Jethro Kuan – *April 28, 2017*

1 Java

1.1 Access Modifier

C: Class; P: Package; SC: Subclass; W: World

Modifier	C	P	SC	W
public	Y	Y	Y	Y
protected	Y	Y	Y	N
none	Y	Y	N	N
private	Y	N	N	N

1.2 Comparable

Implement `Comparable<T>`:

```
int compareTo(T o)
```

1.3 hashCode

- If two objects are `equal`, `hashCode` must return the same result
- hashCode must return the same result hen invoked on the same object more than once

```
int hashCode() {}
boolean equals(Object o) {}
```

2 Algorithmic Analysis

Amortized Cost Algo has amortized cost $T(n)$ if $\forall$ k $\in$ Z, cost of k operations is $\leq kT(n)$

2.1 Master's Theorem

$T(n) = aT(\frac{n}{b}) + f(n)$ where a $\geq$ 1, b > 1

- $f(n) \in O(n^c)$, $c < \log_b a$ then $T(n) \in \Theta(n^{\log_b a})$
- $f(n) \in O(n^c \log^k n)$, $c = \log_b a$ then $T(n) \in \Theta(n^c \log^{k+1} n)$
- $f(n) \in O(n^c), c > \log_b a$ and $\exists k$ st $af(\frac{n}{b}) \leq kf(n)$ then $T(n) \in \Theta(n^{\log_b a})$

2.1.1 Common Ones

- $T(n) = T(n/2) + \Theta(1) = O(\log n)$ (Binary Search)
- $T(n) = 2T(n/2) + \Theta(1) = O(n)$ (Binary Tree Traversal)
- $T(n) = 2T(n/2) + \Theta(\log n) = O(n)$ (Optimal Sorted Matrix Search)
- $T(n) = 2T(n/2) + O(n) = O(n \log n)$ (Merge Sort)

3 Abstract Data Types

Bag Insert(i); Draw()

Stack (LIFO) empty(); peek(); pop(); push(i)

Queue (FIFO) add(); offer(i); peek(); poll(); remove()

Dequeue double-ended queue

4 Searching

- Binary Search $O(\log n)$

```
int binarySearch(int[] arr, int key) {
    int start = 0, end = arr.length - 1;
    int found = -1;
    while(start <= end) {
        int mid = start + (end - start)/2;
        if(arr[mid] < key) {
            start = mid + 1;
        } else if(arr[mid] > key) {
            end = mid - 1;
        } else {
            found = mid;
            // if we want first instance
            end = mid - 1;
            // if we want last instance
            // start = mid + 1;
        }
    }
    return found;
}
```

One sided Binary Search Suppose one side is bounded, eg $[1, \infty)$. Use the sequence $[1,2,4,8,16\ldots, 2^k\ldots]$ If it works for 2^k, then search on $[2^{k-1}, 2^k]$

Peak Finding A[j] in array A is peak if (i) A[j] > A[j-1] (ii) A[j] > A[j+1]. If only one item in array, vacously true

1D Peak Finding $O(\log n)$ D&C

```
if a[n/2] < a[n/2-1] look at 1..n/2-1
else if a[n/2] < a[n/2+1] look at n/2+1..n
else return a[n/2]
```

2D Peak Finding $O(m+n)$ D&C

```
find max in border + cross O(m+n)
if max is peak return
else go into quadrant with higher number
```

5 Sorting

Bubble Sort Stable, In-place, W&A $O(n^2)$, B $O(n)$, S $O(1)$; Invariant : At iteration i , the sub-array A[1 .. i] is sorted and any element in A[i + 1 .. A . size] is greater or equal to any element in A[1 .. i]

Selection Sort In-Place, Unstable; find minimum element and swap. W,A,B $O(n^2)$, S $O(n/1)$; Invariant: a[0. . . i-1] is sorted all entries in a[i..n-1] are larger than or equal to the entries in a[0..i-1]

Insertion Sort In-place, Stable; W $O(n^2)$, B $O(n \log n)$, S $O(n)$; Invariant: The subarray a[i] consists of the original elements in sorted order.

