Statistical learning– Bayesian (Net) Learning

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1: Outline

♢ Bayes net learning
  - ML parameter learning with complete data
  - linear regression

2: Full Bayesian Learning

- View learning as Bayesian updating of a probability distribution over the hypothesis space

- H is the hypothesis variable, values $h_1, h_2, \ldots$, prior $\mathbf{P}(H)$

- jth observation d_j gives the outcome of random variable D_j
  training data $\mathbf{d} = d_1, \ldots, d_N$

- Given the data so far, each hypothesis has a posterior probability:

  $$P(h_i|\mathbf{d}) = \alpha P(\mathbf{d}|h_i)P(h_i)$$

  where $P(\mathbf{d}|h_i)$ is called the likelihood

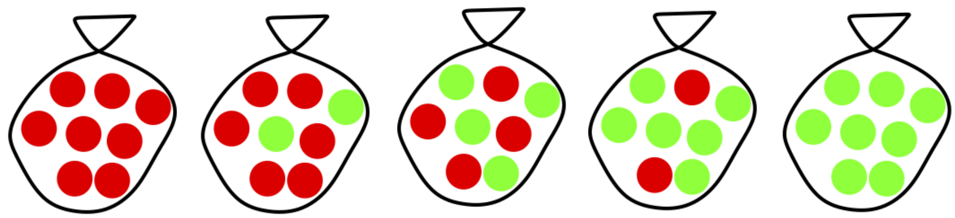
- Predictions use a *likelihood-weighted* average over the hypotheses:

  $$\mathbf{P}(X|\mathbf{d}) = \sum_i \mathbf{P}(X|\mathbf{d}, h_i)P(h_i|\mathbf{d}) = \sum_i \mathbf{P}(X|h_i)P(h_i|\mathbf{d})$$

  $\Rightarrow$ No need to pick one best-guess hypothesis!

3: Example

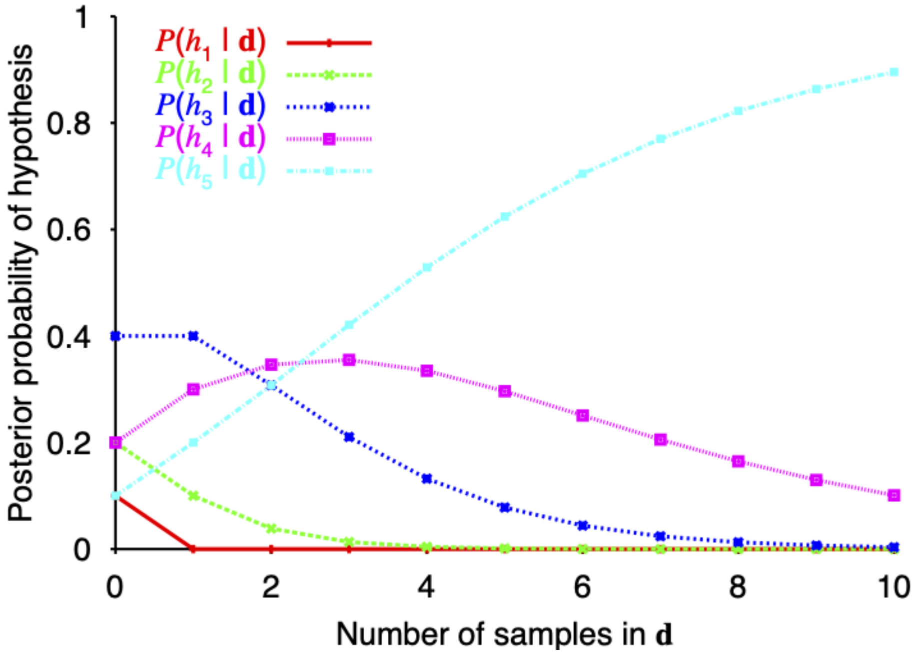
- Suppose there are five kinds of bags of candies:
  - 10% are h_1: 100% cherry candies
  - 20% are h_2: 75% cherry candies + 25% lime candies
  - 40% are h_3: 50% cherry candies + 50% lime candies
  - 20% are h_4: 25% cherry candies + 75% lime candies
  - 10% are h_5: 100% lime candies



