

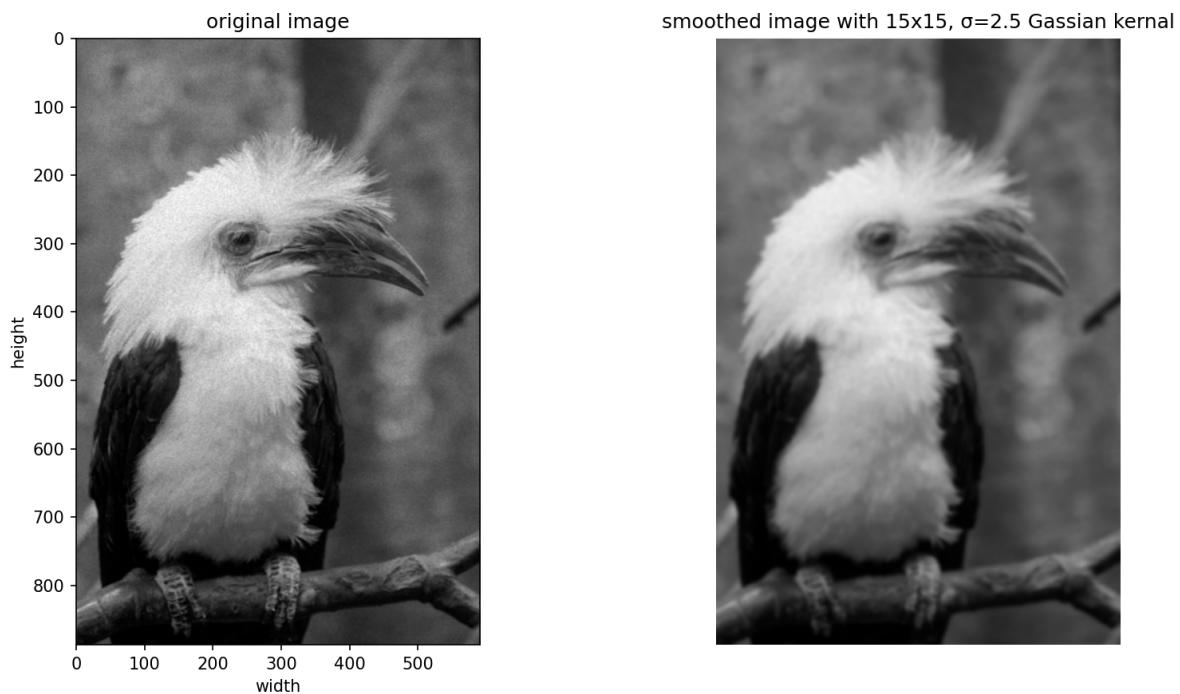
图像处理与可视化: Homework 3

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[HW3-1] 编程实现图像域基于空间滤波器的 (1) 平滑操作、(2) 锐化算法；并把算法应用与图片上，显示与原图的对比差别（备注：实现的代码不能调用某个算法库里面的函数实现平滑或锐化）。