Merge Sort Stable, In Place; W/B $O(n \log n)$, S $O(n)$

Quick Sort In-place, Unstable; W $O(n^2)$, A/B $O(n \log n)$ S $O(\log n)$

6 Geometric Algorithms

6.1 Jarvis March $O(hn)$

1. Find somewhere to start, e.g. y-min coordinate
2. Add point with maximum angle from horizon $O(n)$
3. Keep adding points with maximum angle from previous

6.2 Line Intersection Algorithm $O(n \log n)$

1. Divide into two equal size sets (along vertical line)
2. Recursively find convex hulls (base case 3 points)
3. Merge convex hulls
   (a) Find upper tangent lines
      i. while (u, v, w) clockwise, decrement v
      ii. while (v, w, z) clockwise, increment w
   (b) Find lower tangent lines
      i. while (w, v, u) clockwise, increment v
      ii. while (z, v, u) clockwise, decrement w

6.3 Quick Hull $O(n \log n)$

1. Choose pivot, construct two subproblems, delete interior points
2. recurse on subproblems

7 Trees

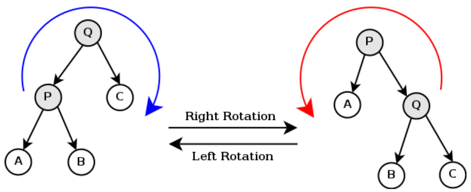
7.1 Binary Trees (height h)

$h(v) = max\,(h(v.left), h(v.right)) + 1$

- BST: left ST < key < right ST
- traversal $O(n)$ IN:LSR, PRE:SLR, POST:LRS
- insert, search, findMax, findMin: $O(h)$
- successor $O(h)$:
  - if hasRightChild, smallest node in right sub-tree
  - else, first parent node that is left child
- delete $O(h)$: switch numChild
  - 0: remove v
  - 1: remove v, connect child(v) to parent(v)
  - 2: swap with successor(v), remove(v)

7.2 AVL Trees (height $h = \log n$)

- **Property**: Every node is height-balanced
- $|v.left.height - v.right.height| \leq 1$



- insert $O(\log n)$:
  - insert key in BST

- walk up, perform max 2 rotations if out-of-balance
- delete(v): ($\log n$ rotations)
  - If v has 2 children, swap with successor
  - delete v, and reconnect children
  - for every ancestor of deleted node
    - rotate if out-of-balance
- Splay Trees: Rotate nodes that are accessed to root. consider using where operations are non-random.

7.3 Augmented Trees

7.3.1 Rank Tree (Order Statistics)

- store weight of tree in each node:
- $w(v) = w(v.left) + w(v.right) + 1$
- select(k) $O(\log n)$: finds node with rank k

```
rank = left.weight + 1;
  if (k == rank)
    return v
  else if (k < rank)
    return left.select(k)
  else return right.select(k-rank)
```

- rank(v) $O(\log n)$: computes rank of node v

```
rank = v.left.weight + 1
  while (v != null) do
    if v is left child do nothing
    if v is right child,
       rank += v.parent.left.weight + 1
    v = v.parent
```

7.3.2 Interval Trees

- Each node is an interval $(m, n), m \leq n$
- Sort by m, augment node with maximum n of children in each node
- search(x) $O(\log n)$:

```
if x in c
  return c
else if c has no left child
  search in right subtree
else if x > max endpoint in c.left
  search in right subtree
else search in left subtree
```

- findAll(x) $O(k \log n)$ for k overlapping intervals

```
search(x)
store it somewhere else
remove interval
repeat until no intervals found
```

7.3.3 Orthogonal Range Searching

1. 1D
   (a) use a binary tree search tree
   (b) store all points in the leaves of the tree, internal nodes store only copies
   (c) each internal node v stores the max of any leaf in the left subtree
   (d) Query Time: $O(k + \log n)$
   (e) Building Tree: $O(n \log n)$
2. k-dim Tree
   (a) each node in the x-tree has a set of points in its subtree
   (b) store the y-tree at each x-node containing all points
   (c) Query Time: $O(k + \log^d n)$
   (d) Building Tree: $O(n \log^{d-1} n)$
   (e) Space: $O(n \log^{d-1} n)$

8 Hash Tables

- n: #items, m: #buckets
- Simple Uniform Hashing: Keys are equally likely to map to every bucket, and are mapped independently
  - $load(ht) = \frac{n}{m}$
  - $E_{\text{search}} = 1 + \frac{n}{m}$
  - Assume $m = \Omega(n)$, $E_{\text{search}} = O(1)$

8.1 Hash Functions

8.1.1 Division

- $h(k) = k \bmod m$, choose m prime

8.1.2 Multiplication

- fix table size: $m = 2^r$, for some r
- fix word size: w, size of key in bits
- fix odd constant A, $A > 2^{w-1}$
- $h(k) = (Ak) \bmod 2^w >> (w - r)$

8.1.3 Rolling Hash

- When key changes by single character

8.2 Chaining

- bucket stores linked list, containing (object, value)
- Worst insert $O(1 + cost(h))$
- Expected search $= 1 + \frac{n}{m} = O(1)$
- Worst search $O(n)$

