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1 Background

Theorem 1. Define $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and

$$\Psi(x) = \int_x^\infty \phi(t)dt$$

for $x \in \mathbb{R}$. Then

$$\left(\frac{1}{x^3} - \frac{1}{x}\right) \phi(x) \leq \Psi(x) \leq \frac{1}{x}\phi(x)$$

Theorem 2. (Borell-TIS inequality) Let f_t be a gaussian process such that $\mathbb{E}[f_t] = 0$ Then on any measurable set D , and $u > 0$,

$$\mathbb{P}[\sup_D f_t > u + \mathbb{E}[\sup_D f_t]] \leq \exp(-u^2/(2\sigma_{max}^2))$$

where

$$\sigma_{max} = \sup_D \mathbf{E}[f_t^2]$$

Theorem 3. (Slepian's inequality) If f and g are as bounded, centered gaussian processes, and

$$\mathbb{E}[(f_t - f_s)^2] \leq \mathbb{E}[(g_t - g_s)^2]$$

then

$$\mathbb{P}[\sup_{t \in D} f_t > u] \leq \mathbb{P}[\sup_{t \in D} g_t > u]$$

2 Supremum of an isotropic GP

Theorem 4. Let f_t be a gaussian process on \mathbb{R} , with $\text{Cov}(f_t, f_u) = C(t-u)$, where $C(0) = 1$, and $C(t) \rightarrow 0$ as $||t|| \rightarrow \infty$. Then supposing f_t is bounded on $[0, 1]$,

$$\mathbb{P}\left(\liminf_{T \rightarrow \infty} \frac{\sup_{[0, T]} f_t}{\sqrt{2 \log(T)}} \geq 1\right) = 1$$

Proof.

It suffices to prove

$$\mathbb{P}\left(1 - \varepsilon \leq \liminf_{T \rightarrow \infty} \frac{\sup_{[0, T]} f_t}{\sqrt{2 \log(T)}}\right) = 1$$

for arbitrary $\varepsilon \in (0, 1)$.

Take $\varepsilon \in (0, 1)$.

Find $\tau > 0$ such that $C(\tau) < \frac{\varepsilon}{2-\varepsilon}$ and find $T_0 > 0$ such that $T > \max\{2\tau, \frac{2-\varepsilon}{\varepsilon} \log(2\tau), e^{\frac{1-C(\tau)}{(1-\varepsilon)^2}}\}$. For each of $n = 1, \dots$, let $T = T_0 + n$, and let $m = \lfloor \frac{T+1}{\tau} \rfloor$. Define $t_k = k\tau$ for $k = 1, \dots, m$. Let Z_1, \dots, Z_m be iid $N(0, 1 - C(\tau))$. For $i \neq j$, we have

$$\mathbb{E}[(Z_i - Z_j)^2] = 2(1 - C(\tau)) \leq 2(1 - C((i - j)\tau)) = \mathbb{E}[(f_{t_i} - f_{t_j})^2]$$

Hence by Slepian's inequality,

$$\mathbb{P}(\sup_{t \in [0, T]} f_t > u) \geq \mathbb{P}(\max_{i \in \{1, \dots, m\}} f_{t_i} > u) \geq \mathbb{P}(\max_{i \in \{1, \dots, m\}} Z_i > u)$$

for all $u > 0$. Thus, taking $u = (1 - \varepsilon)\sqrt{2 \log(T + 1)}$ so that

$$u \leq \left(1 - \frac{\varepsilon}{2}\right) \sqrt{2(1 - C(\tau)) \log\left(\frac{T}{\tau} - 1\right)}$$

and

$$\frac{\sqrt{1 - C(\tau)}}{u} - \frac{(1 - C(\tau))^{3/2}}{u^3} \leq \frac{\sqrt{1 - C(\tau)}}{2u}$$

we have

$$\mathbb{P}(\sup_{t \in [0, T]} f_t < \sqrt{2 \log(T-1)}(1-\epsilon)) \leq \mathbb{P}(\max_{i \in \{1, \dots, m\}} Z_i < u) \quad (1)$$

$$= \left(1 - \Psi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right)^m \quad (2)$$

$$\leq \left(1 - \left(\frac{\sqrt{1-C(\tau)}}{u} - \frac{(1-C(\tau))^{3/2}}{u^3}\right) \phi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right)^m \quad (3)$$

$$\leq \left(1 - \left(\frac{\sqrt{1-C(\tau)}}{2u}\right) \phi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right)^m \quad (4)$$

