Oct 24, 2023 (Due: 08:00 Oct 31, 2023)

1. Let $A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$. Suppose that $X = [x, Ax, \dots, A^{n-1}x]$ is nonsingular. Show that $X^{-1}AX$ is upper Hessenberg.

2. Let $A_0 \in \mathbb{C}^{n\times n}$, μ_0, μ_1, …, $\mu_m \in \mathbb{C}$. Define A_1, A_2, …, A_{m+1} by

$$A_k - \mu_k I = Q_k R_k, \qquad A_{k+1} = R_k Q_k + \mu_k I,$$

for $k \in \{0, 1, \dots, m\}$, where Q_k's are unitary matrices. Show that

$$(A_0 - \mu_0 I)(A_0 - \mu_1 I)\cdots(A_0 - \mu_m I) = (Q_0 Q_1 \cdots Q_m)(R_m \cdots R_1 R_0).$$

3. Let

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Design an algorithm to compute an orthogonal matrix $Q \in \mathbb{R}^{2\times 2}$ such that

$$Q^\top A Q = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}.$$

(H) What happens if the matrix A is complex?

4. Let

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_2 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-1} & a_{n-1} & b_{n-1} \\ & & & c_n & a_n \end{bmatrix} \in \mathbb{R}^{n\times n},$$

with $b_i c_{i+1} > 0$ for $i \in \{1, 2, \dots, n-1\}$. Show that A is diagonalizable, and has real spectrum.

5. Implement the following algorithms for Hessenberg reduction:
(a) using Householder reflections;
(b) using Arnoldi process based on modified Gram–Schmidt orthogonalization.
Randomly generate a few matrices and compute the corresponding Hessenberg decomposition $A = QHQ^\top$. Check the accuracy in terms of $\|Q^*AQ - H\|_{\mathsf{F}}$ and $\|Q^*Q - I\|_{\mathsf{F}}$ for your Hessenberg reduction implementations. What do you observe?

(optional) Perturb the matrix A a little bit. How do Q and H change accordingly?

6. (optional) Let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ & a_{2,2} & a_{2,3} \\ & a_{3,2} & a_{3,3} \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

Design an algorithm to compute an orthogonal matrix $Q \in \mathbb{R}^{3\times 3}$ such that $Q^\top AQ$ is of the form

$$Q^\top AQ = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ & & a_{1,1} \end{bmatrix}.$$