A functional MRI mind-reading game

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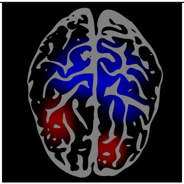
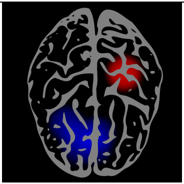
Section 1

Introduction

Functional MRI

Stimuli	Response
	
	

Functional MRI

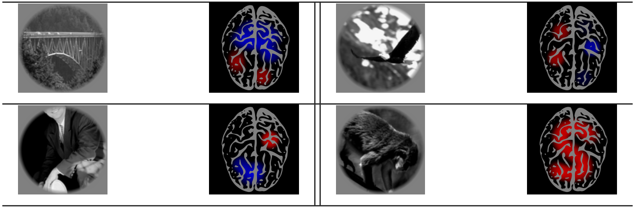
Stimuli x	Response y
$\begin{pmatrix} 1.0 \\ 0 \\ 3.0 \\ 0 \\ -1.2 \end{pmatrix}$	$\begin{pmatrix} 1.2 \\ 0 \\ -1.8 \\ -1.2 \end{pmatrix}$
$\begin{pmatrix} 0 \\ -2.2 \\ -3.1 \\ 4.5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1.2 \\ -1.9 \\ 0.5 \\ 0.6 \end{pmatrix}$

Encoding vs Decoding

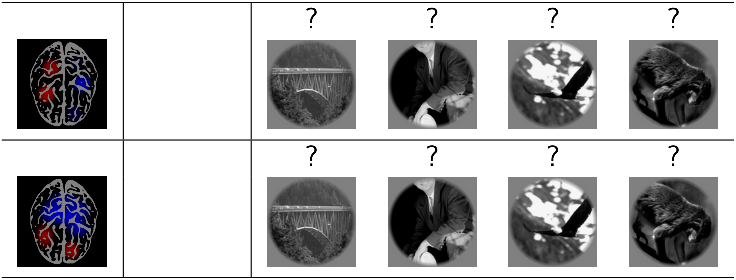
- Encoding: predict y from x.
- Decoding: reconstruct x from y (mind-reading).

A mind-reading game: Classification

Training Data

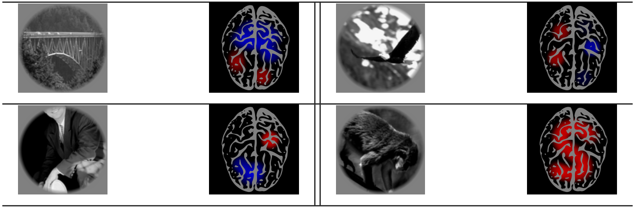


Test Data

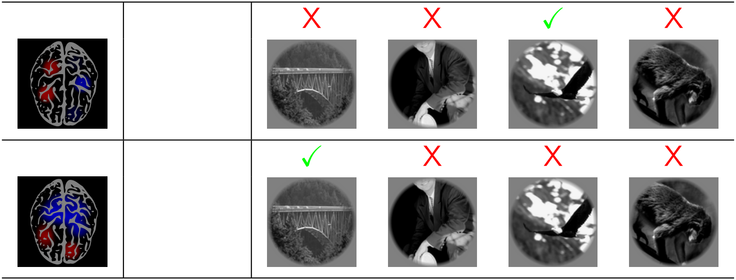


A mind-reading game: Classification

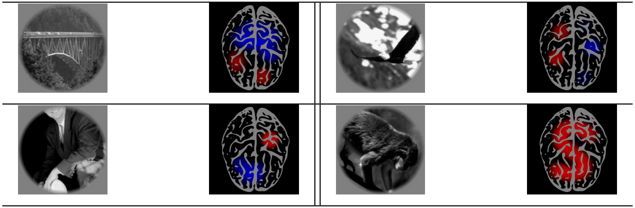
Training Data



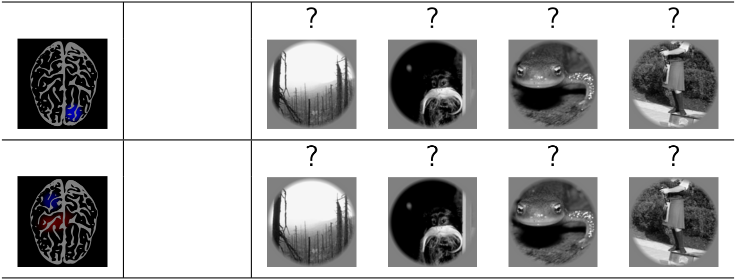
Test Data



Training Data



Test Data *(new images!)*



Statistical formulation I

Training data.

