A functional MRI mind-reading game

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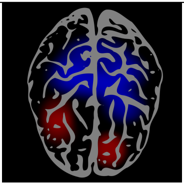
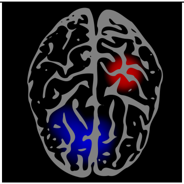
April 1, 2015

Functional MRI

Stimuli	Response
	
	

Functional MRI

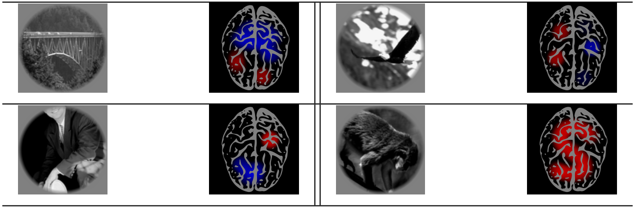
Stimuli x	Response y
$\begin{pmatrix} 1.0 \\ 0 \\ 3.0 \\ 0 \\ -1.2 \end{pmatrix}$	$\begin{pmatrix} 1.2 \\ 0 \\ -1.8 \\ -1.2 \end{pmatrix}$
$\begin{pmatrix} 0 \\ -2.2 \\ -3.1 \\ 4.5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -1.2 \\ -1.9 \\ 0.5 \\ 0.6 \end{pmatrix}$

Encoding vs Decoding

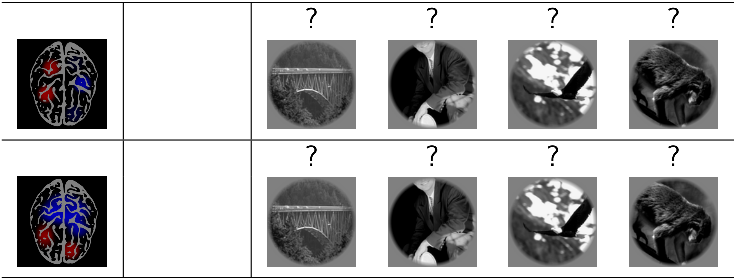
- Encoding: predict y from x.
- Decoding: reconstruct x from y (mind-reading).
  - Classification: label response y by a class from the training data
  - Identification: label response y by a class *outside* of the training data
  - Reconstruction: infer x from y

Classification

Training Data

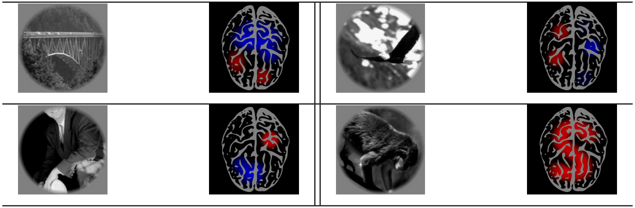


Test Data

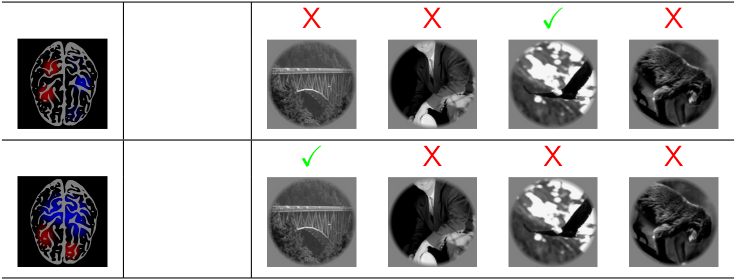


Classification

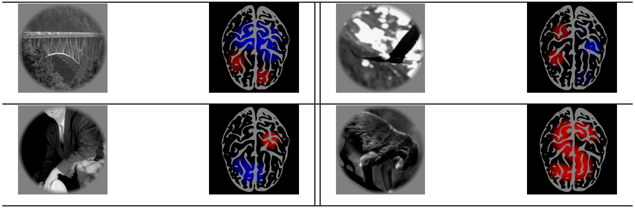
Training Data



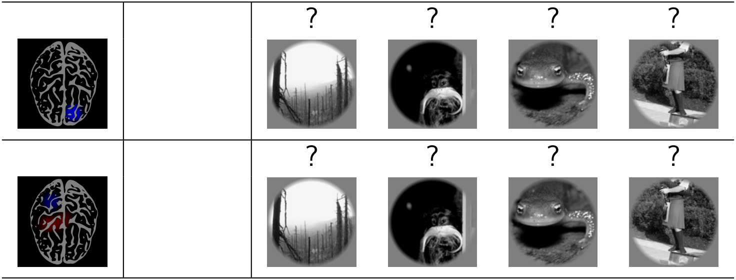
Test Data



Training Data

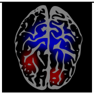
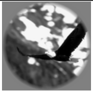
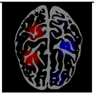
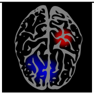
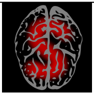


