# A functional MRI mind-reading game

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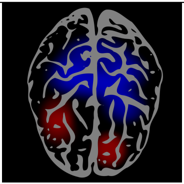
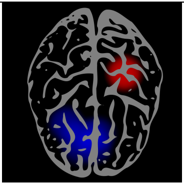
# Functional MRI

| Stimuli | Response |
|---|---|
|  |  |
|  |  |

# Functional MRI

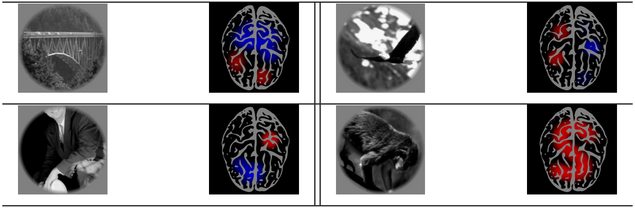
| Stimuli $x$ | Response $y$ |
|---|---|
| $\begin{pmatrix} 1.0 \\ 0 \\ 3.0 \\ 0 \\ -1.2 \end{pmatrix}$ | $\begin{pmatrix} 1.2 \\ 0 \\ -1.8 \\ -1.2 \end{pmatrix}$ |
| $\begin{pmatrix} 0 \\ -2.2 \\ -3.1 \\ 4.5 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} -1.2 \\ -1.9 \\ 0.5 \\ 0.6 \end{pmatrix}$ |

# Encoding vs Decoding

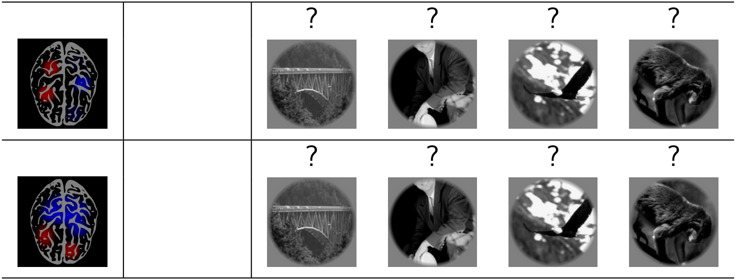
- Encoding: predict $y$ from $x$.
- Decoding: reconstruct $x$ from $y$ (mind-reading).
  - Classification: label response $y$ by a class from the training data
  - Identification: label response $y$ by a class *outside* of the training data
  - Reconstruction: infer $x$ from $y$

# Classification

Training Data

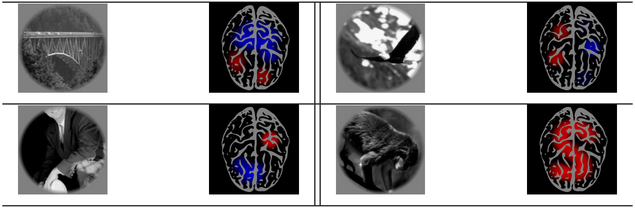


Test Data

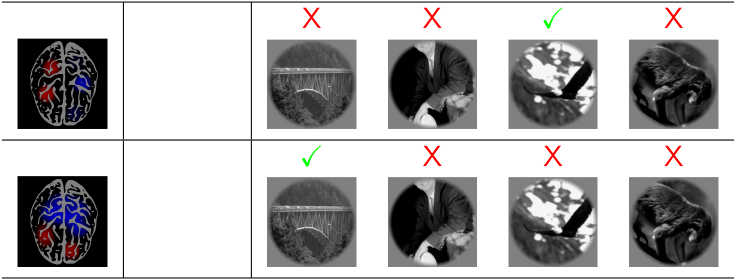


# Classification

Training Data

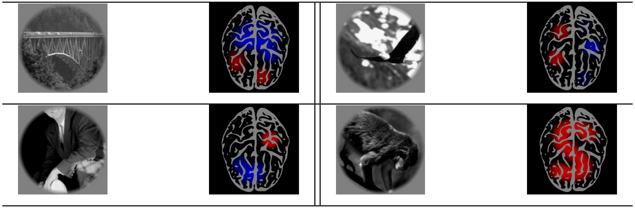


Test Data

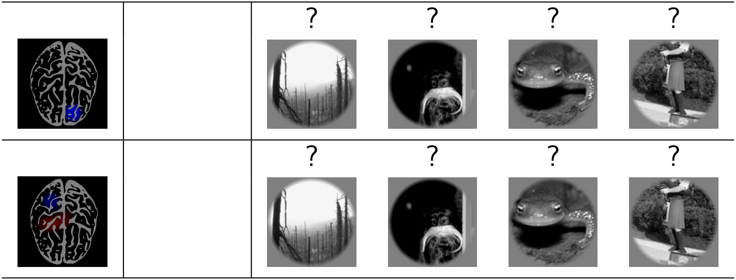


# Identification

## Training Data

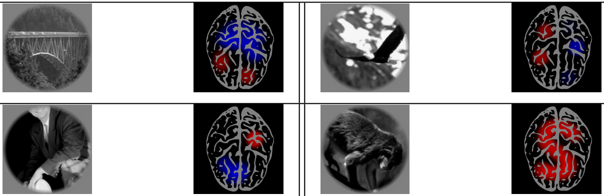


## Test Data *(new images!)*

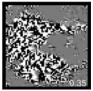
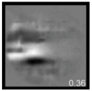
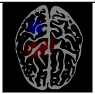
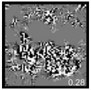
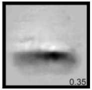


