Sep 12, 2023 (Due: 08:00 Sep 19, 2023)

1. Find the exact LU factorization of the $n \times n$ matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 1 \\ -1 & -1 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 1 \end{bmatrix}.$$

2. Let $A \in \mathbb{C}^{n \times n}$ be Hermitian, i.e., $A^* = A$. Show that after one step of Gaussian elimination, the $(n-1) \times (n-1)$ trailing principal submatrix (known as the *Schur complement*) is still Hermitian.

3. Let $A \in \mathbb{C}^{n \times n}$ be strictly diagonally dominant, i.e.,

$$|a_{ii}| > \sum_{\substack{1 \le j \le n \\ j \ne i}} |a_{ij}|$$

for $1 \le i \le n$. Show that after one step of Gaussian elimination, the $(n-1) \times (n-1)$ Schur complement is still strictly diagonally dominant.

4. Let $A = (a_{ij})$ be a square banded matrix with bandwidth $2b + 1$ (i.e., $a_{ij} = 0$ if $|i - j| > b$). Suppose that all leading principal minors of A are nonzero such that A admits an LU factorization $A = LU$. Show that L and U are also banded, and determine their bandwidths. What is the complexity of computing such an LU factorization?

5. Implement Gaussian elimination with two pivoting strategies—partial pivoting and complete pivoting. Measure the execution time of your program in terms of matrix dimensions and visualize the result by a log–log scale plot.

(optional) Compare the performance of your implementation with a linear solver from a well-developed math library (`linsolve` from MATLAB/Octave, `DGESV` from LAPACK, `numpy.linalg.solve` from NumPy, etc.).

6. (optional) Suppose that you have a BLAS library with an inefficient implementation of `STRSM`. Try to design an efficient `STRSM` subroutine by making use of `SGEMM`. You do not have to really implement it. Just describe your algorithm.

Hint: Search online if you do not know what `STRSM` and `SGEMM` stand for. Useful information can be found at, e.g., `http://www.netlib.org/blas/`.