Test Data *(new images!)*

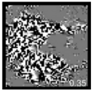
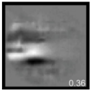
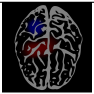
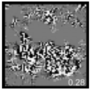
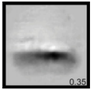


Reconstruction

Training Data

Test Data *(new images!)*

Stimulus	Response	Standard basis	Gabor basis
			
			

Classification vs Identification vs Reconstruction

- Classification is easy: doesn't require domain-specific model
- Identification and reconstruction both require a model relating image features to responses

Difficulty of Identification vs Reconstruction

	High dimensions	Number of candidate stimuli
Identification	Neutral	Hard
Reconstruction	Hard	Easy

Motivating questions

- Under what conditions would it be possible to get performance on reconstruction or identification?
- How can we develop methods which achieve better performance on these tasks?
- Can we interpret the performance metric (prediction error, misclassification error) of a model to draw scientific conclusions? (E.g. which features are important, information content of fMRI scan.)

Classification vs Identification vs Reconstruction

Supervised learning problems

	Misclassification Rate	Prediction error
No covariates	Classification	(nothing to predict)
Covariates (x)	Identification	Regression

- Reconstruction falls under the framework of regression
- With neither labels
- Identification is a new category of supervised learning problem: let's develop a theory!

Section 2

Theory

The problem of identification

Training data.

- Given training classes $S_{\text{train}} = \{\text{train:}1, \ldots, \text{train:}k\}$ where each class train:i has features $x_{\text{train:}i}$.
- For $t = 1, \ldots, T_{\text{train}}$, choose class label $z_{\text{train:}t} \in S_{\text{train}}$; sample a response $y_{\text{train:}t}$ from that class.

Test data.

- Given test classes $S_{\text{test}} = \{\text{test:}1, \ldots, \text{test:}\ell\}$ with features $\{x_{\text{test:}1}, \ldots, x_{\text{test:}\ell}\}$
- Task: for $t = 1, \ldots, T_{\text{test}}$, label $y_{\text{test:}t}$ by class $\hat{z}_{\text{test:}t} \in S_{\text{train}}$; try to minimize misclassification rate

Additional assumptions

- For a point y from class with features x,

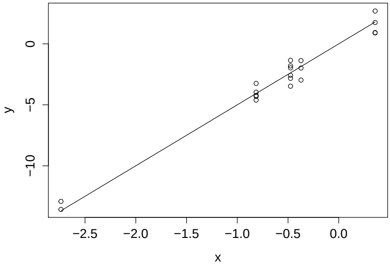
$$y = f(x) + \epsilon$$

  where the noise ϵ is drawn from some distribution and f is an unknown function
- The features for the training and test classes are sampled iid from the same distribution P

