langle 20|\,\Big)\big(\alpha\,|000\rangle+\alpha\,|011\rangle+\alpha\,|022\rangle+\beta\,|110\rangle+\beta\,|121\rangle+\beta\,|102\rangle+\gamma\,|220\rangle$$
$$+\,\gamma\,|201\rangle+\gamma\,|212\rangle\,\big)$$
$$=\frac{1}{\sqrt{3}}\big(\alpha\,|000\rangle+\alpha\,|011\rangle+\alpha\,|022\rangle+\beta\,|120\rangle+\beta\,|101\rangle+\beta\,|112\rangle+\gamma\,|210\rangle+\gamma\,|221\rangle+\gamma\,|202\rangle\,\big);$$

5. Apply Hadamard gate on q_0, meaning we map the Bell states into the computational states of the two qutrits($\Phi_1\to|00\rangle$; $\Phi_2\to|01\rangle$; $\Phi_3\to|02\rangle$; $\Phi_4\to|10\rangle$; $\Phi_5\to|11\rangle$; $\Phi_6\to|12\rangle$; $\Phi_7\to|20\rangle$; $\Phi_8\to|21\rangle$; $\Phi_9\to|22\rangle$)::

$$\hat{H}_3\frac{1}{\sqrt{3}}\big(\alpha\,|000\rangle+\alpha\,|011\rangle+\alpha\,|022\rangle+\beta\,|120\rangle+\beta\,|101\rangle+\beta\,|112\rangle+\gamma\,|210\rangle+\gamma\,|221\rangle+\gamma\,|202\rangle\,\big)=$$
$$=\frac{1}{3}\Big(\,|0\rangle\langle 0|+|0\rangle\langle 1|+|0\rangle\langle 2|+|1\rangle\langle 0|+\omega\,|1\rangle\langle 1|+\omega^2\,|1\rangle\langle 2|+|2\rangle\langle 0|+\omega^2\,|2\rangle\langle 1|+\omega^4\,|2\rangle\langle 2|\,\Big)$$
$$\big(\alpha\,|000\rangle+\alpha\,|011\rangle+\alpha\,|022\rangle+\beta\,|120\rangle+\beta\,|101\rangle+\beta\,|112\rangle+\gamma\,|210\rangle+\gamma\,|221\rangle+\gamma\,|202\rangle\,\big)$$
$$=\frac{1}{3}\big(\alpha\,|000\rangle+\alpha\,|011\rangle+\alpha\,|022\rangle+\alpha\,|100\rangle+\alpha\,|111\rangle+\alpha\,|122\rangle+\alpha\,|200\rangle+\alpha\,|211\rangle+\alpha\,|222\rangle$$
$$+\,\beta\,|020\rangle+\beta\,|001\rangle+\beta\,|012\rangle+\omega\beta\,|120\rangle+\omega\beta\,|101\rangle+\omega\beta\,|112\rangle+\omega^2\beta\,|220\rangle+\omega^2\beta\,|201\rangle+\omega^2\beta\,|212\rangle$$
$$+\,\gamma\,|010\rangle+\gamma\,|021\rangle+\gamma\,|002\rangle+\omega^2\gamma\,|110\rangle+\omega^2\gamma\,|121\rangle+\omega^2\gamma\,|102\rangle+\omega^4\gamma\,|210\rangle+\omega^4\gamma\,|221\rangle+\omega^4\gamma\,|202\rangle\,\big);$$

6. Since Alice will be measuring q_0 and q_1, we make groups:

$$|00\rangle\otimes(\alpha\,|0\rangle+\beta\,|1\rangle+\gamma\,|2\rangle)+|01\rangle\otimes(\gamma\,|0\rangle+\alpha\,|1\rangle+\beta\,|2\rangle)+|02\rangle\otimes(\beta\,|0\rangle+\gamma\,|1\rangle+\alpha\,|2\rangle)$$
$$+\,|10\rangle\otimes(\alpha\,|0\rangle+\beta\omega\,|1\rangle+\gamma\omega^2\,|2\rangle)+|11\rangle\otimes(\gamma\omega^2\,|0\rangle+\alpha\,|1\rangle+\beta\omega\,|2\rangle)+|12\rangle\otimes(\beta\omega\,|0\rangle+\gamma\omega^2\,|1\rangle+\alpha\,|2\rangle)$$
$$+\,|20\rangle\otimes(\alpha\,|0\rangle+\beta\omega^2\,|1\rangle+\gamma\omega^4\,|2\rangle)+|21\rangle\otimes(\gamma\omega^4\,|0\rangle+\alpha\,|1\rangle+\beta\omega^2\,|2\rangle)+|22\rangle\otimes(\beta\omega^2\,|0\rangle+\gamma\omega^4\,|1\rangle+\alpha\,|2\rangle)$$

