

Minimum coefficient of variation for log-concave densities

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1 Problem

Let $f(x)$ be a twice-differentiable convex function on \mathbb{R}^+ such that $\int_0^\infty e^{-f(x)} dx < \infty$. Then, if X is distributed according to

$$\Pr[X < t] = \frac{\int_0^t e^{-f(x)} dx}{\int_0^\infty e^{-f(x)} dx}$$

we say that X has a log-concave distribution on the positive real line.

Define the coefficient of variation of X by

$$\text{CV}[X] = \frac{\mathbf{E}[X]}{\sqrt{\text{Var}[X]}}.$$

Accordingly, define functionals

$$E(f) = \frac{\int_0^\infty x e^{-f(x)} dx}{\int_0^\infty e^{-f(x)} dx}$$

$$E_2(f) = \frac{\int_0^\infty x^2 e^{-f(x)} dx}{\int_0^\infty e^{-f(x)} dx}$$

$$V(f) = E(f)^2 - E_2(f)$$

$$\text{CV}(f) = \frac{E(f)}{\sqrt{V(f)}}$$

so that $E(f) = \mathbf{E}[X]$, $E_2(f) = \mathbf{E}[X^2]$, $V(f) = \text{Var}(X)$ and $\text{CV}(f) = \text{CV}[X]$ for X defined as above.

Now consider the problem of finding the log-concave distribution with the smallest coefficient of variation, i.e.

$$\text{minimize}_f \text{CV}(f) \text{ subject to } f'' \geq 0.$$

Intuitively, if the above optimization has a unique solution, then it should lie on the boundary of the constraint, hence $f''(x) = 0$. This suggests the exponential distribution, corresponding to $f(x) = x/\lambda$, which has a coefficient of variation equal to 1. In the following, we will use the variational calculus to show that $f(x) = x$ is a local minimum of the optimization problem, but we do not have a proof that $f(x) = x$ is a global minimum.

2 Calculus of Variations

The calculus of variations allows one to define gradients of functionals. Given a functional $\Lambda : \mathbb{F} \rightarrow \mathbb{R}$ with function space \mathcal{F} as domain, any function h in the dual space \mathcal{F} such that

$$\lim_{\epsilon \rightarrow 0} \frac{\Lambda(f + \epsilon g) - \Lambda(f)}{\epsilon} = \int h(x)g(x)dx$$

for all $g \in \mathcal{F}$ is called a gradient of Λ at f :

$$h = \nabla \Lambda(f).$$

In our problem, the gradients of the functionals E , V , and CV are given as follows:

$$\begin{aligned} \nabla E(f) &= p_f(x)(E(f) - x) \\ \nabla V(f) &= p_f(x)(E_2(f) - 2E(f)^2 + 2E(f)x - x^2) \\ \nabla \text{CV}(f) &= p_f(x) \left[\frac{E(f) - x}{\sqrt{V(f)}} - \frac{E(f)}{2V(f)^{3/2}} (E_2(f) - 2E(f)^2 + 2E(f)x - x^2) \right]. \end{aligned}$$

where

$$p_f(x) = \frac{e^{-f(x)}}{\int_0^\infty e^{-f(z)}dz}.$$

Letting $h_f = \nabla \text{CV}(f)$, in order to show that f is a local minimum, it suffices to show that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\text{CV}(f + \epsilon g) - \text{CV}(f)) > 0,$$

i.e. that

$$\int_0^\infty h_f(x)g(x) \geq 0$$

for any g such that $g''(x) \geq 0$ for all x such that $f''(x) = 0$.

Define $H_f(x) = \int_0^x h_f(z)dz$ and $\mathbf{H}(x) = \int_0^x H_f(z)dz$. Then we for any $t > 0$, applying integration by parts we have

$$\int_0^t h_f(x)g(x)dx = H_f(x)g(x)|_0^t - \mathbf{H}_f(x)g'(x)|_0^t + \int_0^t \mathbf{H}_f(x)g''(x)dx.$$

Hence, supposing that $H_f(0) = 0$, $\mathbf{H}_f(0) = 0$, and

$$0 = \lim_{x \rightarrow \infty} H_f(x)g(x) = \lim_{x \rightarrow \infty} \mathbf{H}_f(x)g''(x)$$

we have

$$\int_0^\infty h_f(x)g(x) = \int_0^\infty \mathbf{H}_f(x)g''(x).$$

This motivates the following lemma:

Lemma. *Suppose that*

$$0 = \lim_{x \rightarrow \infty} H_f(x) = \lim_{x \rightarrow \infty} \mathbf{H}_f(x).$$

Then

$$\inf_{g: g'' \geq 0} \int_0^\infty g(x)h_f(x)dx < 0.$$

implies

$$\inf_{g: g'' \geq 0} \int_0^\infty g''(x)\mathbf{H}_f(x)dx < 0$$

Proof. Suppose that

$$\inf_{g: g'' \geq 0} \int_0^\infty g(x)h_f(x)dx < 0.$$

Then there exists g with $g'' \geq 0$ such that

$$\int_0^\infty g(x)h_f(x)dx = \delta < 0.$$

Then by continuity of the integral, there exists some $t > 0$ where

$$\int_0^t g(x)h_f(x)dx = \delta/2 = \int_0^\infty g(x)I(x \leq t)h_f(x)dx$$

But observe that we can construct twice-differentiable and convex \tilde{g} such that $\tilde{g}(x)$ and $g(x)I(x \leq t)$ are arbitrarily close. Hence we conclude that there exists \tilde{g} with

$$\int_0^\infty g(x)h_f(x)dx < 0.$$

and

$$0 = \lim_{x \rightarrow \infty} H_f(x)\tilde{g}(x) = \lim_{x \rightarrow \infty} \mathbf{H}_f(x)\tilde{g}(x),$$

which implies

$$\int_0^\infty \tilde{g}''(x)\mathbf{H}_f(x)dx < 0,$$

completing the proof. \square

2.1 Exponential distribution

For $f(x) = x$, the exponential distribution, we have

$$h_f(x) = \frac{1}{2}e^{-x}(x^2 - 4x + 2)$$

$$H_f(x) = \frac{1}{2}e^{-x}(2x - x^2)$$

$$\mathbf{H}_f(x) = \frac{1}{2}e^{-x}(x^2 + 4x).$$

Since $\mathbf{H}_f(x) \geq 0$, it is clear that

$$\inf_{g: g'' \geq 0} \int_0^\infty \mathbf{H}_f(x)g''(x)dx \geq 0.$$

Now since $H_f(x)$ and $\mathbf{H}_f(x)$ satisfy the conditions of the lemma, we can conclude from the contrapositive of the lemma that

$$\inf_{g: g'' \geq 0} \int_0^\infty h_f(x)g(x)dx \geq 0,$$

which thus implies that $f(x) = x$ is a local minimum.