14.6 (a) *Proof.* $< b_n >$ is bounded so let M be an upper bound. Then $b_k < M$ so $|b_k| < |M|$ for all $k \in \mathbb{N}$. $\sum_{n=1}^{\infty} |a_n|$ converges so by Theorem (14.4), it satisfies the Cauchy Criterion. Let $\varepsilon > 0$. Then there is a $N \in \mathbb{N}$ such that for all $n \geq m > N$, $|\sum_{k=m}^{n} |a_k|| < \frac{\varepsilon}{|M|}$. Since $|b_k| < |M|$ for all $k \in \mathbb{N}$, then $|b_k||\sum_{k=m}^{n} |a_k|| = |b_k| \sum_{k=m}^{n} |a_k| = \sum_{k=m}^{n} |a_k||b_k| = \sum_{k=m}^{n} |a_k b_k| < \frac{\varepsilon}{|M|} \cdot |M| = \varepsilon$. By the Triangle Inequality, $|\sum_{k=m}^{n} a_k b_k| \leq \sum_{k=m}^{n} |a_k b_k| < \varepsilon$. Since ε was arbitrary, $\sum_{n=1}^{\infty} a_n b_n$ satisfies the Cauchy Criterion so it converges. □

(b) *Proof.* Corollary (14.7) states that absolutely convergent series are convergent. Since $\sum_{n=1}^{\infty} |a_n|$ is absolutely convergent, (14.7) is a special case of part (a) where $< b_n >=< 1 >$, as we showed that $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n$ converges so absolutely convergent series are convergent. □

14.8 *Proof.* $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge so they satisfy the Cauchy Criterion. Let $\varepsilon > 0$. Then there is a $N \in \mathbb{N}$ such that for all $n \geq m > N$, $|\sum_{k=m}^{n} a_k| < \frac{\varepsilon}{2}$ and $|\sum_{k=m}^{n} b_k| < \frac{\varepsilon}{2}$. So, $|\sum_{k=m}^{n} a_k| + |\sum_{k=m}^{n} b_k| = |\sum_{k=m}^{n} a_k + b_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since a_n and b_n are non-negative for all $n \in \mathbb{N}$, $(a_n + b_n)^2 = a_n^2 + 2a_n b_n + b_n^2 \geq 2a_n b_n \geq a_n b_n$ so $a_n + b_n \geq \sqrt{a_n b_n}$ and hence $\sum_{k=m}^{n} \sqrt{a_k b_k} \leq \sum_{k=m}^{n} a_k + b_k = |\sum_{k=m}^{n} a_k + b_k| < \varepsilon$ for all $n \geq m > N$. Since ε was arbitrary, $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ converges. □

15.3 *Proof.* We will prove by contradiction. Suppose $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges and $p \geq 1$. We will first consider p = 1. Then, $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converges. $\frac{1}{n \log n}$ is a decreasing non-negative function on $[2, +\infty)$ so by the Integral Test, $\int_2^{\infty} \frac{1}{n \log n} = \log n \Big|_2^{\infty} = +\infty$ so the series diverges. Contradiction since $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges so $p \neq 1$. Now consider $p < 1$. Then $\frac{1}{n(\log n)^p}$ is a decreasing non-negative function on $[2, +\infty)$ so by the Integral Test, $\int_2^{\infty} \frac{1}{n(\log n)^p} = \frac{(\log n)^{1-p}}{1-p} \Big|_2^{\infty} = +\infty$ since $1 - p > 0$. Contradiction since $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges so $p > 1$.

Now suppose $p > 1$. $\frac{1}{n(\log n)^p}$ is a decreasing non-negative function on $[2, +\infty)$ so by the Integral Test, $\int_2^{\infty} \frac{1}{n(\log n)^p} = \frac{(\log n)^{1-p}}{1-p} \Big|_2^{\infty} = 0 - \frac{(\log 2)^{1-p}}{1-p} < +\infty$ since 1-p ¡ 0. Therefore, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges. □

15.4 (a) $\sqrt{n} > \log n$ so $\frac{1}{\sqrt{n}} < \frac{1}{\log n}$ for all $n \in [2, \infty)$. Then, $\frac{1}{n} = \frac{1}{\sqrt{n} \cdot \sqrt{n}} < \frac{1}{\sqrt{n} \log n}$. $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges so by the Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \log n}$ diverges as well.

