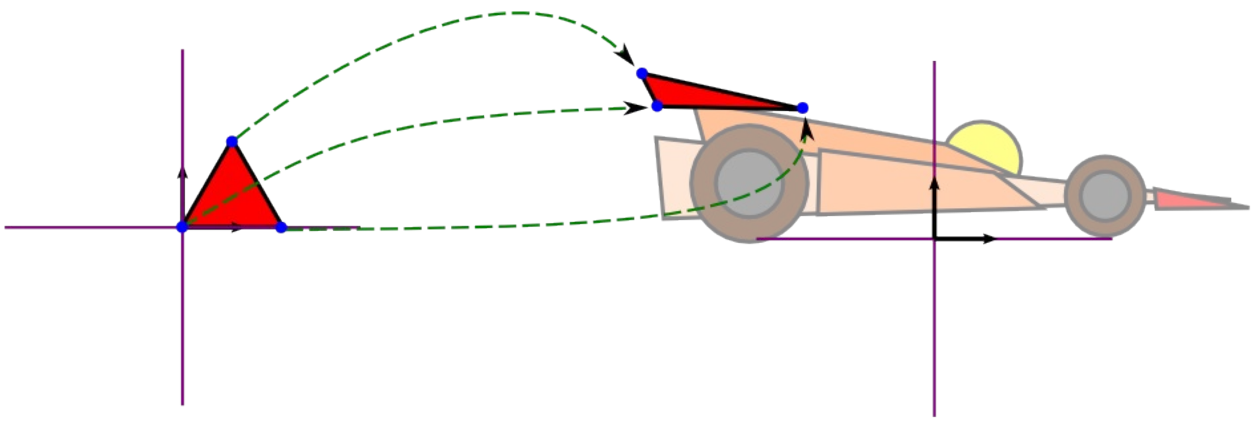
Unidad 4

Espacios y Transformaciones

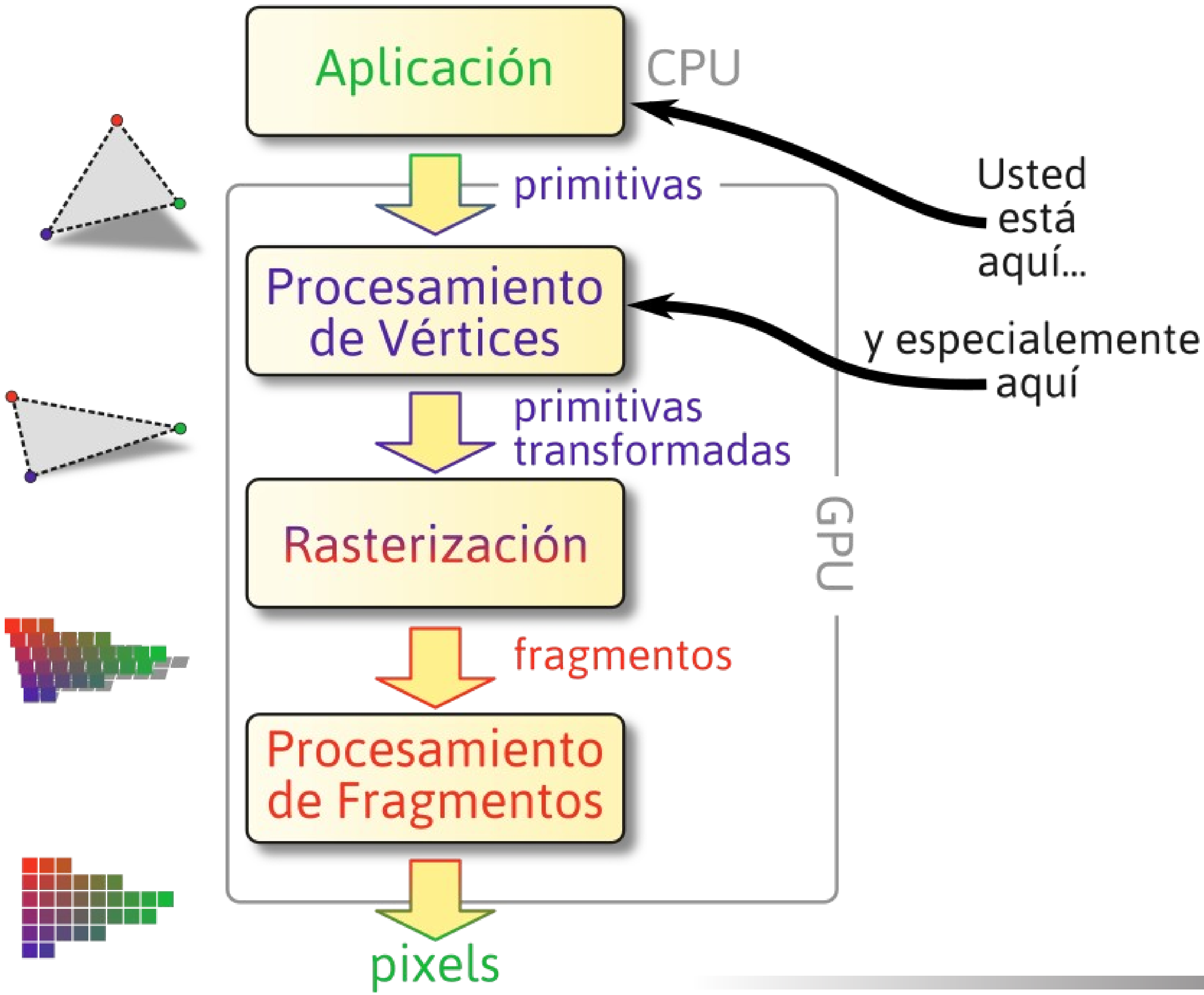
Universidad Nacional del Litoral
Facultad de Ingeniería y Ciencias Hídricas

