HW8: INTRODUCTION TO RELATIONS

There is no branch of mathematics, however abstract, which may not some day be applied to phenomena in real world. — Nicolai Ivanovitch Lobachevsky (1792-1856)

Course: CS 5002
Fall 2021
Due: Tuesday, November 23, 2021

PROBLEMS

Problem 1: Definition of a relation

Let's consider the following **congruence modulo 3** relation R, defined from the set of integers, $\mathbb{Z}$ to the set of integers $\mathbb{Z}$ as follows:

$$m\ R\ n \iff 3|(m-n)$$

(a) (1 point) Is $10\ R\ 1$? Please explain why or why not.

(a)
m = 10, n = 1
10 mod 3 = 1
1 mod 3 = 1
Therefore, 10 R 1 because 10 and 1 are congruent modulo 3.

(b) (1 point) Is $(8, 1) \in R$? Please explain why or why not.

(b)
m = 8, n = 1
8 mod 3 = 2
1 mod 3 = 1
Therefore, $(8, 1) \notin R$ because 8 and 1 are not congruent modulo 3.

(c) (1 point) List five integers n such that $n\ R\ 0$.

(c)
We need to find 5 integers that modulo 3 equals 0.
Therefore, 3, 6, 9, 12, 15 all satisfy the above relation.

(d) (1 point) List five integers n such that $n\ R\ 2$.

(d)
We need to find 5 integers that modulo 3 equals 2.
Therefore, 5, 8, 11, 13, 17 all satisfy the above relation.

Problem 2: Definition of a relation

Let A be the set of all strings of a's and b's of length 4. Let's define a relation R on A as follows: For all $s, t \in A$,

$$s\ R\ t \iff s \text{ has the same first two characters as } t.$$

(a) (2 points) Is $abaa\ R\ abba$?

(a)
The first two characters are both "ab", so abaa R abba.

(b) (2 points) Is $aabb\ R\ bbaa$?

(b)
The first two characters of aabb is "aa", and the first two characters of bbaa is "bb". Therefore, the relation does not hold.

(c) (2 points) Is $aaaa\ R\ aaab$?

(c)
The first two characters are both "aa", so aaaa R aaab.

Problem 3: Combining relations

Let R be the relation $\{(1,3),(1,4),(2,3),(2,4),(2,7)\}$, and let S be the relation $\{(3,5),(4,5),(3,6),(4,6)\}$. Find the composition $S \circ R$.

$$S \circ R = \{(1,5),(2,5),(1,6),(2,6)\}$$

Problem 4: Combining Relations

Let A be the set of all students on our campus, and let B be the set of all courses offered at Northeastern University. Let relation R_1 consist of all ordered pairs (a,b), where student a is required to take course b. Similarly, let relation R_2 consist of all ordered pairs (a,b), where student a has taken course b.

Describe (in words) the ordered pairs in each of the combined relations:

(a) (2 points) $R_1 \cap R_2$

(a)
$R1 \cap R2$ consist of all ordered pairs (a, b), where students a is required to take course b and has already taken course b.

(b) (2 points) $R_1 - R_2$

(b)
$R1 - R2$ consist of all ordered pairs (a, b), where students a is required to take course b and has not taken course b yet.

(c) (2 points) $R_2 - R_1$

(c)
$R2 - R1$ consist of all ordered pairs (a, b), where students a is not required to take course b but has taken course b anyway.

Problem 5: Matrix representation of a relation

Represent each of these relations on the set $\{4, 5, 7\}$ with a matrix, such that the elements of the given set are listed in an increasing order:

(a) (2 points) $\{(4, 4), (4, 5), (4, 7), (5, 7), (7, 4), (7, 5)\}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(b) (2 points) $\{(4,4),(4,5),(5,5),(7,7)\}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) (2 points) $\{(4,4),(4,5),(5,5),(5,7),(7,7)\}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) (2 points) $\{(5,5),(7,7)\}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) (2 points) $\{(4,5),(4,7),(5,7)\}$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Problem 6: Properites of relations

Let R be the "greater than" relation on the set of integers, formally defined as follows:

for all $x, y \in \mathbb{Z}, x \ R \ y \iff x > y$.

Please show your work to determine whether or not the given relation is:

(a) (2 points) Reflexive:

(a)
$x > x$ is not valid for any $x \in \mathbb{Z}$. Therefore, R is not reflexive.

(b) (2 points) Symmetric:

(b)
For $x, y \in \mathbb{Z}$, if x < y, then y cannot be greater than x. Therefore, R is not symmetric.

(c) (2 points) Anti-symmetric:

(c)
For $x, y \in \mathbb{Z}$, if x < y, then y cannot be greater than x. Therefore, R is anti-symmetric.

