Minimum coefficient of variation for log-concave densities

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1 Problem

Let $f(x)$ be a twice-differentiable convex function on $\mathbb{R}^+$ such that $\int_0^\infty e^{-f(x)} < \infty$. Then, if X is distributed according to

$$\Pr[X < t] = \frac{\int_0^t e^{-f(x)}dx}{\int_0^\infty e^{-f(x)}dx}$$

we say that X has a log-concave distribution on the positive real line.

Define the coefficient of variation of X by

$$\mathrm{CV}[X] = \frac{\mathbf{E}[X]}{\sqrt{\mathrm{Var}[X]}}.$$

Accordingly, define functionals

$$E(f) = \frac{\int_0^\infty xe^{-f(x)}dx}{\int_0^\infty e^{-f(x)}dx}$$

$$E_2(f) = \frac{\int_0^\infty x^2e^{-f(x)}dx}{\int_0^\infty e^{-f(x)}dx}$$

$$V(f) = E(f)^2 - E_2(f)$$

$$\mathrm{CV}(f) = \frac{E(f)}{\sqrt{V(f)}}$$

so that $E(f) = \mathbf{E}[X]$, $E_2(f) = \mathbf{E}[X^2]$, $V(f) = \mathrm{Var}(X)$ and $\mathrm{CV}(f) = \mathrm{CV}[X]$ for X defined as above.

Now consider the problem of finding the log-concave distribution with the smallest coefficient of variation, i.e.

$$\text{minimize}_f \ \text{CV}(f) \text{ subject to } f'' \geq 0.$$

Intuitively, if the above optimization has a unique solution, then it should lie on the boundary of the constraint, hence $f''(x) = 0$. This suggests the exponential distribution, corresponding to $f(x) = x/\lambda$, which has a coefficient of variation equal to 1. In the following, we will use the variational calculus to show that $f(x) = x$ is a local minimum of the optimization problem, but we do not have a proof that $f(x) = x$ is a global minimum.

2 Calculus of Variations

The calculus of variations allows one to define gradients of functionals. Given a functional $\Lambda : \mathbb{F} \to \mathbb{R}$ with function space $\mathcal{F}$ as domain , any function h in the dual space $\bar{\mathcal{F}}$ such that

$$\lim_{\epsilon \to 0} \frac{\Lambda(f + \epsilon g) - \Lambda(f)}{\epsilon} = \int h(x) g(x) dx$$

for all $g \in \mathcal{F}$ is called a gradient of Λ at f:

$$h = \nabla\Lambda(f).$$

In our problem, the gradients of the functionals E, V, and CV are given as follows:

$$\nabla E(f) = p_f(x)(E(f) - x)$$

$$\nabla V(f) = p_f(x)(E_2(f) - 2E(f)^2 + 2E(f)x - x^2)$$

$$\nabla \text{CV}(f) = p_f(x) \left[\frac{E(f) - x}{\sqrt{V(f)}} - \frac{E(f)}{2V(f)^{3/2}} \left(E_2(f) - 2E(f)^2 + 2E(f)x - x^2 \right) \right].$$

where

$$p_f(x) = \frac{e^{-f(x)}}{\int_0^\infty e^{-f(z)} dz}.$$

Letting $h_f = \nabla\text{CV}(f)$, in order to show that f is a local minimum, it suffices to show that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}(\text{CV}(f + \epsilon g) - \text{CV}(f)) > 0,$$

i.e. that

$$\int_0^\infty h_f(x)g(x) \geq 0$$

for any g such that $g''(x) \geq 0$ for all x such that $f''(x) = 0$.