Transformación: función o mapeo que hace corresponder cada punto del espacio con otro punto del mismo espacio.

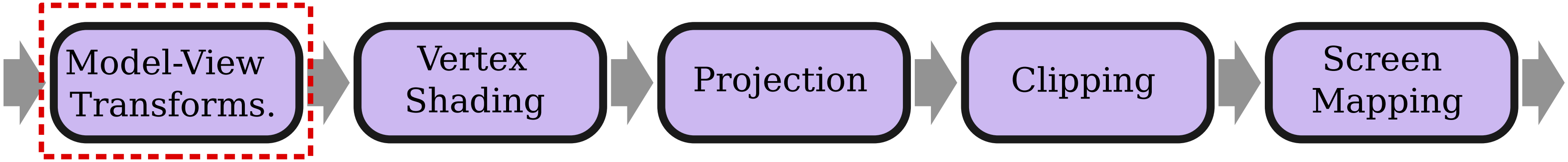
$$\underline{\hat{P}} = \boldsymbol{T}\left(\underline{P}\right)$$

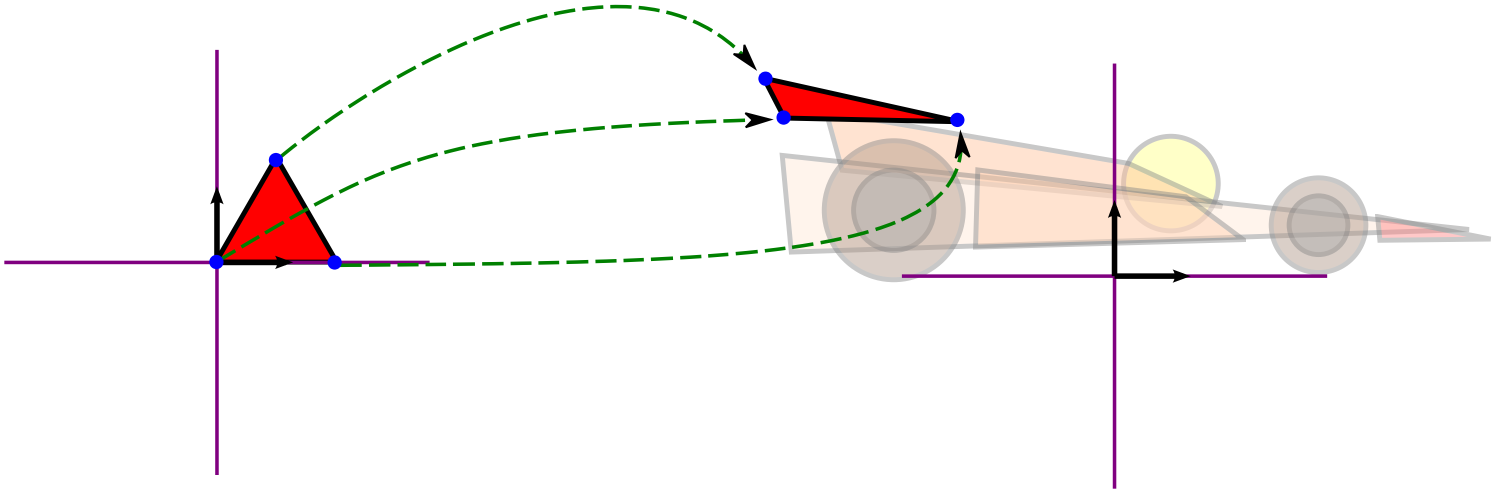


Transformaciones en el Pipeline

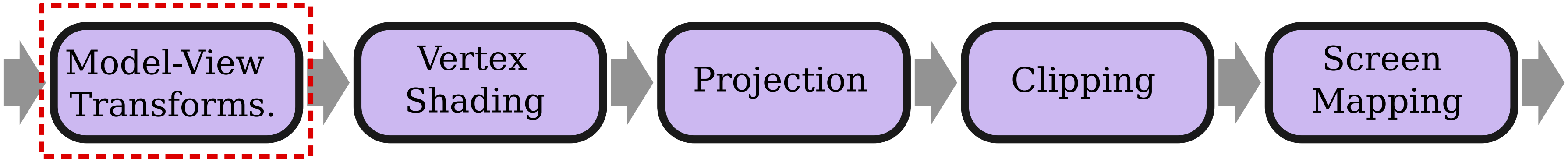


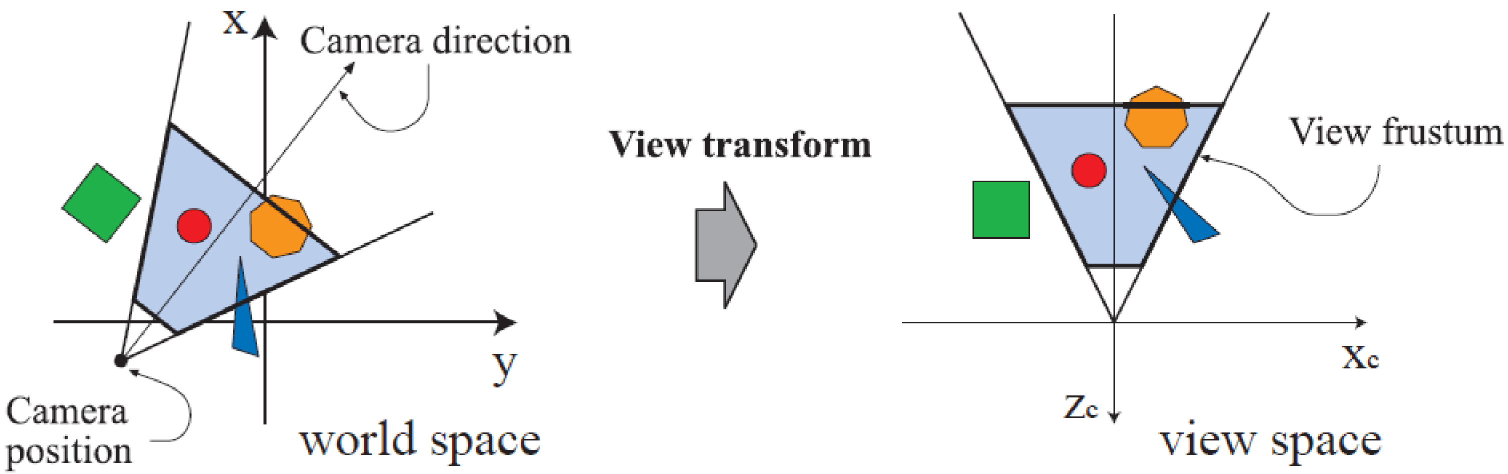
Procesamiento de Vértices: Model Matrix



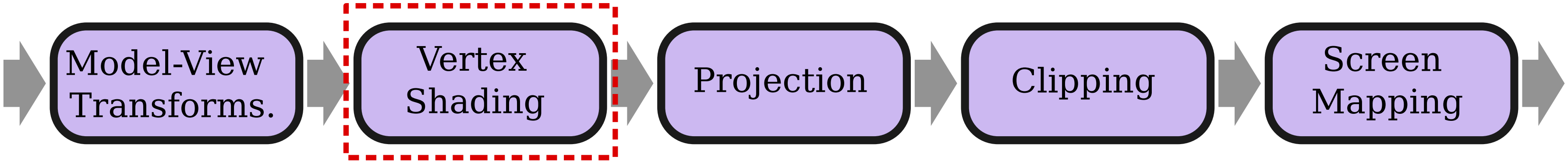


