Detailed derivation of the Grad–Shafranov equation for an axisymmetric tokamak

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Abstract

In this document I derive, step by step, the Grad–Shafranov equation for the magnetohydrodynamic (MHD) equilibrium of a plasma confined in an axisymmetric tokamak. I start from Maxwell's equations in the static regime, from the MHD equilibrium condition $\mathbf{J} \times \mathbf{B} = \nabla p$, and from a convenient decomposition of the magnetic field in terms of the poloidal flux function $\psi(r, z)$ and the toroidal field function $F(\psi)$. The final result is the elliptic equation

$$\Delta^* \psi = -\mu_0 r^2 \frac{dp}{d\psi} - \frac{1}{2} \frac{dF^2}{d\psi},$$

known as the Grad–Shafranov equation, which determines the axisymmetric magnetic equilibrium of the plasma.

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1 Introduction

The equilibrium of a magnetically confined plasma in a tokamak is described, in a first MHD approximation and in the static regime, by the force balance condition

$$\mathbf{J} \times \mathbf{B} = \nabla p, \tag{1}$$

where $\mathbf{B}$ is the total magnetic field, $\mathbf{J}$ the electric current density, and p the plasma pressure. This equation must be satisfied simultaneously with Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{3}$$

In a tokamak with toroidal symmetry it is natural to work in cylindrical coordinates (r, φ, z) and exploit the invariance in the toroidal angle φ. In this context it is possible to introduce a poloidal flux function $\psi(r, z)$ that parametrizes the magnetic surfaces and allows one to write $\mathbf{B}$ and $\mathbf{J}$ in a compact form. The Grad–Shafranov equation is the resulting elliptic equation for $\psi(r, z)$ when (1), (2), and (3) are combined with toroidal symmetry.

In what follows, I derive this equation in detail, explaining the meaning of each step and the physical interpretation of the quantities introduced.

2 Geometry, coordinates, and notation

2.1 Cylindrical coordinates and orthonormal basis

I will use cylindrical coordinates (r, φ, z), where:

- r is the distance to the symmetry axis (toroidal axis),
- φ is the toroidal angle around the axis,

- z is the vertical coordinate.

The associated orthonormal basis is

$$\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\varphi, \hat{\mathbf{e}}_z\}.$$

In these coordinates, the gradient of a scalar function f is

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\,\hat{\mathbf{e}}_\varphi + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_z. \tag{4}$$

Assuming *toroidal symmetry*,

$$\frac{\partial(\cdot)}{\partial \varphi} = 0, \tag{5}$$

the gradient reduces to

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{e}}_r + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_z. \tag{6}$$

The gradient of the angular coordinate φ is

$$\nabla \varphi = \frac{1}{r}\,\hat{\mathbf{e}}_\varphi. \tag{7}$$

2.2 Divergence and curl in cylindrical coordinates

Let $\mathbf{A} = A_r\hat{\mathbf{e}}_r + A_\varphi\hat{\mathbf{e}}_\varphi + A_z\hat{\mathbf{e}}_z$. Its divergence and curl in cylindrical coordinates with toroidal symmetry take the form

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}\,(rA_r) + \frac{\partial A_z}{\partial z}, \tag{8}$$

$$(\nabla \times \mathbf{A})_r = -\frac{\partial A_\varphi}{\partial z}, \tag{9}$$

$$(\nabla \times \mathbf{A})_\varphi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \tag{10}$$

$$(\nabla \times \mathbf{A})_z = \frac{1}{r}\frac{\partial}{\partial r}. \tag{11}$$

These expressions are obtained from the general formulas in cylindrical coordinates by imposing $\partial/\partial\varphi = 0$.

3 Poloidal flux function and poloidal field

3.1 Geometric definition of $\psi(r, z)$

The poloidal flux function $\psi(r, z)$ is defined as the magnetic flux of the total field $\mathbf{B}$ through the disk $S(r)$ in the poloidal plane (r, z) whose boundary is the toroidal circle $\Gamma(r)$:

$$\psi(r, z) = \frac{1}{2\pi}\int_{S(r)} \mathbf{B} \cdot d\mathbf{S}. \tag{12}$$

The factor $1/(2\pi)$ is a convention that makes ψ have dimensions of flux per unit toroidal angle.