- Given training classes $S_{\text{train}} = \{\text{train:}1, \ldots, \text{train:}k\}$ where each class train:i has features $x_{\text{train:}i}$.
- For $t = 1, \ldots, T_{\text{train}}$, choose class label $z_{\text{train:}t} \in S_{\text{train}}$; generate

$$y_{\text{train:}t} = f(x_{z_{\text{train:}t}}) + \epsilon_t$$

  where f is an unknown function, and ϵ_t is i.i.d. from a known or unknown distribution.

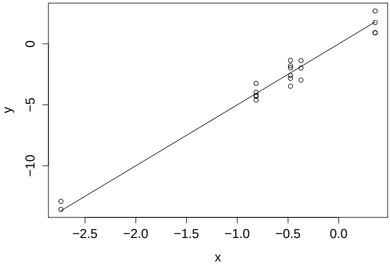
Test data.

- Given test stimuli $S_{\text{test}} = \{\text{test:}1, \ldots, \text{test:}\ell\}$ with features $\{x_{\text{test:}1}, \ldots, x_{\text{test:}\ell}\}$
- Task: for $t = 1, \ldots, T_{\text{test}}$, label $y_{\text{test:}t}$ by stimulus $\hat{z}_{\text{test:}t} \in S_{\text{train}}$; try to minimize misclassification rate

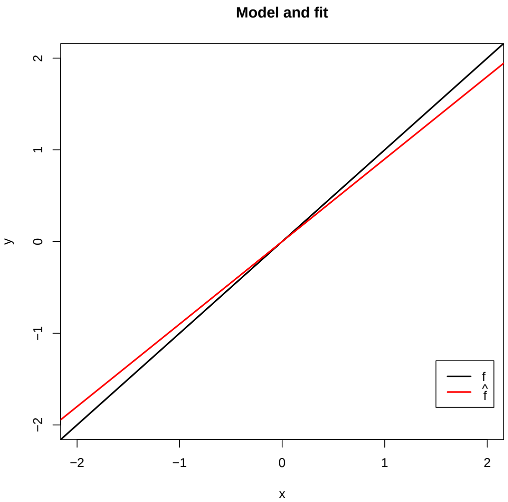
Statistical formulation II

- f is an unknown function
- P is a known or unknown distribution over image features
- *Training data.* Draw $x_{\text{train}:i} \sim P$ for $i = 1\, hdots, k$.
- *Test data.* Draw $x_{\text{train}:i} \sim P$ for $i = 1\, hdots, \ell$.
- Theoretical question: Analyze average misclassification rate when classes are generated this way

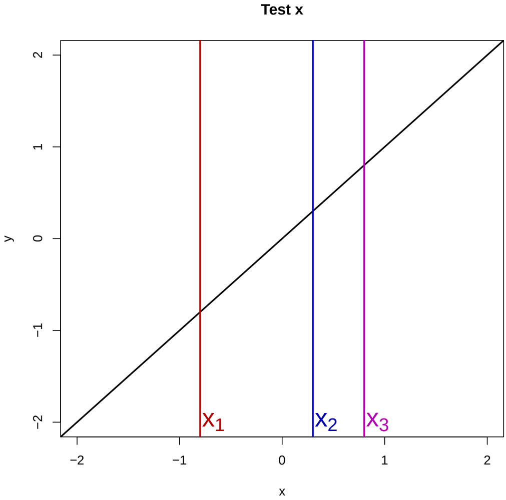
Toy example I



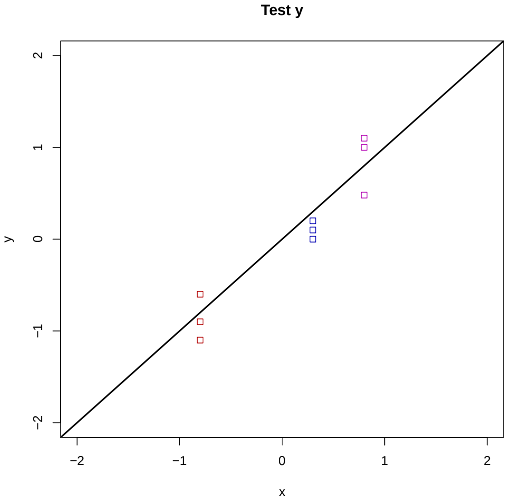
- Features x are one-dimensional real numbers, as are responses y. Parameter β is also a real number.
- Model is linear: $y \sim N(x\beta, \sigma_\epsilon^2)$



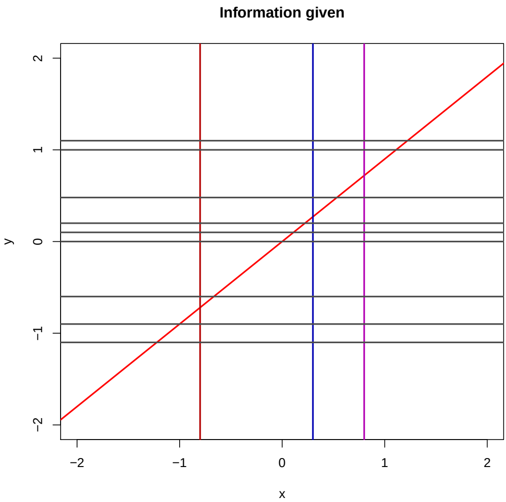
Suppose we estimated $\hat{\beta}$ from training data.



