27.2 *Proof.* Let $A$ be a bounded subset of $\mathbb{R}$. $f$ is continuous on $\mathbb{R}$ so it is continuous on $A \subset \mathbb{R}$. By the Weierstrass Approximation Theorem, there is a sequence of polynomials $< p_n >$ such that $p_n \to f$ uniformly on [-n,n] for all $n \in \mathbb{N}$. Then choose $N \in \mathbb{N}$ such that $A \subset [-N, N]$. Hence, for all $n > N$, $A \subset [-n, n]$ so $p_n \to f$ uniformly on $A$. Since $A$ was arbitrary, $p_n \to f$ uniformly on each bounded subset of $\mathbb{R}$ $\square$

27.4 *Proof.* By the Weierstrass Approximation Theorem, there is a sequence of polynomials $< q_n >$ such that $q_n \to f$ uniformly on $[a, b]$. Then let $s_n = \frac{(f(b)-q_n(b)))-(f(a)-q_n(a))}{b-a}(x-a) + f(a) - q_n(a)$ and $p_n = q_n + s_n$. So, $p_n(a) = q_n(a) + s_n(a) = q_n(a) + f(a) - q_n(a) = f(a)$ and $p_n(b) = q_n(b) + s_n(b) = q_n(b) + f(b) - q_n(b) - f(a) + q_n(a) + f(a) - q_n(a) = f(b)$. Without loss of generality, suppose $f(a) - q_n(a) \leq f(b) - q_n(b)$. Then $s_n$ is monontonic so $s_n(x) \in [s_n(a), s_n(b)]$ for all $x \in [a, b]$. Hence, $s_n(x) \leq |s_n(b)|$ for all $x \in [a, b]$. Let $\varepsilon > 0$. Then $q_n \to f$ uniformly on $[a, b]$ so there is a $N \in \mathbb{N}$ such that if $n > N$, then $|q_n(x) - f(x)| < \varepsilon - |s_n(b)|$ for all $x \in [a, b]$. Therefore, $\varepsilon > |s_n(b)| + |q_n(x) - f(x)| = | - s_n(b)| + |f(x) - q_n(x)| \geq |f(x) - q_n(x) - s_n(b)| = |f(x) - p_n(x)|$. Since $\varepsilon$ was arbitrary, $p_n \to f$ uniformly on [a,b]. $\square$

28.4 (a) *Proof.* Let $a \neq 0$. 1 and $x$ are differentiable at $a$ so by the Quotient Rule, $\frac{1}{x}$ is differentiable at $a \neq 0$. Then by the Chain Rule, $\sin(x)$ is differentiable at $\frac{1}{a}$ so $\sin(\frac{1}{x})$ is differentiable at $a$. $x^2$ is differentiable at $a$ so by the Product Rule, $f(x) = x^2 \sin(\frac{1}{x})$ is differentiable at $a$. Since $a$ was arbitrary, $f(x)$ is differentiable at each $a \neq 0$. $\square$

$f'(x) = \frac{d}{dx} x^2 \sin(\frac{1}{x}) = 2x \sin \frac{1}{x} + x^2 \cdot \frac{d}{dx} \sin \frac{1}{x}$ by the Product Rule. Then $f'(x) = 2x \sin \frac{1}{x} + x^2 \cdot \frac{-1}{x^2} \cos \frac{1}{x}$ by the Chain Rule. Therefore, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ so $f'(a) = 2a \sin \frac{1}{a} - \cos \frac{1}{a}$ for $a \neq 0$.

(b) *Proof.* $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x} - 0}{x} = \lim_{x\to 0} x \sin \frac{1}{x}$. Then $-|x| \leq x \sin \frac{1}{x} \leq |x|$ and $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$ so by the Squeeze Lemma, $\lim_{x\to 0} x \sin \frac{1}{x} = 0 = f'(0)$. Thus, $f$ is differentiable at $x = 0$ and $f'(0) = 0$. $\square$

(c) *Proof.* We will use proof by contradiction. Suppose that $f'$ is continuous at $x = 0$. Then let $< x_n > = < \frac{1}{n} >$. $\lim_{n\to\infty} x_n = 0$ so $\lim_{n\to\infty} f'(x_n) = \lim_{n\to\infty} 2 \cdot \frac{1}{n} \sin(n) - \cos(n) = -\lim_{n\to\infty} \cos(n) \neq 0$ so $\lim_{n\to\infty} f'(x_n) \neq f'(0)$. Contradiction so $f'$ is not continuous at $x = 0$. $\square$

28.8 (a) *Proof.* Let $\varepsilon > 0$. Then if $|x - 0| < \sqrt{\varepsilon}$, $|x^2| < \varepsilon$. Suppose $x$ is rational. Then $|f(x)| = |x^2| < \varepsilon$. Since $\varepsilon$ was arbitrary, $f$ is continuous at $x = 0$. Now suppose that $x$ is irrational. Then $f(x) = 0$ so $|f(x) - f(0)| = |0 - 0| = 0 < \varepsilon$. Since $\varepsilon$ was arbitrary, $f$ is continuous at $x = 0$. $\square$

(b) *Proof.* We will use proof by contradiction. Suppose that $f$ is continuous at some $x_0 \neq 0$. Suppose $x_0$ is rational. Then there is a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| = |f(x) - x_0^2| < \frac{x_0^2}{2}$. By denseness of the irrationals in $\mathbb{R}$, there is an irrational $x_1 \in \text{Ball}(x_0, \delta)$. Then, $f(x_1) = 0$ so $|f(x_1) - x_0^2| = |0 - x_0^2| = x_0^2 > \frac{x_0^2}{2}$. Contradiction so $f$ is not continuous at $x_0$. Since $x_0$ was arbitrary, $f$ is discontinuous at all $x \neq 0$.

