HW4: INTRODUCTION TO FUNCTIONS

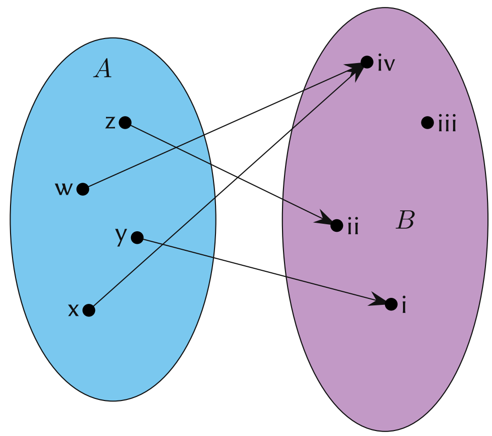
The essential function of art is moral. But a passionate, implicit morality, not didactic. A morality which changes the blood, rather than the mind.– D.H. Lawrence

Course: CS 5002
Fall 2021
Due: October 9, 2021

PROBLEMS

Problem 1: Function definitions

Consider the function f defined by the figure to the right. Find:



(a) (3 points) A = {z, w, y, x} ....

(b) (3 points) B = {iv, iii, ii, i} ....

(c) (4 points) Define $f : A \to B$ represented by the diagram. f = {(z, ii), (w, iv), (y, i), (x, iv)} ....

Problem 2: Domain, target anf range

Please find the domain, target and range of each of these functions.

(a) (3 points) The function that assigns each positive integer its first digit.

Domain: every positive integer in R.
Range: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}.
Target: R.

(b) (3 points) The function that takes a bit string, and assigns the number of ones in the string.

Domain: All bit string of size n, where n is in Z+.
Range: {0, 1, 2, 3, ……., n - 3, n - 2, n - 1, n}.
Target: Z.

(c) (3 points) The function that takes a string of letters, and assigns the number of vowels $\{a, e, i, o, u\}$ it contains.

Domain: All string of letters of size n, where n is in Z+.
Range: {0, 1, 2, 3, ……., n - 3, n - 2, n - 1, n}.
Target: Z.

Problem 3: Valid functions

Determine if each of the following sets of ordered pairs on X is a function from X to X, when $X = \{1, 2, 3, 4, 5\}$. Please explain your answer.

(a) (2 points) $f = \{(2, 5), (5, 4), (2, 1), (3, 2), (4, 4)\}$

(a)
No, f maps x = 2 to both y = 1 and y = 5, so an element in the domain is mapped to two different elements in the target, and f has no mapping for x = 1, so f is not well-defined.

(b) (2 points) $g = \{(5, 1), (4, 5), (1, 5)\}$

(b)
No, g has no mapping for x = 2 and x = 3, so the f is not well-defined.

(c) (2 points) $h = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$

(c)
Yes, h has mapping for every element in the domain, and the range of h is 1 since all the elements in the domain mapped to 1 in the target.

(d) (2 points) $h = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5)\}$

(d)
No, h maps x = 1 to y = 1, y = 2, y = 3, y = 4, y = 5, so an element in the domain is mapped to more than 1 elements in the target, so f is not well-defined.

Problem 4: Valid functions

Determine if f from $\mathbb{Z}$ to $\mathbb{R}$ a valid function. Please explain your answer.

(a) (3 points) $f(x) = x$

(a)
Yes, f maps every integer in the domain to exactly one element in the set of all real numbers, and all integer (Z) is a subset of all real numbers (R), so f(x) = x is in the range of R.

(b) (3 points) $f(x) = \pm x$

(b)
No, f maps every integer in the domain to two elements except when x = 0. For a well-defined function, every element in the domain needs to map to exactly one element in the target of the function. Therefore, f(x) is not well defined.

(c) (3 points) $f(x) = \sqrt{x^2 + 1}$

(c)
Yes, f maps every pair of positive and negative integers to exactly one element in the set of all real numbers, so $f(x) = \sqrt{x^2 + 1}$ is in the range of R. Since Z is a subset of R, so the function is well-defined, and f(x) is in the range of R.

(d) (3 points) $f(x) = \frac{1}{x}$

(d)
No, f maps every integer in the domain to exactly one element except when x = 0. If the denominator of f(x) is 0, the function is not defined. Therefore, this is not a valid function with the domain of Z.

Problem 5: One-to-one functions

Determine if each of the functions $f : \mathbb{Z} \to \mathbb{Z}$ is a **one-to-one** function.

(a) (2 points) $f(x) = x - 1$

(a)
Yes.
Proof:
Assume f(a) = f(b) for all $a, b \in Z$
This follows that a - 1 = b - 1
Adding 1 on both side: a = b
Therefore, if a = b, f(a) = f(b), thus proving that f(x) is injective. ■

(b) (2 points) $f(x) = x^2 + 1$

(b)
No, $f(x) = x^2 + 1$ will map both x = x and x = -x to the same value after the x^2 operation, so $f(x = x) = x^2 + 1$ is the same as $f(x = -x) = x^2 + 1$. The function is not injective.