or in the Bell basis:

$$\Phi_1\otimes(\alpha\,|0\rangle+\beta\,|1\rangle+\gamma\,|2\rangle)+\Phi_2\otimes(\gamma\,|0\rangle+\alpha\,|1\rangle+\beta\,|2\rangle)+\Phi_3\otimes(\beta\,|0\rangle+\gamma\,|1\rangle+\alpha\,|2\rangle)$$
$$+\Phi_4\otimes(\alpha\,|0\rangle+\beta\omega\,|1\rangle+\gamma\omega^2\,|2\rangle)+\Phi_5\otimes(\gamma\omega^2\,|0\rangle+\alpha\,|1\rangle+\beta\omega\,|2\rangle)+\Phi_6\otimes(\beta\omega\,|0\rangle+\gamma\omega^2\,|1\rangle+\alpha\,|2\rangle)$$
$$+\Phi_7\otimes(\alpha\,|0\rangle+\beta\omega^2\,|1\rangle+\gamma\omega^4\,|2\rangle+\Phi_8\otimes(\gamma\omega^4\,|0\rangle+\alpha\,|1\rangle+\beta\omega^2\,|2\rangle+\Phi_9\otimes(\beta\omega^2\,|0\rangle+\gamma\omega^4\,|1\rangle+\alpha\,|2\rangle)$$

7. Alice measures q_0 and q_1 and send results to Bob;

8. Bob performs operations on q_2 based on the results he got from Alice:
$|00\rangle$ - Do nothing;
$|01\rangle$ - Apply X gate;
$|02\rangle$ - Apply X^2 gate;
$|10\rangle$ - Apply Z^2 gate;
$|11\rangle$ - Apply Z^2X^2 gate;
$|12\rangle$ - Apply Z^2X gate
$|20\rangle$ - Apply Z gate;
$|21\rangle$ - Apply ZX^2 gate
$|22\rangle$ - Apply ZX gate

9. Bob measures $q_2 = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$ which should give is the same as q_0.

An important thing to note here is that we have applied the $CNOT_3$ gate two times on the first and second qutrit. This is done in order to return us to the original computational basis that it first started in. Every operator for qutrits needs to be applied a total of three times to be returned to the original state and basis it started, with the Hadamard gate being the only exception, which is always equal to its inverse $\hat{H}_3 = \hat{H}_3^{-1} \rightarrow \hat{H}_3^2 = \hat{I}_3$. So, in order to bring back q_1 from the Bell basis to the computational basis, we apply $CNOT_3$ two times to it, and we would need to apply $\hat{H}_3$ two times, but due to its properties, we only do it once. The ordering here is changed, first it was $\hat{H}_3 CNOT_3$ now it is $CNOT_3^2 \hat{H}_3$. For $d = 4$, we would apply it three times, $d = 5$ four times, and so on. Of course, we could try and apply the inverse of $CNOT_3$ to return to the computational basis, but the inverse of $CNOT_3(q_1, q_2)$ is not $CNOT_3(q_2, q_1)$ and even if it was, we couldn't apply it to q_0 and q_1 since that would mean interacting with q_2 which is no longer in our possession at that point. This is why we use the cyclicity of the operators and the space which contains the states itself. In fact, the inverse of $CNOT_3(q_1, q_2)$ is $CNOT_3(q_1, q_2)^2$, but (since the no-cloning theorem is in effect, meaning that whatever q_0, q_2, and q_1 are, they must either be parallel or orthogonal to each other) we are allowed to instead use $CNOT_3(q_0, q_1)^2$. Or in other words, regardless of the initial states, once we've applied the $CNOT_3$ for the third time, it should bring us back in the initial states. The key that allows this is the Hadamard gate, which through this kind of manipulation, is able to map the superposition of q_0 onto q_2.

