

Differential Geometric Theory of Statistics

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1 Introduction

Statistics is a science which studies methods of inference, from observed data, concerning the probabilistic structure underlying such data. The class of all the possible probability distributions is usually too wide to consider all its elements as candidates for the true probability distribution from which

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the data were derived. Statisticians often assume a statistical model which is a subset of the set of all the possible probability distributions, and evaluate procedures of statistical inference assuming that the model is faithful, i.e., it includes the true distribution. It should, however, be remarked that a model is not necessarily faithful but is approximately so. In either case, it should be very important to know the shape of a statistical model in the whole set of probability distributions. This is the geometry of a statistical model. A statistical model often forms a geometrical manifold, so that the geometry of manifolds should play an important role. Considering that properties of specific types of probability distributions, for example, of Gaussian distributions, of Wiener processes, and so on, have so far been studied in detail, it seems rather strange that only a few theories have been proposed concerning properties of a family itself of distributions. Here, by the properties of a family we mean such geometric relations as mutual distances, flatness or curvature of the family, etc. Obviously it is not a trivial task to define such geometric structures in a natural, useful and invariant manner. Only local properties of a statistical model are responsible for the asymptotic theory of statistical inference. Local properties are represented by the geometry of the tangent spaces of the manifold. The tangent space has a natural Riemannian metric given by the Fisher information matrix in the regular case. It represents only a local property of the model, because the tangent space is nothing but local linearization of the model manifold. In order to obtain larger-scale properties, one needs to define mutual relations of the two different tangent spaces at two neighboring points in the model. This can be done by defining a one-to-one affine correspondence between two tangent spaces, which is called an affine connection in differential geometry. By an affine connection, one can consider local properties around each point beyond the linear approximation. The curvature of a model can be obtained by the use of this connection. It is clear that such a differential-geometrical concept provides a tool convenient for studying higher-order asymptotic properties of inference. However, by connecting local tangent spaces further, one can obtain global relations. Hence, the validity of the differential-geometrical method is not limited within the framework of asymptotic theory.

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- 3 Higher-Order Asymptotic Theory of Statistical Inference in Curved Exponential Family
- 4 Information, Sufficiency and Ancillarity Higher Order Theory
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