Oct 10, 2023 (Due: 08:00 Oct 17, 2023)

1. Write a program to compute the QR factorization of a general complex matrix $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ with CGS and MGS, with and without reorthogonalization. Visualize the loss of orthogonality $|Q^*Q - I_n|$.

2. Generate a few tall-skinny matrices with condition numbers varying from 10^0 to 10^{15}. Visualize the loss of orthogonality $\|Q^*Q - I_n\|_\mathsf{F}$ and the residual norm $\|A - QR\|_\mathsf{F}$ for Householder-QR, Cholesky-QR, CGS, MGS, etc.

3. Let $A \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{n\times m}$. Suppose that for any $b \in \mathbb{C}^m$, $x = Xb$ is always a minimizer of the least squares problem $\min_x \|Ax - b\|_2$. Show that $AXA = A$ and $(AX)^* = AX$.

4. Generate a few least squares problems with condition numbers varying from 10^0 to 10^{15}. Compare the accuracy of the solutions produced by the following methods:
(a) solve the normal equation $A^*Ax = A^*b$ through the Cholesky factorization of A^*A;
(b) solve the augmented system

$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix};$$

(c) solve the equation $Rx = Q^*b$ through Householder-QR;
(d) solve the equation $Rx = Q^*b$ through MGS.

5. Find the "best" straight line that approximately passes through the data set $\{(n, \ln n) \in \mathbb{R}^2 \colon n \in \{2, 3, 4, 5, 6, 7\}\}$. Visualize your result and clarify in what sense your solution is the best.

6. (H) Describe how to use cyclic reduction to solve a linear system if the coefficient matrix is a diagonally dominant five-diagonal matrix.

7. (H, optional) Implement Householder-QR with the B-inner product, where B is positive definite. Compare it with CGS/MGS/Cholesky-QR by visualizing the loss of orthogonality $\|Q^*BQ - I_n\|_\mathsf{F}$ and the residual norm $\|A - QR\|_\mathsf{F}$.