# Reconstruction

Training Data



Test Data *(new images!)*

| Stimulus | Response | Standard basis | Gabor basis |
|---|---|---|---|
|  |  |  |  |
|  |  |  |  |

# Classification vs Identification vs Reconstruction

- Classification is easy: doesn't require domain-specific model
- Identification and reconstruction both require a model relating image features to responses

*Difficulty of Identification vs Reconstruction*

| | High dimensions | Number of candidate stimuli |
|---|---|---|
| Identification | Neutral | Hard |
| Reconstruction | Hard | Easy |

# Motivating questions

- Under what conditions would it be possible to get performance on reconstruction or identification?
- How can we develop methods which achieve better performance on these tasks?
- Can we interpret the performance metric (prediction error, misclassification error) of a model to draw scientific conclusions? (E.g. which features are important, information content of fMRI scan.)

# Classification vs Identification vs Reconstruction

*Supervised learning problems*

| | Misclassification Rate | Prediction error |
|---|---|---|
| No covariates | Classification | (nothing to predict) |
| Covariates ($x$) | Identification | Regression |

- Reconstruction is regression $x \sim y$
- Does there already exist statistical theory for identification?
- Next: a toy model for identification

Section 2

# Theory

# The problem of identification

*Training data.*

- Given training classes $S_{\text{train}} = \{\text{train:}1, \ldots, \text{train:}k\}$ where each class train:$i$ has features $x_{\text{train:}i}$.
- For $t = 1, \ldots, T_{\text{train}}$, choose class label $z_{\text{train:}t} \in S_{\text{train}}$; sample a response $y_{\text{train:}t}$ from that class.

*Test data.*

- Given test classes $S_{\text{test}} = \{\text{test:}1, \ldots, \text{test:}\ell\}$ with features $\{x_{\text{test:}1}, \ldots, x_{\text{test:}\ell}\}$
- Task: for $t = 1, \ldots, T_{\text{test}}$, label $y_{\text{test:}t}$ by class $\hat{z}_{\text{test:}t} \in S_{\text{train}}$; try to minimize misclassification rate

# Additional assumptions

- For a point $y$ from class with features $x$,

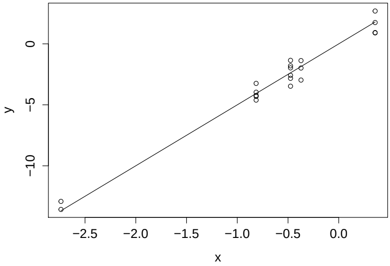
$$y = f(x) + \epsilon$$

  where the noise $\epsilon$ is drawn from some distribution and $f$ is an unknown function
- The features for the training and test classes are sampled iid from the same distribution $P$

