homework 20241205
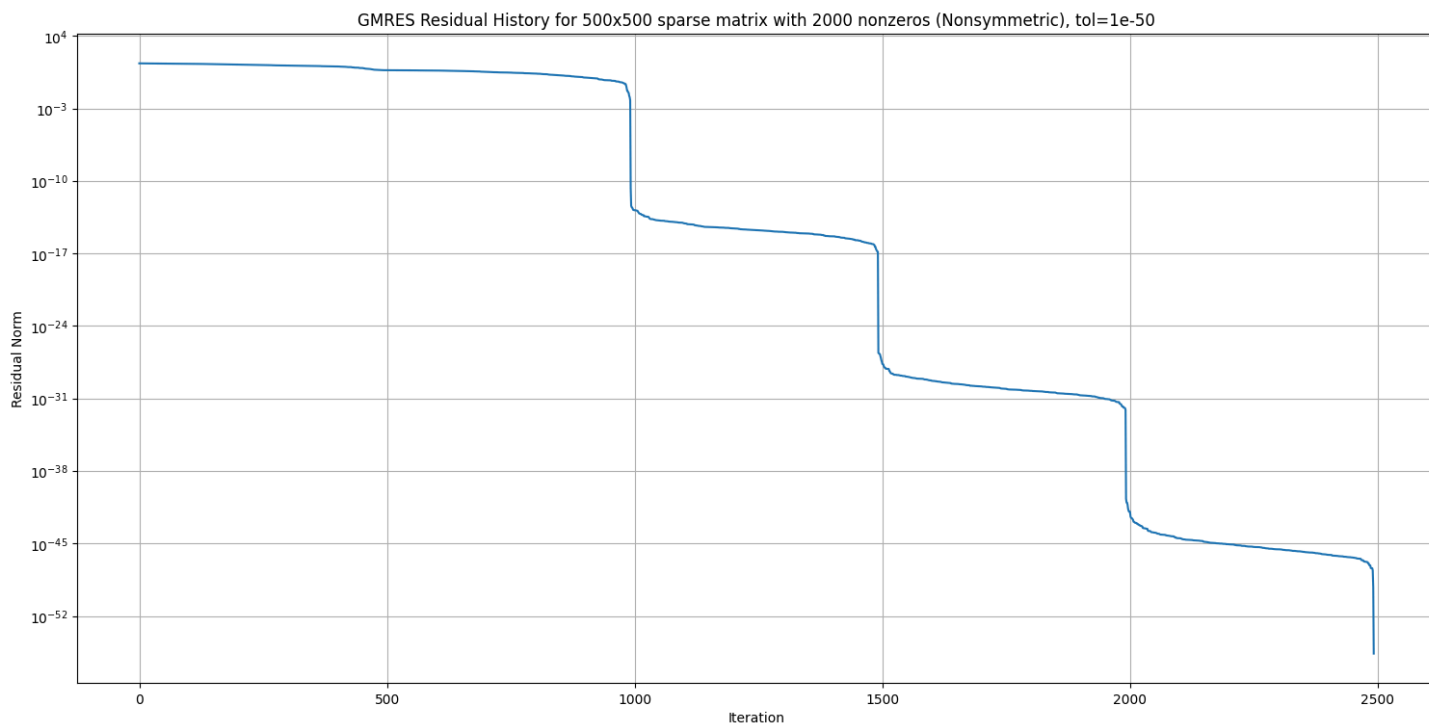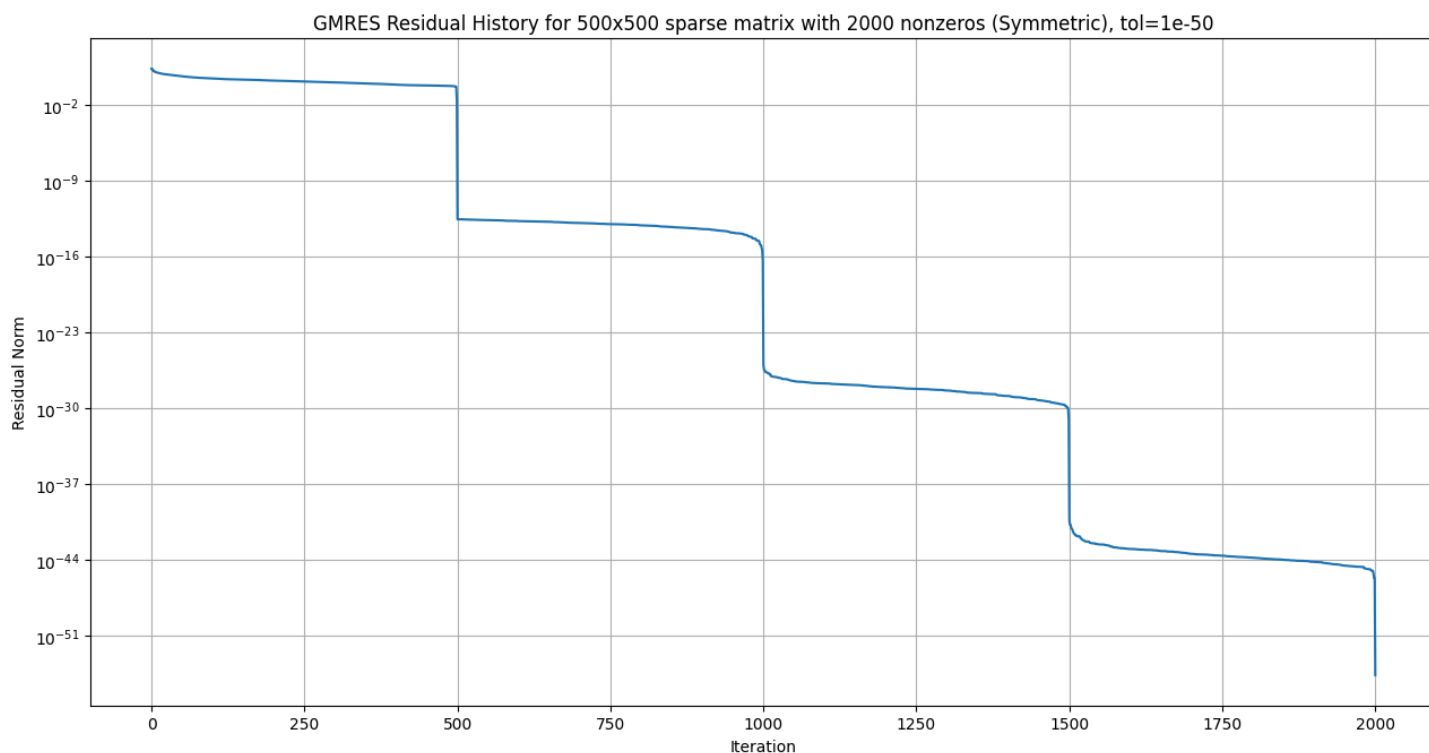
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第一题

