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1 Background

Theorem (Borell-TIS inequality) Let f_t be a gaussian process such that $\mathbb{E}[f_t] = 0$ Then on any set D, and $u > 0$,

$$\mathbb{P}[\sup_D f_t > u + \mathbb{E}[\sup_D f_t]] \leq \exp(-u^2/(2\sigma^2_{max}))$$

where

$$\sigma_{max} = \sup_D \mathbf{E}[f_t^2]$$

Theorem (Slepian's inequality) If f and g are as bounded, centered gaussian processes, and

$$\mathbb{E}[(f_t - f_s)^2] \leq \mathbb{E}[(g_t - g_s)^2]$$

then

$$\mathbb{P}[\sup_{t \in D} f_t > u] \leq \mathbb{P}[\sup_{t \in D} g_t > u]$$

Theorem. Let f be centered stationary Gaussian process on a compact group T. Then the following three conditions are equivalent: (i) f_t is continuous (ii) f_t is bounded (iii)

$$\int_0^\infty \sqrt{H(\varepsilon)} d\varepsilon < \infty$$

2 Supremum of an isotropic GP

Let f_t be a gaussian process on $\mathbb{R}^D$, with $\mathrm{Cov}(f_t, f_u) = C(t-u)$, where $C(0) = 1$, and $C(t) \to 0$ as $||t|| \to \infty$. Then if f_t is bounded on an interval,

$$\mathbb{P}\left(\lim_{T \to \infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} = 1\right) = 1$$

Sketch of proof:

1. Lower bound.

Fix T. Let $\delta = exp(\sqrt{\log(T)})$ and consider $t_1, ..., t_{(2T/\delta)^D}$ on a square lattice of spacing δ on $[-T, T]^D$. Let $Z_{t_1}, \ldots, Z_{t_{(2T/\delta)^D}}$ iid $N(0, 1 - C(\delta))$. By

Slepian's inequality, $\mathbb{P}(\max_{t_1,\ldots,t_{(2T/\delta)^D}} f_t > u) \geq \mathbb{P}(\max_{t_1,\ldots,t_{(2T/\delta)^D}} Z_t > u)$. Now note that as $T \to \infty$,

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{\max Z_t}{\sqrt{2D\log(T)}} = 1\right) = 1$$

This suggests, and with some more detailed analysis, implies that

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} \geq 1\right) = 1$$

1. Upper bound.

Partition $[-T,T]^D$ into hypercubes of edge length 1. By union bound and Borrell-TIS inequality,

$$\mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D \mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D e^{-u^2/2}$$

This can be used to show

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} \leq 1\right) = 1$$

3 One-dimensional isotropic gaussian processes

Now that

$$C_\kappa(t) = [1 - \kappa|t|]_+$$

is a covariance kernel. This implies the following:

Suppose $C(0) = 1$, $-C''(t) > 0$ for $t > 0$ and $-\int_0^\infty tC'(t) < \infty$. Then $C(t)$ is a covariance kernel for a gaussian process. This is because one can find a mixture density $\rho(k)$ such that

$$C(t) = \int_0^\infty C_\kappa(t)\rho(\kappa)d\kappa$$

4 Nonexistence of unbounded isotropic GP

We attempt to show that if $C(t) = 0$, $C(t) \to \infty$ as $t \to \infty$, and $C(t) > 0$ for $t \neq 0$, that f_t is continuous and therefore bounded on an interval. Otherwise,

$$\int_0^\infty \sqrt{H(\varepsilon)}d\varepsilon \to \infty$$

This in turn implies something like

$$\lambda(\{t : C(t) > 1 - \epsilon\}) \sim \exp(-\exp(1/t))$$

We can try to derive a contradiction. Let $\rho(\kappa)$ be the reflected Fourier transform of $C(t)$, so

$$C(t) = \int_0^\infty \rho(\kappa)\cos(\kappa t)$$

Since $C(0) = 1$, $\rho(\kappa)$ is a probability distribution. We know that ρ has no first moment because $C(t)$ is nondifferentiable at 0.

It must be that $-C'(t) \geq \mathrm{const}/t^2$ as $t \to 0$. Meanwhile

$$-C'(t) = \int_0^\infty \rho(\kappa)\kappa\sin(\kappa t)d\kappa$$

We have to show that the intergral on the right can't possible tend to const/t as $t \to 0$.

We know that $\kappa\sin(\kappa t)$ is positive from 0 to $\frac{\pi}{t}$. Hence we have

$$\int_0^{\frac{\pi}{t}} \kappa\sin(\kappa t)\rho(\kappa)d\kappa \sim \mathrm{const}1/t$$

for t small. It remains to show that the rest of the integral

$$\int_{\frac{\pi}{t}}^\infty \kappa\sin(\kappa t)\rho(\kappa)d\kappa \sim \mathrm{const}1/t$$

can't be too large. For any particular t in $[0, \delta]$ it could be large, but it may be possible to show that the average value of the above interval is small for $t \in [\delta/2, \delta]$, say. Then we would have to show that this suffices to establish that $\int_0^\infty \sqrt{H(\varepsilon)}d\varepsilon \to \infty$.

5 References

Adler RF, Taylor J. *Random Fields and Geometry*