8.3 Open Addressing

- One item per slot, probe sequence of buckets until find only one
- $h(key, i) : U \mapsto 1..m$, i is no. of collisions
- *search*: keep probing until empty bucket, or exhausted entire table
- *delete*: set key to tombstone value, so probe sequence still works
- *insert*: on deleted cell, overwrite, else find next available slot
- good hash function:
  1. $h(key, i)$ enumerates all possible buckets
  2. Simple Uniform Hashing
- *Linear*: $h(k, i) = h(k) + i$, Clustering
- *Double*: $h(k, i) = f(k) + i \cdot g(k) \bmod m$
- Insert, Search: $\frac{1}{1-\alpha}$ where $\alpha = \frac{n}{m} \leq 1$
- good: saves space, rare mem alloc, better cache perf
- bad: sensitive to hash, load

8.4 Cuckoo Hashing

- Resolving hash collisions with worst-case constant lookup time
- Lookup: inspection of just two locations in the hash table
- Insertion: Insert in one of two cells if empty; else kick out other keys to second location.
- If infinite loop, hash function is rebuilt in place

8.5 Table resizing

- Scan old table $O(m_1)$, create new table $O(m_2)$, insert each element \$O(1), total $O(m_1 + m_2 + n)$
- $O(n)$ amor: if $n == m$, $m = 2m$, if $n < \frac{m}{4}$, $m = \frac{m}{2}$

8.6 Fingerprint Hash Table (FHT)

- Vector of 0/1 bits
- no false negatives, but has false positives. $P_{\text{no FP}} = \left(\frac{1}{e}\right)^{n/m}$

8.7 Bloom Filter

- use n hash functions. More space per item, but require n collisions for false positive.
- $P_{\text{coll}} = \left(1 - e^{-kn/m}\right)^k$
- Two hash functions, $h(k)$ and $t(k)$, two tables T_1 and T_2
- *insert*: $T_1[h(k)] = 1$, $T_2[h(k)] = 1$
- *search*: if $T_1[h(k)]$ and $T_2[h(k)]$ both 1 return true

9 Graphs

Type	Space	v,w	any	all
List	$O(V + E)$	slow	fast	fast
Mat	$O(V^2)$	fast	slow	slow

9.1 Simple search

- BFS/DFS do not explore all paths

9.1.1 BFS $O(V + E)$

```
bfs(root)
  Q.enqueue(root)

  while Q is not empty:
    current = Q.dequeue()
    visit(current)
    for each node n adj to current
      if n not visited
        n.parent = current
        Q.enqueue(n)
```

9.1.2 DFS $O(V + E)$

- Same as BFS, but use stack instead of queue

9.1.3 Topological Sort (DAG)

- Post-order DFS
- Kahn's Algorithm (first append all nodes with no incoming edges to result set, remove edges connected to these nodes and repeat, also O(V+E))

9.2 SSSP

9.2.1 Bellman-Ford $O(EV)$

```
do V number of times
  for (Edge e : graph)
    relax(e)
```

- can terminate early if no improvement
- can detect negative cycle: perform V times, then perform once more, if have changes it has negative cycle
- if all weights are the same, use BFS

9.2.2 Dijkstra $O(E \log V)$

- Doesn't work with negative edge weights
- can terminate once end is found

```
add start to PQ
dist[i] = INF for all i
dist[start] = 0
while PQ not empty
  w = pq.dequeue()
  for each edge e connected to w
    if edge is improvement
      update pq[w] O(logn)
      update dist[w]
```

9.2.3 DAG

- Toposort, relax in order

9.3 Heap

- implements priority queue, is a complete binary tree
- priority of parent > priority of child
- insert: create new leaf, `bubbleUp`
- decreaseKey: update priority, `bubbleDown`
- delete: swap with leaf, delete, and then `bubble`
- store in array:
  - $left(x) = 2x + 1$
  - $right(x) = 2x + 2$
  - $parent(x) = \lfloor (x-1)/2 \rfloor$

9.3.1 Heap Sort

1. Heapify (insert n items) O(n log n)
2. Extract from heap n times (O(n log n))
3. **Improvement**: recursively join 2 heaps and bubble root down (base case single node) O(n)
4. O(n log n) worst case, deterministic, in-place

9.3.2 UFDS (weighted)

- union(p,q) $O(\log n)$
  - find parent of p and q
  - make root of smaller tree root of larger tree
- find(k) $O(\log n)$
  - search up the tree, return the root
  - (PC): update all traversed nodes parent to root
- WU with PC, union and find $O(\alpha(m, n))$

