8.2 (a) The limit is equal to 0.

Proof. Let $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. If $n > N$, then $n > \frac{1}{\varepsilon}$ so $\varepsilon > \frac{1}{n} = \frac{n}{n^2} > \frac{n}{n^2+1} = |\frac{n}{n^2+1} - 0|$. Since ε was arbitrary,

$$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

□

(c) The limit is equal to $\frac{4}{7}$.

Proof. Let $\varepsilon > 0$. Let $N = \frac{6}{\varepsilon}$. If $n > N$, then $n > \frac{6}{\varepsilon}$ so $\varepsilon > \frac{6}{n}$. $7n - 5 > n$ for all $n \in \mathbb{N}$ so $\varepsilon > \frac{6}{n} > \frac{6}{7n-5} = \frac{42}{7(7n-5)} > \frac{41}{7(7n-5)} = |\frac{4n+3}{7n-5} - \frac{4}{7}|$. Since ε was arbitrary,

$$\lim_{n\to\infty} \frac{4n+3}{7n-5} = \frac{4}{7}$$

□

8.4 *Proof.* First let us consider the case where $M = 0$. $|t_n| \leq M = 0$ so $t_n = 0$ for all $n \in \mathbb{N}$. Then $s_n t_n = 0$ for all $n \in \mathbb{N}$ so clearly

$$\lim_{n\to\infty} s_n t_n = 0$$

Then, let us consider the case where $M \neq 0$. Note that $M > 0$ since $0 \leq |t_n| \leq M$ and we have dealt with the case where $M = 0$ above. Let $\varepsilon > 0$. Then there is some N_1 such that if $n > N_1$, then $|s_n - 0| < \frac{\varepsilon}{M}$. $|t_n| \leq M$ for all $n \in \mathbb{N}$ so $|s_n||t_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon$. Since $|s_n||t_n| = |s_n t_n| = |s_n t_n - 0|$, $|s_n t_n - 0| < \varepsilon$. Since ε was arbitrary,

$$\lim_{n\to\infty} s_n t_n = 0$$

□

8.8 (a) *Proof.* Let $\varepsilon > 0$. Let $N = \frac{1-\varepsilon^2}{2\varepsilon}$. If $n > N$, then $n > \frac{1-\varepsilon^2}{2\varepsilon}$ so $2n\varepsilon > 1 - \varepsilon^2$. Hence, we have that $1 < \varepsilon^2 + 2n\varepsilon$ and so $1 + n^2 < \varepsilon^2 + 2n\varepsilon + n^2 = (\varepsilon + n)^2$. Therefore, $\sqrt{n^2+1} < \varepsilon + n$ and hence $\sqrt{n^2+1} - n < \varepsilon$ so $|\sqrt{n^2+1} - n - 0| < \varepsilon$. Since ε was arbitrary,

$$\lim_{n\to\infty} (\sqrt{n^2+1} - n) = 0$$

□

8.10 *Proof.* Let $\varepsilon > 0$. Let $\lim_{n\to\infty}(s_n) = a + \varepsilon > a$. Then there is some N such that if $n > N$, then $|s_n - (a+\varepsilon)| < \varepsilon$ so $-\varepsilon < s_n - (a+\varepsilon) < \varepsilon$ and hence $-\varepsilon + (a+\varepsilon) < s_n < \varepsilon + (a+\varepsilon)$. As a result, we get that $a < s_n < a + 2\varepsilon$ so $s_n > a$. Since ε was arbitrary, we have therefore shown that there exists a number N such that $n > N$ implies $s_n > a$. □

9.4 (a) The first four terms are 1, $\sqrt{2}$, $\sqrt{\sqrt{2}+1}$, and $\sqrt{\sqrt{\sqrt{2}+1}+1}$

(b) Assuming that $< s_n >$ converges, let $\lim_{n\to\infty}(s_n) = s > 0$ since all the terms are positive. Since $\lim_{n\to\infty}(s_n) = \lim_{n\to\infty}(s_{n+1})$, we get that $\lim_{n\to\infty}(s_{n+1}) = s$ so $\lim_{n\to\infty}(\sqrt{s_n+1}) = s$. Hence, $s^2 = [\lim_{n\to\infty}(\sqrt{s_n+1})]^2 = \lim_{n\to\infty}(\sqrt{s_n+1}) \cdot \lim_{n\to\infty}(\sqrt{s_n+1}) = \lim_{n\to\infty}(s_n+1)$ by limit properties. Then, we get that $s^2 = \lim_{n\to\infty}(s_n) + 1 = s + 1$ and so $s^2 - s - 1 = 0$. Therefore, $s = \frac{1}{2}(a \pm \sqrt{5})$. Since $s > 0$, $s = \frac{1}{2}(a + \sqrt{5})$

9.6 (a) Since $a = \lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(x_{n+1})$ and $\lim_{n\to\infty}(x_{n+1}) = \lim_{n\to\infty}(3x_n^2) = 3\lim_{n\to\infty}(x_n^2) = 3\lim_{n\to\infty}(x_n) \cdot \lim_{n\to\infty}(x_n) = 3a \cdot a = 3a^2$ by limit properties, we then have that $3a^2 = a$. Therefore, $a = 0$ or $a = \frac{1}{3}$

(b) No, the limit does not exist as $\lim_{n\to\infty}(x_n) = +\infty$. For any $M > 0$, there is a N such that if $n > N$, then $s_n > M$ since the terms of the sequence get larger and larger without bound as n increases.