Define $H_f(x) = \int_0^x h_f(z)dz$ and $\boldsymbol{H}(x) = \int_0^x H_f(z)dz$. Then we for any $t > 0$, applying integration by parts we have

$$\int_0^t h_f(x)g(x)dx = H_f(x)g(x)|_0^t - \boldsymbol{H}_f(x)g'(x)|_0^t + \int_0^t \boldsymbol{H}_f(x)g''(x)dx.$$

Hence, supposing that $H_f(0) = 0$, $\boldsymbol{H}_f(0) = 0$, and

$$0 = \lim_{x\to\infty} H_f(x)g(x) = \lim_{x\to\infty} \boldsymbol{H}_f(x)g'(x)$$

we have

$$\int_0^\infty h_f(x)g(x) = \int_0^\infty \boldsymbol{H}_f(x)g''(x).$$

This motivates the following lemma:

Lemma. *Suppose that*

$$0 = \lim_{x\to\infty} xH_f(x) = \lim_{x\to\infty} \boldsymbol{H}_f(x),$$

and also that there exists $x^* > 0$ *such that* $\inf_{x\geq x^*} h_f(x) \geq 0$. *Then*

$$\inf_{g:g''\geq 0} \int_0^\infty g(x)h_f(x)dx < 0.$$

implies

$$\inf_{g:g''\geq 0} \int_0^\infty g''(x)\boldsymbol{H}_f(x)dx < 0$$

Proof. Suppose that

$$\inf_{g:g''\geq 0} \int_0^\infty g(x)h_f(x)dx < 0.$$

Then there exists g with $g'' \geq 0$ such that

$$\int_0^\infty g(x)h_f(x)dx = \delta < 0.$$

Now consider piecewise functions $\tilde{g}$ of the form

$$\tilde{g}(x) = \begin{cases} g(x) & \text{for } x < t^* \\ j(x) & \text{for } x \in [t^*, t^* + \epsilon] \\ j(t^* + \epsilon) + (x - t^* - \epsilon)g'(t^* + \epsilon) & \text{for } x \geq t^* + \epsilon \end{cases}$$

where j is chosen so that $j(x) \leq g(x)$ and so that $\tilde{g}$ is twice-differentiable and convex, Since $\tilde{g}(x) \leq g(x)$ for $x \geq t^*$, and also since $h_f(x) \geq 0$ for $x \geq t^*$, we conclude that $\tilde{g}(x)h_f(x) \leq g(x)h_f(x)$ pointwise, and hence

$$\int_0^\infty \tilde{g}(x)h_f(x)dx \leq \int_0^\infty g(x)h_f(x) < 0.$$

Furthermore, we can modify $\tilde{g}$ to be twice-differentiable while preserving the above property. Hence we conclude that there exists $\tilde{g}$ with

$$\int_0^\infty \tilde{g}(x)h_f(x)dx < 0.$$

and also

$$0 = \lim_{x\to\infty} H_f(x)\tilde{g}(x) = \lim_{x\to\infty} \boldsymbol{H}_f(x)\tilde{g}'(x).$$

This allows us to conclude that

$$\int_0^\infty \tilde{g}(x)h_f(x)dx = \int_0^\infty \tilde{g}''(x)\boldsymbol{H}_f(x)dx < 0,$$

completing the proof. □

2.1 Exponential distribution

For $f(x) = x$, the exponential distribution, we have

$$h_f(x) = \frac{1}{2}e^{-x}(x^2 - 4x + 2)$$

$$H_f(x) = \frac{1}{2}e^{-x}(2x - x^2)$$

$$\boldsymbol{H}_f(x) = \frac{1}{2}e^{-x}(x^2 + 4x).$$

Since $\boldsymbol{H}_f(x) \geq 0$, it is clear that

$$\inf_{g:g''\geq 0} \int_0^\infty \boldsymbol{H}_f(x)g''(x)dx \geq 0.$$

Now since $H_f(x)$ and $\boldsymbol{H}_f(x)$ satisfy the conditions of the lemma, we can conclude from the contrapositive of the lemma that

$$\inf_{g:g''\geq 0} \int_0^\infty h_f(x)g(x)dx \geq 0,$$

which thus implies that $f(x) = x$ is a local minimum.