HW5: INTRODUCTION TO COUNTING

I'm not counting any chickens. —Jeff Bridges

Course: CS 5002
Fall 2021
Due: October 16, 2021

PROBLEMS

Problem 1: Factorials

(a) (2 points)

$$\binom{42}{40}$$

$$\binom{42}{40} = \frac{42!}{40!\,(42-40)!} = \frac{41 \cdot 42}{1 \cdot 2} = 861$$

(b) (2 points)

$$\binom{12}{7}$$

$$\binom{12}{7} = \frac{12!}{7!\,(12-7)!} = \frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{95040}{120} = 792$$

(c) (2 points)

$$\binom{5}{4}$$

$$\binom{5}{4} = \frac{5!}{4!\,(5-4)!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 1} = 5$$

Problem 2: Strings and counting

(a) (3 points) How many different bit strings of length 7 are there?

For each bit of string, there are two possibilities, 0 and 1. For a length of 7, the total number of different bit strings are $2^7 = 128$.

 Points: ______________ out of 9

(b) (4 points) How many bit strings of length six start with 11 or end with a 0?

For bit string of length six start with 11: $2^4 = 16$.
For bit string of length six end with 0: $2^5 = 32$
For bit string that start with 11 and end with 0: $2^3 = 8$
For bit string of length six start with 11 or end with 0: 16 + 32 - 8 = 40.

Problem 3: Counting without repetitions

For this group of questions, assume that repetitions are not permitted.

(a) (3 points) How many 3-digit numbers can be formed from the six digits 1, 3, 5, 7, 9, 0?

$$P(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 4 \cdot 5 \cdot 6 = 120$$

(b) (3 points) How many of these numbers are less that 500?

Since the first digit cannot be greater than 5, the only three choices for the first digit are 0, 1 and 3.
For the other two digits, the number of ways is:

$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20$$

Therefore, the total number of three digits number that are less than 500 are 3 * 20 = 60.

(c) (3 points) How many are even?

For the number to be even, the last digit needs to be 0.
Therefore, the number of ways is:

$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 4 \cdot 5 = 20$$

(d) (3 points) How many are odd?

Using permutation by complement.
Odd number = 3-digit number - even number = 120 - 20 = 100

(e) (3 points) How many are multiples of 5?

Multiple of 5 ends with either 5 or 0.
For the other two digits of the number:

$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20$$

Therefore, the total number of three digits number that are multiple of 5 are 2 * 20 = 40.

Problem 4: Quiz

Assume a quiz has four problems, and:

- the first problem has four true/false questions,
- the second problem requires choosing one of four alternatives, and
- the answer to the third problem is an integer ≥ 15 and ≤ 20
- the fourth problem is a multiple choice question with five alternatives.

How many different ways is it possible to answer the quiz? Please explain your work.

The first problem: $2^4 = 16$
The second problem: 4
The third problem: From 15 to 20, inclusive, there are 6 numbers.
The fourth problem: 5
Total number of different ways is $16 \cdot 4 \cdot 6 \cdot 5 = 1920$

Problem 5: More bit strings

How many bit strings of length 4 do not have two consecutive 0?

Case 1: First and second are 0. Number of ways: $2^2 = 4$
Case 2: Second and third are 0. Number of ways: $2^2 = 4$
Case 3: Third and fourth are 0. Number of ways: $2^2 = 4$
Case 1 ∩ Case 2: 000X, where X is either 1 or 0, so there are 2 options.
Case 2 ∩ Case 3: X000, where X is either 1 or 0, so there are 2 options.
Case 1 ∩ Case 3: 0000, only 1 intersection.
Case 1 ∩ Case 2 ∩ Case 3: 0000, only 1 possible intersected string among three.
Therefore, Case1 U Case2 U Case3 = 4 + 4 + 4 - 2 - 2 - 1 + 1 = 8.
Total number of ways for a 4-digit bit string: $2^4 = 16$.
Using permutation by complement, bit strings of length 4 that do not have two consecutive 0:
16 - 8 = 8.

Problem 6: Permutations

(a) (4 points) Find the number m of permutations that can be formed from all the letters of the word UNCOPYRIGHTABLE.

There is no repetition in the word UNCOPYRIGHTABLE, so the different number of permutations is m = 15!

(b) (4 points) Find the number m of permutations that can be formed from all the letters of the word HITCHHIKER.

HITCHHIKER contains the following character and count:
H (3) I (2) T (1) C (1) K (1) E (1) R (1)
Therefore, the number of permutations is:

$$\binom{10}{3}\binom{7}{2}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \frac{10!}{3!\,2!\,1!\,1!\,1!\,1!\,1!} = \frac{10!}{12} = 302400$$

(c) (5 points) How many different signals, each consisting of eight flags hung in a vertical line, can be formed from five identical red flags and four identical blue flags?