9.4 MST

- acyclic subset of edges that connects all nodes, and has minimum weight

9.4.1 Properties

1. Cutting edge in MST results in 2 MSTs
2. **Cycle Poperty**: $\forall$ cycle, max weight edge is not in MST
3. **Cut Property**: $\forall$ partitions, min weight edge across cut is in MST

9.4.2 Prim's $O(E \log V)$

```
T = {start}
enqueue start's edges in PQ
while PQ not empty
  e = PQ.dequeue()
  if (vertex v linked with e not in T)
    T = T U {v, e}
  else
    ignore edge
MST = T
```

9.4.3 Kruskal's $O(E \log V)$

- Use union find
- connect two nodes if they are in the same blue tree

```
Sort E edges by increasing weight
T = {}
for (i = 0; i < edgeList.length; i++)
  if adding e = edgelist[i] does
  not form a cycle
    add e to T
    else ignore e
MST = T
```

9.4.4 Boruvka's $O(E \log V)$

```
T = { one-vertex trees }
While T has more than one component:
 For each component C of T:
   Begin with an empty set of edges S
   For each vertex v in C:
     Find the cheapest edge from v
     to a vertex outside of C, and
     add it to S
   Add the cheapest edge in S to T
 Combine trees connected by edges
MST = T
```

9.4.5 Variants

1. Same weight: BFS/DFS $O(E)$
2. Edges have weight $1..k$:
   - Kruskal's
     - Bucket sort Edges $O(E)$
     - Union/check $O(\alpha(V))$
     - Total cost: $O(\alpha(V)E)$
   - Prim's
     - Use array of size k as PQ, each slot holds linked list of nodes
   - insert/remove nodes $O(V)$
   - decreaseKey $O(E)$
3. Directed MST
   - $\forall$ node except root, add minimum incoming edge $O(E)$
4. MaxST
   - negate all weights, run MST algo

9.4.6 MST Problems

1. How do I add an edge (A,B) of weight k into graph G and find MST quickly?
   - Use cycle property; max edge in any cycle is not in MST
   - only add (A,B) if k is not the max weight edge
   - O(V + E) time to find max edge along A → B with DFS
2. Given an undirected graph with K power plants, find the minimum cost to connect all other sites.
   - run Prim's, use super source
   - weight of new edges are zero
   - this is a single MST
3. How do I make Kruskal run faster when sorting?
   - Store edges in separate linked lists
   - To process edges in increasing weight, process all edges in one linked list then the next
   - Time: $O(E)$ or $O(E\alpha(m, n))$
   - Space: $O(E)$, need to store all E edges
4. Minimum Bottleneck Spanning Tree (MBST)
   - General idea: If I use some edge e that is not in the MST to replace some edge e' in the MST, then my max. edge is max (max edge on original MST, e).
   - Intuitively, my MST would then fulfill the condition of MBST.
   - Note: Every MST is an MBST, but not every MBST is an MST
5. Find maximum distance between 2 vertices in MST
   - Bruteforce: perform DFS starting from every single location since there is only one path from any node to another
   - DFS: $O(V + E)$, doing it V times, $O(V(V + E)) = O(V^2)$ since $E = V - 1$
   - Space: $O(V)$, need to store all the edges in MST

9.5 Floyd-Warshall (APSP)

- Shortest paths have optimal substructure
- Shortest paths have overlapping subproblems
- Idea: gradually allow usage of intermediate vertices
- Invariant: At step k, shortest path via nodes 0 to k are correct

```
// precondition: A[i][j] contains weight
//  of edge (i,j) or inf if no edge
int[][] APSP(A) {
  // len = # vertices
  // clone A into S
  for(int k = 0; k < len; k++)
      for(int i = 0; i < len; i++)
          for(int j = 0; j < len; j++)
              S[i][j] =
                  Math.min(S[i][j],
                           S[i][k] + S[k][j]);
  return S;
}
```

9.6 Network Flow

k-edge connected Source and target are k-edge connected if there are k edge disjoint paths(don't share edges) from source to target.

