HW9: INTRODUCTION TO INDUCTION AND RECURSION

"This inductively justifies the conclusion that induction cannot justify any conclusions." - David Deutsch

Course: CS 5002
Fall 2021
Due: Saturday, December 4, 2021

PROBLEMS

Problem 1: Sequences

What are the terms a_0 , a_1 , a_2 , and a_3 of the sequence $\{a_n\}$, where $\{a_n\}$ equals to:

(a) (2 points) $(-2)^n$

(a)
$a_0 = (-2)^0 = 1$
$a_1 = (-2)^1 = -2$
$a_2 = (-2)^2 = 4$
$a_3 = (-2)^3 = -8$

(b) (2 points) 3

(b)
$a_0 = 3$
$a_1 = 3$
$a_2 = 3$
$a_3 = 3$

(c) (2 points) $7 + 4^n$

(c)
$a_0 = 7 + 4^0 = 8$
$a_1 = 7 + 4^1 = 11$
$a_2 = 7 + 4^2 = 23$
$a_3 = 7 + 4^3 = 71$

(d) (2 points) $2^n + (-2)^n$

(d)

$a_0 = 2^0 + (-2)^0 = 2$

$a_1 = 2^1 + (-2)^1 = 0$

$a_2 = 2^2 + (-2)^2 = 8$

$a_3 = 2^3 + (-2)^3 = 0$

Problem 2: Sequences and Word Problems

Please list the first 10 terms of each of these sequences.

(a) (2 points) The sequence that begins with 2, and in which each successive term is 3 more than the preceding term.

2, 5, 8, 11, 14, 17, 20, 23, 26, 29.

(b) (2 points) The sequence that lists the odd positive integers in an increasing order, listing each odd integer twice.

1, 1, 3, 3, 5, 5, 7, 7, 9, 9

(c) (2 points) The sequence whose first term is 2, second term is 4, and each succeeding term is the sum of the two preceding terms.

2, 4, 6, 10, 16, 26, 42, 68, 110, 178

(d) (2 points) The sequence whose nth term is the sum of the first n positive integers.

1, 3, 6, 10, 15, 21, 28, 36, 45, 55

(e) (2 points) The sequence whose n-th term is defined as: $3n - n^2$.

2, 2, 0, -4, -10, -18, -28, -40, -54, -70

Problem 3: Sums

What are the values of these sums, where set $S = 1, 3, 5, 7$?

(a) (3 points) $\sum_{j \in S} j^2$

(a)

$$\sum_{j \in S} j^2 = 1^2 + 3^2 + 5^2 + 7^2 = 84$$

(b) (3 points) $\sum_{j \in S} 1$

(b)

$$\sum_{j \in S} j^2 = 1 + 1 + 1 + 1 = 4$$

Problem 4: Double Sums

Compute each of these double sums.

(a) (3 points) $\sum_{i=0}^{3}\sum_{j=0}^{2}(3i+2j)$

(a)

$$\sum_{i=0}^{3}\sum_{j=0}^{2}(3i+2j)$$
$$= (0+0)+(3+0)+(6+0)+(9+0)+(0+2)+(3+2)+(6+2)$$
$$+(9+2)+(0+4)+(3+4)+(6+4)+(9+4)$$
$$= 0+3+6+9+2+5+8+11+4+7+10+13 = 78$$

(b) (3 points) $\sum_{i=1}^{3}\sum_{j=0}^{2} j$

(b)

$$\sum_{i=1}^{3}\sum_{j=0}^{2} j = (0+1+2)\cdot(3-1+1) = 9$$

Problem 5: Recurrence Relations I

Find the first five terms for each of these recurrence relations, with the corresponding initial conditions:

(a) (2 points) $a_n = -a_{n-1}, a_0 = 5$

(a)

$a_0 = 5$

$a_1 = -5$

$a_2 = 5$

$a_3 = -5$

$a_4 = 5$

(b) (2 points) $a_n = a_{n-1} + 3, a_0 = 1$

(b)

$a_0 = 1$

$a_1 = 4$

$a_2 = 7$

$a_3 = 10$

$a_4 = 13$

(c) (2 points) $a_n = a_{n-1} - n, a_0 = 4$

(c)

$a_0 = 4$

$a_1 = 3$

$a_2 = 1$

$a_3 = -2$

$a_4 = -6$

(d) (2 points) $a_n = 2a_{n-1} - 3, a_0 = -1$

(d)

