



# Homework 8 Solutions

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## Problem 1

Let  $\hat{x} - x = r$ . Then  $\|r\| = O(\epsilon)$ , and

$$\begin{aligned}\hat{x}^* A \hat{x} &= (x + r)^* A (x + r) \\ &= x^* A x + r^* A x + x^* A r + r^* A r \\ &= \lambda(1 + r^* x + x^* r) + r^* A r,\end{aligned}$$

$$\begin{aligned}\hat{x}^* \hat{x} &= (x + r)^* (x + r) \\ &= 1 + r^* x + x^* r + r^* r.\end{aligned}$$

Hence

$$\begin{aligned}\left\| \frac{\hat{x}^* A \hat{x}}{\hat{x}^* \hat{x}} - \lambda \right\| &= \left\| \frac{r^* A r - \lambda r^* r}{1 + r^* x + x^* r + r^* r} \right\| \\ &\leq \|r^* A r\| + \|\lambda r^* r\| \\ &= (\|A\| + |\lambda|) \|r\|^2 \\ &= (\|A\| + |\lambda|) O(\epsilon^2)\end{aligned}$$

## Problem 2

Let

$$Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix},$$

in which  $c = \cos \theta$ ,  $s = \sin \theta$ . Since

$$[x, y] Q = [cx - sy, sx + cy],$$

we have

$$\begin{aligned} 0 &= (cx - sy)^T (sx + cy) \\ &= sc(\|x\|^2 - \|y\|^2) + x^T y (c^2 - s^2), \end{aligned}$$

Suppose  $x^T y \neq 0$ , we have

$$\frac{\|y\|^2 - \|x\|^2}{x^T y} = \frac{c^2 - s^2}{cs} = \frac{1}{t} - t,$$

in which  $t = \tan \theta$ . Let the left hand side be  $2d$ , then solve the equation  $t^2 + 2dt - 1 = 0$ , get

$$t = -d \pm \sqrt{d^2 + 1}$$

we choose the  $t$  that has **a smaller modulo**. Then

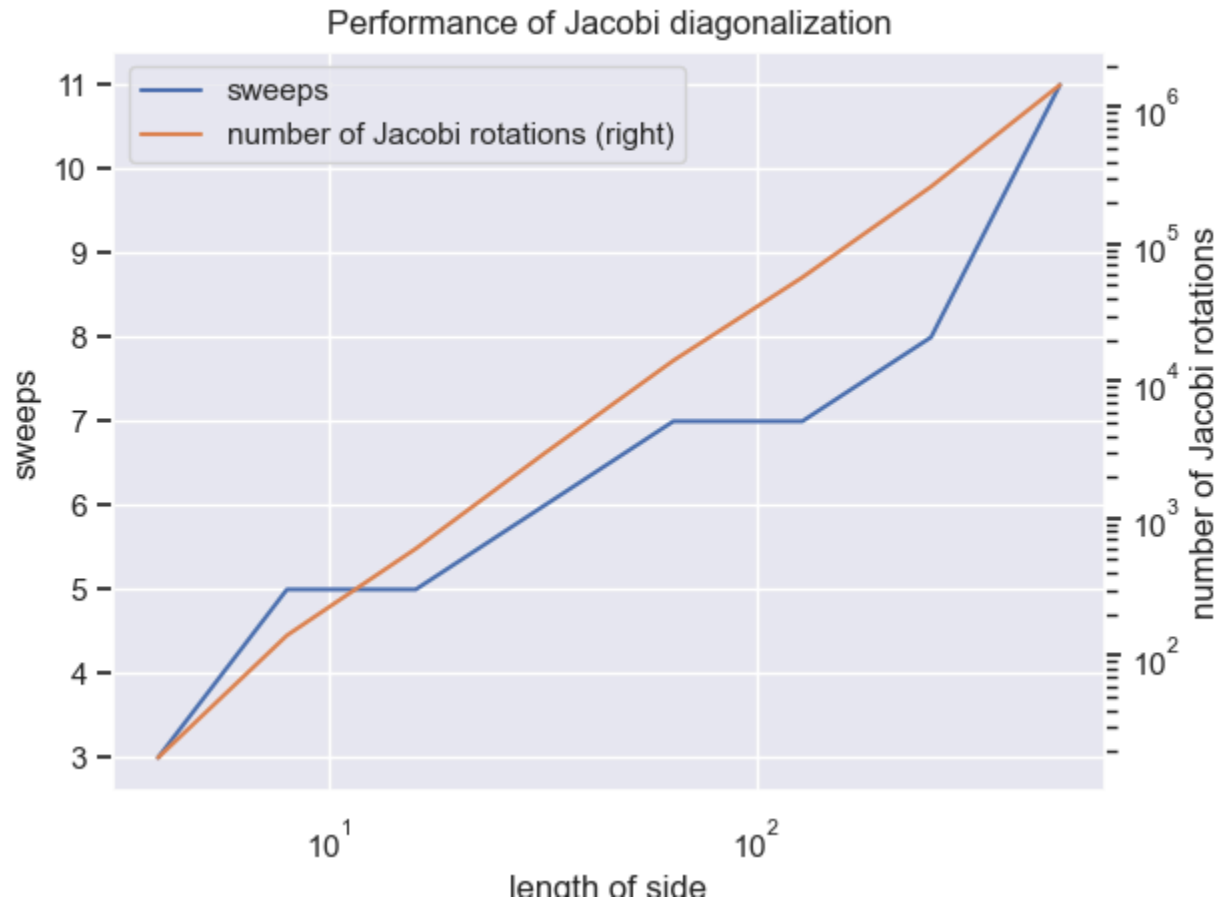
$$\begin{aligned} c &= \frac{1}{\sqrt{2}\sqrt{d^2 + 1} - 2|d|\sqrt{d^2 + 1}}, \\ s &= \sqrt{1 - c^2}. \end{aligned}$$