Now suppose that $x_0$ is irrational. Then there is a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| = |f(x) - 0| < x_0^2$. By denseness of the rationals in $\mathbb{R}$, there is a rational $x_2$ such that $x_0 < x_2 < x_0 + \delta$ so $f(x_2) = x_2^2 > x_0^2$ if $x_0 > 0$. If $x_0 < 0$, then there is a rational $x_2$ such that $x_0 - \delta < x_2 < x_0$ so $f(x_2) = x_2^2 > x_0^2$. Then $|f(x_2) - f(x_0)| = x_2^2 > x_0^2$. Contradiction so $f$ is not continuous at $x_0$. Since $x_0$ was arbitrary, $f$ is discontinuous at all $x \neq 0$. $\square$

(c) *Proof.* Let $\varepsilon > 0$. If $x$ is rational and nonzero, then $|x - 0| < \varepsilon$ implies $\varepsilon > |x| = |\frac{x^2}{x}| = |\frac{f(x)}{x} - 0|$. If $x$ is irrational, then $f(x) = 0$ so $\frac{f(x)}{x} = 0$ and hence $|\frac{f(x)}{x}| < \varepsilon$ for all irrational $x$. Since $\varepsilon$ was arbitrary, $\lim_{x\to 0} \frac{f(x)}{x} = 0$. Hence, $f$ is differentiable at $x = 0$. $\square$

29.4 *Proof.* Let $h(x) = f(x)e^{g(x)}$. Then $f(a) = 0$ so $h(a) = f(a)e^{g(a)} = 0$ and $f(b) = 0$ so $h(b) = f(b)e^{g(b)} = 0$. Then $g(x)$ is differentiable so $e^{g(x)}$ is differentiable by the Chain Rule. Thus $h(x)$ is differentiable and $h'(x) = f'(x)e^{g(x)} + f(x)g'(x)e^{g(x)} = e^{g(x)}(f'(x) + f(x)g'(x))$. By Rolle's Theorem, there is some $x \in (a, b)$ such that $h'(x) = e^{g(x)}(f'(x) + f(x)g'(x)) = 0$ so there is some $x \in (a, b)$ such that $f'(x) + f(x)g'(x) = 0$. $\square$

29.10 (a) *Proof.* Let $g(x) = x^2 \sin \frac{1}{x}$. Then by Exercise (28.4) above, $g'(0) = 0$. Hence, $f'(x) = g'(x) + \frac{d}{dx}(\frac{x}{2}) = g'(x) + \frac{1}{2}$ so $f'(0) = g'(0) + \frac{1}{2} = \frac{1}{2} > 0$ □

(b) *Proof.* We will use proof by contradiction. Suppose $f$ is increasing on an open interval $I$ containing 0. Then $f'(x) \geq 0$ for all $x \in I$. For $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} + \frac{x}{2}$. Then there is some $x_0 \in I$ such that $\sin \frac{1}{x_0} = 0$ and $\cos \frac{1}{x_0} = 1$ since $\sin \frac{1}{x}$ and $\cos \frac{1}{x}$ fluctuate between -1 and 1 as $x$ approaches 0. Then $f'(x_0) = 0 - 1 + \frac{1}{4\pi} = \frac{1}{4\pi} - 1 < 0$. Contradiction so $f$ is not increasing on $I$. Since $I$ was arbitrary, $f$ is not increasing on any open interval containing 0. □

(c) Corollary 29.7(i) requires a positive derivative throughout the interval, while in part (a) of the exercise above we only showed it was positive at one point.

29.14 *Proof.* We will first show that $f(x) \geq x$ for all $x \geq 0$ using contradiction. Suppose $f(x) < x$ for some $x \geq 0$. Then let $g(x) = f(x) - x$ so $g(x) < 0$ for some $x \geq 0$. $f$ and $x$ are differentiable on $\mathbb{R}$ so $g$ is differentiable on $\mathbb{R}$ and $g'(x) = f'(x) - 1 < 0$ for some $x \geq 0$ so $f'(x) < 1$ for some $x \geq 0$. Contradiction as $1 \leq f'(x)$ for all $x \in \mathbb{R}$. Therefore, $f(x) \geq x$ for all $x \geq 0$.

Now we will show that $f(x) \leq 2x$ for all $x \geq 0$ using contradiction. Suppose $f(x) > 2x$ for some $x \geq 0$. Then let $h(x) = f(x) - 2x$ so $h(x) > 0$ for some $x \geq 0$. $f$ and $2x$ are differentiable on $\mathbb{R}$ so $h$ is differentiable on $\mathbb{R}$ and $h'(x) = f'(x) - 2 > 0$ for some $x \geq 0$ so $f'(x) > 2$ for some $x \geq 0$. Contradiction as $f'(x) \leq 2$ for all $x \in \mathbb{R}$. Therefore, $f(x) \leq 2x$ for all $x \geq 0$.

Therefore, $x \leq f(x) \leq 2x$ for all $x \geq 0$ □