Homework 4 Solutions

Weihao Li

Fudan University

Problem 1

In the complex case, let v denote the Householder vector and $P = I - 2(v^*v)^{-1}vv^*$ the Householder reflection. Suppose we are given $x \in \mathbb{C}^n$ and want Px to be a multiple of e_1.Then v must in $span\{x, e_1\}$. Setting

$$v = x + \alpha e_1, \alpha \in \mathbb{C}^n$$

gives

$$Px = \left(\frac{|\alpha^2| - ||x||_2^2 + \alpha\bar{x_1} - \bar{\alpha}x_1}{||x + \alpha e_1||_2^2}\right)x - 2\alpha\frac{v^*x}{v^*v}e_1$$

If the first entry of x: $x_1 = |x_1|e^{i\theta}$ (obviously θ is unique), then we have two choice for α: $\alpha = \pm||x||_2 e^{i\theta}$.(In implementation we use $t/abs(t)$ to compute the angle of a complex number t)

Let $x' = Px$ and x_1' denote the first entry of Px. If we do not care about sign of the real part of x_1' , we simply choosing $\alpha = ||x||_2 e^{i\theta}$ because it can maximize the norm of v and avoid possible cancellation(since it has the same direction of x_1 and not the opposite one), while resulting in $Px = -e^{i\theta}||x||_2 e_1$. Pay attention that the sign of $\mathrm{Re}(x_1')$ is opposite of x_1 and not always negative or positive.

However, if we want the the sign of $\mathrm{Re}(x_1')$ to be positive, we can not just choose $\alpha = -||x||_2 e^{i\theta}$ (just try the real case $x_1 = e^{i\pi}|x_1| = -|x_1|$!), but always choose α that has negative real part. That is, when $\mathrm{Re}(e^{i\theta}) < 0$, choose $\alpha = ||x||_2 e^{i\theta}$, and when $\mathrm{Re}(e^{i\theta}) \geq 0$ choose $\alpha = -||x||_2 e^{i\theta}$, and use

$$x_1 + \alpha = e^{i\theta}(|x_1| - ||x||_2) = -e^{i\theta}\left(\frac{|x_2|^2 + \cdots + |x_n|^2}{|x_1| + ||x||_2}\right)$$

in computation to avoid possible cancellation when x is close to a multiple of e_1.

Problem 2

Suppose we are use Householder triangularization on a $m \times n$ matrix A.

Time complexities:

Forming a Householder vector of length k : $O(k)$.

Applying a size k Householder matrix to a $k \times t$ sub-matrix: $O(kt)$, since

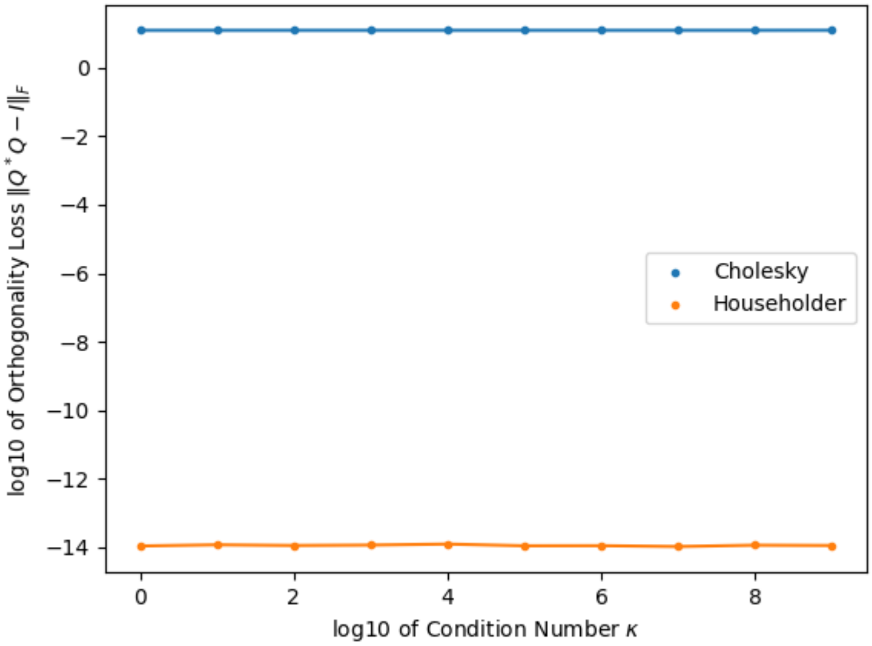
$PA = (I - \beta vv^*)A = A - (\beta v)v^*A$ only involves vector-matrix products.