(c) (2 points) $f(x) = x^3$

(c)
Yes.
Proof:
Assume f(a) = f(b) for all $a, b \epsilon Z$
This follows that $a^3 = b^3$
This leads to that $\sqrt[3]{a} = \sqrt[3]{b}$
The above equation only holds if a = b
Therefore, if a = b, f(a) = f(b), thus proving that f(x) is injective. ■

(d) (2 points) $f(x) = \lceil \frac{x}{2} \rceil$

(d)
No, f(1) = $\left\lceil \frac{1}{2} \right\rceil = 1$, and f(2) = $\left\lceil \frac{2}{2} \right\rceil = 1$, so f does not map every integer to a unique integer in the range of f. The function f(x) = $\left\lceil \frac{x}{2} \right\rceil$ is not injective.

Problem 6: Onto functions

All but two of the following statements are correct ways to express the fact that a function is **onto**. Find the two that are incorrect. Please justify your answer.

(a) (1 point) f is onto $\leftrightarrow$ every element in its codomain is the image of some element in its domain.

Correct.

(b) (1 point) f is onto $\leftrightarrow$ every element in its domain has a corresponding image in its codomain.

Incorrect, the statement says that for every element in its domain, there is a corresponding image in its target, so it is possible that some elements in the target do not have a mapping in the domain, so it is not surjective.

(c) (1 point) f is onto $\leftrightarrow \forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Correct.

(d) (1 point) f is onto $\leftrightarrow \forall x \in X, \exists y \in Y$ such that $f(x) = y$.

Incorrect, the statement says that ∀x ∈ X, ∃y ∈ Y, so it is possible that some elements in Y do not have a mapping in the domain X, so it is not surjective.

(e) (1 point) f is onto $\leftrightarrow$ the range of f is the same as the codomain of f.

Correct.

Problem 7: One-to-one correspondence

Determine whether or not the following functions are bijections from $\mathbb{R}$ to $\mathbb{R}$:

(a) (4 points) $f(x) = -3x + 4$

(a)
A function f has an inverse iff f is a bijection.
Assume: f(x) = y

$$y = -3x + 4$$
$$y - 4 = -3x$$
$$x = \frac{y-4}{-3} = \frac{4-y}{3}$$

For every element y in R, $\frac{4-y}{3}$ is unique in R, so the inverse of f(x) is $f^{-1} = \frac{4-x}{3}$ is well-defined from R to R.
If the inverse of f(x) is well-defined, f(x) is bijective.

(b) (4 points) $f(x) = x^2 + 1$

(b)
A function f has an inverse iff f is a bijection.
Assume: f(x) = y

$$y = x^2 + 1$$
$$y - 1 = x^2$$
$$x = \sqrt{y-1}$$

For every element y < 1 in R, $\sqrt{y-1}$ is not defined, so the inverse of f(x) is $f(x) = \sqrt{x-1}$ is not well-defined from R to R.
If the inverse of f(x) is not well-defined, f(x) is not bijective.

(c) (4 points) $f(x) = \frac{x^2+1}{x^2+2}$

(c)
A function f has an inverse iff f is a bijection.
Assume: f(x) = y

$$y = \frac{x^2+1}{x^2+2} = \frac{x^2+2}{x^2+2} - \frac{1}{x^2+2} = 1 - \frac{1}{x^2+2}$$
$$1 - y = \frac{1}{x^2+2}$$
$$\frac{1}{1-y} = x^2 + 2$$
$$x = \sqrt{\frac{1}{1-y} - 2}$$

For every element y = 1in R, $\sqrt{\frac{1}{1-y} - 2}$ is not defined because the denominator cannot be 0, so the inverse of f(x) is $f(x) = \sqrt{\frac{1}{1-x} - 2}$ is not well-defined from R to R.
If the inverse of f(x) is not well-defined, f(x) is not bijective.

(d) (4 points) $f(x) = x^3$

(d)
A function f has an inverse iff f is a bijection.
Assume: f(x) = y

$$y = x^3$$
$$x = \sqrt[3]{y}$$

For every element y in R, $\sqrt[3]{y}$ is unique in R, so the inverse of f(x) is $f^{-1} = \sqrt[3]{y}$ from R to R is well-defined.
If the inverse of f(x) is well-defined, f(x) is bijective.

(e) (4 points) $f(x) = x^5 + 1$

(e)
A function f has an inverse iff f is a bijection.
Assume: f(x) = y

$$y = x^5 + 1$$
$$x = \sqrt[5]{y-1}$$

For every element y in R, $\sqrt[5]{y-1}$ is unique in R, so the inverse of f(x) is $f^{-1} = \sqrt[5]{y-1}$ from R to R is well-defined.
If the inverse of f(x) is well-defined, f(x) is bijective.

