- Proof of Case 4 of L'Hopital's Theorem: $s = +\infty, L \in \mathbb{R}, \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$

Proof. $lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$ so there is a $M \in \mathbb{R}$ such that if $x > M$, then $\frac{f'(x)}{g'(x)} < L_1$ for all $L_1 > L$. Hence, there is an open interval (b, ∞) where $g'(x) \neq 0$ for all $x \in (b, \infty)$ so by the Intermediate Value Theorem for derivatives, g' is either strictly positive or strictly negative in (b, ∞). Without loss of generality, consider g' to be negative so g is strictly decreasing in (b, ∞). Note that the other case follows by a symmetric argument using $g \to -g$. Since $\lim_{x\to\infty} g(x) = 0$, then by possibly increasing b we can insure g is strictly positive on (M, ∞). Let $\alpha_1 = \max\{M, b\}$. Then for all $x, y \in (\alpha_1, \infty)$ with $x > y$ there is a $z \in (y, x)$ such that $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)}$ by the Generalized Mean Value Theorem. Therefore, $x, y \in (\alpha_1, \infty)$ with $x > y$ implies $\frac{f(x)-f(y)}{g(x)-g(y)} < L_1$. Then $\frac{f(y)}{g(y)} = \lim_{x\to\infty} \frac{f(x)-f(y)}{g(x)-g(y)} < L_1$ for all $y \in (\alpha_1, \infty)$. By a similar argument for all $L_2 < L$, there is a α_2 such that $x \in (\alpha_2, \infty)$ implies $\frac{f(x)}{g(x)} > L_2$. Since L_1 and L_2 were arbitrary, by the definition of the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ □

30.2 (a) $\lim_{x\to 0} \frac{x^3}{\sin x - x} = \lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \lim_{x\to 0} \frac{6x}{-\sin x} = \lim_{x\to 0} \frac{6}{-\cos x} = -6$

(b) $\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\cos^{-2} x - 1}{3x^2} = \lim_{x\to 0} \frac{2\cos^{-3} x \sin x}{6x} = \lim_{x\to 0} \frac{\frac{\tan x}{\cos^2 x}}{3x} = \lim_{x\to 0} \frac{\tan x}{3x\cos^2 x} = \lim_{x\to 0} \frac{\sec^2 x}{3\cos^2 x - 6x\cos x \sin x} = \frac{1}{3-0} = \frac{1}{3}$

(c) $\lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{1+1-0} = 0$

(d) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = \exp \lim_{x\to 0} \ln(\cos x)^{\frac{1}{x^2}} = \exp \lim_{x\to 0} \frac{1}{x^2} \ln \cos x = \exp \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x} = \exp \lim_{x\to 0} \frac{-\tan x}{2x} = \exp \lim_{x\to 0} \frac{-\sec^2(x)}{2} = \exp(\frac{-1}{2}) = \frac{1}{\sqrt{e}}$

30.7 (a) *Proof.* $\cos x \sin x \geq -1$ so $f(x) \geq x - 1$. Then by the Squeeze Lemma, $\lim_{x\to\infty} f(x) \geq \lim_{x\to\infty} x - 1 = \infty$. Likewise, $g(x) = e^{\sin x}(x + \cos x \sin x) \geq e^{\sin x}(x-1) \geq e^{-1}(x-1) = \frac{x-1}{e}$. Then by the Squeeze Lemma, $\lim_{x\to\infty} f(x) \geq \lim_{x\to\infty} \frac{x-1}{e} = \infty$. Therefore, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ □

(b) $f'(x) = 1 + \cos^2(x) - \sin^2(x) = 2\cos^2(x)$ and $g'(x) = e^{\sin x}\cos x(x + \cos x \sin x) + e^{\sin x} f'(x) = e^{\sin x}\cos x(x + \cos x \sin x + 2\cos x) = e^{\sin x}\cos x(f(x) + 2\cos x)$

(c) $\frac{f'(x)}{g'(x)} = \frac{2\cos^2 x}{e^{\sin x}\cos x(f(x)+2\cos x)} = \frac{2e^{-\sin x}\cos x}{f(x)+2\cos x}$ if $\cos x \neq 0$ and $x > 3$

(d) $\left|\frac{2e^{-\sin x}\cos x}{f(x)+2\cos x}\right| \leq \left|\frac{2e^{-\sin x}}{2\cos x + f(x)}\right| \leq \left|\frac{2e^1}{2\cos x + f(x)}\right| \leq \left|\frac{2e}{f(x)-2}\right|$.

