HW6: INTRODUCTION TO DISCRETE PROBABILITY

Statistically, the probability of any one of us being here is so small that you'd think the mere fact of existing would keep us all in a contented dazzlement of surprise. —Lewis Thomas

Course: CS 5002
Fall 2021
Due: Saturday, November 6, 2021

PROBLEMS

Problem 1: Simple Random Experiments

Consider the following simple experiment: A fair die is tossed, and its face value is observed. If the number on the face value is even, value 1 is assigned to some random variable X. If, on the other hand, an odd number is observed, value 0 is assigned to X.

(a) (1 point) What is the range of X?

(b) (3 points) Find probabilities $P(X = 1)$ and $P(X = 0)$.

(a)
There are only two results for a single toss of the fair die, either even or odd. Therefore, the range of X is 0 and 1.

(b)
A fair die has 6 facets, and each facet has the same probability. Since the odd and even events are 3 each, P (X = 1) = 3/6 = 1/2 and P (X = 0) = 3/6 = 1/2.

Consider now the second simple experiment, where some coin is tossed three times. We assume that the tosses are independent, and the probability of a head is p. Let Y be the random variable representing the number of heads observed.

(a) (1 point) What is the range of Y?

(b) (5 points) Find the probabilities $P(Y = 0)$, $P(Y = 1)$, $P(Y = 2)$, and $P(Y = 3)$.

(a)
For three tosses, the worst case is that no head is observed, and the best case is that all three tosses are head. Therefore, the range of Y is 0, 1, 2, and 3.

(b)
All the possibilities:
For each toss, there are two possibilities, which are head or tail, so the total number of events are
$2^3 = 8$.
P (Y = 0) = 1 / 8, since only TTT has no head in it.
P (Y = 1) = 3 / 8, which are HTT, THT, TTH.
P (Y = 2) = 3 / 8, which are HHT, HTH, THH.
P (Y = 3) = 1 / 8, since only HHH has all heads in it.

Problem 2: Bitstrings and Palindromes

(a) (6 points) What is the probability that a 16-bit binary string is a **palindrome**? Please explain your work.

(a)
For a 16-bit binary string, each bit can only be 0 or 1. For a string to be palindrome, it needs to be the same when reading forward and backward. So, the first and last bit need to be the same, either 0 or 1. Therefore, there are a total of $2^{(16/2)} = 2^8$.
Therefore, the possibility is:

$$P(16 - bit\ palindrome) = \frac{2^8}{2^{16}} = \frac{1}{2^8}$$

(b) (4 points) What is the probability that a 15-bit binary string is a **palindrome**? Please explain your work.

(b)
For a 15-bit binary string, each bit can only be 0 or 1. For a string to be palindrome, it needs to be the same when reading forward and backward. So, the first and last bit need to the same, either 0 or 1. The middle index can be either 0 or 1. Therefore, there are a total of $2^{(14/2)} * 2 = 2^8$.
Therefore, the possibility is:

$$P(15 - bit\ palindrome) = \frac{2^8}{2^{15}} = \frac{1}{2^7}$$

Problem 3: Roulette and Probabilities

In roulette, a wheel with 64 numbers is spun. Of the 64 numbers, 30 are red, and 30 are black. The other four numbers are neither black nor red, and their values are 0 and 00 (two instances of each). The probability that, when a wheel is spun, it lands on any particular number is $\frac{1}{64}$.

(a) (2 points) What is the probability that the wheel lands on a red number?

(a)
There are a total of 30 red numbers, so the probability to land on a red number is:

$$P(red) = 30 \cdot \frac{1}{64} = \frac{30}{64} = \frac{15}{32}$$

(b) (2 points) What is the probability that the wheel lands on 0 or 00?

(b)
There are 2 (00) and 2 (0), so the total possibility to land on 0 or 00 is:

$$P(00 \; or \; 0) = P(0) + P(00) = 2 \cdot \frac{1}{64} + 2 \cdot \frac{1}{64} = \frac{4}{64} = \frac{1}{16}$$

(c) (3 points) What is the probability that the wheel lands on a black number twice in a row?

(c)
The possibility to land on black is $\frac{30}{64} = \frac{15}{32}$
Therefore, the probability to land on a black number twice in a row:

$$P(two \; black \; in \; a \; row) = \frac{15}{32} \cdot \frac{15}{32} = \frac{225}{1024}$$

(d) (3 points) What is the probability that in five spins, the wheel never lands on either 0 or 00?

(d)
The possibility to not land on either 0 or 00 is $\frac{60}{64} = \frac{15}{16}$
For five spins, the possibility to not land on either 0 or 00:

$$P(Not \; land \; on \; 0 \; or \; 00 \; five \; times) = \left(\frac{15}{16}\right)^5$$

Problem 4: Probabilities and Ice Cream

Let's assume your best friend has decided to surprise you with a dessert consisting of four scoops of ice cream, where each of the scoops is equally as likely to be either a **vanilla scoop** or a **hazelnut scoop**. Let VVHH indicate that the first two scoops are vanilla and the second two scoops are hazelnut. Similarly, let HVHV indicate that the first scoop is hazelnut, the second is vanilla, and so on.