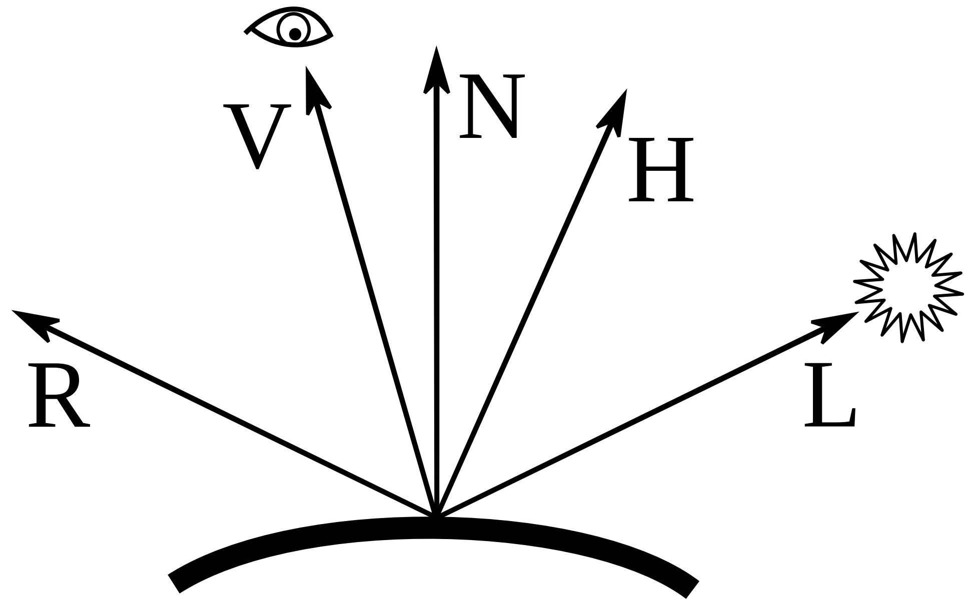
Procesamiento de Vértices: View Matrix



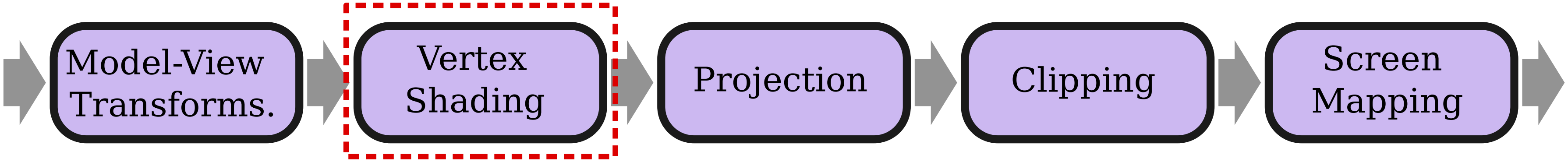


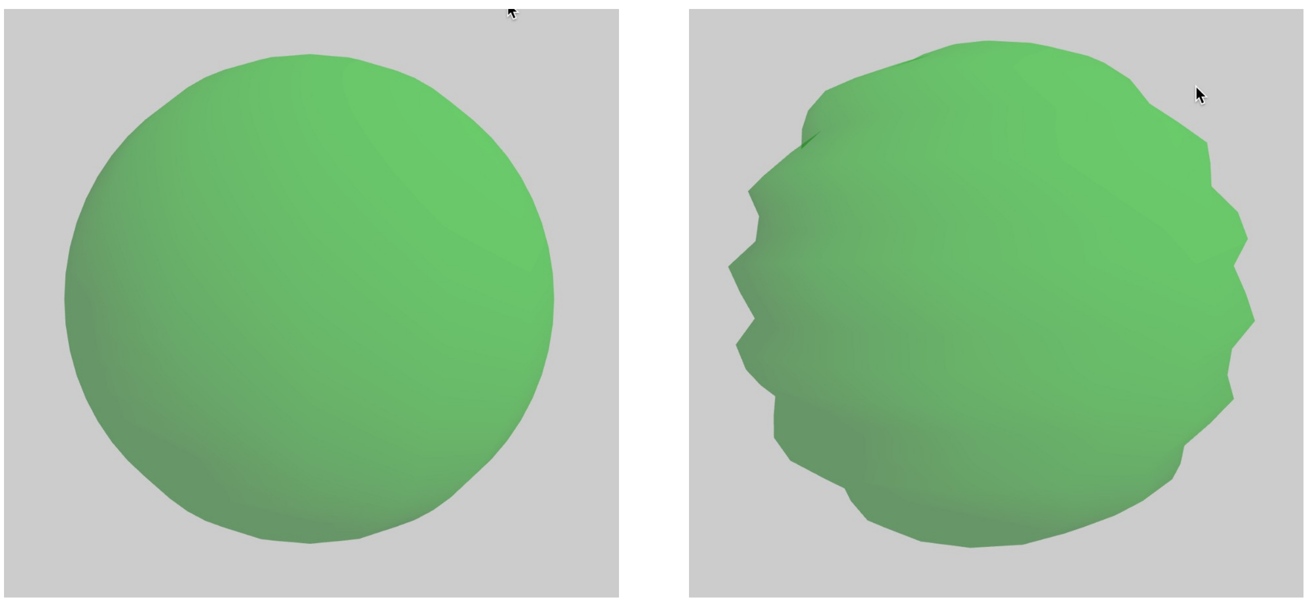
Procesamiento de Vértices: Vertex Shading