Then $\lim_{x\to\infty} \left|\frac{2e}{f(x)-2}\right| = 0$ so by the Squeeze Lemma, $\lim_{x\to\infty} \left|\frac{2e^{-\sin x}\cos x}{f(x)+2\cos x}\right| = 0$.

Therefore, $\lim_{x\to\infty} \frac{2e^{-\sin x}\cos x}{f(x)+2\cos x} = 0$. But, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f(x)}{e^{\sin x} f(x)} = \lim_{x\to\infty} \frac{1}{e^{\sin x}}$ since $\lim_{x\to\infty} f(x) = \infty$. Then, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{e^{\sin x}}$, which does not exist.

31.2 *Proof.* $f(x) = \sinh(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j = \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} x^{2j+1} = \sum_{j=0}^{\infty} \frac{\frac{1}{2}(1-(-1)^j)}{j!} x^j = \frac{1}{2}\sum_{j=0}^{\infty}(\frac{x^j}{j!} - \frac{(-x)^j}{j!}) = \frac{1}{2}(e^x - e^{-x})$. Then there is some y between 0 and x such that $R_n(x) = f^{(n)}(y)\frac{x^n}{n!} \leq \left|\frac{\cosh(x)x^n}{n!}\right|$ so $|R_n(x)| \leq \frac{|\cosh(x)||x|^n}{n!}$. Then $\lim_{n\to\infty} \frac{|\cosh(x)||x|^n}{n!} = 0$ by the Ratio Test so $\lim_{n\to\infty} |R_n(x)| = 0$ by the Squeeze Lemma for all $x \in \mathbb{R}$. Thus, $\sum_{j=0}^{\infty} \frac{\frac{1}{2}(1-(-1)^j)}{j!} x^j$ converges pointwise to $\sinh(x)$ for all $x \in \mathbb{R}$. $\cosh(x) = f'(x) = \frac{1}{2}(e^x + e^{-x})$ so by Theorem (26.5), $f'(x)$ also converges pointwise to $\cosh(x)$ for all $x \in \mathbb{R}$ □

31.4 (a) Let $f_a(x) = \begin{cases} e^{-\frac{1}{x-a}} & x > a \\ 0 & x \leq a \end{cases}$

Then $f_a(x) = f(x-a)$ which is infinitely differentiable on $\mathbb{R}$.

(b) Let $g_b(x) = \begin{cases} e^{-\frac{1}{b-x}} & x < b \\ 0 & x \geq b \end{cases}$

Then $g_b(x) = g(b-x)$ which is infinitely differentiable on $\mathbb{R}$.

(c) Let $h_{a,b}(x) = \begin{cases} e^{-\frac{1}{x-a}-\frac{1}{b-x}} & a < x < b \\ 0 & x \leq a, x \geq b \end{cases}$
Then $h_{a,b}(x) = f_a(x)g_b(x)$ which is a product of infinitely differentiable functions on $\mathbb{R}$ and so it is also infinitely differentiable on $\mathbb{R}$.

(d) Let $h^*_{a,b}(x) = \begin{cases} 0 & x \leq a \\ \frac{e^{-\frac{1}{x-a}}}{e^{-\frac{1}{x-a}}+e^{-\frac{1}{b-x}}} & a < x < b \\ 1 & x \geq b \end{cases}$
Then $h^*_{a,b}(x) = \frac{f_a(x)}{f_a(x)+g_b(x)}$ which is a quotient of infinitely differentiable functions on $\mathbb{R}$ and so it is also infinitely differentiable on $\mathbb{R}$. Note that $(f_a + g_b)(x) > 0$ for all $x \in \mathbb{R}$ so the denominator is never zero.

32.2 (a) $U(f) = \inf\{\sum_{k=1}^n \lambda(A_k) \sup f(A_k)\}$ where $A_k = [t_{k-1}, t_k]$ is a segment of a partition P of $[0, b]$. Then $U(f) = \inf\{\sum_{j=1}^n (t_k - t_{k-1})t_k\} = \inf\{\sum_{k=1}^n t_k^2 - t_k t_{k-1}\}$. Then $t_k = \frac{kb}{n}$ so $U(f) = \inf\{\sum_{j=1}^n \frac{k^2b^2}{n^2} - \frac{kb}{n} \cdot \frac{(k-1)b}{n}\} = \inf\{\frac{b^2}{n^2}\sum_{k=1}^n k^2 - k(k-1)\} = \inf\{\frac{b^2}{n^2}\sum_{k=1}^n k\} = \inf\{\frac{b^2}{n^2} \cdot \frac{n(n+1)}{2}\} = \inf\{\frac{b^2(n+1)}{2n}\} = \frac{b^2}{2}$ for large n. Since k and P were arbitrary, then $U(f) = \frac{b^2}{2}$.
$L(f) = \sup\{\sum_{k=1}^n \lambda(A_k) \inf f(A_k)\}$. For any subset A_k of $[0, b]$ there is an irrational $x_0 \in A_k$ by denseness so $\inf f(A_k) = 0$ for all k. Therefore, $L(f) = 0$.

