12.2 *Proof.* Suppose $\limsup_{n\to\infty} |s_n| = 0$. Then $\lim_{N\to\infty} \sup \{|s_n| : n > N\} = 0$. For all $n > N$, $\sup\{|s_n| : n > N\} \geq |s_n| \geq 0$ so by the Squeeze Lemma, $\lim_{n\to\infty} |s_n| = 0$. Let $\varepsilon > 0$. Then there is some $N_1 > N$ such that for all $n > N_1 > N$, $||s_n| - 0| < \varepsilon$ so $||s_n|| = |s_n| = |s_n - 0| < \varepsilon$. Since ε was arbitrary, $\lim_{n\to\infty} s_n = 0$.
Now suppose $\lim_{n\to\infty} s_n = 0$. Let $\varepsilon > 0$. Then there is some $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $|s_n - 0| = |s_n| < \varepsilon$ so ε is an upper bound on $|s_n|$. Then for all $n > N > N_1$, $|s_n| \leq \sup\{|s_n| : n > N\} < \varepsilon$. Since ε was arbitrary, $\lim_{N\to\infty} \sup\{|s_n| : n > N\} = 0$ so $\limsup_{n\to\infty} |s_n| = 0$.
Therefore, $\limsup_{n\to\infty} |s_n| = 0$ if and only if $\lim_{n\to\infty} s_n = 0$. □

12.4 *Proof.* $< s_n >$ is bounded so $\sup\{s_n : n > N\}$ exists. Likewise, $< t_n >$ is bounded so $\sup\{t_n : n > N\}$ exists. Then for all $n > N$, $s_n \leq \sup\{s_n : n > N\}$ and $t_n \leq \sup\{t_n : n > N\}$. Hence, $s_n + t_n \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$ for all $n > N$. Since $\sup\{s_n : n > N\} + \sup\{t_n : n > N\}$ is an upper bound on $\{s_n + t_n : n > N\}$, $\sup\{s_n + t_n : n > N\}$ exists. Since it is the least upper bound, $\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$. Therefore, by (9.9c), $\limsup_{n\to\infty} s_n + t_n \leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n$. □

12.8 *Proof.* $< s_n >$ is bounded so $\sup\{s_n : n > N\}$ exists. Likewise, $< t_n >$ is bounded so $\sup\{t_n : n > N\}$ exists. Then for all $n > N$, $s_n \leq \sup\{s_n : n > N\}$ and $t_n \leq \sup\{t_n : n > N\}$. Since $< s_n >$ and $< t_n >$ are sequences of non-negative numbers, $s_n t_n \leq \sup\{s_n : n > N\}\cdot\sup\{t_n : n > N\}$ for all $n > N$. Since $\sup\{s_n : n > N\}\cdot\sup\{t_n : n > N\}$ is an upper bound on $\{s_n t_n : n > N\}$, $\sup\{s_n t_n : n > N\}$ exists. Since it is the least upper bound, $\sup\{s_n t_n : n > N\} \leq \sup\{s_n : n > N\}\sup\{t_n : n > N\}$. Therefore, by (9.9c), $\limsup_{n\to\infty} s_n t_n \leq \limsup_{n\to\infty} s_n \cdot \limsup_{n\to\infty} t_n$. □

12.12 (a) We will first show that $\limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} s_n$. Since $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$ for all $n, N \in \mathbb{N}$ where $n > N$, $\sigma_n \leq \frac{1}{n}(s_1 + s_2 + \cdots + s_N + v_N + v_N + \cdots + v_N = \frac{1}{n}\sum_{i=1}^{N}(s_i) + \frac{n-N}{n} v_N$ where $v_N = \sup\{s_n : n > N\}$. Applying $\limsup_{n\to\infty}$ to both sides, we get $\limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty}(\frac{1}{n}\sum_{i=1}^{N}(s_i) + \frac{n-N}{n} v_N) = v_N$. Then applying $\lim_{N\to\infty}$ to both sides, we get that $\lim_{N\to\infty}(\limsup_{n\to\infty} \sigma_n) = \limsup_{n\to\infty} \sigma_n \leq \lim_{N\to\infty} v_N = \limsup_{n\to\infty} s_n$.
Now, we will show that $\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} \sigma_n$. Since $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$ for all $n, N \in \mathbb{N}$ where $n > N$, $\sigma_n \geq \frac{1}{n}(s_1 + s_2 + \cdots + s_N + u_N + u_N + \cdots + u_N = \frac{1}{n}\sum_{i=1}^{N}(s_i) + \frac{n-N}{n} u_N$ where $u_N = \inf\{s_n : n > N\}$. Applying $\liminf_{n\to\infty}$ to both sides, we get $\liminf_{n\to\infty} \sigma_n \geq \liminf_{n\to\infty}(\frac{1}{n}\sum_{i=1}^{N}(s_i) + \frac{n-N}{n} u_N) = u_N$. Then applying $\lim_{N\to\infty}$ to both sides, we get that $\lim_{N\to\infty}(\liminf_{n\to\infty} \sigma_n) = \liminf_{n\to\infty} \sigma_n \geq \lim_{N\to\infty} u_N = \liminf_{n\to\infty} s_n$.
Since $\liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n$, we have shown that $\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} s_n$

(b) If $\lim_{n\to\infty} s_n$ exists and is equal to some $s \in \mathbb{R}$, then $\liminf_{n\to\infty} s_n$ and $\limsup_{n\to\infty} s_n$ exist and are equal to s as well. Then, $s \leq \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n \leq s$ so $s = \liminf_{n\to\infty} \sigma_n = \limsup_{n\to\infty} \sigma_n$. Therefore, $\lim_{n\to\infty} \sigma_n$ exists and $\lim_{n\to\infty} \sigma_n = s = \lim_{n\to\infty} s_n$. Since s was arbitrary, the claim holds whenever $\lim_{n\to\infty} s_n$ exists.

(c) Let $< s_n > = < 2 + (-1)^n > = < 1, 3, 1, 3, \cdots >$. Then, it is obvious that $\lim_{n\to\infty} s_n$ does not exist. Let $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n) = \frac{1}{n}((2 + (-1)^1) + (2 + (-1)^2) + \cdots + (2 + (-1)^n)) = \frac{1}{n}(2n + ((-1)^1 + (-1)^2 + \cdots + (-1)^n)) = \frac{1}{n}(2n + (-1)^n)$. Since $< \sigma_n > = < \frac{1}{n}(2n + (-1)^n) >$, $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty}(2 + \frac{(-1)^n}{n}) = \lim_{n\to\infty}(2) + \lim_{n\to\infty}(\frac{(-1)^n}{n}) = 2 + 0 = 2$ by limit properties.
Therefore, we have given an example where $\lim_{n\to\infty} \sigma_n$ exists, but $\lim_{n\to\infty} s_n$ does not exist.