In [1]:

```
import time
import numpy as np
import matplotlib.pyplot as plt
```

Common code for all questions

In [2]:

```
def accuracy(x,y):
    '''To measure SER using accuracy'''
    return (x == y).mean()

def progress_bar(a,b,ET='NA'):
    if not ET is 'NA':
        ET = (b-a)*ET/(a+0.0001)
        ET = int(ET*10)/10
    return print('\rETime left {4}s [{2}] completed {0}/{1} = {3:.1f}%'.format(
                        a,b,'*'*a + '.'*(b-a),100*a/b,ET),end='\t')

def gen_rand_base4(n):
    '''generate array of n symbols each symbol can be [0,3] with probab 0.25'''
    np.random.randint(low=0,high=4,size=n)

def PAM4gen(n,sigma_I=1):
    '''
    generate I(k) ~ Unif( -3d, -d, d, 3d); d  = sqrt(1/5)
    generate -3 -1 1 3 randomly scaled by 1/5
    x = 0 1 2 3     y = 2x-3 = -3 -1 1 3
    I = y/root(5) = 0.4x - 0.6'''
    rand0_3 = np.random.randint(low=0,high=4,size=n)
    return base4toPAM4(rand0_3,sigma_I)

def base4toPAM4(x,sigma_I=1):
    '''map [0,1,2,3] to [-3*d,-d,d,3*d]'''
    d   = np.sqrt(sigma_I/5)
    a,b = 2*d,3*d
    return a*x - b

def PAM4tobase4(x,sigma_I=1):
    '''map [-3*d,-d,d,3*d] to [0,1,2,3]'''
    d   = np.sqrt(sigma_I/5)
    a,b = 2*d,3*d
    return (x + b)/a
```

Common code for all questions

In [3]:

```
f_chnl0       = np.array([0.8,-1,0.6])/np.sqrt(2)#channel through which message is sent
PAM4_alphabet = base4toPAM4(np.arange(4))# array of possible PAM4 symbols
init_pad      = PAM4_alphabet[[0,0]]     # the initial memory of channel is assumed as
 (-3*d,-3*d)

def channel_out(x,f_chnl=f_chnl0,init_pad=init_pad):
    '''function to calculate the output of a filter'''
    x = np.hstack(( init_pad,x ))
    z = np.convolve(x,f_chnl)
    z = z[len(init_pad):len(z)-len(f_chnl)+1]
    return z

def nearest_symbol(x):
    '''return the nearest valid PAM4 symbol'''
    dist = np.abs(x - PAM4_alphabet.reshape(-1,1))
    ii   = np.argmin(dist,axis=0)
    return PAM4_alphabet[ii]
```

energy of channel = (0.8 x 0.8 + 1 x 1 + 0.6 x 0.6)/2 = 2/2 = 1

variance(I(k) = 1

SNR = (energy of the channel * variance(I)/noise_variance)

=>SNR = 1*1/noise_variance

=>noise_variance = 1/SNR

a) Viterbi Algorithm

In [4]:

```
class viterbi_decoder(object):
    def __init__(self,f_chnl=f_chnl0,tail_bits=PAM4_alphabet[[0,0]],init_pad=PAM4_alpha
bet[[0,0]]):
        self.L_chnl      = len(f_chnl)      #length of the channel
        self.mem         = self.L_chnl - 1  # memory of the channel
        self.f_chnl      = f_chnl           # coefficients of the channel
        self.tail_bits   = tail_bits        # tails bits for viterbi algorithm
        self.init_pad    = init_pad         # initial L-1 bits stored in the channel

    def trellis_gen(self,rseq):
        '''function to generate the trellis and fill the cumulative metrics'''

        init_memory = tuple(self.init_pad)  # initial L-1 bits stored in the channel
        stage_0      = {init_memory:[0,-1]}# first stage of the trellis
        stages       = [stage_0] + [dict() for i in range(len(rseq))]#create an empty
 trellis
        f_chnl_part  = self.f_chnl[1:]     #last L-1 coefficients of channel

        for stage_index in range(len(rseq)):#for loop to compute the metrics of next st
age
            r_true    = rseq[stage_index]
            nxt_stage = stages[stage_index+1]

            tail_index = stage_index - len(rseq) + self.mem
            tmp_alphabet = PAM4_alphabet if tail_index<0 else [self.tail_bits[tail_inde
x]]#to constrain the trellis for tail symbols

            for memory,cm in stages[stage_index].items():
                transition_metric_part0 = r_true - np.matmul(f_chnl_part,memory)#partia
l computation of transition metric for efficiency
                for jPAM4 in tmp_alphabet:
                    tmp_memory           = (jPAM4, *memory[:-1])
                    tmp_cm               = cm[0] + (transition_metric_part0 - self.f_ch
nl[0]*jPAM4)**2##cumulative metric calc
                    if nxt_stage.get(tmp_memory,[1e23,-1])[0] > tmp_cm:##choosing the n
ode which gives the min cumulative metric
                        nxt_stage[tmp_memory] = [tmp_cm,memory[-1]]##[cumulative metri
c,survivor in previous stage]
        return stages

    def decode_inf_delay(self,rseq):
        '''infinite delay Viterbi Algorithm equivalent to MMSE'''
        '''rseq = [r(0) r(1)  ... r(k-2) r(k-1) r(k)]'''
        stages = self.trellis_gen(rseq)#compute all stages in the trellis
        msg = np.zeros(len(rseq))
        symb_index = len(rseq)-1

        #find the node in the last stage with minimum cumulative metric
        #will have only one node in the last stage since we are using known tail symbol
s
        min_key = -1
        min_cm  = float('inf')
        for k,v in stages[-1].items():
            if min_cm > v[0]:
                min_key, min_cm = k, v[0]

        curr_memory = min_key
        for stage_index in range(len(stages)-1,0,-1):#traversing backward along the sur
vivor sequence
```

```
            msg[symb_index] = (curr_memory[0])
            curr_memory = (*curr_memory[1:],stages[stage_index][curr_memory][1])#find t
he previous symbol in the suvivor sequence
            symb_index -= 1
        return np.array(msg), min_cm

    def decode_finite_delay(self,stages,delta):
        msg = np.zeros(len(stages)-1)

        #decoding the first N-Delta symbols of the sequence using fintie delay decoding
        for symb_index in range(len(stages)-1-delta):
            min_key = -1
            min_cm  = float('inf')
            for k,v in stages[symb_index + delta + 1].items():
                if min_cm > v[0]:
                    min_key = k
                    min_cm  = v[0]

            curr_memory = min_key
            for stage_index in range(symb_index + delta + 1,symb_index+1,-1):
                curr_memory = (*curr_memory[1:],stages[stage_index][curr_memory][1])
            msg[symb_index] = curr_memory[0]

        #decoding the last Delta symbols using infinite delay decoding
        min_key = -1
        min_cm  = float('inf')
        for k,v in stages[-1].items():
            if min_cm > v[0]:
                min_key, min_cm = k, v[0]

        curr_memory = min_key
        for stage_index in range(len(stages)-1,max(len(stages)-1-delta,0),-1):
            msg[stage_index-1] = (curr_memory[0])
            curr_memory = (*curr_memory[1:],stages[stage_index][curr_memory][1])
        return np.array(msg), min_cm

    def transition_metric(self,r_true,tap_delay_line):
        '''redundant function'''
        return (r_true - np.matmul(self.f_chnl,tap_delay_line))**2
```