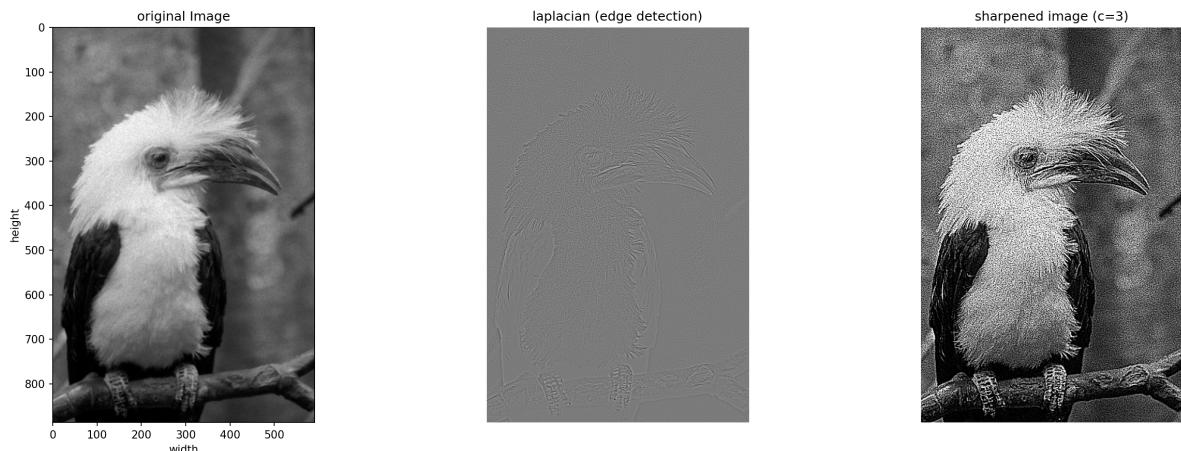
(1) 平滑操作

- 原始图像 `3-1.png` 是一张经过噪音污染的照片，接下来分别展示使用平滑算法、锐化原始图像和锐化平滑后的图像



- `comparison_smooth_7_3-1.png` 是原始图像 `3-1.png` 和 使用 $15 \times 15, \sigma = 2.5$ 的高斯核进行平滑后的图像 `smooth_7_3-1` 的并排显示（数字 7 指的是核大小 15 减 1 的一半，选取核大小的时候遵循三个标准差准则，例如 7 是标准差 2.5 的 3 倍 7.5 的向下取整），平滑后的图像失去了很多细节，但好在噪音也一定程度上被去除了

(2) 锐化算法



- `comparison_sharpen_3_3-1.png` 是原始图像 `3-1.png`, 对原始图像 `3-1.png` 进行拉普拉斯核滤波处理并进行归一化和最终锐化并进行截断的结果; 可以看到, 由于原图像受到噪音污染, 拉普拉斯滤波处理后的图像无法明显地检测出边缘; 最终, 原始图像的噪声同边缘一样被锐化了