(c) In (a), we assumed that the limit exists by claiming that $a = \lim_{n\to\infty}(x_n)$. However this is not a valid assumption so we cannot say that $a = \lim_{n\to\infty}(x_n)$.

9.12 (a) *Proof.* Assuming that the limit exists, $L = \lim_{n\to\infty} |\frac{s_{n+1}}{s_n}| < 1$. By denseness, let $a \in \mathbb{R}$ such that $L < a < 1$. Let $\varepsilon = a - L > 0$. Then there is some $N_0 \in \mathbb{N}$ such that if $n > N_0$, then $||\frac{s_{n+1}}{s_n}| - L| < \varepsilon$. Note that if $n > N_0$, then we can further restrict n so that $n \geq N_0 + 1 = N$. Hence $-\varepsilon < \frac{|s_{n+1}|}{|s_n|} - L < \varepsilon$ for $n \geq N$. We then see that $L - \varepsilon < \frac{|s_{n+1}|}{|s_n|} < L + \varepsilon$ so $|s_n| \cdot (L - \varepsilon) < |s_{n+1}| < |s_n| \cdot (L + \varepsilon) = |s_n| a$ so we get that $|s_{n+1}| < |s_n| a$ for $n \geq N$. Note that we then have $|s_n| < |s_{n-1}| a$, $|s_{n-1}| < |s_{n-1}| a$, and so on. Therefore, $|s_n| < |s_{n-1}| a < |s_{n-2}| a^2 < \cdots < |S_N| a^{n-N}$. By limit properties, we get that $\lim_{n\to\infty} |s_N| a^{n-N} = 0$. Therefore, since a was arbitrary, we can apply the Squeeze Lemma to get

$$\lim_{n\to\infty} |s_n| = 0$$

□

(b) *Proof.* Suppose $L > 1$. Let $t_n = \frac{1}{|s_n|}$. Then, $\lim_{n\to\infty} |\frac{t_{n+1}}{t_n}| = \lim_{n\to\infty} |\frac{s_n}{s_{n+1}}| = \frac{1}{\lim_{n\to\infty} |\frac{s_{n+1}}{s_n}|} = \frac{1}{L} < 1$. By part (a), then $\lim_{n\to\infty}(t_n) = 0$ so by Theorem 9.10

$$\lim_{n\to\infty} |s_n| = +\infty$$

□

9.18 (a) *Proof.* We will prove by induction. Note that $1 + a = \frac{1-a^{1+1}}{1-a} = \frac{(1-a)(1+a)}{1-a} = 1 + a$ since $a \neq 1$ so the claim holds for n = 1. Now suppose the claim holds for some $n \in \mathbb{N}$. Then, $1 + a + a^2 + \cdots + a^n + a^{n+1} = \frac{1-a^{n+1}}{1-a} + a^{n+1}$ by inductive hypothesis above. Then, this is equal to $\frac{1-a^{n+1}+(1-a)a^{n+1}}{1-a} = \frac{1-a^{n+2}}{1-a}$ so the the claim holds for n+1. Since n was arbitrary, the claim holds for n+1 whenever it holds for n. Therefore by induction, the claim holds for all $n \in \mathbb{N}$. □

(b) $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a}$ by part (a). By limit properties since $|a| < 1$, this is equal to

$$\frac{1 - \lim_{n\to\infty} a^{n+1}}{1 - a} = \frac{1}{1 - a}$$

(c) $\lim_{n\to\infty}(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}) = \frac{1}{1-\frac{1}{3}}$ by part (b), where $a = \frac{1}{3}$ (note that $|a| = |\frac{1}{3}| < 1$). Therefore, the limit is equal to $\frac{3}{2}$.

(d) We will first consider $a = 1$. Since $a^k = 1^k = 1$ for all $k \in \mathbb{N}_{\leq n}$, then $\lim_{n\to\infty}(1+a+a^2+\cdots+a^n) = \lim_{n\to\infty}((n+1)\cdot 1) = +\infty$. Now we will consider $a > 1$. $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a}$ by part (a). By limit properties, this is equal to $\frac{1-\lim_{n\to\infty} a^{n+1}}{1-a}$. Since $a > 1$, $0 < \frac{1}{a} < 1$ so by Theorem 9.7(b), $\lim_{n\to\infty}(\frac{1}{a})^n = 0$. Then, $\lim_{n\to\infty}(\frac{1}{a^n}) = 0$ so by Theorem 9.10, this implies that $\lim_{n\to\infty} a^n = +\infty$ where $a > 1$. Since $\lim_{n\to\infty} a^{n+1} = \lim_{n\to\infty}(a \cdot a^n) = a \cdot \lim_{n\to\infty} a^n = +\infty$ by limit properties where $a > 1$. Therefore for $a \geq 1$,

$$\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) = +\infty$$