CS 5002, FALL 2021, DAYTIME SECTION - MIDTERM A

I can feel the paradise before my world implodes...

Course: CS 5002
Fall 2021
Due: 11:59am PT on Friday, October 22, 2021

Problem 1: Propositional Logic and Boolean Algebra

(a) Let propositions p and q be defined as: (5)

$p :=$ "I am feeling sick." and

$q :=$ "I am getting some rest, and some homemade chicken soup."

Using logical operators, and propositions p and q, please define compound statements that describes the following sentences.

1. I am getting some rest, and some homemade chicked soup, but I am not feeling sick.

$q \wedge \neg p$

2. If I am feeling sick, I am getting some rest, and some homemade chicken soup.

$p \rightarrow q$

3. If I am getting rest, and some homemade chicken soup, then I am feeling sick.

$q \rightarrow p$

4. I only get rest and some homemade chicken soup if and only if I am feeling sick.

$p \leftrightarrow q$

5. I am not feeling sick, and I am not getting any rest, nor any homemade soup.

$\neg p \wedge \neg q$

(b) Consider the pair of circuits represented below. Please evaluate if the given circuits are logical equivalent. Show your work. (10)

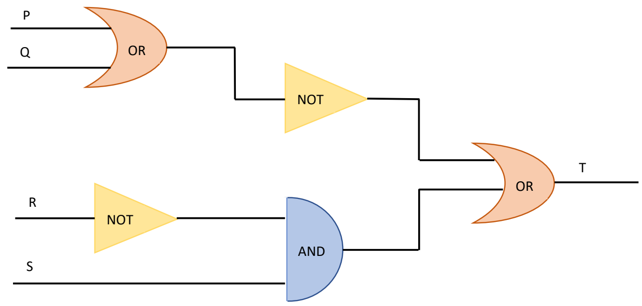

Figure 1: The first circuit used in Problem 1, part (b).

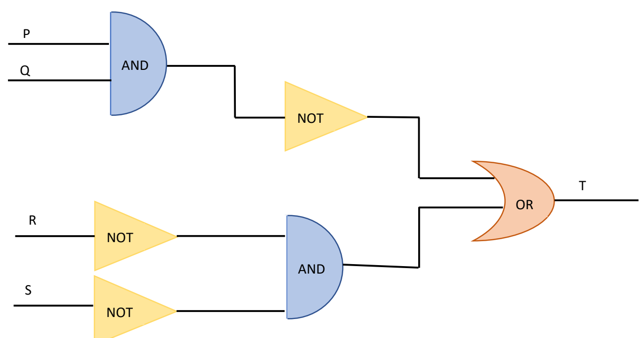

Figure 2: The second circuit used in Problem 1, part (b).

Figure 1: $T = \overline{P+Q} + \bar{R}S$
Figure 2: $T = \overline{PQ} + \bar{R}\bar{S}$

P	Q	R	S	$\overline{P+Q}$	$\bar{R}S$	$\overline{PQ}$	$\bar{R}\bar{S}$	$\overline{P+Q} + \bar{R}S$	$\overline{PQ} + \bar{R}\bar{S}$
1	1	1	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
1	1	0	1	0	1	0	0	1	0
1	1	0	0	0	0	0	1	0	1
1	0	1	1	0	0	1	0	0	1
1	0	1	0	0	0	1	0	0	1
1	0	0	1	0	1	1	0	1	1
1	0	0	0	0	0	1	1	0	1
0	1	1	1	0	0	1	0	0	1
0	1	1	0	0	0	1	0	0	1
0	1	0	1	0	1	1	0	1	1
0	1	0	0	0	0	1	1	0	1
0	0	1	1	1	0	1	0	1	1
0	0	1	0	1	0	1	0	1	1
0	0	0	1	1	1	1	0	1	1
0	0	0	0	1	0	1	1	1	1

Therefore, the two circuits are not logically equivalent.

(c) Find the value of the output signal T of the circuit represented in Figure 2 for input signals $P = Q = R = 1$ and $S = 0$. (2)

From the above table in part (b), when P = Q = R = 1 and S = 0,

$$T = \overline{PQ} + \bar{R}\bar{S} = 0$$

(d) For the table represented below construct a Boolean expression that would have the given table as its truth table. (3)

P	Q	R	S
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

Sum up all the expressions when S = 1:

$S = PQR + PQ\bar{R} + P\bar{Q}R + \bar{P}Q\bar{R} + \bar{P}\bar{Q}R + \bar{P}\bar{Q}\bar{R}$

$= PQ(R + \bar{R}) + \bar{P}\bar{R}(Q + \bar{Q}) + P\bar{Q}R + \bar{P}\bar{Q}R$ $(Distributive\ Laws)$