- `comparison_sharpen_3_smooth_7_3-1.png` 是原始图像的平滑处理 `smooth_7_3-1.png`, 进行拉普拉斯核滤波处理并进行归一化 `lapacian_3_smooth_7_3-1.png` 和最终锐化并进行截断 `sharpen_3_smooth_7_3-1.png` 的并排显示; 结果显示, 拉普拉斯核滤波处理检测出了更明显的边缘, 锐化算法在经过平滑预处理的图像上也实现了较好的效果

```
# HW3-1.py
import numpy as np
from PIL import Image
import os
import matplotlib.pyplot as plt
import time
import math

### (1) smooth
def spatial_filter_smooth_one_channel(N, sigma, channel_array):
    """
    Parameter:
        N: kernel size (2N+1)x(2N+1)
        sigma: standard devia
        channel_array: single channel array
    Return:
        smooth_channel_array: smoothed channel array with N padding
    """

    height, width = channel_array.shape
    kernel_size = 2 * N + 1

    # construct gaussion kernel
    kernel = np.zeros((kernel_size, kernel_size))
    for i in range(kernel_size):
        for j in range(kernel_size):
            x, y = i - N, j - N
            kernel[i, j] = np.exp(-(x**2 + y**2) / (2 * sigma**2))
    kernel /= kernel.sum()

    # pad with edge's pixel
    padded = np.pad(channel_array, pad_width=N, mode='edge')
    smooth_channel_array = np.zeros((height, width))
    for i in range(kernel_size):
        for j in range(kernel_size):
            if i < N or i > height - N or j < N or j > width - N:
                continue
            smooth_channel_array[i - N, j - N] = padded[i, j]
    return smooth_channel_array
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        smooth_channel_array += kernel[i, j] * padded[i: i + height, j: j +
width]

    return smooth_channel_array.astype(channel_array.dtype)

def spatial_filter_smooth(N, sigma, image_path, save=True):
    """
    Parameter:
        N: kernel radius
        sigma: gaussian sigma
        image_path
        save
    """
    img = Image.open(image_path)
    img_array = np.array(img)
    # if gray
    start_time = time.time()
    if len(img_array.shape) == 2:
        smooth_array = spatial_filter_smooth_one_channel(N, sigma, img_array)
    # if rgb
    else:
        smooth_array = np.zeros_like(img_array)
        for channel in range(img_array.shape[2]):
            smooth_array[:, :, channel] = spatial_filter_smooth_one_channel(
                N, sigma, img_array[:, :, channel])
    print(f"spatial filter smooth of {image_path}: {time.time() - start_time:.2f}s")
    # save
    if save:
        image_dir = "image"
        file_name = os.path.basename(image_path)
        output_name = "smooth_" + str(N) + "_" + file_name
        save_path = os.path.join(image_dir, output_name)

        result_img = Image.fromarray(smooth_array)
        result_img.save(save_path)

    return smooth_array

### (2) sharpen
def sharpen_one_channel(c, channel_array):
    """
    Parameter:
        c: for the laplacian kernal defined here, c is a positive number
    sharpen one channel using Laplacian filter
    Return:
        sharpen_channel_array: sharpened channel array after trimming
        laplacian = normalized \nable^{2} f
    """
    height, width = channel_array.shape
    laplacian_kernel = np.array([
        [-1, -1, -1],
        [-1, 8, -1],
        [-1, -1, -1]
    ], dtype=np.float64)

    kernel_size = 3

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pad_width = kernel_size // 2
padded_channel = np.pad(channel_array, pad_width=pad_width, mode='edge')

# for actual sharpen: dont normalize laplacian and trim sharpened channel
array
laplacian = np.zeros((height, width))
for i in range(kernel_size):
    for j in range(kernel_size):
        laplacian += laplacian_kernel[i, j] * padded_channel[i:i+height,
j:j+width]

sharpen_channel_array = channel_array + c * laplacian
sharpen_channel_array = np.clip(sharpen_channel_array, 0, 255)

# for demonstrate and comparison: normalize laplacian to [0, 255] for
visualization
laplacian_min = (-laplacian).min()
laplacian_max = (-laplacian).max()
if laplacian_max - laplacian_min > 0:
    normalized_laplacian = (-laplacian - laplacian_min) / (laplacian_max -
laplacian_min) * 255
else:
    normalized_laplacian = np.zeros_like(-laplacian)

return sharpen_channel_array, normalized_laplacian

def spatial_filter_sharpen(c, image_path, save=True):

    img = Image.open(image_path)
    img_array = np.array(img)
    start_time = time.time()
    # if grey
    if len(img_array.shape) == 2:
        sharpen_array, laplacian_array = sharpen_one_channel(c, img_array)
    # if RGB
    else:
        sharpen_array = np.zeros_like(img_array)
        laplacian_array = np.zeros_like(img_array)
        for channel in range(img_array.shape[2]):
            sharpen_array[:, :, channel], laplacian_array[:, :, channel] =
sharpen_one_channel(
                c, img_array[:, :, channel])
    print(f"spatial filter sharpen of {image_path}: {time.time() -
start_time:.2f}s")
    # save
    if save:
        image_dir = "image"
        file_name = os.path.basename(image_path)
        # save sharpened image
        sharpen_name = "sharpen_" + str(c) + "_" + file_name
        sharpen_path = os.path.join(image_dir, sharpen_name)
        sharpen_img = Image.fromarray(sharpen_array.astype(np.uint8))
        sharpen_img.save(sharpen_path)
        # save laplacian image
        laplacian_name = "laplacian_" + str(c) + "_" + file_name
        laplacian_path = os.path.join(image_dir, laplacian_name)
        if len(laplacian_array.shape) == 3 and laplacian_array.shape[2] == 4:

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        laplacian_array = laplacian_array[:, :, :3]
        laplacian_img = Image.fromarray(laplacian_array)
        laplacian_img.save(laplacian_path)

    return sharpen_array, laplacian_array

### plot
def smooth_comparison(image_path, N, sigma, save_original=True):

    img = Image.open(image_path)
    img_array = np.array(img)

    # smooth
    smooth_result = spatial_filter_smooth(N, sigma, image_path, save_original)
    # plot
    fig, axes = plt.subplots(1, 2, figsize=(12, 6))

    im1 = axes[0].imshow(img_array, cmap='gray' if len(img_array.shape) == 2
else None)
    axes[0].set_title('original image')
    axes[0].set_xlabel('width')
    axes[0].set_ylabel('height')
    axes[0].axis('on')

    im2 = axes[1].imshow(smooth_result.astype(np.uint8), cmap='gray' if
len(img_array.shape) == 2 else None)
    axes[1].set_title(f'smoothed image with {2*N+1}x{2*N+1}, σ={sigma} Gaussian
kernel')
    axes[1].axis('off')

    plt.tight_layout()
    # save
    image_dir = "image"
    file_name = os.path.basename(image_path)
    name, _ = os.path.splitext(file_name)
    plt.savefig(os.path.join(image_dir, f'comparison_smooth_{N}_{name}.png'),
bbox_inches='tight', dpi=150)
    plt.close()

def sharpen_comparison(image_path, c, save_original):

    img = Image.open(image_path)
    img_array = np.array(img)

    # sharpen
    sharpen_result, laplacian_result = spatial_filter_sharpen(c, image_path,
save_original)
    # plot
    fig, axes = plt.subplots(1, 3, figsize=(18, 6))

    im1 = axes[0].imshow(img_array, cmap='gray' if len(img_array.shape) == 2
else None)
    axes[0].set_title('original Image')
    axes[0].set_xlabel('width')
    axes[0].set_ylabel('height')
    axes[0].axis('on')

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if len(laplacian_result.shape) == 3 and laplacian_result.shape[2] == 4:
    laplacian_result = laplacian_result[:, :, :3]
im2 = axes[1].imshow(laplacian_result, cmap='gray' if
len(laplacian_result.shape) == 2 else None)
axes[1].set_title(f'laplacian (edge detection)')
axes[1].axis('off')

# sharpened image
im3 = axes[2].imshow(sharpen_result.astype(np.uint8), cmap='gray' if
len(img_array.shape) == 2 else None)
axes[2].set_title(f'sharpened image (c={c})')
axes[2].axis('off')

plt.tight_layout()
# save
image_dir = "image"
file_name = os.path.basename(image_path)
name, _ = os.path.splitext(file_name)
plt.savefig(os.path.join(image_dir, f'comparison_sharpen_{c}_{name}.png'),
bbox_inches='tight', dpi=150)
plt.close()

if __name__ == "__main__":
    # smooth
    sigma = 2.5
    N = math.floor(3 * sigma)
    # sharpen
    c = 3
    # result comparison
    image_path = os.path.join("image", "3-1.png")
    smooth_comparison(image_path, N, sigma, save_original = True)
    sharpen_comparison(image_path, c, save_original = False)
    smooth_image = os.path.join("image", f"smooth_{N}_3-1.png")
    sharpen_comparison(smooth_image, c, save_original = True)

