Finite Element Methods: Solving the 1D Kuramoto-Sivashinsky Equation

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Abstract

This report presents the implementation of finite element methods to solve the 1D 4^{th} order non-linear Kuramoto–Sivashinsky equation describing flame-front dynamics. The project employs high-order finite elements and implicit time-stepping discretization for the numerical solver. Key results include the evolution of the flame front over time, visualized through contour and snapshot plots.

1 Introduction

The Kuramoto-Sivashinsky equation models flame-front dynamics. It exhibits chaotic behavior, making it a challenging problem for numerical solvers. This work relies on Crank-Nicholson time-marching and utilizes high-order Lagrangian finite element methods using Dolfinx and PETSc to solving the equation, and simulating the results.

2 Methods

2.1 Differential Equation

The Kuramoto-Sivashinsky equation:

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0 \tag{1}$$

Here:

- u: Flame-front amplitude
- ν: Diffusion parameter
- uu_x: Nonlinear advection term
- u_{xx}: 2^{nd} order diffusion term
- u_{xxxx}: 4^{th} order diffusion term

2.2 Weak Formulation

The weak form integrates the equation against a test function v, with stabilization to handle high-order terms:

$$\int_\Omega \left(\frac{u^{n+1} - u^n}{\Delta t} v + \theta \mathcal{L}(u^{n+1}) v + (1-\theta)\mathcal{L}(u^n) v + \mathcal{N}(u^{n+1}) v + \text{Stabilization} \right) dx = 0 \tag{2}$$

2.3 Numerical Implementation

- **Domain:** [0, 32] with 256 elements.
- **Function Space:** Continuous Galerkin (CG) with degree 4 polynomials.
- **Initial Condition:**

$$u(x,0) = 4.0 \exp\left(-0.5 \left(\frac{x - 14.55}{2} \right)^2 \right) \sin\left(\frac{2\pi x}{32} \right) \tag{3}$$

- **Dirichlet Boundary Conditions:** $u(0) = u(L) = 0$.
- **Time-Marching:** Implicit Crank-Nicholson method with $\theta = 0.5$, $\Delta t = 0.005$, and a total simulation time $T = 100$.
- **Stabilization Term:**

$$h^2 \int_\Omega \nabla u_{n+1} \cdot \nabla v \, dx,$$

$$h = \frac{1}{N}$$

  where N is the number of elements in the grid.

2.4 Initial Conditions

The initial condition stated above introduces an initial flame instability or perturbation to the system. This initial perturbation evolves under the dynamics of the equation, leading to flame-front oscillations.

2.5 Solver Configuration

- **Nonlinear Solvers:** Newton's method.
- **Linear Solver:** GMRES with ILU preconditioner.
- **Convergence Criteria:** $rtol = 10^{-2}, atol = 10^{-3}$.

3 Results

3.1 Evolution of the Flame Front

The contour plot (Figure 1) illustrates the normalized amplitude over time and space. The displacements get smaller and smaller over time. Other notable features are periodic doubling pattern of the solution.