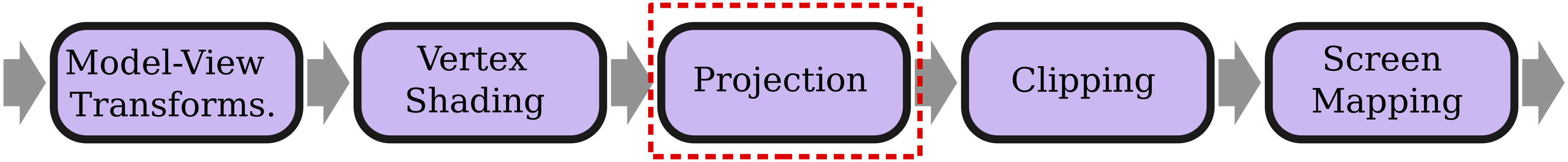


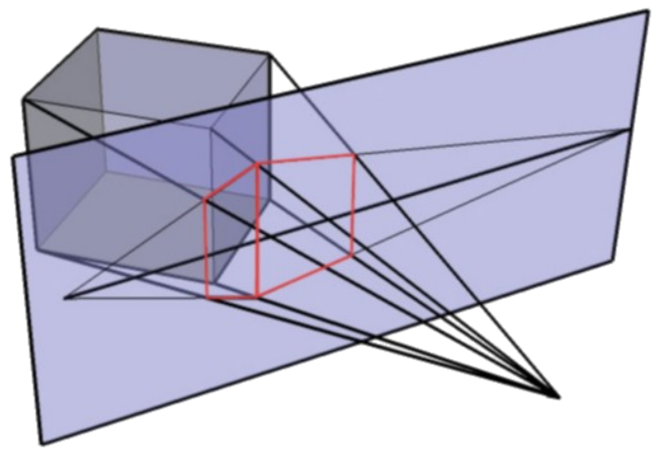
Procesamiento de Vértices: Vertex Shader



