Sep 19, 2023 (Due: 08:00 Sep 26, 2023)

1. Assume that b, δb, x, δx satisfy

$$Ax = b, \qquad A(x + \delta x) = b + \delta b,$$

where

$$A = \begin{bmatrix} 610 & 987 \\ 987 & 1597 \end{bmatrix}.$$

Construct examples such that
(1) $\|\delta b\|_\infty / \|b\|_\infty$ is very small while $\|\delta x\|_\infty / \|x\|_\infty$ is very large;
(2) $\|\delta b\|_\infty / \|b\|_\infty$ is very large while $\|\delta x\|_\infty / \|x\|_\infty$ is very small.

2. Let $Z \in \mathbb{C}^{n\times n}$ and

$$A = \begin{bmatrix} I_n & Z \\ 0 & I_n \end{bmatrix}.$$

Find $\kappa_{\mathsf{F}}(A) = \|A\|_{\mathsf{F}} \|A^{-1}\|_{\mathsf{F}}$.

3. Let $PA = LU$ be the LU factorization of an $n \times n$ matrix A with partial pivoting. Show that $\|U\|_\infty \le 2^{n-1} \|A\|_\infty$.

4. It can be shown that Gaussian elimination without pivoting is numerically stable for solving strictly diagonally dominant linear systems, in the sense that the growth factor is bounded. Give a concrete upper bound on the growth factor.

5. In the lecture we have provided an upper bound for

$$\frac{\|(A + \delta A)^{-1} b - A^{-1} b\|}{\|A^{-1} b\|}.$$

What happens if b is also perturbed? More precisely, provide a tight upper bound for

$$\frac{\|(A + \delta A)^{-1} (b + \delta b) - A^{-1} b\|}{\|A^{-1} b\|}.$$

6. (H) Discuss how to solve a quadratic equation $ax^2 + bx + c = 0$ in a stable manner and provide a rounding error analysis. You may assume that the coefficients are real, and $b^2 \ge 4ac$.

7. (optional) It can be shown that Gaussian elimination with partial pivoting is numerically stable for solving nonsingular tridiagonal linear systems, in the sense that the growth factor is bounded. Give a concrete upper bound on the growth factor.

8. (optional) Provide an rounding error analysis for solving triangular linear system with multiple right-hand-sides.