Geometrically, each surface $\psi = \text{const}$ corresponds to a magnetic surface (a torus) formed by rotating a closed curve in the (r, z) plane around the tokamak symmetry axis.

3.2 Relation between ψ and the poloidal components of B

The poloidal magnetic field is the projection of $\mathbf{B}$ onto the (r, z) plane:

$$\mathbf{B}_p = B_r \hat{\mathbf{e}}_r + B_z \hat{\mathbf{e}}_z. \tag{13}$$

From the definition (12) and applying Stokes' theorem to the curve $\Gamma(r)$, one obtains that the poloidal components of $\mathbf{B}$ are related to ψ by

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad B_z = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{14}$$

These relations automatically guarantee that $\nabla \cdot \mathbf{B} = 0$ in the poloidal subspace, since one can check that

$$\nabla \cdot \mathbf{B}_p = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial B_z}{\partial z} = 0, \tag{15}$$

after substituting (14).

3.3 Compact form: $\mathbf{B}_p = \nabla\psi \times \nabla\varphi$

Using (6) and (7), we write

$$\nabla\psi = \frac{\partial \psi}{\partial r}\,\hat{\mathbf{e}}_r + \frac{\partial \psi}{\partial z}\,\hat{\mathbf{e}}_z, \tag{16}$$

$$\nabla\varphi = \frac{1}{r}\,\hat{\mathbf{e}}_\varphi. \tag{17}$$

The cross product is

$$\nabla\psi \times \nabla\varphi = \left(\frac{\partial \psi}{\partial r}\,\hat{\mathbf{e}}_r + \frac{\partial \psi}{\partial z}\,\hat{\mathbf{e}}_z\right) \times \left(\frac{1}{r}\hat{\mathbf{e}}_\varphi\right) \tag{18}$$

$$= \frac{1}{r}\left[\frac{\partial \psi}{\partial r}\,(\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\varphi) + \frac{\partial \psi}{\partial z}\,(\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\varphi)\right] \tag{19}$$

$$= \frac{1}{r}\left[\frac{\partial \psi}{\partial r}\,\hat{\mathbf{e}}_z - \frac{\partial \psi}{\partial z}\,\hat{\mathbf{e}}_r\right]. \tag{20}$$

Comparing with (14), we see that

$$\mathbf{B}_p = B_r \hat{\mathbf{e}}_r + B_z \hat{\mathbf{e}}_z = \nabla\psi \times \nabla\varphi. \tag{21}$$

This compact expression will be very useful later.

4 Full decomposition of the magnetic field

4.1 Poloidal part and toroidal part

The total magnetic field is decomposed as

$$\mathbf{B} = \mathbf{B}_p + \mathbf{B}_\varphi. \tag{22}$$

The poloidal part has already been written in (21). The toroidal part is written as

$$\mathbf{B}_\varphi = B_\varphi \,\hat{\mathbf{e}}_\varphi. \tag{23}$$

We introduce the *toroidal field function* $F(\psi)$ by defining

$$\mathbf{B}_\varphi = F(\psi)\,\nabla\varphi \quad \Rightarrow \quad B_\varphi = \frac{F(\psi)}{r}. \tag{24}$$

The quantity $F(\psi)$ is related to the poloidal current enclosed by the magnetic surfaces; in many texts one writes $F(\psi) = RB_\varphi$ (where R is the major radius) and interprets it as the contribution of the external toroidal field and of the plasma poloidal current.

4.2 Final form of B

With these definitions, the full decomposition is

$$\boxed{\mathbf{B} = \nabla\psi \times \nabla\varphi + F(\psi)\,\nabla\varphi.} \tag{25}$$

This form clearly separates:

- the *poloidal geometry* of the field (contained in ψ),
- the *toroidal structure* (contained in $F(\psi)$).

5 Elliptic operator Δ^* and toroidal component of the current

5.1 Definition of the operator Δ^*

We define the operator

$$\Delta^*\psi := r^2\nabla \cdot \left(\frac{1}{r^2}\nabla\psi\right). \tag{26}$$

This operator is analogous to the 2D Laplacian, but adapted to the toroidal geometry. Using (6) and (8), we have

$$\nabla \cdot \left(\frac{1}{r^2}\nabla\psi\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial z}\right) \tag{27}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial z^2}. \tag{28}$$

Therefore,

$$\Delta^*\psi = r^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial z^2}\right] \tag{29}$$

$$= r\,\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2}. \tag{30}$$

The first term expands as

$$r\,\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = r\left[-\frac{1}{r^2}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2}\right] = -\frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}, \tag{31}$$

and thus

$$\boxed{\Delta^*\psi = \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}.} \tag{32}$$

This operator is the one that appears in the Grad–Shafranov equation.