4.3.4 Teleportation protocol

On the figure below, you can see the algorithm that is used to simulate the quantum teleportation with qutrits. It is done in Mathematica, and the end result is given as a 27×1 vector, where the states are numbered from $1 - 27$ in the following order:

$\|000\rangle = 1$	$\|100\rangle = 10$	$\|200\rangle = 19$
$\|001\rangle = 2$	$\|101\rangle = 11$	$\|201\rangle = 20$
$\|002\rangle = 3$	$\|102\rangle = 12$	$\|202\rangle = 21$
$\|010\rangle = 4$	$\|110\rangle = 13$	$\|210\rangle = 22$
$\|011\rangle = 5$	$\|111\rangle = 14$	$\|211\rangle = 23$
$\|012\rangle = 6$	$\|112\rangle = 15$	$\|212\rangle = 24$
$\|020\rangle = 7$	$\|120\rangle = 16$	$\|220\rangle = 25$
$\|021\rangle = 8$	$\|121\rangle = 17$	$\|221\rangle = 26$
$\|022\rangle = 9$	$\|122\rangle = 18$	$\|222\rangle = 27$

We run the protocol for three cases $\alpha = 1$, $\beta = 1$ and $\gamma = 1$.

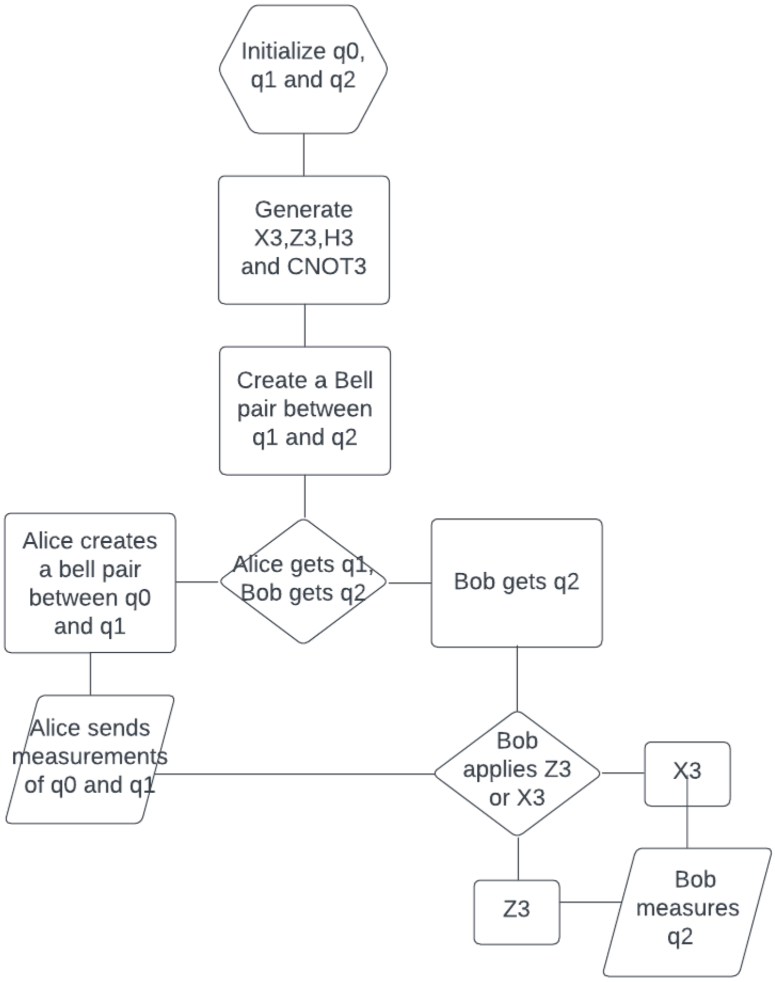

Fig. 10: Block diagram for qutrit teleportation

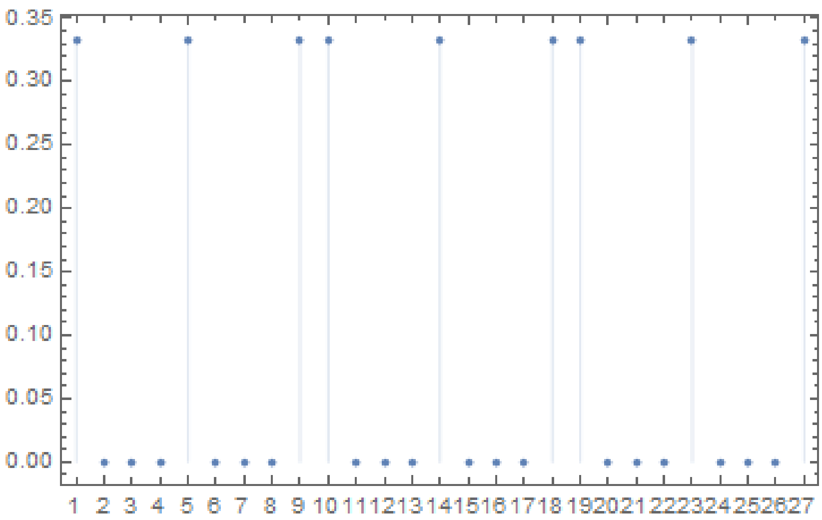

Fig. 11: Results from Mathematica simulation for $\alpha = 1$

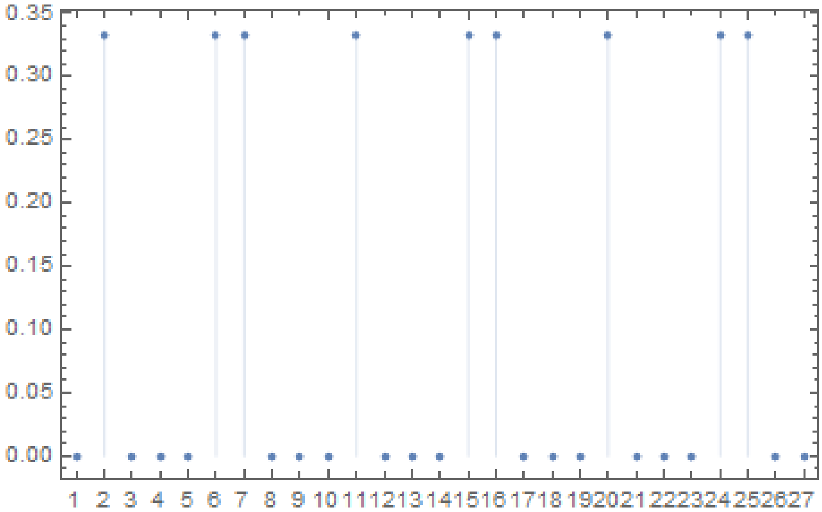

Fig. 12: Results from Mathematica simulation for $\beta = 1$

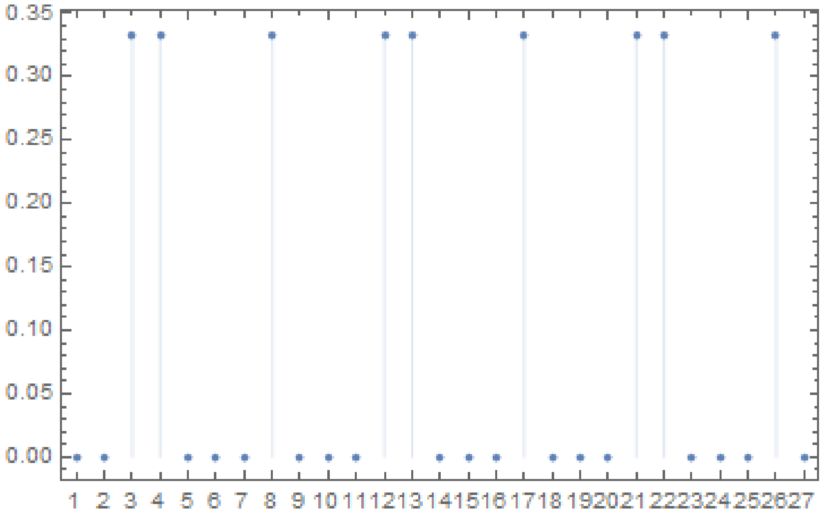

Fig. 13: Results from Mathematica simulation for $\gamma = 1$

4.4 Models for quantum encryption key

Here we present a classical text cipher, and adapt it in such a way, that we can teleport that information through. If at any point someone intercepts the quantum channel, they would have a chance to guess the right state.

