Charles Zheng

1 Background

Theorem 1. Define $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and

$$\Psi(x) = \int_{x}^{\infty} \phi(t)dt$$

for $x \in \mathbb{R}$. Then

$$\left(\frac{1}{x^3} - \frac{1}{x}\right)\phi(x) \le \Psi(x) \le \frac{1}{x}\phi(x)$$

Theorem 2. (Borell-TIS inequality) Let f_t be a gaussian process such that $\mathbb{E}[f_t] = 0$ Then on any measurable set D, and u > 0,

$$\mathbb{P}[\sup_{D} f_t > u + \mathbb{E}[\sup_{D} f_t]] \le \exp(-u^2/(2\sigma_{max}^2))$$

where

$$\sigma_{max} = \sup_{D} \mathbf{E}[f_t^2]$$

Theorem 3. (Slepian's inequality) If f and g are as bounded, centered gaussian processes, and

$$\mathbb{E}[(f_t - f_s)^2] \le \mathbb{E}[(g_t - g_s)^2]$$

then

$$\mathbb{P}[\sup_{t \in D} f_t > u] \le \mathbb{P}[\sup_{t \in D} g_t > u]$$

2 Supremum of an isotropic GP

Theorem 4. Let f_t be a gaussian process on \mathbb{R} , with $Cov(f_t, f_u) = C(t-u)$, where C(0) = 1, and $C(t) \to 0$ as $||t|| \to \infty$. Then supposing f_t is bounded on [0, 1],

$$\mathbb{P}\left(\lim_{T\to\infty}\frac{\sup_{[0,T]}f_t}{\sqrt{2\log(T)}}=1\right)=1$$

Proof.

It suffices to prove

$$\mathbb{P}\left(1 - \varepsilon \le \liminf_{T \to \infty} \frac{\sup_{[0,T]} f_t}{\sqrt{2\log(T)}} \le \limsup_{T \to \infty} \frac{\sup_{[0,T]} f_t}{\sqrt{2\log(T)}} \le 1 + \varepsilon\right) = 1$$

for arbitrary $\varepsilon \in (0,1)$.

Take $\varepsilon \in (0,1)$.

Find $\tau>0$ such that $C(\tau)<\frac{\varepsilon}{2-\varepsilon}$ and find $T_0>0$ such that $T>\max\{2\tau,\frac{2-\varepsilon}{\varepsilon}\log(2\tau),e^{\frac{1-C(\tau)}{(1-\varepsilon)^2}}\}$. For each of $n=1,\ldots,$, let $T=T_0+n$, and let let $m=\lfloor\frac{T+1}{\tau}\rfloor$. Define $t_k=k\tau$ for $k=1,\ldots,m$. Let Z_1,\ldots,Z_m be iid $N(0,1-C(\tau))$. For $i\neq j$, we have

$$\mathbb{E}[(Z_i - Z_j)^2] = 2(1 - C(\tau)) \le 2(1 - C((i - j)\tau)) = \mathbb{E}[(f_{t_i} - f_{t_i})^2]$$

Hence by Slepian's inequality,

$$\mathbb{P}(\sup_{t \in [0,T]} f_t > u) \ge \mathbb{P}(\max_{i \in \{1,...,m\}} f_{t_i} > u) \ge \mathbb{P}(\max_{i \in \{1,...,m\}} Z_i > u)$$

for all u > 0. Thus, taking $u = (1 - \varepsilon)\sqrt{2\log(T+1)}$ so that

$$u \le \left(1 - \frac{\varepsilon}{2}\right) \sqrt{2(1 - C(\tau))\log\left(\frac{T}{\tau} - 1\right)}$$

and

$$\frac{\sqrt{1 - C(\tau)}}{u} - \frac{(1 - C(\tau))^{3/2}}{u^3} \le \frac{\sqrt{1 - C(\tau)}}{2u}$$