代码文件 **GMRES.py**

第二题

2. $x_{k+1} = x_k + r_k \alpha_k$

两边同乘 A: $A x_{k+1} = A x_k + A r_k \alpha_k$

用 b 同减在两边: $b - A x_{k+1} = b - A x_k - A r_k \alpha_k$

由定义 $r_k = b - A x_k$, $r_{k+1} = b - A x_{k+1}$

有 $r_{k+1} = r_k - A r_k \alpha_k = 0$

故 $A r_k = \frac{1}{\alpha_k} r_k$. 其中 $\alpha_k = \frac{r_k^T r_k}{r_k^T A r_k}$ 且 $r_k \neq 0$. 否则不会有 r_{k+1}

说明 r_k 是 A 的特征向量, 对应特征值是 $\frac{r_k^T A r_k}{r_k^T r_k}$.

3. $f(x_{k+1}) \leq (1-k^{-1})f(x_k)$

$\Leftrightarrow f(x_{k+1}) - f(x_k) \leq -k^{-1}f(x_k)$

$f(x_{k+1}) - f(x_k) = f(x_k + \alpha_k p_k) - f(x_k)$

$\qquad = (x_k + \alpha_k p_k)^T A(x_k + \alpha_k p_k) - 2b^T(x_k + \alpha_k p_k)$

$\qquad \ominus \ x_k^T A x_k + 2b^T x_k$

$\qquad = 2x_k^T A \alpha_k p_k + \alpha_k^2 p_k^T A p_k - 2b^T \alpha_k p_k$

将 $b = x_k + A x_k$ 代入 $\qquad = 2x_k^T A \alpha_k p_k + \alpha_k^2 p_k^T A p_k - 2x_k^T \alpha_k p_k - 2x_k^T A \alpha_k p_k$

$\qquad = \alpha_k^2 p_k^T A p_k - 2x_k^T \alpha_k p_k$ ， $\alpha_k = \dfrac{p_k^T x_k}{p_k^T A p_k}$ 代入

再取 $p_k = x_k$ $\qquad = \ominus \dfrac{(x_k^T x_k)^2}{x_k^T A x_k}$ ．

不等式 $\Leftrightarrow \dfrac{(x_k^T x_k)^2}{x_k^T A x_k} \geq k^{-1} f(x_k)$

$f(x_k) = [A^{-1}(b - x_k)]^T A[A^{-1}(b - x_k) - 2b^T[A^{-1}(b - x_k)]$

$\qquad = b^T A^{-1} b - 2b^T A^{-1} x_k + x_k^T A^{-1} x_k - 2b^T A^{-1} b + 2b^T A^{-1} x_k$

$\qquad = x_k^T A^{-1} x_k - b^T A^{-1} b < x_k^T A^{-1} x_k$． A 正定 $\Rightarrow A^{-1}$ 也正定．

不等式 $\Leftarrow \dfrac{(x_k^T x_k)^2}{x_k^T A x_k} \geq k^{-1} x_k^T A^{-1} x_k \geq k^{-1} f(x_k)$、

$\qquad \Leftarrow k^{-1} \geq \left(\dfrac{x_k^T A^{-1} x_k}{x_k^T x_k}\right) \left(\dfrac{x_k^T A x_k}{x_k^T x_k}\right)$

$\qquad \Leftarrow \max\limits_{x_k \neq 0} \dfrac{x_k^T A^{-1} x_k}{x_k^T x_k} \cdot \max\limits_{x_k \neq 0} \dfrac{x_k^T A x_k}{x_k^T x_k} \geq$ 右式 得证．

此号由 $\|A\|_2 = \max\limits_{x \neq 0} \dfrac{x^T A x}{x^T x}$ 可得．

第四题

代码文件 **SD.py**

A special linear equations, with Steepest Descent Iteration 10 times