Viterbi algorithm takes about 5-8 mins(depending on PC) to compute SER for all 36 combinations of noise variance and Delta

check the progress bar for time left

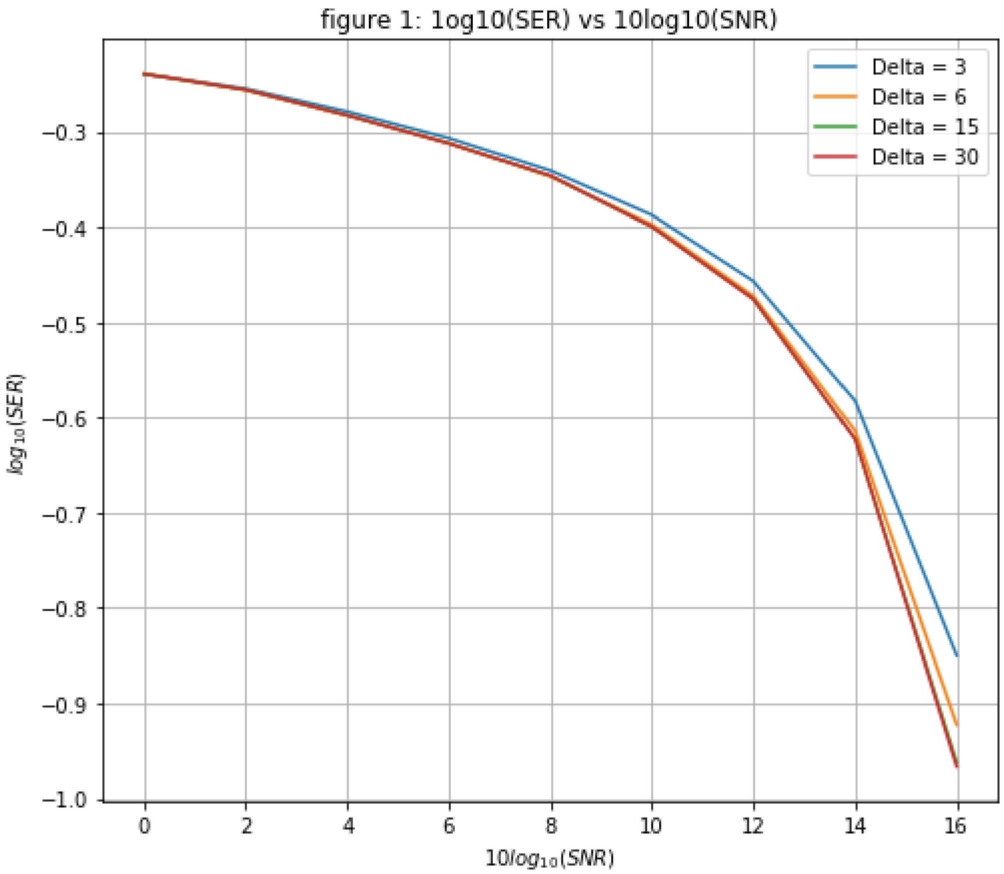
In [5]:

```
time1 = time.time()

Delta_arr    = [3,6,15,30]
SNR_arr      = 10**(np.arange(0,16.001,2)/10)
sigma_v2_arr = 1/SNR_arr                         # noise variance = 1/SNR
sigma_v_arr  = np.sqrt(sigma_v2_arr)    # noise standard deviation = sqrt(noise varianc
e)
tail_bits    = PAM4_alphabet[[1,2]]     # known tail bits(symbols) for the viterbi algo
rithm can be changed manually here
init_pad     = np.array([0,0])#PAM4_alphabet[[0,0]] #initial memory of the channel

viterbi0 = viterbi_decoder(f_chnl0,tail_bits,init_pad)

msg_PAM4  = PAM4gen(100000)## generate I(k) ~ Unif(-3*d,-d,d,3*d); d = sqrt(1/5)
msg_PAM4  = np.hstack([msg_PAM4, tail_bits])#add known tail bits

tx_seq    = channel_out(msg_PAM4,f_chnl0,init_pad)#output of the channel without noise
SER_VA    = np.zeros(( len(Delta_arr),len(sigma_v_arr) ))

display_progress = 1
print()
if display_progress: progress_bar(0,len(sigma_v_arr))
for ind_sigma_v,sigma_v in enumerate(sigma_v_arr):
    noise     = np.random.randn(len(msg_PAM4))*sigma_v  #generate noise
    rx_seq    = tx_seq + noise                          #add noise to channel output
    trellis   = viterbi0.trellis_gen(rx_seq)# common trellis generation for different v
alues of delta to simulate faster

    for ind_Delta,Delta in enumerate(Delta_arr):#Decoding with different delays
        est_seq = viterbi0.decode_finite_delay(trellis.copy(),Delta)[0]
        SER_VA[ind_Delta][ind_sigma_v] = 1 - accuracy(msg_PAM4,est_seq)
    if display_progress: progress_bar(ind_sigma_v+1,len(sigma_v_arr),time.time()-time1)

time2 = time.time()
print('\ntime for {} symbols = {:.2f} secs = {:.3f} mins'.format(len(msg_PAM4),time2-ti
me1,(time2-time1)/60))
```

```
ETime left 0.0s [*********] completed 9/9 = 100.0%
time for 100002 symbols = 538.06 secs = 8.968 mins
```

In [6]:

```
plt.figure(figsize=(8,7))
plt.title('figure 1: log10(SER) vs 10log10(SNR)')
for ind_i,Delta in enumerate(Delta_arr):
    # plt.plot(10*np.log10(SNR_arr),SER_VA[ind_i]*100,label='Delta = %d'%(Delta))
    plt.plot(10*np.log10(SNR_arr),np.log10(SER_VA[ind_i]),label='Delta = %d'%(Delta))
plt.xlabel('$10log_{10}(SNR)$')
plt.ylabel('$log_{10}(SER)$')
plt.legend()
plt.grid()
plt.savefig('figure1')
```



In [7]:

```
'''Check if the SER of Delta = 15 and Delta =30 are equal'''
print(np.allclose(SER_VA[2],SER_VA[3],atol=1e-3))#check if max difference is  < 0.1 per
cent
print(np.abs(SER_VA[2]-SER_VA[3]),np.abs(SER_VA[2]-SER_VA[3]).max())
```

```
False
[0.0000000e+00 3.9999200e-05 4.9999000e-05 2.2999540e-04 1.9999600e-04
 9.9998000e-06 5.9998800e-05 9.9998000e-06 1.1199776e-03] 0.00111997760044
80189
```