(b) f is not integrable on $[0, b]$ because $U(f) \neq L(f)$

32.6 *Proof.* Let $\varepsilon > 0$. Let $U_n = U(f, P_n)$ and $L_n = L(f, Q_n)$. Then $\lim_{n\to\infty}(U_n - L_n) = 0$ so there is a $N \in \mathbb{N}$ such that if $n > N$, then $\varepsilon > |U_n - L_n| = U_n - L_n$ by Lemma (32.3). Thus, $U_{N+1} - L_{N+1} < \varepsilon$. Let $R = P_{N+1} \cup Q_{N+1}$. Then by Lemma (32.2), $U_{N+1} \geq U(f, R)$ and $L_{N+1} \leq L(f, R)$ so $-L_{N+1} \geq -L(f, R)$. Therefore, $\varepsilon > U_{N+1} - L_{N+1} \geq U(f, R) - L(f, R)$. Since ε was arbitrary and f is bounded on $[a, b]$, f is integrable by Theorem (32.5). Then $L_n \leq L(f) \leq \int_a^b f \leq U(f) \leq U_n$ so $0 \leq \int_a^b f - L_n \leq U_n - L_n$ and $|\int_a^b f - U_n| = U_n - \int_a^b f \leq U_n - L_n$. Since $\lim_{n\to\infty}(U_n - L_n) = 0$, by the Squeeze Lemma $\lim_{n\to\infty} L_n = \int_a^b f = \lim_{n\to\infty} U_n$ □

33.8 (a) *Proof.* f and g are integrable so $f + g$ and $f - g$ are integrable. Then by Exercise (33.7), $(f + g)^2$ and $(f - g)^2$ are integrable so $(f + g)^2 - (f - g)^2 = 4fg$ is integrable. Therefore, fg is integrable. □

(b) *Proof.* By Exercise (17.8), $\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$ and $\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$. Then f and g are integrable so $f + g$ and $f - g$ are integrable. Thus, $\frac{1}{2}(f + g)$ and $\frac{1}{2}|f - g|$ are integrable so $\min(f, g)$ and $\max(f, g)$ are integrable. □

33.10 *Proof.* Let $\varepsilon > 0$. Then f is continuous on $[\frac{\varepsilon}{12}, 1]$ so it is integrable on $[\frac{\varepsilon}{12}, 1]$. Hence, there is a partition P_1 of $[\frac{\varepsilon}{12}, 1]$ such that $|U(f, P_1) - L(f, P_1)| < \frac{\varepsilon}{3}$. Similarly, f is continuous on $[-1, -\frac{\varepsilon}{12}]$ so it is integrable on $[-1, -\frac{\varepsilon}{12}]$. Hence, there is a partition P_2 of $[-1, -\frac{\varepsilon}{12}]$ such that $|U(f, P_2) - L(f, P_2)| < \frac{\varepsilon}{3}$. Then on $[-\frac{\varepsilon}{12}, \frac{\varepsilon}{12}]$ for any interval $A_j \subset [-\frac{\varepsilon}{12}, \frac{\varepsilon}{12}]$, $\sup f(A_j) = 1$ and $\inf f(A_j) = -1$. Let $P = P_1 \cup P_2$. So, $U(f, P) = \sum_{k=1}^n \lambda(t_l, t_{k-1}) \sup f(A_k) = U(f, P_2) + 2 \cdot \frac{\varepsilon}{12} \cdot 1 + U(f, P_1)$. Likewise, $L(f, P) = L(f, P_2) + 2 \cdot \frac{\varepsilon}{12} \cdot (-1) + L(f, P_1)$. Then, $|U(f, P) - L(f, P)| = |U(f, P_2) + \frac{\varepsilon}{6} + U(f, P_1) - L(f, P_2) + \frac{\varepsilon}{6} - L(f, P_1)| \leq |U(f, P_2) - L(f, P_2)| + |U(f, P_1) - L(f, P_1)| + \frac{\varepsilon}{3}$ by the Triangle Inequality. Thus, $|U(f, P) - L(f, P)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Since ε was arbitrary, f is integrable on $[-1, 1]$ □