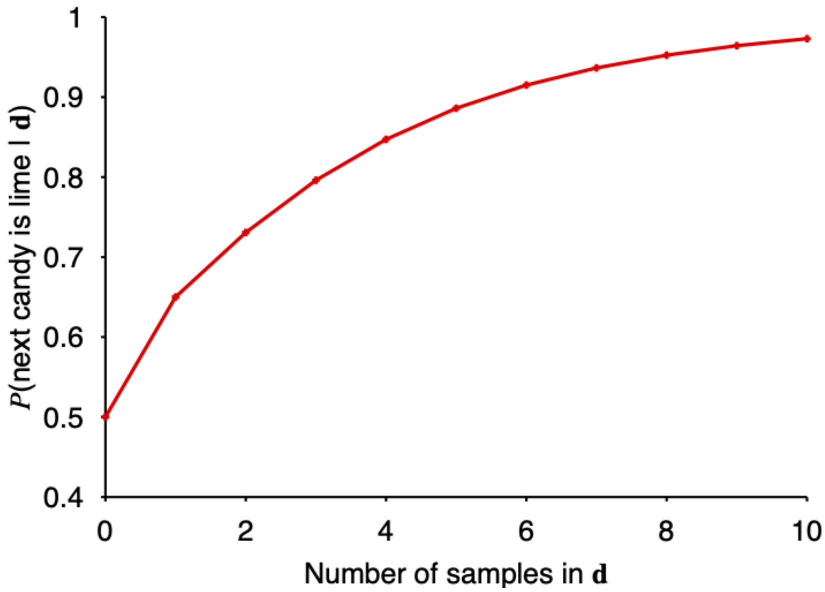
- Then we observe candies drawn from some bag:

- **Questions:**
  1. What kind of bag is it?
  2. What flavor will the next candy be?

4: Posterior prob. of hypotheses



5: Prediction probability



6: MAP approximation

- Summing over the hypothesis space is often intractable
  (e.g., 18,446,744,073,709,551,616 Boolean functions of 6 attributes)

- **Maximum a posteriori** (MAP) learning: choose h_{MAP} maximizing $P(h_i|\mathbf{d})$
  i.e., maximize $P(\mathbf{d}|h_i)P(h_i)$ or $\log P(\mathbf{d}|h_i) + \log P(h_i)$

- Log terms can be viewed as (negative of)

  bits to encode data given hypothesis + bits to encode hypothesis

- This is the basic idea of minimum description length (MDL) learning

- For deterministic hypotheses, $P(\mathbf{d}|h_i)$ is 1 if consistent, 0 otherwise

  $\Rightarrow$ MAP = simplest consistent hypothesis

7: ML approximation

- For large data sets, prior becomes irrelevant

- **Maximum likelihood** (ML) learning

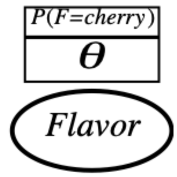
  $\Rightarrow$ Choose $h_{\rm ML}$ maximizing $P(\mathbf{d}|h_i)$

  i.e., simply get the best fit to the data; identical to MAP for uniform prior (which is reasonable if all hypotheses are of the same complexity)

- ML is the "standard" (non-Bayesian) statistical learning method

8: ML learning in Bayes nets

- Bag from a new manufacturer; what fraction θ of cherry candies?



- Any θ is possible: continuum of hypotheses h_θ
  θ is a parameter for this simple (binomial) family of models
- Suppose we unwrap N candies, c cherries and $\ell = N - c$ limes
- These are i.i.d. (independent, identically distributed) observations, so

$$P(\mathbf{d}|h_\theta) = \prod_{j=1}^{N} P(d_j|h_\theta) = \theta^c \cdot (1-\theta)^\ell$$

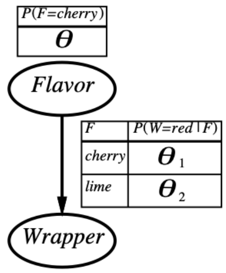
- Maximize this w.r.t. θ — which is easier for the **log-likelihood**:

$$\begin{aligned} L(\mathbf{d}|h_\theta) &= \log P(\mathbf{d}|h_\theta) = \sum_{j=1}^{N} \log P(d_j|h_\theta) = c\log\theta + \ell\log(1-\theta) \\ \frac{dL(\mathbf{d}|h_\theta)}{d\theta} &= \frac{c}{\theta} - \frac{\ell}{1-\theta} = 0 \qquad \Rightarrow \qquad \theta = \frac{c}{c+\ell} = \frac{c}{N} \end{aligned}$$

- Seems sensible, but causes problems with 0 counts!

