Neural networks

Neural networks

1: Outline

◇ Brains

◇ Neural networks

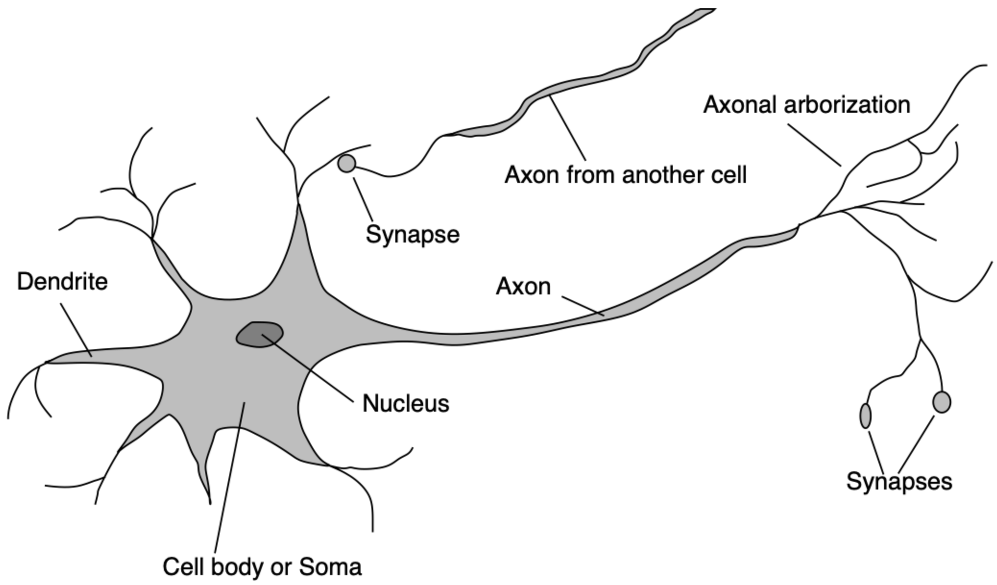
◇ Perceptrons

◇ Multilayer perceptrons

◇ Applications of neural networks

2: Brains

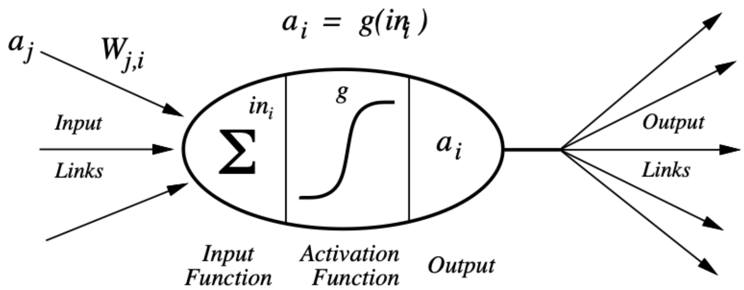
10^{11} neurons of >20 types, 10^{14} synapses, 1ms–10ms cycle time
Signals are noisy "spike trains" of electrical potential



3: McCulloch–Pitts "unit"

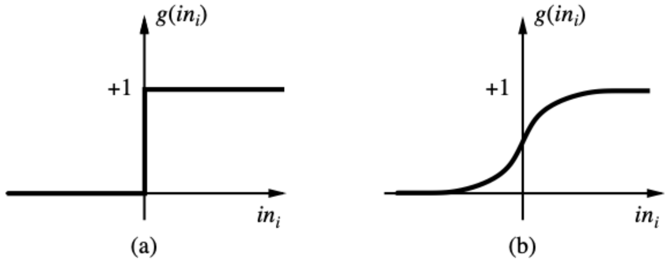
Output is a "squashed" linear function of the inputs:

$$a_i \leftarrow g(in_i) = g\left(\sum_j W_{j,i} a_j\right)$$



A gross oversimplification of real neurons, but its purpose is
to develop understanding of what networks of simple units can do.