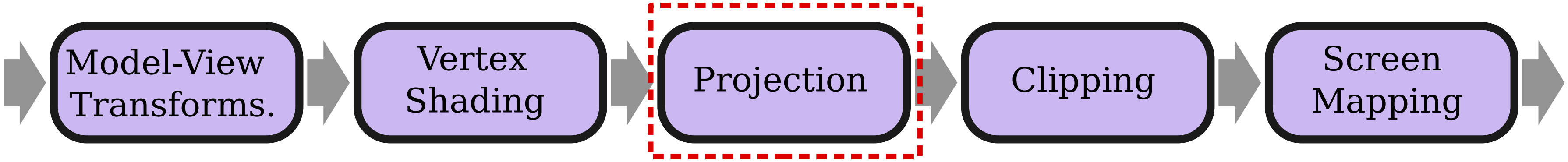


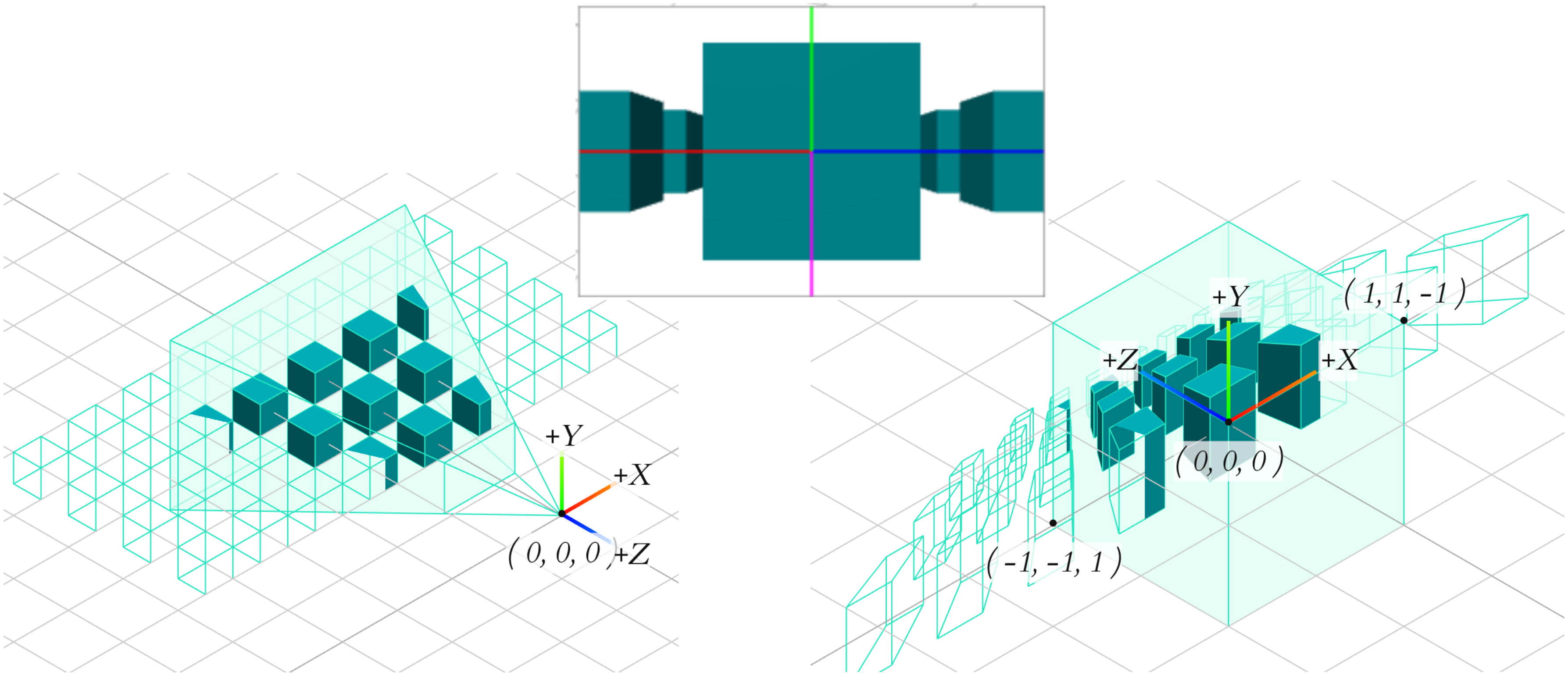
Procesamiento de Vértices: Projection



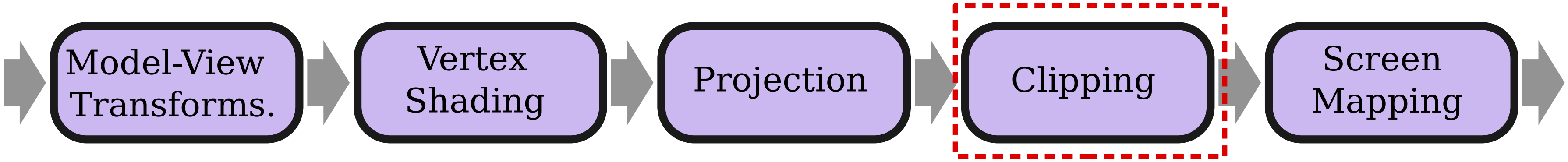


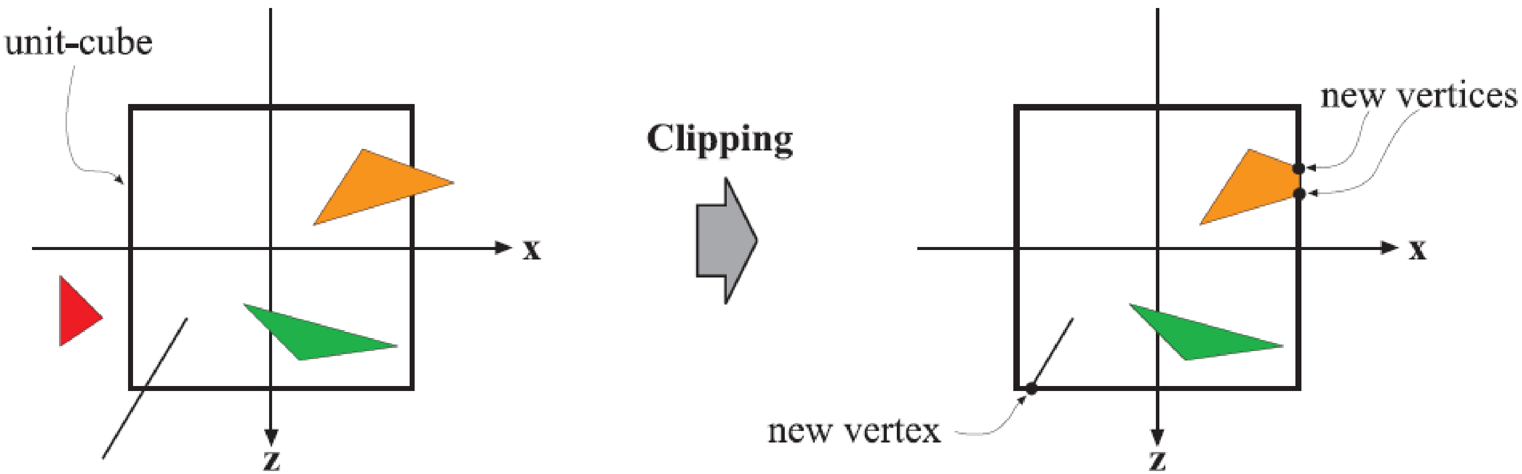
Procesamiento de Vértices: Projection Matrix