(a) (2 points) How many ice cream desserts consisting of four scoops, of either vanilla or hazelnut flavor, are possible?

(a)
Since the order of choice matters, and for each scoop, there are two possible choices to make, either vanilla or hazelnut. Therefore, the total number of choices are $2^4 = 16$.

(b) (4 points) What is the probability that exactly one of the four scoops is vanilla flavored?

(b)
There are only four permutations, VHHH, HVHH, HHVH, HHHV, so the possibility is:

$$P(only\ one\ vanilla) = \frac{4}{16} = \frac{1}{4}$$

(c) (4 points) What is the probability that at least two scoops are hazelnut flavored?

(c)
If there is no hazelnut in the flavor, there is only one arrangement, VVVV.
If there is only one hazelnut in the flavor, there are four arrangements from part (b).
So, there are 16 - 4 - 1 = 11 arrangements that have at least 2 hazelnut flavors:

$$P(At\ least\ two\ H) = \frac{11}{16}$$

Problem 5: Socks and Probability

Some mornings are busy, and getting dressed, and especially getting a matching pair of socks can be difficult. Let's see what might happen...

(a) (2 points) Some drawer contains 10 blue socks and 10 purple socks, all unmatched. Let's assume we're taking socks out of the drawer at random, one at the time, and in the dark. How many socks must we take out to be sure that we have at least two socks of the same color?

(a)
Based on the pigeonhole theory, there are total of two colors, so, when taking 3 socks out of the drawer, we can guarantee that there is at least a pair of the same color.

(b) (4 points) Being sure that we have a matching pair of socks is great, but we will take our chances. What is the probability that we take out two socks, and that they are a matching pair?

(b)

$$P(a\ matching\ pair\ in\ two\ socks) = P(both\ purple) + P(both\ blue) = \frac{10}{20}\cdot\frac{9}{19}+\frac{10}{20}\cdot\frac{9}{19}$$
$$=\frac{180}{380}=\frac{9}{19}$$

(c) (4 points) We didn't really like our odds, so we go to a store and buy 10 more pairs of socks - 10 blue socks and 10 green socks. Now we have 20 blue socks, 10 purple socks and 10 green socks in the drawer. How many socks do we need to draw out of the drawer to be sure that we have a pair of blue socks?

(c)
We need to first draw all the purple and green socks, so that makes it 20 socks. Then, the rest of the socks are blue socks, so we need to draw 22 socks to make sure that we have a pair of blue socks.

(d) (5 points) The drawer still contains 40 socks - 20 blue, 10 purple and 10 green socks. Let's assume that we know that the first sock drawn was purple. What is the probability that the next sock we draw is not purple?

(d)
After we take out the first sock as purple, there is only 9 purple socks left in drawer. There are 30 socks that are not purple left in the drawer. Therefore, the possibility that the next sock we draw is not purple is:

$$P(\,second\ not\ purple) = \frac{30}{39} = \frac{10}{13}$$

Problem 6: Dice and Expected Value

(a) (5 points) When a pair of balanced dice are rolled and the sum of the numbers showing face up is computed, the result can be any number from 2 to 12, inclusive. What is the expected value of the sum?

(a)
There are total of 6 * 6 = 36 possibilities.
P (2) = 1/36
P (3) = 2/36
P (4) = 3/36
P (5) = 4/36
P (6) = 5/36
P (7) = 6/36
P (8) = 5/36
P (9) = 4/36
P (10) = 3/36
P (11) = 2/36
P (12) = 1/36
Define A to be the outcome of the two dices roll:

$$E[A] = \frac{1}{36}\cdot 2 + \frac{2}{36}\cdot 3 + \frac{3}{36}\cdot 4 + \frac{4}{36}\cdot 5 + \frac{5}{36}\cdot 6 + \frac{6}{36}\cdot 7 + \frac{5}{36}\cdot 8 + \frac{4}{36}\cdot 9 + \frac{3}{36}\cdot 10 + \frac{2}{36}\cdot 11 + \frac{1}{36}\cdot 12 = 7$$

Problem 7: Fair Coin Gambling

(a) (10 points) A person pays \$1 to play the following game. The person tosses a fair coin four times. If no heads occur, the person pays an additional \$2, if one head occurs, the person pays additional \$1, if two heads occur, the person just loses the initial dollar, if three heads occur, the person wins \$3, and if four heads occur, the person wins \$4. What is the person's expected loss or gain?

