Concave function problem

Charles Zheng

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Problem

Find twice-differentiable f such that

- $f''(x) \leq 0$ (concavity)
- $f'(x) = -\alpha$ for $x \leq x_1$,
- $f'(x) = -\beta$ for $x \geq x_2$,
- $f(x_1) = y_1$ and $f(x_2) = y_2$.

Does such a function exist for all $y_2 > y_1$, $\alpha_1 < \alpha_2$?

Polynomial approach

Without loss of generality, let $x_1 = 0$, $x_2 = 1$, $y_1 = 0$ and define $y = y_2$.

Let us assume that a solution exists such that f is equal to a k-degree polynomial on $[0, 1]$.

$$f(x) = \sum_{i=0}^{k} a_i x^i.$$

Then the conditions on the coefficients $a_0, ..., a_k$ are as follows.

- $a_0 = 0$
- $a_1 = -\alpha$
- $a_2 = 0$
- $\sum_i a_i = y$
- $\sum_i i a_i = -\beta$
- $\sum_i i(i-1) a_i = 0$
- $\sum_i i(i-1) a_i x^{i-2} \leq 0$ for $x \in [0, 1]$.

The only free parameters are $a_3, ..., a_k$, and we have 3 linear constraints on the parameters and 1 inequality constraint. We must have at least 3 free parameters, so we need $k \geq 5$ The inequality constraint is quite difficult to handle theoretically, since it involves checking the roots of polynomials. Therefore, we will proceed on a case-by-case basis, starting with $k = 5$.

Case k=5

The conditions for $k = 5$ are written

- $a_3 + a_4 + a_5 = y + \alpha$
- $3a_3 + 4a_4 + 5a_5 = -\beta + \alpha$
- $6a_3 + 12a_4 + 20a_5 = 0$
- $6a_3 + 12a_4 x + 20a_5 x^2 \leq 0$ for $x \in [0, 1]$

By minimizing the inequality constraint, we rewrite it as follows:

Either: * $a_4 \leq 0$, $a_4 \geq -\frac{5}{3}a_5$, and

a_3, a_4, a_5 are uniquely determined by the boundary conditions.