There are two ways to arrange 8 flags hung, 5 red and 3 blue or 4 red and 4 blue.
Therefore, the number of permutations is:

$$\binom{8}{5}\binom{3}{3} + \binom{8}{4}\binom{4}{4} = \frac{8!}{5!\,3!} + \frac{8!}{4!\,4!} = 56 + 70 = 126$$

Problem 7: Combinations

(a) (3 points) Find the number m of committees of five that can be formed from nine people.

$$m = \binom{9}{5} = \frac{9!}{5!\,(9-5)!} = \frac{9!}{5!\,4!} = \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4} = 126$$

(b) (3 points) Consider a scenario where 15 students who are eligible for a scholarship. Find the number m of ways a group of four students can be selected from the 15 eligible students.

$$m = \binom{15}{4} = \frac{15!}{4!\,(15-4)!} = \frac{15!}{4!\,11!} = \frac{12 \cdot 13 \cdot 14 \cdot 15}{1 \cdot 2 \cdot 3 \cdot 4} = 1365$$

(c) (4 points) A bag contains 5 red marbles and 6 white marbles. Find the number m of ways that four marbles can be drawn from the bag.

$$m = \binom{11}{4} = \frac{11!}{4!\,(11-4)!} = \frac{11!}{4!\,7!} = \frac{8 \cdot 9 \cdot 10 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4} = 330$$

Problem 8: Teammates in a row

A team consists of five boys and five girls. Find the number of ways they can sit in a row if:

(a) (4 points) The boys and girls are each to sit together (all girls together, all boys together)

We can view the five boys and five girls as two whole segments, so the question asks how to permute 2 segments.

$$P(2,2) = 2! = 2$$

Within each segment, we can permute boys and girls differently, each with:

Girls: $P(5,5) = 5! = 120$

Boys: $P(5,5) = 5! = 120$

Therefore, the total number of ways to make all the boys and girls sit together are:

120 * 120 * 2 = 28800

(b) (5 points) Just the girls are to sit together.

We can view the girl as one segment and five boys as five segments, so there are total of 6 segments.

$$P(6,6) = 6! = 720$$

Within the girl segment, the number of ways to permute girls are:

$$P(5,5) = 5! = 120$$

Therefore, the total number of ways to make all the girls sit together are:

120 * 720 = 86400

Problem 9: University admissions

Some university receives 1525 applications for a graduate program in computer science. Of the received applications, 589 applicants majored in computer science, 300 majored in business, and 72 majored in both computer science and business. How many of the applicants did not major either in computer science or in business?

Major in computer science = C

Major in business = B

$n(U) = 1525$ $n(C) = 589$ $n(B) = 300$ $n(C \cap B) = 72$

$n(C \cup B)^c$

$= n(U) - n(C \cup B)$

$= n(U) - (n(C) + n(B) - n(C \cap B))$

$= 1525 - (589 + 300 - 72)$

$= 1525 - 817$

$= 708$

Problem 10: Odd integers

How many odd four-digit integers have the property that their digits, read left to right, are in strictly decreasing order?

For a four-digit number to be odd, it can only be ending with 1, 3, 5, 7, 9.

If the number ends in 1, there are 8 remaining digits to choose from:

$$\binom{8}{3} = \frac{8!}{3!\,5!} = 56$$

If the number ends in 3, there are 6 remaining digits to choose from:

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20$$

If the number ends in 5, there are 4 remaining digits to choose from:

$$\binom{4}{3} = \frac{4!}{3!\,1!} = 4$$

The number cannot be 7 or 9, because there won't be enough digits to form a four-digit number.

Total number of odd strictly decreasing four-digit numbers: 56 + 20 + 4 = 80

Problem 11: Digits and averages

How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

The middle digit can only be an integer if the sum of the two other digits are even numbers. Therefore, if the first digit and the last digit are both even, then the first digit has 4 choices (2, 4, 6, 8) and the last digit has 5 choices (0, 2, 4, 6, 8), so the total number of ways are 4 * 5 = 20 if both digits are even. If both digits are odd, then the first digit has 5 choices (1, 3, 5, 7, 9) and the last digit has 5 choices (1, 3, 5, 7, 9), so the total number of ways are 5 * 5 = 25. Therefore, the total number of ways to satisfy the conditions are 20 + 25 = 45.

Problem 12: Lab assignments

Some lab section has 12 students, who are supposed to break up into 4 groups of 3 students each. A Teaching Assistant (TA) has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

How many group assignments are possible?

There are same number of students in each group. For all the groups, the way to permute these groups does not matter, and for within each group, the way to permute these students in each group does not matter, so the number of ways is the total ways to permute all 12 students divided by the ways to permute 4 groups and to permute 3 students within each group.
If the student number is 12, group number is 4.
n = 12, m = 4

$$T = \frac{n!}{\left(\left(\frac{n}{m}\right)!\right)^m \cdot m!} = \frac{12!}{(3!)^4 \cdot 4!} = \frac{12!}{6^4 \cdot 4!} = 15400$$

Therefore, there are a total of 15400 ways to assign 12 students to 4 groups of 3 students each.

Question	Points	Score
Factorials	6	
Strings and counting	7	
Counting without repetitions	15	
Quiz	7	
More bit strings	5	
Permutations	13	
Combinations	10	
Teammates in a row	9	
University admissions	7	
Odd integers	7	
Digits and averages	7	
Lab assignments	7	
Total:	100	

SUBMISSION DETAILS

Things to submit:

- Please submit this assignment as a .pdf named "CS5002_[lastname]_HW5.pdf" through Canvas by 11:59pm PT on Saturday, October 16, 2021.
- Please make sure your name is in the document as well (e.g., written on the top of the first page).