4: Activation functions

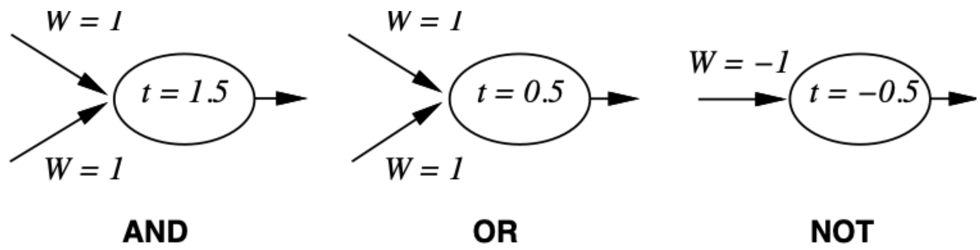


(a) is a step function or threshold function

(b) is a sigmoid function $1/(1+e^{-x})$

Changing the bias weight $W_{0,i}$ moves the threshold location

5: Implementing logical functions



McCulloch and Pitts: every Boolean function can be implemented

Network Structure

Feed-forward networks:
- single-layer perceptrons
- multi-layer perceptrons

Feed-forward networks implement functions, have no internal state **Recurrent networks**:

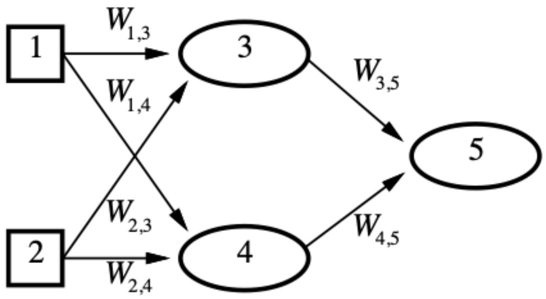
– Hopfield networks have symmetric weights ($W_{i,j} = W_{j,i}$)
$g(x) = \text{sign}(x)$, $a_i = \pm 1$; **holographic associative memory**

– Boltzmann machines use stochastic activation functions,

$\approx$ MCMC in Bayes nets – recurrent neural nets have directed cycles with delays

$\Rightarrow$ have internal state (like flip-flops), can oscillate etc.

6: Feed-forward example

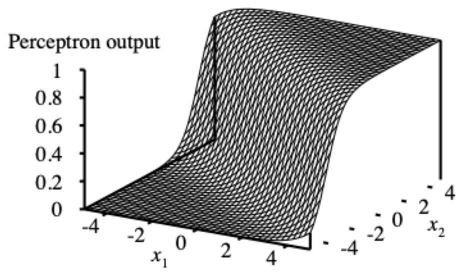


Feed-forward network = a parameterized family of nonlinear functions:

$$
\begin{aligned}
a_5 &= g(W_{3,5} \cdot a_3 + W_{4,5} \cdot a_4) \\
&= g(W_{3,5} \cdot g(W_{1,3} \cdot a_1 + W_{2,3} \cdot a_2) + W_{4,5} \cdot g(W_{1,4} \cdot a_1 + W_{2,4} \cdot a_2))
\end{aligned}
$$

Adjusting weights changes the function: **do learning this way!**

7: Single-layer perceptrons



Output units all operate separately—no shared weights

Adjusting weights moves the location, orientation, and steepness of cliff

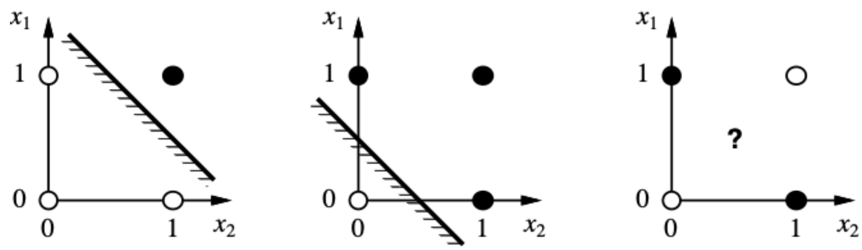
8: Expressiveness of perceptrons

Consider a perceptron with $g =$ step function (Rosenblatt, 1957, 1960)

