24.4 (a) $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \begin{cases} 0 & x \in [0,1) \\ \frac{1}{2} & x = 1 \\ 1 & x \in (1,\infty) \end{cases}$

(b) No. By the contrapositive to Theorem (24.3), f is not continuous on [0,1] so $< f_n >$ does not converge uniformly to f on [0,1].

(c) No. By the contrapositive to Theorem (24.3), f is not continuous on $[0,\infty)$ so $< f_n >$ does not converge uniformly to f on $[0,\infty)$.

24.10 (a) *Proof.* Let $\varepsilon > 0$. Then $f_n \to f$ uniformly so there is a $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $x \in S$. Then $g_n \to g$ uniformly so there is a $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|g_n(x) - g(x)| < \frac{\varepsilon}{2}$ for all $x \in S$. Then let N $= \max\{N_1, N_2\}$ so if $n > N$, then $\varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} > |f_n(x) - f(x)| + |g_n(x) - g(x)| \geq |f_n(x) + g_n(x) - f(x) - g(x)|$ for all $x \in S$ by the Triangle Inequality. Since ε was arbitrary, $f_n + g_n \to f + g$ uniformly on S. $\square$

(b) No, I don't believe so. This will be proven in the next question.

24.11 (a) $f_n(x) - f(x) = 0$ for all $x \in \mathbb{R}$ so $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = 0$. Then by Theorem (24.4), $f_n \to f$ uniformly.
$\lim_{n\to\infty} \sup\{|g_n(x) - g(x)| : x \in \mathbb{R}\} = \lim_{n\to\infty} \frac{1}{n} = 0$. Then by Theorem (24.4), $g_n \to g$ uniformly.

(b) *Proof.* $f_n g_n(x) = \frac{x}{n}$ and $fg(x) = x \cdot 0 = 0$ for all $x \in \mathbb{R}$. We will use proof by contradiction. Let $\varepsilon > 0$. Suppose $< f_n g_n >$ converges uniformly to fg so there is a $N \in \mathbb{N}$ such that if $n > N$, then $|f_n g_n(x) - fg(x)| = |\frac{x}{n} - 0| < \varepsilon$ for all $x \in \mathbb{R}$. Specifically, $|\frac{x}{N+1}| < \varepsilon$ for all $x \in \mathbb{R}$. But for $x > \varepsilon(N+1) > 0$, $\frac{x}{N+1} = |\frac{x}{N+1}| > \varepsilon$. Contradiction so $< f_n g_n >$ does not converge uniformly to fg. $\square$

25.6 (a) $\sum_{k=1}^{\infty} a_k x^k \leq \sum_{k=1}^{\infty} |a_k| x^k \leq \sum_{k=1}^{\infty} |a_k| \cdot 1 < \infty$ for $x \in [-1,1]$ so by the Weierstrass M-test, $\sum_{k=1}^{\infty} a_k x^k$ converges uniformly on $[-1,1]$. Then $g_k(x) = a_k x^k$ is a continuous function on $[-1,1]$ so the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on $[-1,1]$ by Theorem (25.5). Thus, the series converges uniformly on $[-1,1]$ to a continuous function.

(b) Yes, by part (a) since $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} |\frac{1}{n^2}| = \sum_{k=1}^{\infty} \frac{1}{n^2} < \infty$ because it's a p-series with p = $2 > 1$ so $\sum_{k=1}^{\infty} \frac{1}{n^2} x^k$.

25.8 $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^2 \cdot 2^{n+1}}}{\frac{x^n}{n^2 \cdot 2^n}} \right| = \lim_{n\to\infty} \left| \frac{n^2 x}{2(n+1)^2} \right| = \frac{|x|}{2} < 1$ so $|x| < 2$. Thus, the radius of convergence is 2.

$\frac{x^n}{n^2 \cdot 2^n} \leq \left| \frac{x^n}{n^2 \cdot 2^n} \right| \leq \left| \frac{2^n}{n^2 \cdot 2^n} \right| \leq |\frac{1}{n^2}| = \frac{1}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since it's a p-series with p $= 2 > 1$ so by the Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{x^n}{n^2 \cdot 2^n}$ converges uniformly on [-2,2]. Then $g_k(x) = \frac{x^k}{k^2 \cdot 2^k}$ is a continuous function on $[-2,2]$ so the series $\sum_{k=1}^{\infty} g_k$ represents a continuous function on $[-2,2]$ by Theorem (25.5). Thus, the series converges uniformly on $[-2,2]$ to a continuous function.

