Homework 7 Solutions

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Problem 1

Since

$$I = X^{-1}X = X^{-1}[x, Ax, A^2x, \dots, A^{n-1}x] = [X^{-1}x, X^{-1}Ax, X^{-1}A^2x, \dots, X^{-1}A^{n-1}x],$$

we have

$$X^{-1}A^kx = e_{k+1}, k = 0, 1, \dots, n-1.$$

Hence

$$\begin{aligned} X^{-1}AX &= [X^{-1}Ax, X^{-1}A^2x, \dots, X^{-1}A^nx] \\ &= [e_2, e_3, \dots, e_{n-1}, X^{-1}A^nx], \end{aligned}$$

which is upper Hessenberg.

Problem 2

We use induction on m.
When $m = 1$, $A_0 - \mu_0 I = Q_0 R_0$ is obvious.
Assume $m = m_0$ is given, we prove case $m = m_0 + 1$ is right.
Use inductive assumption on A_1, we have

$$(A_1 - \mu_1 I)(A_1 - \mu_2 I)\dots(A_1 - \mu_{m_0+1} I) = (Q_1 Q_2 \dots Q_{m_0+1})(R_{m_0+1} \dots R_2 R_1).$$

Since $Q_0^*(A_0 - \mu_k I)Q_0 = A_1 - \mu_k I$ for any $k = 0, 1, \dots, m_0 - 1$,

$$(A_0 - \mu_1 I)(A_0 - \mu_2 I)\dots(A_0 - \mu_{m_0+1} I) = Q_0(A_1 - \mu_1 I)(A_1 - \mu_2 I)\dots(A_1 - \mu_{m_0+1} I)Q_0^*,$$

and by the commutativity of A_0 and I, we have

$$\begin{aligned}\prod_{k=0}^{m_0+1}(A_0 - \mu_k I) &= (A_0 - \mu_1 I)(A_0 - \mu_2 I)\dots(A_0 - \mu_{m_0+1} I)(A_0 - \mu_0 I) \\ &= Q_0(A_1 - \mu_1 I)(A_1 - \mu_2 I)\dots(A_1 - \mu_{m_0+1} I)Q_0^*(A_0 - \mu_0 I) \\ &= (Q_0 Q_1 \dots Q_{m_0+1})(R_{m_0+1} \dots R_1)Q_0^* Q_0 R_0 \\ &= (Q_0 Q_1 \dots Q_{m_0+1})(R_{m_0+1} \dots R_1 R_0),\end{aligned}$$

and we finished the proof.$\square$

Problem 3

See pseudocode below.

```
ExchangeDiagonal(A)
   a = A[0, 0]
   A[1, 0] = 1e-4   //perturbation
   final_Q = I
   while abs(A[1, 0]) > 1e-15:
      Q, R = qr_decomposition(A - a * I)
      final_Q = final_Q @ Q
      A = R @ Q + a * I
   return final_Q
```

Note. When running algorithm above, we found that the element in the c's place keep changing its sign. If you want the element at the c's place in the final $Q^T AQ$ also close to c , you can add a condition in the `while` statement.

```
initial A:

[[4.e+00 7.e+00]

 [1.e-06 8.e+00]]

iterration 1, element at c's place: 1e-06

iterration 2, element at c's place: 6.999999753846153

iterration 3, element at c's place: -6.9999989999996695

iterration 4, element at c's place: 6.999998999999999

iterration 5, element at c's place: -6.999998999999999

iterration 6, element at c's place: 6.999998999999999

final A:

[[8.00000175e+00 6.99999900e+00]

 [2.76180950e-32 3.99999825e+00]]
```

Problem 4

Let $D = diag(d_1, d_2, \dots, d_n)$ be a diagonal matrix with **positive** entries. With $b_i c_{i+1} > 0$ for $i = 1, 2, \dots, n-1$, we can let

$$\begin{aligned} d_1 &= 1 \\ d_2 &= d_1 \sqrt{\frac{b_1}{c_2}} \\ d_3 &= d_2 \sqrt{\frac{b_2}{c_3}} \\ &\dots \\ d_n &= d_{n-1} \sqrt{\frac{b_{n-1}}{c_n}}. \end{aligned}$$

By calculation we can show that DAD^{-1} is real symmetric, hence it can be diagonalizable and has real spectrum.

Problem 5

see `hessenberg.py`. The accuracy of Arnoldi process seems to be better than Householder.

```
size of A: 100 * 100.

Upper Hessenberg via Householder:

norm of difference between H and Q^TAQ: 2.584e+01

orthogonal loss of Q: 8.779e-15

Upper Hessenberg via Arnoldi process:

norm of difference between H and Q^TAQ: 3.371e-11

orthogonal loss of Q: 7.750e-12
```