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1 Background

Theorem (Borell-TIS inequality) Let f_t be a gaussian process such that $\mathbb{E}[f_t] = 0$ Then on any set D, and $u > 0$,

$$\mathbb{P}[\sup_D f_t > u + \mathbb{E}[\sup_D f_t]] \leq \exp(-u^2/(2\sigma^2_{max}))$$

where

$$\sigma_{max} = \sup_D \mathbf{E}[f_t^2]$$

2 Supremum of an isotropic GP

Let f_t be a gaussian process on $\mathbb{R}^D$, with $\text{Cov}(f_t, f_u) = C(t-u)$, where $C(0) = 1$, and $C(t) \to 0$ as $||t|| \to \infty$. Then if f_t is bounded on an interval,

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} = 1\right) = 1$$

Sketch of proof:

1. Lower bound.

Fix T. Let $\delta = exp(\sqrt{\log(T)})$ and consider $t_1, ..., t_{(2T/\delta)^D}$ on a square lattice of spacing δ on $[-T,T]^D$. Let $Z_{t_1}, \ldots, Z_{t_{(2T/\delta)^D}}$ iid $N(0, 1-C(\delta))$. By Slepian's inequality, $\mathbb{P}(\max_{t_1,\ldots,t_{(2T/\delta)^D}} f_t > u) \geq \mathbb{P}(\max_{t_1,\ldots,t_{(2T/\delta)^D}} Z_t > u)$. Now note that as $T \to \infty$,

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{\max Z_t}{\sqrt{2D\log(T)}} = 1\right) = 1$$

This suggests, and with some more detailed analysis, implies that

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} \geq 1\right) = 1$$

1. Upper bound.

Partition $[-T,T]^D$ into hypercubes of edge length 1. By union bound and Borell-TIS inequality,

$$\mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D \mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D e^{-u^2/2}$$

This can be used to show

$$\mathbb{P}\left(\limsup_{T\to\infty}\frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}}\leq 1\right)=1$$

3 Nonexistence of unbounded isotropic GP