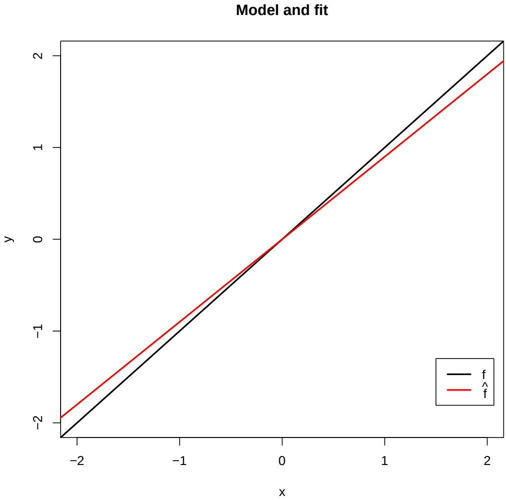
$$x_{\text{train}:i} \sim P$$

$$x_{\text{train}:i} \sim P$$

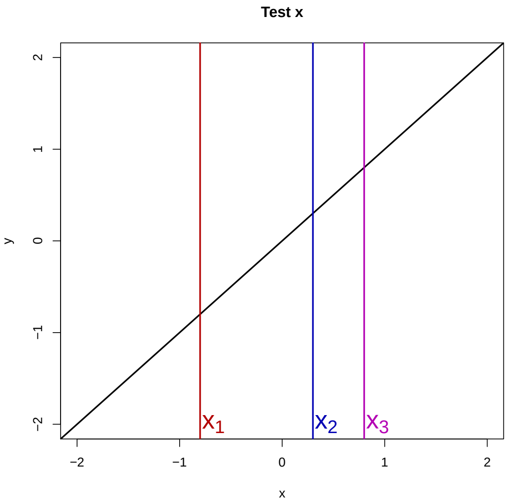
Toy example I



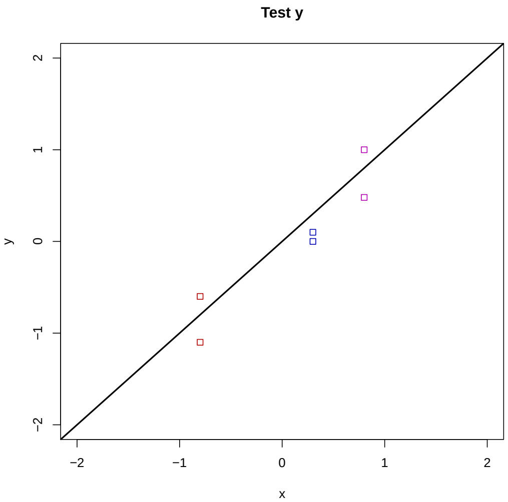
- Features x are one-dimensional real numbers, as are responses y. Parameter β is also a real number.
- Model is linear: $y \sim N(x\beta, \sigma_\epsilon^2)$



Suppose we estimated $\hat{\beta}$ from training data.

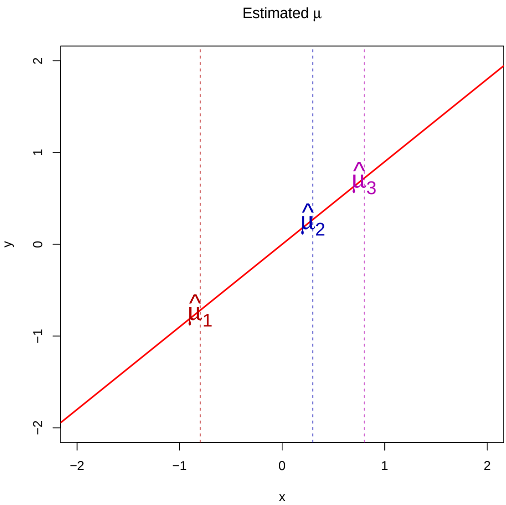


Generate features $x_{\text{test}:1}, \ldots, x_{\text{test}:\ell}$ iid $N(0, \sigma_x^2)$.

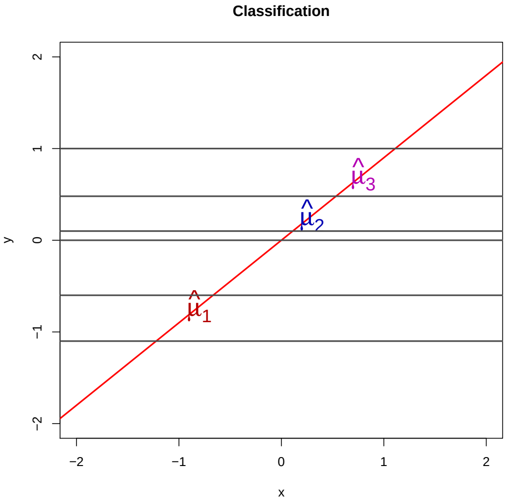


Hidden labels $z_{\text{test}:t}$ are iid uniform from S_{train}.
Generate $y_{\text{test}:t} \sim N(\beta x_{z_{\text{test}:t}}, \sigma_\epsilon^2)$