5.2 Relation $\Delta^*\psi = -\mu_0 r J_\varphi$

From Ampère's law,

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}. \tag{33}$$

We take the toroidal component of the curl:

$$(\nabla\times\mathbf{B})_\varphi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}. \tag{34}$$

Substituting the values of B_r and B_z from (14),

$$\frac{\partial B_r}{\partial z} = \frac{\partial}{\partial z}\left(-\frac{1}{r}\frac{\partial\psi}{\partial z}\right) = -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}, \tag{35}$$

$$\frac{\partial B_z}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial\psi}{\partial r}. \tag{36}$$

Thus,

$$(\nabla\times\mathbf{B})_\varphi = -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \left[\frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial\psi}{\partial r}\right] \tag{37}$$

$$= -\frac{1}{r}\left[\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right] \tag{38}$$

$$= -\frac{1}{r}\,\Delta^*\psi. \tag{39}$$

From Ampère's law,

$$(\nabla\times\mathbf{B})_\varphi = \mu_0 J_\varphi, \tag{40}$$

so that

$$\boxed{J_\varphi = -\frac{1}{\mu_0 r}\,\Delta^*\psi \qquad \Longleftrightarrow \qquad \Delta^*\psi = -\mu_0 r J_\varphi.} \tag{41}$$

This relation connects the operator Δ^* with the toroidal current.

6 Poloidal current and relation with $F(\psi)$

6.1 Direct computation of J from B

Using (25), we write

$$\mathbf{B} = \mathbf{B}_p + \mathbf{B}_\varphi = \nabla\psi \times \nabla\varphi + F(\psi)\,\nabla\varphi. \tag{42}$$

The current is

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B}. \tag{43}$$

We can compute the poloidal components using the formulas (11). In particular, the toroidal component is already known from (41). We are now interested in the poloidal part $\mathbf{J}_p = (J_r, J_z)$.

From (11), the relevant components are

$$\mu_0 J_r = -\frac{\partial B_\varphi}{\partial z}, \tag{44}$$

$$\mu_0 J_z = \frac{1}{r}\frac{\partial}{\partial r}. \tag{45}$$

Since $B_\varphi = F(\psi)/r$, we get

$$\frac{\partial B_\varphi}{\partial z} = \frac{\partial}{\partial z}\left(\frac{F(\psi)}{r}\right) = \frac{1}{r}F'(\psi)\frac{\partial \psi}{\partial z}, \tag{46}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial r}\big(F(\psi)\big) = F'(\psi)\frac{\partial \psi}{\partial r}, \tag{47}$$

which leads to

$$J_r = -\frac{1}{\mu_0 r}F'(\psi)\frac{\partial \psi}{\partial z}, \qquad J_z = \frac{1}{\mu_0 r}F'(\psi)\frac{\partial \psi}{\partial r}. \tag{48}$$

6.2 Proportionality $\mathbf{J}_p \propto \mathbf{B}_p$

Recall that

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad B_z = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{49}$$

Therefore,

$$\mathbf{B}_p = B_r\hat{\mathbf{e}}_r + B_z\hat{\mathbf{e}}_z = \frac{1}{r}\left(\frac{\partial \psi}{\partial r}\hat{\mathbf{e}}_z - \frac{\partial \psi}{\partial z}\hat{\mathbf{e}}_r\right). \tag{50}$$

Comparing with (48), we can write

$$\mathbf{J}_p = J_r\hat{\mathbf{e}}_r + J_z\hat{\mathbf{e}}_z = \frac{F'(\psi)}{\mu_0}\,\mathbf{B}_p. \tag{51}$$

That is, the poloidal component of the current is parallel to the poloidal component of the magnetic field, with a proportionality factor $F'(\psi)/\mu_0$.