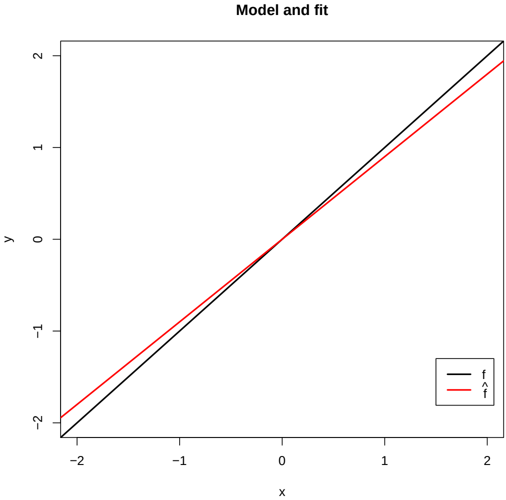
$$x_{\text{train}:i} \sim P$$

$$x_{\text{train}:i} \sim P$$

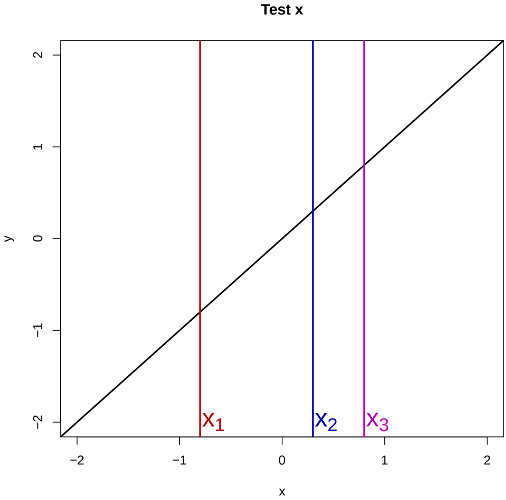
# Toy example I



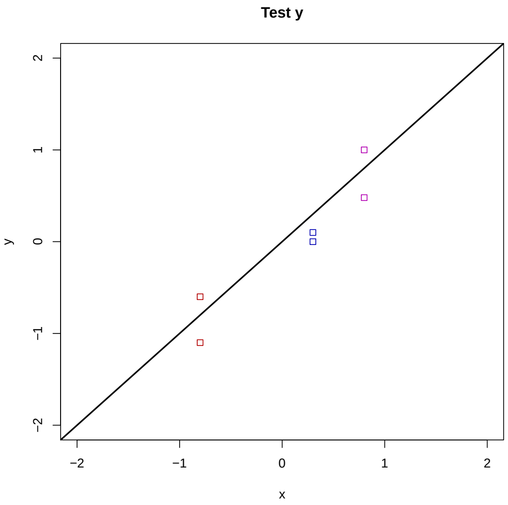

- Features $x$ are one-dimensional real numbers, as are responses $y$. Parameter $\beta$ is also a real number.
- Model is linear: $y \sim N(x\beta, \sigma_\epsilon^2)$




Suppose we estimated $\hat{\beta}$ from training data.

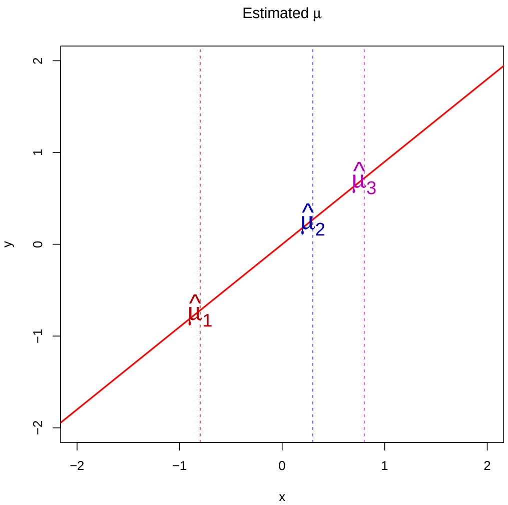



Generate features $x_{\text{test}:1}, \ldots, x_{\text{test}:\ell}$ iid $N(0, \sigma_x^2)$.

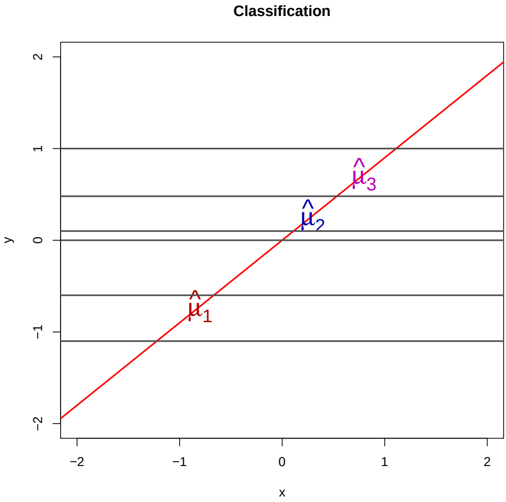



Hidden labels $z_{\text{test}:t}$ are iid uniform from $S_{\text{train}}$.
Generate $y_{\text{test}:t} \sim N(\beta x_{z_{\text{test}:t}}, \sigma^2_\epsilon)$