(b) $\log n > 1$ for all $n \geq 3$ so $\frac{\log n}{n} > \frac{1}{n}$. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so by the Comparison Test, $\sum_{n=2}^{\infty} \frac{\log n}{n}$ diverges as well.

(c) $\frac{1}{n(\log n)(\log \log n)}$ is a non-negative and decreasing function on $[4, +\infty)$ so $\int_4^{\infty} \frac{1}{n(\log n)(\log \log n)} = \log(\log(\log n)) \Big|_4^{\infty} = +\infty$. Therefore, $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)}$ diverges by the Integral Test.

(d) For all $n \in [2, +\infty)$, $\log n < \sqrt{n}$ so $\frac{\log n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$. $\frac{3}{2} > 1$ so $\sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges. Therefore, by the Comparison Test, $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ converges as well.

15.6 (a) Let $a_n = \frac{1}{n}$. Then $\sum_{n=1}^{\infty} a_n$ is the Harmonic Series so it diverges but $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(b) *Proof.* Suppose $\sum_{n=1}^{\infty} a_n$ converges. Let $\varepsilon > 0$. Then there is a $N \in \mathbb{N}$ such that for all $m \geq n > N$, $|\sum_{k=m}^{n} a_k| < \sqrt{\varepsilon}$ so $|\sum_{k=m}^{n} a_k| \cdot |\sum_{k=m}^{n} a_k| < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$. $|\sum_{k=m}^{n} a_k^2| < |\sum_{k=m}^{n} a_k| \cdot |\sum_{k=m}^{n} a_k| < \varepsilon$. Since ε was arbitrary, $\sum_{n=1}^{\infty} a_n^2$ satisfies the Cauchy Criterion so it converges. □

(c) Let $a_n = \frac{(-1)^n}{\sqrt{n}}$. $\lim_{n\to\infty} a_n = 0$ and terms are decreasing in magnitude so by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges. However, $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is the Harmonic Series.

16.4 (a) $.2 = \frac{1}{5}$

(b) $.0\overline{2} = \frac{2}{100}\sum_{n=0}^{\infty}\frac{1}{10^n} = \frac{2}{100}\cdot\frac{10}{9} = \frac{2}{90}$
(c) $.\overline{02} = \frac{2}{100}\sum_{n=0}^{\infty}\frac{1}{100^n} = \frac{2}{100}\cdot\frac{100}{99} = \frac{2}{99}$
(d) $3.\overline{14} = 3 + \frac{14}{100}\sum_{n=0}^{\infty}\frac{1}{100^n} = 3 + \frac{14}{100}\cdot\frac{100}{99} = \frac{314}{99}$
(e) $.\overline{10} = \frac{1}{10}\sum_{n=0}^{\infty}\frac{1}{100^n} = \frac{1}{10}\cdot\frac{100}{99} = \frac{1}{99}$
(f) $.1\overline{492} = \frac{1}{10} + \frac{492}{10000}\sum_{n=0}^{\infty}\frac{1}{1000^n} = \frac{1}{10} + \frac{492}{10000}\cdot\frac{1000}{999} = \frac{1491}{9990}$

16.6 $\frac{1}{7} = .\overline{142857}$
$\frac{2}{7} = .\overline{285714}$
$\frac{3}{7} = .\overline{428571}$
$\frac{4}{7} = .\overline{571428}$
$\frac{5}{7} = .\overline{714285}$
$\frac{6}{7} = .\overline{857142}$
Note that each fraction repeats the same numerals in a different order, starting with the smallest such repeating numeral and moving up

16.8 *Proof.* Let $s_n = 0.d_1^{(n)}d_2^{(n)}d_3^{(n)}\ldots$ for some $n \in \mathbb{N}$. Let $i \in \mathbb{N}$. Suppose $d_i^{(n)} \neq 6$. Then $e_i = 6 \neq d_i^{(n)}$. Now suppose $d_i^{(n)} = 6$. Then $e_i = 7 \neq d_i^{(n)}$. Therefore, $e_i \neq d_i^{(n)}$. Since i was arbitrary, $e_i \neq d_i^{(n)}$ for all $i \in \mathbb{N}$ and so $y = 0.e_1e_2e_3... \neq s_n$. Since n was arbitrary, $y \neq s_n$ for all n.

□