Charles Zheng

1 Background

Theorem (Borell-TIS inequality) Let f_t be a gaussian process such that $\mathbb{E}[f_t] = 0$ Then on any set D, and $u > 0$,

$$\mathbb{P}[\sup_D f_t > u + \mathbb{E}[\sup_D f_t]] \leq \exp(-u^2/(2\sigma_{max}^2))$$

where

$$\sigma_{max} = \sup_D \mathbf{E}[f_t^2]$$

Theorem (Slepian's inequality) If f and g are as bounded, centered gaussian processes, and

$$\mathbb{E}[(f_t - f_s)^2] \leq \mathbb{E}[(g_t - g_s)^2]$$

then

$$\mathbb{P}[\sup_{t \in D} f_t > u] \leq \mathbb{P}[\sup_{t \in D} g_t > u]$$

Theorem. Let f be centered stationary Gaussian process on a compact group T. Then the following three conditions are equivalent: (i) f_t is continuous (ii) f_t is bounded (iii)

$$\int_0^\infty \sqrt{H(\varepsilon)} d\varepsilon < \infty$$

2 Example of unbounded isotropic GP

Note that

$$C_\kappa(t) = [1 - \kappa|t|]_+$$

is a covariance kernel. This implies the following:

Suppose $C(0) = 1$, $-C'(t) > 0$ for $t > 0$ but $-C'(t)$ decreasing for $t > 0$. Then $C(t)$ is a covariance kernel for a gaussian process. This is because one can find a mixture density $\rho(k)$ such that

$$C(t) = \int_0^\infty C_\kappa(t)\rho(\kappa)d\kappa$$

Then define

$$C(t) = \begin{cases} 1 - \frac{1}{\sqrt{\log(-|t|)}} & \text{for } t \in [-\exp(-\sqrt{3/2}), \exp(\sqrt{3/2})] \\ \left[1 - \frac{1}{\sqrt{3/2}} - \frac{t}{2\exp(-\sqrt{3/2})(3/2)^{3/2}}\right]_+ & \text{for } |t| > \exp(\sqrt{3/2}) \end{cases}$$

It follows that by Theorem 1 that f_t is unbounded.

3 Supremum of an isotropic GP

Let f_t be a gaussian process on $\mathbb{R}^D$, with $\text{Cov}(f_t, f_u) = C(t-u)$, where $C(0) = 1$, and $C(t) \to 0$ as $||t|| \to \infty$. Then supposing f_t is bounded on an interval,

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} = 1\right) = 1$$

Sketch of proof:

1. Lower bound.

Fix T. Let $\delta = exp(\sqrt{\log(T)})$ and consider $t_1, ..., t_{(2T/\delta)^D}$ on a square lattice of spacing δ on $[-T,T]^D$. Let $Z_{t_1}, \ldots, Z_{t_{(2T/\delta)^D}}$ iid $N(0, 1-C(\delta))$. By Slepian's inequality, $\mathbb{P}(\max_{t_1,\ldots,t_{(2T/\delta)^D}} f_t > u) \geq \mathbb{P}(\max_{t_1,\ldots,t_{(2T/\delta)^D}} Z_t > u)$. Now note that as $T \to \infty$,

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{\max Z_t}{\sqrt{2D\log(T)}} = 1\right) = 1$$

This suggests, and with some more detailed analysis, implies that

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} \geq 1\right) = 1$$

1. Upper bound.

Partition $[-T,T]^D$ into hypercubes of edge length 1. By union bound and Borrell-TIS inequality,

$$\mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D \mathbb{P}(\sup_{[-T,T]^D} f_t \geq u) \leq (2T)^D e^{-u^2/2}$$

This can be used to show

$$\mathbb{P}\left(\limsup_{T\to\infty} \frac{\sup_{[-T,T]^D} f_t}{\sqrt{2D\log(T)}} \leq 1\right) = 1$$

4 References

Adler RF, Taylor J. *Random Fields and Geometry*