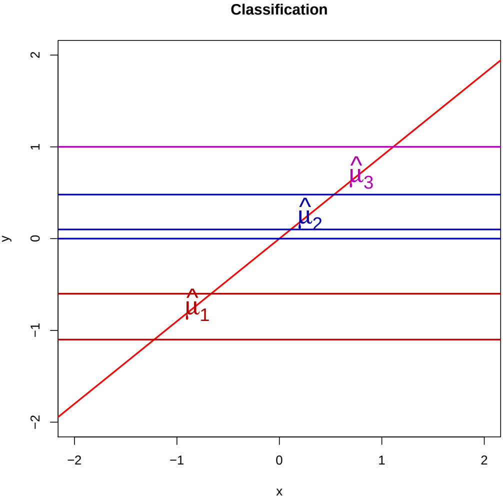

Classify $\hat{y}_{\text{test}:t}$



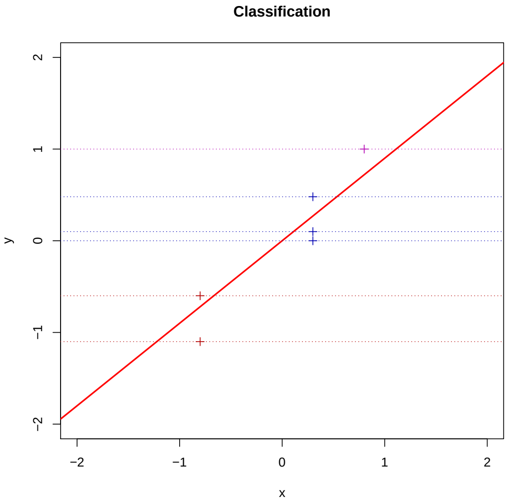

$$\hat{\mu}_{\text{test}:i} = \hat{\beta} x_{\text{test}:i}$$

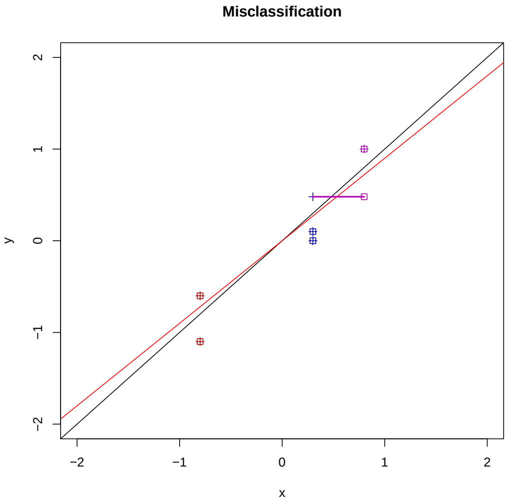



$$\hat{z}_{\text{test}:t} = \text{argmin}_z \ell_{\hat{\mu}_z}(y_{\text{test}:t})$$

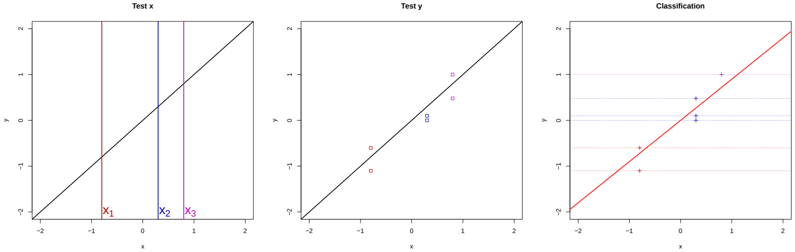



$$\hat{z}_{\text{test}:t} = \text{argmin}_z(\hat{\mu}_z - y_{\text{test}:t})^2$$


Classification
y
x
−2
−1
0
1
2
−2
−1
0
1
2


Misclassification
y
x
2
1
0
−1
−2
−2
−1
0
1
2

# Toy example I




- Generate features $x_{\text{test}:1}, \ldots, x_{\text{test}:\ell}$ iid $N(0, \sigma_x^2)$.
- Hidden labels $z_{\text{test}:t}$ are iid uniform from $S_{\text{train}}$. Generate $y_{\text{test}:t} \sim N(\beta x_{z_{\text{test}:t}}, \sigma_\epsilon^2)$
- Classify $\hat{y}_{\text{test}:t}$ by maximum likelihood assuming $\hat{\beta}$ is correct. Thus:

$$\hat{z}_{\text{test}:t} = \text{argmin}_z (\hat{\beta} x_z - y_{\text{test}:t})^2$$

# Toy example I: Questions

1. We know the prediction error is minimized when $\hat{\beta} = \beta$. Is it also true that misclassification error in the mind-reading game is minimized when $\hat{\beta} = \beta$?
2. Even if the answer to 1. is yes, should we estimate $\hat{\beta}$ using the same methods as in least-squares regression?

# Question 1: Outline

We will find an answer to question 1 as follows

- Write an explicit expression for the misclassification rate as a function of $\hat{\beta}$
- Take the derivative of that expression with respect to $\hat{\beta}$ at the true $\beta$
- Does that derivative equal zero?
- If so, look at second derivatives, lower bounds, etc.

*Write an explicit expression for the misclassification rate*

- The expected misclassification error is the same if we take $T_{\text{test}} = 1$. Then let $(x_*, y_*)$ be the feature-response pair in the test set, where

$$y_* = x_* \beta + \epsilon_*$$

- Denote the features for the incorrect classes as $x_1, \ldots, x_{\ell-1}$.
- Let $\delta = \hat{\beta} - \beta$.