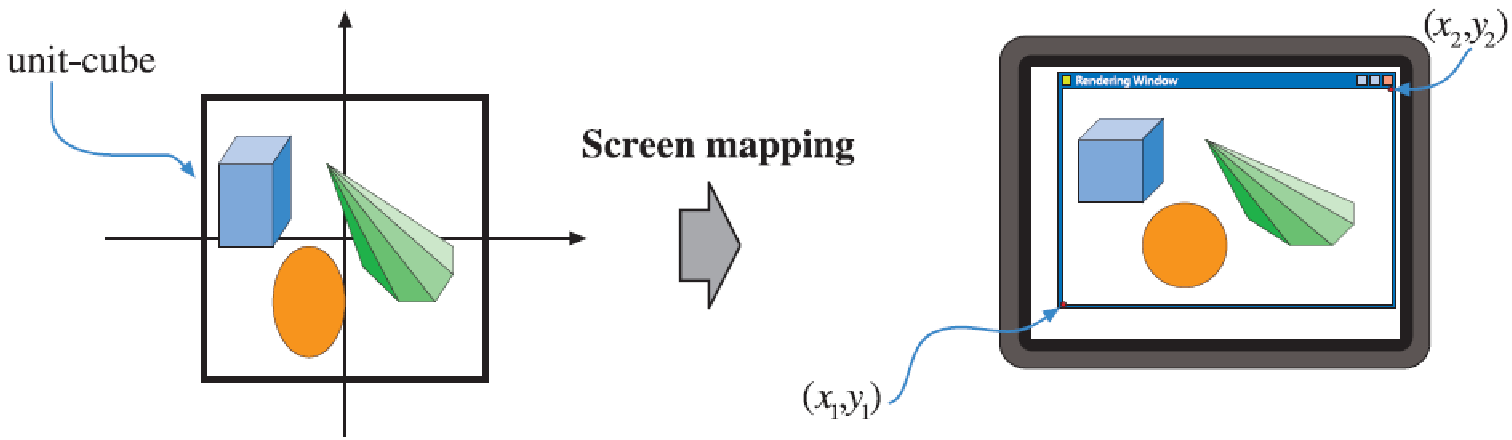
Procesamiento de Primitivas: Clipping



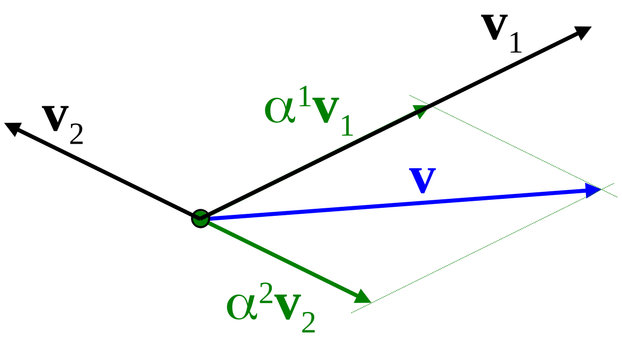


Procesamiento de Vértices y Primitivas

Model-View Transforms. → Vertex Shading → Projection → Clipping → Screen Mapping



Espacio Vectorial – Combinación Lineal



$$\mathbf{v}_1 = \begin{vmatrix} v_1^{\,1} \\ v_1^{\,2} \\ v_1^{\,3} \\ \cdots \\ \cdots \\ v_1^{\,n} \end{vmatrix} \in \mathbb{R}^n$$

n componentes
en
n filas

$$\{\mathbf{v}_i\} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$$

Conjunto de m vectores

$$\mathbf{v} = \sum_{i=1}^{m} \alpha^i \, \mathbf{v}_i$$

combinación lineal de m vectores = vector

superíndice = fila $\downarrow$

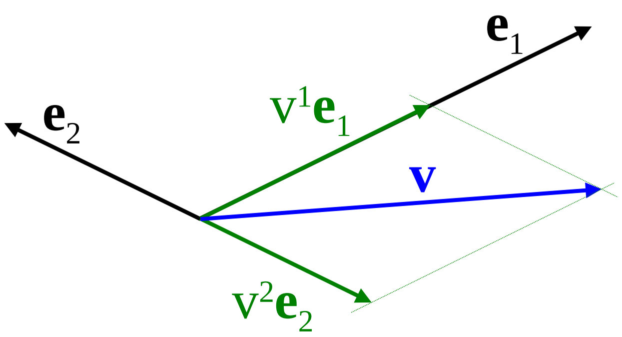
$$\begin{vmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \\ \cdots \\ \alpha^m \end{vmatrix}$$

m factores

m vectores

subíndice = columna o ítem $\longrightarrow$ $\begin{vmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \ldots \mathbf{v}_m \end{vmatrix} \begin{vmatrix} \sum_{i=1}^{m} \alpha^i \mathbf{v}_i \end{vmatrix}$