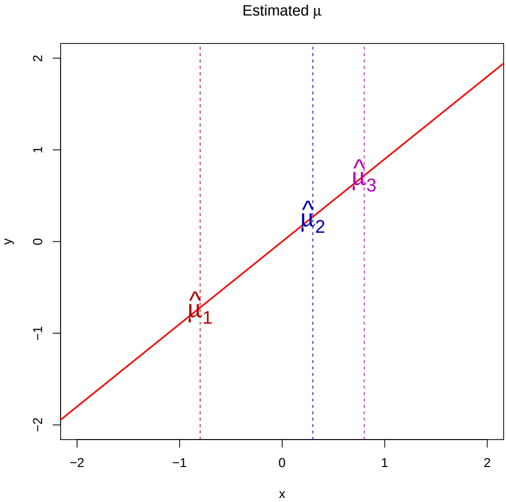
Generate features $x_{\text{test}:1}, \ldots, x_{\text{test}:\ell}$ iid $N(0, \sigma_x^2)$.



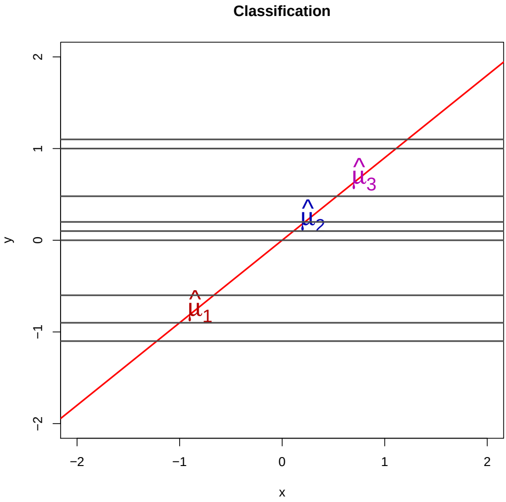
Hidden labels $z_{\text{test}:t}$ are iid uniform from S_{train}.
Generate $y_{\text{test}:t} \sim N(\beta x_{z_{\text{test}:t}}, \sigma_\epsilon^2)$



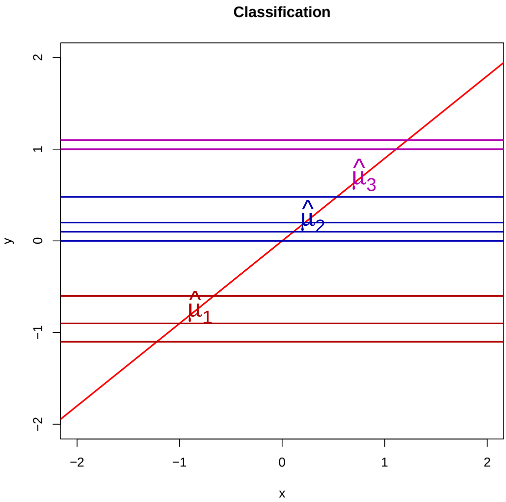
Classify $\hat{y}_{\text{test}:t}$



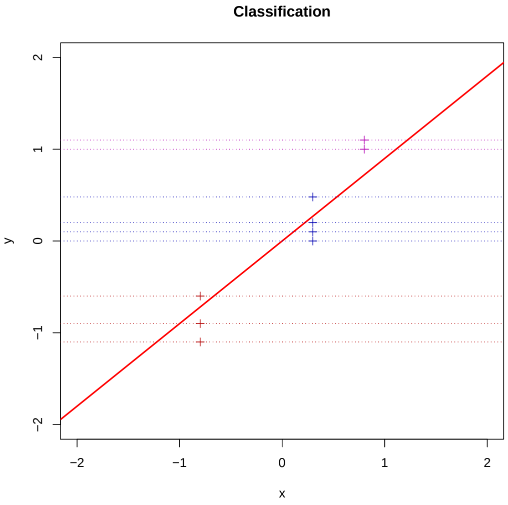
$$\hat{\mu}_{\text{test}:i} = \hat{\beta} x_{\text{test}:i}$$