*Write an explicit expression for the misclassification rate (cont.)*

- Ignore the possibility of ties. The response $y_*$ is misclassified if and only if

$$\min_{i=1,\ldots,\ell-1} |y_* - x_i\hat{\beta}| < |y_* - x_*\hat{\beta}|$$

equivalently

$$\cup_{i=1,\ldots,\ell-1} E_i$$

where $E_i$ is the event that

$$|y_* - x_i\hat{\beta}| < |y_* - x_*\hat{\beta}|$$

*Write an explicit expression for the misclassification rate (cont.)*

- Use the following conditioning

$$\mathbf{E}[\text{misclassification}] = \mathbf{E}[\mathbf{E}[\Pr_{x_1,\ldots,x_\ell}[\cup_i E_i]|x_* = x, \epsilon_* = \epsilon]]$$

- Use the fact that events $E_i$ are independent and have the same probability, thus:

$$\mathbf{E}[\text{misclassification}] = 1 - \mathbf{E}[\mathbf{E}[(1 - \Pr[E_1])^{\ell-1}|x_* = x, \epsilon_* = \epsilon]]$$

- Next: write an expression for $\Pr[E_1]$

*Write an expression for* $\Pr[E_1]$.

- $E_1$ can also be written as the event

$$|x_*\beta + \epsilon_* - x_1(\beta + \delta)| < |-\delta x_* + \epsilon_*|$$

- Conditioning on $\epsilon_*$ and $x_*$, we have

$$\Pr[E_1] = \left|\Phi\left(\frac{x_*}{\sigma_x}\right) - \Phi\left(\frac{x_*(\beta - \delta) + 2\epsilon_*}{\sigma_x(\beta + \delta)}\right)\right|$$

An exact expression for expected misclassification is therefore

$$1 - \int_\epsilon \left[\int_x \left(1 - \left|\Phi\left(\frac{x}{\sigma_x}\right) - \Phi\left(\frac{x(\beta-\delta)+2\epsilon}{\sigma_x(\beta+\delta)}\right)\right|\right)^{\ell-1} d\Phi(\tfrac{x}{\sigma_x})\right] d\Phi(\tfrac{\epsilon}{\sigma_\epsilon})$$

*Take the derivative of the expression with respect to $\delta$*

Fix $\epsilon > 0$. The derivative of the inner integral wrt $\delta = 0$ is proportional to

$$\int_x (1 - \Phi(\frac{x\beta + 2\epsilon}{\sigma_x \beta}) + \Phi(\frac{x}{\sigma_x}))\phi(\frac{x\beta + 2\epsilon}{\sigma_x \beta})(x + \frac{\epsilon}{\beta})\phi(\frac{x}{\sigma_x})dx$$

*Is the derivative zero?*

*Is the derivative zero?*

Note that

$$\phi\left(\frac{x\beta + 2\epsilon}{\sigma_x\beta}\right)\phi\left(\frac{x}{\sigma_x}\right) \propto \phi\left(\frac{\sqrt{2}(x+\frac{\epsilon}{\beta})}{\sigma_x}\right)$$

which is the density of a normal variate with mean $-\epsilon/\beta$

But now note that the other terms

$$\left(1 - \Phi\left(\frac{x\beta + 2\epsilon}{\sigma_x\beta}\right) + \Phi\left(\frac{x}{\sigma_x}\right)\right)\left(x - \frac{\epsilon}{\beta}\right)$$

are antisymmetric about $x = -\frac{\epsilon}{\beta}$.

Thus by symmetry, the derivative of the inner integral $\delta = 0$ vanishes. The same argument works for $\epsilon < 0$, hence the misclassification rate is stationary at $\hat{\beta} = \beta$.

*(We'll skip the second derivative checking, etc.)*

# Question 1: Remarks

- $\epsilon$ can have any distribution
- The distribution of $x$ is important for the proof: we need

$$\phi\left(\frac{x\beta + 2\epsilon}{\sigma_x \beta}\right)\phi\left(\frac{x}{\sigma_x}\right) \propto \phi\left(\frac{\sqrt{2}(x + \frac{\epsilon}{\beta})}{\sigma_x}\right)$$

  This can only work if $\phi(x) = \exp(-Cx^2)h(|x|)$
- Conjecture: $\hat{\beta} = \beta$ **only if**

$$\phi(x) = \exp(-Cx^2)h(x)$$

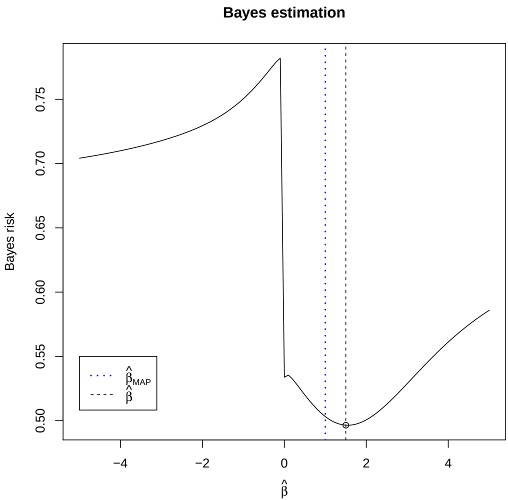
  where $h(x) = h(-x)$.
- Generalize to higher dimensions?

