Homework 12 Solutions

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Problem 1

Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_0 = 0.$$

Then by the Arnoldi process,

$$v_1 = r = b - Ax_0 = b, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

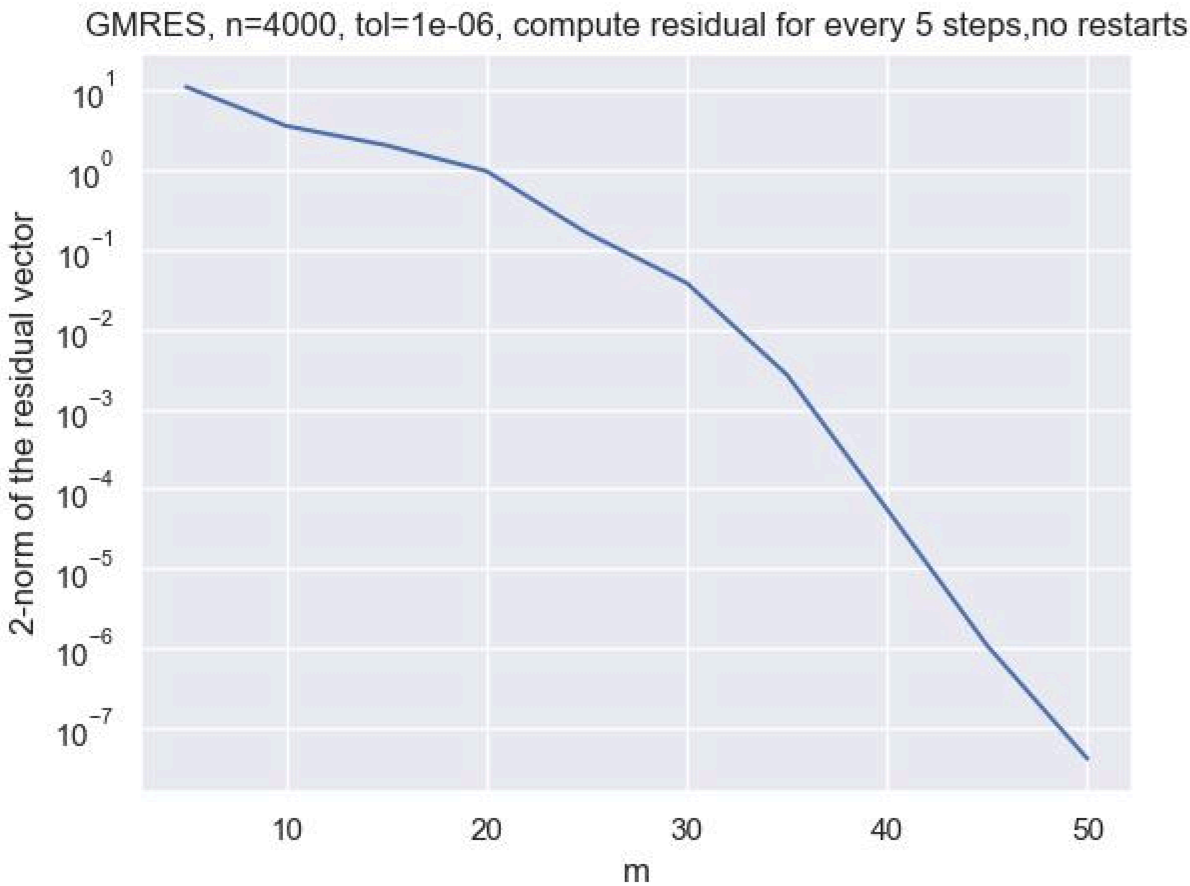
Let $V = [v_1, v_2, v_3]$, then $H = V^T AV$ is a upper Hessenberg matrix:

$$H = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

which is singular.

Problem 2

see `gmres.py`.



Problem 3

Let x_k be the approximate solution at kth iterate. Then

$$\begin{aligned} r_k &= b - Ax_k \neq 0, \\ 0 = r_{k+1} &= b - Ax_{k+1} \\ &= b - A\left(x_k + r_k \frac{r_k^T r_k}{r_k^T A r_k}\right) \\ &= r_k - \frac{r_k^T r_k}{r_k^T A r_k} A r_k. \end{aligned}$$

Hence r_k is an eigenvector of A, with eigenvalue $r_k^T A r_k / (r_k^T r_k)$.$\square$

Problem 4

Put

$$
\begin{aligned}
f(x_{k+1}) - f(x_k) &= f\left(x_k + r_k \frac{r_k^T r_k}{r_k^T A r_k}\right) - f(x_k) \\
&= -\frac{(r_k^T r_k)^2}{r_k^T A r_k},
\end{aligned}
$$

and since $x_k = A^{-1}(b - r_k)$ and A^{-1} is also positive definite, we have

$$
\begin{aligned}
f(x_k) &= x_k^T A x_k - 2b^T x_k \\
&= r_k^T A^{-1} r_k - b^T A^{-1} b \\
&\geq r_k^T A^{-1} r_k.
\end{aligned}
$$

Then we only need to prove

$$
-\frac{(r_k^T r_k)^2}{r_k^T A r_k} \leq -\kappa^{-1} r_k^T A^{-1} r_k,
$$

which is equivalent to

$$
\frac{r_k^T A^{-1} r_k}{r_k^T r_k} \frac{r_k^T A r_k}{r_k^T r_k} \leq \kappa = ||A||_2 ||A^{-1}||_2 = \frac{\lambda_n}{\lambda_1}.
$$

By the Rayleigh quotient theorem, $\forall r \in \mathbb{R}^n / \{0\}$,

$$
\begin{aligned}
max\ \frac{r^T A r}{r^T r} &= \lambda_n, \\
max\ \frac{r^T A^{-1} r}{r^T r} &= \frac{1}{\lambda_1},
\end{aligned}
$$

hence we finish the proof.□