Oct 17, 2023 (Due: 08:00 Oct 24, 2023)

1. Suppose that a tall-skinny matrix $A \in \mathbb{R}^{m\times n}$ is upper bidiagonal (i.e., $a_{i,j} \neq 0$ only if $i - j \in \{0, -1\}$). Design an algorithm based on Givens rotations to solve the ridge regression problem

$$\min \|Ax - b\|_2^2 + \lambda\|x\|_2^2,$$

where λ is a given positive number.

2. Consider the constrained least squares problem

$$\min_{Cx=d} \|Ax - b\|_2.$$

In the lecture we use the QR factorization of C^* to eliminate the linear constraint $Cx = d$ and reduce the problem to a standard least squares one. Design an algorithm based on Gaussian elimination to eliminate the linear constraint.

3. The least squares problem $\min\|Ax - b\|_2$ is equivalent to an augmented linear system

$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

which is Hermitian and indefinite. Similar augmented linear systems exist for the constrained least squares problem

$$\min_{Cx=d} \|Ax - b\|_2.$$

Can you figure it out?

4. Randomly generate a 1000×1000 matrix A with positive entries. Use the power method to compute $\rho(A)$. Visualize the convergence history.

5. Randomly generate a 200×200 real symmetric matrix A with known spectrum (e.g., $A = Q\Lambda Q^\top$, where Λ is known and Q is a randomly generated orthogonal matrix). Choose one eigenvalue of A, and use inverse iteration and Rayleigh quotient iteration to compute this eigenvalue. Visualize the convergence history and report the execution time (preferably with detailed profiling).

6. (optional) Investigate the behavior of the power method applied to the following matrices:

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda & 1 \\ 0 & -\lambda \end{bmatrix},$$

where $\lambda \in \mathbb{C}$ is a given constant.

7. (optional) Show that Rayleigh quotient iteration converges locally quadratically in the generic case.

(Hint: Have a look at Exercise 6.32 in the text book.)