That's how  $Q$  is constructed.

### Problem 3

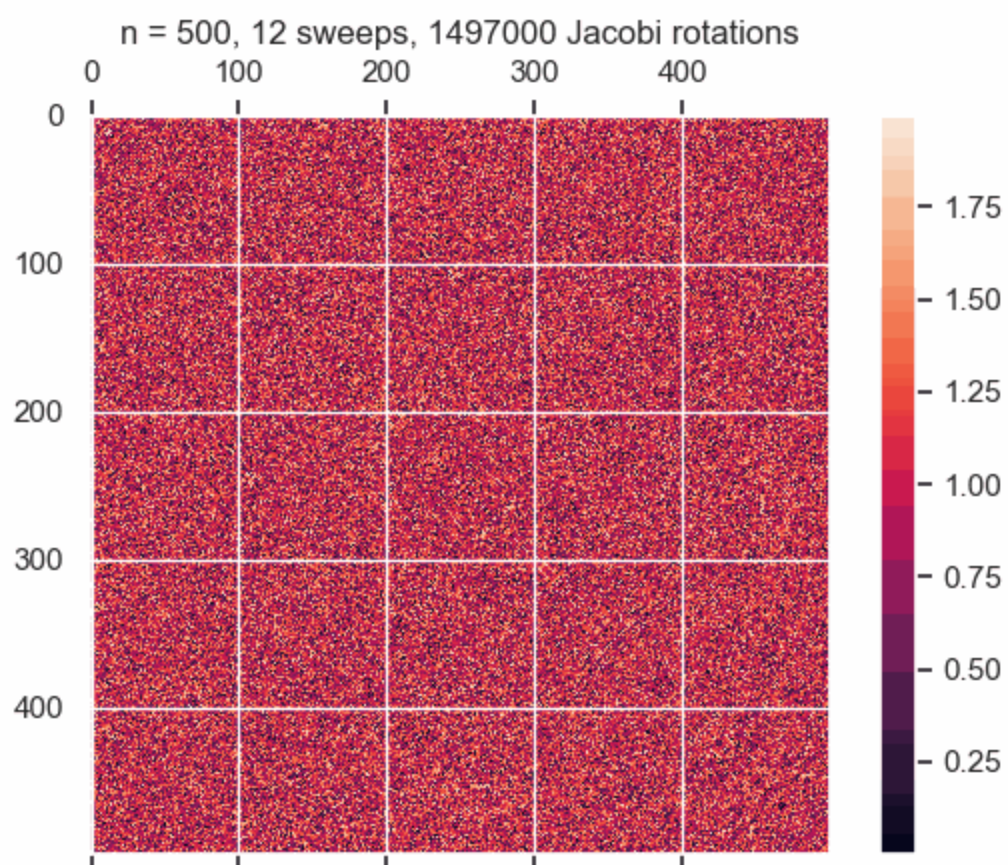
see `jacobi.py`

Performance:



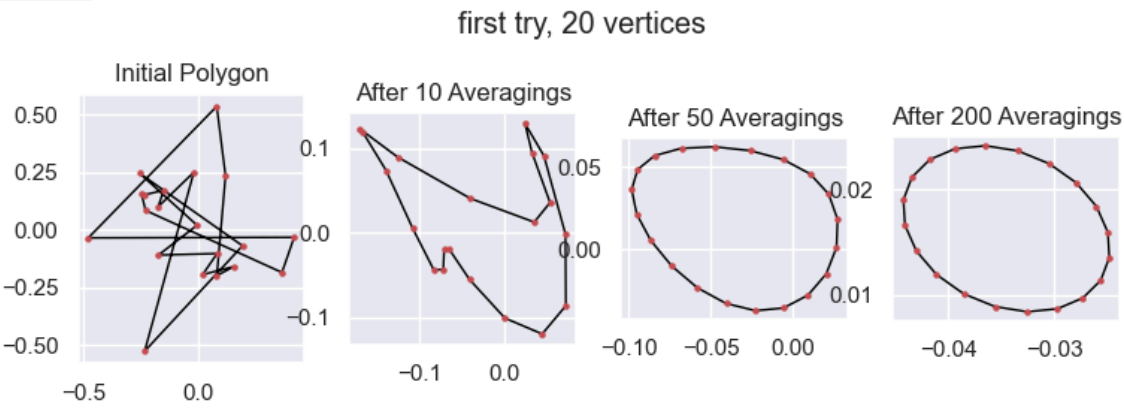
When length of side of the matrix is over 1000, the computation get incredibly slow, with sometimes overflow when computing the  $2 \times 2$  symmetric Schur decomposition.