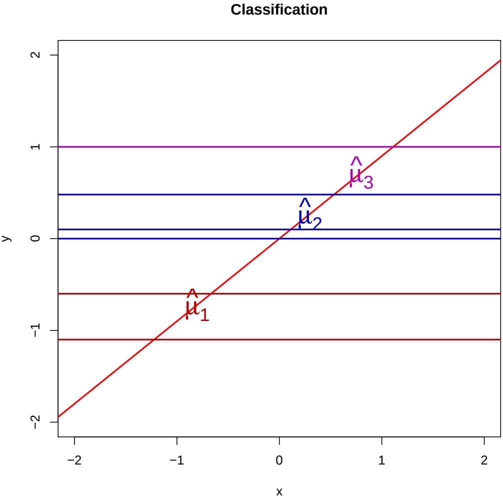
Classify $\hat{y}_{\text{test}:t}$



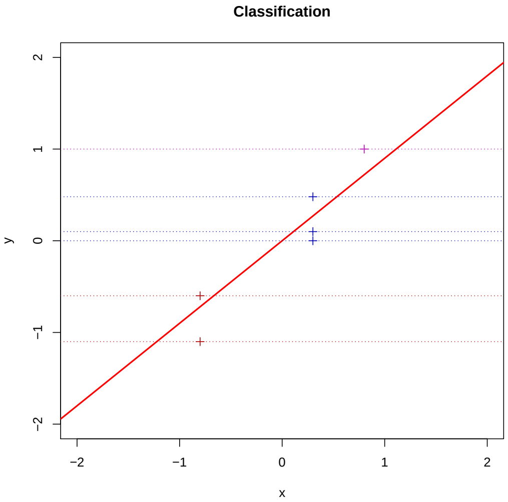
$$\hat{\mu}_{\text{test}:i} = \hat{\beta} x_{\text{test}:i}$$

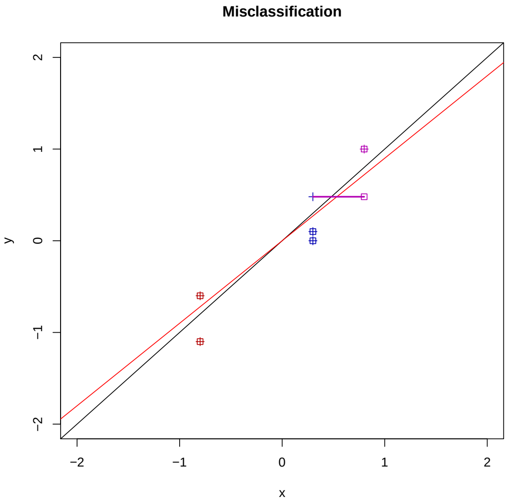


$$\hat{z}_{\text{test}:t} = \text{argmin}_z \ell_{\hat{\mu}_z}(y_{\text{test}:t})$$



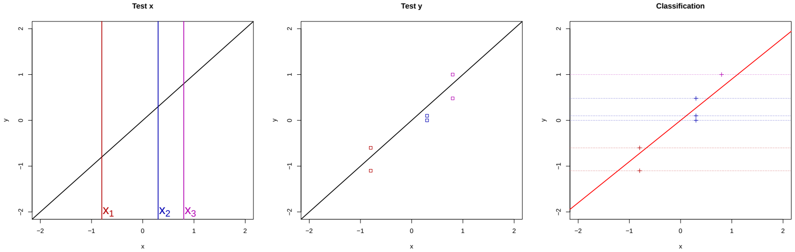
$$\hat{z}_{\text{test}:t} = \text{argmin}_z(\hat{\mu}_z - y_{\text{test}:t})^2$$


Classification
2
1
0
−1
−2
y
−2
−1
0
1
2
x



Misclassification
y
x
2
1
0
−1
−2
−2
−1
0
1
2

Toy example I



- Generate features $x_{\text{test}:1}, \ldots, x_{\text{test}:\ell}$ iid $N(0, \sigma_x^2)$.
- Hidden labels $z_{\text{test}:t}$ are iid uniform from S_{train}. Generate $y_{\text{test}:t} \sim N(\beta x_{z_{\text{test}:t}}, \sigma_\epsilon^2)$
- Classify $\hat{y}_{\text{test}:t}$ by maximum likelihood assuming $\hat{\beta}$ is correct. Thus:

$$\hat{z}_{\text{test}:t} = \text{argmin}_z (\hat{\beta} x_z - y_{\text{test}:t})^2$$

Toy example I: Questions

1. We know the prediction error is minimized when $\hat{\beta} = \beta$. Is it also true that misclassification error in the mind-reading game is minimized when $\hat{\beta} = \beta$?
2. Even if the answer to 1. is yes, should we estimate $\hat{\beta}$ using the same methods as in least-squares regression?

