Nov 21, 2023 (Due: 08:00 Nov 28, 2023)

1. Find an example of linear system such that Jacobi method converges while Gauss–Seidel method diverges. Justify your claim.

2. Find an example of positive definite linear system such that Gauss–Seidel method converges while Jacobi method diverges. Justify your claim.

3. Let $B \in \mathbb{C}^{n\times n}$, $g \in \mathbb{C}^n$. Suppose that $\rho(B) = 0$. Show that the iterative scheme

$$x^{(k+1)} = Bx^{(k)} + g$$

converges to the solution of $x = Bx + g$ for any initial guess $x^{(0)}$ within at most n iterations.

4. Let B, $M \in \mathbb{C}^{n\times n}$, $g \in \mathbb{C}^n$. Suppose that both M and $M - B^*MB$ are positive definite. Show that the iterative scheme

$$x^{(k+1)} = Bx^{(k)} + g$$

converges to the solution of $x = Bx + g$ for any initial guess $x^{(0)}$.

5. Numerically solve the 2D Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

over the unit square $[0, 1]^2$ with boundary conditions

$$u(0, y) = u(1, y) = u(x, 1) = 0, \qquad u(x, 0) = \sin(\pi x).$$

Use Jacobi method or Gauss–Seidel method to solve the discretized system. Visualize the solution and the convergence history.

6. (H) Implement the scaling and squaring algorithm for computing (combined with Padé approximation) for computing the matrix exponential. Test the accuracy with different combinations of the parameters.

7. (optional) Let $A \in \mathbb{C}^{n\times n}$ be Hermitian with positive diagonal entries. Suppose that the Gauss–Seidel method on $Ax = b$ converges for any b. Show that A is positive definite.
Hint: Show that $A - \bigl(I - (D - L)^{-1}A\bigr)^* A \bigl(I - (D - L)^{-1}A\bigr)$ is positive definite.