25.10 (a) $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{1+x^{n+1}}}{\frac{x^n}{1+x^n}} \right| = \lim_{n\to\infty} \left| \frac{x+x^{n+1}}{1+x^{n+1}} \right| = |x| < 1$. Therefore, the radius of convergence is 1 so the series converges for all $x \in [0,1)$.

(b) $\frac{x^n}{1+x^n} \leq a^n$ for all $x \in [0,a)$. Since $0 < a < 1$, $\sum_{n=1}^{\infty} a^n < \infty$ so by the Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{x^n}{1+x^n}$ converges uniformly on [0,a] for each a such that $0 < a < 1$.

(c) *Proof.* Suppose the series converges uniformly to f on [0,1). $\lim_{n\to\infty} \frac{x^n}{1+x^n} = 0$ so $f(x) = 0$ for $x \in [0,1)$. Then let $\varepsilon = \frac{1}{4}$ so there is a $N \in \mathbb{N}$ such that for all $n > N$, $\left| \frac{x^n}{1+x^n} - 0 \right| < \frac{1}{4}$. Then, $\frac{x^{N+1}}{1+x^{N+1}} < \frac{1}{4}$ for all $x \in [0,1)$. But this fails for values of x close to 1 since $\lim_{x\to 1} \frac{x^{N+1}}{1+x^{N+1}} = \frac{1}{1+1} = \frac{1}{2} > \frac{1}{4}$. Contradiction so the series does not converge uniformly on [0,1). $\square$

26.2 (a) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. Since we are in the interval of convergence (-1,1), we can differentiate term-by-term so $\sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$. Multiply both sides by x to get $\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$ for $|x| < 1$.

(b) $|x| = |\frac{1}{2}| < 1$ so by part (a), $\sum_{n=0}^{\infty} n(\frac{1}{2})^n = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$.

(c) $|x| = |\frac{1}{3}| < 1$ so by part (a), $\sum_{n=0}^{\infty} n(\frac{1}{3})^n = \frac{\frac{1}{3}}{(1-\frac{1}{3})^2} = \frac{3}{4}$.
$|x| = |\frac{-1}{3}| < 1$ so by part (a), $\sum_{n=0}^{\infty} n(\frac{-1}{3})^n = \frac{\frac{-1}{3}}{(1-\frac{-1}{3})^2} = \frac{-3}{16}$.

26.4 (a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ for $x \in \mathbb{R}$

(b) $F(x) = \int_0^x e^{-t^2} dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$ for $x \in \mathbb{R}$. The last step uses the fact that we can integrate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$ term-by-term in its interval of convergence $[0, x]$.

26.6 (a) The interval of convergence for these series is $(-\infty, +\infty)$ so we can differentiate term-by-term.
$s'(x) = 1 - \frac{3x^3}{3!} + \frac{5x^5}{5!} - \cdots = 1 - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots = c(x)$
$c'(x) = 0 - \frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \cdots = -(x - \frac{x^3}{3!} + \frac{x^5}{5!}) = -s(x)$

(b) $(s^2 + c^2)' = 2s \cdot s' + 2c \cdot c' = 2s \cdot c + 2c \cdot (-s) = 0$

(c) By part (b), $s^2 + c^2 = k$ for some $k \in \mathbb{R}$. At $x = 0$, $s(0) = 0$ and $c(0) = 1$ so $s(0)^2 + c(0)^2 = 1 = k$. Therefore, $s^2 + c^2 = 1$.

26.8 (a) $\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$ for $|y| < 1$ so if we let $y = -x^2$, $\sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1+x^2}$ for $|-x^2| < 1$, which is equivalent to $|x| < 1$. Thus, $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$ for $x \in (-1, 1)$.

(b) $\arctan(x) = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $|x| < 1$. Note that we are within the interval of convergence so we can integrate term-by-term.

(c) At $x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ which converges by the Alternating Series Test so by Abel's Theorem, $f(x) = \arctan(x)$ is continuous at $x = 1$. Then, $\arctan(1) = \frac{\pi}{4} = \lim_{x \to 1} \arctan(x)$. By part (b), this is equal to $\lim_{x \to 1} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ so $\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

(d) At $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ which converges by the Alternating Series Test so by Abel's Theorem, $f(x) = \arctan(x)$ is continuous at $x = -1$. Then, $\arctan(-1) = \frac{-\pi}{4} = \lim_{x \to -1} \arctan(x)$. By part (b), this is equal to $\lim_{x \to -1} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ so $\pi = -4 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$.