Nov 7, 2023 (Due: 08:00 Nov 14, 2023)

1. Let $\hat{x} \in \mathbb{R}^n$ be an approximate eigenvector of a real symmetric matrix A such that $\|\hat{x}\|_2 = 1$. Show that there exists a real symmetric matrix ΔA such that

$$(A + \Delta A)\hat{x} = \hat{x}\hat{\lambda}, \qquad \|\Delta A\|_2 = \|A\hat{x} - \hat{x}\hat{\lambda}\|_2,$$

where $\hat{\lambda} = \hat{x}^\top A\hat{x}$.

2. Let $D \in \mathbb{R}^{n\times n}$ be diagonal with distinct eigenvalues, $z \in \mathbb{R}^n$ be a vector with no zero entries, and $\rho \in \mathbb{R} \setminus \{0\}$. Suppose that (λ, u) is an eigenpair of $D + \rho zz^\top$. Show that $\lambda I - D$ is nonsingular, and $z^\top u \neq 0$.

How to handle the abnormal case (i.e., if any of the three assumptions are violated) in practice?

3. Under the same assumptions of Exercise 1, show that $(\lambda I - D)^{-1}z$ is an eigenvector of $D + \rho zz^\top$.

4. Randomly generate a relatively small (e.g., 6×6) real symmetric matrix of the form $A = \operatorname{diag}\{d_1, d_2, \dots, d_n\} + zz^\top$. Visualize the function

$$f(\lambda) = 1 - \sum_{i=1}^{n} \frac{z_i^2}{\lambda - d_i}.$$

Highlight the eigenvalues of A in the plot and make sure they match the roots of $f(\lambda)$. For simplicity, you may compute the eigenvalues of A by existing functions from math libraries (e.g., `eig` from MATLAB/Octave).

5. (optional) Let $D = \operatorname{diag}\{d_1, \dots, d_n\}$ be a real diagonal matrix. Let α_1, …, α_n be real scalars that satisfy

$$d_n < \alpha_n < \cdots < d_i < \alpha_i < d_{i-1} < \alpha_{i-1} < \cdots < d_1 < \alpha_1.$$

Show that the α_i's are exact eigenvalues of $D + uu^\top$, where entries of the real vector u are defined by

$$u_i = \left(\frac{\prod_{1\le j\le n}(\alpha_j - d_i)}{\prod_{1\le j\le n,\, j\neq i}(d_j - d_i)} \right)^{1/2}, \qquad (1 \le i \le n).$$