Problem 8: Composition of functions

Below are three functions defined from $\mathbb{R}$ to $\mathbb{R}$.

$$f(x) = 5 - x$$
$$g(x) = 3x$$
$$h(x) = x + 4$$

Find the following compositions:

(a) (3 points) $f \circ g$

$$f \circ g = f(g(x)) = 5 - 3x$$

(b) (3 points) $g \circ f$

$$g \circ f = g\big(f(x)\big) = 3(5 - x) = 15 - 3x$$

(c) (4 points) $f \circ g \circ h$

$$f \circ g \circ h = f\Big(g\big(h(x)\big)\Big) = 5 - 3(x + 4) = 5 - 3x - 12 = -3x - 7$$

(d) (4 points) $f \circ h \circ f$

$$f \circ h \circ f = f\Big(h\big(f(x)\big)\Big) = 5 - (5 - x + 4) = x - 4$$

Problem 9: Invertible functions

Prove that some function $f(x) = ax + b$ from $\mathbb{R}$ to $\mathbb{R}$, where a and b are constants, is invertible when $a \neq 0$, and find its inverse.

Proof:
Assume f(x) = y

$$y = ax + b$$
$$y - b = ax$$
$$x = \frac{y - b}{a}$$

If a = 0, x is not defined because the denominator of a fraction cannot be 0. ■
The inverse of function f(x) is not defined when a = 0. The inverse of f(x) is $f^{-1}(x) = \frac{x-b}{a}$ when $a \neq 0$ from R to R.

Problem 10: Integer functions

Let x be a real number. Show that:

$$\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$$

Proof:
Let $x = n + \epsilon$, where n is an integer and $0 \leq \epsilon < 1$.
Case 1: $0 \leq \epsilon < \frac{1}{3}$
$3x = 3n + 3\epsilon$ and $\lfloor 3x \rfloor = 3n$ because $0 \leq 3\epsilon < 1$.
Similarly, $\lfloor x \rfloor = n$ because $0 \leq \epsilon < \frac{1}{3}$
$x + \frac{1}{3} = n + \frac{1}{3} + \epsilon$, so $\left\lfloor x + \frac{1}{3} \right\rfloor = n$ because $0 \leq \frac{1}{3} + \epsilon < 1$.
$x + \frac{2}{3} = n + \frac{2}{3} + \epsilon$, so $\left\lfloor x + \frac{2}{3} \right\rfloor = n$ because $0 \leq \frac{2}{3} + \epsilon < 1$.
$\lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor = 3n$, so $\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor$
Case 2: $\frac{1}{3} \leq \epsilon < \frac{2}{3}$
$3x = 3n + 3\epsilon$ and $\lfloor 3x \rfloor = 3n + 1$ because $1 \leq 3\epsilon < 2$.
Similarly, $x + \frac{1}{3} = n + \frac{1}{3} + \epsilon$, so $\left\lfloor x + \frac{1}{3} \right\rfloor = n$ because $0 \leq \frac{1}{3} + \epsilon < 1$.
$x + \frac{2}{3} = n + \frac{2}{3} + \epsilon$, so $\left\lfloor x + \frac{2}{3} \right\rfloor = n + 1$ because $1 \leq \frac{2}{3} + \epsilon < 2$.
$\lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor = n + n + (n + 1) = 3n + 1$, so $\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor$
Case 3: $\frac{2}{3} \leq \epsilon < 1$
$3x = 3n + 3\epsilon$ and $\lfloor 3x \rfloor = 3n + 2$ because $2 \leq 3\epsilon < 3$.
Similarly, $x + \frac{1}{3} = n + \frac{1}{3} + \epsilon$, so $\left\lfloor x + \frac{1}{3} \right\rfloor = n + 1$ because $1 \leq \frac{1}{3} + \epsilon < 2$.
$x + \frac{2}{3} = n + \frac{2}{3} + \epsilon$, so $\left\lfloor x + \frac{2}{3} \right\rfloor = n + 1$ because $1 \leq \frac{2}{3} + \epsilon < 2$.
$\lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor = n + (n + 1) + (n + 1) = 3n + 2$, so $\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor$
In conclusion, for all $x \in R$, $\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor$. ■

Question	Points	Score
Function definitions	10	
Domain, target anf range	9	
Valid functions	8	
Valid functions	12	
One-to-one functions	8	
Onto functions	5	
One-to-one correspondence	20	
Composition of functions	14	
Invertible functions	8	
Integer functions	6	
Total:	100	

SUBMISSION DETAILS

Things to submit:

- Please submit this assignment as a .pdf named "CS5002_[lastname]_HW4.pdf" through Canvas by 11:59pm PT on Saturday, October 9, 2021.
- Please make sure your name is in the document as well (e.g., written on the top of the first page).