7 MHD equilibrium: $p = p(\psi)$ and $F = F(\psi)$

7.1 Proof that $p = p(\psi)$

The MHD equilibrium equation is

$$\mathbf{J} \times \mathbf{B} = \nabla p. \tag{52}$$

We take the scalar product with $\mathbf{B}$:

$$\mathbf{B} \cdot (\mathbf{J} \times \mathbf{B}) = \mathbf{B} \cdot \nabla p. \tag{53}$$

But $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = 0$ for any pair of vectors $\mathbf{a}, \mathbf{b}$, so

$$\mathbf{B} \cdot (\mathbf{J} \times \mathbf{B}) = 0, \tag{54}$$

and therefore

$$\mathbf{B} \cdot \nabla p = 0. \tag{55}$$

This means that the pressure gradient is perpendicular to the magnetic field: p is constant along field lines.

On the other hand, from (25) we have

$$\begin{aligned} \mathbf{B} \cdot \nabla\psi &= \big(\nabla\psi \times \nabla\varphi\big) \cdot \nabla\psi + F(\psi)\, \nabla\varphi \cdot \nabla\psi \qquad (56) \\ &= 0 + 0 = 0, \qquad (57) \end{aligned}$$

because $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $\nabla\varphi \cdot \nabla\psi = 0$ (no dependence on φ).

Consequently, field lines remain on surfaces $\psi =$ const. Since p is constant along field lines and these lines live on level sets of ψ, we conclude that the constant-pressure surfaces coincide with the magnetic surfaces:

$$\boxed{p = p(\psi).} \tag{58}$$

7.2 Proof that $F = F(\psi)$

Let us consider again the equilibrium equation (52). We take the toroidal component (component in the direction $\hat{\mathbf{e}}_\varphi$):

$$(\mathbf{J} \times \mathbf{B})_\varphi = (\nabla p)_\varphi. \tag{59}$$

Since $p = p(r, z)$ and does not depend on φ, we have

$$(\nabla p)_\varphi = \frac{1}{r}\frac{\partial p}{\partial \varphi} = 0. \tag{60}$$

Therefore,

$$(\mathbf{J} \times \mathbf{B})_\varphi = 0. \tag{61}$$

Writing $\mathbf{J}_p = (J_r, J_z)$ and $\mathbf{B}_p = (B_r, B_z)$, this component is

$$(\mathbf{J} \times \mathbf{B})_\varphi = J_r B_z - J_z B_r = 0, \tag{62}$$

which implies that

$$\mathbf{J}_p \parallel \mathbf{B}_p. \tag{63}$$

We have already seen in (51) that

$$\mathbf{J}_p = \frac{F'(\psi)}{\mu_0}\,\mathbf{B}_p, \tag{64}$$

which is consistent with this condition.

On the other hand, from the compact form (25) and the general expression (65) that can be obtained for $\mathbf{J}$:

$$\mathbf{J} = -\frac{1}{\mu_0}\Delta^*\psi\,\nabla\varphi + \frac{1}{\mu_0}\nabla F \times \nabla\varphi, \tag{65}$$

the poloidal part of the current is

$$\mathbf{J}_p = \frac{1}{\mu_0}\nabla F \times \nabla\varphi. \tag{66}$$

Similarly,

$$\mathbf{B}_p = \nabla\psi \times \nabla\varphi. \tag{67}$$

The condition $\mathbf{J}_p \parallel \mathbf{B}_p$ can be written as

$$\nabla F \times \nabla\varphi = \alpha(r,z)\,\nabla\psi \times \nabla\varphi \tag{68}$$

for some scalar $\alpha(r,z)$. We now take the cross product with $\nabla\varphi$ on both sides:

$$(\nabla F \times \nabla\varphi) \times \nabla\varphi = \alpha(r,z)\,(\nabla\psi \times \nabla\varphi) \times \nabla\varphi. \tag{69}$$

Using the vector identity

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{b} = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{b} - |\mathbf{b}|^2\,\mathbf{a}, \tag{70}$$

and the fact that $\nabla F \cdot \nabla\varphi = 0$ and $\nabla\psi \cdot \nabla\varphi = 0$ (toroidal symmetry), we obtain

$$(\nabla F \times \nabla\varphi) \times \nabla\varphi = -|\nabla\varphi|^2\,\nabla F, \tag{71}$$

$$(\nabla\psi \times \nabla\varphi) \times \nabla\varphi = -|\nabla\varphi|^2\,\nabla\psi. \tag{72}$$