$a_0 = -1$

$a_1 = -5$

$a_2 = -13$

$a_3 = -29$

$a_4 = -61$

(e) (2 points) $a_n = (n+1)a_{n-1}, a_0 = 2$

(e)
$a_0 = 2$
$a_1 = 4$
$a_2 = 12$
$a_3 = 48$
$a_4 = 240$

Problem 6: Recurrence Relations II

Find $f(1), f(2), f(3)$, and $f(4)$ if $f(n)$ is defined recursively, with initial condition $f(0) = 1$:

(a) (2 points) $f(n+1) = 3f(n)$

(a)
$f(0) = 1$
$f(1) = 3f(0) = 3$
$f(2) = 3f(1) = 9$
$f(3) = 3f(2) = 27$
$f(4) = 3f(3) = 81$

(b) (3 points) $f(n+1) = f(n)^2 + f(n) + 1$

(b)
$f(0) = 1$
$f(1) = f(0)^2 + f(0) + 1 = 1 + 1 + 1 = 3$
$f(2) = f(1)^2 + f(1) + 1 = 9 + 3 + 1 = 13$
$f(3) = f(2)^2 + f(2) + 1 = 169 + 13 + 1 = 183$
$f(4) = f(3)^2 + f(3) + 1 = 33489 + 183 + 1 = 33673$

Problem 7: Induction I

Use mathematical induction to show that

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^n} = 1 - \frac{1}{3^n}$$

whenever n is a positive integer.

Proof:
By induction on n, when n is positive integer.
Base case: n = 1
When n = 1, the left side of the equation is $\frac{2}{3}$.
When n = 1, the right side of the equation is $1 - \frac{1}{3} = \frac{2}{3}$.
Therefore, when n = 1, the equation holds.
Inductive step: Suppose that for positive integer k, $\frac{2}{3} + \cdots + \frac{2}{3^k} = 1 - \frac{1}{3^k}$, then we will show that

$$\frac{2}{3} + \cdots + \frac{2}{3^{k+1}} = 1 - \frac{1}{3^{k+1}}$$

Starting with the left side of the equation to be proven:

$$\frac{2}{3} + \cdots + \frac{2}{3^{k+1}} = \frac{2}{3} + \cdots + \frac{2}{3^k} + \frac{2}{3^{k+1}} = 1 - \frac{1}{3^k} + \frac{2}{3^{k+1}} = 1 - \frac{(3^{k+1}) + 2(3^k)}{3^k 3^{k+1}} = 1 - \frac{(3^k)}{3^k 3^{k+1}}$$

$$= 1 - \frac{1}{3^{k+1}}$$

Therefore, $\frac{2}{3} + \cdots + \frac{2}{3^{k+1}} = 1 - \frac{1}{3^{k+1}}$ ■

Problem 8: Induction II

Use mathematical induction to prove that 9 divides

$$n^3 + (n+1)^3 + (n+2)^3$$

whenever n is a positive integer.

Proof:
We need to prove that $n^3 + (n+1)^3 + (n+2)^3 = 9k, \textit{where } k \textit{ is a positive integer}$.
By induction on n, when n is a positive integer.
Base case: n = 1
When n = 1, the left side of the equation is 1 + 8 + 27 = 36.
9 divides 36.
Inductive step: Suppose that for positive integer k, $k^3 + (k+1)^3 + (k+2)^3$, then we will show that 9 divides $(k+1)^3 + (k+2)^3 + (k+3)^3$.
Since 9 divides $k^3 + (k+1)^3 + (k+2)^3$, if 9 divides $(k+1)^3 + (k+2)^3 + (k+3)^3 - (k^3 + (k+1)^3 + (k+2)^3)$, we know that 9 divides $(k+1)^3 + (k+2)^3 + (k+3)^3$.

$$(k+1)^3 + (k+2)^3 + (k+3)^3 - (k^3 + (k+1)^3 + (k+2)^3) = 9(k^2 + 3k + 3)$$

Therefore, since the difference is divisible by 9, $(k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by 9 from the inductive step. ■

Problem 9: Induction III

Please prove by induction:

$$\sum_{k=1}^{n}(k \cdot k!) = (n+1)! - 1$$

Proof:

By induction on n.

Base case: n = 1

When n = 1, the left side of the equation is $\sum_{k=1}^{1}(k \cdot k!) = 1$

When n = 1, the right side of the equation is $(1+1)! - 1 = 2 - 1 = 1$

Therefore, when n = 1, the equation holds.

