10.6 (a) *Proof.* Let $\varepsilon > 0$. Let $N = log_2(\frac{2}{\varepsilon})$. Then if $n > N$, $n > log_2(\frac{2}{\varepsilon})$. So, $n > 1 + log_2(\frac{1}{\varepsilon})$ and thus $2^{n-1} > \frac{1}{\varepsilon}$. Rearranging, we see that $\varepsilon > \frac{1}{2^{n-1}} = \frac{\frac{1}{2^n}}{1-\frac{1}{2}} = \sum_{i=n}^{\infty} 2^{-i} > \sum_{i=n}^{n+k-1} 2^{-i}$ for some $k \in \mathbb{Z}^+$. Then, this is equal to $2^{-(n+k-1)} + 2^{-(n+k-2)} + \cdots + 2^{-n} > |s_{n+k} - s_{n+k-1}| + |s_{n+k-1} - s_{n+k-2}| + \cdots + |s_{n+1} - s_n| \geq |s_{n+k} - s_{n+k-1} + s_{n+k-1} - s_{n+k-2} + \cdots + s_{n+1} - s_n|$ by the Triangle Inequality. Let $m = n + k$ so $\varepsilon > |s_{n+k} - s_n| = |s_m - s_n|$. Since k and n were arbitrary, m is also arbitrary so this inequality holds for all $m, n > N$. Since ε was arbitrary, $< s_n >$ is a Cauchy sequence and hence convergent by Theorem 10.11. $\square$

(b) No. We will show this by giving a counter-example. Let $s_n = \sum_{i=1}^{n} \frac{1}{i}$. Then, $|s_{n+1} - s_n| = |\frac{1}{n+1}| = \frac{1}{n+1} < \frac{1}{n}$ for all $n \in \mathbb{N}$. But, s_n is the Harmonic Series and so it diverges to positive infinity. Thus, it is not a Cauchy sequence. Therefore, the result in (a) is not true if we only assume that $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$.

10.8 *Proof.* Let $< s_n >$ be an increasing sequence of positive numbers. Let $n \in \mathbb{N}$. Then since $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$, $\sigma_{n+1} = \frac{1}{n+1}(s_1 + s_2 + \cdots + s_n + s_{n+1}) = \frac{1}{n+1}(n\sigma_n + s_{n+1}) = \sigma_n - \frac{1}{n+1}\sigma_n + \frac{1}{n+1}s_{n+1}$. Since $< s_n >$ is increasing, $s_i < s_{n+1}$ for all $i \in \mathbb{N}_{<n+1}$ so $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n) < \frac{1}{n}(s_{n+1} + s_{n+1} + \cdots + s_{n+1}) = \frac{1}{n} \cdot n s_{n+1} = s_{n+1}$. Thus, $\frac{1}{n+1}\sigma_n < \frac{1}{n+1}s_{n+1}$ so $-\frac{1}{n+1}\sigma_n + \frac{1}{n+1}s_{n+1} > 0$. Then, $\sigma_{n+1} - \sigma_n = -\frac{1}{n+1}\sigma_n + \frac{1}{n+1}s_{n+1} > 0$ and thus $\sigma_{n+1} > \sigma_n$. Since n was arbitrary, this is true for all $n \in \mathbb{N}$. Since $< s_n >$ was arbitrary, $< \sigma_n >$ is an increasing sequence. $\square$

10.10 (a) $s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$, $s_3 = \frac{1}{3}(\frac{2}{3}+1) = \frac{5}{9}$, and $s_4 = \frac{1}{3}(\frac{5}{9}+1) = \frac{14}{27}$.

(b) *Proof.* We will prove by induction. Note that $s_1 = 1 > \frac{1}{2}$ so the claim holds for $n = 1$. Suppose the claim holds for some $n \in \mathbb{N}$. Then, $s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$ so the claim holds for n+1. Since n was arbitrary, the claim holds for n+1 whenever it holds for n. Therefore, by induction, $s_n > \frac{1}{2}$ for all n. $\square$

(c) By definition, $s_{n+1} = \frac{1}{3}(s_n + 1)$ so $3s_{n+1} = s_n + 1$. By Part (b) above, $s_n > \frac{1}{2}$ for all n so $2s_n > 1$ and thus $s_n + 1 < s_n + 2s_n = 3s_n$. Thus, $3s_{n+1} = s_n + 1 < 3s_n$ and so $s_{n+1} < s_n$ for all n. Therefore, $< s_n >$ is a decreasing sequence.