$= PQ + \bar{P}\bar{R} + P\bar{Q}R + \bar{P}\bar{Q}R$ $(Complement\ Laws)$

$= P(Q + \bar{Q}R) + \bar{P}(\bar{R} + \bar{Q}R)$ $(Distributive\ Laws)$

$= P\big((Q + \bar{Q})(Q + R)\big) + \bar{P}\big((\bar{R} + \bar{Q})(\bar{R} + R)\big)$ $(Distribute\ Laws)$

$= P(Q + R) + \bar{P}(\bar{R} + \bar{Q})$ $(Complement\ Laws)$

$= PQ + PR + \bar{P}\bar{R} + \bar{P}\bar{Q}$ $(Distributive\ Laws)$

Problem 2: Proofs

(a) Prove the following statement either by exhaustion or by counterexample: (8)

If x is a positive integer such that $3 \leq x \leq 6$, then $(x^2 + 7x - 5) \geq 25$.

Proof:

I will prove this by exhaustion.

Since x is a positive integer from 3 to 6, we can list all the possibilities:

$If\ x = 3, x^2 + 7x - 5 = 9 + 21 - 5 = 25 \geq 25$

$If\ x = 4, x^2 + 7x - 5 = 16 + 28 - 5 = 39 \geq 25$

$If\ x = 5, x^2 + 7x - 5 = 25 + 35 - 5 = 55 \geq 25$

$If\ x = 6, x^2 + 7x - 5 = 36 + 42 - 5 = 73 \geq 25$

We have proved by exhaustion that $x^2 + 7x - 5$ is greater than or equal to 25 for all the positive integers from 3 to 6, inclusive.■

(b) Prove the following statement using a direct proof technique, or give a counterexample. (12)

For every integer value x, if x is even, then $(x+2)(x+3)$ is even.

Proof:
I will prove this using a direct proof.
Since, x is even, assume x = 2k for some integer k.
Then, $(x+2)(x+3) = (2k+2)(2k+3) = 4k^2 + 10k + 6 = 2(2k^2 + 5k + 3)$
Since, k is an integer, $2k^2 + 5k + 3$ is also an integer, so $(x+2)(x+3)$ is even because an integer multiplied by 2 is also an even integer. ■

Problem 3: Sets

(a) Consider the following sets $A = \{5, 7, 8, 9, 10, 16, 23\}$ and $B = \{3, 5, 7, 10, 23\}$. (12)

Write the members of the resulting sets for the following set operations:

- $A \cap B$

$A \cap B = \{5, 7, 10, 23\}$

- $A \cup B$

$A \cup B = \{3, 5, 7, 8, 9, 10, 16, 23\}$

- $A - B$

$A - B = \{8, 9, 16\}$

- $B - A$

$$B - A = \{3\}$$

- $B \times C$, where C is a set defined as $C = \{x, y\}$

$$B \times C = \{(3,x), (5,x), (7,x), (10,x), (23,x), (3,y), (5,y), (10,y), (23,y)\}$$

(b) In a group of 90 first-semester ALIGN students, 35 students are taking CS 5001, and 33 are taking CS 5002. How many students are taking only CS 5001? How many students are taking only CS 5002? (8)

(Note: you can assume that no first-semester ALIGN students are taking any other courses, and that no ALIGN student is not taking any of the listed courses.)

U = 90 first semester ALIGN students
A = Students taking CS5001
B = Students taking CS5002
Corrections (Piazza): 28 students are not taking either CS5001 or CS5002.
Based on the problem statement, we can have:

$$n(U) = 90, n(A) = 35, n(B) = 33, n(A \cup B)^c = 28$$
$$n(A \cup B) = n(U) - n(A \cup B)^c = n(A) + n(B) - n(A \cap B) = 90 - 28 = 62$$
$$n(A \cap B) = n(A) + n(B) - 62 = 35 + 33 - 62 = 6$$
$$|A - B| = n(A) - n(A \cap B) = 35 - 6 = 29$$
$$|B - A| = n(A) - n(A \cap B) = 33 - 6 = 27$$

Therefore, there are 29 students and 27 students who only take CS5001 and CS5002, respectively.

Problem 4: Functions

(a) Consider a function $f(x)$ that takes the last four digits of a person's social security number (SSN), represented as **xxxx** , where x is an integer between 0 and 9, and turns those four digits into their sum. For example, given an input $x = 1234$, function $f(x)$ returns 1+2+3+4 = 10. (8)

- What is the domain and range of the given function?