Question 1: Outline

We will find an answer to question 1 as follows

- Write an explicit expression for the misclassification rate as a function of $\hat{\beta}$
- Take the derivative of that expression with respect to $\hat{\beta}$ at the true β
- Does that derivative equal zero?
- If so, look at second derivatives, lower bounds, etc.

Write an explicit expression for the misclassification rate

- The expected misclassification error is the same if we take $T_{\text{test}} = 1$. Then let (x_*, y_*) be the feature-response pair in the test set, where

$$y_* = x_* \beta + \epsilon_*$$

- Denote the features for the incorrect classes as $x_1, \ldots, x_{\ell-1}$.
- Let $\delta = \hat{\beta} - \beta$.

Write an explicit expression for the misclassification rate (cont.)

- Ignore the possibility of ties. The response y_* is misclassified if and only if

$$\min_{i=1,\ldots,\ell-1} |y_* - x_i\hat{\beta}| < |y_* - x_*\hat{\beta}|$$

equivalently

$$\cup_{i=1,\ldots,\ell-1} E_i$$

where E_i is the event that

$$|y_* - x_i\hat{\beta}| < |y_* - x_*\hat{\beta}|$$

Write an explicit expression for the misclassification rate (cont.)

- Use the following conditioning

$$\mathbf{E}[\text{misclassification}] = \mathbf{E}[\mathbf{E}[\Pr_{x_1,\ldots,x_\ell}[\cup_i E_i]|x_* = x, \epsilon_* = \epsilon]]$$

- Use the fact that events E_i are independent and have the same probability, thus:

$$\mathbf{E}[\text{misclassification}] = 1 - \mathbf{E}[\mathbf{E}[(1 - \Pr[E_1])^{\ell-1}|x_* = x, \epsilon_* = \epsilon]]$$

- Next: write an expression for $\Pr[E_1]$

Write an expression for $\Pr[E_1]$.

- E_1 can also be written as the event

$$|x_*\beta + \epsilon_* - x_1(\beta + \delta)| < | - \delta x_* + \epsilon_*|$$

- Conditioning on ϵ_* and x_*, we have

$$\Pr[E_1] = \left|\Phi\left(\frac{x_*}{\sigma_x}\right) - \Phi\left(\frac{x_*(\beta - \delta) + 2\epsilon_*}{\sigma_x(\beta + \delta)}\right)\right|$$

An exact expression for expected misclassification is therefore

$$1 - \int_\epsilon \left[\int_x \left(1 - \left| \Phi\left(\frac{x}{\sigma_x}\right) - \Phi\left(\frac{x(\beta-\delta)+2\epsilon}{\sigma_x(\beta+\delta)}\right) \right| \right)^{\ell-1} d\Phi(\frac{x}{\sigma_x}) \right] d\Phi(\frac{\epsilon}{\sigma_\epsilon})$$

Take the derivative of the expression with respect to δ

Fix $\epsilon > 0$. The derivative of the inner integral wrt $\delta = 0$ is proportional to

$$\int_x (1 - \Phi(\frac{x\beta + 2\epsilon}{\sigma_x \beta}) + \Phi(\frac{x}{\sigma_x}))\phi(\frac{x\beta + 2\epsilon}{\sigma_x \beta})(x + \frac{\epsilon}{\beta})\phi(\frac{x}{\sigma_x})dx$$

Is the derivative zero?
The derivative of the inner integral wrt $\delta = 0$ is proportional to

$$\int_x (1 - \Phi(\frac{x\beta + 2\epsilon}{\sigma_x \beta}) + \Phi(\frac{x}{\sigma_x}))\phi(\frac{x\beta + 2\epsilon}{\sigma_x \beta})(x + \frac{\epsilon}{\beta})\phi(\frac{x}{\sigma_x})dx$$

