Homework 11 Solutions

Weihao Li
Fudan University

Problem 1

Let

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix},$$

then $\rho(D^{-1}(L+U)) = 0$ and $\rho((D-L)^{-1}U) = 2$, hence Jacobi method converges while Gauss–Seidel method diverges.$\square$

Problem 2

Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix},$$

then $\rho(D^{-1}(L+U)) = \frac{5}{4}$ and $\rho((D-L)^{-1}U) = \frac{1}{2}$, hence Gauss–Seidel method converges while Jacobi method diverges. $\square$

Problem 3

Let the exact solution be x_{sol}, and the error $y^{(k)} = x^{(k)} - x_{sol}$.
Then

$$y^{(k+1)} = By^{(k)}, k = 0, 1, \ldots$$

and $B^n = O$ since $\rho(B) = 0$, hence $y^{(n)} = B^n y^{(0)} = 0$, and $x^{(n)} = x_{sol}$. $\square$

Problem 4

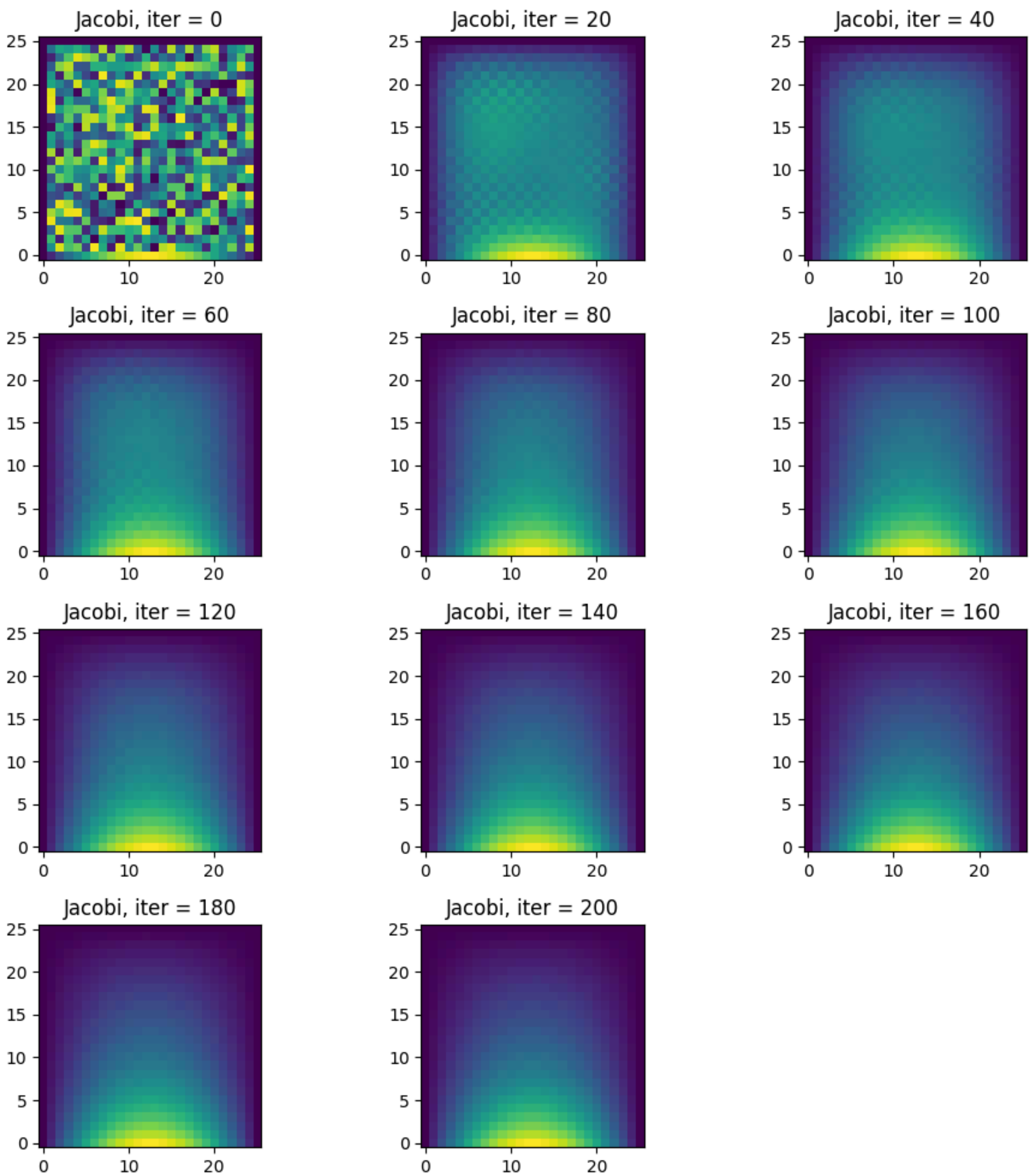
For every eigenpair (λ, v) of B, we have