SER values for Delta = 15 and Delta = 30 are almost same at every SNR

max difference between SER(Delta = 15) and SER(Delta = 30) is **0.09** percent

so for any value of Delta >= 15 the SER is almost same at every SNR

SER(Delta = 3) > SER(Delta = 10) > SER(Delta = 15) = SER(Delta = 30) at every SNR

SER decreases(monotonically) with increasing Delta at every SNR

SER decreases with increasing SNR for every Delta

difference in SER is more clear at high SNR

Linear MMSE Equalizer

In [8]:

```
def nearest_symbol(x):
    dist = np.abs(x - PAM4_alphabet.reshape(-1,1))
    ii   = np.argmin(dist,axis=0)
    return PAM4_alphabet[ii]

def R_fxn(N,sigma_v2,f_chnl=f_chnl0):
    '''Calculate R Autocorrelation matrix for LMMSE equalizer'''
    if N>len(f_chnl):
        f_chnl = np.hstack([f_chnl,np.zeros(N-len(f_chnl))])
    f_chnl = f_chnl[:N]
    row = [(f_chnl[i:]*f_chnl[:N-i]).sum() for i in range(N)]
    R   = np.diag(np.full(N,fill_value=sigma_v2))
    for i in range(N):
        i1 = N -i
        for j in range(i1):
            R[j][j+i] = R[j+i][j] = R[j+i][j] + row[i]
    return R

def P_fxn(N,Delta,f_chnl=f_chnl0):
    '''Calculate P for LMMSE equalizer'''
    L = len(f_chnl)
    P = np.zeros(N)
    for i in range(N):
        j = Delta - i
        if j >=0 and j<=L-1:
            P[i] = f_chnl[j]
    return P
```

In [9]:

```
f_chnl0 = np.array([0.8,-1,0.6])/np.sqrt(2)
sigma_I = 1 ##variance of msg signal = var[I(k)]
PAM4_alphabet = base4toPAM4(np.arange(4),sigma_I=sigma_I)
init_pad      = np.array([0,0])#PAM4_alphabet[[0,0]] ##Initial memory(L-1 bits) stored
 in the channel

SNR  = 10#( in dB)
SNR  = 10**(SNR/10)
sigma_v2 = 1/SNR
sigma_v  = np.sqrt(sigma_v2)
msg_PAM4 = PAM4gen(int(1e5),sigma_I=sigma_I)
tx_seq   = channel_out(msg_PAM4, f_chnl0, init_pad)       #noiseless output of channel
rx_seq   = tx_seq + sigma_v*np.random.randn(len(msg_PAM4))#add noise with SNR 10dB to t
he channel output
```

B1, B2, B3

In [10]:

```
N,Delta = 3,0
for (N,Delta) in [(3,0),(10,0),(10,5)]:
    R,P         = R_fxn(N,sigma_v2,f_chnl0), P_fxn(N,Delta,f_chnl0)  #calculate
    R_inv       = np.linalg.inv(R)
    W_opt       = np.matmul(R_inv,P)
    W_opt_flat = W_opt.ravel()# flattened view of W_opt

    est_msg     = channel_out(rx_seq,W_opt_flat)
    Jmin        = np.square(msg_PAM4[:len(msg_PAM4)-Delta] - est_msg[Delta:]).mean()
    print('N = {}, Delta = {}'.format(N,Delta))
    print('Jmin simulated   {:.4f}\nJmin theoritical {:.4f}'.format(Jmin, sigma_I**2 -
np.matmul(W_opt_flat,P) ))
    print('w_opt:',W_opt_flat)
    print()
```

```
N = 3, Delta = 0
Jmin simulated   0.4596
Jmin theoritical 0.4577
w_opt: [0.95869858 0.80157854 0.30092483]

N = 10, Delta = 0
Jmin simulated   0.4477
Jmin theoritical 0.4447
w_opt: [ 0.98162681  0.84173676  0.31296526 -0.08206659 -0.20034628 -0.137
12618
 -0.03402374  0.02698237  0.03503563  0.01640834]

N = 10, Delta = 5
Jmin simulated   0.3362
Jmin theoritical 0.3369
w_opt: [ 0.05175672  0.18258482  0.29532074  0.17546218 -0.36100637  0.589
39739
  0.64385394  0.30512781  0.01904098 -0.05445636]
```

B4

calculated **Jmin(N,Delta)** varying **N from 2 to 40 and Delta from 0 to 40**.

since N = 1(memory less equalizer) is just scaling the signal it is not considered

In [11]:

```
N_max = 40 + 1
Jmin_arr  = np.full(( N_max,N_max ),fill_value=3.0)
for N in range(2,N_max):
    for Delta in range(N_max):
         R,P              = R_fxn(N,sigma_v2,f_chnl0), P_fxn(N,Delta,f_chnl0)
         W_opt_flat       = np.matmul(np.linalg.inv(R),P).ravel()
         Jmin_arr[N][Delta] = sigma_I**2 - W_opt_flat @ P
```

