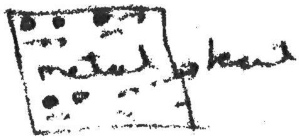
Private Notes (1) MA106 01

A tourist's guide to partial differential Equations

→ The Heat Equation → $\boxed{\frac{dT}{dt} = \alpha \nabla^2 T}$

→ imagine you have a piece of metal



→ How is the heat distributed across it at any one moment → what's the temperature at any one point

→ we want to know how this distribution will change over time → as heat flows from warmer spots to cooler ones

The 1D Example →

lets say you have 2 different rods at 2 different temperatures along the x-axis.

→ now lets bring these two rods into contact, → the heat will flow from the hot rod to the cold one and over time the temperature will be equal.

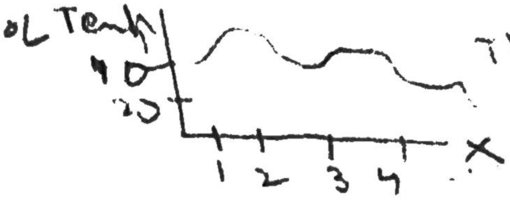
→ but how?, what will the temperature dist be at each point in time → can we do the mathematical modelling for it? → Math model

$$\frac{dT}{dt}(x,t) = \alpha \cdot \frac{d^2T}{dx^2}(x,t)$$

→ Mathematical Modelling

→ Building up the Heat equation

→ Currently we are imagining a rod along the x-axis, with each point of that rod being given by the value on the x-axis



The rod is the x-axis
$x \leq 10$

→ The temperature is a function of this rod
so $T(x)$

However the temperature is changing over time, so really we have another dimension to consider

$T(x,t)$ changes

→ since we have an entire function changing over time and the function has multiple inputs, there are more than 1 rates of change happening.

$\frac{dT}{dx}$ (temperature change with respect to x)

$\frac{dT}{dt}$ (temperature change with respect to t)

→ so essentially our heat transfer is controlled by

$\frac{dT}{dt}$ (small change to temperature with small change in time)

AND

$\frac{dT}{dx}$ (small change to temperature with small change along x - spatial dimension)

→ Moreover, the change in the time dimension depends upon how the function changes with respect to space.

So, the modelling of how the heat eq/ changes with time depends on how it changes with space, i.e they are proportional.

$$\underbrace{\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}}_{\text{Partial differential equation}}$$

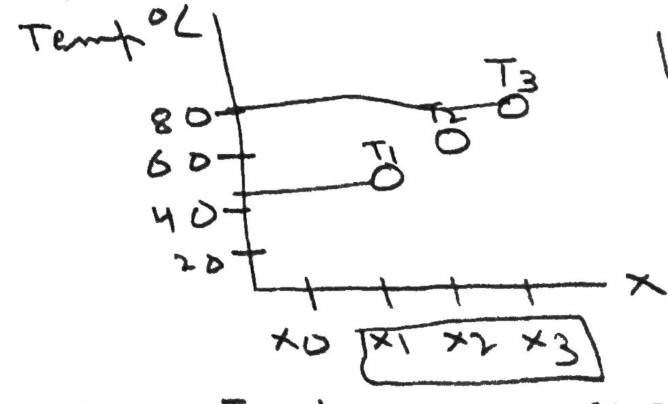
→ change with respect to t is proportional to the second partial derivative with respect to x

→ with just 2 inputs, 1 for x, and 1 for t in the 1D PDE

$$\boxed{\frac{\partial T}{\partial t}(x,t) = k \cdot \frac{\partial^2 T}{\partial x^2}(x,t)}$$

→ Where does this come from?

Lets look at a discrete number of things



$$\left(\frac{T_1 + T_3}{2} - T_2\right) > 0 \text{ or } < 0$$

→ to model how T_2 changes, we can average out its neighbouring particles → $\left(\frac{T_1+T_3}{2} - T_2\right)$ → if

> 0 → T_2 will heat up proportional to ~~difference~~

< 0 → T_2 will cool down proportional to difference

so → $$\frac{\partial T_2}{\partial t} = \alpha\left(\frac{T_1 + T_3}{2} - T_2\right)$$

$$= \alpha\left(\frac{T_1+T_3}{2} - \frac{2T_2}{2}\right) \to \frac{\alpha}{2}\left(T_1+T_3-2T_2\right)$$

$$\to \frac{\alpha}{2}\left[(T_3-T_2)+(T_1-T_2)\right]$$

$$\frac{dT_2}{dt} \to \frac{\alpha}{2}\left[\underbrace{(T_3-T_2)}_{\Delta T_2} - \underbrace{(T_2-T_1)}_{\Delta T_1}\right]$$

if $\to \Delta T_2 - \Delta T_1 > 0 \to T_2$ heats up
$\to \Delta T_2 - \Delta T_1 < 0 \to T_2$ cools down

