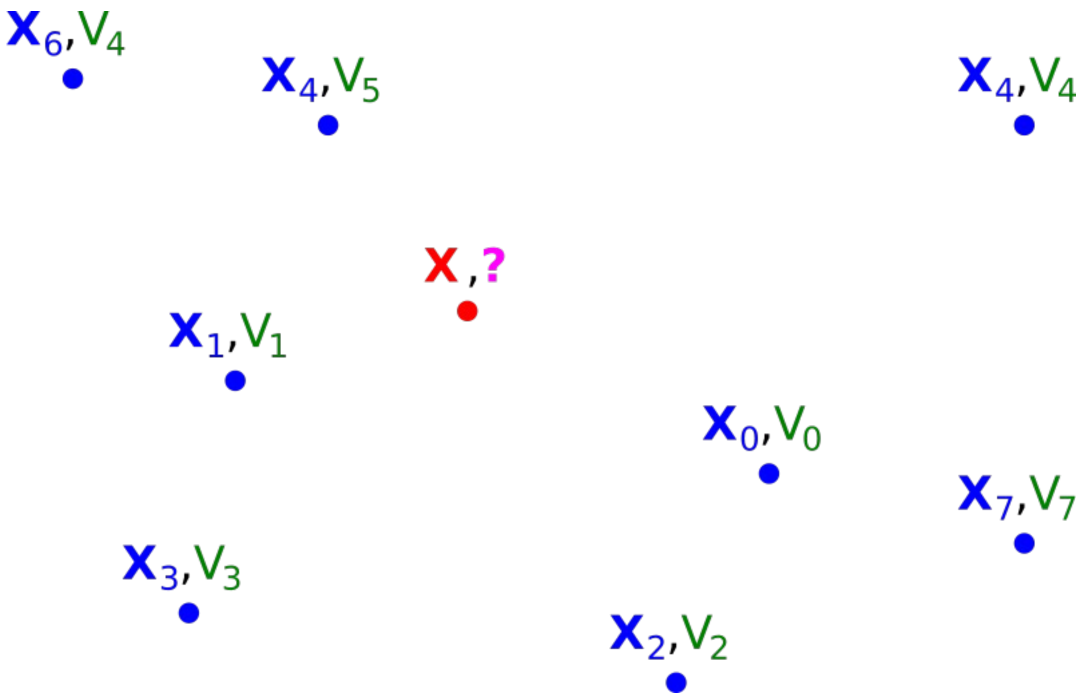
Unidad 2

Interpolación

Universidad Nacional del Litoral
Facultad de Ingeniería y Ciencias Hídricas

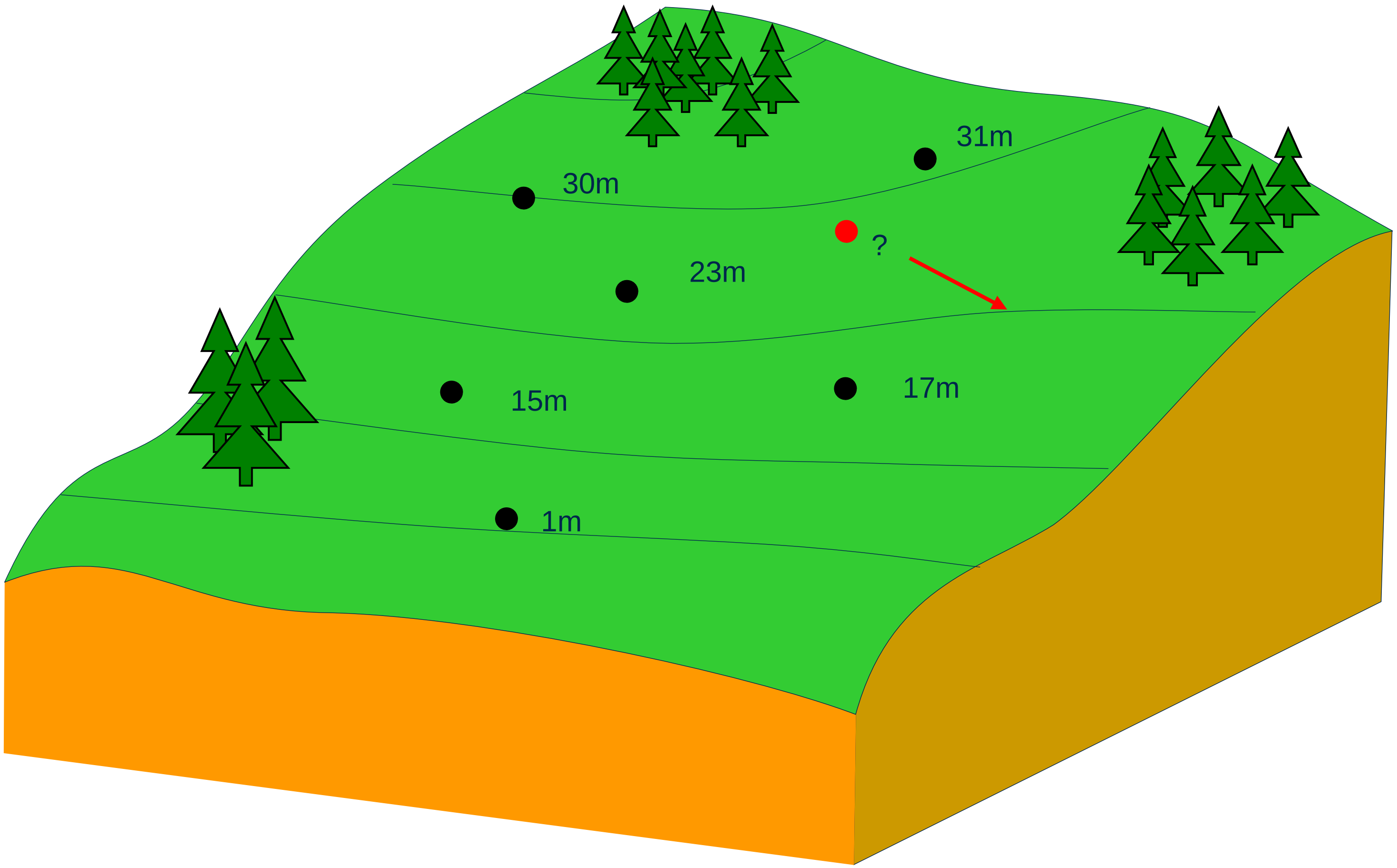
Interpolación de Datos



Conjunto finito de Nodos fijos, se conocen
posición y valor(es) asociado(s)

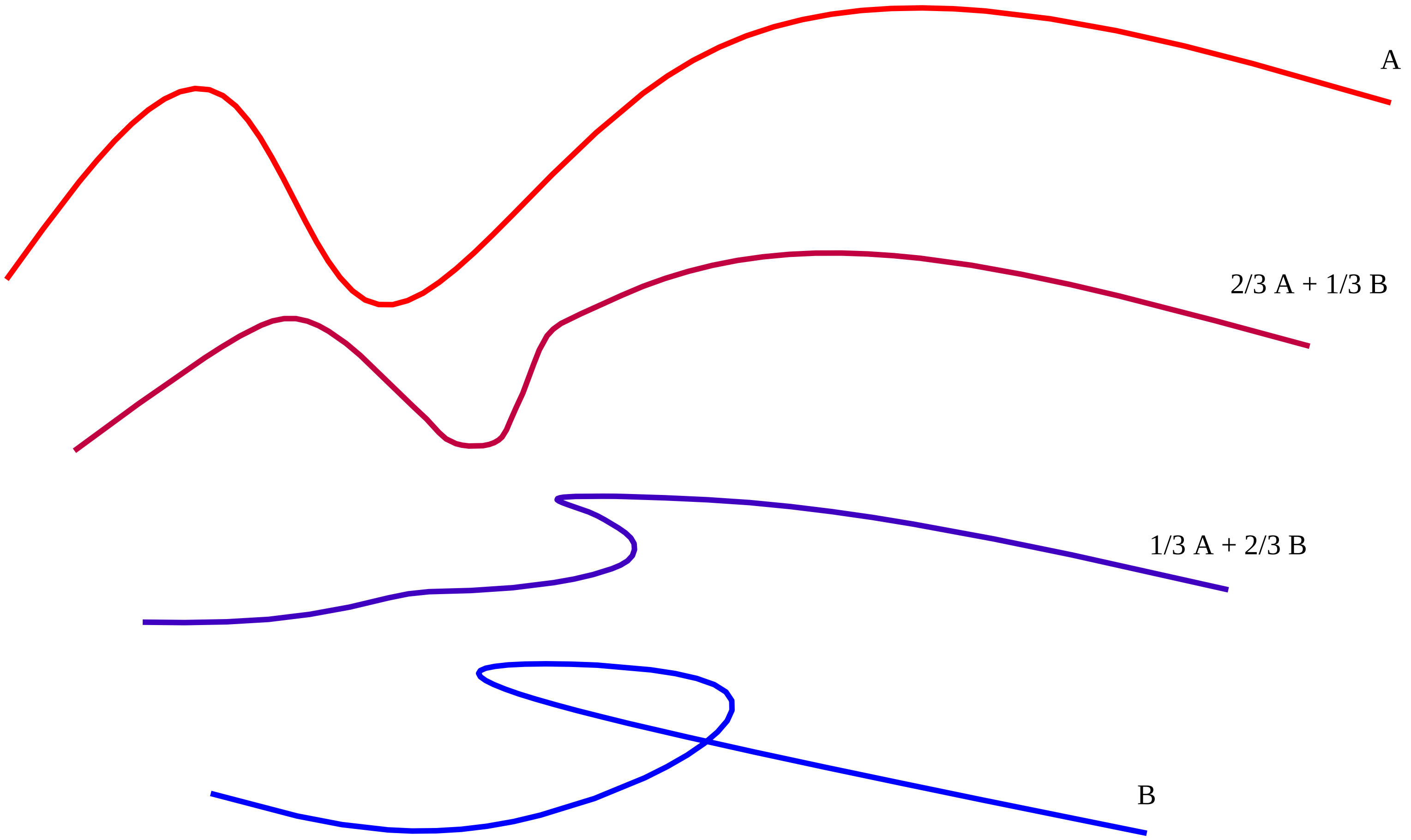
Se pretende “encontrar” el **valor** en un punto cualquiera

Ejemplo: Interpolación de Alturas

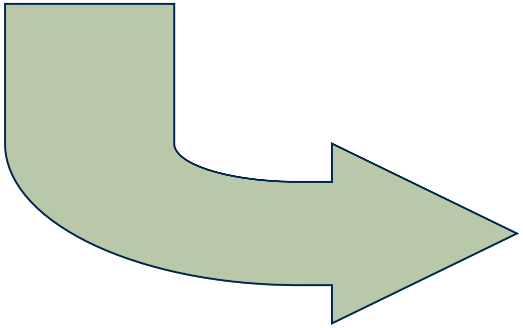


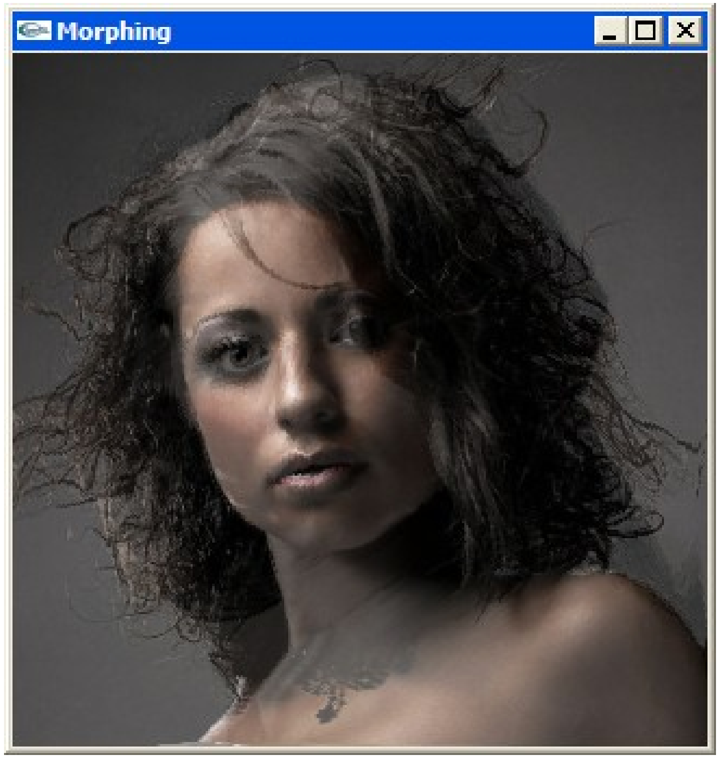
Ejemplo: Interpolación de “Píxeles”