we have

$$\begin{split} \mathbb{P}(\sup_{t \in [0,T]} f_t < \sqrt{2 \log(T-1)} (1-\epsilon)) &\leq \mathbb{P}(\max_{i \in \{1,\dots,m\}} Z_i < u) \qquad (1) \\ &= \left(1 - \Psi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right)^m \qquad (2) \\ &\leq \left(1 - \left(\frac{\sqrt{1-C(\tau)}}{u} - \frac{(1-C(\tau))^{3/2}}{u^3}\right) \phi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right)^m \\ &\leq \left(1 - \left(\frac{\sqrt{1-C(\tau)}}{2u}\right) \phi\left(\frac{u}{\sqrt{1-C(\tau)}}\right)\right)^m \\ &\leq \exp\left(-m\left(\frac{\sqrt{1-C(\tau)}}{2u}\right) \phi\left(\left(1 - \frac{\varepsilon}{2}\right)\sqrt{2\log\left(\frac{T}{\tau} - 1\right)}\right)\right) \\ &\leq \exp\left(-m\left(\frac{\sqrt{1-C(\tau)}}{2u}\right) \phi\left(\left(1 - \frac{\varepsilon}{2}\right)\sqrt{2\log\left(\frac{T}{\tau} - 1\right)}\right)\right) \\ &= \exp\left(-\frac{m}{\sqrt{2\pi}\left(\frac{T}{\tau} - 1\right)^{\left(1 - \frac{\varepsilon}{2}\right)^2}}\left(\frac{\sqrt{1-C(\tau)}}{2u}\right)\right) \\ &\leq \exp\left(-\frac{\frac{T}{\tau} - 1}{\sqrt{2\pi}\left(\frac{T}{\tau} - 1\right)^{\left(1 - \frac{\varepsilon}{2}\right)^2}}\left(\frac{\sqrt{1-C(\tau)}}{2u}\right)\right) \\ &= \exp\left(-\frac{\left(\frac{T}{\tau} - 1\right)^{1-\left(1 - \frac{\varepsilon}{2}\right)^2}}{\sqrt{2\pi}}\left(\frac{\sqrt{1-C(\tau)}}{2(1-\varepsilon)\sqrt{2\log(T_0 + n - 1)}}\right)\right) \\ &= \exp\left(-\frac{\left(\frac{T_0 + n}{\tau} - 1\right)^{1-\left(1 - \frac{\varepsilon}{2}\right)^2}}{\sqrt{2\pi}}\left(\frac{\sqrt{1-C(\tau)}}{2(1-\varepsilon)\sqrt{2\log(T_0 + n - 1)}}\right)\right) \end{split}$$

Hence

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{t \in [0,T]} (1-\varepsilon)\sqrt{2\log(T_0 + n - 1)}\right) < \infty$$
 (11)

which by Borel-Cantelli, and the fact that

$$\liminf_{T \to \infty} \frac{\sup_{[0,T]} f_t}{\sqrt{2 \log(T)}} \le \liminf_{n \to \infty} \frac{\sup_{[0,T_0+n]} f_t}{\sqrt{2 \log(T_0+n-1)}}$$

implies

$$\mathbb{P}\left(\liminf_{T\to\infty} \frac{\sup_{[0,T]} f_t}{\sqrt{2\log(T)}} \ge 1\right) = 1 \tag{12}$$

Let $\mu = \mathbb{E}[\sup_{[0,1]} f_t]$, so that

$$\mathbb{P}[\sup_{t \in [0,1]} f_t > u] \le e^{-(u-\mu)^2/2}$$

for all $u > \mu$. Note that

$$\mathbb{P}\left(\limsup \frac{\sup_{t \in [0,T]} f_t}{\sqrt{2 \log T}} > 1 + \varepsilon\right) \le \mathbb{P}\left(\sum_{n=1}^{\infty} \mathbb{1}\left\{\sup_{t \in [0,n]} f_t > (1+\varepsilon)\sqrt{2 \log(n-1)}\right\} = \infty\right)$$
(13)

$$\leq \mathbb{P}\left(\sum_{n=1}^{\infty} \mathbb{1}\left\{\sup_{t \in [n]} st\right\}\right)$$

Meanwhile,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{t \in [n-1,n]} f_t > (1+\varepsilon)\sqrt{2\log(n-1)}\right) \le \sum_{n=1}^{\infty} e^{-(\sqrt{2\log(n-1)-\mu})^2/2} < \infty$$

Hence by the Borel-Cantelli lemma and (??),

$$\mathbb{P}\left(\limsup \frac{\sup_{t \in [0,T]} f_t}{\sqrt{2\log T}} > 1 + \varepsilon\right) = 1$$

Combining this with (??) yields the desired result. \square .

3 References

Adler RF, Taylor J. Random Fields and Geometry