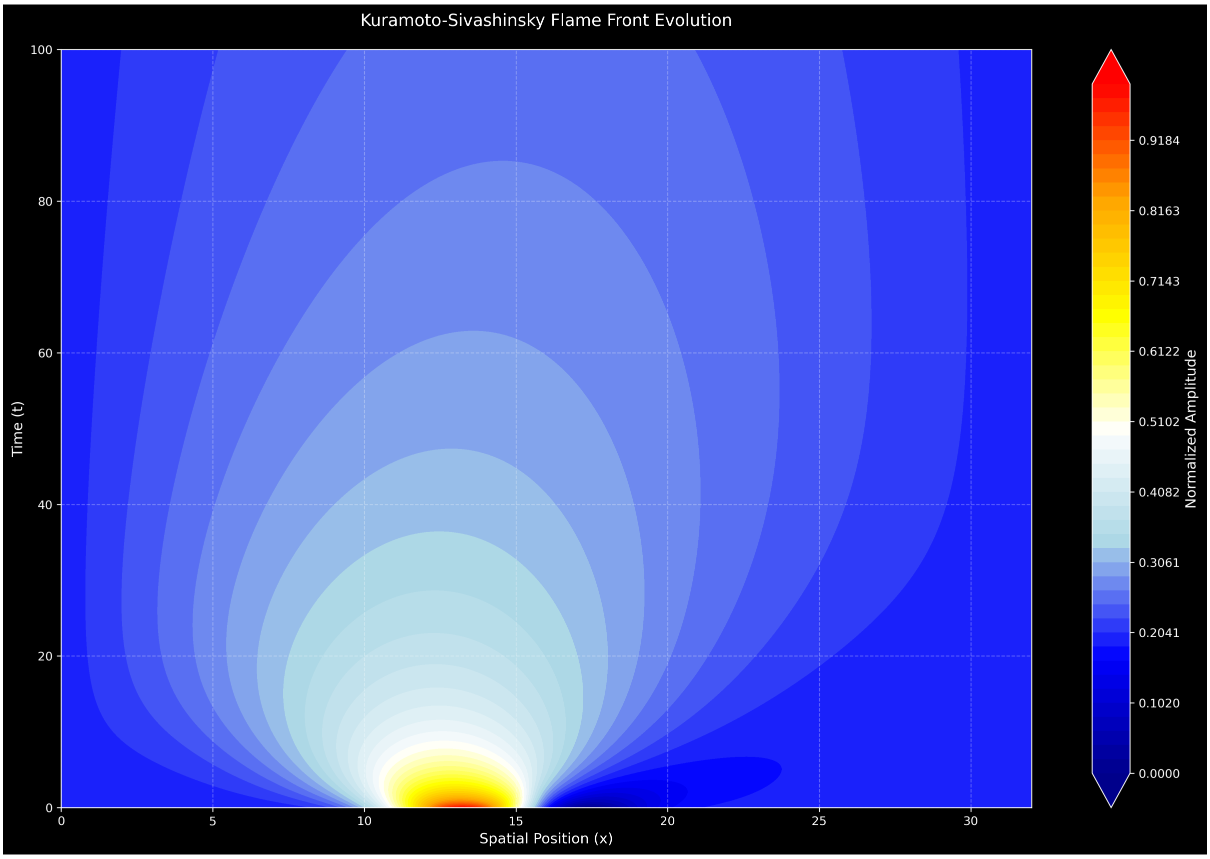

Figure 1: Kuramoto-Sivashinsky Flame Front Evolution.

3.2 Statistics

- Maximum value: 1.6673
- Minimum value: -0.3922
- Final time reached: 100.0 s

4 Verification

4.1 Snapshots of the Solution

Figure 2 shows flame-front snapshots at selected times $(0, T/3, 2T/3, T)$. Highlights the transition from initial perturbations to stabilized dynamics.