```

- 直接在本目录下运行 `hw3-1.py`，处理过的图像将保存至本目录下的 `image` 文件夹中

[HW3-2] 证明：

(1) 证明冲击串 (impulse train) 的傅里叶变换后的频域表达式也是一个冲击串。

冲激串 $S_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$ 是周期为 ΔT 的周期函数，可展开为傅里叶级数：

$$S_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n}{\Delta T} t}$$

其中 c_n 为： $c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} S_{\Delta T}(t) e^{-j \frac{2\pi n}{\Delta T} t} dt$

在一个周期 $[-\Delta T/2, \Delta T/2]$ 内， $S_{\Delta T}(t)$ 仅包含 $\delta(t)$ ，利用 δ 函数的取样特性：

$$c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j \frac{2\pi n}{\Delta T} t} dt = \frac{1}{\Delta T} e^0 = \frac{1}{\Delta T}$$

故 $S_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{\Delta T} e^{j \frac{2\pi n}{\Delta T} t}$

$$\mathcal{F}(u) = \mathcal{F}\{S_{\Delta T}(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \frac{1}{\Delta T} e^{j \frac{2\pi n}{\Delta T} t}\right\} = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \mathcal{F}\left\{e^{j 2\pi \left(\frac{n}{\Delta T}\right) t}\right\} \quad (1)$$

上式中的傅里叶变换较难计算，故转而计算 δ 冲激函数的傅里叶逆变换： $\mathcal{F}^{-1}\{\delta(\mu - \frac{n}{\Delta T})\}$

$$\mathcal{F}^{-1}\{\delta(\mu - \frac{n}{\Delta T})\} = \int_{-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T}) e^{j2\pi\mu t} d\mu = e^{j2\pi(\frac{n}{\Delta T})t}$$

$$\text{根据傅里叶变换的可逆性, } \mathcal{F}\left\{e^{j2\pi\frac{n}{\Delta T}t}\right\} = \delta(\mu - \frac{n}{\Delta T}) \quad (2)$$

将(2)代入(1): $S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$, 即冲击串的傅里叶变换后的频域表达式也是一个冲击串

(2) 证明实信号f(x)的离散频域变换结果是共轭对称的。

即证明, 设 $f_n (n = 0, 1, \dots, M-1)$ 为一维离散实信号, $f_n = f_n^*$, 其离散傅里叶变换 (DFT) 为 $F_m (m = 0, 1, \dots, M-1)$, 则 F_m 是共轭对称的, 即 $F_m^* = F_{M-m}$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j\frac{2\pi mn}{M}}, \quad F_m^* = \left(\sum_{n=0}^{M-1} f_n e^{-j\frac{2\pi mn}{M}} \right)^* = \sum_{n=0}^{M-1} f_n^* \left(e^{-j\frac{2\pi mn}{M}} \right)^*$$

$$\text{代入 } f_n^* = f_n, \quad (e^{-j\theta})^* = e^{j\theta}, \quad \text{故 } F_m^* = \sum_{n=0}^{M-1} f_n e^{j\frac{2\pi mn}{M}}$$

$$\text{进一步变换指数项 } e^{j\frac{2\pi mn}{M}} = \cos\left(\frac{2\pi mn}{M}\right) + j \sin\left(\frac{2\pi mn}{M}\right)$$