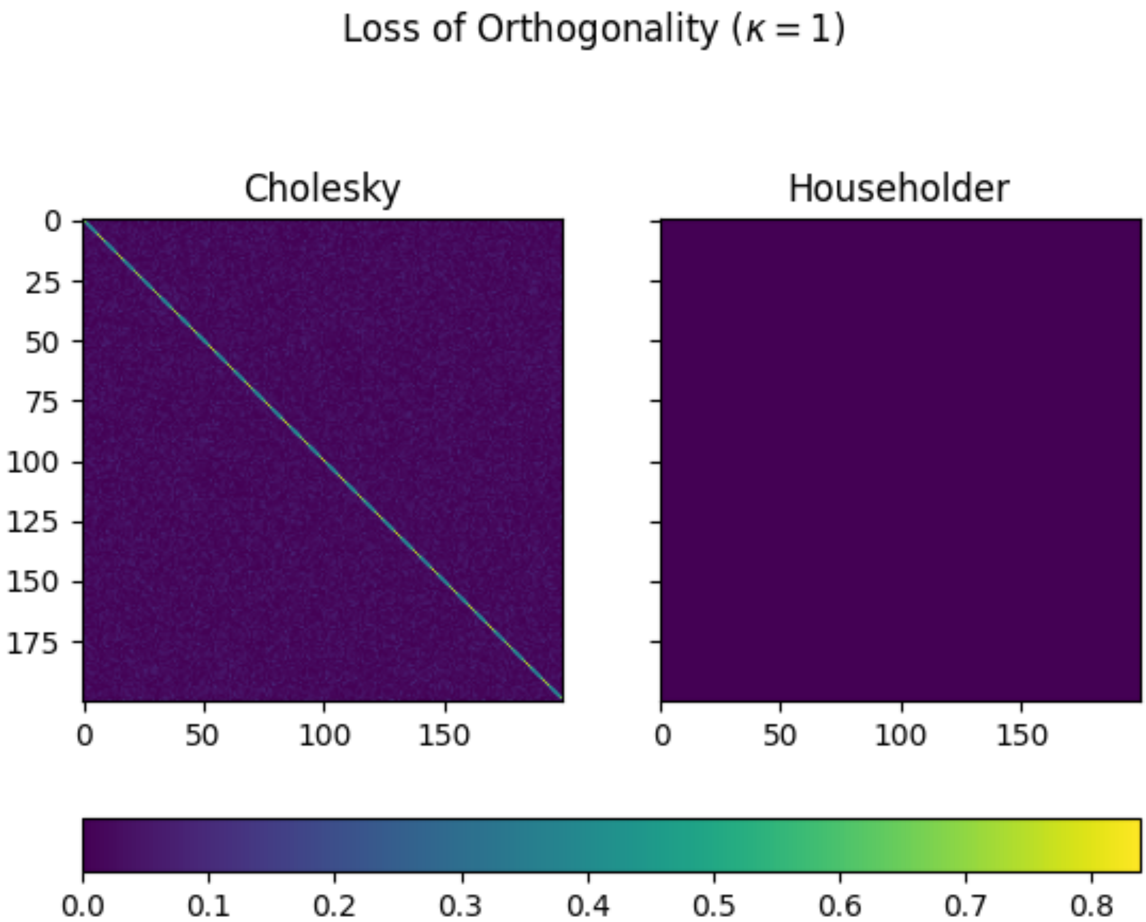
So the time complexity of whole Householder triangularization is about $O(mn^2)$.

For forming the orthogonal/unitary matrix Q, the process is same as applying Householder matrix reversely to size m Identity matrix, so the time complexity of forming Q is about $O(m^2n)$.

Problem 3

See `qrdecomp.py` . The visualization of loss of orthogonality is shown below.





Problem 4

First, use Givens rotation to eliminate the first column of A:

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & & & \\ \times & & \times & & \\ \hat{\times} & & & \times & \\ \hat{\times} & & & & \times \end{bmatrix} \to \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & & & \\ \hat{\times} & & \times & & \\ \hat{\times} & & & \times & \times \\ 0 & & & \times & \times \end{bmatrix} \to \begin{bmatrix} \times & \times & \times & \times & \times \\ \hat{\times} & \times & & & \\ \hat{\times} & & \times & \times & \times \\ 0 & & \times & \times & \times \\ 0 & & & \times & \times \end{bmatrix} \to \cdots$$

(Schematic diagram, use $\times$ to denote non-zero entries and $\hat{\times}$ the entries to generalize Givens rotation for next step)
After $n-2$ Givens rotations, we transform A into a Hessenberg matrix:

$$G_{n-2}\ldots G_1 A \sim \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & & \times & \times & \times \\ 0 & & & \times & \times \end{bmatrix}.$$

Then use another $n-1$ Givens rotation to eliminate entries on the first subdiagonal. The algorithm requires about $O(n^2)$ flops.

Problem 5

Lemma: A upper triangular orthogonal matrix must be a diagonal matrix.
For orthogonal matrix $Q \in \mathbb{R}^{n\times n}$, perform Householder triangularization on Q:

$$H_n \ldots H_2 H_1 Q = R,$$

in which R is an upper triangular matrix with non-negative diagonal, and H_i are the Householder matrices that produce zeros on the i th column of Q.
Since H_i and Q are all orthogonal, R is orthogonal too, then by the lemma and that R has non-negative diagonal entries, $R = I_n$. Then

$$Q = H_1^T H_2^T \ldots H_n^T = H_1 H_2 \ldots H_n.$$

When $\det(Q) = 1$, perform Givens triangularization on Q:

$$G_{n(n-1)/2}\ldots G_1 Q = R,$$

in which R is an upper triangular matrix, and G_i are the Givens rotations.
From the same reasoning above, R is a diagonal matrix with 1 or -1 as diagonal entries.
Since $\det(Q) = 1$ and $\det(G_i) = 1$ for all G_i, we have $\det(R) = 1$, then R only have an even number of -1 on its diagonal, which means R can be factorized into finitely many $G(i, k, \pi)$.
Then by the inverse of Givens rotation is also a Givens rotation, we conclude that
Q can be factorized as the product of finitely many Givens rotations.