Ejemplo: Interpolación de Curvas

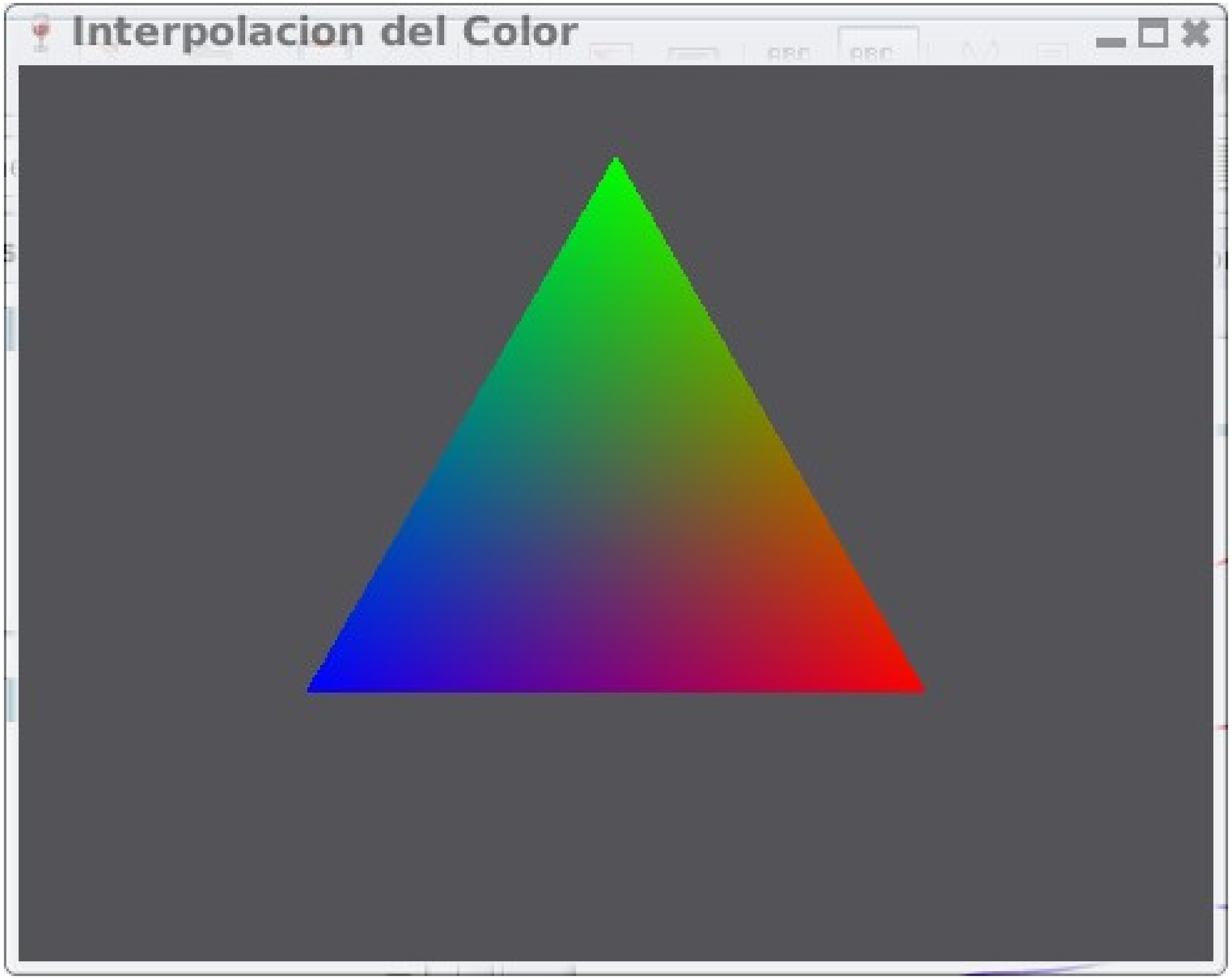


Ejemplo: Interpolación de Imágenes

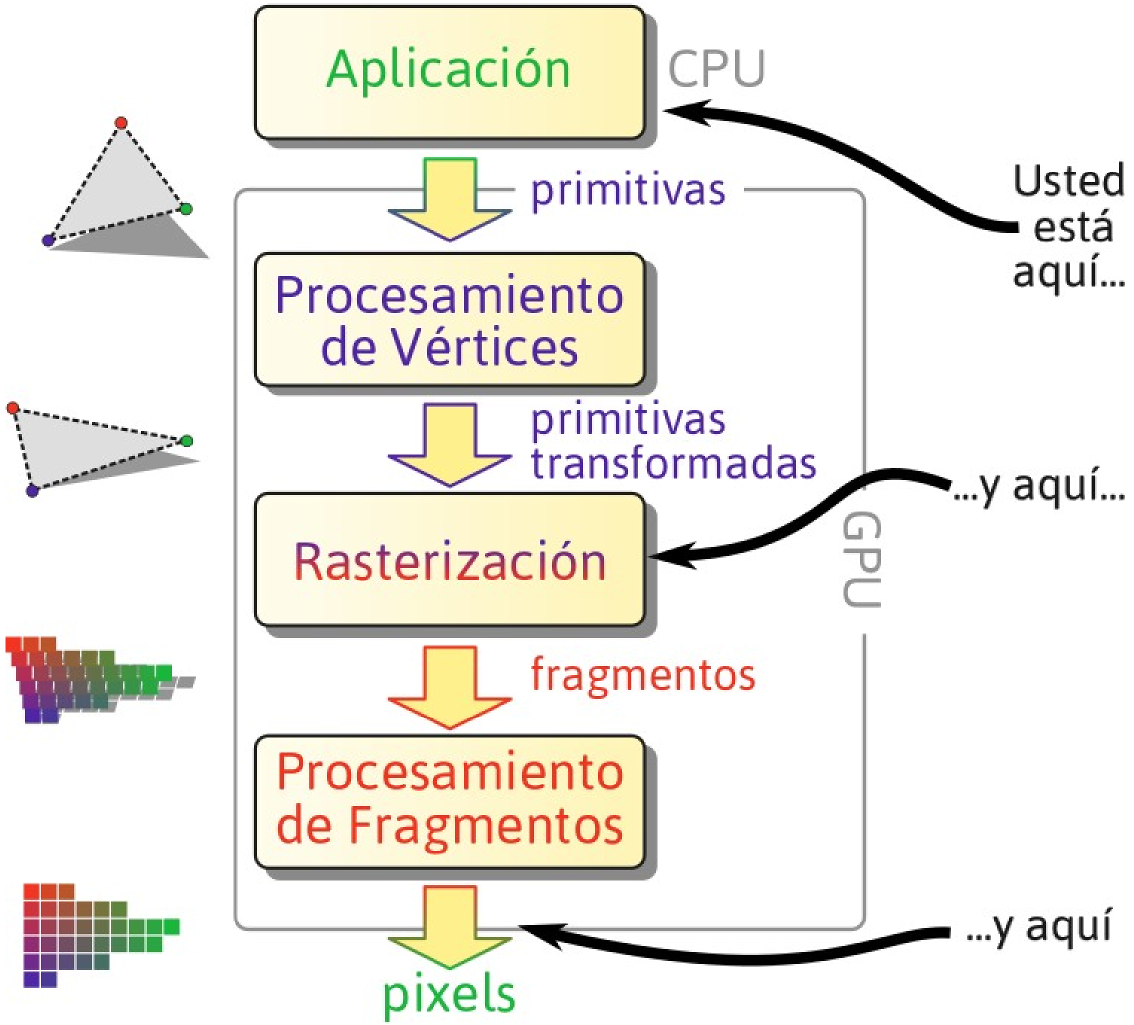




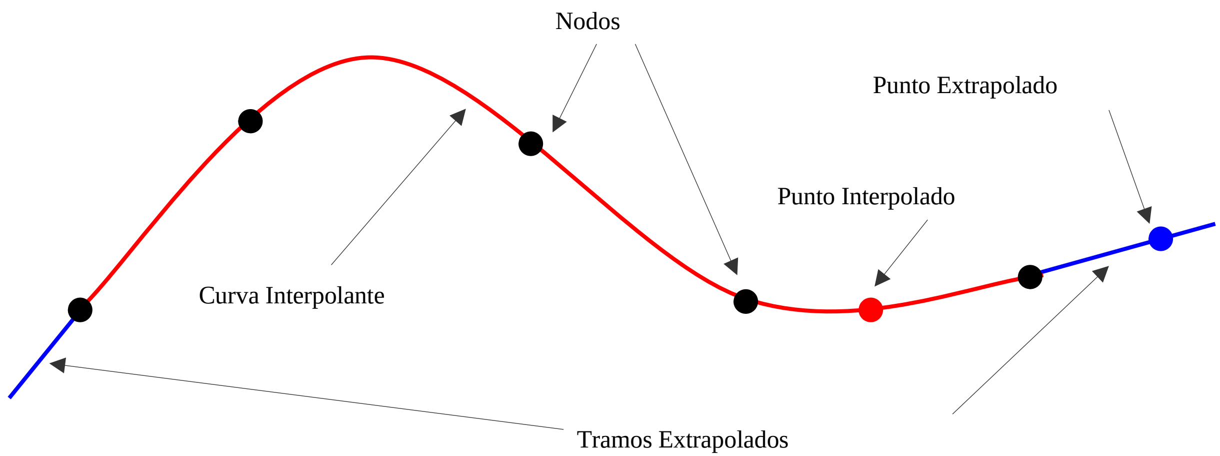
Ejemplo: Interpolación de Colores

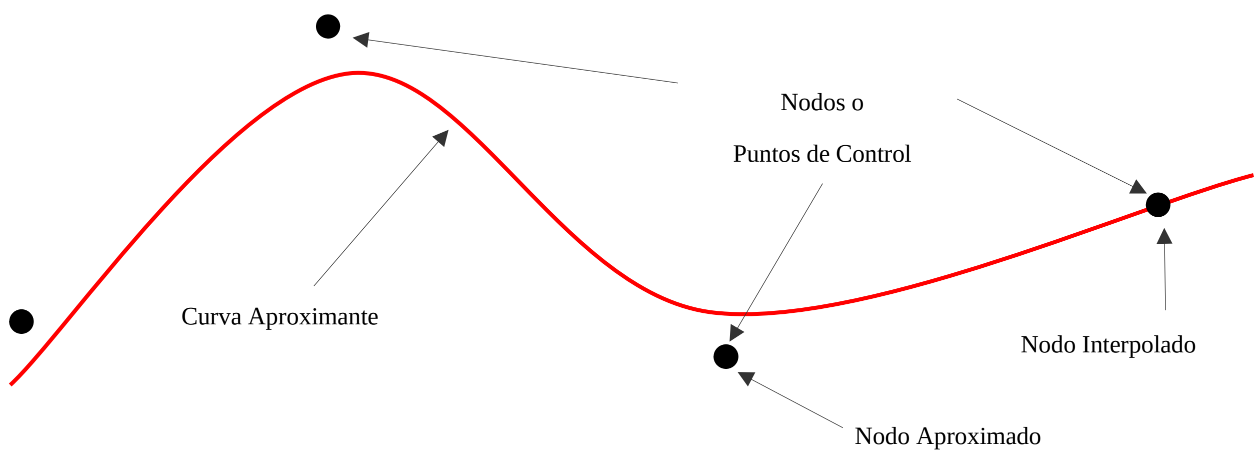


Usted Está Aquí

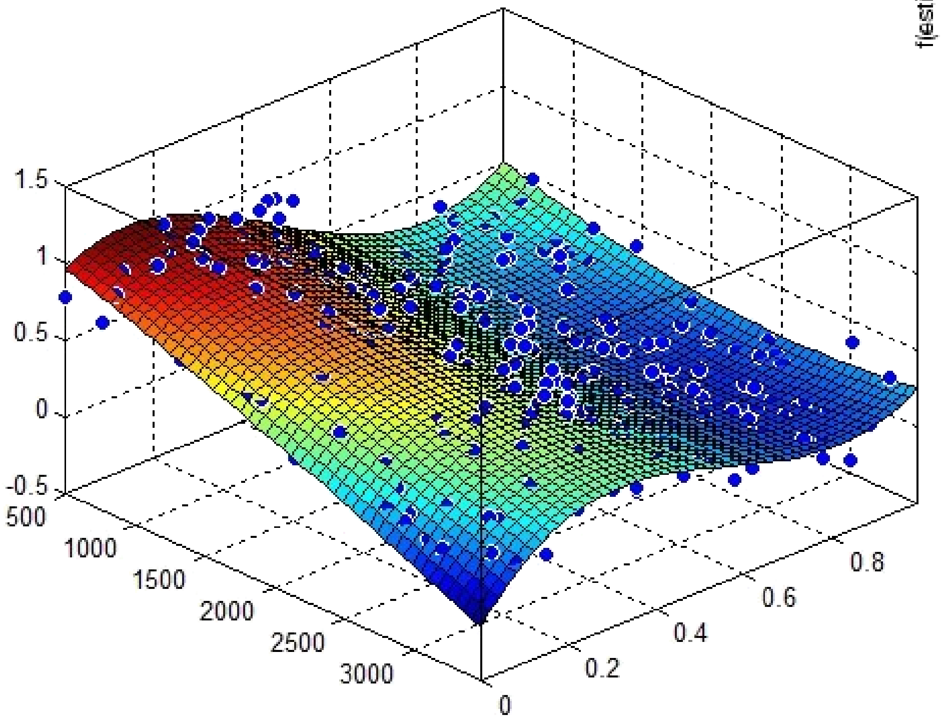


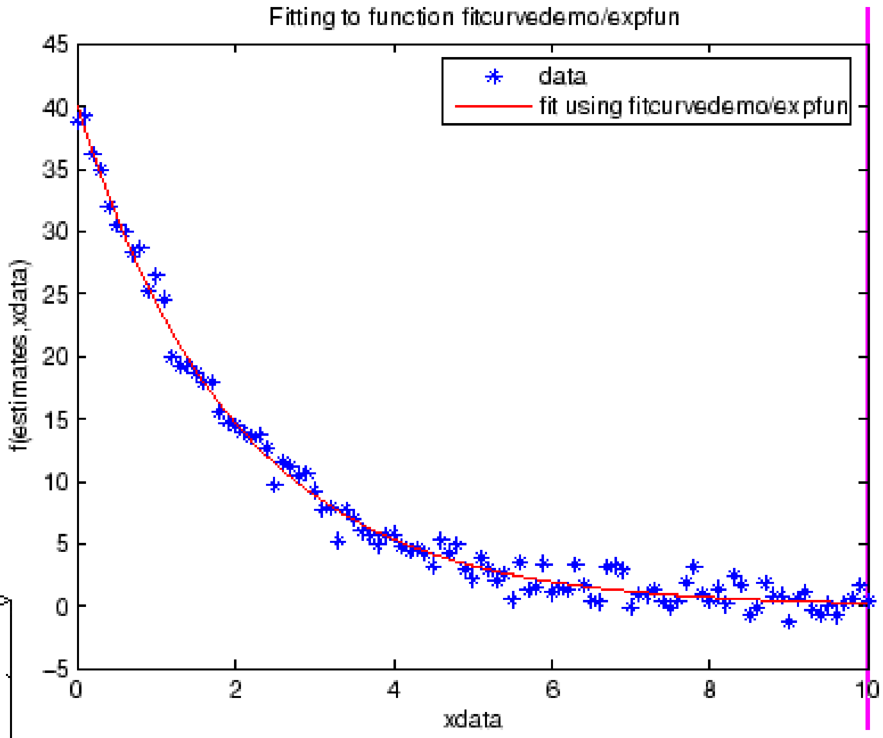
Interpolacion vs. Extrapolación vs. Aproximación



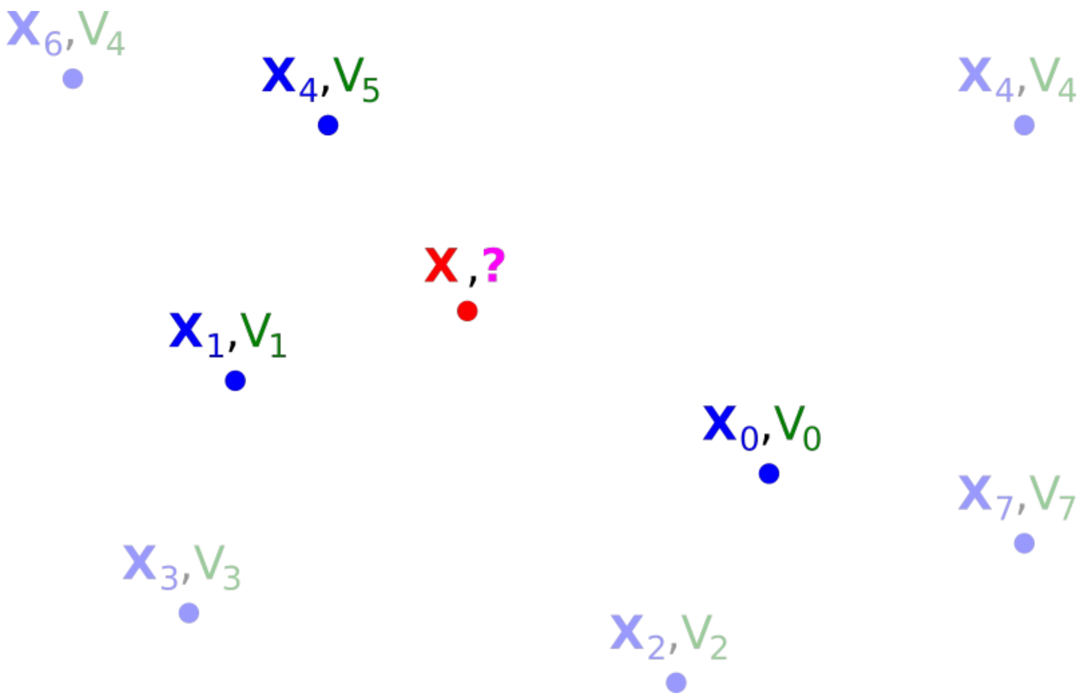


Ejemplo: Aproximación





Método de Interpolación



¿Cuáles nodos conocidos utilizar?

¿Cómo calcular el nuevo valor a partir de esos nodos y sus valores?

Promedio Ponderado

Para valores reales:

$$v(\boldsymbol{x}) = \sum_i \alpha(\boldsymbol{x}, \boldsymbol{x_i}) v_i = \sum_i \alpha^i(\boldsymbol{x}) v_i \qquad \text{con} \qquad \sum_i \alpha^i = 1$$

Normalmente:

$$\alpha^i \sim \frac{1}{|\boldsymbol{x_i} - \boldsymbol{x}|}$$

Métodos locales:

$$\text{muy lejanos} \Rightarrow = 0$$

¿Cuáles nodos conocidos utilizar?