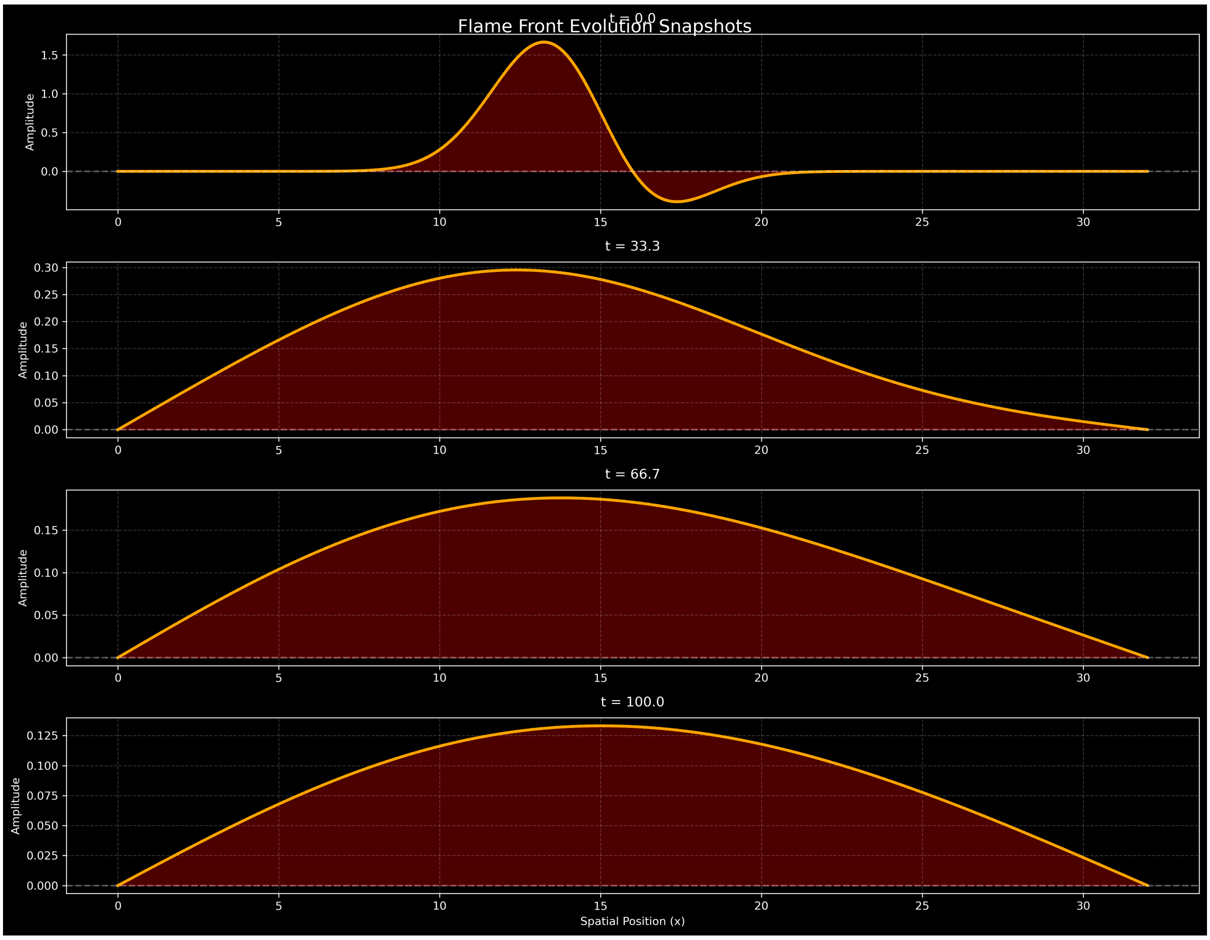

Figure 2: Kuramoto-Sivashinsky Flame Front Evolution.

The vertical displacements gradually decrease in amplitude as demonstrated by the solution snapshots plotted over time. Due to the boundary condition constraints set at $u(0) = 0$ and $u(L) = 0$, we get a nice sinusoidal plot to interpret the results. To visualize more chaos inherent in the Kuramoto-Sivashinsky equation, it's recommended to lift the boundary constraints at either $x = 0$ or $x = L$. Please check github to see more chaotic plots.

4.2 Energy Plot

The energy at time t is typically defined as:

$$E(t) = \int_{\Omega} u(x,t)^2 \, dx,$$

where Ω is the spatial domain.