In turn,

$$\phi\left(\frac{x\beta + 2\epsilon}{\sigma_x \beta}\right)\phi\left(\frac{x}{\sigma_x}\right) \propto \phi\left(\frac{\sqrt{2}(x + \frac{\epsilon}{\beta})}{\sigma_x}\right)$$

which is the density of a normal variate with mean $-\epsilon/\beta$
But now note that the other terms

$$\left(1 - \Phi\left(\frac{x\beta + 2\epsilon}{\sigma_x \beta}\right) + \Phi\left(\frac{x}{\sigma_x}\right)\right)\left(x - \frac{\epsilon}{\beta}\right)$$

are symmetric about $x = -\frac{\epsilon}{\beta}$.
Thus by symmetry, the derivative of the inner integral $\delta = 0$ vanishes. The same argument works for $\epsilon < 0$, hence the misclassification rate is stationary at $\hat{\beta} = \beta$.

Question 1: Remarks

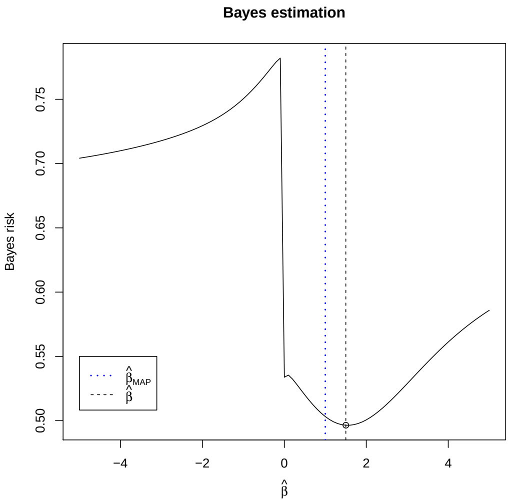
- (We'll skip the second derivative checking, etc.)
- The proof still works for ϵ with any distribution
- However, we cannot really relax the condition that x is gaussian, since we need
$$\phi\left(\frac{x\beta + 2\epsilon}{\sigma_x \beta}\right)\phi\left(\frac{x}{\sigma_x}\right) \propto \phi\left(\frac{\sqrt{2}(x + \frac{\epsilon}{\beta})}{\sigma_x}\right)$$
This can only work if $\phi(x) = \exp(-Cx^2)h(|x|)$
- The proof probably generalizes to higher dimensions
- Conjecture: $\hat{\beta} = \beta$ *only if* $x \sim N(\mu, \Sigma)$

Toy example I: Estimation

- Second question: what about estimation?
- Take a Bayesian viewpoint: suppose we have a prior distribution for β
- For *least-squares regression*, we would use $\hat{\beta} = \int \beta p_{posterior}(\beta) d\beta$, the posterior mean.
- For *identification*, we would choose

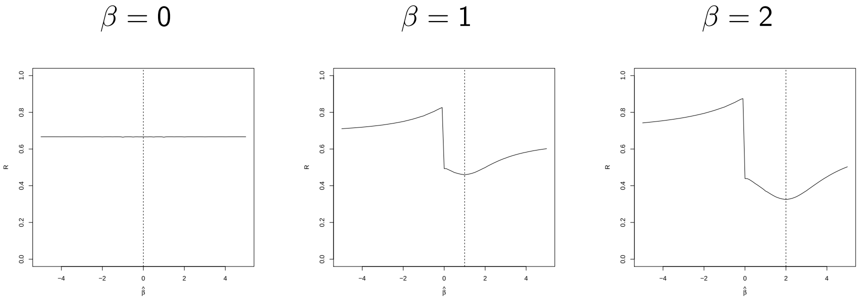
$$\hat{\beta} = \text{argmin}_{\hat{\beta}} \int R(\beta; \hat{\beta}) p_{posterior}(\beta) d\beta$$

  where R is the expected misclassification rate.
- How will these differ?



Point estimate for identification (black dashed) is larger than posterior mean (blue dotted)

Toy example I: Estimation



Risk function is more sensitive for large β.

Estimation: questions

- Is the optimal $\hat{\beta}$ for identification is in general "larger" than the optimal $\hat{\beta}$ for regresssion, in a frequentist (e.g. minimax) sense?
- Lasso/Ridge penalized regression models are commonly used for identification
- Hypothesis: the optimal λ for identifying x from y will be smaller (hence produce less sparse $\hat{\beta}$) than the optimal λ for regression $y \sim x$.