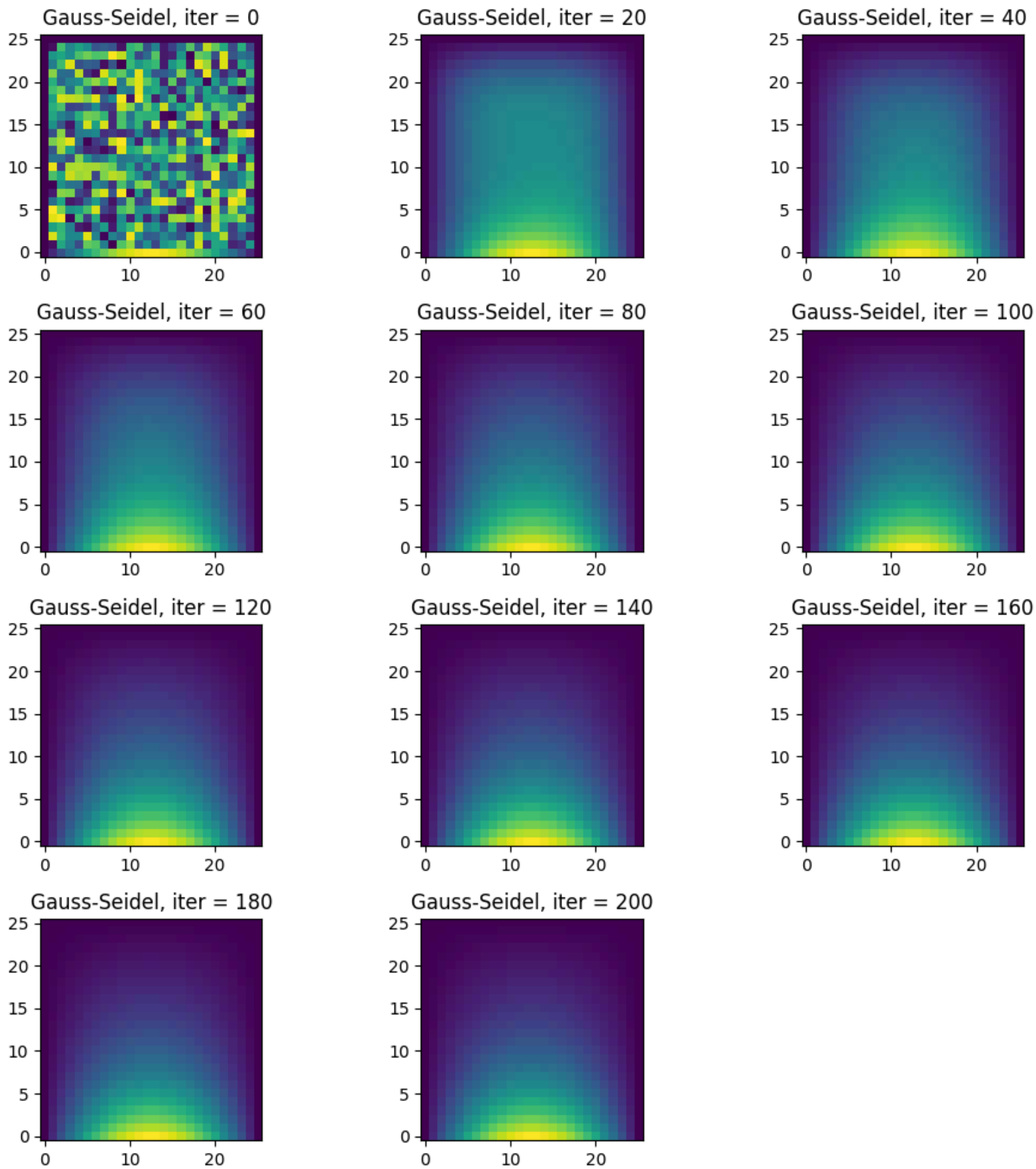
$$\begin{aligned} v^*Mv &> 0 \\ v^*(M - B^*MB)v &= v^*Mv - (Bv)^*MBv \\ &= v^*Mv(1 - |\lambda|^2) > 0, \end{aligned}$$

