Homework 3 Solutions

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(Numerical Algorithms with Case Studies I)

Problem 1.

(1)Let $x=(0.01,0)^T$, $\delta x=(61,-37.7)^T$, then $b=(6.1,9.87)$, and $\delta b=(0.1,0.1)^T$.

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty}=6100.0, \qquad \frac{\|\delta b\|_\infty}{\|b\|_\infty}=0.0101317$$

(2)Let $x=(9.87,-6.1)^T$, $\delta x=(0.1,0.1)^T$, then $b=(1.687539^{-13},-1.000000^{-2})$, and $\delta b=(159.7,$ $258.4)^T$.

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty}=0.0101317, \qquad \frac{\|\delta b\|_\infty}{\|b\|_\infty}=1.531224^{15}$$

Problem 2. For $A=\left(\begin{smallmatrix} I_n & Z \\ O & I_n \end{smallmatrix}\right)$, it's easy to use method of undetermined coefficients to draw

$$A^{-1}=\begin{pmatrix} I_n & -Z \\ O & I_n \end{pmatrix}.$$

Then

$$\kappa_F(A)=\|A\|_F\|A^{-1}\|_F=2n+\|Z\|_F^2.$$

Problem 3. The first step of LU factorization with partial pivoting, is to find permutation matrix P_1 and Gaussian transformation L_1 that

$$L_1P_1A=U_1,$$

which L_1 takes the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\widetilde{a_{21}}}{\widetilde{a_{11}}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\widetilde{a_{n1}}}{\widetilde{a_{11}}} & 0 & \cdots & 1 \end{pmatrix},$$

every element's absolute value in L_1 is less or equal than 1. Then $\|L_1\|_\infty \leqslant 2$. Put $\|P_1\|_\infty=1$, then

$$\|U_1\|_\infty \leqslant \|L_1\|_\infty\|P_1\|_\infty\|A_1\|_\infty \leqslant 2\|A\|_\infty$$

The kth step $(k=2,3,\cdots,n-1)$ of LU factorization with partial pivoting takes the form

$$L_kP_kU_{k-1}=U_k.$$

By same process, $\|L_k\|_\infty \leqslant 2$ and $\|P_k\|_\infty=1$.Then

$$\|U_k\|_\infty \leqslant 2\|U_{k-1}\|_\infty.$$

Put them together, after the LU factorization, we conclude that

$$\|U\|_\infty = \|U_{n-1}\|_\infty \leqslant 2^{n-1}\|A\|_\infty$$

Problem 4. Put A as a strictly diagonally dominating matrix by columns. When reducing the first column of A ($L_1 A^{(1)} = A^{(2)}$), we have

$$\sum_{i=2}^{n} |l_{i1}| = \sum_{i=2}^{n} \left|a_{i1}^{(1)}\right| / \left|a_{11}^{(1)}\right| < 1,$$

in which l_{ij} are entries of L_1.

Hence for the jth column modulus sum of the Schur complement in $A^{(2)}$,

$$\begin{aligned}\sum_{i=2}^{n} \left|a_{ij}^{(2)}\right| &\leqslant \sum_{i=2}^{n} \left(\left|a_{ij}^{(1)}\right| + |l_{i1}|\left|a_{1j}^{(1)}\right|\right) \\ &= \sum_{i=2}^{n} \left|a_{ij}^{(1)}\right| + \left|a_{1j}^{(1)}\right| \sum_{i=2}^{n} |l_{i1}| \\ &\leqslant \sum_{i=2}^{n} \left|a_{ij}^{(1)}\right| + \left|a_{1j}^{(1)}\right| = \sum_{i=1}^{n} \left|a_{ij}^{(1)}\right|, j = 2, \cdots, n.\end{aligned}$$

And it can be proved by the same way that Before the rth step of Gaussian elimination,

$$\sum_{i=r}^{n} \left|a_{ij}^{(r)}\right| \leqslant \sum_{i=r-1}^{n} \left|a_{ij}^{(r-1)}\right|, j = r, \cdots, n. \tag{1}$$