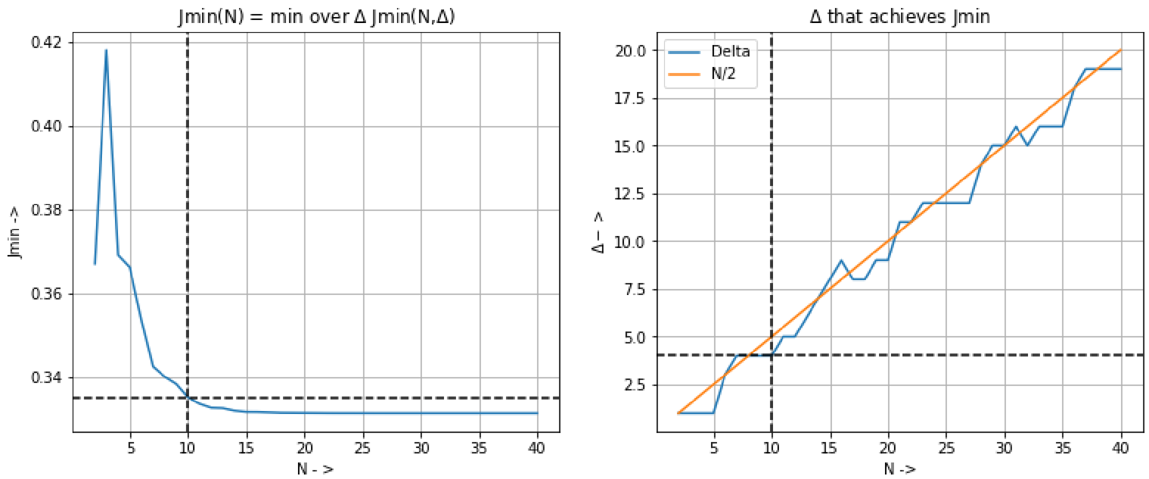
In [12]:

```
print(Jmin_arr.min())
N_arr = np.arange(N_max)
print('N values and the Delta for which Jmin is optimal')
print(np.vstack([N_arr,Jmin_arr.argmin(axis=1)])[:,2:])

plt.figure(figsize=(13,5))
plt.subplot(121)
plt.title('Jmin(N) = min over $\Delta$ Jmin(N,$\Delta$)')
plt.plot(N_arr[2:],Jmin_arr.min(axis=1)[2:])
plt.axvline(x=10,c='k',ls='--')
plt.axhline(y=Jmin_arr[10].min(),c='k',ls='--')
plt.grid()
plt.xlabel('N - >')
plt.ylabel('Jmin ->')

plt.subplot(122)
plt.title('$\Delta$ that achieves Jmin')
plt.plot(N_arr[2:],Jmin_arr.argmin(axis=1)[2:],label='Delta')
plt.plot(N_arr[2:],N_arr[2:]/2,label='N/2')
plt.axvline(x=10,c='k',ls='--')
plt.axhline(y=Jmin_arr.argmin(axis=1)[10],c='k',ls='--')
plt.xlabel('N ->')
plt.ylabel('$\Delta ->$')
plt.grid()
plt.legend()
```

```
0.33145583339352125
N values and the Delta for which Jmin is optimal
[[ 2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
  26 27 28 29 30 31 32 33 34 35 36 37 38 39 40]
 [ 1  1  1  1  3  4  4  4  4  5  5  6  7  8  9  8  8  9  9 11 11 12 12 12
  12 12 14 15 15 16 15 16 16 16 18 19 19 19 19]]
```

Out[12]:

```
<matplotlib.legend.Legend at 0x29d8def6160>
```



The value of Delta which minimizes Jmin for a fixed N is close to N/2(right plot)

From the above plots we can see that Jmin(N) decreases with N but almost saturates after N = 20

Jmin(N=10,Delta=4) = 0.3352; Jmin(N=20,Delta=9) = 0.3315; Jmin(N=40,Delta=19) = 0.3314;

we can choose N=40,Delta=19 for optimal filter but the gain in performance compared is only 0.004(Decrease in Jmin)

which is not worth 4 times the hardware complexity(from N = 10 to N = 40)

so N = 10,Delta =4 is chosen as the optimal filter

In [13]:

```
print(Jmin_arr[10][4],Jmin_arr[20][9],Jmin_arr[40][19],Jmin_arr.min())
print('best possible Jmin is {}'.format(Jmin_arr.min()))
print('Jmin for the optimal filter chosen is {}'.format(Jmin_arr[10][4]))
```

```
0.33523140612210733 0.3315061922677629 0.33145583339352125 0.3314558333935
2125
best possible Jmin is 0.33145583339352125
Jmin for the optimal filter chosen is 0.33523140612210733
```

In [14]:

```
N, Delta  = 10, 4
#### N, Delta  = 20, 9
#### N, Delta  = 60, 30

# SNR_arr    = np.logspace(0,16/10,9)
SNR_arr      = 10**(np.arange(0,16.0001,2)/10)
SER_LE       = np.zeros(len(SNR_arr))

sigma_v2_arr  = 1/SNR_arr
sigma_v_arr   = np.sqrt(sigma_v2_arr)

msg_PAM4 = PAM4gen(int(1e5),sigma_I=sigma_I)
tx_seq   = channel_out(msg_PAM4, f_chnl0, init_pad)

for ind,sigma_v2 in enumerate(sigma_v2_arr):
    sigma_v     = np.sqrt(sigma_v2)
    R,P         = R_fxn(N,sigma_v2,f_chnl0), P_fxn(N,Delta,f_chnl0)
    R_inv       = np.linalg.inv(R)
    W_opt       = R_inv@P
    W_opt_flat = W_opt.ravel()# flattened view of W_opt

    rx_seq      = tx_seq + sigma_v*np.random.randn(len(msg_PAM4))
    est_msg     = channel_out(rx_seq,W_opt_flat)
    est_msg     = nearest_symbol(est_msg)
    msg_PAM4[:len(msg_PAM4)-Delta] - est_msg[Delta:]
    SER_LE[ind] = 1 - accuracy(msg_PAM4[:len(msg_PAM4)-Delta], est_msg[Delta:])
print('SER:',SER_LE,'\naccuracy:',1-SER_LE)
```

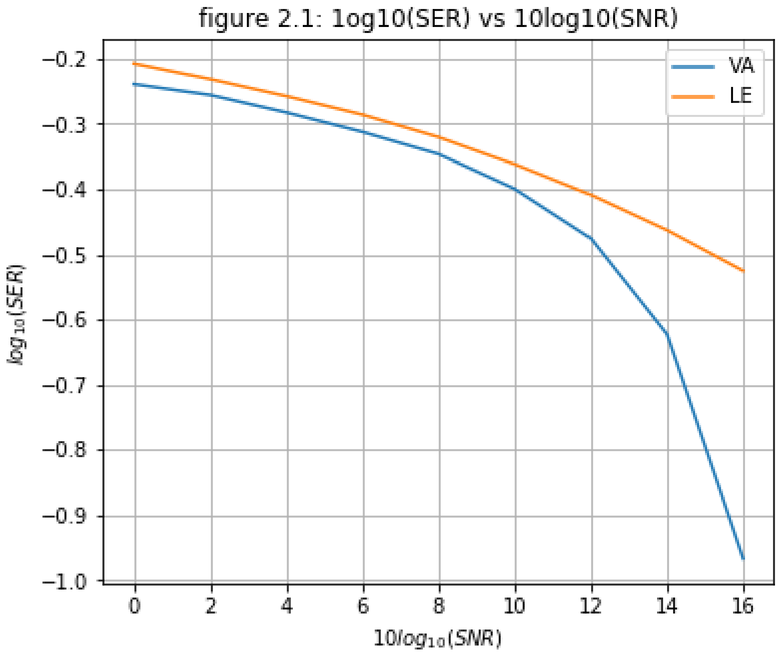
```
SER: [0.61944478 0.58638346 0.55232209 0.51734069 0.47823913 0.43448738
 0.3900956  0.34440378 0.29836193]
accuracy: [0.38055522 0.41361654 0.44767791 0.48265931 0.52176087 0.565512
62
 0.6099044  0.65559622 0.70163807]
```

In [15]:

```
plt.figure(figsize=(6,5))
plt.title('figure 2.1: log10(SER) vs 10log10(SNR)')

# plt.plot(10*np.log10(SNR_arr),SER_VA[3]*100, label='VA')
# plt.plot(10*np.log10(SNR_arr),SER_LE*100,    label='LE')

plt.plot(10*np.log10(SNR_arr),np.log10(SER_VA[3]),label='VA')
plt.plot(10*np.log10(SNR_arr),np.log10(SER_LE)   ,label='LE')

plt.xlabel('$10log_{10}(SNR)$')
plt.ylabel('$log_{10}(SER)$')
plt.grid()
plt.legend()
```