¿Cómo calcular el nuevo valor a partir de esos nodos y sus valores?

Interpolación Afín

1) Obtener **x** por combinación afín de los $\mathbf{x_i}$:

$$\boldsymbol{x} = \sum_i \alpha^i(\boldsymbol{x})\,\boldsymbol{x_i} \qquad \text{con} \qquad \sum_i \alpha^i = 1$$

Las coordenadas de **x** y de los $\mathbf{x}_i$ son conocidas,

las incógnitas son los **pesos**

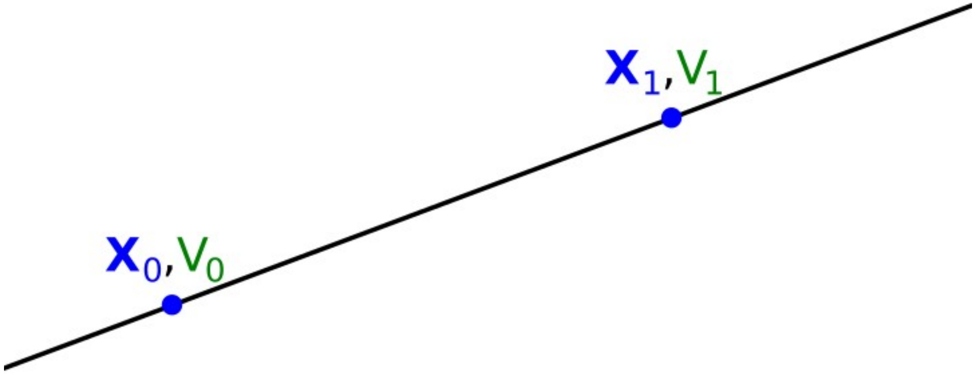
2) Calcular v utilizando los **pesos** del paso 1:

$$v = \sum_i \alpha^i(\boldsymbol{x})\,v_i$$

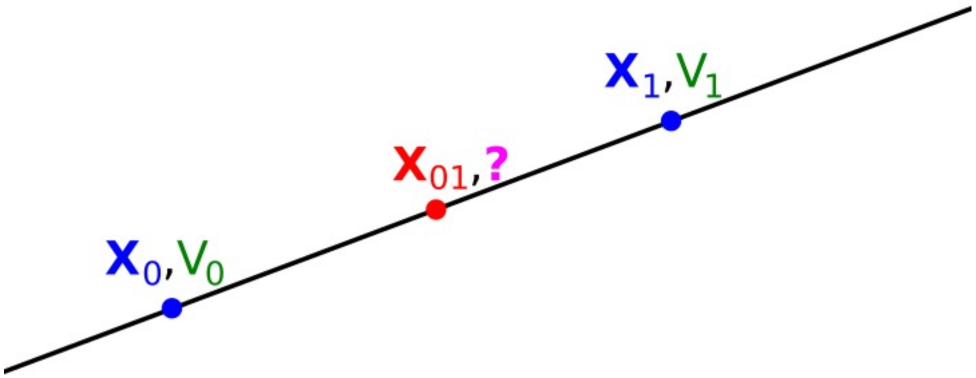
Los valores reciben el mismo tratamiento que las coordenadas.

Resta definir cómo obtener los pesos.

Interpolación Afín

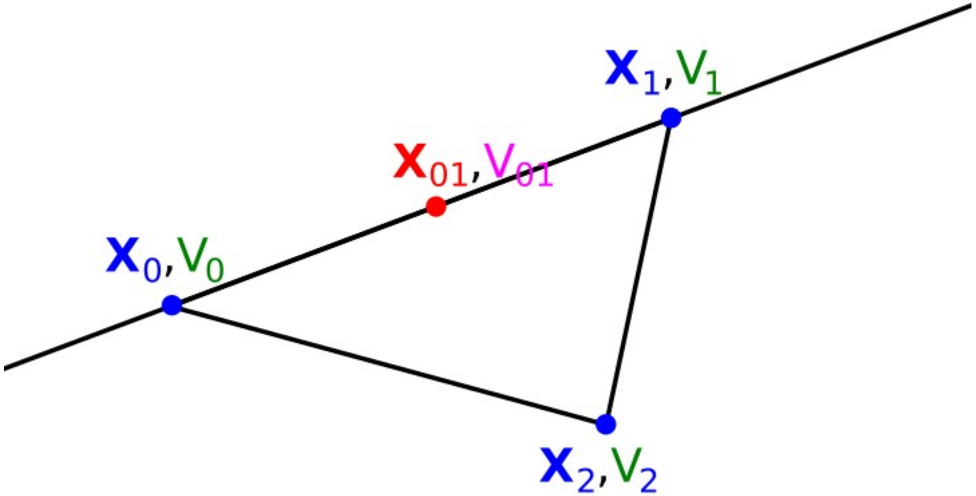


Interpolación Afín



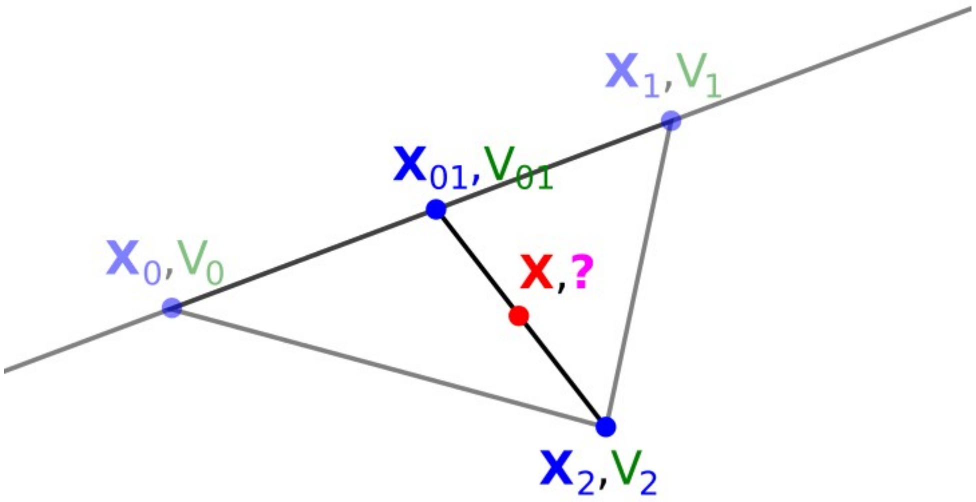
$$x_{01} = \alpha^0 x_0 + \alpha^1 x_1 \quad \text{con} \quad \alpha^0 + \alpha^1 = 1$$

Interpolación Afín



$$\boldsymbol{x}_{01} = \alpha^0 \boldsymbol{x}_0 + \alpha^1 \boldsymbol{x}_1 \quad \text{con} \quad \alpha^0 + \alpha^1 = 1$$