利用三角函数的周期性:

$$\cos\left(\frac{2\pi mn}{M}\right) = \cos\left(\frac{2\pi mn}{M} - 2\pi n\right) = \cos\left(2\pi n - \frac{2\pi mn}{M}\right) = \cos\left(\frac{2\pi(M-m)n}{M}\right)$$

$$\sin\left(\frac{2\pi mn}{M}\right) = \sin\left(\frac{2\pi mn}{M} - 2\pi n\right) = -\sin\left(2\pi n - \frac{2\pi mn}{M}\right) = -\sin\left(\frac{2\pi(M-m)n}{M}\right)$$

$$\text{代回指数项: } e^{j\frac{2\pi mn}{M}} = \cos\left(\frac{2\pi(M-m)n}{M}\right) - j \sin\left(\frac{2\pi(M-m)n}{M}\right) = e^{-j\frac{2\pi(M-m)n}{M}}$$

$$\text{代回 } F_m^* = \sum_{n=0}^{M-1} f_n e^{-j\frac{2\pi(M-m)n}{M}}, \quad \text{而 } F_{M-m} = \sum_{n=0}^{M-1} f_n e^{-j\frac{2\pi(M-m)n}{M}}$$

$$\text{故 } F_m^* = F_{M-m}$$

(3) 证明二维变量的离散频域/傅里叶变换的卷积定理。

设 $f(x, y)$ 和 $h(x, y)$ 为 $M \times N$ 的二维离散信号, 先给出一些基本公式:

- 二维离散傅里叶变换:

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

- 二维离散傅里叶逆变换:

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

- 二维循环卷积:

$$f(x, y) * h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x-m, y-n)$$

$$\textcircled{1} \text{ 证明: } f(x, y) * h(x, y) \iff F(u, v) H(u, v), \quad \mathcal{F}\{f(x, y) * h(x, y)\} = F(u, v) H(u, v)$$

$$\mathcal{F}\{f(x, y) * h(x, y)\} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} (f(x, y) * h(x, y)) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x-m, y-n) \right) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

交换求和顺序:

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) \left(\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} h(x-m, y-n) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})} \right)$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) \left(\sum_{p=0}^{M-1} \sum_{q=0}^{N-1} h(x-m, y-n) e^{-j2\pi(\frac{u(x-m)}{M} + \frac{v(y-n)}{N})} e^{-j2\pi(\frac{um}{M} + \frac{vn}{N})} \right)$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) e^{-j2\pi(\frac{um}{M} + \frac{vn}{N})} \left(\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} h(x-m, y-n) e^{-j2\pi(\frac{u(x-m)}{M} + \frac{v(y-n)}{N})} \right)$$

令 $p = x - m$, $q = y - n$, 由于假定的周期性, 求和标号可以平移到原来的一个周期内:

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) e^{-j2\pi(\frac{um}{M} + \frac{vn}{N})} \left(\sum_{p=0}^{M-1} \sum_{q=0}^{N-1} h(p, q) e^{-j2\pi(\frac{up}{M} + \frac{vq}{N})} \right)$$

括号内是二维离散傅里叶变换:

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) e^{-j2\pi(\frac{um}{M} + \frac{vn}{N})} H(u, v)$$

$$= H(u, v) \left(\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) e^{-j2\pi(\frac{um}{M} + \frac{vn}{N})} \right)$$

$$= F(u, v) H(u, v)$$

$$\textcircled{2} \text{ 证明: } f(x, y)h(x, y) \iff \frac{1}{MN} F(u, v) * H(u, v), \quad \mathcal{F}\{f(x, y)h(x, y)\} = \frac{1}{MN}(F * H)(u, v)$$

$$\mathcal{F}\{f(x, y)h(x, y)\} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)h(x, y) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

将 $h(x, y)$ 用二维离散傅里叶逆变换表达 (为避免混淆, 使用 m, n 作为求和变量):

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \left(\frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} H(m, n) e^{j2\pi(\frac{mx}{M} + \frac{ny}{N})} \right) e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})}$$

交换求和顺序, 提取 $\frac{1}{MN}$:

$$= \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} H(m, n) \left(\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{j2\pi(\frac{mx}{M} + \frac{ny}{N})} e^{-j2\pi(\frac{ux}{M} + \frac{vy}{N})} \right)$$

$$= \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} H(m, n) \left(\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(\frac{(u-m)x}{M} + \frac{(v-n)y}{N})} \right)$$

括号内是二维离散傅里叶变换的表达式:

$$= \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} H(m, n) F(u - m, v - n)$$

$$= \frac{1}{MN} (H * F)(u, v) = \frac{1}{MN} (F * H)(u, v)$$