(d) (2 points) Transitive:

(d)
For $x, y, z \in \mathbb{Z}$, if x < y, y < z, then x < z. Therefore, R is transitive.

Problem 7: Properites of relations

Let A be a Cartesian product $\mathbb{Z} \times \mathbb{Z}$, and let F be a relation defined on A as follows:

For all (x_1, y_1) and $(x_2, y_2) \in A : (x_1, y_1) \ F \ (x_2, y_2) \iff x_1 = x_2$

Please show your work to determine whether or not the given relation is:

(a) (2 points) Reflexive:

(a)
For (x1, y1), x1 = x1. Therefore, F is reflexive.

(b) (2 points) Symmetric:

(b)
If x1 = x2, then x2 = x1. Therefore, F is symmetric.

(c) (2 points) Anti-symmetric:

(c)
If x1 = x2, then it is not possible for $x2 \neq x1$. Therefore, F is not anti-symmetric.

(d) (2 points) Transitive:

(d)
If x1 = x2, x2 = x3, then x1 = x3. Therefore, F is transitive.

Problem 8: Matrix representation and properties of relations

Represent each of these relations on the set $\{0, 3, 7\}$ with a matrix, such that the elements of the given set are listed in an increasing order:

Use matrix representation of a relation to determine whether the following relations are reflexive and symmetric.

$\{(0,3),(0,7),(3,7),(7,7)\}$

(a) (2 points) Matrix representation?

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (2 points) Reflexive?

(b)
No, because the relation does not contain (0, 0), (3, 3), so the relation is not reflexive.

(c) (2 points) Symmetric?

(c)
No, there is no (3, 0) for (0, 3), no (7, 0) for (0, 7), no (7, 3) for (3, 7). Therefore, the relation is not symmetric.

$\{(0,0),(3,3),(7,7)\}$

(a) (2 points) Matrix representation?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (2 points) Reflexive?

(b)
Yes, because for 0, 3, and 7, there is (0, 0), (3, 3), and (7, 7), so the relation is reflexive.

(c) (2 points) Symmetric?

(c)
Yes, there is neither aRb nor bRa when a, b is any arbitrary number in the set of {0, 3, 7}, so the relation is symmetric.

Problem 9: Equivalence relations

Consider the following relations defined on set $\{1, 4, 5, 6\}$. Show your work to determine whether or not the given relations are equivalence relations.

(a) (4 points) $\{(1, 1), (1, 4), (5, 6), (6, 6)\}$

(a)
Reflexive:
There is no (5, 5), so the relation is not reflexive.
Symmetric:
There is no (4, 1) for (1, 4), (6, 5) for (5, 6), so the relation is not symmetric.
Transitive:
There are no three numbers connected, so the relation is transitive.
Therefore, the given relation is not an equivalence relation because equivalence relation needs to be reflexive, symmetric and transitive.

(b) (4 points) $\{(1, 1), (1, 4), (1, 5), (1, 6), (4, 4), (4, 5), (4, 6), (5, 5), (5, 6), (6, 6)\}$

(b)
Reflexive:
There is a self-loop for every number in the set, so the relation is reflexive.
Symmetric:
There is no (4, 1) for (1, 4), (5, 1) for (1, 5), and (6, 1) for (1, 6), (5, 4) for (4, 5), (6, 4) for (4, 6), (6, 5) for (5, 6), so the relation is not symmetric.
Transitive:
For every three connected number, it can form a transitive relation, so the relation is transitive.
Therefore, the given relation is not an equivalence relation because equivalence relation needs to be reflexive, symmetric and transitive.

Problem 10: Partial ordering

Consider the following relations on set $\{0, 3, 5, 7\}$. Show your work to determine which of the given relations are partial orderings.

(a) (4 points) $\{(0, 0), (3, 3), (5, 5), (7, 7)\}$

(a)
Reflexive:
Every number is a self-loop, so the relation is reflexive.
Symmetric:
There are no connections between any number at all, so the relation is both anti-symmetric and symmetric.
Transitive:
There are simply no connections between any number at all, so the relation is transitive.
Therefore, the relation is a partial ordering because it is reflexive, anti-symmetric, and transitive.