Interpolación Afín



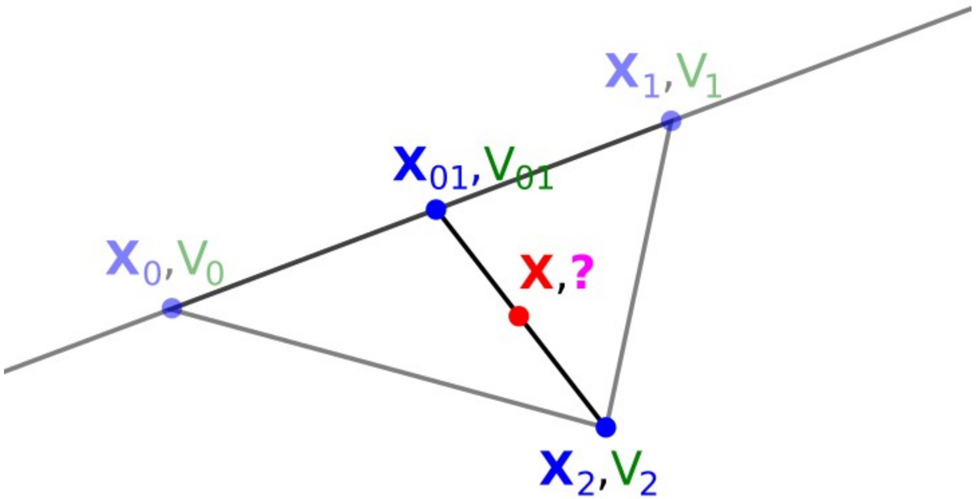
$$\boldsymbol{x}_{01} = \alpha^0 \boldsymbol{x}_0 + \alpha^1 \boldsymbol{x}_1 \quad \text{con} \quad \alpha^0 + \alpha^1 = 1$$

$$\boldsymbol{x} = \beta^1 \boldsymbol{x}_{01} + \beta^2 \boldsymbol{x}_2 \quad \text{con} \quad \beta^1 + \beta^2 = 1$$

$$\boldsymbol{x} = \beta^1 \left(\alpha^0 \boldsymbol{x}_0 + \alpha^1 \boldsymbol{x}_1 \right) + \beta^2 \boldsymbol{x}_2$$

$$\boldsymbol{x} = \beta^1 \alpha^0 \boldsymbol{x}_0 + \beta^1 \alpha^1 \boldsymbol{x}_1 + \beta^2 \boldsymbol{x}_2$$

Interpolación Afín



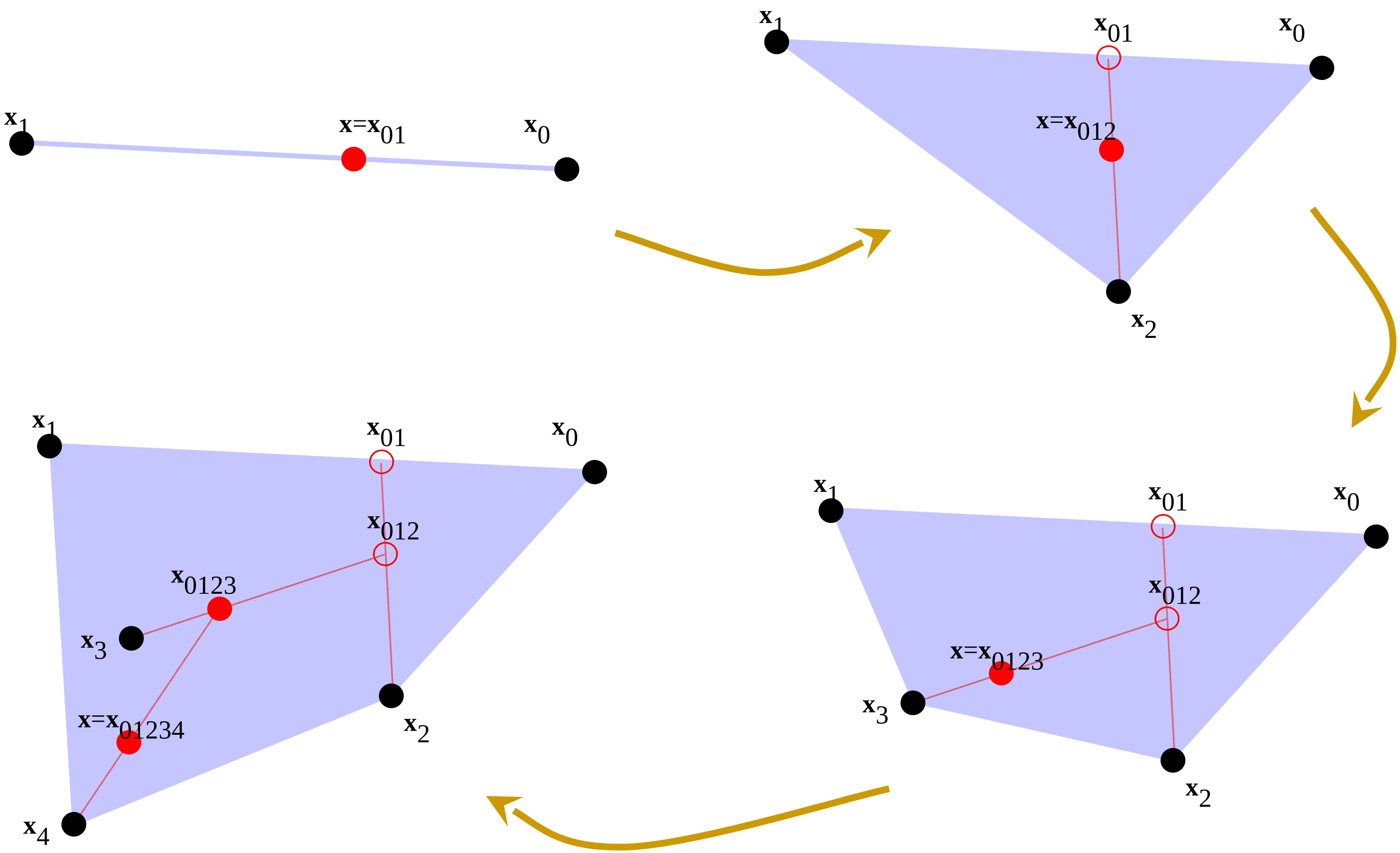
$$\boldsymbol{x}_{01} = \alpha^0 \boldsymbol{x}_0 + \alpha^1 \boldsymbol{x}_1 \quad \text{con} \quad \alpha^0 + \alpha^1 = 1$$

$$\boldsymbol{x} = \beta^1 \boldsymbol{x}_{01} + \beta^2 \boldsymbol{x}_2 \quad \text{con} \quad \beta^1 + \beta^2 = 1$$

$$\boldsymbol{x} = \beta^1 (\alpha^0 \boldsymbol{x}_0 + \alpha^1 \boldsymbol{x}_1) + \beta^2 \boldsymbol{x}_2$$

$$\boldsymbol{x} = \underbrace{\beta^1 \alpha^0}_{\gamma^0} \boldsymbol{x}_0 + \underbrace{\beta^1 \alpha^1}_{\gamma^1} \boldsymbol{x}_1 + \underbrace{\beta^2}_{\gamma^2} \boldsymbol{x}_2 \quad \text{con} \quad \gamma^0 + \gamma^1 + \gamma^2 = 1$$

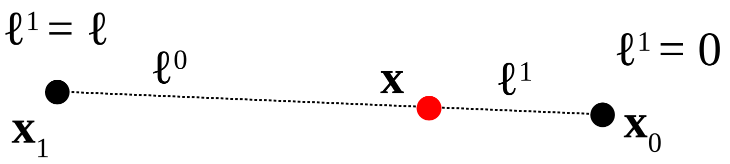
Combinación Convexa - Envoltorio Convexo



Coordenadas Baricéntricas

$$\ell^1 = \alpha^0 \ell^1{}_0 + \alpha^1 \ell^1{}_1$$

$$\ell^1 = \alpha^0 0 + \alpha^1 \ell; \quad \Rightarrow \alpha^1 = \ell^1 / \ell$$

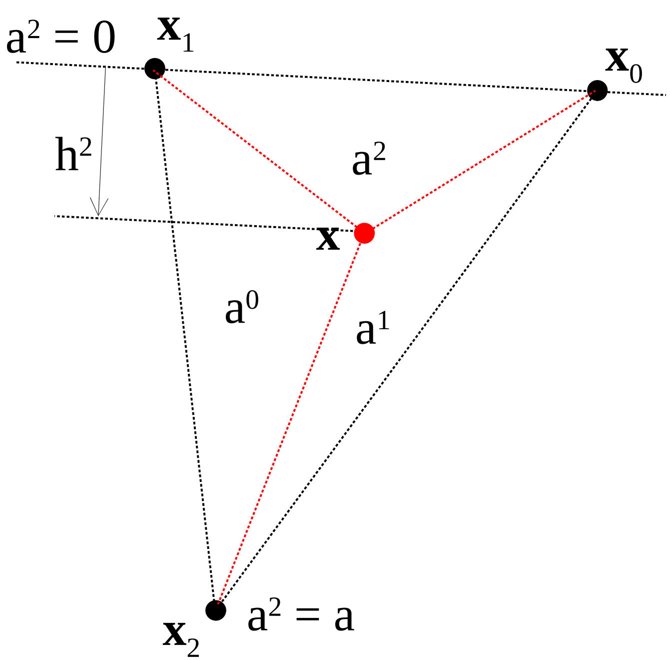


$$\boldsymbol{\ell} = (\mathbf{x}_1 - \mathbf{x}_0) \qquad\qquad \ell = |\boldsymbol{\ell}|$$

$$\ell^1 = (\mathbf{x} - \mathbf{x}_0) \cdot \boldsymbol{\ell} / \ell \qquad \ell^0 = (\mathbf{x}_1 - \mathbf{x}) \cdot \boldsymbol{\ell} / \ell$$

$$\alpha^1 = (\mathbf{x} - \mathbf{x}_0) \cdot \boldsymbol{\ell} / \ell^2 \qquad \alpha^0 = (\mathbf{x}_1 - \mathbf{x}) \cdot \boldsymbol{\ell} / \ell^2$$

Coordenadas Baricéntricas

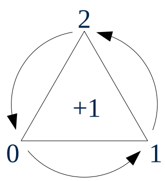


$$a^2 = \alpha^0 a^2_0 + \alpha^1 a^2_1 + \alpha^2 a^2_2$$

$$a^2 = \alpha^0 0 + \alpha^1 0 + \alpha^2 a \quad \Rightarrow \alpha^2 = a^2/a$$

$$\mathbf{a} = (\mathbf{x}_1 - \mathbf{x}_0) \times (\mathbf{x}_2 - \mathbf{x}_0)$$

$$\mathbf{a}^i = (\mathbf{x}_{i+1} - \mathbf{x}) \times (\mathbf{x}_{i+2} - \mathbf{x}) \quad (i \,\%\, 3)$$



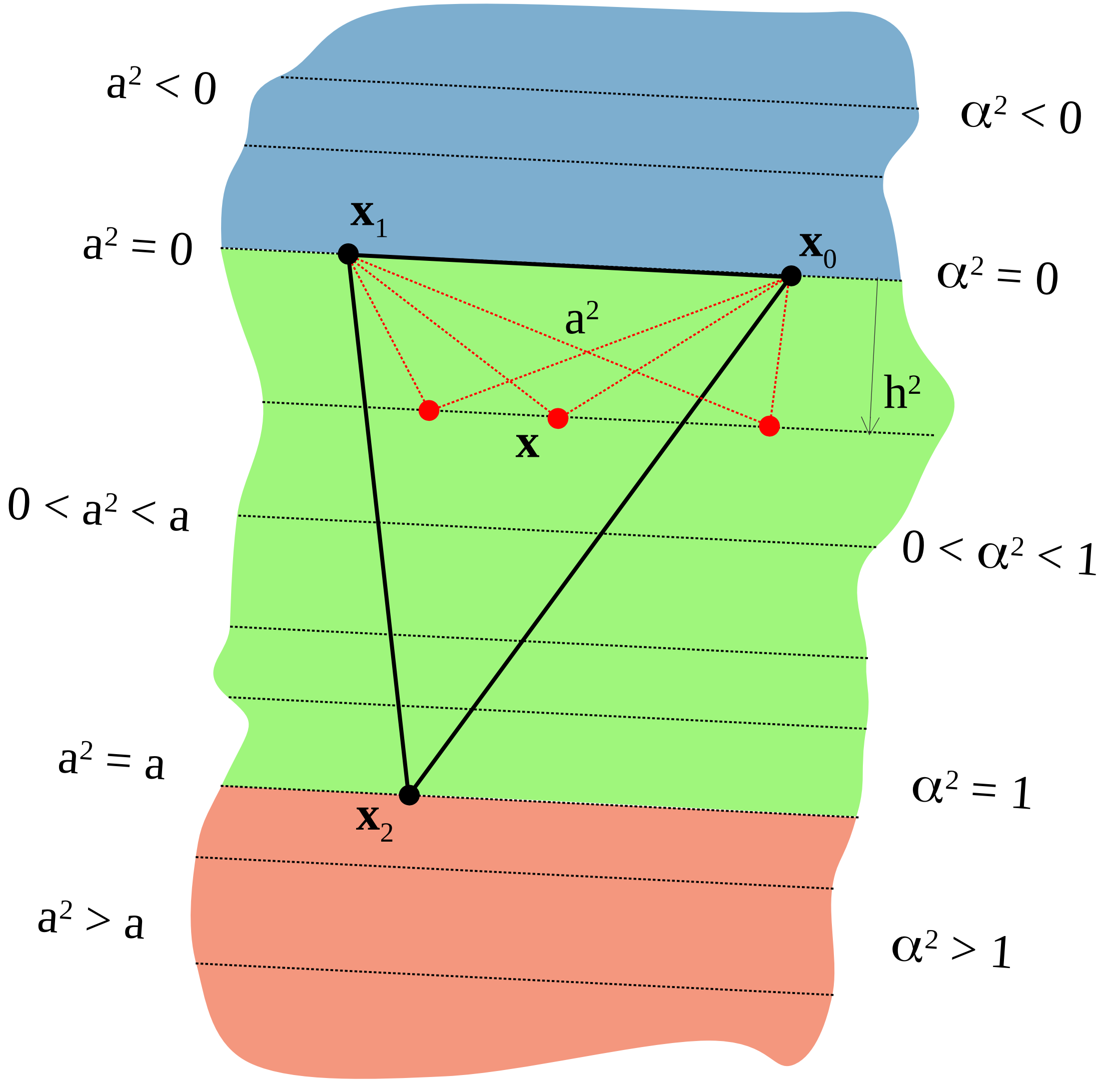
$$\alpha^i = \mathbf{a}^i \cdot \mathbf{a}/\mathbf{a}^2$$

$$v = ((\mathbf{x}_1 - \mathbf{x}_0) \times (\mathbf{x}_2 - \mathbf{x}_0)) \cdot (\mathbf{x}_3 - \mathbf{x}_0)$$