Out[15]:

```
<matplotlib.legend.Legend at 0x29d8e152048>
```



Observations Linear Equalizer

from the above plot we can see that VA with Delta = 30 is always signficantly better than optimal(almost) Linear Equalizer for any SNR

Since **Viterbi Algorithm (Delta =30) has low SER than optimal LE(N=10,Delta=4) at every SNR(especially at high SNR)**

SER difference between Optimal viterbi and Optimal LE is very high at high SNR

Decision Feedback Equalizer

In [16]:

```
class filter(object):
    '''simulates a filter with memory'''
    def __init__(self,imp_res):
        self.imp_res        = np.array(imp_res)
        self.tap_delay_line = np.zeros(self.imp_res.shape)

    def out(self,input):
        self.tap_delay_line[1:] = self.tap_delay_line[:-1]
        self.tap_delay_line[0]  = input
        return np.matmul(self.tap_delay_line,self.imp_res)

class DFE(object):
    def __init__(self,wff,wfb):
        '''simulates decision feedback equalizer with the given coefficients'''
        self.reset(wff,wfb)

    def reset(self,wff=None,wfb=None):
        if not wff is None: self.wff = wff
        if not wfb is None: self.wfb = wfb
        self.ff  = filter(self.wff)
        self.fb  = filter(self.wfb)
        self.past_output = 0

    def out(self,input):
        tmp = self.ff.out(input) - self.fb.out(self.past_output)
        tmp = self.nearest_symbol(tmp)
        self.past_output = tmp
        return tmp

    def seq_out(self,rx_seq):
        tmp = np.zeros(len(rx_seq))
        for i in range(len(tmp)):
            tmp[i] = self.out(rx_seq[i])
        return tmp

    def nearest_symbol(self,x):
        dist = np.abs(x - PAM4_alphabet.reshape(-1,1))
        ii   = np.argmin(dist,axis=0)
        return PAM4_alphabet[ii]

# ff = filter([1,1,0])
# for i in range(1,4):
#     print(ff.out(i))
```

In [17]:

```
def corr_fxn2(i,j,f_chnl):
    '''returns E[r(k-i) x I(k-j)]'''
    #r(k-i) I(k-j)  j is less than equal to i
    diff = j - i
    if diff >=0 and diff<= len(f_chnl) - 1:
        return f_chnl[diff]
    return 0

def R_fxn2(N1,N2,sigma_v2,f_chnl=f_chnl0):
    '''R matrix for Decision Feedback Equailzer'''
    A1 = R_fxn(N1,sigma_v2,f_chnl)
    A4 = np.diag(np.full(N2,fill_value=1))
    A2 = np.zeros(( N1,N2 ))
    A3 = np.zeros(( N2,N1 ))

    for i in range(N1):
        for j in range(N2):
            A3[j][i] = A2[i][j] = -corr_fxn2(i,Delta+1+j,f_chnl)
    # print(A2,'\n',A3)
    return np.vstack([ np.hstack([A1,A2]), np.hstack([A3,A4]) ])

def P_fxn2(N1,N2,Delta,f_chnl=f_chnl0):
    '''P matrix for Decision Feedback Equailzer'''
    L = len(f_chnl)
    P = np.zeros(N1+N2)
    for i in range(N1):
        P[i] = corr_fxn2(i,Delta,f_chnl)#P[i] = E[r(k-i)I(k-Delta)]
    return P
```

In [18]:

```
f_chnl0 = np.array([0.8,-1,0.6])/np.sqrt(2)
sigma_I = 1
PAM4_alphabet = base4toPAM4(np.arange(4),sigma_I=sigma_I)
init_pad = PAM4_alphabet[[0,0]]

SNR  = 10#( in dB)
SNR  = 10**(SNR/10)
sigma_v2 = 1/SNR
sigma_v  = np.sqrt(sigma_v2)
msg_PAM4 = PAM4gen(int(1e5),sigma_I=sigma_I)
# msg_PAM4 = PAM4gen(int(3e1),sigma_I=sigma_I)
tx_seq   = channel_out(msg_PAM4, f_chnl0, init_pad)
rx_seq   = tx_seq + sigma_v*np.random.randn(len(msg_PAM4))
```

C1 C2

In [19]:

```
N1,N2,Delta = 6,4,0
for (N1,N2,Delta) in [(6,4,0),(6,4,3)]:
    R,P          = R_fxn2(N1,N2,sigma_v2,f_chnl0), P_fxn2(N1,N2,Delta,f_chnl0)
    R_inv        = np.linalg.inv(R)
    W_opt        = np.matmul(R_inv,P)
    W_opt_flat = W_opt.ravel()# flattened view of W_opt
    W_ff,W_fb  = np.hsplit(W_opt_flat,[N1])
    # print(W_ff,W_fb)

    DFE0         = DFE(W_ff,W_fb)
    est_msg      = DFE0.seq_out(rx_seq)
    # msg_PAM4   = nearest_symbol(msg_PAM4)
    Jmin         = np.square(msg_PAM4[:len(msg_PAM4)-Delta] - est_msg[Delta:]).mean()
    print('N1 = {}, N2 = {}, Delta = {}'.format(N1,N2,Delta))
    print('Jmin simulated   {:.4f}\nJmin theoritical {:.4f}'.format(Jmin, sigma_I**2 -
W_opt_flat @ P))
    print('w_opt:',W_opt_flat)
    print()
```

```
N1 = 6, N2 = 4, Delta = 0
Jmin simulated   0.5642
Jmin theoritical 0.2381
w_opt: [ 1.34687006e+00  7.25080752e-16  1.99103042e-16  4.66147705e-17
  3.66317007e-17  2.71738925e-17 -9.52380952e-01  5.71428571e-01
  2.21935568e-16  8.33033345e-17]

N1 = 6, N2 = 4, Delta = 3
Jmin simulated   0.4447
Jmin theoritical 0.1796
w_opt: [ 6.07156583e-02 -2.99095927e-02 -3.65516413e-01  1.01585039e+00
 -2.15427018e-16  6.70935082e-16 -8.73390177e-01  4.30988818e-01
 -5.74628545e-16  3.05788119e-16]
```

C3

calculated Jmin(N,Delta) varying N1 from 1 to 9 and Delta from 0 to 10.

since N1 = 1(memory less equalizer) is just scaling the signal it is not considered

