Homework 9 Solutions

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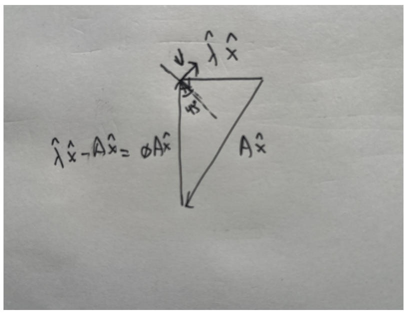
Problem 1

Apply $\hat{x}^T$ on the left side of $(A + \Delta A)\hat{x} = \hat{x}\hat{\lambda}$, we have $\hat{x}^T \Delta A\hat{x} = 0$, or $\hat{x} \perp \Delta A\hat{x}$.



Then we use this property and a Householder reflection to construct a ΔA that satisfies the two condition.
Let

$$
\begin{aligned}
v &= \frac{1}{\sqrt{2}}\left(\frac{\hat{\lambda}\hat{x} - A\hat{x}}{||\hat{\lambda}\hat{x} - A\hat{x}||_2} - \hat{x}\right), \\
H &= I - 2vv^T, \\
\Delta A &= ||\hat{\lambda}\hat{x} - A\hat{x}||_2 H.
\end{aligned}
$$

It's not hard to show that

$$(A + \Delta A)\hat{x} = \hat{x}\hat{\lambda}$$

and

$$||\Delta A||_2 = ||\hat{\lambda}\hat{x} - A\hat{x}||_2 \cdot 1 = ||\hat{\lambda}\hat{x} - A\hat{x}||_2.$$

$\square$

Problem 2

if $z^T u = 0$, then $Du = \lambda u$, since D have distinct eigenvalues, $u = \alpha e_i$ for some $i \in$

$\{1, 2, \ldots, n\}, \alpha \neq 0$.
Then $z^T u = \alpha z^T e_i = 0$, which is contradictory with that z has no zero entries.
If $\lambda I - D$ is singular, then $e_i^T (\lambda I - D) = 0$ for some $i \in \{1, 2, \ldots, n\}$.
Hence

$$0 = e_i^T (\lambda I - D) u = \rho e_i^T z^T u,$$

but $z^T u \neq 0$, then $e_i^T z = 0$, which is contradictory with that z has no zero entries.

For the abnormal cases:

1. If we want to eliminate zeros we don't want in z, say $z_i = 0$, we find some $z_j \neq 0$, and just use a Givens rotation $G(i, j)$ on the row i and j. We can show that $G^T D G = D$, so that won't change D.
2. Suppose $D = diag(d_1, d_2, \ldots, d_n)$. If d_i is equal(or very close to) d_j, we use a Givens rotation $G(i, j)$ to produce zero on z_j. Do this for all closed entries so after that $w = V^T z = (w_1, w_2, \ldots, w_n)$ (V is compound of Givens rotations) meet the condition that if $w_i w_j \neq 0$, then $d_i \neq d_j$. Then if w has r non-zero elements, use a permutation P to move all non-zero element of w to its front: $Pw = (w_{(1)}, w_{(2)}, \ldots, w_{(r)}, 0, \ldots, 0)$. Let $w' = (w_{(1)}, w_{(2)}, \ldots, w_{(r)})$, then

$$PV^T (D + \rho z z^T) V P^T = \begin{bmatrix} D_1 + \rho w' w'^T & 0 \\ 0 & D_2 \end{bmatrix},$$

and all entries in D_1 are distinct, all entries in w' are non-zero, which satisfy the assumptions.

Problem 3

By using elementary transformation on matrix

$$\begin{bmatrix} \lambda I - D & z \\ z^T & \frac{1}{\rho} \end{bmatrix},$$

we can show that when λ is an eigenvalue of $D + \rho z z^T$, then

$$f(\lambda) = 1 + \rho z^T (D - \lambda I) z = 0.$$

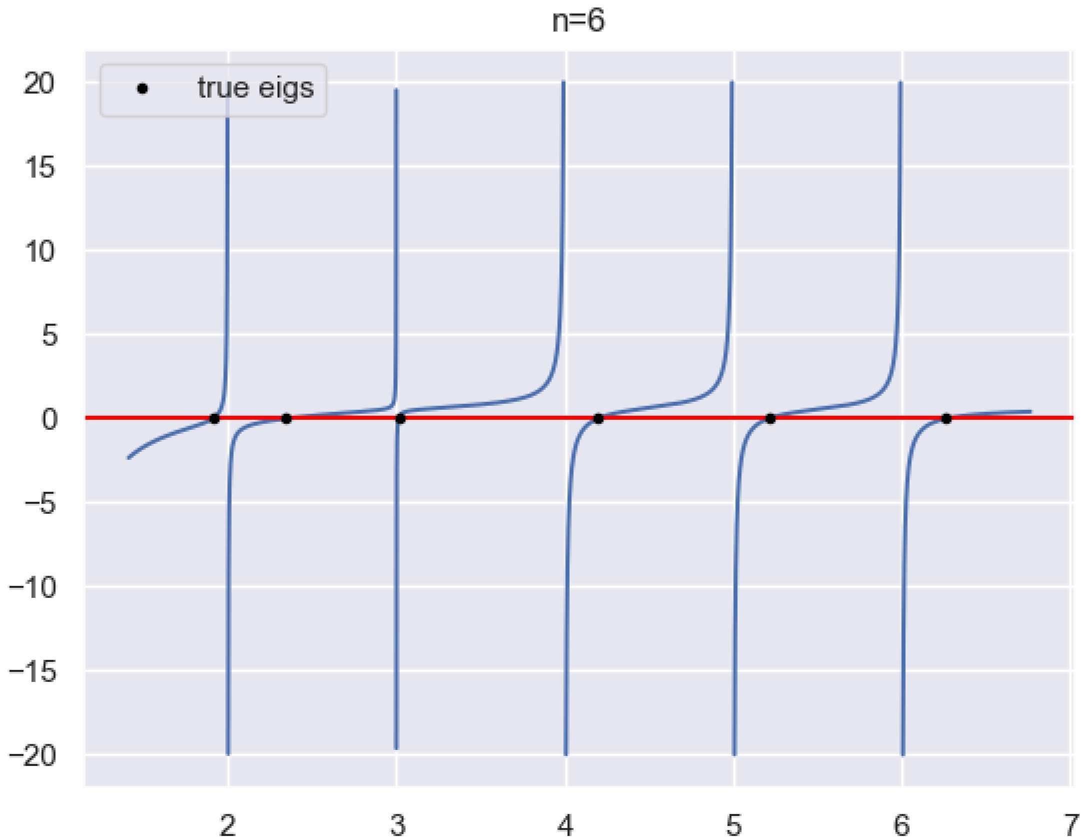
Hence

$$\begin{aligned} (D + \rho z z^T)(\lambda I - D)^{-1} z &= -z + (\lambda I + \rho z z^T)(\lambda I - D)^{-1} z \\ &= \lambda(\lambda I - D)^{-1} z - z + \rho z (z^T (\lambda I - D)^{-1} z) \\ &= \lambda(\lambda I - D)^{-1} z - z + z \\ &= \lambda(\lambda I - D)^{-1} z, \end{aligned}$$

and that's the conclusion.$\square$

Problem 4

see `prob4.py`


n=6
true eigs
20
15
10
5
0
−5
−10
−15
−20
2
3
4
5
6
7