In previous homework, we have proved that for an order n strictly diagonally dominant matrix A, after a step of Gaussian elimination, the order $n-1$ Schur complement is still strictly diagonally dominant. So the diagonal terms therefore dominate in exactly the same way in all $A^{(r)}$ and,from (1), the sum of the modulus of the elements of any column of $\mathrm{A}^{(\mathrm{r})}$ decreases as r increases. Hence

$$\begin{aligned}\max_{i,j,r} \left|a_{ij}^{(r)}\right| = \max_{i,r} \left|a_{ii}^{(r)}\right| &\leqslant \max_{i,r} \sum_{i=r}^{n} \left|a_{ij}^{(r)}\right| \\ &\leqslant \max_{j} \sum_{i=1}^{n} \left|a_{ij}^{(1)}\right| \\ &\leqslant 2\max_{i} \left|a_{ii}^{(1)}\right|\end{aligned}$$

Then we have an upper bound $\rho_n \leqslant 2$.

Further more, put

$$\sum_{i \neq j}^{n} \left|a_{ij}^{(1)}\right| = \sigma_j \left|a_{jj}^{(1)}\right|, 0 \leqslant \sigma_j < 1$$

and let $\sigma = \max_{1 \leqslant j \leqslant n} \sigma_j$. Then

$$\begin{aligned}\max_{i,j,r} \left|a_{ij}^{(r)}\right| &\leqslant \max_{j} \sum_{i=1}^{n} \left|a_{ij}^{(1)}\right| \\ &= \max_{j} (1+\sigma_j) \left|a_{jj}^{(1)}\right| \\ &\leqslant \max_{j} (1+\sigma_j) \max_{i} |a_{ii}{}^{(1)}| = (1+\sigma) \max_{i} \left|a_{ii}^{(1)}\right|\end{aligned}$$

Then we have a tighter upper bound $\rho_n \leqslant 1+\sigma$, but σ is up to A.

Problem 5. Let $Ax = b$ and $(A+\delta A)(x+\delta x) = b+\delta b$. Then we can see

$$(A+\delta A)\delta x = \delta b - \delta A x,$$

and the term to be estimated is

$$\frac{\|(A+\delta A)^{-1}(b+\delta b) - A^{-1}b\|}{\|A^{-1}b\|} = \frac{\|\delta x\|}{\|x\|}.$$

Then

$$\begin{aligned}
\|\delta x\| &= \|(A+\delta A)^{-1}(\delta b - \delta A x)\| \\
&\leqslant \|A^{-1}\|\|(I+A^{-1}\delta A)^{-1}\|\|\delta b - \delta A x\| \\
&= \|A^{-1}\|\left\|\sum_{k=0}^{\infty}(-A^{-1}\delta A)^k\right\|\|\delta b - \delta A x\| \\
&\leqslant \frac{\|A^{-1}\|(\|\delta b\| + \|\delta A x\|)}{1-\|A^{-1}\delta A\|} \\
&= \frac{\|A\|\|A^{-1}\|}{1-\|A^{-1}\delta A\|}\cdot\frac{\|\delta b\| + \|\delta A\|\|x\|}{\|A\|} \qquad (2)
\end{aligned}$$

after using Neumann series. To guarantee feasibility, we need to show $\rho(A^{-1}\delta A) < 1$. Since $A+\delta A$ is invertible, use Kahan formula:

$$\|\delta A\|_2 > \frac{1}{\|A^{-1}\|_2}.$$

Hence

$$\rho(A^{-1}\delta A) \leqslant \|A^{-1}\delta A\|_2 \leqslant \|A^{-1}\|_2\|\delta A\|_2 < 1.$$

Divide $\|x\|$ on the both side of (2) and use $\|b\| = \|Ax\| \leqslant \|A\|\|x\|$, we have

$$\frac{\|\delta x\|}{\|x\|} \leqslant \frac{\|A\|\|A^{-1}\|}{1-\|A^{-1}\delta A\|}\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right).$$

Let $\kappa(A) = \|A\|\|A^{-1}\|$, then

$$\frac{\|\delta x\|}{\|x\|} \leqslant \frac{\kappa(A)}{1-\kappa(A)\frac{\|\delta A\|}{\|A\|}}\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right)$$