(b) (4 points) $\{(0, 0), (0, 3), (0, 5), (3, 0), (3, 3), (3, 5), (5, 0), (5, 5), (7, 7)\}$

(b)
Reflexive:
Every number is a self-loop, so the relation is reflexive.
Symmetric:
(0, 3) has (3, 0)
(0, 5) has (5, 0)
(3, 5) does not have (5, 3)
Therefore, the relation is neither symmetric nor anti-symmetric.
Transitive:
(0, 3), (3, 0), (0, 0)
(0, 5), (5, 0), (0, 0)
(3, 0), (0, 0), (3, 0)
(3, 0), (0, 5), (3, 5)
(5, 0), (0, 0), (5, 0)
(5, 0), (0, 3), no (5, 3)
(3, 5), (5, 0), (3, 0)
(3, 5), (5, 5), (3, 5)
Therefore, the relation is not transitive.
Therefore, the relation is not a partial ordering because it is reflexive, but not anti-symmetric and not transitive.

Problem 11: Equivalence relation

Define three equivalence relations on the set of all ALIGN students in your cohort.

For two students a and b, aRb, the age of a is equal to the age of b.
For two students a and b, aRb, the GPA of a is equal to the GPA of b.
For two students a and b, aRb, the years of working experience of a is equal to the years of working experience of b before they entered ALIGN program.

Problem 12: Equivalence relation

Let R be the relation on the set of ordered pairs of positive integers, such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalance relation.

Proof:
An equivalence relation is reflexive, symmetric, and transitive.
Reflexive:
$R\ is\ reflexive\ since\ \big((a,b),(a,b)\big) \in R\ because\ ab = ba$
Symmetric:
$R\ is\ symmetric\ since\ if \big((a,b),(c,d)\big) \in R, then\ ad = bc, which\ means\ that\ cb = da, so\ \big((c,d),(a,b)\big) \in R$
Transitive:
$If\ \big((a,b),(c,d)\big) \in R\ and\ \big((c,d),(e,f)\big) \in R, then\ ad = bc, cf = de, multiplying\ both\ sideof\ the\ equations\ to\ get\ acdf = bcde, which\ gives\ af = be, so\ ((a,b),(e,f)) \in R$
Therefore, R is equivalence relation because it is reflexive, symmetric, and transitive.■

Problem 13: Partial ordering

Let R be a relation on the set of people such that xRy if x and y are people, and x is older than y. Show that R is not a partial ordering.

Proof:
A partial ordering is reflexive, transitive, and anti-symmetric.
Reflexive:
For x and x are people, x cannot be older than x, so that R is not reflexive.
Anti-symmetric:
x and y are two different people, so if x is older than y, y cannot be older than x, so R is anti-symmetric.
Transitive:
If x is older than y, y is older than z, then x is older than z, so R is transitive.
Therefore, R is not a partial ordering because it is not reflexive even though it is transitive and anti-symmetric.■

Problem 14: Circular relation

A relation R is said to be **circular** if aRb and bRC imply cRa. Show that R is reflexive and circular if and only if it is an equivalence relation.

Proof:
To prove this iff statement, we must prove the following two statements:

1. If R is reflexive and circular, then it is an equivalence relation.
2. If R is an equivalence relation, R is reflexive and circular.

Prove 1:
For some arbitrary $a, b \in \mathbb{R}$, so that aRb since R is reflexive, we want to prove that bRa. We have bRb, and since it is circular aRb and bRb, we get bRa. So, aRb and bRa, R is symmetric.
For some arbitrary $a, b, c \in \mathbb{R}$, so that aRb and bRc, we want to prove that aRc. Since R is circular, we know that cRa, and since we have already proved that R is symmetric, we have aRc. Therefore, R is transitive.
Since R is reflexive, symmetric and transitive, R is an equivalence relation.
Prove 2:
Since R is an equivalence relation, we know R is reflexive, symmetric and transitive. R is already reflexive, so we just need to prove it is circular. For some arbitrary $a, b, c \in \mathbb{R}$, so that aRb and bRc, because R is transitive, we have aRc, because R is symmetric, we have cRa, so R is circular. Therefore, R is reflexive and circular.
We have proved that both statement 1 and 2 are correct, so the R is reflexive and circular if and only if R is an equivalence relation.■

Question	Points	Score
Definition of a relation	4	
Definition of a relation	6	
Combining relations	5	
Combining Relations	6	
Matrix representation of a relation	10	
Properites of relations	8	
Properites of relations	8	
Matrix representation and properties of relations	12	
Equivalence relations	8	
Partial ordering	8	
Equivalence relation	6	
Equivalence relation	5	
Partial ordering	8	
Circular relation	6	
Total:	100	

SUBMISSION DETAILS

Things to submit:

- Please submit this assignment as a .pdf named "CS5002_[lastname]_HW8.pdf" through Canvas by 11:59pm PT on Tuesday, November 23, 2021 (Please notice the unusual due day).
- Please make sure your name is in the document as well (e.g., written on the top of the first page).