

Charles Zheng

## 1 Background

**Theorem (Borell-TIS inequality)** Let  $f_t$  be a gaussian process such that  $\mathbb{E}[f_t] = 0$  Then on any set  $D$ , and  $u > 0$ ,

$$\mathbb{P}[\sup_D f_t > u + \mathbb{E}[\sup_D f_t]] \leq \exp(-u^2/(2\sigma_{max}^2))$$

where

$$\sigma_{max} = \sup_D \mathbf{E}[f_t^2]$$

**Theorem (Slepian's inequality)** If  $f$  and  $g$  are as bounded, centered gaussian processes, and

$$\mathbb{E}[(f_t - f_s)^2] \leq \mathbb{E}[(g_t - g_s)^2]$$

then

$$\mathbb{P}[\sup_{t \in D} f_t > u] \leq \mathbb{P}[\sup_{t \in D} g_t > u]$$

**Theorem.** Let  $f$  be centered stationary Gaussian process on a compact group  $T$ . Then the following three conditions are equivalent: (i)  $f_t$  is continuous (ii)  $f_t$  is bounded (iii)

$$\int_0^\infty \sqrt{H(\varepsilon)} d\varepsilon < \infty$$

## 2 Isotropic GPs are bounded

Take centered isotropic  $f_t$  on the real line with  $C(t) > 0$  for  $t \neq 0$ . We will show that  $f_t$  is bounded on the interval  $[0, T]$ .

Otherwise, it follows from theorem ?? that for some constant  $C$  and  $\varepsilon_0 > 0$ , we have

$$H(\varepsilon) > \frac{C}{\varepsilon^4}$$

for all  $\varepsilon < \varepsilon_0$ . We have

$$H(\varepsilon) = \log(N(\varepsilon)) \leq \log(T) - \log(\lambda(\{t : C(t) > 1 - \frac{\varepsilon}{2}\}))$$

### 3 Supremum of an isotropic GP

Let  $f_t$  be a gaussian process on  $\mathbb{R}^D$ , with  $\text{Cov}(f_t, f_u) = C(t - u)$ , where  $C(0) = 1$ , and  $C(t) \rightarrow 0$  as  $\|t\| \rightarrow \infty$ . Then if  $f_t$  is bounded on an interval,

$$\mathbb{P} \left( \lim_{T \rightarrow \infty} \frac{\sup_{[-T, T]^D} f_t}{\sqrt{2D \log(T)}} = 1 \right) = 1$$

Sketch of proof:

1. *Lower bound.*

Fix  $T$ . Let  $\delta = \exp(\sqrt{\log(T)})$  and consider  $t_1, \dots, t_{(2T/\delta)^D}$  on a square lattice of spacing  $\delta$  on  $[-T, T]^D$ . Let  $Z_{t_1}, \dots, Z_{t_{(2T/\delta)^D}}$  iid  $N(0, 1 - C(\delta))$ . By Slepian's inequality,  $\mathbb{P}(\max_{t_1, \dots, t_{(2T/\delta)^D}} f_t > u) \geq \mathbb{P}(\max_{t_1, \dots, t_{(2T/\delta)^D}} Z_t > u)$ . Now note that as  $T \rightarrow \infty$ ,

$$\mathbb{P} \left( \lim_{T \rightarrow \infty} \frac{\max Z_t}{\sqrt{2D \log(T)}} = 1 \right) = 1$$

This suggests, and with some more detailed analysis, implies that

$$\mathbb{P} \left( \limsup_{T \rightarrow \infty} \frac{\sup_{[-T, T]^D} f_t}{\sqrt{2D \log(T)}} \geq 1 \right) = 1$$

1. *Upper bound.*

Partition  $[-T, T]^D$  into hypercubes of edge length 1. By union bound and Borrell-TIS inequality,

$$\mathbb{P} \left( \sup_{[-T, T]^D} f_t \geq u \right) \leq (2T)^D \mathbb{P} \left( \sup_{[-T, T]^D} f_t \geq u \right) \leq (2T)^D e^{-u^2/2}$$

This can be used to show

$$\mathbb{P} \left( \limsup_{T \rightarrow \infty} \frac{\sup_{[-T, T]^D} f_t}{\sqrt{2D \log(T)}} \leq 1 \right) = 1$$

### 4 One-dimensional isotropic gaussian processes

Now that

$$C_\kappa(t) = [1 - \kappa|t|]_+$$

is a covariance kernel. This implies the following:

Suppose  $C(0) = 1$ ,  $-C''(t) > 0$  for  $t > 0$  and  $-\int_0^\infty tC''(t) < \infty$ . Then  $C(t)$  is a covariance kernel for a gaussian process. This is because one can find a mixture density  $\rho(k)$  such that

$$C(t) = \int_0^\infty C_\kappa(t)\rho(\kappa)d\kappa$$

## 5 References

Adler RF, Taylor J. *Random Fields and Geometry*