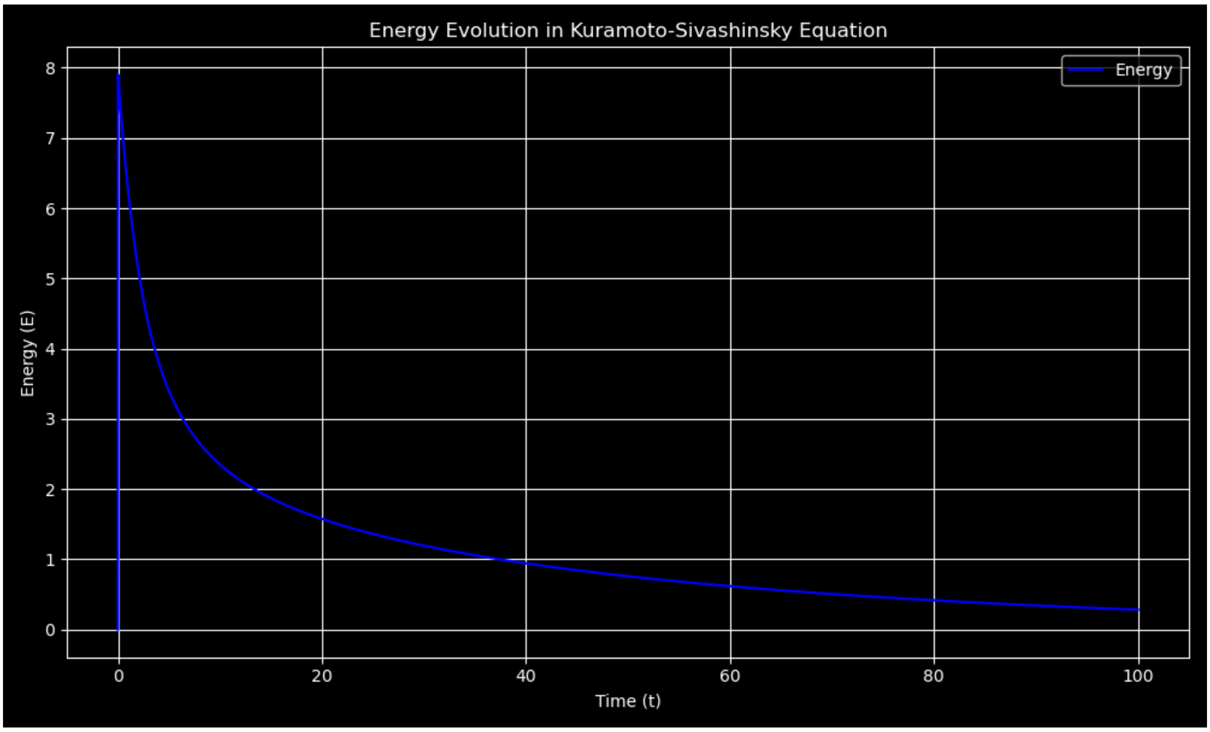

Figure 3: Kuramoto-Sivashinsky Flame Front Evolution.

5 Discussion

The energy gradually decreases to 0 as expected due to the Dirichlet boundary conditions acting as energy sinks. Fixed boundary conditions dissipate oscillations or perturbations that might otherwise sustain chaotic behavior. Eventually, the solution $u(x,t)$ becomes zero everywhere (trivial steady state). Thus, the integral $E(t) = \int_{\Omega} u(x,t)^2 \, dx$ converges to 0.

6 Conclusion

This project demonstrates the efficacy of finite element methods in solving the Kuramoto-Sivashinsky equation. The results provide insights into the chaotic and stabilizing behaviors of the system.

7 File Appendix

1. `ks.py` - File contains PDE solver
2. `ks_energy_evolution.png` - Energy Evolution Plot.
3. `ks_flame_evolution_dolfinx.png` - Flame Evolution Plot.
4. `ks_flame_snapshots_dolfinx.png` - Flame Snapshots at Selected times.
5. `ks_flame_evolution_dolfinx_u0=0.png` - Flame Evolution for Dirichlet boundary condition only at $u(0) = 0$
6. `ks_flame_evolution_dolfinx_uL=0.png` - Flame Evolution for Dirichlet boundary condition only at $u(L) = 0$

8 How to Run Code

1. Github Repo - `https://github.com/h7474/finite-elements-project`
2. Run Kuramoto-Sivashinsky solver with **python ks.py**