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1 Background

Theorem (Borell-TIS inequality) Let f_t be a gaussian process such that $\mathbb{E}[f_t] = 0$ Then on any set D, and $u > 0$,

$$\mathbb{P}[\sup_D f_t > u + \mathbb{E}[\sup_D f_t]] \leq \exp(-u^2/(2\sigma_{max}^2))$$

where

$$\sigma_{max} = \sup_D \mathbf{E}[f_t^2]$$

Theorem (Slepian's inequality) If f and g are as bounded, centered gaussian processes, and

$$\mathbb{E}[(f_t - f_s)^2] \leq \mathbb{E}[(g_t - g_s)^2]$$

then

$$\mathbb{P}[\sup_{t\in D} f_t > u] \leq \mathbb{P}[\sup_{t\in D} g_t > u]$$

Theorem. Let f be centered stationary Gaussian process on a compact group T. Then the following three conditions are equivalent: (i) f_t is continuous (ii) f_t is bounded (iii)

$$\int_0^\infty \sqrt{H(\varepsilon)} d\varepsilon < \infty$$

2 Isotropic GPs are locally bounded

Take centered isotropic f_t on the real line with $C(t) > 0$ for $t \neq 0$. We will show that f_t is bounded on the interval $[0, T]$. Let $\bar{C}(t) = 1 - C(t)$.

Otherwise, it follows from theorem 1 that for some constant C and $\varepsilon_0 > 0$, we have

$$H(\varepsilon) > \frac{C}{\varepsilon^4}$$

for all $\varepsilon < \varepsilon_0$. We have

$$H(\varepsilon) = \log(N(\varepsilon)) \leq \log(T) - \log(\lambda(\{t : C(t) > 1 - \frac{\varepsilon}{2}\}))$$

Therefore for some $\epsilon_0 > 0$, some other constant C,

$$\lambda(\{t : \bar{C}(t) < \epsilon\}) < \exp(-\frac{C}{\epsilon^2})$$

for all $\epsilon < \epsilon_0$.

Yet, from the triangle inequality, we have

$$||f_{-t} - f_0||_2 + ||f_0 - f_t||_2 \leq ||f_{-t} - f_t||_2$$

or

$$2\sqrt{\bar{C}(t)} \leq \sqrt{\bar{C}(2t)}$$

so $4\bar{C}(t) \geq \bar{C}(2t)$ for all $t > 0$.

This means that

$$\lambda(\{t : \bar{C}(t) < \epsilon\}) \geq 2\lambda(\{t : \bar{C}(t) < 4\epsilon\})$$

for sufficiently small t.

3 Supremum of an isotropic GP

Let f_t be a gaussian process on $\mathbb{R}^D$, with $\mathrm{Cov}(f_t, f_u) = C(t-u)$, where $C(0) = 1$, and $C(t) \to 0$ as $||t|| \to \infty$. Then since we know f_t is bounded on an interval,

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} = 1\right) = 1$$

Sketch of proof:

1. Lower bound.

Fix T. Let $\delta = exp(\sqrt{\log(T)})$ and consider $t_1, ..., t_{(2T/\delta)^D}$ on a square lattice of spacing δ on $[-T,T]^D$. Let $Z_{t_1}, \ldots, Z_{t_{(2T/\delta)^D}}$ iid $N(0, 1-C(\delta))$. By Slepian's inequality, $\mathbb{P}(\max_{t_1,...,t_{(2T/\delta)^D}} f_t > u) \geq \mathbb{P}(\max_{t_1,...,t_{(2T/\delta)^D}} Z_t > u)$. Now note that as $T \to \infty$,

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{\max Z_t}{\sqrt{2D\log(T)}} = 1\right) = 1$$

This suggests, and with some more detailed analysis, implies that

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} \geq 1\right) = 1$$

1. Upper bound.

Partition $[-T,T]^D$ into hypercubes of edge length 1. By union bound and Borrell-TIS inequality,

$$\mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D \mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D e^{-u^2/2}$$

This can be used to show

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} \leq 1\right) = 1$$

4 One-dimensional isotropic gaussian processes

Now that

$$C_\kappa(t) = [1-\kappa|t|]_+$$

is a covariance kernel. This implies the following:

Suppose $C(0) = 1$, $-C''(t) > 0$ for $t > 0$ and $-\int_0^\infty tC'(t) < \infty$. Then $C(t)$ is a covariance kernel for a gaussian process. This is because one can find a mixture density $\rho(k)$ such that

$$C(t) = \int_0^\infty C_\kappa(t)\rho(\kappa)d\kappa$$

5 References

Adler RF, Taylor J. *Random Fields and Geometry*