ΔT_2 [$\to$ since the difference of avgs is greater, T_2 will change

ΔT_1 [

x_1 x_2 x_3 Therefore

$$\to \underbrace{\Delta T_2 - \Delta T_1}_{\text{second difference}} = \Delta\Delta T_1 \to \frac{dT_2}{dt} = \frac{\alpha}{2}\left(\Delta\Delta T_1\right)$$

now this is very similar to the original equation

$$\to \frac{dT_2}{dt} = \frac{\alpha}{2}\Delta\Delta T_1 \to \text{discrete}$$

so when we go from discrete to the continous this turns into $\to \dfrac{dT}{dt} = K \cdot \underbrace{\dfrac{d^2T}{dx^2}}$

where K would be const any proportional constant between adj particles — the ~~second difference~~ is replaced by the <u>second derivative</u>

$\to$ so instead of calculating temperature avgs between two fixed points x_1, x_2 $\to$ we are now shrinking that step to a "very small change in x" dx

$\to$ so now an arbitrary point will change depending on the temperatures of the infinately small dx's adjacent to it.

Therefore we have now built up (5)

$$\boxed{\frac{dT}{dt} = \alpha \cdot \frac{d^2T}{dx^2}}$$

The heat equation in 1D x

(P.S for 3D $\Rightarrow \frac{dT}{dt} = \alpha \left[\nabla^2 T\right]$)

References:

3blue1brown - "But what is a Partial Differential Equation?" Youtube.com

ROUGH NOTES, ONLY FOR UNDESTAND.

→ How does the temperature distribution from a 1D rod change over time → From the Equation →

$$\frac{dT}{dt}(x,t) = K \cdot \frac{d^2T}{dx^2}(x,t)$$

What this means: the rate at which temperature at a given point changes over time, depends on the second derivative of that temperature at that point with respect to space.

→ constraints $T(x,t)$ must satisfy

1) PDE → $\frac{dT}{dt}(x,t) = K \cdot \frac{d^2T}{dx^2}(x,t)$
2) Boundary condition
3) Initial condition

→ Lets consider $T(x,0) = \sin(x) \rightarrow \frac{dT}{dx} = \cos(x) > \frac{d^2T}{dx^2} = -\sin(x)$

$$\frac{dT}{dt}(x,0) = -K \cdot \underbrace{T(x,0)}_{\sin(x)}$$

$$T(x, 5\Delta t) = c^5 \cdot \sin(x)$$

$$T' = -KT$$

$$\int \frac{T'}{T}\, dE = \int -K\, dt$$

$\ln T = -Kt \rightarrow \underbrace{T = Ce^{-Kt}}$ → This is newton law of cooling

where C is the initial temperature.

→ There extrapolating this logic to our PDE

$$T(x,t) = \sin(x)\, e^{-Kt}$$

→ initial distribution

$$\underbrace{-K\sin(x)e^{-Kt}}_{\frac{dT}{dt}} = \cancel{K} \cdot \underbrace{-\sin(x)\, e^{-Kt}}_{\frac{d^2T}{dx^2}}$$

→ satisfies the PDE

The function must also satisfy the boundary condition
→ since there is no heat transfer from the edges (1)
of the rod ∴ the rate of change is 0 $\forall_t$ t > 0.

$$ (2) \rightarrow \frac{dT}{dx}(0,t) = \frac{dT}{dx}(L,t) = 0 \quad \forall_t \ t > 0 $$

→ our function which satisfies the PDE

$$ T(x,t) = \sin(x)\, e^{-kt} $$

→ we can change our function to cos(x) and it will still satisfy the PDE.

$$ -k\cos(x)\, e^{-kt} = k \cdot -\cos(x) \cdot e^{-kt} $$

$$ \text{and now } \frac{dT}{dx} = -\sin(x)\, e^{-kt} = 0 \text{ at } x = 0 $$

However, for it to satisfy the equation on the right hand side we need to change the frequency s.t the function is flat at x = L

→ This lowest frequency will be $\frac{\pi}{L}$ (or the first harmonic)

→ the boundary is satisfied by $k\frac{\pi}{L} \cdot \forall_{k \in N}$

→ therefore now we have an infinite set of Functions T(x,t) that satisfy both (1) and (2)

$$ T(x,t) = \cos(n(\pi/L) \cdot x)\, e^{-k(n\frac{\pi}{L})^2 t} $$

$$ \forall n \in \mathbb{N}, n \neq 0 \quad \forall t > 0 $$

→ The heat equation is also linear i.e for any two solution $T_1, T_2 \rightarrow T_3 = T_1 + T_2$ is also a solution

$$ T(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} $$

solves the PDE //