(d) By parts (b) and (c) above, $< s_n >$ is a decreasing sequence that is bounded below and so its limit exists by Theorem 10.2. Let $s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1}$. Then, $\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty}(\frac{1}{3}(s_n + 1)) = \frac{1}{3}\lim_{n\to\infty} s_n + \frac{1}{3}$ by limit properties so $s = \frac{1}{3}s + \frac{1}{3}$ so $3s = s + 1$ and hence $s = 2$. Therefore, $\lim_{n\to\infty} s_n = \frac{1}{2}$.

11.2 (a) $< a_{n_k} >=< 1, 1, 1, \cdots >$ is a monotone subsequence of $< a_n >$. $< b_n >$, $< c_n >$, and $< d_n >$ are already monotone subsequences so the subsequence consisting of the entire sequence is a monotone subsequence.

(b) The subsequential limits of the sequences are 1 and -1 for $< a_n >$, 0 for $< b_n >$, $+\infty$ for $< c_n >$, and $\frac{6}{7}$ for $< d_n >$.

(c) $\limsup_{n\to\infty} a_n = 1$, $\liminf_{n\to\infty} a_n = -1$, $\limsup_{n\to\infty} b_n = \liminf_{n\to\infty} b_n = 0$, $\limsup_{n\to\infty} c_n = \liminf_{n\to\infty} c_n = +\infty$, $\limsup_{n\to\infty} d_n = \liminf_{n\to\infty} d_n = \frac{6}{7}$.

(d) $< a_n >$ diverges and its limit is undefined, $< b_n >$ converges to 0, $< c_n >$ diverges to $+\infty$, and $< d_n >$ converges to $\frac{6}{7}$

(e) $< a_n >$, $< b_n >$, and $< d_n >$ are bounded. $< c_n >$ is not bounded.

11.4 (a) $< w_{n_k} >=< 4, 16, 64, \cdots >=< 4^k >$ is a monotone subsequence of $< w_n >$. $< x_{n_k} >=< 5, 5, 5, \cdots >$ is a monotone subsequence of $< x_n >$. $< y_{n_k} >=< 0, 0, 0, \cdots >$ is a monotone subsequence of $< y_n >$. $< z_{n_k} >=< 0, 0, 0, \cdots >$ is a monotone subsequence of $< z_n >$.

(b) The subsequential limits of the sequences are $+\infty$ and $-\infty$ for $< w_n >$, 5 and 5^{-1} for $< x_n >$, 2 and 0 for $< y_n >$, and 0 and $\pm\infty$ for $< z_n >$.

(c) $\limsup_{n\to\infty} w_n = +\infty$, $\liminf_{n\to\infty} w_n = -\infty$, $\limsup_{n\to\infty} x_n = 5$, $\liminf_{n\to\infty} x_n = 5^{-1}$, $\limsup_{n\to\infty} y_n = 2$, $\liminf_{n\to\infty} y_n = 0$, $\limsup_{n\to\infty} z_n = +\infty$, and $\liminf_{n\to\infty} z_n = -\infty$.

(d) All four sequences diverge with undefined limits.

(e) $< x_n >$ and $< y_n >$ are bounded. $< w_n >$ and $< z_n >$ are not bounded.

11.6 *Proof.* Let $< a_n >$ be a sequence and $< b_k >=< a_{n_k} >$ be a subsequence of $< a_n >$. Then, let $< c_m >=< b_{k_m} >$ be a subsequence of $< b_k >$. By definition (3) of 11.1, there are natural functions σ and ρ given by $\sigma(k) = n_k$ and $\rho(m) = k_m$ for $k, m \in \mathbb{N}$. The functions σ and ρ "select" an infinite subset of $\mathbb{N}$ in order. We can then define the subsequence of a corresponding to σ as $b = a \circ \sigma$ and the subsequence of b corresponding to ρ as $c = b \circ \rho$. Then, $c_m = c(m) = b \circ \rho(m) = b(\rho(m)) = b(k_m) = a \circ \sigma(k_m) = a(\sigma(k_m)) = a(n_{k_m}) = a_{n_{k_m}}$. Hence, c is a subsequence of a since $c = a \circ (\sigma \circ \rho)$ where $\sigma \circ \rho$ is an increasing function mapping $\mathbb{N}$ into $\mathbb{N}$. Since a, b, and c were arbitrary, the claim holds for any sequence and its subsequences. $\square$

11.10 (a) S = $\{0\} \cup \{\frac{1}{k} | k \in \mathbb{N}\}$

(b) S is the set of subsequential limits, so by Theorem 11.2, S is the same as the set of accumulation points. Thus, $\limsup_{n\to\infty} s_n = 1$ and $\liminf_{n\to\infty} s_n = 0$.