Can represent AND, OR, NOT, majority, etc., but not XOR

Represents a linear separator in input space:

$$\sum_j W_j x_j > 0 \quad \text{or} \quad \mathbf{W} \cdot \mathbf{x} > 0$$



(a) x_1 **and** x_2 (b) x_1 **or** x_2 (c) x_1 **xor** x_2

Minsky & Papert (1969) pricked the neural network balloon

9: Perceptron learning - 1

Learn by adjusting weights to reduce error on training set

The squared error for an example with input $\mathbf{x}$ and true output y is

$$E = \frac{1}{2}Err^2 \equiv \frac{1}{2}(y - h_W(\mathbf{x}))^2 \ ,$$

Perform optimization search by gradient descent:

$$\begin{aligned}\frac{\partial E}{\partial W_j} &= Err \times \frac{\partial Err}{\partial W_j} = Err \times \frac{\partial}{\partial W_j}\left(y - g\left(\sum_{j=0}^{n} W_j x_j\right)\right) \\ &= -Err \times g'(in) \times x_j\end{aligned}$$

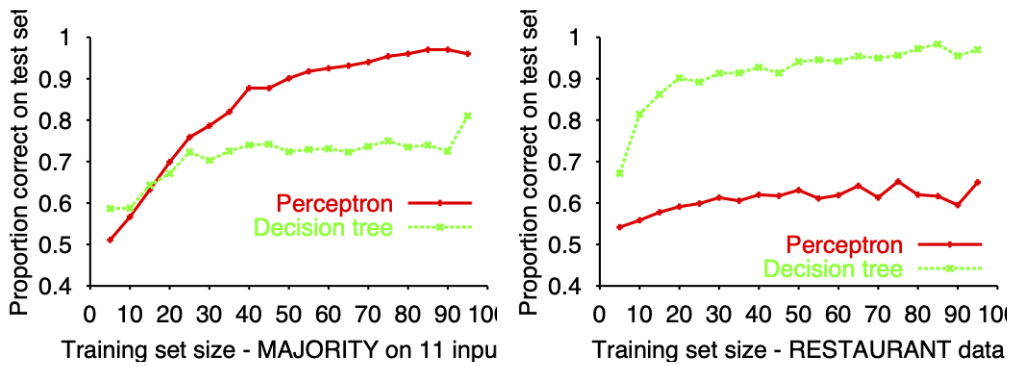
Simple weight update rule:

$$W_j \leftarrow W_j + \alpha \times Err \times g'(in) \times x_j$$

e.g., a positive error $\Rightarrow$ increase network output $\Rightarrow$ increase weights on positive inputs, decrease on negative inputs

10: Perceptron learning - 2

Perceptron learning rule converges to a consistent function
for any linearly separable data set



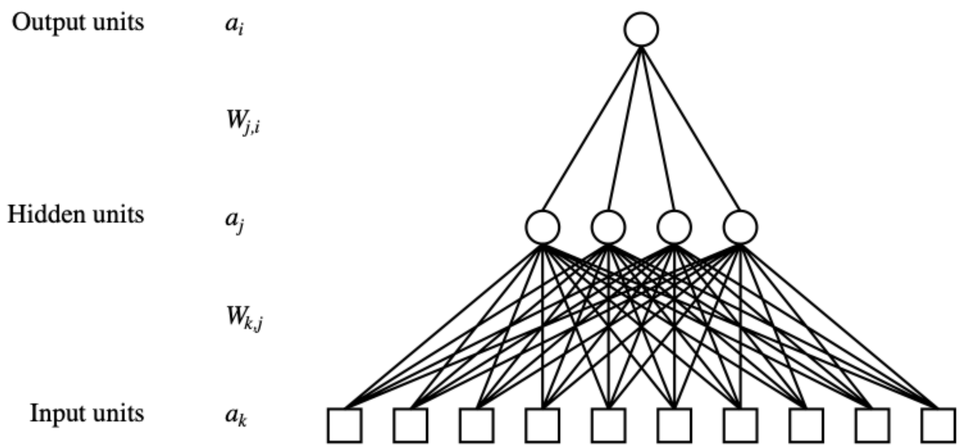
Perceptron learns majority function easily, DTL is hopeless

Point: majority is representable, but only as a very large DT and DTL won't learn that without a very large data set

DTL learns restaurant function easily, perceptron cannot represent it.