Convergence History(Below in fact is a gif! See `convergence_history-1.gif` in the package.):



Problem 4

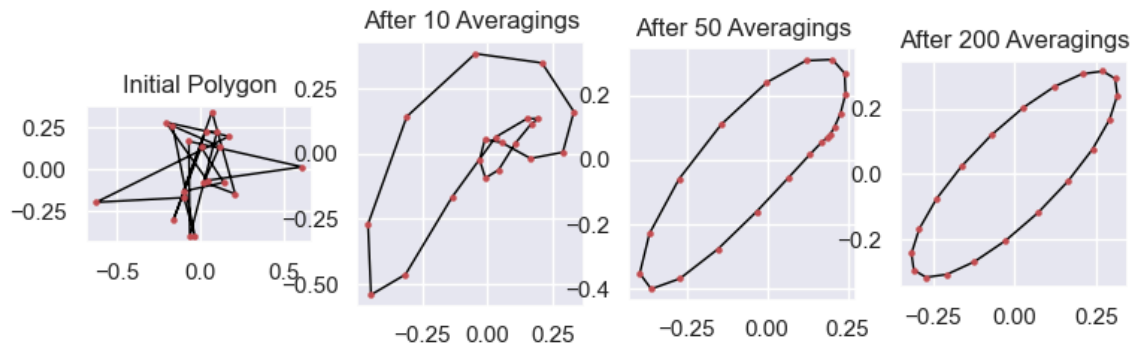
see `FRPE.py`



100 trials,  $\delta = 0.001$

n		Average $k_\delta$
10.0		125.74
20.0		484.41
40.0		1806.94
80.0		6847.79

second try, 20 vertices, with normalizations



third experiment, 12 vertices, even is red and odd is blue