Inductive step: suppose that for positive integer k, $\sum_{k=1}^{n}(k \cdot k!) = (n+1)! - 1$, then we will show that $\sum_{k=1}^{n+1}(k \cdot k!) = (n+2)! - 1$.

$$\sum_{k=1}^{n+1}(k \cdot k!) = \sum_{k=1}^{n}(k \cdot k!) + (n+1) \cdot (n+1)! = (n+1)! - 1 + (n+1) \cdot (n+1)!$$
$$= (n+2)(n+1)! - 1 = (n+2)! - 1$$

Therefore, $\sum_{k=1}^{n+1}(k \cdot k!) = (n+2)! - 1.$■

Problem 10: Induction IV

Please prove by induction that for all integers $n \geq 0$ it holds that:

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$$

Proof:
By induction on n.
Base case: n = 0.
When n = 0, the left side of the equation is $\sum_{i=1}^{1} i \cdot 2^i = 2$.
When n = 0, the right side of the equation is $0 \cdot 2^2 + 2 = 2$.
Therefore, when n = 0, the equation holds.
Inductive step: Suppose that for positive integer k, $\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$, then we will show that $\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$.

$$\sum_{i=1}^{k+2} i \cdot 2^i = \sum_{i=1}^{k+1} i \cdot 2^i + (k+2) \cdot 2^{(k+2)} = k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{(k+2)} = (2k+2) \cdot 2^{k+2} + 2$$
$$= 2(k+1) \cdot 2^{k+2} + 2 = (k+1)2^{k+3} + 2$$

Therefore, $\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$. ■

Problem 11: Standing in Line and Induction

Use mathematical induction to prove that if n people stand in a line, where n is a positive integer, and if the first person in the line is a woman and the last person in line is a man, then somewhere in the line there is a woman directly in front of a man.

Proof:
By induction on n.
Base case: n = 2, the first person is a woman, the last person in the line is a man.
There is a woman directly in front of the last man.
Inductive step: Suppose there are k people, if there is a woman directly in front of a man, then for k + 1 people, there is also a woman directly in front of a man.
There are three cases:
Case 1: The new person is before the woman that is directly in front of a man. In this case, the woman is still directly in front of the man.
Case 2: The new person is after the man that is directly after a woman. In this case, the woman is still directly in front of the man.
Case 3: The new person is added between the woman and the man who are next to each other. In this case, if the new person is a woman, she is directly in front of the man. If the new person is a man, he is directly behind the woman. In both cases, there is still a woman directly in front of a man.
Therefore, for any positive integer $n(n \geq 2)$, there is always a woman directly in front of a man in the line.

Problem 12: Induction and Postage Stamps

Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for n $\geq$ 18.

(a) (3 points) Show statements $P(18), P(19), P(20)$, and $P(21)$ are true, completing the basis step of the proof.

(a)
P(18) can be formed by two 7-cent and one 4-cent.
P(19) can be formed by three 4-cent and one 7-cent.
P(20) can be formed by five 4-cent.
P(21) can be formed by three 7-cent.

(b) (3 points) What is the inductive hypothesis of the proof?

(b)
The inductive hypothesis is:
For all $k \geq 18$, $P(k)$ is true.

(c) (2 points) What do you need to prove in the inductive step?

(c)
In order to prove that P(k + 1) is true, we need to prove that P(k – 3) is true.

(d) (5 points) Complete the inductive step for k $\geq$ 21.

(d)
Inductive step: For $k \geq 21, assume\ P(j) is\ true\ for\ any\ j\ in\ range\ 18\ through\ k.$
Since $k \geq 21$, then $k - 3 \geq 18$. Therefore, k – 3 is in the range 18 through k and by the inductive hypothesis, k – 3 cents can be combined by 7-cent and 4-cent.
When an additional 4-cent is purchased, the cent is (k – 3) + 4 = k + 1. Therefore, P(k + 1) is true.

(e) (2 points) Explain why these steps show that this statement is true whenever $n \geq 18$.

(e)
Since we have proved that P(18), P(19), P(20), and P(21) are all true. The inductive hypothesis assumes that all $k \geq 18$, P(k) is true. Using the four base cases, we can assume that if any P(k – 3) is true, then adding one 4-cent stamp would still make P(k + 1) true for $k \geq 21$. Therefore, we have proved that for $k \geq 18$, P(k) is true.

Question	Points	Score
Sequences	8	
Sequences and Word Problems	10	
Sums	6	
Double Sums	6	
Recurrence Relations I	10	
Recurrence Relations II	5	
Induction I	7	
Induction II	8	
Induction III	7	
Induction IV	8	
Standing in Line and Induction	10	
Induction and Postage Stamps	15	
Total:	100	

SUBMISSION DETAILS

Things to submit:

- Please submit this assignment as a .pdf named "CS5002_[lastname]_HW9.pdf" through Canvas by 11:59pm PT on Saturday, December 4, 2021.
- Please make sure your name is in the document as well (e.g., written on the top of the first page).