4.4.1 Caesar's cipher key

When applying a Caesar cipher, each letter of the text is replaced by one that is shifted, to the left(or the right) by a constant integer. Today for the text we use on computers, we use the so called ASCII code, where each letter is represented by a byte, meaning it has eight bits, or in short $2^8 - 1$ possible combinations, giving us 255 characters. The letters "A-Z" are numbered from "35-90", and the letters "a-z" are "97-122".Caesar had a very simple idea with this, he would write a message, then shift the entire text by a certain integer and he would send the encrypted message, along with the integer(key), and nobody except him and the people who knew how to use the key. But with modern day computers, such encryption is easily broken.

4.4.2 Quantum Caesar's cipher key

The idea we have is, we could teleport the key for the encryption through a quantum channel. Any information that is being teleported using current day quantum computers is bound to have errors, meaning whatever message is sent, it will need to be repeated multiple times, in order to guarantee a higher chance for success(for qubits it works fine, but qutrits will have more error). A key works great in this case, it can be small, repeatable, and anyone who doesn't know how to use it is useless, even if they manage to steal it. However, the beautiful thing about using quantum entanglement as the source of encryption gives us another advantage, it creates a unique key, that upon being used once is instantly destroyed, and can never decode the information again. With all of this in mind, this is how the encryption protocol would be:

1. Get some text, and convert it to ASCII;
2. For each letter of the text, increase the value of the ASCII code by a random number from $[0, d-1]$;
3. Transform key from ASCII to binary;
4. Prepare quantum states that are related to the integers from step 3;
5. Apply quantum teleportation protocol on the states;
6. Send text and measurement result;
7. Apply gates and attain cipher key;
8. Use key on text, you should get the same text as the one that was originally encrypted.

4.4.3 Simulated model of teleporting QCC key

On the figure below you can see the algorithm that will be used for the Quantum Caesar cipher protocol. Results shown are for the teleportation of "brane", using Qiskit's platform. The text translated in binary is represented by 21 zeroes and 19 ones, which is the exact amount you see being detected for the arbitrary qubit.

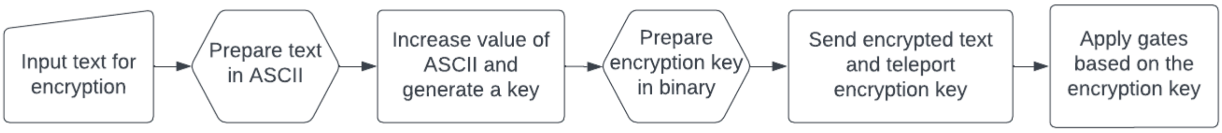

Fig. 14: Block diagram for quantum encryption protocol

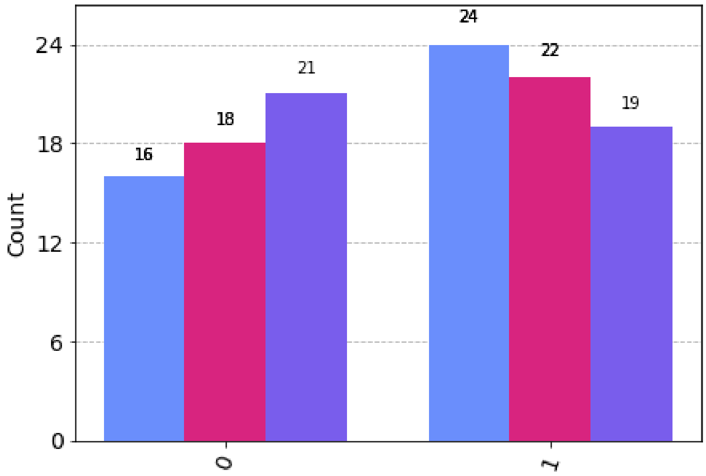

Fig. 15: Results from teleporting "brane" on a simulator

Conclusion

In essence we have developed a method for iterative construction of operators and states in d-dimensional Hilbert space. Derivations are shown using both Dirac notation, and matrix mechanics, which help perform the operations on a computer. Visual representations were also developed in order to more easily compute the operations done on the qudits. Quantum entanglement was shown as a fundamental property of physics through statistical correlation between measurements. We then use this property to develop and execute quantum teleportation protocols on the Qiskit platform, as well as a local quantum simulator. Results are shown for qubits and qutrits and there does not seem to be any indication that the method would be any different for higher dimensional qudits. At the end, a simple quantum encryption model is discussed, highlighting the various applications of this generalised theory.

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