9: Multiple parameters

- Red/green wrapper depends probabilistically on flavor:



- Likelihood for, e.g., cherry candy in green wrapper:

$$
\begin{aligned}
& P(F = cherry, W = green | h_{\theta,\theta_1,\theta_2}) \\
&= P(F = cherry | h_{\theta,\theta_1,\theta_2}) P(W = green | F = cherry, h_{\theta,\theta_1,\theta_2}) \\
&= \theta \cdot (1 - \theta_1)
\end{aligned}
$$

N candies:

$c =$ total cherries; $l =$ total limes;
r_c red-wrapped cherries; g_c green-wrapped cherries;
r_l red-wrapped limes, g_l green-wrapped limes:

$$P(\mathbf{d}|h_{\theta,\theta_1,\theta_2}) = \theta^c (1-\theta)^\ell \cdot \theta_1^{r_c} (1-\theta_1)^{g_c} \cdot \theta_2^{r_\ell} (1-\theta_2)^{g_\ell}$$

$$
\begin{aligned}
L &= [c \log \theta + \ell \log(1-\theta)] + [r_c \log \theta_1 + g_c \log(1-\theta_1)] \\
&+ [r_\ell \log \theta_2 + g_\ell \log(1-\theta_2)]
\end{aligned}
$$

10: Multiple parameters contd.

Derivatives of L contain only the relevant parameter:

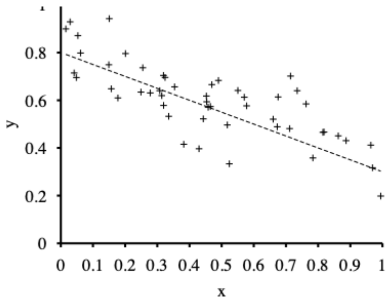
$$\frac{\partial L}{\partial \theta} = \frac{c}{\theta} - \frac{\ell}{1-\theta} = 0 \qquad \Rightarrow \quad \theta = \frac{c}{c+\ell}$$

$$\frac{\partial L}{\partial \theta_1} = \frac{r_c}{\theta_1} - \frac{g_c}{1-\theta_1} = 0 \qquad \Rightarrow \quad \theta_1 = \frac{r_c}{r_c+g_c}$$

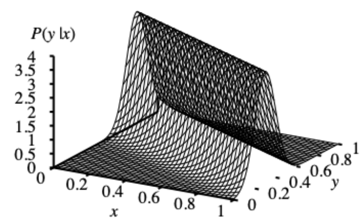
$$\frac{\partial L}{\partial \theta_2} = \frac{r_\ell}{\theta_2} - \frac{g_\ell}{1-\theta_2} = 0 \qquad \Rightarrow \quad \theta_2 = \frac{r_\ell}{r_\ell+g_\ell}$$

With complete data, **parameters can be learned separately**

11: Linear Gaussian model



Regression Sample



Regression Model

Maximizing $P(y|x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-(\theta_1 x+\theta_2))^2}{2\sigma^2}}$ w.r.t. θ_1, θ_2

over the set of training data is the same as

minimizing $E = \sum_{j=1}^{N} (y_j - (\theta_1 x_j + \theta_2))^2$

That is, **minimizing the sum of squared errors** gives the ML solution for a linear fit **assuming Gaussian noise of fixed variance**

12: Summary

- Full Bayesian learning gives best possible predictions but is intractable
- MAP learning balances complexity with accuracy on training data
- Maximum likelihood assumes uniform prior, OK for large data sets

1. Choose a parameterized family of models to describe the data
   this requires substantial insight and sometimes new models
2. Write down the likelihood of the data as a function of the parameters
   may require summing over hidden variables, i.e., inference
3. Write down the derivative of the log likelihood w.r.t. each parameter
4. Find the parameter values such that the derivatives are zero
   may be hard/impossible; modern optimization techniques help