17.2 (a) $(f+g)(x) = \begin{cases} x^2+4 & x \geq 0 \\ x^2 & x < 0 \end{cases}$

$(fg)(x) = \begin{cases} 4x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$(f \circ g)(x) = 4$ since $g(x) \geq 0$ for all $x \in \mathbb{R}$

$(g \circ f)(x) = \begin{cases} 16 & x \geq 0 \\ 0 & x < 0 \end{cases}$

The domain of each of these 4 functions is $\mathbb{R}$

(b) fg and $f \circ g$ are continuous.

17.6 *Proof.* Let $f = \frac{p}{q}$ be a rational function, where p and q are polynomial functions. By Exercise (17.5), p and q are continuous on $\{x \in \mathbb{R} \mid g(x) \neq 0\} \subseteq \mathbb{R}$. By Theorem (17.4), $f = \frac{p}{q}$ is continuous at all points in its domain so it is a continuous function. Since f was arbitrary, every rational function is continuous. $\square$

17.10 (a) *Proof.* It suffices to find a sequence $< x_n >$ that converges to 0 but $< f(x_n) >$ doesn't converge to $f(0) = 0$. Let $< x_n >=< \frac{1}{n} >$. Then $\lim_{n\to\infty} x_n = 0$ but $\lim_{n\to\infty} f(x_n) = 1 \neq 0$. Therefore, f is discontinuous at 0. $\square$

(b) *Proof.* It suffices to find a sequence $< x_n >$ that converges to 0 but $< g(x_n) >$ doesn't converge to $g(0) = 0$. Let $< x_n >=< \frac{1}{\pi(2n+\frac{1}{2})} >$. Then $\lim_{n\to\infty} x_n = 0$ but $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \sin(\pi(2n+\frac{1}{2}) = 1 \neq 0$. Therefore, g is discontinuous at 0. $\square$

(c) *Proof.* It suffices to find a sequence $< x_n >$ that converges to 0 but $< sgn(x_n) >$ doesn't converge to $\text{sgn}(0) = 0$. Let $< x_n >=< \frac{1}{n} >$. Then $\lim_{n\to\infty} x_n = 0$ but $\lim_{n\to\infty} sgn(x_n) = \lim_{n\to\infty} \frac{\frac{1}{n}}{|\frac{1}{n}|} = 1 \neq 0$. Therefore, sgn is discontinuous at 0. $\square$

17.14 *Proof.* Let $x_0 \in \mathbb{Q}$. Let $\varepsilon = f(x_0)$ and $\delta > 0$. By denseness there are irrational numbers in the interval $(x_0 - \delta, x_0 + \delta)$. Let $x_1 \in (x_0 - \delta, x_0 + \delta)$ be an irrational number so $|x_1 - x_0| < \delta$ and $f(x_1) = 0$. Thus, $|f(x_1) - f(x_0)| = |f(x_0)| = f(x_0) = \varepsilon$. Since δ was arbitrary, f is discontinuous at x_0. Since x_0 was arbitrary, f is discontinuous at each point of $\mathbb{Q}$.
Now consider $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Let $\varepsilon > 0$ and $\delta = \min\{|y - x_0| : y \in \mathbb{Q}\}$. Then if $|x - x_0| < \delta$, x is irrational so $f(x) = 0$ and hence $|f(x) - f(x_0)| = |0 - 0| = 0 < \varepsilon$. Since ε was arbitrary, f is continuous at x_0. Since x_0 was arbitrary, f is discontinuous at each point of $\mathbb{R} \setminus \mathbb{Q}$. $\square$

18.2 If [a,b] is replaced with (a,b), the proof of (18.1) breaks down since we cannot conclude the x_0 found by the Bolzano-Weierstrass Theorem must belong to (a, b) as the interval is open and not closed like in the original proof for Theorem (18.1). As a counter-example, consider $f(x) = \frac{1}{x}$ on $(0, 1)$. Let $x_n = \frac{1}{2n}$ so $f(x_n) = 2n > n$ but $\lim_{n\to\infty} x_n = 0 \notin (0, 1)$.

18.4 *Proof.* Since $|x - x_0|$ is continuous and positive on S as $x_0 \notin S$, $f(x) = \frac{1}{|x-x_0|}$ is a continuous function on S. Since $\lim_{n\to\infty} f(x_n) = \frac{1}{\lim_{n\to\infty} |x_n - x_0|} = +\infty$, f is unbounded. Therefore, we have shown there exists an unbounded continuous function f on S. $\square$

18.10 *Proof.* Let $g(x) = f(x+1) - f(x)$ on [0,1]. f is continuous on [0,2] so $f(x+1)$ is continuous for $x \in [-1, 1]$. Thus, g is continuous on $[0, 2] \cap [-1, 1] = [0, 1]$. $g(1) = f(2) - f(1)$ and $g(0) = f(1) - f(0) = f(1) - f(2) = -g(1)$ so we have two cases:
<u>Case 1</u> If $g(1) \neq 0$, then one of $g(0)$ and $g(1)$ is positive and the other negative so by the Intermediate Value Theorem, there is some $x_0 \in [0, 1]$ such that $g(x_0) = f(x_0+1) - f(x_0) = 0$ so $f(x_0+1) = f(x_0)$. Therefore, there exist $x = x_0$ and $y = x_0 + 1$ in [0,2] such that $f(x) = f(y)$ and $|y - x| = 1$.
<u>Case 2</u> If $g(1) = 0$, then $g(0) = 0$ as well so by the Intermediate Value Theorem, there is some $x_0 \in [0, 1]$ such that $g(x_0) = f(x_0 + 1) - f(x_0) = 0$ so $f(x_0 + 1) = f(x_0)$. Therefore, there exist $x = x_0$ and $y = x_0 + 1$ in [0,2] such that $f(x) = f(y)$ and $|y - x| = 1$. $\square$