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1 Background

Theorem 1. *Define* $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ *and*

$$\Psi(x) = \int_x^\infty \phi(t)dt$$

for $x \in \mathbb{R}$. *Then*

$$\left(\frac{1}{x^3} - \frac{1}{x}\right)\phi(x) \leq \Psi(x) \leq \frac{1}{x}\phi(x)$$

Theorem 2. ***(Borell-TIS inequality)*** *Let* f_t *be a gaussian process such that* $\mathbb{E}[f_t] = 0$ *Then on any measurable set* D, *and* $u > 0$,

$$\mathbb{P}[\sup_D f_t > u + \mathbb{E}[\sup_D f_t]] \leq \exp(-u^2/(2\sigma_{max}^2))$$

where

$$\sigma_{max} = \sup_D \boldsymbol{E}[f_t^2]$$

Theorem 3. ***(Slepian's inequality)*** *If* f *and* g *are as bounded, centered gaussian processes, and*

$$\mathbb{E}[(f_t - f_s)^2] \leq \mathbb{E}[(g_t - g_s)^2]$$

then

$$\mathbb{P}[\sup_{t\in D} f_t > u] \leq \mathbb{P}[\sup_{t \in D} g_t > u]$$

2 Supremum of an isotropic GP

Theorem 4. *Let* f_t *be a gaussian process on* $\mathbb{R}$, *with* $Cov(f_t, f_u) = C(t-u)$, *where* $C(0) = 1$, *and* $C(t) \to 0$ *as* $||t|| \to \infty$. *Then supposing* f_t *is bounded on* $[0,1]$,

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{\sup_{[0,T]} f_t}{\sqrt{2\log(T)}} = 1\right) = 1$$

Proof.

It suffices to prove

$$\mathbb{P}\left(1 - \varepsilon \leq \lim_{T\to\infty} \frac{\sup_{[0,T]} f_t}{\sqrt{2\log(T)}} \leq 1 + \varepsilon\right) = 1$$

for arbitrary $\epsilon \in (0,1)$.

Take $\epsilon \in (0,1)$.

Find $\tau > 0$ such that $C(\tau) < \frac{\varepsilon}{2-\varepsilon}$ and find $T_0 > 0$ such that $T > \max\{2\tau, \frac{2-\varepsilon}{\varepsilon}\log(2\tau), e^{\frac{1-C(\tau)}{(1-\varepsilon)^2}}\}$. For each of $n = 1,\ldots,$, let $T = T_0 + n - 1$, and let let $m = \lfloor \frac{T+1}{\tau} \rfloor$. Define $t_k = k\tau$ for $k = 1,\ldots,m$. Let $Z_1,\ldots,Z_m$ be iid $N(0, 1-C(\tau))$. We have

$$\mathbb{E}[(Z_i - Z_j)^2] \leq 2(1-C(\tau)) \leq 2(1-C((i-j)\tau)) = \mathbb{E}[(f_{t_i} - f_{t_j})^2]$$

Hence by Slepian's inequality,

$$\mathbb{P}(\sup_{t\in[0,T]} f_t > u) \geq \mathbb{P}(\max_{i\in\{1,\ldots,m\}} f_{t_i} > u) \geq \mathbb{P}(\max_{i\in\{1,\ldots,m\}} Z_i > u)$$

for all $u > 0$. Thus, taking $u = (1-\varepsilon)\sqrt{2\log(T+1)}$ so that

$$u \leq \left(1 - \frac{\varepsilon}{2}\right)\sqrt{2(1-C(\tau))\log\left(\frac{T}{\tau} - 1\right)}$$

and

$$\frac{\sqrt{1-C(\tau)}}{u} - \frac{(1-C(\tau))^{3/2}}{u^3} \leq \frac{\sqrt{1-C(\tau)}}{2u}$$

we have

$$\begin{aligned}
\mathbb{P}(\sup_{t\in[0,T]} f_t < \sqrt{2\log(T-1)}(1-\epsilon)) &\leq \mathbb{P}(\max_{i\in\{1,\ldots,m\}} Z_i < u) && (1)\\
&= \left(1 - \Psi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right)^m && (2)\\
&\leq \left(1 - \left(\frac{\sqrt{1-C(\tau)}}{u} - \frac{(1-C(\tau))^{3/2}}{u^3}\right)\phi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right)^m && (3)\\
&\leq \left(1 - \left(\frac{\sqrt{1-C(\tau)}}{2u}\right)\phi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right)^m && (4)\\
&\leq \exp\left(-m\left(\frac{\sqrt{1-C(\tau)}}{2u}\right)\phi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right) && (5)
\end{aligned}$$

Now note that as $T \to \infty$,

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{\max Z_t}{\sqrt{2D\log(T)}} = 1\right) = 1$$

This suggests, and with some more detailed analysis, implies that

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} \geq 1\right) = 1$$

1. Upper bound.

Partition $[-T,T]^D$ into hypercubes of edge length 1. By union bound and Borrell-TIS inequality,

$$\mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D \mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D e^{-u^2/2}$$

This can be used to show

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} \leq 1\right) = 1$$

3 References

Adler RF, Taylor J. *Random Fields and Geometry*