Charles Zheng EE 378b HW 3

1. a. Since the space of unit vectors is compact, there exists x^* such that $||M||_2 = ||Mx^*||_2$. For the same reason, there exist unit vectors x°, y° such that

$$\langle x^\circ, My^\circ \rangle = \max_{||x||=||y||=1} \langle x, My \rangle$$

Letting $y^* = Mx^*/||M||_2$, we have $||y||_2 = 1$. Therefore

$$||M||_2 = y^* M x^* \leq \max_{||y||=||x||=1} \langle y, Mx \rangle$$

Also, by Cauchy-Schwarz we have

$$\max_{||x||=||y||=1} \langle x, My \rangle = (x^\circ)^* M y^\circ \leq ||x^\circ||_2 ||My^\circ||_2 \leq ||My^\circ||_2 \leq \max_{||y||=1} ||My||_2 = ||M||_2$$

Having shown that

$$||M||_2 \leq \max_{||x||=||y||=1} \langle x, My \rangle \leq ||M||_2$$

we conclude that the definitions are equivalent

b. Let the SVD of M be written $M = UDV^T$ where $D = diag(\sigma_1, \ldots, \sigma_n)$. Let v_1 be the first column of V, then $||v_1||_2 = 1$ and

$$||Mv_1||_2 = ||UDV^T v_1||_2 = ||UDe_1||_2 = ||De_1||_2 = \sigma_1$$

Hence

$$\sigma_1 = ||Mv_1||_2 \leq \max_{||x||=1} ||Mx||_2 = ||M||_2$$

Meanwhile for any unit vector x, defining $y = V^T x$ we have $||y||_2 \leq 1$. Then

$$\max_{||x||=1} ||Mx||_2 = \max_{||x||=1} ||UDV^T x||_2 \leq \max_{||x||=1} ||DV^T x||_2 \leq \max_{||y||=1} ||Dy||_2$$

But defining $a_i = y_i^2$,

$$\max_{||y||=1} ||Dy||_2^2 = \max_{||y||=1} \sum_{i=1}^n \sigma_i^2 y_i^2 = \max_{\sum a_i = 1, a_i \geq 0} \sum_{i=1}^n \sigma_i^2 a_i$$

is maximized by $a = e_1$, hence $\max_{||y||=1} ||Dy||_2 = \sigma_1$.

Having shown that

$$\sigma_1 \leq ||M||_2 \leq \sigma_1$$

we conclude that the two definitions are equivalent.

2. a. From 1a we have

$$||M^*||_2 = \max_{||x||=||y||=1} \langle x, M^* y\rangle = \max_{||x||=||y||=1} \langle y, Mx\rangle = ||M||_2$$

b. We have

$$||AB||_2 = \max_{||x||=1} ||ABx||_2 = \max_{y=Bx \text{ for some } ||x||=1} ||Ay||_2$$

Meanwhile, if $y = Bx$, and $||x||_2 = 1$, we have $||y||_2 \leq ||B||_2$. Therefore the set $\{y : y = Bx$ for some x such that $||x|| = 1\}$ is contained in the set $\{y : ||y||_2 \leq ||B||_2\}$. Hence

$$\max_{y=Bx \text{ for some } ||x||=1} ||Ay||_2 \leq \max_{||y||=||B||_2} ||Ay||_2 = ||A||_2||B||_2$$

3. i.

$$||aM||_2 = \max_{||x||=1} ||aMx||_2 = \max_{||x||=1} |a|||M_2x||_2 = |a| \max_{||x||=1} ||Mx||_2 = a||M||_2$$

ii.

$$||A+B||_2 = \max_{||x||=1} ||Ax+Bx||_2 \leq \max_{||x||=1} ||Ax||_2+||Bx||_2 \leq \max_{||x||=1} ||Ax||_2+\max_{||x||=1} ||Bx||_2 = ||A||_2+||B||_2$$

iii. Proof of contrapositive: If $M \neq 0$, then some row M_i is nonzero. But then $||Me_i||_2 = ||M_i||_2 > 0$, so $||M||_2 > 0$.

4.

5.