# Toy example I: Estimation

- Second question: what about estimation?
- Take a Bayesian viewpoint: suppose we have a prior distribution for $\beta$
- For *least-squares regression*, we would use $\hat{\beta} = \int \beta p_{posterior}(\beta) d\beta$, the posterior mean.
- For *identification*, we would choose

$$\hat{\beta} = \text{argmin}_{\hat{\beta}} \int R(\beta; \hat{\beta}) p_{posterior}(\beta) d\beta$$

where $R$ is the expected misclassification rate.
- How will these differ?

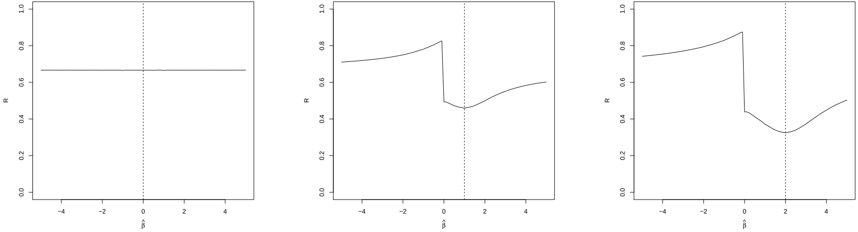



Point estimate for identification (black dashed) is larger than posterior mean (blue dotted)

# Toy example I: Estimation

*Why the upward bias?*

$\beta = 0$ $\beta = 1$ $\beta = 2$




Risk function is more sensitive for large $\beta$.

# Estimation: questions

- Is the optimal $\hat{\beta}$ for identification is in general "larger" than the optimal $\hat{\beta}$ for regresssion, in a frequentist (e.g. minimax) sense?
- Lasso/Ridge penalized regression models are commonly used for identification
- Hypothesis: the optimal $\lambda$ for identifying $x$ from $y$ will be smaller (hence produce less sparse $\hat{\beta}$) than the optimal $\lambda$ for regression $y \sim x$.

# Generalizing to higher dimensions

*Model fitting*

- $x$ is $p$-dimensional column vector, $y$ is $q$-column vector
- Using training data, learn a model

$$y = B^T x + b^T + \epsilon$$

  where $B$ is a $p \times q$ matrix and $b$ is a $q$-row vector.
- Using residuals from training data, estimate $\hat{\Sigma}_\epsilon$

*Identification*

- For each test class feature $x_{\text{test}:i}$, compute the predicted mean response

$$\mu_{\text{test}:i} = B^T x_{\text{test}:i} + b^T$$

- (MLE) Label a new response $y_*$ with test class $z$ that minimizes

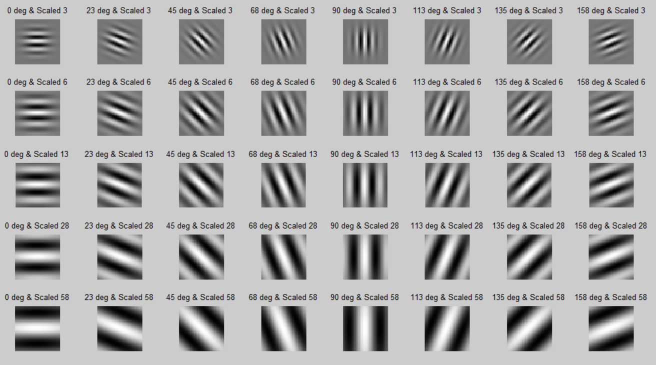
$$(\mu_z - y_*)^T \hat{\Sigma}_\epsilon^{-1} (\mu_z - y_*)$$

Section 3

# Experiments

# Data

- From Kay *et al.* paper
- 1750 images with averaged responses from 2 repeats
- Responses $y$: 1331 voxels from the most basic visual subsystem, V1
- Features $x$: 10921 image features based on Gabor filters



# Regression vs Identification

*Partition*

- Randomly partition into training set (1725) and test set (25)

*Model fitting via lasso*

- Notation: $Y = (y_{\text{train}:1}, \ldots, y_{\text{train}:1725})^T$, $X = (x_{z_{\text{train}:1}}, \ldots, x_{z_{\text{train}:1725}})^T$
- Fix $\lambda$. Fit a separate Lasso regression for each voxel:

$$\text{minimize} \frac{1}{2}||Y_i - \hat{\beta}^{(i)}X + \hat{\beta}_0^{(i)}||^2 + \lambda||\hat{\beta}||_1$$

