21.10 (a) $f:(0,1)\to[0,1]\ x\mapsto \frac{1}{2}\sin(2\pi x)+\frac{1}{2}$ is a continuous function that maps (0,1) onto [0,1]

(b) $g:(0,1)\to\mathbb{R}\ x\mapsto \tan(\pi(x-\frac{1}{2}))$ is a continuous function that maps (0,1) onto $\mathbb{R}$

(c) $h:[0,3]\to[0,1]\ x\mapsto \begin{cases} x & x\in[0,1] \\ 1 & x\in(1,3]\end{cases}$ is a continuous function that maps $[0,1]\cup[2,3]$ onto [0,1]

(d) Suppose there is a continuous function f mapping [0,1] onto (0,1) or $\mathbb{R}$. [0,1] is closed and bounded so by Heine-Borel Theorem, it is compact so f([0,1]) must also be compact. But, neither (0,1) nor $\mathbb{R}$ are compact so no such f exists.

21.12 Let $S_n=(n-1,n)$ for $n\in\mathbb{N}$. Then $<S_n>=<(0,1),(1,2),(2,3),\cdots>$ is an infinite disjoint sequence of subsets of $\mathbb{R}$. Since interior(closure(S_n))=interior([n-1,n])$\neq\varnothing$ so it is not nowhere dense and hence of second category in $\mathbb{R}$

22.4 (a) Closure(E) $=E\cup(\{0\}\times[-1,1])$

(b) By Exercise (22.3), it suffices to prove that E is connected. Let $(s,\sin(\frac{1}{s}))$ and $(t,\sin(\frac{1}{t}))$ be elements of E. Then there is a continuous function $f:[s,t]\to E\ x\mapsto(x,\sin(\frac{1}{x}))$ such that $f(s)=(s,\sin(\frac{1}{s}))$ and $f(t)=(t,\sin(\frac{1}{t}))$. Since s and t were are arbitrary, E is path-connected and therefore E is connected.

(c) Suppose Closure(E) is path-connected. Then there is a continuous function $f:[0,1]\to$ Closure(E) where f $=(f_1,f_2)$. Let $t_0=\inf\{t\in[0,1]:f_1(t)>0\}$. By continuity of f, f_1 and f_2 are continuous so there is a $\delta>0$ such that $t_0<t<t_0+\delta$ implies $|f_2(t)-f_2(t_0)|<1$. Then let $t_1\in(t_0,t_0+\delta)$ so $f_1(t_1)>0$. By continuity and the Intermediate Value Theorem, $f_1([t_0,t_1])=[0.f_1(t_1)]$. $f_2(t)=\sin(\frac{1}{f_1(t)})$ for all t where $f_1(t)\neq0$ so $f_2([t_0,t_1])=[-1,1]$. Contradiction to $|f_2(t)-f_2(t_0)|<1$ for all $t\in(t_0,t_0+\delta)$. Therefore, Closure(E) is not path connected.

22.6 (a) d(f,g) $=\sup\{|f(x)-g(x)|:x\in S\}\geq0$. d($f$,$g$) $=0\leftrightarrow f=g$
d(f,g) $=\sup\{|f(x)-g(x)|:x\in S\}=\sup\{|g(x)-f(x)|:x\in S\}=$ d(g,f)
d(f,h) $=\sup\{|f(x)-h(x)|:x\in S\}=\sup\{|f(x)-g(x)+g(x)-h(x)|:x\in S\}\leq\sup\{|f(x)-g(x)|:x\in S\}+\sup\{g(x)-h(x)|:x\in S\}=$ d(f,g) + d(g,h).
Since all three of these properties hold, d is a distance function so $C(S)$ is a metric space.

(b) If the functions in $C(S)$ are unbounded, then the difference between two functions may be undefined so d may be undefined and thus $C(S)$ may then no longer be a metric space.

23.2 (a) $\lim_{n\to\infty}\left|\frac{\sqrt{n+1}x^{n+1}}{\sqrt{n}x^n}\right|=|x|<1$ so the radius of convergence is 1. At $x=\pm1$, the summation diverges by the Preliminary Test so the interval of convergence is (-1,1).