In [20]:

```
# N1+N2 = 10
Delta_max = 10
Jmin_DFE_arr = np.full(( 10,Delta_max ),fill_value=10.0,dtype=np.float)
for N1 in range(1,10): #N1 from 1 to 9
    N2 = 10 - N1       #N2 from 9 to 1
    for Delta in range(Delta_max):
        R,P        = R_fxn2(N1,N2,sigma_v2,f_chnl0), P_fxn2(N1,N2,Delta,f_chnl0)
        R_inv      = np.linalg.inv(R)
        W_opt      = np.matmul(R_inv,P)
        W_opt_flat = W_opt.ravel()# flattened view of W_opt
        tmp        = 1 - np.matmul(W_opt_flat,P)#sigma_I**2 - W_opt_flat @ P
        tmp        = tmp if tmp>0 else 10# if Jmin < 0 set it 10(since -ve Jmin implies
formula is invalid)
        Jmin_DFE_arr[N1][Delta] = tmp
```

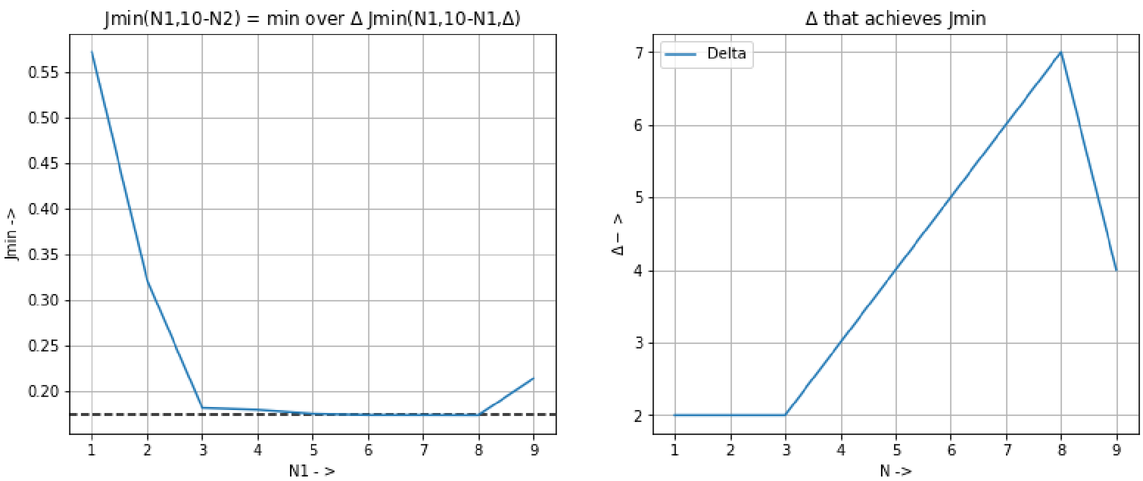
In [21]:

```
print(Jmin_DFE_arr.min())
N_arr = np.arange(10)
print(np.vstack([N_arr,Jmin_DFE_arr.argmin(axis=1)])[:,1:])
print(np.vstack([N_arr,Jmin_DFE_arr.min(axis=1)])[:,1:])

plt.figure(figsize=(13,5))
plt.subplot(121)
plt.title('Jmin(N1,10-N2) = min over $\Delta$ Jmin(N1,10-N1,$\Delta$)')
plt.axhline(y=Jmin_DFE_arr.min(),c='k',ls='--')
plt.plot(N_arr[1:],Jmin_DFE_arr.min(axis=1)[1:])
plt.grid()
plt.xlabel('N1 - >')
plt.ylabel('Jmin ->')
plt.subplot(122)
plt.title('$\Delta$ that achieves Jmin')
plt.plot(N_arr[1:],Jmin_DFE_arr.argmin(axis=1)[1:],label='Delta')
# plt.plot(N_arr[1:],N_arr[1:]/2,label='N/2')
plt.xlabel('N ->')
plt.ylabel('$\Delta ->$')
plt.grid()
plt.legend()
```

```
0.17372072544483863
[[1 2 3 4 5 6 7 8 9]
 [2 2 2 3 4 5 6 7 4]]
[[1.         2.         3.         4.         5.         6.
  7.         8.         9.        ]
 [0.57142857 0.3218126  0.18163147 0.17957867 0.17525337 0.17387811
  0.17385976 0.17372073 0.21390243]]
```

Out[21]:

```
<matplotlib.legend.Legend at 0x29d8de3d4e0>
```



In [22]:

```
N1_Jmin = Jmin_DFE_arr.min(axis=1).argmin()
Delta_N1_Jmin = Jmin_DFE_arr[N1_Jmin].argmin()
print('optimal filter for DFE is N1={} N2={} Delta={} Jmin={}'.format(
    N1_Jmin,10-N1_Jmin,Delta_N1_Jmin,Jmin_DFE_arr.min()))
print()
```

```
optimal filter for DFE is N1=8 N2=2 Delta=7 Jmin=0.17372072544483863
```

From the above plots(left) we can see that Jmin(N1,10-N2) is minimum at N1 = 8,N2 = 2,

value of Delta which minimises Jmin for N1 = 8 is 7

The optimal DFE filter for N1+N2 = 10 is N1=8 N2=2 Delta=7 Jmin=0.17372072544483863

C4

In [23]:

```
N1, N2, Delta  = 8, 2, 7

# SNR_arr    = np.logspace(0,16/10,9)
SNR_arr     = 10**(np.arange(0,16.0001,2)/10)
SER_DFE      = np.zeros(len(SNR_arr))

sigma_v2_arr  = 1/SNR_arr
sigma_v_arr   = np.sqrt(sigma_v2_arr)

msg_PAM4 = PAM4gen(int(1e5),sigma_I=sigma_I)
tx_seq   = channel_out(msg_PAM4, f_chnl0, init_pad)

for ind,sigma_v2 in enumerate(sigma_v2_arr):
    sigma_v     = np.sqrt(sigma_v2)
    R,P         = R_fxn2(N1,N2,sigma_v2,f_chnl0), P_fxn2(N1,N2,Delta,f_chnl0)
    R_inv       = np.linalg.inv(R)
    W_opt       = R_inv@P
    W_opt_flat = W_opt.ravel()# flattened view of W_opt
    W_ff,W_fb  = np.hsplit(W_opt_flat,[N1])

    rx_seq      = tx_seq + sigma_v*np.random.randn(len(msg_PAM4))

    DFE0        = DFE(W_ff,W_fb)
    est_msg     = DFE0.seq_out(rx_seq)
    Jmin        = np.square(msg_PAM4[:len(msg_PAM4)-Delta] - est_msg[Delta:]).mean()

    SER_DFE[ind] = 1 - accuracy(msg_PAM4[:len(msg_PAM4)-Delta], est_msg[Delta:])
print('SER',SER_LE,'\naccuracy',1-SER_LE)
```