$$\leq \exp\left(-m \left(\frac{\sqrt{1-C(\tau)}}{2u}\right) \phi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right) \quad (5)$$

$$\leq \exp\left(-m \left(\frac{\sqrt{1-C(\tau)}}{2u}\right) \phi\left(\left(1 - \frac{\epsilon}{2}\right) \sqrt{2 \log\left(\frac{T}{\tau} - 1\right)}\right)\right) \quad (6)$$

$$= \exp\left(-\frac{m}{\sqrt{2\pi} \left(\frac{T}{\tau} - 1\right)^{(1-\frac{\epsilon}{2})^2}} \left(\frac{\sqrt{1-C(\tau)}}{2u}\right)\right) \quad (7)$$

$$\leq \exp\left(-\frac{\frac{T}{\tau} - 1}{\sqrt{2\pi} \left(\frac{T}{\tau} - 1\right)^{(1-\frac{\epsilon}{2})^2}} \left(\frac{\sqrt{1-C(\tau)}}{2u}\right)\right) \quad (8)$$

$$= \exp\left(-\frac{\left(\frac{T}{\tau} - 1\right)^{1-(1-\frac{\epsilon}{2})^2}}{\sqrt{2\pi}} \left(\frac{\sqrt{1-C(\tau)}}{2(1-\epsilon)\sqrt{2 \log(T-1)}}\right)\right) \quad (9)$$

$$= \exp\left(-\frac{\left(\frac{T_0+n}{\tau} - 1\right)^{1-(1-\frac{\epsilon}{2})^2}}{\sqrt{2\pi}} \left(\frac{\sqrt{1-C(\tau)}}{2(1-\epsilon)\sqrt{2 \log(T_0+n-1)}}\right)\right) \quad (10)$$

Hence

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{t \in [0, T]} (1 - \varepsilon) \sqrt{2 \log(T_0 + n - 1)}\right) < \infty \quad (11)$$

which by Borel-Cantelli, and the fact that

$$\liminf_{T \rightarrow \infty} \frac{\sup_{[0, T]} f_t}{\sqrt{2 \log(T)}} \leq \liminf_{n \rightarrow \infty} \frac{\sup_{[0, T_0 + n]} f_t}{\sqrt{2 \log(T_0 + n - 1)}}$$

implies the result \square .

Theorem 5. *Let f_t be a gaussian process on \mathbb{R} , with $\text{Cov}(f_t, f_u) = C(t - u)$, where $C(0) = 1$, and $C(t) \rightarrow 0$ as $||t|| \rightarrow \infty$. Then supposing f_t is bounded on $[0, 1]$,*

$$\lim_{T \rightarrow \infty} \mathbb{P}\left(1 - \varepsilon \leq \lim_{T \rightarrow \infty} \frac{\sup_{[0, T]} f_t}{\sqrt{2 \log(T)}} \leq 1 + \varepsilon\right) = 1$$

for all $\varepsilon \in (0, 1)$.

Proof.

The lower bound follows from the previous result. Let $\mu = \mathbb{E}[\sup_{[0, 1]} f_t]$, so that

$$\mathbb{P}\left[\sup_{t \in [0, 1]} f_t > u\right] \leq e^{-(u - \mu)^2 / 2}$$

for all $u > \mu$. For any $T > 0$, letting $n = \lceil T \rceil$ using union bound and isotropy, we have

$$\mathbb{P}\left[\sup_{t \in [0, T]} f_t > u\right] \leq \mathbb{P}\left[\max_{i=\{1, \dots, n\}} \sup_{t \in [i-1, i]} f_t > u\right] \leq n \mathbb{P}\left[\sup_{t \in [0, 1]} f_t > u\right] \leq n e^{-(u - \mu)^2 / 2}$$

for all $u > \mu$. Therefore, defining $u = (1 + \varepsilon) \sqrt{2 \log(T)}$, and taking n large enough so that $u > \mu$, we get

$$\mathbb{P}\left[\sup_{t \in [0, T]} f_t > u\right] \leq n e^{-(u - \mu)^2 / 2} = \frac{n}{(n + 1)^{(1 + \varepsilon)^2}} \exp\left(-\mu(1 + \varepsilon) \sqrt{2 \log(n + 1)} - \frac{\mu^2}{2}\right)$$

which tends to 0 as $T \rightarrow \infty$. \square

3 References

Adler RF, Taylor J. *Random Fields and Geometry*