33.14 (a) *Proof.* By the Intermediate Value Theorem, there is a $x \in (a, b)$ such that:
$f(x)g(x) = \frac{1}{b-a}\int_a^b f(t)g(t)dt$. Therefore, there is a $x \in (a, b)$ such that $\int_a^b f(t)g(t)dt = (b - a)f(x)g(x) = f(x)[(b - a) \cdot g(x)] = f(x)\int_a^b g(t)dt$ □

(b) *Proof.* Theorem (33.9) is the special case where $g(x) = 1$. By part (a), there is a $x \in (a, b)$ such that $\int_a^b f(t) \cdot 1dt = f(x)\int_a^b 1dt = f(x)(b - a)$. Therefore, there is a $x \in (a, b)$ such that $f(x) = \frac{1}{b-a}\int_a^b f(t)dt$, which is Theorem (33.9). □

(c) *Proof.* The left-hand side is $\int_{-1}^{1} t^2 dt = \frac{1}{3}|_{-1}^{1} = \frac{1}{3} - \frac{-1}{3} = \frac{2}{3}$. However, the right-hand side is $f(x)\int_{-1}^{1} g(t)dt = x(\frac{1}{2}t^2)|_{-1}^{1} = x(\frac{1}{2} - \frac{1}{2}) = x \cdot 0$. However, there is no $x \in (-1, 1)$ such that $\frac{2}{3} = x \cdot 0$. Therefore, the conclusion in part (a) does not hold in this case. $\square$

34.2 (a) Let $F(x) = \int_0^x e^{x^2}$. Then, $\lim_{x\to 0} \frac{F(x)}{x} = F'(0) = e^0 = 1$

(b) Let $F(x) = \int_3^x e^{x^2}$ so $F(3) = 0$ and $F'(x) = e^{x^2}$ by the Fundamental Theorem of Calculus. Then, $\lim_{h\to 0} \frac{\int_3^{h+3} e^{h^2}}{h} = \lim_{h\to 0} \frac{F(h+3)}{h} = \lim_{h\to 0} \frac{F(h+3)-0}{h} = \lim_{h\to 0} \frac{F(h+3)-F(3)}{h} = F'(3) = e^{3^2} = e^9$

34.6 *Proof.* $G(x) = \int_0^{\sin x} f(u)du = \int_0^x f(\sin t)\cos t dt$ by the Chain Rule. f and cos are continuous on $\mathbb{R}$ so their product $f \cdot \cos$ is also continuous on $\mathbb{R}$ and thus by the Fundamental Theorem of Calculus, G is differentiable on $\mathbb{R}$ and $G'(x) = f(\sin x)\cos x$. $\square$

34.12 *Proof.* f is continuous on $[a.b]$ so let $g = f$. Then $\int_a^b f(x)^2 dx = 0$. f is continuous on $[a, b]$ so $f^2 = f \cdot f$ is continuous and nonnegative on $[a, b]$. Thus, by Theorem (33.4b), $f^2(x) = f(x)^2 = 0$ for all $x \in [a, b]$ Therefore, $f(x) = 0$ for all $x \in [a, b]$. $\square$

36.4 (a) $\int_0^1 \ln x dx = \lim_{a\to 0^+} \int_a^1 \ln x dx = \lim_{a\to 0^+} (x \ln x - x)|_a^1 = \lim_{a\to 0^+} (0-1-a \ln a + a) = \lim_{a\to 0^+} (a(1-\ln a)) - 1 = \lim_{a\to 0^+} \frac{1-\ln a}{\frac{1}{a}} - 1 = \lim_{a\to 0^+} \frac{\frac{-1}{a}}{\frac{-1}{a^2}} - 1 = \lim_{a\to 0^+} a - 1 = 0 - 1 = -1$

(b) $\int_2^\infty \frac{\ln x}{x} dx = \lim_{b\to\infty} \int_2^b \frac{\ln x}{x} dx = \lim_{b\to\infty} (\frac{(\ln x)^2}{2}|_2^b) = \lim_{b\to\infty} \frac{(\ln b)^2 - (\ln 2)^2}{2} = \infty$

(c) $\int_0^\infty \frac{1}{1+x^2} dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2} dx = \lim_{b\to\infty} (\arctan x)|_0^b = \lim_{b\to\infty} (\arctan b - \arctan 0)$
$= \lim_{b\to\infty} \arctan b - 0 = \frac{\pi}{2}$