(b) $\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)^{\sqrt{n+1}}}}{\frac{x^n}{n^{\sqrt{n}}}}\right|=\lim_{n\to\infty}\left|\frac{n^{\sqrt{n}}}{(n+1)^{\sqrt{n+1}}}x\right|=\sqrt{\lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}x^2\right|}=\sqrt{\lim_{n\to\infty}\left|(\frac{n}{n+1})^{n+1}\cdot\frac{x^2}{n}\right|}=\sqrt{\lim_{n\to\infty}\left|\frac{x^2}{n}\right|}=0<1$ for all x so the interval of convergence is $\mathbb{R}$ and the radius of convergence is $+\infty$

(c) $\lim_{n\to\infty}\left|\frac{x^{(n+1)!}}{x^{n!}}\right|=\lim_{n\to\infty}\left|x^{n!(n+1-1)}\right|<1$ if $|x|<1$ so the radius of convergence is 1. At $x=\pm1$, the summation diverges by the Preliminary Test so the interval of convergence is (-1,1).

(d) $\lim_{n\to\infty}\left|\frac{\frac{3^{n+1}x^{2n+2+1}}{\sqrt{n+1}}}{\frac{s^nx^{2n+1}}{\sqrt{n}}}\right|=\lim_{n\to\infty}\left|\frac{3\sqrt{n}}{\sqrt{n+1}}x^2\right|=|3x^2|<1$ so $|x|<\frac{1}{\sqrt{3}}$ and hence the radius of convergence is $\frac{1}{\sqrt{3}}$. At $x=\pm\frac{1}{\sqrt{3}}$, the summation diverges by the Preliminary Test so the interval of convergence is $(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$

23.4 (a) $\limsup_{n\to\infty}\frac{4+2(-1)^n}{5}=\lim_{N\to\infty}\sup\{\frac{4+2(-1)^n}{5}:n>N\}=\frac{4+2}{5}=\frac{6}{5}$
$\liminf_{n\to\infty}\frac{4+2(-1)^n}{5}=\lim_{N\to\infty}\inf\{\frac{4+2(-1)^n}{5}:n>N\}=\frac{4-2}{5}=\frac{2}{5}$

$\limsup_{n\to\infty}\left|\frac{(\frac{4+2(-1)^{n+1}}{5})^{n+1}}{(\frac{4+2(-1)^n}{5})^n}\right| = \lim_{N\to\infty}\sup\{\left|\frac{(4+2(-1)^{n+1})^{n+1}}{5(4+2(-1)^n)^n} : n > N\right|\} = +\infty$

$\liminf_{n\to\infty}\left|\frac{(\frac{4+2(-1)^{n+1}}{5})^{n+1}}{(\frac{4+2(-1)^n}{5})^n}\right| = \lim_{N\to\infty}\inf\{\left|\frac{(4+2(-1)^{n+1})^{n+1}}{5(4+2(-1)^n)^n} : n > N\right|\} = 0$

(b) Neither $\lim_{n\to\infty} a_n$ nor $\lim_{n\to\infty}(-1)^n a_n$ exist so neither series converges by the Preliminary Test.

(c) From part(a), $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{6}{5}$ so the radius of convergence is $\frac{5}{6}$. If $x = \pm\frac{6}{5}$, the summation diverges by the Preliminary Test so the interval of convergence is $(-\frac{5}{6}, \frac{5}{6})$.

23.6 (a) $\sum_{n=0}^{\infty} a_n R^n$ converges. $R^n = |(-R)^n|$ so $\sum_{n=0}^{\infty} a_n|(-R)^n|$ converges. Hence, $\sum_{n=0}^{\infty} a_n(-R)^n$ is absolutely convergent so it is convergent by Corollary (14.7).

(b) Consider the power series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}x^n$ Then, $\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{n+1}x^{n+1}}{\frac{(-1)^n}{n}x^n}\right| = |x| < 1$ so the radius of convergence is 1. At $x = 1$, $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ converges by the Alternating Series Test since $\lim_{n\to\infty}\frac{1}{n} = 0$. At $x = -1$, $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}(-1)^n = \sum_{n=0}^{\infty}\frac{1}{n}$ is the Harmonic Series so it diverges. Thus, the interval of convergence is (-1,1].

23.8 (a) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{\sin(nx)}{n}$. Since $-1 \leq \sin(nx) \leq 1$, by the Squeeze Lemma, $0 = \lim_{n\to\infty}\frac{-1}{n} \leq \lim_{n\to\infty}\frac{\sin(nx)}{n} \leq \lim_{n\to\infty}\frac{1}{n} = 0$ so $\lim_{n\to\infty}\frac{\sin(nx)}{n} = 0$.

(b) $f_n'(x) = \cos(nx)$. Then, $\lim_{n\to\infty}\cos(nx) = \lim_{n\to\infty}\cos(n\pi)$ at $x = \pi$. Since $\cos(n\pi) = 1$ for all even n and -1 for all odd n, $\cos(n\pi) = (-1)^n$ so $\lim_{n\to\infty}(-1)^n$ doesn't exist. Thus, $\lim_{n\to\infty} f_n'(x)$ need not exist.