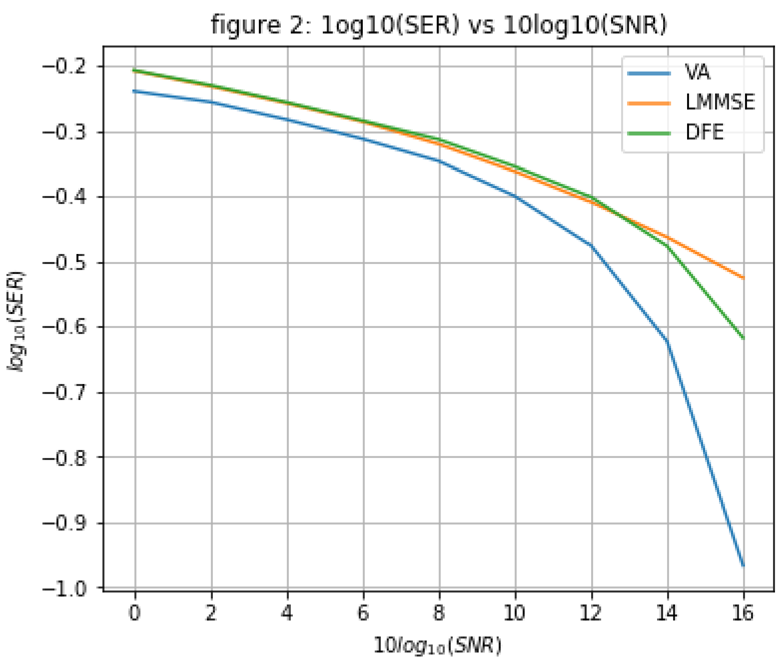
```
SER [0.61944478 0.58638346 0.55232209 0.51734069 0.47823913 0.43448738
 0.3900956  0.34440378 0.29836193]
accuracy [0.38055522 0.41361654 0.44767791 0.48265931 0.52176087 0.5655126
2
 0.6099044  0.65559622 0.70163807]
```

In [24]:

```
plt.figure(figsize=(6,5))
plt.title('figure 2: log10(SER) vs 10log10(SNR)')

## plt.plot(10*np.log10(SNR_arr),SER_VA[3]*100, label='VA')
## plt.plot(10*np.log10(SNR_arr),SER_LE*100,    label='LMMSE')
## plt.plot(10*np.log10(SNR_arr),SER_DFE*100,   label='DFE')

plt.plot(10*np.log10(SNR_arr),np.log10(SER_VA[3]),label='VA')
plt.plot(10*np.log10(SNR_arr),np.log10(SER_LE)   ,label='LMMSE')
plt.plot(10*np.log10(SNR_arr),np.log10(SER_DFE)  ,label='DFE')

plt.xlabel('$10log_{10}(SNR)$')
plt.ylabel('$log_{10}(SER)$')
plt.grid()
plt.legend()

# plt.savefig('figure2')
```

Out[24]:

```
<matplotlib.legend.Legend at 0x29da380bd30>
```



Observations DFE

Viterbi with Delta = 30 **is better** than both LMMSE and DFE at all SNRs

At low SNR the performance of optimal LE and optimal DFE is almost same.

At high SNR DFE is significantly better than LE this is probably because of **less error propagation** (from nearest neighbour decisions)**at High SNR**

since The nearest neighbour decision is more accurate at high SNR

Bonus Question

10 tap LE N = 10 Delta = 4

10 tap DFE N1 = 8, N2 = 2, Delta = 7

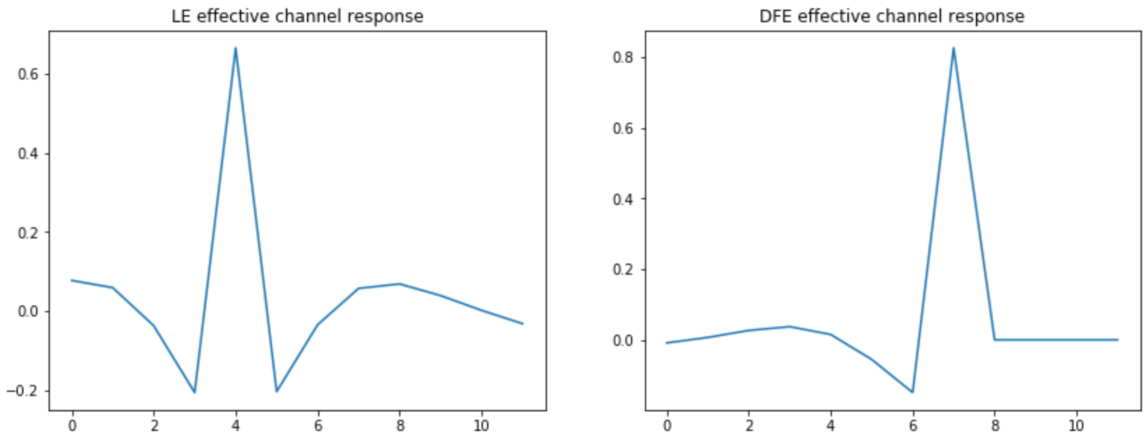
Effective channel response(from combining channel and Equalizer)

In [25]:

```
Delta_LE        = 4
N1, Delta_DFE   = 8, 7


SNR = 10**(10/10)
sigma_v2  = 1/SNR
sigma_v   = np.sqrt(sigma_v2)

R_LE,P_LE         = R_fxn(10,sigma_v2,f_chnl0),  P_fxn(10,Delta_LE,f_chnl0)
W_opt_LE          = np.matmul(np.linalg.inv(R_LE),P_LE).ravel()

R_DFE,P_DFE       = R_fxn2(N1,10-N1,sigma_v2,f_chnl0), P_fxn2(N1,10-N1,Delta_DFE,f_chnl0
)
W_opt_DFE         = np.matmul(np.linalg.inv(R_DFE),P_DFE).ravel()
Wff, Wfb          = np.hsplit(W_opt_DFE,[N1])

eff_res_LE        = np.convolve(f_chnl0, W_opt_LE)
eff_res_DFE       = np.convolve(f_chnl0, Wff)

kk  =  Delta_DFE + 1
eff_res_DFE       = np.hstack(( eff_res_DFE,np.zeros(12-len(eff_res_DFE)) ))
eff_res_DFE      -= np.hstack(( np.zeros(kk),Wfb,np.zeros(2 + N1 - kk) ))# removing post
cursor ISI by subtracting Wfb

plt.figure(figsize=(14,5))
plt.subplot(121)
plt.title('LE effective channel response')
plt.plot(eff_res_LE,label='eff_LE')
plt.subplot(122)
plt.title('DFE effective channel response')
plt.plot(eff_res_DFE,label='eff_DFE')
# plt.legend()
# plt.figure()
# plt.title('eff_res_DFE - eff_res_LE')
# plt.plot(eff_res_DFE - eff_res_LE)
```