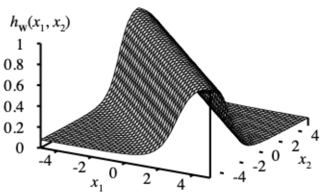
11: Multilayer perceptrons

Layers are usually fully connected;
numbers of hidden units typically chosen by hand

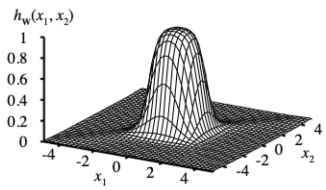


12: Expressiveness of MLPs

All continuous functions w/ 2 layers, all functions w/ 3 layers



(a) ridge



(b) bump

Combine two opposite-facing threshold functions to make a ridge

Combine two perpendicular ridges to make a bump

Add bumps of various sizes and locations to fit any surface

Proof requires exponentially many hidden units

13: Back-propagation learning

Output layer: same as for single-layer perceptron,

$$W_{j,i} \leftarrow W_{j,i} + \alpha \times a_j \times \Delta_i$$

where $\Delta_i = Err_i \times g'(in_i)$

Hidden layer: **back-propagate** the error from the output layer:

$$\Delta_j = g'(in_j) \sum_i W_{j,i} \Delta_i \ .$$

Update rule for weights in hidden layer:

$$W_{k,j} \leftarrow W_{k,j} + \alpha \times a_k \times \Delta_j \ .$$

(Most neuroscientists deny that back-propagation occurs in the brain)

14: Back-propagation derivation -1

The squared error on a single example is defined as

$$E = \frac{1}{2}\sum_i (y_i - a_i)^2 \ ,$$

where the sum is over the nodes in the output layer.

$$\begin{aligned}
\frac{\partial E}{\partial W_{j,i}} &= -(y_i - a_i)\frac{\partial a_i}{\partial W_{j,i}} = -(y_i - a_i)\frac{\partial g(in_i)}{\partial W_{j,i}} \\
&= -(y_i - a_i)g'(in_i)\frac{\partial in_i}{\partial W_{j,i}} = -(y_i - a_i)g'(in_i)\frac{\partial}{\partial W_{j,i}}\left(\sum_j W_{j,i}a_j\right) \\
&= -(y_i - a_i)g'(in_i)a_j = -a_j\Delta_i
\end{aligned}$$