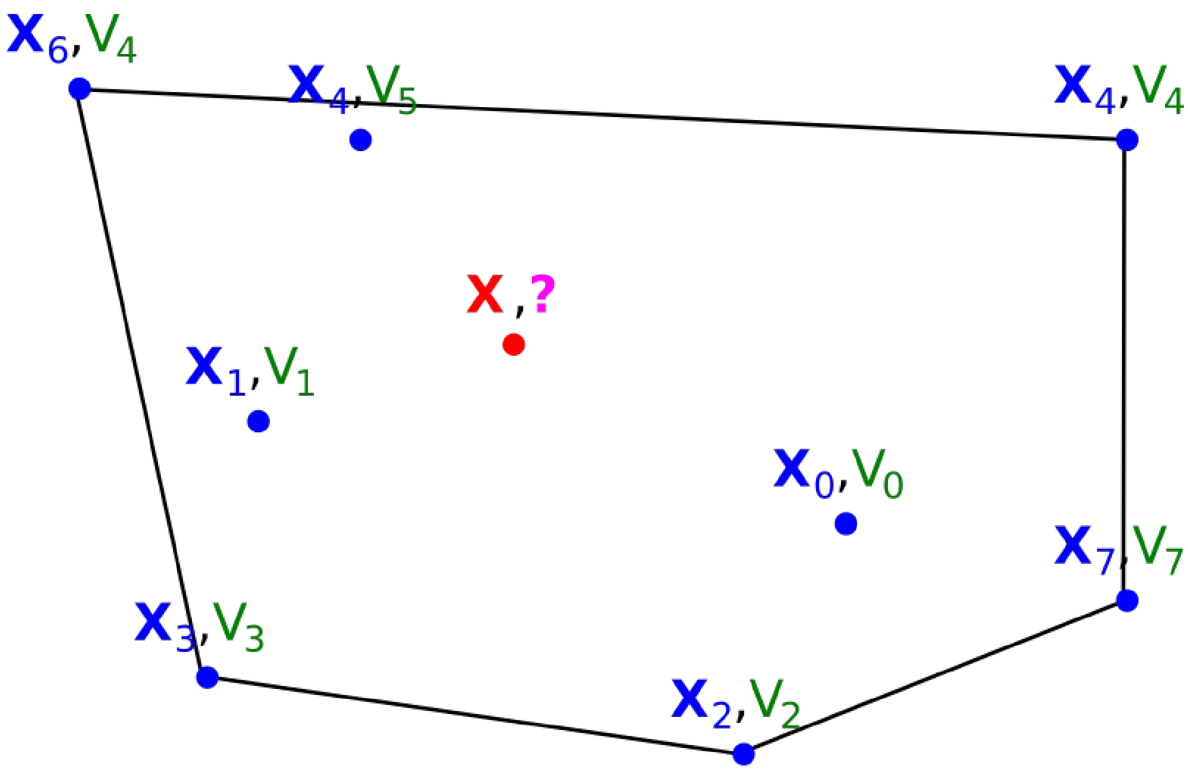
$$\alpha^i = ((\mathbf{x}_{i+1} - \mathbf{x}) \times (\mathbf{x}_{i+2} - \mathbf{x})) \cdot (\mathbf{x}_{i+3} - \mathbf{x}) \,/\, v \qquad (i \,\%\, 4)$$

Lineas Isoparamétricas

$\alpha^2 = a^2/a.$

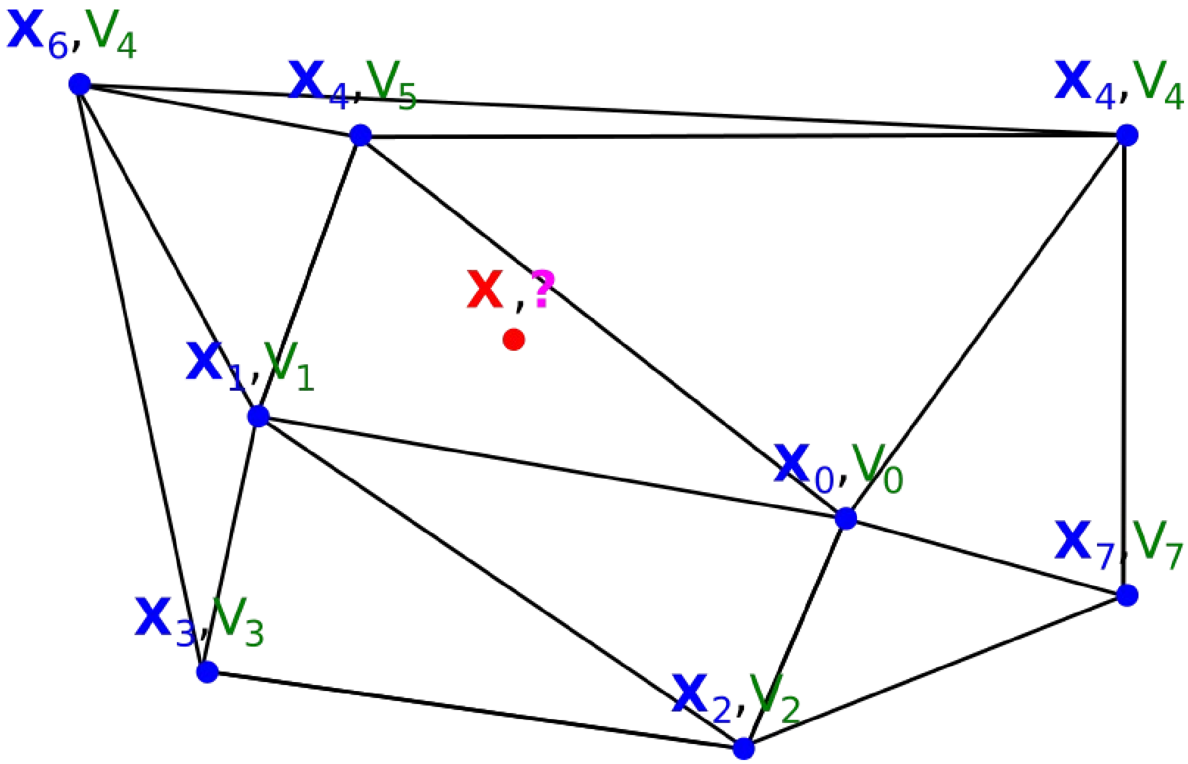


Interpolación Afín de “Muchos” Puntos



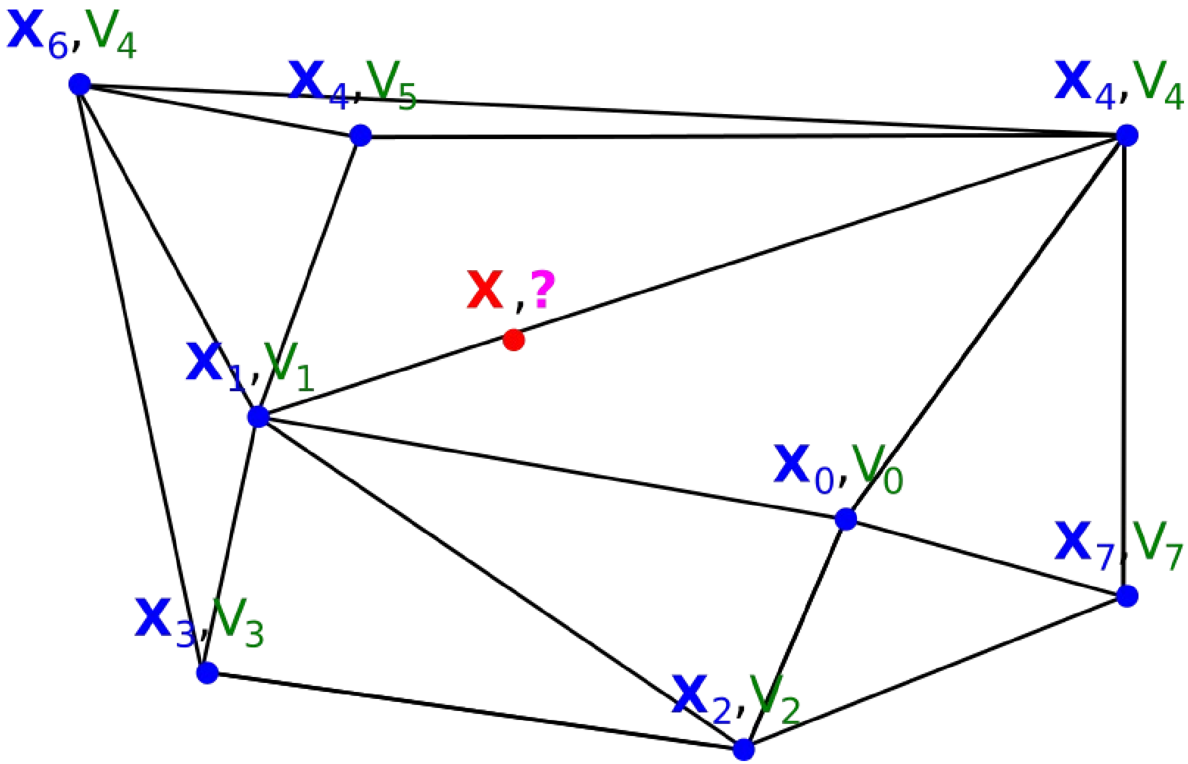
Resta definir cómo obtener los pesos.

Interpolación Afín de “Muchos” Puntos



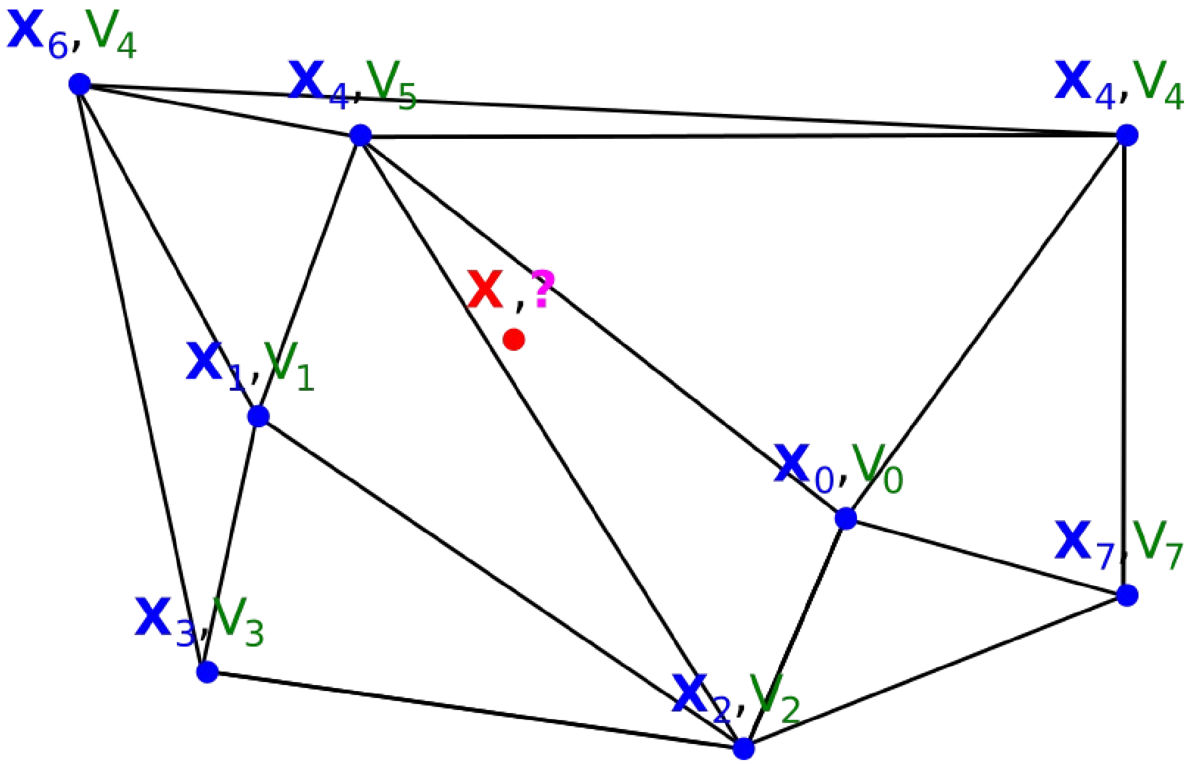
Posible método: armar una triangulación y utilizar sólo los tres nodos del triángulo que lo contiene