Out[25]:

```
[<matplotlib.lines.Line2D at 0x29d8ca62438>]
```



The post cursor ISI is very less for DFE because of FB

Residual Inter symbol Interference calculation

Residual ISI = **sum of square** of coefficients**(except cursor** i.e, the coefficient with highest magnitude**)** of the effective channel

cursor is the coefficient with highest magnitude and it will be at **k = Delta**

In [26]:

```python
print('Residual ISI values are:')
for i in ['eff_res_LE', 'eff_res_DFE']:
    tmp = eval(i).copy()
    tmp[tmp == tmp.max()] = 0
    # print(tmp)
    # print(i,np.square(tmp).sum(),np.var(tmp),np.abs(tmp).sum(),(tmp).sum())
    print('{}\tISI {}'.format(i,np.square(tmp).sum()))
```

```
Residual ISI values are:
eff_res_LE      ISI 0.1059923099919701
eff_res_DFE     ISI 0.027764447085367894
```

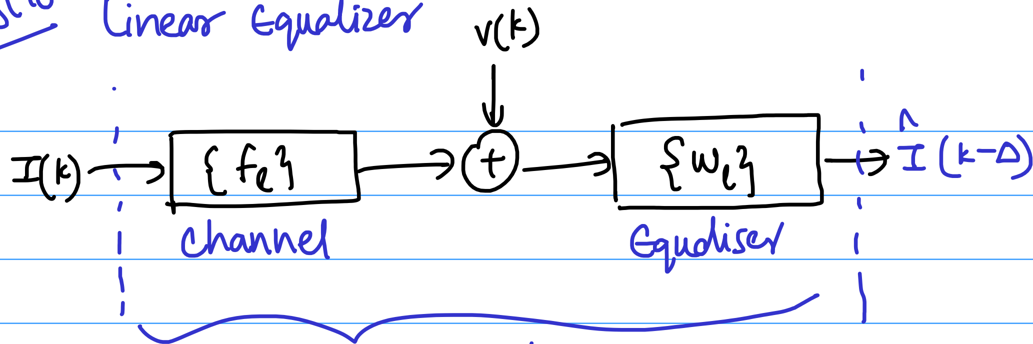
LE ISI = 0.1059923099919701

DFE ISI = 0.027764447085367894

Bonus question

EE17B061
P. Harsha vardhan

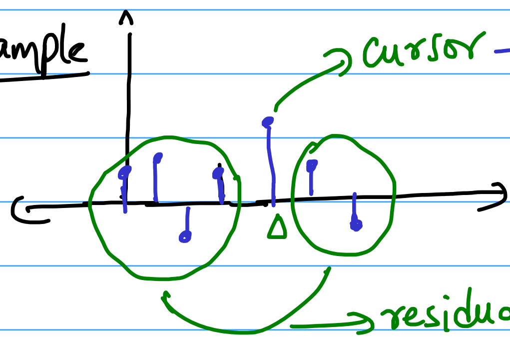
Linear Equalizer



Effective channel response
→ Supposed to be $\delta[k-\Delta]$ → discrete time impulse response
But will have non-zero values at $k=\Delta$

residual ISI = sum of squares of coefficients of effective channel at $k \neq \Delta$

Example



cursor → coefficient at Δ

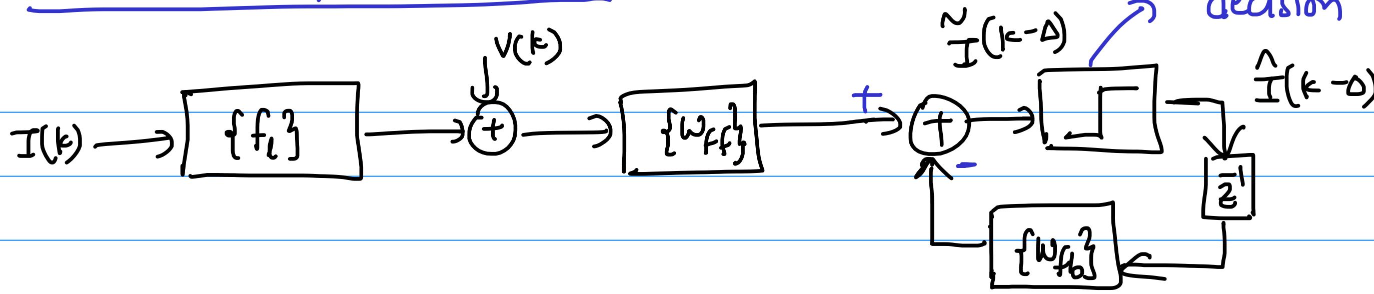
residual ISI = sum of squares of these coefficients.

Effective channel response for LE

= Channel * Equalizer
* → Convolution

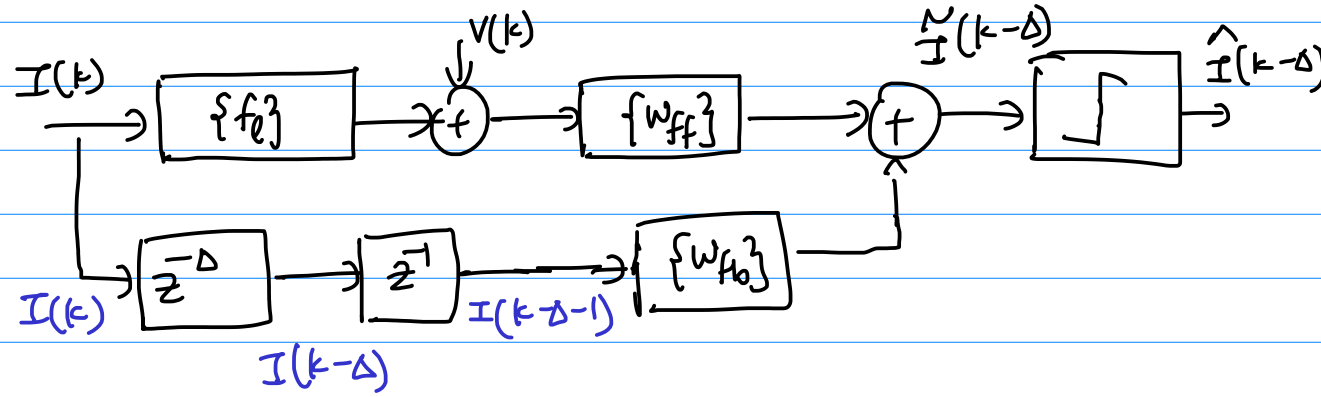
= Convolution($\{f_l\}, \{w_l\}$)

Effective impulse response for DFE



Assuming $\hat{I}(k-\Delta) = I(k-\Delta)$

we can consider equivalent impulse response as



$(\because \hat{I}(k-\Delta) = I(k-\Delta))$

effective channel response

Convolution

$$= \{f_l\} * \{w_{ff}\} - \{w_{fb}\} \times z^{-(\Delta+1)}$$

$$= \text{Convolve}(\{f_l\}, \{w_{ff}\}) - (\{w_{fb}\} \text{ delayed by } \Delta+1)$$

Ex:

$\{f_l\} * \{w_{ff}\} =$

⊖

delayed $\{w_{fb}\}$