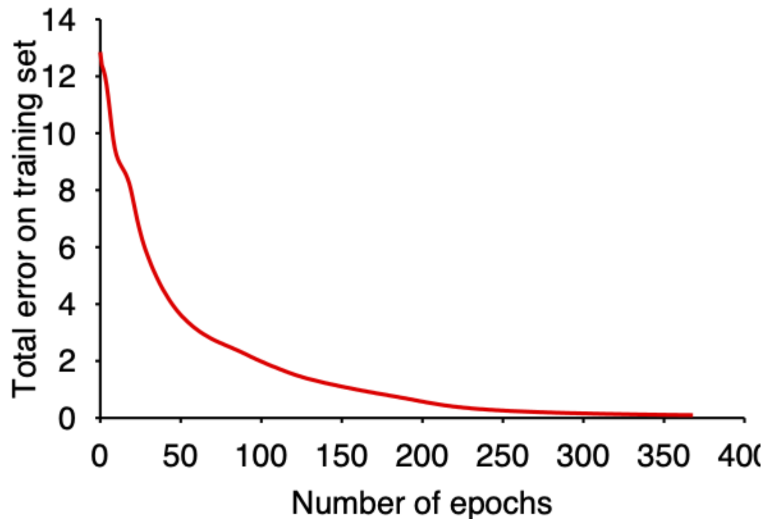
15: Back-propagation derivation - 2

$$
\begin{aligned}
\frac{\partial E}{\partial W_{k,j}} &= -\sum_i (y_i - a_i)\frac{\partial a_i}{\partial W_{k,j}} = -\sum_i (y_i - a_i)\frac{\partial g(in_i)}{\partial W_{k,j}} \\
&= -\sum_i (y_i - a_i) g'(in_i)\frac{\partial in_i}{\partial W_{k,j}} = -\sum_i \Delta_i \frac{\partial}{\partial W_{k,j}}\left(\sum_j W_{j,i} a_j\right) \\
&= -\sum_i \Delta_i W_{j,i}\frac{\partial a_j}{\partial W_{k,j}} = -\sum_i \Delta_i W_{j,i}\frac{\partial g(in_j)}{\partial W_{k,j}} \\
&= -\sum_i \Delta_i W_{j,i} g'(in_j)\frac{\partial in_j}{\partial W_{k,j}} \\
&= -\sum_i \Delta_i W_{j,i} g'(in_j)\frac{\partial}{\partial W_{k,j}}\left(\sum_k W_{k,j} a_k\right) \\
&= -\sum_i \Delta_i W_{j,i} g'(in_j) a_k = -a_k \Delta_j
\end{aligned}
$$

16: Back-propagation learning - 3

At each epoch, sum gradient updates for all examples and apply

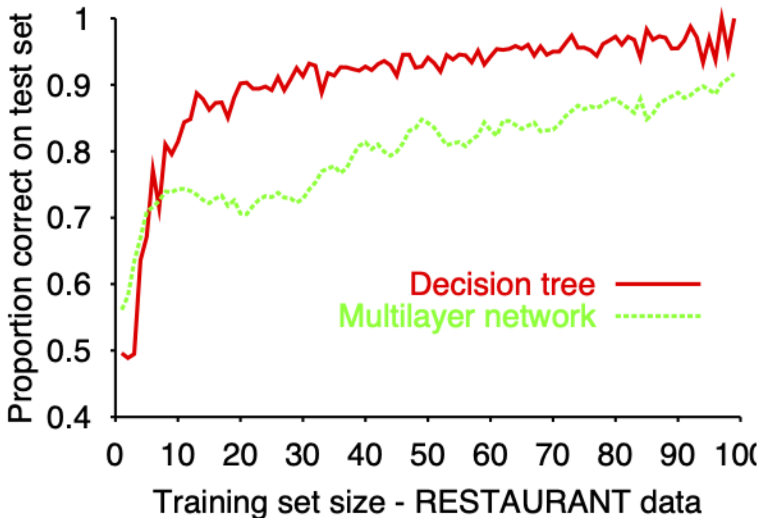
Training curve for 100 restaurant examples: finds exact fit



Typical problems: slow convergence, local minima

17: Back-propagation learning - 4

Learning curve for MLP with 4 hidden units:

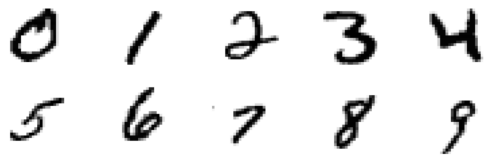


MLPs are quite good for complex pattern recognition tasks,
but resulting hypotheses cannot be understood easily

Point: This makes MLPs ineligible for some tasks, such as credit card and loan approvals, where law requires clear unbiased criteria

18: Handwritten digit recognition

EASY EXAMPLES



HARD EXAMPLES

3-nearest-neighbor = 2.4% error
400–300–10 unit MLP = 1.6% error
LeNet: 768–192–30–10 unit MLP = 0.9% error

Current best (kernel machines, vision algorithms) ≈ 0.6% error

19: Summary

Most brains have lots of neurons; each neuron $\approx$ linear–threshold unit (?)

Perceptrons (one-layer networks) insufficiently expressive

Multi-layer networks are sufficiently expressive; can be trained by gradient descent, i.e., error back-propagation

Many applications: speech, driving, handwriting, fraud detection, etc.

Engineering, cognitive modelling, and neural system modelling
subfields have largely diverged