Interpolación Afín de “Muchos” Puntos



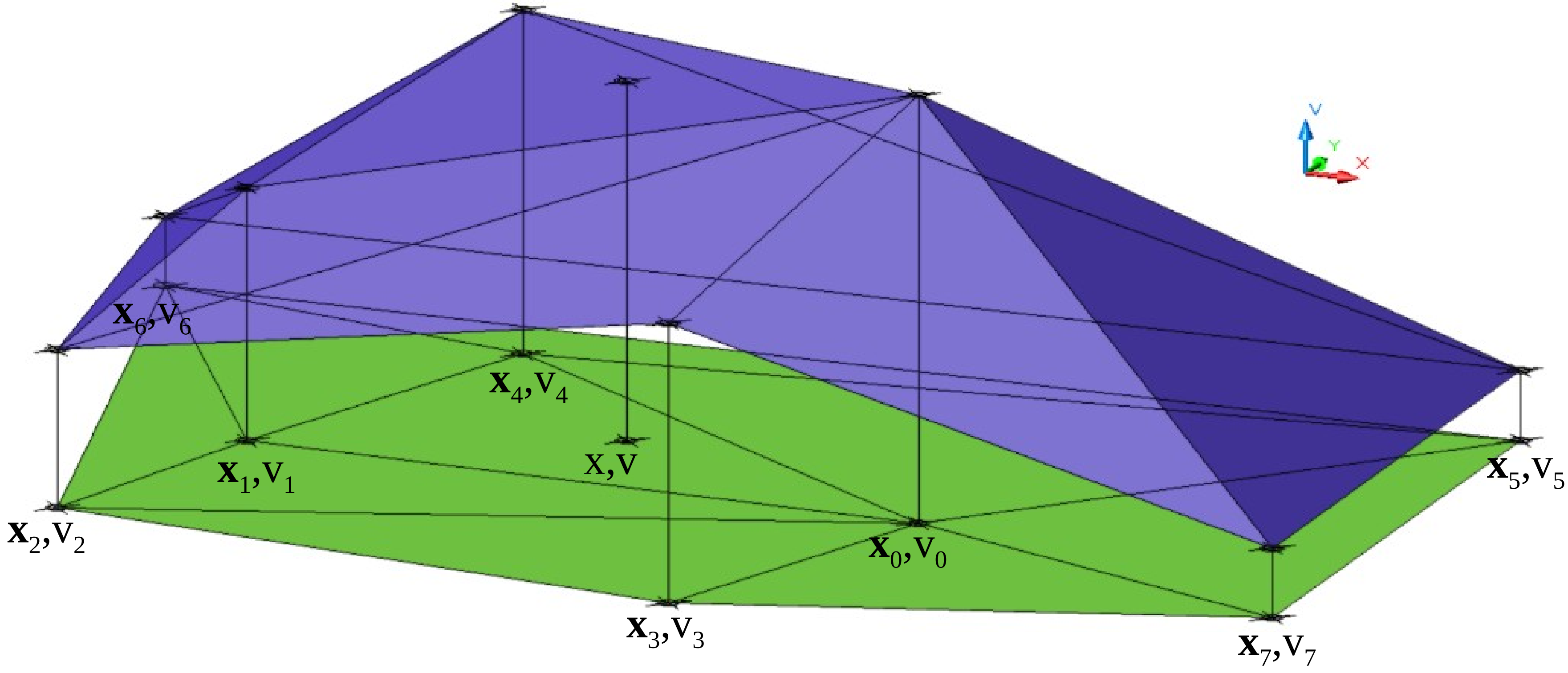
Sigue habiendo varias formas!

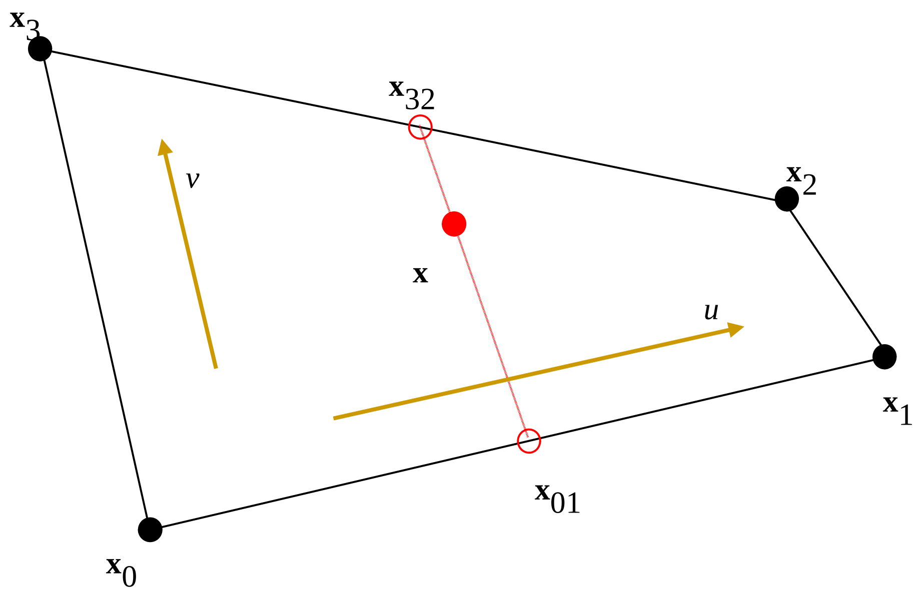
Interpolación Afín de “Muchos” Puntos



Sigue habiendo varias formas!

Interpolación Lineal por Tramos





$$x_{01}=(1-u)\,x_0+u\,x_1$$

$$x_{32}=(1-u)\,x_3+u\,x_2$$

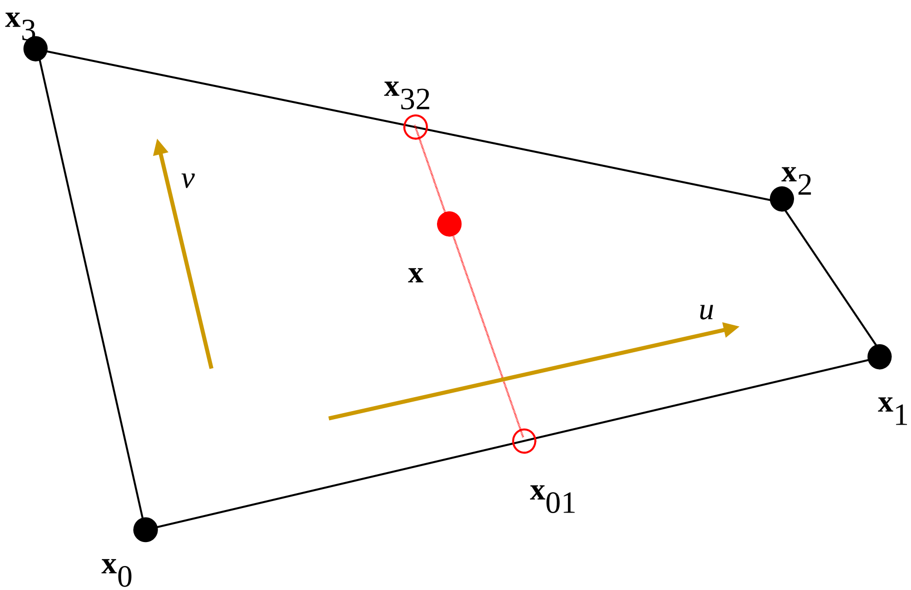
$$x=(1-v)\,x_{01}+v\,x_{32}$$

$$x=(1-v)\underbrace{\left((1-u)\,x_0+u\,x_1\right)}_{x_{01}}+v\underbrace{\left((1-u)\,x_3+u\,x_2\right)}_{x_{32}}$$

$$x=\underbrace{(1-v)(1-u)}_{\alpha_0}\,x_0+\underbrace{(1-v)\,u}_{\alpha_1}\,x_1+\underbrace{v\,(1-u)}_{\alpha_3}\,x_3+\underbrace{v\,u}_{\alpha_2}\,x_2$$

con $\quad \alpha^0+\alpha^1+\alpha^2+\alpha^3=1$

Interpolación Bilineal



$$x_{01} = (1-u)\,x_0 + u\,x_1$$

$$x_{32} = (1-u)\,x_3 + u\,x_2$$

$$x = (1-v)\,x_{01} + v\,x_{32}$$

$$x = (1-v)\underbrace{\left((1-u)\,x_0 + u\,x_1\right)}_{x_{01}} + v\underbrace{\left((1-u)\,x_3 + u\,x_2\right)}_{x_{32}}$$

$$x = \underbrace{(1-v-u+vu)}_{\alpha_0}\,x_0 + \underbrace{(u-vu)}_{\alpha_1}\,x_1 + \underbrace{(v-vu)}_{\alpha_3}\,x_3 + \underbrace{vu}_{\alpha_2}\,x_2$$

con $\quad \alpha^0 + \alpha^1 + \alpha^2 + \alpha^3 = 1$

Interpolación Hiperbólica

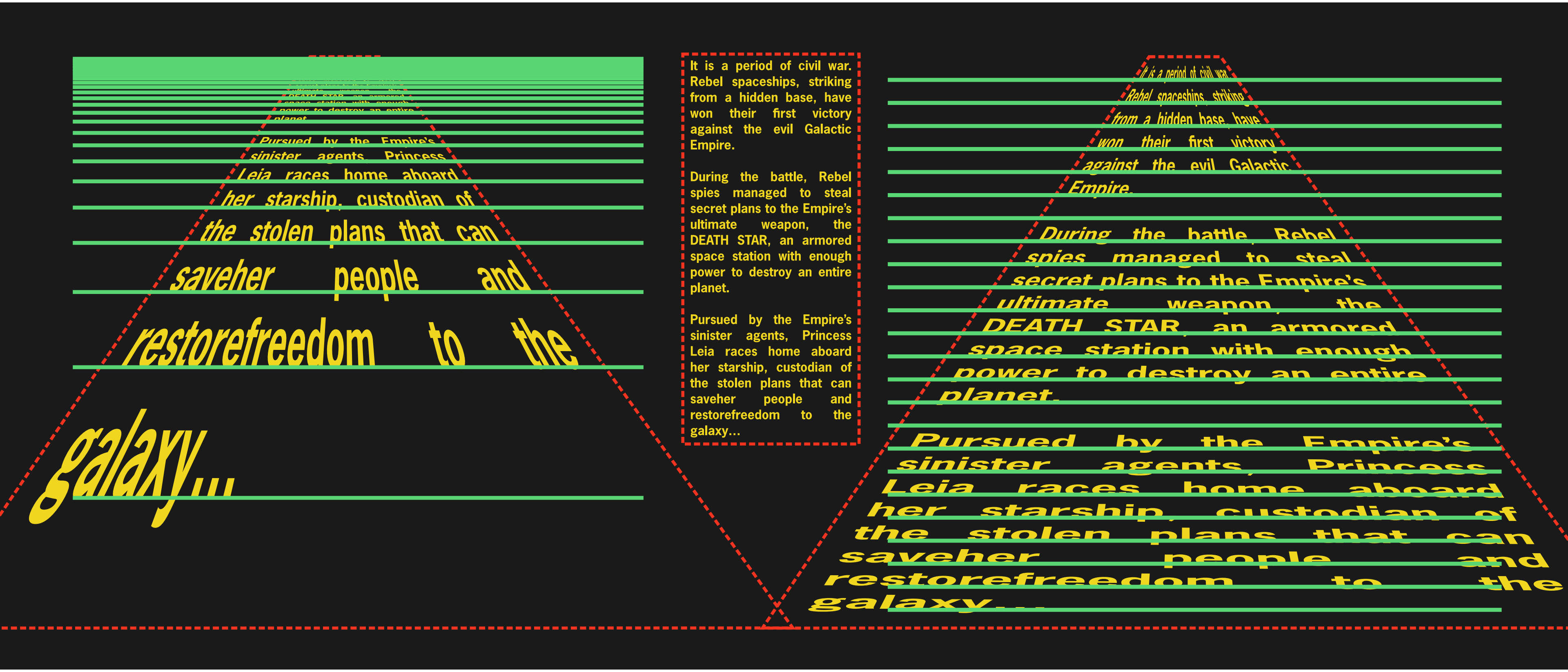
$$\{w\boldsymbol{x}, w\} = \alpha^0\{w_0\boldsymbol{x_0}, w_0\} + \alpha^1\{w_1\boldsymbol{x_1}, w_1\}$$

$$\{w\boldsymbol{x}, w\} = \{\alpha^0 w_0\boldsymbol{x_0}, \alpha^0 w_0\} + \{\alpha^1 w_1\boldsymbol{x_1}, \alpha^1 w_1\}$$

$$\{w\boldsymbol{x}, w\} = \{\alpha^0 w_0\boldsymbol{x_0} + \alpha^1 w_1\boldsymbol{x_1}, \alpha^0 w_0 + \alpha^1 w_1\} \qquad \Rightarrow w = \alpha^0 w_0 + \alpha^1 w_1$$

