21.10 (a) $f : (0,1) \to [0,1]$ $x \mapsto \frac{1}{2}\sin(2\pi x) + \frac{1}{2}$ is a continuous function that maps (0,1) onto [0,1]

(b) $g : (0,1) \to \mathbb{R}$ $x \mapsto \tan(\pi(x - \frac{1}{2}))$ is a continuous function that maps (0,1) onto $\mathbb{R}$

(c) $h : [0,3] \to [0,1]$ $x \mapsto \begin{cases} x & x \in [0,1] \\ 1 & x \in (1,3] \end{cases}$ is a continuous function that maps $[0,1] \cup [2,3]$ onto [0,1]

(d) Suppose there is a continuous function f mapping [0,1] onto (0,1) or $\mathbb{R}$. [0,1] is closed and bounded so by Heine-Borel Theorem, it is compact so f([0,1]) must also be compact. But, neither (0,1) nor $\mathbb{R}$ are compact so no such f exists.

21.12 Let $S_n = (n-1,n)$ for $n \in \mathbb{N}$. Then $< S_n >=< (0,1),(1,2),(2,3),\cdots >$ is an infinite disjoint sequence of subsets of $\mathbb{R}$. Since interior(closure(S_n))=interior([n-1,n])$\neq \varnothing$ so it is not nowhere dense and hence of second category in $\mathbb{R}$

22.4 (a) Closure(E) $= E \cup (\{0\} \times [-1,1])$

(b) By Exercise (22.3), it suffices to prove that E is connected. Let $(s, \sin(\frac{1}{s}))$ and $(t, \sin(\frac{1}{t}))$ be elements of E. Then there is a continuous function $f : [s,t] \to E$ $x \mapsto (x, \sin(\frac{1}{x}))$ such that $f(s) = (s, \sin(\frac{1}{s}))$ and $f(t) = (t, \sin(\frac{1}{t}))$. Since s and t were are arbitrary, E is path-connected and therefore E is connected.

(c) Suppose Closure(E) is path-connected. Then there is a continuous function $f : [0,1] \to$ Closure(E) where f $= (f_1, f_2)$. Let $t_0 = \inf\{t \in [0,1] : f_1(t) > 0\}$. By continuity of f, f_1 and f_2 are continuous so there is a $\delta > 0$ such that $t_0 < t < t_0 + \delta$ implies $|f_2(t) - f_2(t_0)| < 1$. Then let $t_1 \in (t_0, t_0 + \delta)$ so $f_1(t_1) > 0$. By continuity and the Intermediate Value Theorem, $f_1([t_0,t_1]) = [0.f_1(t_1)]$. $f_2(t) = \sin(\frac{1}{f_1(t)})$ for all t where $f_1(t) \neq 0$ so $f_2([t_0,t_1]) = [-1,1]$. Contradiction to $|f_2(t) - f_2(t_0)| < 1$ for all $t \in (t_0, t_0 + \delta)$. Therefore, Closure(E) is not path connected.

22.6 (a) d(f,g) $= \sup\{|f(x) - g(x)| : x \in S\} \geq 0$. d($f$,$g$) $= 0 \leftrightarrow f = g$
d(f,g) $= \sup\{|f(x) - g(x)| : x \in S\} = \sup\{|g(x) - f(x)| : x \in S\} =$ d(g,f)
d(f,h) $= \sup\{|f(x)-h(x)| : x \in S\} = \sup\{|f(x)-g(x)+g(x)-h(x)| : x \in S\} \leq \sup\{|f(x)-g(x)| : x \in S\} + \sup\{g(x) - h(x)| : x \in S\} =$ d(f,g) + d(g,h).
Since all three of these properties hold, d is a distance function so $C(S)$ is a metric space.

(b) If the functions in $C(S)$ are unbounded, then the difference between two functions may be undefined so d may be undefined and thus $C(S)$ may then no longer be a metric space.

23.2 (a) $\lim_{n\to\infty} \left|\frac{\sqrt{n+1}x^{n+1}}{\sqrt{n}x^n}\right| = |x| < 1$ so the radius of convergence is 1. At $x = \pm 1$, the summation diverges by the Preliminary Test so the interval of convergence is (-1,1).