Max flow st-cut property with minimum capacity(outgoing from s, ignore incoming to s)

Min cut Let S be the nodes reachable from the source in the residual graph. $T =$ all other nodes, S → T is minimum cut

Augmenting Path path in residual graph from s to t that has no 0 weight edges

9.6.1 Ford-Fulkerson

1. Start with 0 flow
2. While there exists augmenting path:
   - find an augmenting path
   - compute bottleneck (min edge)
   - increase flow on the path by bottleneck capacity

Time Complexity:

- DFS: $O(|F|E)$
- BFS(Edmonds-Karp, shortest augmenting path): $O(VE^2)$
- Dinitz: $O(V^2E)$

9.7 Graph Algorithms on Trees

9.7.1 Min Vertex Cover

- Idea: transform tree into DAG, run DP
- only two possiblities for each vertex; taken or not

```
int MVC(int v, int flag) {
    int ans = 0;
    if (memo[v][flag] != -1)
        return memo[v][flag];
    else if (leaf[v]) //if v is leaf
        ans = flag;
    else if (flag == 0) {
        ans = 0;
        for(child : adjList[v]) {
            ans+= MVC(child, 1);
        }
    }
    else if (flag == 1) {
        for (child : adjList[v]) {
            ans += min(MVC(child,1),
                       MVC(child,0));
        }
    }
}
```

9.7.2 SSSP

- On a weighted tree, any graph traversal algorithm (eg. DFS, BFS) can obtain the shortest path to any vertice in $O(V)$
- Weight of shortest path between two vertices is the sum of the weights of edges on the unique path

9.7.3 ASSP

- Run SSSP on V vertices in total $O(V^2)$, compared to $O(V^3)$ FW algorithm

9.7.4 Diameter

- Originally, run FW in $O(V^3)$ and do an $O(V^2)$ all-pairs check, to total $O(V^2)$.
- Now, only need 2 $O(V)$ traversals: DFS/BFS from any vertex s to find the furthest vertex x. Then do a DFS/BFS one more time from vertex x to find furthest vertex y. Length of unique path along x to y is the diameter of the tree.

10 Parallel Algorithms

10.1 Parallel Fibonacci

```
parallelFib(n) {
  if(n < 2) then
  return n;
  x = spawn parallelFib(n - 1);
  y = spawn parallelFib(n - 2);
  sync;
  return x + y;
}
```

- Critical Path: T_∞, Parallelism $= T_1/T_\infty$
- $T_\infty(n) = max(T_\infty(n-1), T_\infty(n-2)) + O(1) = O(n)$
- $T_p > T_1/p$
- $T_p > T_\infty$, cannot run slower on more processors
- Goal: $T_p = (T_1/p) + T_\infty$, T_1/p is the parallel part, T_∞ is the sequential part

10.2 Matrix Addition

Before: • Work analysis: $T_1(n) = O(n^2)$
• critical path analysis: $T_\infty(n) = O(n^2)$ After:

```
pMatAdd(A,B,C,i,j,n)
  if(n == 1)
    C[i,j] = A[i,j] + B[i,j];
  else:
    spawn pMatAdd(A,B,C,i,j,n/2);
    spawn pMatAdd(A,B,C,i,j + n/2,n/2);
    spawn pMatAdd(A,B,C,i + n/2,j,n/2);
    spawn pMatAdd(A,B,C,i + n/2,j + n/2,n/2);
    sync;
```

- Work Analysis: $T_1(n) = 4T_1(n/2) + O(1) = O(n^2)$
- Critical Path Analysis: $T_\infty(n) = T_\infty(n/2) + O(1) = O(\log n)$

10.3 Parallelized Merge Sort

```
pMerge(A[1..k], B[1..m], C[1..n])
  if (m > k) then pMerge(B, A, C);
  else if (n==1) then C[1] = A[1];
  else if (k==1) and (m==1) then
    if (A[1] <= B[1]) then
      C[1] = A[1]; C[2] = B[1];
    else
      C[1] = B[1]; C[2] = A[1];
  else
    // binary search for j where
    // B[j] <= A[k/2] <= B[j+1]
    spawn pMerge(A[1..k/2],
                 B[1..j],
                 C[1..k/2+j])
    spawn pMerge(A[k/2+1..l],
                 B[j+1..m],
                 C[k/2+j+1..n])
    synch;

pMergeSort(A, n)
  if (n=1) then return;
  else
    X = spawn pMergeSort(A[1..n/2], n/2)
    Y = spawn pMergeSort(A[n/2+1, n], n/2)
    synch;
    A = spawn pMerge(X, Y);
```

Critical Path Analysis:

- $T_\infty(n)$: critical path of parallel merge for A,B of length n
- $k > n/2$ elements in A, $n-k$ elements in B, $k/2 + (n-k) < 3n/4$
- $T_\infty(n)T_\infty(3n/4) + O(\log n)O(\log^2 n)$
- $T_1(n) = T_1(\alpha n) + T_1((1-\alpha)n) + O(\log n) \approx 2T_1(n/2) + O(\log n) = O(n)$
- Parallel Sorting: $T_\infty(n) = T_\infty(n/2) + O(log^2 n) = O(log^3 n)$