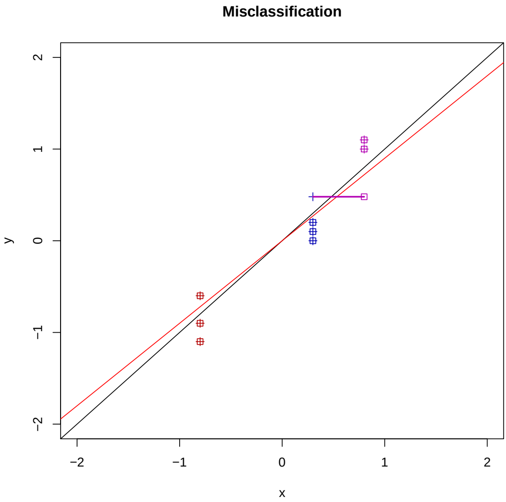


$$\hat{z}_{\text{test}:t} = \text{argmin}_z \ell_{\hat{\mu}_z}(y_{\text{test}:t})$$

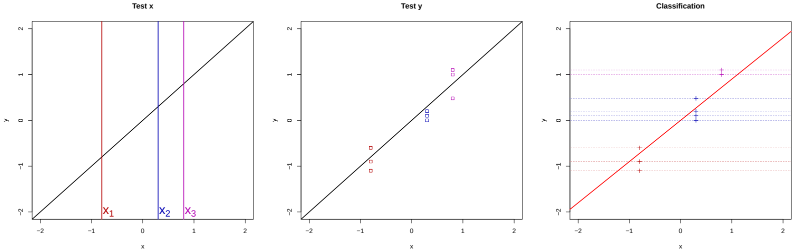


$$\hat{z}_{\text{test}:t} = \text{argmin}_z (\hat{\mu}_z - y_{\text{test}:t})^2$$


Classification
y
x
−2
−1
0
1
2
−2
−1
0
1
2


Misclassification
y
x
−2
−1
0
1
2
−2
−1
0
1
2

Toy example I



- Generate features $x_{\text{test}:1}, \ldots, x_{\text{test}:\ell}$ iid $N(0, \sigma_x^2)$.
- Hidden labels $z_{\text{test}:t}$ are iid uniform from S_{train}. Generate $y_{\text{test}:t} \sim N(\beta x_{z_{\text{test}:t}}, \sigma_\epsilon^2)$
- Classify $\hat{y}_{\text{test}:t}$ by maximum likelihood assuming $\hat{\beta}$ is correct. Thus:

$$\hat{z}_{\text{test}:t} = \text{argmin}_z (\hat{\beta} x_z - y_{\text{test}:t})^2$$

Toy example I: Questions

1. We know the prediction error is minimized when $\hat{\beta} = \beta$. Is it also true that misclassification error in the mind-reading game is minimized when $\hat{\beta} = \beta$?
2. Even if the answer to 1. is yes, should we estimate $\hat{\beta}$ using the same methods as in least-squares regression?

Toy example I: Analysis

- The expected misclassification error is the same if we take $T_{\text{test}} = 1$. Then let (x_*, y_*) be the feature-response pair in the test set, where

$$y_* = x_* \beta + \epsilon_*$$

- Denote the features for the incorrect classes as $x_1, \ldots, x_{\ell-1}$.

Ignore the possibility of ties. The response y_* is misclassified if and only if

$$\min_{i=1,\ldots,\ell-1} |y_* - x_i\hat{\beta}| < |y_* - x_*\hat{\beta}|$$

equivalently

$$\cup_{i=1,\ldots,\ell-1} E_i$$

where E_i is the event

$$|x_*\beta + \epsilon_* - x_i\hat{\beta}| < |x_*(\beta - \hat{\beta}) + \epsilon_*|$$

with probability

$$\Pr[E_i] = \left| \Phi\left(\frac{x_*\hat{\beta}}{\sqrt{\hat{\beta}\sigma_x^2}}\right) - \Phi\left(\frac{x_*(2\beta - \hat{\beta}) + 2\epsilon_*}{\sqrt{\hat{\beta}\sigma_x^2}}\right) \right|$$

Toy example I: Analysis

- Use the following conditioning

$$\mathbf{E}[\text{misclassification}] = \mathbf{E}[\mathbf{E}[\Pr_{x_1,\ldots,x_\ell}[\cup_i E_i]|x_* = x, \epsilon_* = \epsilon]]$$

- An exact expression for expected misclassification is therefore

$$1 - \int_x \int_\epsilon \left(1 - \left|\Phi\left(\frac{x\hat{\beta}}{\sqrt{\hat{\beta}\sigma_x^2}}\right) - \Phi\left(\frac{x(2\beta-\hat{\beta})+2\epsilon}{\sqrt{\hat{\beta}\sigma_x^2}}\right)\right|\right)^{\ell-1} d\Phi(\tfrac{\epsilon}{\sigma_\epsilon})d\Phi(\tfrac{x}{\sigma_x})$$

- Question 1: Is this minimized at $\hat{\beta} = \beta$?