(a)
There are total of $2^4 = 16$ possibilities.

$$P(no\ head) = \frac{C(4,0)}{16} = \frac{1}{16}$$

$$P(1\ head) = \frac{C(4,1)}{16} = \frac{4}{16} = \frac{1}{4}$$

$$P(2\ heads) = \frac{C(4,2)}{16} = \frac{6}{16} = \frac{3}{8}$$

$$P(3\ heads) = \frac{C(4,3)}{16} = \frac{4}{16} = \frac{1}{4}$$

$$P(all\ heads) = p(no\ head) = \frac{1}{16}$$

Define A to be the expected loss or gain of a four toss of a fair coin:

$$E[A] = \frac{1}{16}\cdot(-2-1) + \frac{1}{4}\cdot(-1-1) + \frac{3}{8}\cdot(-1) + \frac{1}{4}\cdot(3-1) + \frac{1}{16}\cdot(4-1) = -\frac{3}{8}$$

Problem 8: Drug Screengin and Probability

A drug-screening test is used in a large population of people of whom %4 actually use drugs. Suppose that false positive rate is $\%3$, and the false negative rate is $\%2$. That is, a person who uses drugs tests positive $\%98$ of the time, and a person who does not use drugs tests negative $\%97$ of the time.

(a) (5 points) What is the probability that a randomly chosen person who tests positive for drugs actually uses drugs?

D represents people using drugs, P represents people tested positive.

P(D) = 0.04

$P(P|\bar{D}) = 0.03$

$P(\bar{P}|D) = 0.02$

(a)

$$P(D|P) = \frac{P(P|D)P(D)}{P(P|D)P(D) + P(P|\bar{D})P(\bar{D})} = \frac{(1-0.02)(0.04)}{(1-0.02)(0.04) + (0.03)(1-0.04)} = 0.58$$

(b) (5 points) What is the probability that a randomly chosen person who tests negative for drugs does not use drugs?

(b)

$$P(\bar{D}|\bar{P}) = \frac{P(\bar{P}|\bar{D})P(\bar{D})}{P(\bar{P}|\bar{D})P(\bar{D}) + P(\bar{P}|D)P(D)} = \frac{(1-0.03)(1-0.04)}{(1-0.03)(1-0.04) + (0.02)(0.04)} = \frac{1164}{1165}$$

Problem 9: Multiple Choice Exams and Probability

A student taking a multiple-choice exam does not know the answers to two questions. All have five choices for the answer. For one of the two questions, the student can eliminate two answer choices as incorrect, but he has no idea about the other answer choices. For the other question, the student has not clue about the correct answer at all. Assume that the student choice of an answer to one of the questions does not affect the his choice of an answer on the other question.

(a) (3 points) What is the probability that the student will answer both questions correctly?

(a)
The two events are independent,
P (first question correct) = 1/3
P (second question correct) = 1/5
P (both questions correct) = 1/3 * 1/5 = 1/15

(b) (4 points) What is the probability that the student will answer exactly one question correctly?

(b)
P (exactly one of the questions correct)
= P (answer only first question correct) + P (answer only second question correct)
= 1/3 * 4/5 + 2/3 * 1/5
= 4/15 + 2/15
= 6/15 = 2/5

(c) (3 points) What is the probability that the student will answer neither question correctly?

(c)
P (neither question correct)
= 1 - P (two questions correct) - P(exactly one question correct)
= 1 - 1/15 - 6/15
= 8/15

Problem 10: Birthdays and Probability

There are n persons in a room.

(a) (10 points) What is the probability that at least two persons have the same birthday?

For this problem, we are going to assume that the year is not a leap year. Thus, each year has exactly 365 days.
The chance of 2 people having different birthdays is:

$$1 - \frac{1}{365} = \frac{364}{365}$$

If there are n people in total, the different ways to choose 2 people from n people are C (n, 2).
We basically need to make C (n, 2) times comparison to determine whether they are all different.

The possibility for every pair among n people is different is: $\left(\frac{364}{365}\right)^{C(n,2)}$

So, the probability that at least two person have the same birthday:

$$P(At\ least\ two\ people\ same\ birthday) = 1 - \left(\frac{364}{365}\right)^{C(n,2)} = 1 - \left(\frac{364}{365}\right)^{n(n-1)/2}$$

Question	Points	Score
Simple Random Experiments	10	
Bitstrings and Palindromes	10	
Roulette and Probabilities	10	
Probabilities and Ice Cream	10	
Socks and Probability	15	
Dice and Expected Value	5	
Fair Coin Gambling	10	
Drug Screengin and Probability	10	
Multiple Choice Exams and Probability	10	
Birthdays and Probability	10	
Total:	100	

SUBMISSION DETAILS

Things to submit:

- Please submit this assignment as a .pdf named "CS5002_[lastname]_HW6.pdf" through Canvas by 11:59pm PT on Saturday, November 6, 2021.
- Please make sure your name is in the document as well (e.g., written on the top of the first page).