Therefore,

$$-|\nabla\varphi|^2\,\nabla F = -\alpha(r,z)\,|\nabla\varphi|^2\,\nabla\psi, \tag{73}$$

and, since $|\nabla\varphi|^2 \neq 0$, we conclude that

$$\nabla F = \alpha(r,z)\,\nabla\psi. \tag{74}$$

That is, ∇F is parallel to $\nabla\psi$, which implies that F is a function of ψ only:

$$\boxed{F = F(\psi).} \tag{75}$$

8 Explicit computation of $\mathbf{J} \times \mathbf{B}$ and the Grad–Shafranov equation

8.1 Components of B and J

We collect the expressions obtained:

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad B_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad B_\varphi = \frac{F(\psi)}{r}, \tag{76}$$

$$J_r = -\frac{1}{\mu_0 r} F'(\psi)\frac{\partial \psi}{\partial z}, \qquad J_z = \frac{1}{\mu_0 r} F'(\psi)\frac{\partial \psi}{\partial r}, \qquad J_\varphi = -\frac{1}{\mu_0 r}\Delta^* \psi. \tag{77}$$

Thus,

$$\mathbf{B} = B_r \hat{\mathbf{e}}_r + B_\varphi \hat{\mathbf{e}}_\varphi + B_z \hat{\mathbf{e}}_z, \tag{78}$$

$$\mathbf{J} = J_r \hat{\mathbf{e}}_r + J_\varphi \hat{\mathbf{e}}_\varphi + J_z \hat{\mathbf{e}}_z. \tag{79}$$

8.2 Cross product $\mathbf{J} \times \mathbf{B}$ in components

In the cylindrical orthonormal basis, the cross product of two vectors **J** and **B** is

$$\mathbf{J} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\varphi & \hat{\mathbf{e}}_z \\ J_r & J_\varphi & J_z \\ B_r & B_\varphi & B_z \end{vmatrix}, \tag{80}$$

so

$$(\mathbf{J} \times \mathbf{B})_r = J_\varphi B_z - J_z B_\varphi, \tag{81}$$

$$(\mathbf{J} \times \mathbf{B})_\varphi = J_z B_r - J_r B_z, \tag{82}$$

$$(\mathbf{J} \times \mathbf{B})_z = J_r B_\varphi - J_\varphi B_r. \tag{83}$$

Let us compute explicitly $(\mathbf{J} \times \mathbf{B})_r$ and $(\mathbf{J} \times \mathbf{B})_z$.

Radial component.

$$\begin{aligned}(\mathbf{J} \times \mathbf{B})_r &= J_\varphi B_z - J_z B_\varphi \qquad (84)\\ &= \left(-\frac{1}{\mu_0 r}\Delta^* \psi\right)\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) - \left(\frac{1}{\mu_0 r} F'(\psi)\frac{\partial \psi}{\partial r}\right)\left(\frac{F(\psi)}{r}\right) \qquad (85)\\ &= -\frac{1}{\mu_0 r^2}\frac{\partial \psi}{\partial r}\,\Delta^* \psi - \frac{1}{\mu_0 r^2}\frac{\partial \psi}{\partial r}\,F(\psi)F'(\psi). \qquad (86)\end{aligned}$$

Factoring out,

$$(\mathbf{J} \times \mathbf{B})_r = \frac{1}{\mu_0 r^2}\left[-\Delta^* \psi - F(\psi)F'(\psi)\right]\frac{\partial \psi}{\partial r}. \tag{87}$$

Vertical component.

$$
\begin{aligned}
(\mathbf{J}\times\mathbf{B})_z &= J_r B_\varphi - J_\varphi B_r && (88)\\
&= \left(-\frac{1}{\mu_0 r}F'(\psi)\frac{\partial\psi}{\partial z}\right)\left(\frac{F(\psi)}{r}\right) - \left(-\frac{1}{\mu_0 r}\Delta^*\psi\right)\left(-\frac{1}{r}\frac{\partial\psi}{\partial z}\right) && (89)\\
&= -\frac{1}{\mu_0 r^2}\frac{\partial\psi}{\partial z}\,F(\psi)F'(\psi) - \frac{1}{\mu_0 r^2}\frac{\partial\psi}{\partial z}\,\Delta^*\psi. && (90)
\end{aligned}
$$