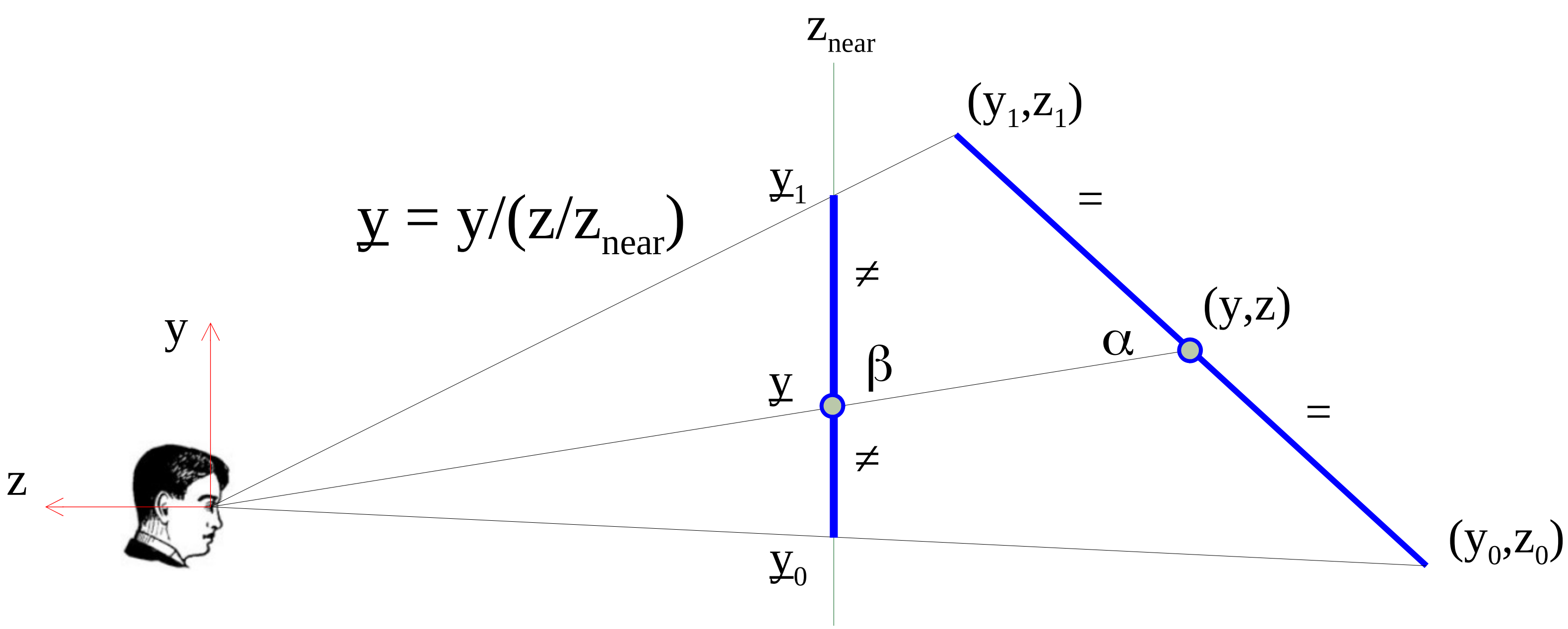
$$\boldsymbol{x} = \frac{\alpha^0 w_0}{w}\boldsymbol{x_0} + \frac{\alpha^1 w_1}{w}\boldsymbol{x_1} \qquad \Rightarrow \qquad \beta_i = \frac{\alpha^i w_i}{\sum_j \alpha^j w_j} \qquad \wedge \qquad \alpha^i = \frac{\beta^i w}{w_i}$$

En una galaxia muy muy lejana...



Interpolación Hiperbólica

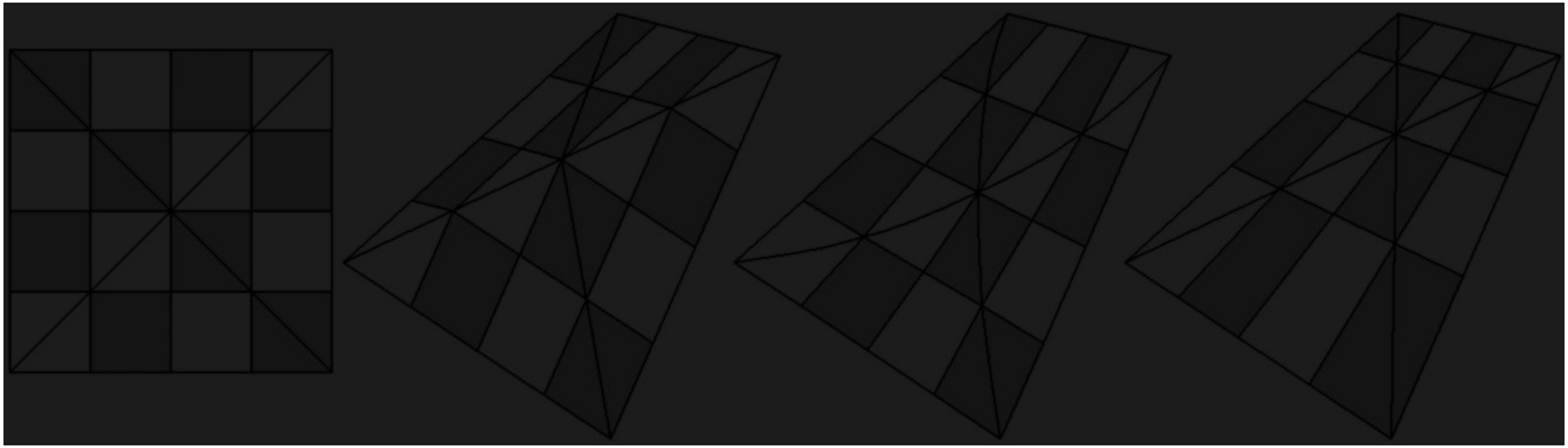
Interpolación Hiperbólica



Sigue siendo combinación afín, pero distinta:

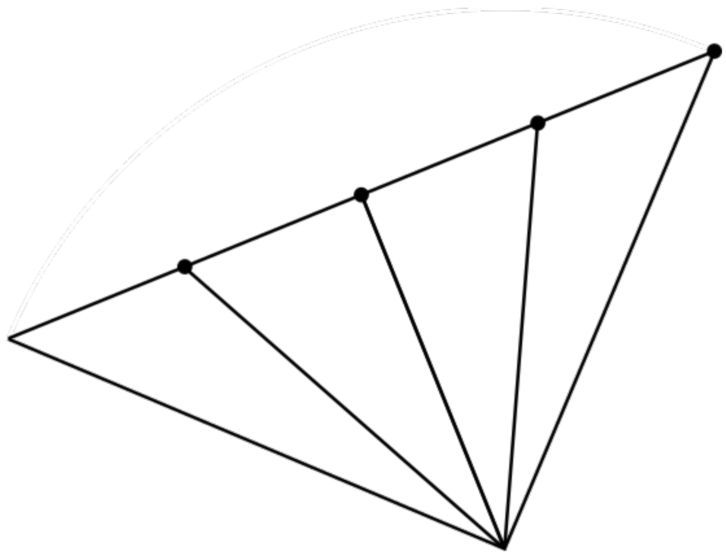
$$b^0 + b^1 = a^0\, w_0/w + a^1\, w_1/w = w/w = 1$$

Interpolación en Cuadriláteros

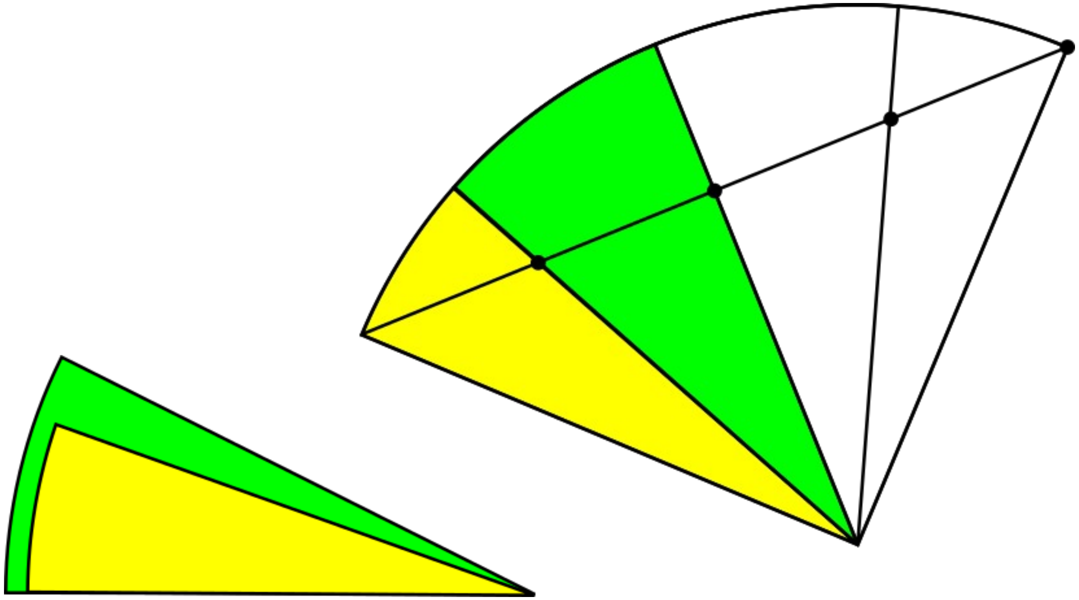


Original Dos Lineales Bilineal Hiperbólica

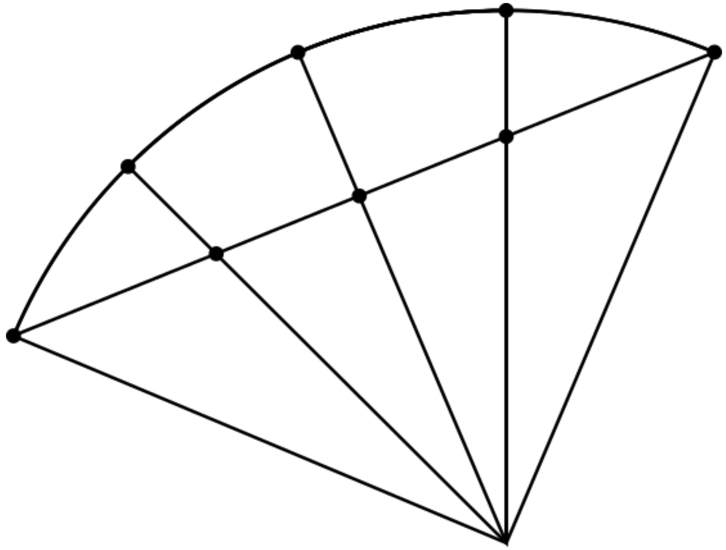
Interpolación Esférica



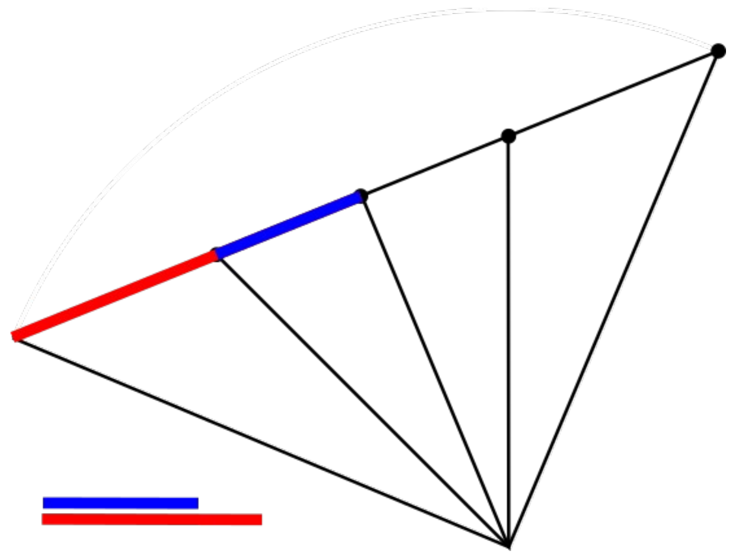
Interpolación Esférica



Interpolación Esférica

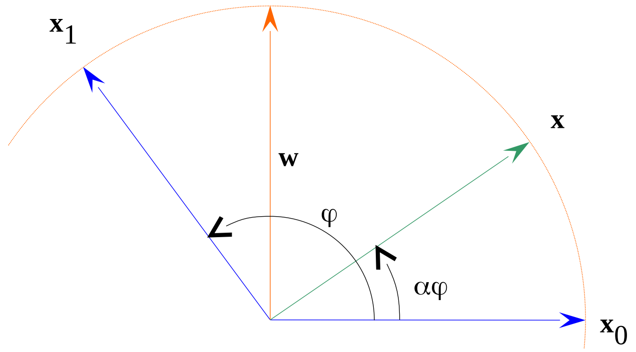


Interpolación Esférica



Interpolación Esférica: SLERP

$\varphi = \mathrm{acos}(\mathbf{x}_0 \cdot \mathbf{x}_1)$



$\mathbf{x} = \cos(\alpha\varphi)\mathbf{x}_0 + \mathrm{sen}(\alpha\varphi)\mathbf{w}$

