19.2 (b) *Proof.* Let $x, y \in [0,3]$ so $|x+y| = x+y \leq 6$. Let $\varepsilon > 0$ and $|x-y| < \frac{\varepsilon}{6}$. So $|x^2 - y^2| = |x-y| \cdot |x+y| < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$. Since ε was arbitrary, f is uniformly continuous on $[0,3]$. ☐

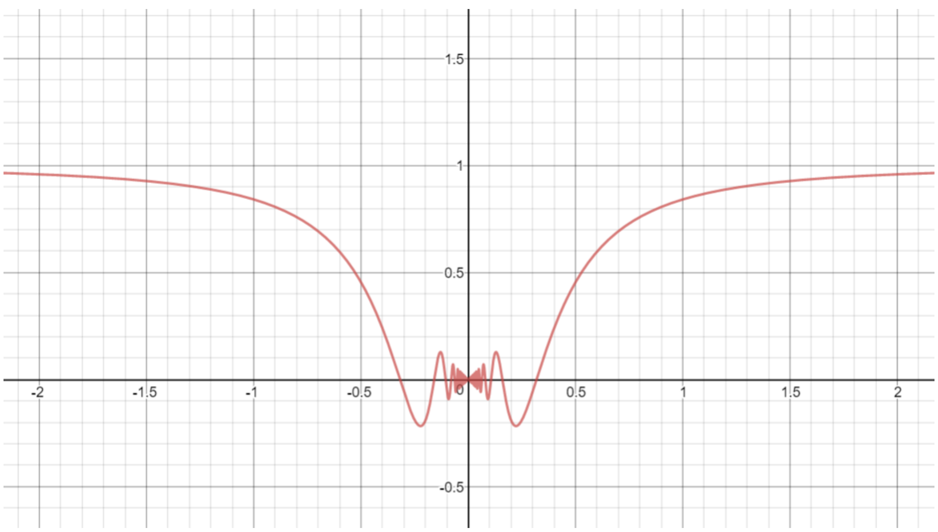
19.8 (a) *Proof.* Let $x, y \in \mathbb{R}$. Without loss of generality, let $y \leq x$. Then by the Mean Value Theorem, there is some $z \in [y,x]$ such that $\frac{d}{dz}\sin(z) = \cos(z) = \frac{\sin(x)-\sin(y)}{x-y}$ so $|\sin(x)-\sin(y)| = |x-y| \cdot |\cos(z)| \leq |x-y| \cdot 1$ since $|\cos(z)| \leq 1$. Since x and y were arbitrary, $|\sin(x) - \sin(y)| \leq |x-y|$ for all $x, y \in \mathbb{R}$. ☐

(b) Let $x, y \in \mathbb{R}$ and $\varepsilon > 0$. Then if $|x-y| < \varepsilon$, $|\sin(x) - \sin(y)| \leq |x-y| < \varepsilon$ by Part (a) above. Since ε was arbitrary, $\sin(x)$ is uniformly continuous on $\mathbb{R}$.

19.10 (a) *Proof.* Let $x_0 \in \mathbb{R}$. Let $\varepsilon > 0$ and $\delta = \min\{1, \sqrt{\varepsilon - 2x_0^2 - 2x_0}\}$. Then $|x - x_0| < \delta$ implies that $|x-x_0| < \sqrt{\varepsilon - 2x_0^2 - 2x_0}$ so $|x-x_0|^2 + 2x_0^2 + 2x_0 < \varepsilon$. We can also see that $|x - x_0| < 1$ so $x_0 < x < x_0 + 1$. Thus, $\varepsilon > x^2 + x_0^2 - 2x \cdot x_0 + 2x_0 \cdot (x_0+1) > x^2 + x_0^2 - 2x \cdot x_0 \; + 2x \cdot x_0 = x^2 + x_0^2 \geq |x^2 \sin(\frac{1}{x})| + |x_0^2 \sin(\frac{1}{x_0})| \geq |x^2 \sin(\frac{1}{x}) - x_0^2 \sin(\frac{1}{x_0})|$ by the Triangle Inequality. Since ε was arbitrary, g is continuous at x_0. Since x_0 was arbitrary, g is continuous on $\mathbb{R}$. ☐

(b) *Proof.* Let $[a,b]$ be a bounded subset on $\mathbb{R}$. By Part (a), g is continuous on $\mathbb{R}$ so it must be continuous on $[a,b] \subset \mathbb{R}$. By Theorem (19.2), it follows that g is uniformly continuous on $[a,b]$. Since $[a,b]$ was arbitrary, g is uniformly continuous on any bounded subset of $\mathbb{R}$. ☐

