13.10 (a) *Proof.* We will use proof by contradiction. Suppose $S \neq \emptyset$ is an open subset of $\{\frac{1}{n} : n \in \mathbb{N}\}$. Then $S = \{\frac{1}{n}$: some n $\in \mathbb{N}\}$. Let n_0 be the smallest such n. Such a n_0 must exist since any nonempty subset of $\mathbb{N}$ has a minimum element in the subset. Then max(S) $= \frac{1}{n_0} \in S$. Since S is open, there is some $\varepsilon > 0$ such that Ball($\frac{1}{n_0}, \varepsilon$) is in S. However, since $\frac{1}{n_0}$ is the maximum of S, $\frac{1}{n_0} + \varepsilon > \frac{1}{n_0}$ so no such open ball can exist. Contradiction so S is not open. Since S was arbitrary, $\emptyset$ is the largest open set contained within $\{\frac{1}{n} : n \in \mathbb{N}\}$ so interior($\{\frac{1}{n} : n \in \mathbb{N}\}$) $= \emptyset$. □

(b) *Proof.* We will use proof by contradiction. Suppose $A \neq \emptyset$ is an open subset of $\mathbb{Q}$. Let $a \in A$. Since A is open, there is some $\varepsilon > 0$ such that Ball(a,ε) is in A. By denseness, there is always some irrational number between a and $a + \varepsilon$ so no such open ball can exist. Since a was arbitrary, contradiction so A is not open. Since A was arbitrary, $\emptyset$ is the largest open set contained within $\mathbb{Q}$ so interior($\mathbb{Q}$) $= \emptyset$. □

(c) *Proof.* We will use proof by contradiction. Suppose $A \neq \emptyset$ is an open subset of F. Then A is in the intersection of some F_i's. Since $1 \in F_i$ for all $i \in \mathbb{N}$, $1 \in A$. Since A is open, there is some $\varepsilon > 0$ such that Ball(1, ε) is in A. Since max(A) = 1, a+ε > a so no such open ball can exist. Contradiction so A is not open. Since A was arbitrary, $\emptyset$ is the largest open set contained within F so interior(F) $= \emptyset$. □

13.13 *Proof.* E is a compact subset of $\mathbb{R}$ so by the Heine-Borel Theorem, E is bounded and closed. Since E is bounded and nonempty, sup(E) and inf(E) exist by completeness. Then by Homework (10.7) and (11.11), there is an E-valued sequence $< a_n >$ such that $\lim_{n\to\infty} a_n$ =sup(E). Since E is closed, sup(E) $\in$ E. Likewise, there is an E-valued sequence $< b_n >$ such that $\lim_{n\to\infty} b_n$ =inf(E). Since E is closed, inf(E) $\in$ E. Therefore, sup(E) and inf(E) belong to E. □

14.12 (a) $\sum \frac{1}{n}$ diverges so $\sum \frac{1}{2n}$ diverges as well. Since $\frac{n-1}{n^2} > \frac{n-\frac{n}{2}}{n^2} = \frac{1}{2n}$, by the Comparison Test, $\sum \frac{n-1}{n^2}$ diverges.

(b) $\lim_{n\to\infty}(-1)^n \neq 0$ so $\sum(-1)^n$ diverges by the Preliminary Test.

(c) $\frac{3n}{n^3} = \frac{3}{n^2}$. $\sum \frac{1}{n^2}$ converges so $\sum \frac{3}{n^2} = 3\sum \frac{1}{n^2}$ converges as well.

(d)
$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{3n^3} \right| = \frac{1}{3} < 1$$

So by the Ratio Test, the series converges.

(e)
$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n^2} \right| = 0 < 1$$

So by the Ratio Test, the series converges.

(f) $\lim_{n\to\infty} |\frac{1}{n^n}|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$ so by the Root Test, the series converges.

(g)
$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)}{2^{n+1}}}{\frac{n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2n} \right| = \frac{1}{2} < 1$$

So by the Ratio Test, the series converges.