Section 3

Experiments

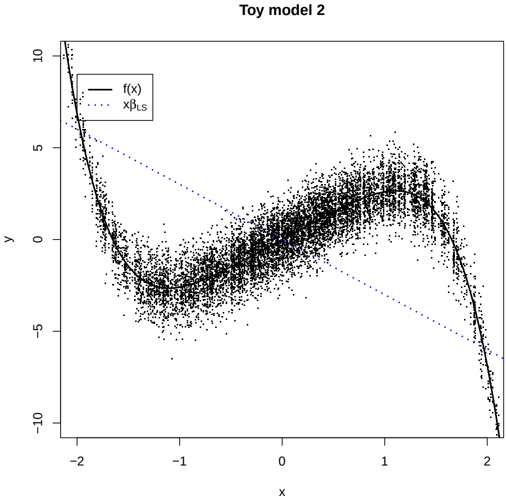
Section 4

Nonlinear toy example

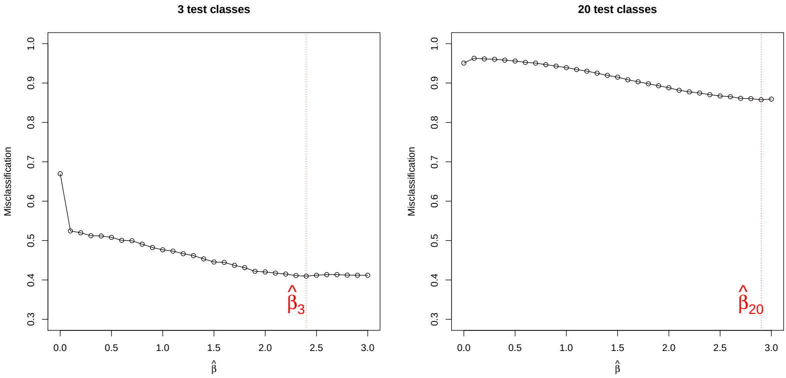
More questions

3. What happens if the true regression function f is nonlinear, but we restrict $\hat{f}$ to be linear?
4. What happens when the number of classes ℓ increases? What if ℓ increases while σ_ϵ^2 decreases?

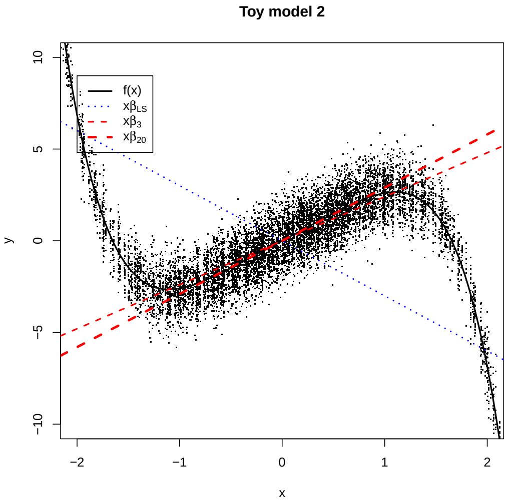
Toy example IIa



Toy example IIa



Effect of increasing ℓ.


Toy model 2
f(x)
xβLS
xβ3
xβ20
y
x
10
5
0
−5
−10
−2
−1
0
1
2

Why is this?

- We can relate identification to regression with a different loss function
- Least squares loss

$$\mathbf{E}[(y - \hat{y})^2]$$

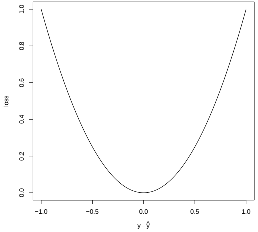
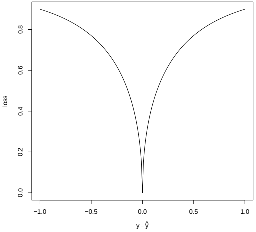
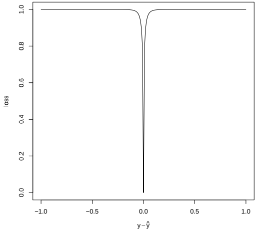
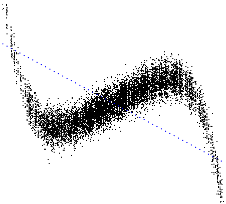
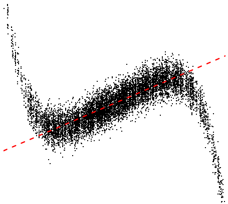
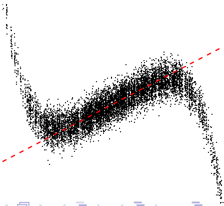
- Identification loss