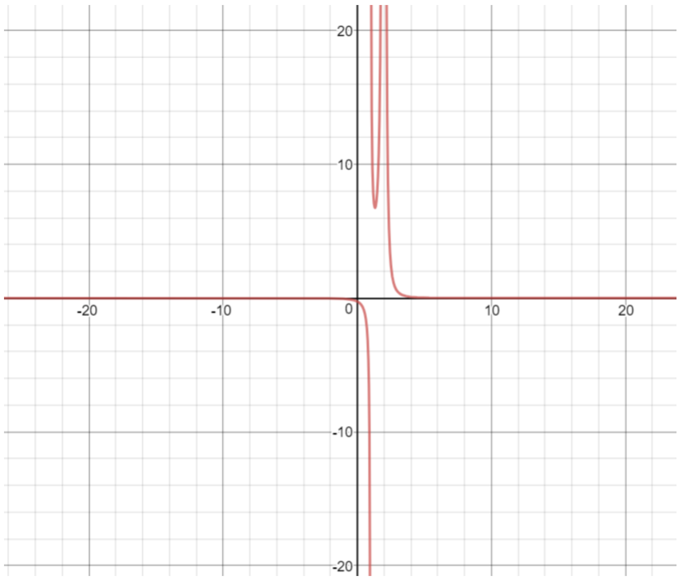
(c) *Proof.* For all $x \in [-1,1] \subset \mathbb{R}$, g is uniformly continuous by Part (b).
For all $x \in (1, +\infty)$. g is differentiable on $(1,+\infty)$ and $g'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$ so $g'(x)$ is bounded below by 1 and bounded above by 2. Therefore, by Theorem (19.6), g is uniformly continuous on $(1, +\infty)$.
For all $x \in (-\infty, -1)$. g is differentiable on $(-\infty,-1)$ and $g'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$ so $g'(x)$ is bounded below by 1 and bounded above by 2. Therefore, by Theorem (19.6), g is uniformly continuous on $(-\infty, -1)$.
Therefore, g is uniformly continuous on $(-\infty,-1) \cup [-1,1] \cup (1,+\infty) = \mathbb{R}$. ☐



20.4

$\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 1$
$\lim_{x\to0^-} f(x) = 0$, $\lim_{x\to0^+} f(x) = 0$, and $\lim_{x\to0} f(x) = 0$

20.8 $\lim_{x\to\infty} x\sin(\frac{1}{x}) = \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = \lim_{\frac{1}{x}\to0^+} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 1$ by Example (19.9).
$\lim_{x\to-\infty} x\sin(\frac{1}{x}) = \lim_{x\to-\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = \lim_{\frac{1}{x}\to0^-} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 1$ by Example (19.9).
For $x \in (0,+\infty)$, $-1 \leq \sin(\frac{1}{x}) \leq 1$ so $-x \leq x\sin(\frac{1}{x}) \leq x$. Thus, $0 = \lim_{x\to0^+} -x \leq \lim_{x\to0^+} x\sin(\frac{1}{x}) \leq \lim_{x\to0^+} x = 0$ so by the Squeeze Lemma, $\lim_{x\to0^+} x\sin(\frac{1}{x}) = 0$.
For $x \in (-\infty,0)$, $-1 \leq \sin(\frac{1}{x}) \leq 1$ so $-x \geq x\sin(\frac{1}{x}) \geq x$. Thus, $0 = \lim_{x\to0^-} -x \geq \lim_{x\to0^-} x\sin(\frac{1}{x}) \geq \lim_{x\to0^-} x = 0$ so by the Squeeze Lemma, $\lim_{x\to0^-} x\sin(\frac{1}{x}) = 0$.
From previous parts, $\lim_{x\to0^-} x\sin(\frac{1}{x}) = \lim_{x\to0^+} x\sin(\frac{1}{x}) = 0$ so by Thm. (20.10), $\lim_{x\to0} x\sin(\frac{1}{x}) = 0$

20.12 (a) 

(b) $\lim_{x \to 2^+} f(x) = +\infty$
$\lim_{x \to 2^-} f(x) = +\infty$
$\lim_{x \to 1^+} f(x) = +\infty$
$\lim_{x \to 1^-} f(x) = -\infty$

(c) Neither limit exists.