1.2 *Proof.* We will prove by induction. Note that $3 = 4(1)^2 - 1$ so formula holds for n=1. Now suppose it holds for some $n \in \mathbb{N}$. Then $3+11+\ldots+(8n-5)+(8(n+1)-5) = 4n^2-n+(8(n+1)-5) = 4n^2+7n+3$. This is equal to $4(n^2+2n+1)-n-1 = 4(n+1)^2-(n+1)$ so the formula holds for n+1. Since n was arbitrary, the formula holds for n+1 whenever it holds for n. Therefore, by induction the formula is true for all $n \in \mathbb{N}$. □

1.6 *Proof.* We will prove by induction. Note that $11^1 - 4^1 = 7$ is divisible by 7, so the claim holds for n=1. Now suppose it holds for some $n \in \mathbb{N}$. Then $11^{n+1} - 4^{n+1} = (7+4)11^n - 4(4)^n = 7(11)^n + 4(11^n - 4^n)$. $7(11)^n$ is clearly divisible by 7. Likewise, $11^n - 4^n$ is divisible by 7 due to the inductive hypothesis. Thus, $7(11)^n + 4(11^n - 4^n)$ is divisible by 7, so the claim holds for n+1. Since n was arbitrary, the claim holds for n+1 whenever it holds for n. Therefore, by induction the claim is true for all $n \in \mathbb{N}$. □

1.8a *Proof.* We will prove by induction. Note that $2^2 = 4 > 3 = 2+1$ so the claim holds for n=2. Now suppose it holds for some $n \in \mathbb{Z}_{\geq 2}$. Then $(n+1)^2 = n^2+2n+1 > n+1+2n+1$ by the inductive hypothesis. Thus, $(n+1)^2$ is greater than $3n+2 > n+2 = (n+1)+1$ and so the claim holds for n+1. Since n was arbitrary, the claim holds for n+1 whenever it holds for n. Therefore, by induction the claim is true for all $n \in \mathbb{Z}_{\geq 2}$. □

1.8b *Proof.* We will prove by induction. Note that $4! = 24 > 16 = 4^2$ so the claim holds for n=5. Now suppose it holds for some $n \in \mathbb{Z}_{\geq 4}$. Then $(n+1)! = (n+1)n! > (n+1)n^2$ by the inductive hypothesis. So, n! is greater than n^3+n^2, which is greater than $(n+1)(n+1)$ by Part (a), which proved that $n^2 > n+1$. Thus, $(n+1)! > (n+1)^2$ so the claim holds for n++1. Since n was arbitrary, the claim holds for n+1 whenever it holds for n. Therefore, by induction the claim is true for all $n \in \mathbb{Z}_{\geq 4}$. □

1.12a For $\underline{n=1}$, $(a+b)^1 = a+b = \binom{1}{0}a^1 + \binom{1}{1}b^1$
For $\underline{n=2}$, $(a+b)^2 = a^2+2ab+b^2 = \binom{2}{0}a^2 + \binom{2}{1}a^1b^1 + \binom{2}{2}b^2$
For $\underline{n=3}$, $(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}b^3$

1.12b

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} && (1)\\
&= \frac{n!(n-k+1)}{k!(n-k)!(n-k+1)} + \frac{n!(k)}{(k-1)!(n-k+1)!(k)} && (2)\\
&= \frac{n!(n-k+1+k)}{k!(n-k+1)!} && (3)\\
&= \frac{(n+1)!}{k!(n-k+1)!} && (4)\\
&= \binom{n+1}{k} && (5)
\end{aligned}$$

Therefore, $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for k = 1, 2, ..., n.

1.12c *Proof.* We will prove by induction. Note that the Binomial Theorem holds for n = 1, as demonstrated in Part (a). Now suppose the Binomial Theorem holds for some $n \in \mathbb{N}$. Then, $(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$. This is then equal to $\sum_{i=0}^{n}\binom{n}{i}a^{n-i+1}b^i + \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^{i+1}$, which can be simplified to $a^{n+1}+b^{n+1}+\sum_{i=1}^{n}(\binom{n}{i}+\binom{n}{i-1})a^{n-i+1}b^i$. By Part (b), the sum of the binomial coefficients simplifies from $\binom{n}{i}+\binom{n}{i-1}$ to $\binom{n+1}{i}$. Thus, $(a+b)^{n+1} = \sum_{i=0}^{n+1}\binom{n+1}{i}a^{n-i+1}b^i$ and so the Binomial Theorem holds for n+1. Since n was arbitrary, the Binomial Theorem holds for n+1 whenever it holds for n. Therefore, by induction, the Binomial Theorem is true for all $n \in \mathbb{N}$. □

2.4 *Proof.* Let $x = \sqrt[3]{5-\sqrt{3}}$. Then $x^3 = 5-\sqrt{3}$ and $x^3 - 5 = -\sqrt{3}$ so $x^6 - 10x^3 + 16 = 0$. By the Rational Root Theorem, the only possible rational solutions are $\pm 1, \pm 2, \pm 4, \pm 8$, and ± 16. By substitution, none of these are roots so $\sqrt[3]{5-\sqrt{3}}$ must be irrational. □

2.8 By the Rational Root Theorem, the only possible rational solutions are ± 1. If x = 1, $(1)^8 - 4(1)^5 + 13(1)^3 - 7(1) + 1 = 4 \neq 0$. If x = -1, $(-1)^8 - 4(-1)^5 + 13(-1)^3 - 7(-1) + 1 = 0$. Thus, the only rational solution is x = -1.