$$\mathbf{E}[1 - \Pr[|y - \hat{y}'| < |y - \hat{y}|]^{\ell-1}]$$

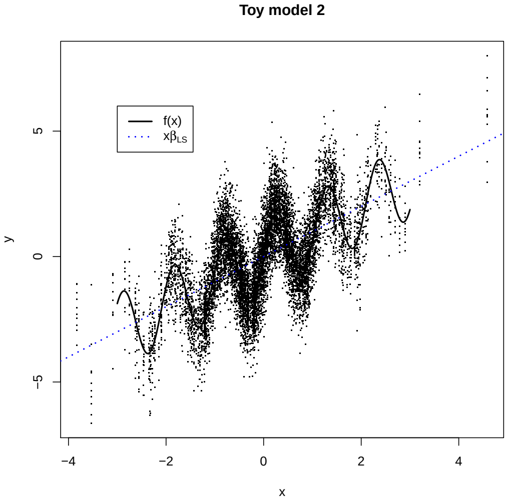
where $\hat{y}'$ is the predicted value for a randomly drawn x.

Why is this?

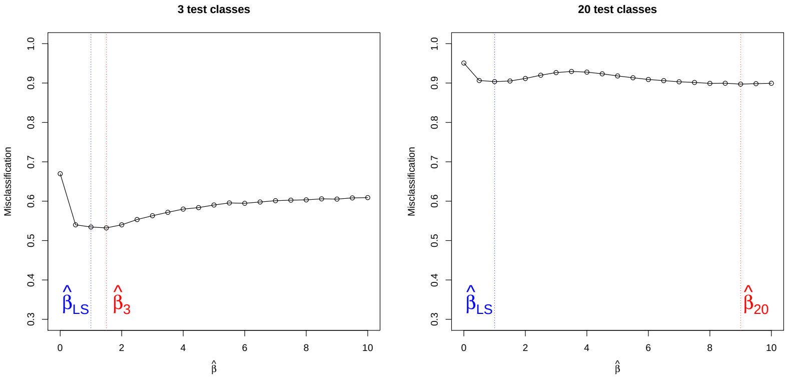
Identification loss more closely resembles 0-1 loss as ℓ increases.

Squared error	$\ell = 3$	$\ell = 20$
		
		

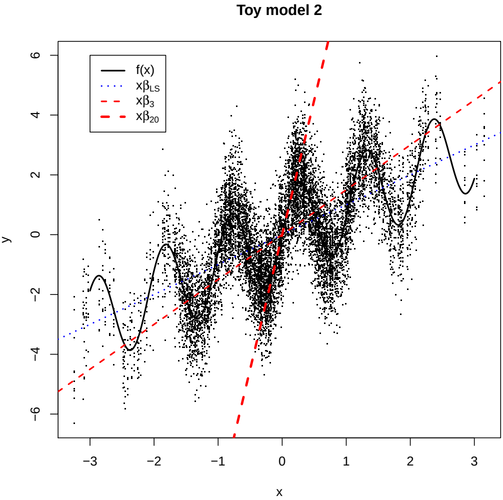
Toy example IIb



Toy example IIb



Effect of increasing ℓ.



Effect of increasing ℓ: global trends will become ignored in favor of locally linear trends!

Implications

- "The model is always wrong"
- Statistical methods should be robust to small deviations from the model
- Even when minor nonlinearities exist in the model, identification performance fails to reflect global fit

Conclusions

- The problem of *decoding*, predicting x from y, is of interest to many neuroscientists
- Different formulations of the decoding problem: classification, identification, and reconstruction (regression) have different properties and advantages
- Statistical theory can help with training the models *and* with interpreting the results

In particular...

- Identification is similar to regression $y \sim x$ in a special case, but can benefit from less sparse estimates.
- Identification can lead to counterintuitive results when there are nonlinearities and ℓ is large

References

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