(b) $\lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)^{\sqrt{n+1}}}}{\frac{x^n}{n^{\sqrt{n}}}}\right| = \lim_{n\to\infty} \left|\frac{n^{\sqrt{n}}}{(n+1)^{\sqrt{n+1}}}x\right| = \sqrt{\lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}x^2\right|} = \sqrt{\lim_{n\to\infty}\left|(\frac{n}{n+1})^{n+1} \cdot \frac{x^2}{n}\right|} = \sqrt{\lim_{n\to\infty}\left|\frac{x^2}{n}\right|} = 0 < 1$ for all x so the interval of convergence is $\mathbb{R}$ and the radius of convergence is $+\infty$

(c) $\lim_{n\to\infty} \left|\frac{x^{(n+1)!}}{x^{n!}}\right| = \lim_{n\to\infty} \left|x^{n!(n+1-1)}\right| < 1$ if $|x| < 1$ so the radius of convergence is 1. At $x = \pm 1$, the summation diverges by the Preliminary Test so the interval of convergence is (-1,1).

(d) $\lim_{n\to\infty} \left|\frac{\frac{3^{n+1}x^{2n+2+1}}{\sqrt{n+1}}}{\frac{s^n x^{2n+1}}{\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\frac{3\sqrt{n}}{\sqrt{n+1}}x^2\right| = |3x^2| < 1$ so $|x| < \frac{1}{\sqrt{3}}$ and hence the radius of convergence is $\frac{1}{\sqrt{3}}$. At $x = \pm\frac{1}{\sqrt{3}}$, the summation diverges by the Preliminary Test so the interval of convergence is $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

23.4 (a) $\limsup_{n\to\infty} \frac{4+2(-1)^n}{5} = \lim_{N\to\infty} \sup\{\frac{4+2(-1)^n}{5} : n > N\} = \frac{4+2}{5} = \frac{6}{5}$
$\liminf_{n\to\infty} \frac{4+2(-1)^n}{5} = \lim_{N\to\infty} \inf\{\frac{4+2(-1)^n}{5} : n > N\} = \frac{4-2}{5} = \frac{2}{5}$

$\limsup_{n\to\infty}\left|\frac{(\frac{4+2(-1)^{n+1}}{5})^{n+1}}{(\frac{4+2(-1)^{n}}{5})^{n}}\right| = \lim_{N\to\infty}\sup\{\left|\frac{(4+2(-1)^{n+1})^{n+1}}{5(4+2(-1)^{n})^{n}} : n > N\right|\} = +\infty$

$\liminf_{n\to\infty}\left|\frac{(\frac{4+2(-1)^{n+1}}{5})^{n+1}}{(\frac{4+2(-1)^{n}}{5})^{n}}\right| = \lim_{N\to\infty}\inf\{\left|\frac{(4+2(-1)^{n+1})^{n+1}}{5(4+2(-1)^{n})^{n}} : n > N\right|\} = 0$

(b) Neither $\lim_{n\to\infty} a_n$ nor $\lim_{n\to\infty}(-1)^n a_n$ exist so neither series converges by the Preliminary Test.

(c) From part(a), $\limsup_{n\to\infty}|a_n|^{\frac{1}{n}} = \frac{6}{5}$ so the radius of convergence is $\frac{5}{6}$. If $x = \pm\frac{6}{5}$, the summation diverges by the Preliminary Test so the interval of convergence is $(-\frac{5}{6}, \frac{5}{6})$.

23.6 (a) $\sum_{n=0}^{\infty} a_n R^n$ converges. $R^n = |(-R)^n|$ so $\sum_{n=0}^{\infty} a_n|(-R)^n|$ converges. Hence, $\sum_{n=0}^{\infty} a_n(-R)^n$ is absolutely convergent so it is convergent by Corollary (14.7).

(b) Consider the power series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}x^n$ Then, $\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{n+1}x^{n+1}}{\frac{(-1)^n}{n}x^n}\right| = |x| < 1$ so the radius of convergence is 1. At $x = 1$, $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ converges by the Alternating Series Test since $\lim_{n\to\infty}\frac{1}{n} = 0$. At $x = -1$, $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}(-1)^n = \sum_{n=0}^{\infty}\frac{1}{n}$ is the Harmonic Series so it diverges. Thus, the interval of convergence is (-1,1].

23.8 (a) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{\sin(nx)}{n}$. Since $-1 \leq \sin(nx) \leq 1$, by the Squeeze Lemma, $0 = \lim_{n\to\infty}\frac{-1}{n} \leq \lim_{n\to\infty}\frac{\sin(nx)}{n} \leq \lim_{n\to\infty}\frac{1}{n} = 0$ so $\lim_{n\to\infty}\frac{\sin(nx)}{n} = 0$.

(b) $f_n'(x) = \cos(nx)$. Then, $\lim_{n\to\infty}\cos(nx) = \lim_{n\to\infty}\cos(n\pi)$ at $x = \pi$. Since $\cos(n\pi) = 1$ for all even n and -1 for all odd n, $\cos(n\pi) = (-1)^n$ so $\lim_{n\to\infty}(-1)^n$ doesn't exist. Thus, $\lim_{n\to\infty} f_n'(x)$ need not exist.