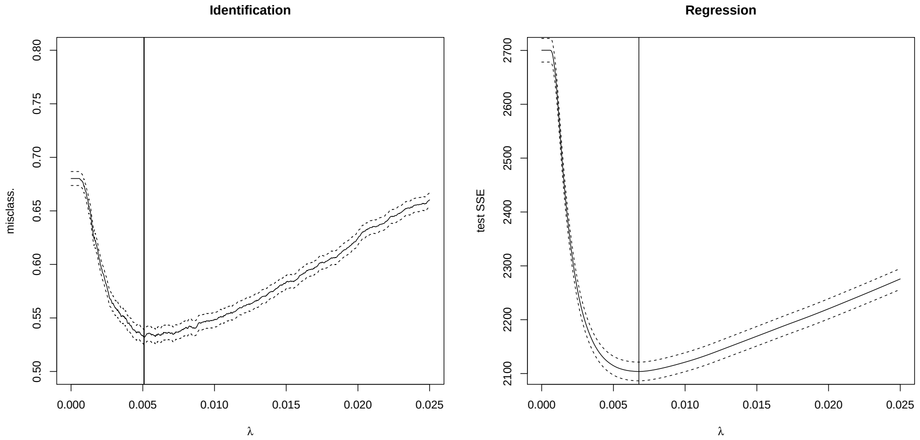
- Let $B = (\hat{\beta}^{(1)}, \ldots, \hat{\beta}^{(1331)})$, $b = (\hat{\beta}_0^{(1)}, \ldots, \hat{\beta}_0^{(1331)})$

*Performance on test set*

- Regression: use test labels to predict $\hat{y}$
- Identification: for test responses $y$, estimate label using MLE

Perform this experiment for $\lambda = 0, 0.1, \ldots, 2$

# Results




Results do not support hypothesis... what else is going on?

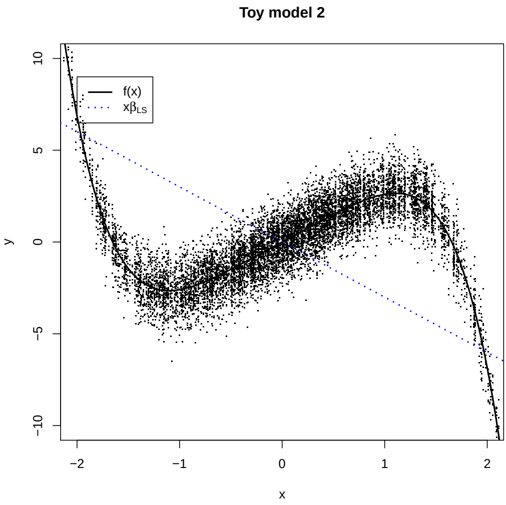
Section 4

# Nonlinear toy example

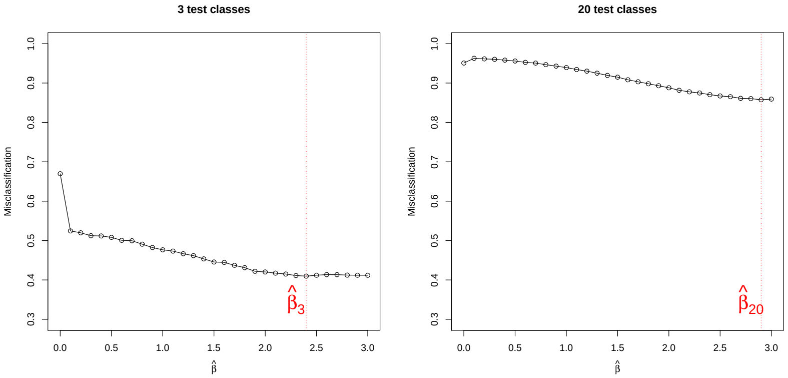
# More questions

3. What happens if the true regression function $f$ is nonlinear, but we restrict $\hat{f}$ to be linear?
4. What happens when the number of classes $\ell$ increases? What if $\ell$ increases while $\sigma_\epsilon^2$ decreases?

# Toy example IIa

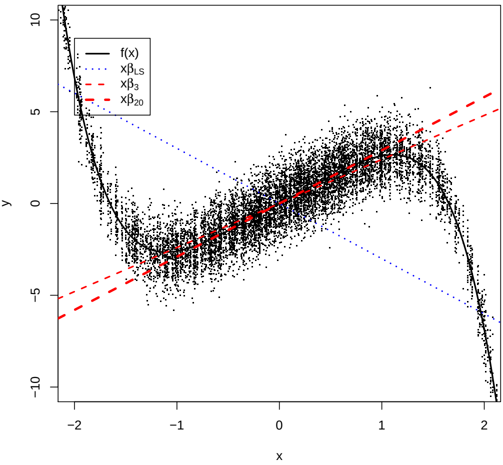


# Toy example IIa




Effect of increasing $\ell$.

