Dec 12, 2023 (Due: 08:00 Dec 19, 2023)

1. Suppose that $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Show that when applying CG to solve $Ax = b$, CG converges within k iterates if A has k distinct eigenvalues.

2. Derive the computational scheme of the preconditioned conjugate gradient (PCG) method.

3. Use the conjugate gradient method to solve the 2D Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

over the unit square $[0,1]^2$ with boundary conditions

$$u(0,y) = u(1,y) = u(x,1) = 0, \qquad u(x,0) = \sin(\pi x).$$

Visualize the solution and the convergence history.

(optional) Use IChol-based preconditioning to accelerate the convergence of CG.

4. Implement the Lanczos algorithm for symmetric eigenvalue problems. Test it with some examples of size several thousands. What do you observe for the convergence of Ritz pairs and the orthogonality of the Lanczos vectors?

5. Suppose that you have a black-box function that computes $v^\top f(A)v$, where $A \in \mathbb{R}^{n\times n}$ is symmetric and $v \in \mathbb{R}^n$. Make use of this black-box function to compute $u^\top f(A)v$.

6. (optional) Implement the Lanczos algorithm for computing the matrix functional $v^* f(A)v$, where A is Hermitian. Test it with some matrices of size several thousands and some sufficiently smooth functions. Plot the convergence history.

7. (optional) Implement the preconditioned inverse iteration. Test it with the exact shift-and-invert preconditioner and compare the result with that produced by the usual inverse iteration.

Can you improve the preconditioned inverse iteration for symmetric eigenvalue problems using the Rayleigh–Ritz procedure?