The domain of the given function is the set of all last four digits integer of SSN from 0000 to 9999.
Domain = {0000, 0001, 0002, ..., 9997, 9998, 9999}
The range of the given function is the sum of all possible last four digits integer of SSN from 0000 to 9999. The minimum possible value is the sum of four 0, and the maximum possible value is the sum of four 9.
Range = {0, 1, 2, 3, 4, 5, ..., 30, 31, 32, 33, 34, 35, 36}

- Is the given function a bijection? Please justify your answer.

For a function to be a bijection, it must be an injection and a surjection. The given function, however, is not an injection, because it is possible that multiple last four digits add up to the same value in the range of this function. For example, 0222, and 0006 both add up to 6 in the range, so f(0222) = f(0006) when 0222 ≠ 0006. Therefore, the given function is not a bijection.

(b) Consider the following function $f(x)$, defined on a set of real numbers, $f : \mathbb{R} \rightarrow \mathbb{R}$: (7)

$$f(x) = 7x^3 - 2x + 5$$

- Examine if the given function $f(x)$ is bijective.
- If possible, find inverse of $f(x)$.

To show that the given function is bijective, we need to show that it is both injective and surjective. In the given case, however, the function $f(x) = 7x^3 - 2x + 5$ is not injective. The derivative of the function is $f'(x) = 21x^2 - 2$, and when the derivative is 0, $x \approx \pm 0.3086$, so the function will increase from negative infinity until -0.3086, and then decrease until 0.3086, and then increase again. Therefore, from the range f(-0.3086) to f(0.3086), there is overlap between f(x) outputs when the inputs x are different. Therefore, the given function is not bijective.

Corrections (Piazza): No need to find the inverse.
Since the function is not bijective, it does not have an inverse function.

(c) Consider the following functions $f(x)$ and $g(x)$ defined over the set of integers, $f : \mathbb{Z} \rightarrow \mathbb{Z}$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$: (5)

$$\begin{aligned} f(x) &= x + 7 \\ g(x) &= 2x^2 - x + 3 \end{aligned}$$

Find compositions:

- $f \circ g$.
- $g \circ f$.

$$f \circ g = f(g(x)) = (2x^2 - x + 3) + 7 = 2x^2 - x + 10$$

$$g \circ f = g(f(x)) = 2(x+7)^2 - (x+7) + 3 = 2(x^2 + 14x + 49) - x - 7 + 3$$
$$= 2x^2 + 28x + 98 - x - 10 = 2x^2 + 27x + 88$$

Problem 5: Counting

(a) Consider the following function $f : A \to B$, where domain A is a set with cardinality $|A| = n$, and target B is a set with cardinality $|B| = m$. (7)

Find the number of possible valid functions $f : A \to B$ (mappings from set A to set B) we can generate.

Since f is a valid function, every element in the domain A must be mapped to an element of target B. Therefore, each element in domain A has m elements to choose from target B. Therefore, total number of functions will be m^n.

(b) Consider the following function $f : A \to B$, where domain A is a set with cardinality $|A| = n$, and target B is a set with cardinality $|B| = m$. (7)

Find the number of possible **injective functions** $f : A \to B$ we can generate.

If n > m, based on the pigeonhole principle, it is impossible to have injective function from A to B because there are at least two different values that map to the same element in target B.
If n <= m, since the function f is injective, that means if $a, b \in A$ and $a \neq b$, then $f(a) \neq f(b)$. So, for the first element in the domain A, it can take m values, and for the second element in the domain A, it can take m - 1 values, then we keep going until we get to the nth element in the domain A, and this element can take m - (n - 1) values in target B.
Therefore, the number of possible injective functions if n <= m is: m(m - 1)(m - 2)...(m - (n - 1)).

(c) Consider the following function $f : A \to B$, where domain A is a set with cardinality $|A| = n$, and target B is a set with cardinality $|B| = n$. (6)
Find the number of possible **bijections** $f : A \to B$ we can generate.

Since the function f is a bijection, for first element in the domain A, there are m elements in domain b to choose from, and for the nth element in the domain A, there is only 1 element to choose from. So, a bijection from domain A to target B when both cardinalities are n is precisely a permutation of n elements. Therefore, the total number of possible bijective functions is n!.

Question	Points	Score
Propositional Logic and Boolean Algebra	20	
Proofs	20	
Sets	20	
Functions	20	
Counting	20	
Total:	100	

SUBMISSION DETAILS

Submit your answers to the midterm problems on Cavas as a pdf. You can do this by printing out this assignment, hand-writing your answers, and then scanning in, or by using a program such as Preview on the Mac to annotate the pdf. Things to submit:

- Submit the following on Canvas for the Midterm:
  - The written parts of this assignment as a .pdf named "CS5002_[lastname]_Midterm.pdf". For example, my file would be named "CS5002_Bonaci_Midterm.pdf". (There should be no brackets around your name).