$\mathbf{x}_1 = \cos(\varphi)\mathbf{x}_0 + \mathrm{sen}(\varphi)\mathbf{w} \quad \Rightarrow \mathbf{w}$

$\mathbf{x} = \cos(\alpha\varphi)\mathbf{x}_0 + (\mathrm{sen}(\alpha\varphi)/\mathrm{sen}(\varphi))(\mathbf{x}_1 - \cos(\varphi)\mathbf{x}_0)$

$\mathrm{sen}(\varphi)\mathbf{x} = \cos(\alpha\varphi)\mathrm{sen}(\varphi)\mathbf{x}_0 + \mathrm{sen}(\alpha\varphi)\,(\mathbf{x}_1 - \cos(\varphi)\mathbf{x}_0) =$

$= [\cos(\alpha\varphi)\mathrm{sen}(\varphi) - \mathrm{sen}(\alpha\varphi)\cos(\varphi)]\mathbf{x}_0 + \mathrm{sen}(\alpha\varphi)\mathbf{x}_1 =$

$= \mathrm{sen}(\varphi - \alpha\varphi)\mathbf{x}_0 + \mathrm{sen}(\alpha\varphi)\mathbf{x}_1 \quad = \quad \mathrm{sen}[(1-\alpha)\varphi]\mathbf{x}_0 + \mathrm{sen}(\alpha\varphi)\mathbf{x}_1$

$\mathbf{X} = \mathrm{slerp}(\mathbf{x}_0, \mathbf{x}_1, \alpha) = \{\mathrm{sen}[(1-\alpha)\varphi]\mathbf{x}_0 + \mathrm{sen}(\alpha\varphi)\mathbf{x}_1\} / \mathrm{sen}(\varphi)$

Cuaterniones

Complejos: $c = a+b\mathbf{i}$ con $\mathbf{i}^2 = -1$

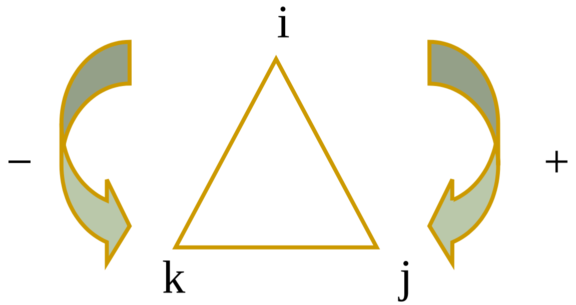
Base: $\{ 1, \mathbf{i} \}$ Componentes: $\{ a, b \}$

Cuaterniones: $q = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ con $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$

y con $\mathbf{i}.\mathbf{j} = \mathbf{k}$, $\mathbf{j}.\mathbf{k} = \mathbf{i}$, $\mathbf{k}.\mathbf{i} = \mathbf{j}$

y también $\mathbf{j}.\mathbf{i} = -\mathbf{k}$, $\mathbf{k}.\mathbf{j} = -\mathbf{i}$, $\mathbf{i}.\mathbf{k} = -\mathbf{j}$

Base: $\{ 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \}$ Componentes: $\{ a, b, c, d \}$



Representación Escalar-Vector:

$q = <d,\mathbf{u}>;\mathbf{u}= \{a,b,c\}$

Vector o cuaternión puro: $\mathbf{u}= <0,\mathbf{u}>$

Escalar: $d = <d,\mathbf{0}>$

Cuaterniones

Producto distribuido (todos contra todos)

$$q_1q_2 = a_1a_2ii + a_1b_2ij + a_1c_2ik + a_1d_2i1 + b_1a_2ji + b_1b_2jj + b_1c_2jk + b_1d_2j1 +$$
$$+ c_1a_2ki + c_1b_2kj + c_1c_2kk + c_1d_2k1 + d_1a_21i + d_1b_21j + d_1c_21k + d_1d_211$$

$$q_1q_2 = -a_1a_21 + a_1b_2k - a_1c_2j + a_1d_2i - b_1a_2k - b_1b_21 + b_1c_2i + b_1d_2j +$$
$$+ c_1a_2j - c_1b_2i - c_1c_21 + c_1d_2k + d_1a_2i + d_1b_2j + d_1c_2k + d_1d_21$$

$$q_1q_2 = d_1d_2 - (a_1a_2 + b_1b_2 + c_1c_2) + d_1(a_2i + b_2j + c_2k) + d_2(a_1i + b_1j + c_1k) +$$
$$+ i(b_1c_2 - c_1b_2) + j(c_1a_2 - a_1c_2) + k(a_1b_2 - b_1a_2)$$

$$q_1q_2 = <(d_1d_2 - \mathbf{u}_1 \cdot \mathbf{u}_2), (d_1\mathbf{u}_2 + d_2\mathbf{u}_1 + \mathbf{u}_1 \times \mathbf{u}_2)>$$

Cuaterniones

Producto distribuido (todos contra todos)

$q_1 q_2 = <(d_1 d_2 - \mathbf{u}_1 \cdot \mathbf{u}_2), (d_1 \mathbf{u}_2 + d_2 \mathbf{u}_1 + \mathbf{u}_1 \times \mathbf{u}_2)>$

$q_1 q_2 \neq q_2 q_1$ sólo porque: $\mathbf{u}_1 \times \mathbf{u}_2 \neq \mathbf{u}_2 \times \mathbf{u}_1$

(si $\mathbf{u}_1 \times \mathbf{u}_2 = 0$ (paralelos) $\Rightarrow q_1 q_2 = q_2 q_1$)

Conjugado:

$q^* = <d, -\mathbf{u}>$ $\qquad$ $q\,q^* = q^* q = a^2 + b^2 + c^2 + d^2 = d^2 + \mathbf{u}^2$

Cuaterniones

Norma o Magnitud Módulo:

$$\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{d^2 + \mathbf{u}^2}$$

Inverso:

$$q^{-1} = q^* / \|q\|^2 \qquad qq^{-1} = q^{-1}q = 1$$

Normalización

$$q_1 = q/\|q\| \qquad q_1^{-1} = q_1^*$$

Rotación con Cuaterniones

Expresión inicial:

$\underline{\mathbf{v}} = q\mathbf{v}q^{-1} = q < 0,\mathbf{v}> q^{-1}$

$q^{-1} = (||q||q_1)^{-1} = ||q||^{-1}q_1^{-1} \quad \Rightarrow \quad (||q||q_1)\mathbf{v}(||q||q_1)^{-1} = q_1\mathbf{v}q_1^{-1}$

Sigue con cuaterniones unitarios:

$q = < \cos(\alpha);\mathbf{u}\mathrm{sen}(\alpha) > \quad (||\mathbf{u}||=1) \qquad ||q||^2 = \cos^2(\alpha)+\mathbf{u}^2\mathrm{sen}^2(\alpha) = 1$

Rotación con Cuaterniones

Expresión : $\underline{\mathbf{v}} = q\mathbf{v}q^{-1} = <\cos(\alpha);\mathbf{u}\,\text{sen}(\alpha)><0,\mathbf{v}><\cos(\alpha); -\mathbf{u}\,\text{sen}(\alpha)>$

Aplicado a un vector paralelo a **u**: $\mathbf{v}// = \gamma\mathbf{u}$:

$\underline{\mathbf{v}}// = <\cos(\alpha);\mathbf{u}\,\text{sen}(\alpha)> <0; \gamma\mathbf{u}> <\cos(\alpha); -\mathbf{u}\,\text{sen}(\alpha)>$

$= <-\gamma\,\text{sen}(\alpha); \gamma\cos(\alpha)\mathbf{u}><\cos(\alpha); -\mathbf{u}\,\text{sen}(\alpha)>$

$= <[\gamma\,\text{sen}(\alpha)\cos(\alpha)-\gamma\,\text{sen}(\alpha)\cos(\alpha)]; \gamma\mathbf{u}[\cos^2(\alpha) + \text{sen}^2(\alpha)]>$

$= <0; \gamma\mathbf{u}>$

$=\mathbf{v}//$

Un vector paralelo a **u** no cambia.

Rotación con Cuaterniones

Expresión : $\underline{\mathbf{v}} = q\mathbf{v}q^{-1} = \langle\cos(\alpha);\mathbf{u}\,\mathrm{sen}(\alpha)\rangle\langle 0,\mathbf{v}\rangle\langle\cos(\alpha);-\mathbf{u}\,\mathrm{sen}(\alpha)\rangle$

Aplicado a un vector perpendicular a **u**: $\mathbf{v}_\perp \cdot \mathbf{u} = 0$ y usando $\mathbf{w} = \mathbf{u} \times \mathbf{v}_\perp$:

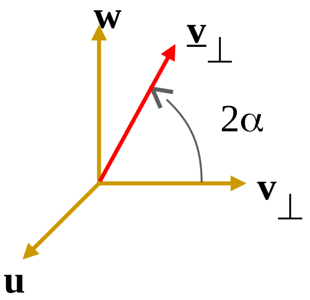
$$\underline{\mathbf{v}}_\perp = \langle\cos(\alpha);\mathbf{u}\,\mathrm{sen}(\alpha)\rangle\langle 0;\mathbf{v}_\perp\rangle\langle\cos(\alpha);-\mathbf{u}\,\mathrm{sen}(\alpha)\rangle$$

$$= \langle 0;\cos(\alpha)\mathbf{v}_\perp + \mathrm{sen}(\alpha)\mathbf{w}\rangle\langle\cos(\alpha);-\mathbf{u}\,\mathrm{sen}(\alpha)\rangle$$

$$= \langle 0;\cos^2(\alpha)\mathbf{v}_\perp + \mathrm{sen}(\alpha)\cos(\alpha)\mathbf{w} + \mathrm{sen}(\alpha)\cos(\alpha)\mathbf{w} - \mathrm{sen}^2(\alpha)\mathbf{v}_\perp\rangle$$

$$= \langle 0;\{[\cos^2(\alpha) - \mathrm{sen}^2(\alpha)]\mathbf{v}_\perp + 2\,\mathrm{sen}(\alpha)\cos(\alpha)\mathbf{w}\}\rangle$$

$$= \langle 0;[\cos(2\alpha)\mathbf{v}_\perp + \mathrm{sen}(2\alpha)\mathbf{w}]\rangle$$



Un vector perpendicular a **u** gira 2α.

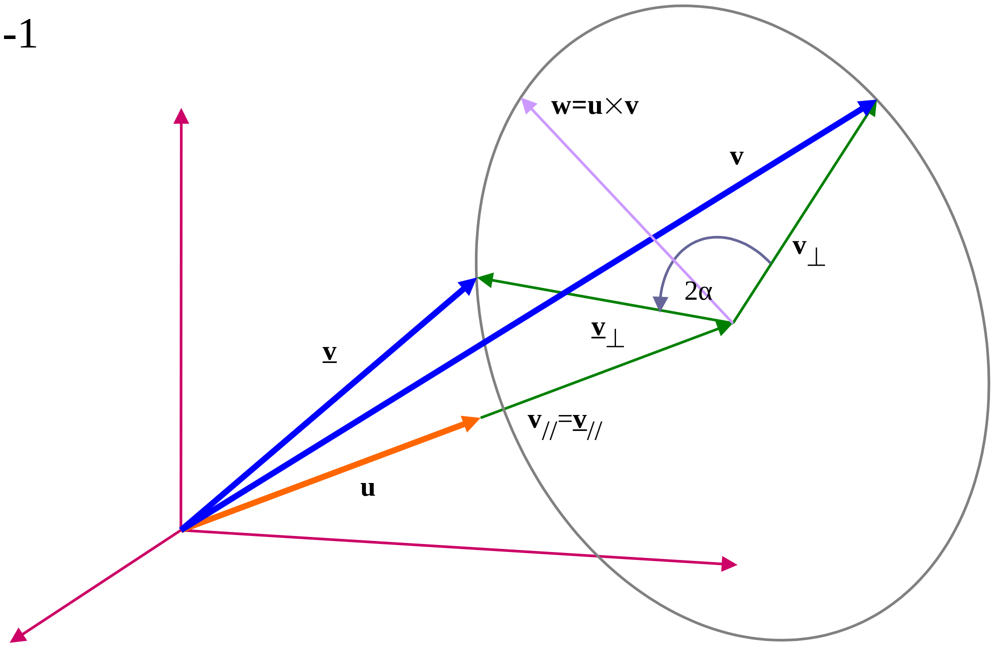
Rotación con Cuaterniones

Transformación :

$\underline{\mathbf{v}}= q\mathbf{v}q^{-1} = <\cos(\alpha);\mathbf{u}\mathrm{sen}(\alpha)>< 0,\mathbf{v}><\cos(\alpha); -\mathbf{u}\mathrm{sen}(\alpha)>$

El resultado es un vector. Es una rotación de ángulo 2α alrededor de**u**.

$q_1 q_2 q_2^{-1} q_1^{-1} = 1 \Rightarrow (q_1 q_2)^{-1} = q_2^{-1} q_1^{-1}$



Combinación:

$q_1(q_2\mathbf{v}q_2^{-1})\, q_1^{-1} = (q_1 q_2)\mathbf{v}(q_1 q_2)^{-1}$

Interpolación: Slerp en S_3