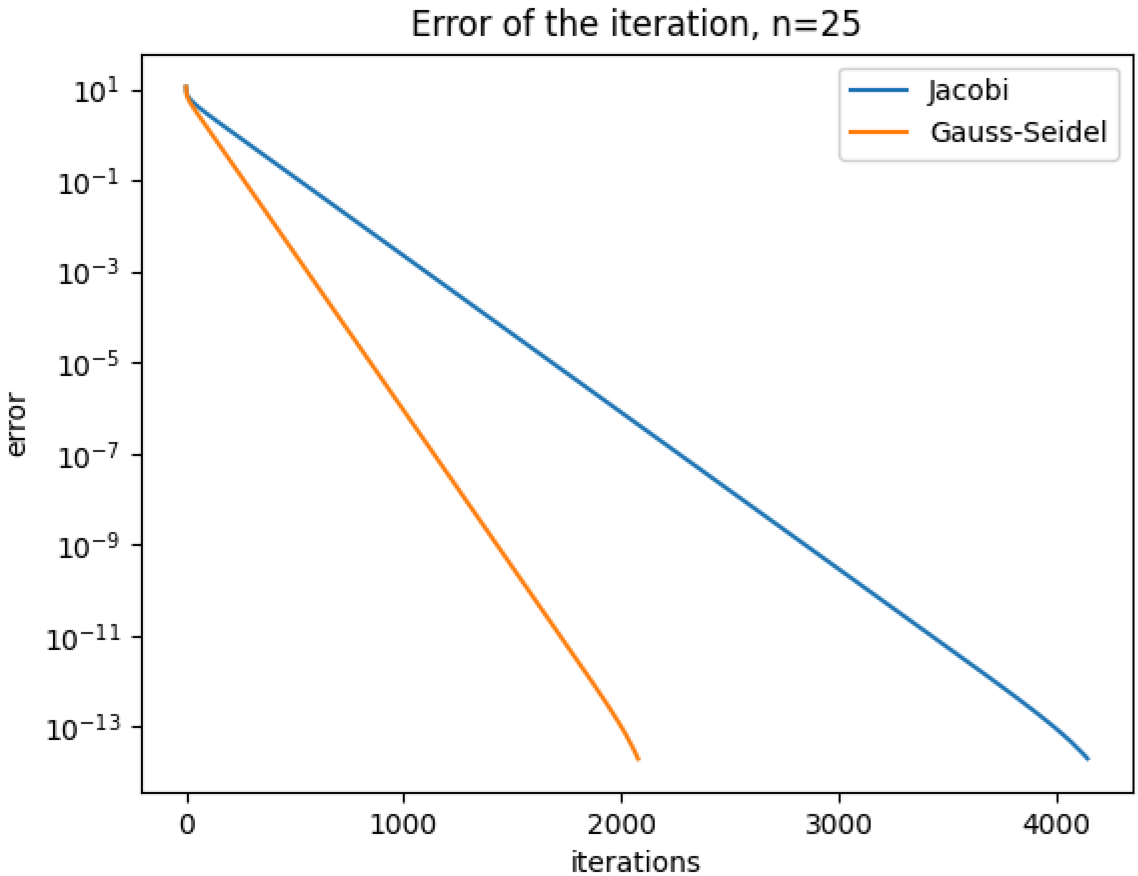
hence $|\lambda| < 1$, and $\rho(B) < 1$, hence the iterative scheme converges to a solution for any initial guess.$\square$

Problem 5

see `prob05.py`.


Jacobi, iter = 0
Jacobi, iter = 20
Jacobi, iter = 40
Jacobi, iter = 60
Jacobi, iter = 80
Jacobi, iter = 100
Jacobi, iter = 120
Jacobi, iter = 140
Jacobi, iter = 160
Jacobi, iter = 180
Jacobi, iter = 200
0
5
10
15
20
25
0
10
20


Gauss-Seidel, iter = 0
Gauss-Seidel, iter = 20
Gauss-Seidel, iter = 40
Gauss-Seidel, iter = 60
Gauss-Seidel, iter = 80
Gauss-Seidel, iter = 100
Gauss-Seidel, iter = 120
Gauss-Seidel, iter = 140
Gauss-Seidel, iter = 160
Gauss-Seidel, iter = 180
Gauss-Seidel, iter = 200



Error of the iteration, n=25
Jacobi
Gauss-Seidel
error
iterations
10^{1}
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^{-13}
0
1000
2000
3000
4000