Toy model 2



# Why is this?

- We can relate identification to regression with a different loss function
- Least squares loss

$$\mathbf{E}[(y - \hat{y})^2]$$

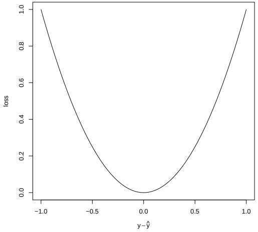
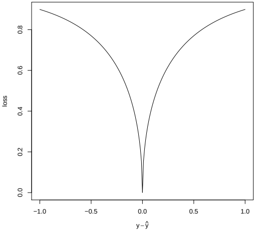
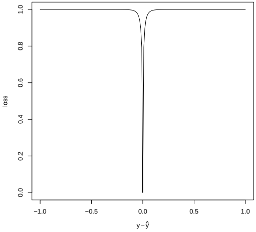
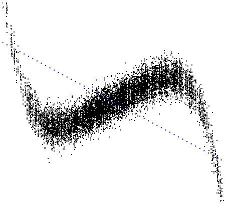
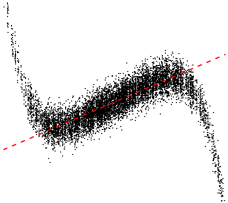
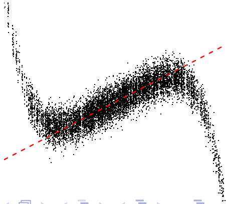
- Identification loss

$$\mathbf{E}[1 - \Pr[|y - \hat{y}'| < |y - \hat{y}|]^{\ell-1}]$$

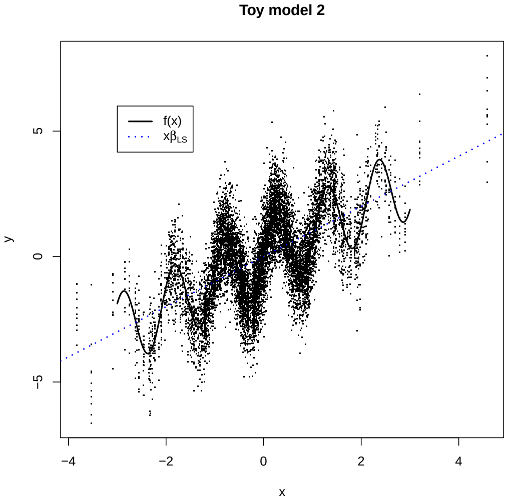
where $\hat{y}'$ is the predicted value for a randomly drawn $x$.

# Why is this?

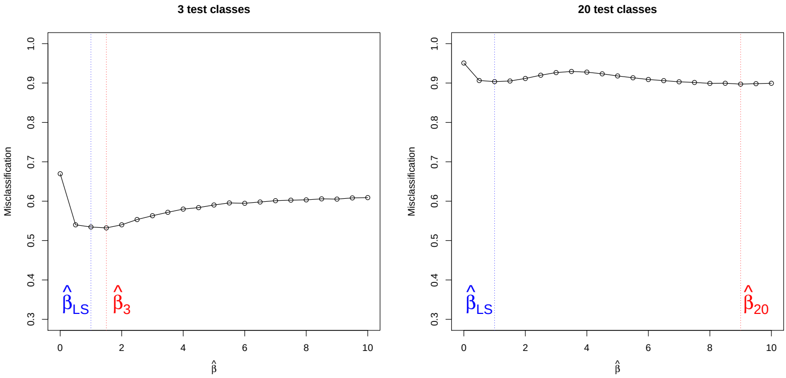
Identification loss more closely resembles 0-1 loss as $\ell$ increases.

| Squared error | $\ell = 3$ | $\ell = 20$ |
|---|---|---|
|   |   |   |
|  |  |  |

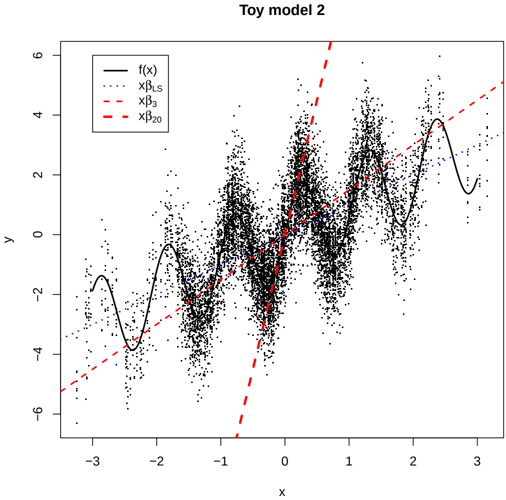
# Toy example IIb



# Toy example IIb




Effect of increasing $\ell$.




Effect of increasing $\ell$: global trends will become ignored in favor of locally linear trends!

# Implications

- “The model is always wrong”
- Statistical methods should be robust to small deviations from the model
- Even when minor nonlinearities exist in the model, identification performance fails to reflect global fit

# Conclusions

- The problem of *decoding*, predicting $x$ from $y$, is of interest to many neuroscientists
- Different formulations of the decoding problem: classification, identification, and reconstruction (regression) have different properties and advantages
- Statistical theory can help with training the models *and* with interpreting the results

*In particular...*

- Identification is similar to regression $y \sim x$ in a special case, but can benefit from less sparse estimates.
- Identification can lead to counterintuitive results when there are nonlinearities and $\ell$ is large

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