Factoring out,

$$
(\mathbf{J}\times\mathbf{B})_z = \frac{1}{\mu_0 r^2}\left[-\Delta^*\psi - F(\psi)F'(\psi)\right]\frac{\partial\psi}{\partial z}. \tag{91}
$$

From (87) and (91) we deduce that

$$
\mathbf{J}\times\mathbf{B} = (\mathbf{J}\times\mathbf{B})_r\,\hat{\mathbf{e}}_r + (\mathbf{J}\times\mathbf{B})_z\,\hat{\mathbf{e}}_z = -\frac{\Delta^*\psi + F(\psi)F'(\psi)}{\mu_0 r^2}\,\nabla\psi. \tag{92}
$$

That is,

$$
\boxed{\mathbf{J}\times\mathbf{B} = -\frac{\Delta^*\psi + F(\psi)F'(\psi)}{\mu_0 r^2}\,\nabla\psi.} \tag{93}
$$

8.3 Comparison with $\nabla p(\psi)$ and final result

From the MHD equilibrium (52) and the fact that $p = p(\psi)$, we have

$$
\mathbf{J}\times\mathbf{B} = \nabla p(\psi) = \frac{\mathrm{d}p}{\mathrm{d}\psi}\,\nabla\psi. \tag{94}
$$

Comparing with (92),

$$
-\frac{\Delta^*\psi + F(\psi)F'(\psi)}{\mu_0 r^2}\,\nabla\psi = \frac{\mathrm{d}p}{\mathrm{d}\psi}\,\nabla\psi. \tag{95}
$$

In the region where $\nabla\psi \neq 0$ we can cancel $\nabla\psi$, obtaining

$$
-\frac{\Delta^*\psi + F(\psi)F'(\psi)}{\mu_0 r^2} = \frac{\mathrm{d}p}{\mathrm{d}\psi}. \tag{96}
$$

Multiplying by $-\mu_0 r^2$,

$$
\Delta^*\psi + F(\psi)F'(\psi) = -\mu_0 r^2\,\frac{\mathrm{d}p}{\mathrm{d}\psi}. \tag{97}
$$

Finally,

$$
\boxed{\Delta^*\psi = -\mu_0 r^2\frac{\mathrm{d}p}{\mathrm{d}\psi} - F(\psi)F'(\psi).} \tag{98}
$$

Using that

$$
F(\psi)F'(\psi) = \frac{1}{2}\frac{\mathrm{d}F^2(\psi)}{\mathrm{d}\psi}, \tag{99}
$$

we can write the equation in a more standard way as

$$
\boxed{\Delta^*\psi = -\mu_0 r^2\frac{\mathrm{d}p}{\mathrm{d}\psi} - \frac{1}{2}\frac{\mathrm{d}F^2(\psi)}{\mathrm{d}\psi}.} \tag{100}
$$

This is the *Grad–Shafranov equation* for the axisymmetric MHD equilibrium of a tokamak.

9 Final physical comments

Some important remarks about the Grad–Shafranov equation (100):

- The operator Δ^* on the left-hand side is an elliptic operator in the (r, z) coordinates that plays a role analogous to the 2D Laplacian, but modified by the toroidal geometry. It measures the "curvature" of the poloidal flux function ψ.

- The right-hand side contains two *free functions* of the problem, $p(\psi)$ and $F(\psi)$:

  - $p(\psi)$ describes how the plasma pressure varies between magnetic surfaces.
  - $F(\psi)$ describes the toroidal field profile and is related to the enclosed poloidal current.

  MHD theory establishes that both are functions of ψ, but does not fix their explicit forms. These must be determined from additional physical considerations (heating methods, current drive, transport, stability, etc.) or via reconstruction from experimental data.

- Once $p(\psi)$ and $F(\psi)$ are specified, the Grad–Shafranov equation, together with appropriate boundary conditions (for example, values of ψ at the vacuum vessel wall), allows one to determine the flux function $\psi(r, z)$ and thus the complete magnetic equilibrium of the plasma:

  - the geometry of the magnetic surfaces,
  - the poloidal field $\mathbf{B}_p$,
  - the toroidal field $\mathbf{B}_\varphi$,
  - the current density $\mathbf{J}$.

In this way, the Grad–Shafranov equation is the central equation for describing axisymmetric MHD equilibrium in tokamaks and other toroidal confinement devices.