Independencial Lineal - Base



$$\mathbf{v} = \begin{vmatrix} v^1 \\ v^2 \\ v^3 \\ \cdots \\ \cdots \\ v^n \end{vmatrix} \in \mathbb{R}^n$$

n componentes

$$\{\mathbf{e}_i\} \text{ LI} \Leftrightarrow \left(\sum_{i=1}^{n} \alpha^i \, \mathbf{e}_i = 0 \Rightarrow \alpha^i = 0, \ \forall i \right)$$

Independencia Lineal (LI)

$$\cancel{(\alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2 = 0 \Rightarrow \mathbf{e}_2 = -\alpha^1/\alpha^2 \, \mathbf{e}_1)}$$

no se puede despejar uno en función del resto

$$\{\mathbf{e}_i\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$$

base = n vectores LI

$$\mathbf{v} = \sum_{i=1}^{n} v^i \, \mathbf{e}_i$$

vector = combinación lineal de n vectores base

Transformación Lineal

Es **Lineal** sii preserva la combinación lineal:

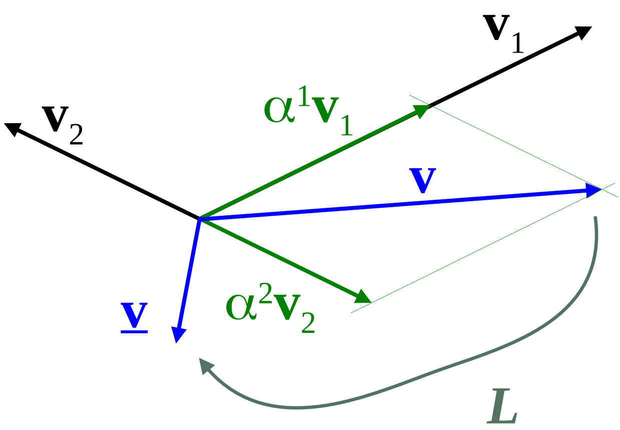
$$\boldsymbol{T}\left(\alpha\,\underline{P}_1+\beta\,\underline{P}_2\right)=\alpha\,\boldsymbol{T}\left(\underline{P}_1\right)+\beta\,\boldsymbol{T}\left(\underline{P}_2\right)$$

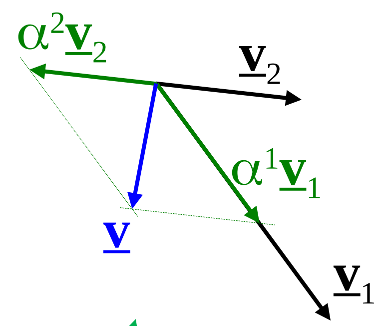
$$\underline{\hat{P}}=\underline{\underline{\boldsymbol{M}}}\;\underline{P}$$

Puede representarse como matriz. Aplicar la transformación equivale a premultiplicar por la matriz correspondiente.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} Ax+By \\ Cx+Dy \end{bmatrix}$$

Transformación Lineal





$$\mathbf{v} = \sum_{i=1}^{n} \alpha^i \, \mathbf{v}_i$$

$$\underline{\mathbf{v}} = \boldsymbol{L}(\mathbf{v}) = \boldsymbol{L}\Big(\sum_{i=1}^{n} \alpha^i \, \mathbf{v}_i\Big) \qquad \boldsymbol{L} \text{ es lineal} \Leftrightarrow \boldsymbol{L}\Big(\sum_{i=1}^{n} \alpha^i \, \mathbf{v}_i\Big) = \sum_{i=1}^{n} \alpha^i \, \boldsymbol{L}(\mathbf{v}_i)$$

$$\underline{\mathbf{v}} = \boldsymbol{L}(\mathbf{v}) = \boldsymbol{L}\Big(\sum_{i=1}^{n} \alpha^i \, \mathbf{v}_i\Big) = \sum_{i=1}^{n} \alpha^i \, \boldsymbol{L}(\mathbf{v}_i) = \sum_{i=1}^{n} \alpha^i \, \underline{\mathbf{v}}_i$$

Transformación Lineal

Dos formas de ver una transformación:

Vector original: $\underline{v} = \sum v_i \underline{e_i}$

(1) Componentes transformadas: $\underline{\hat{v}} = \sum \hat{v}_i \underline{e_i}$

(2) Base transformada: $\underline{\hat{v}} = \sum v_i \underline{\hat{e}_i}$

$$\underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} \hat{e}_x^1 & \hat{e}_x^2 \\ \hat{e}_y^1 & \hat{e}_y^2 \end{bmatrix}$$

Composición de Transformaciones

Aplicación de sucesivas transformaciones. Ejemplo:

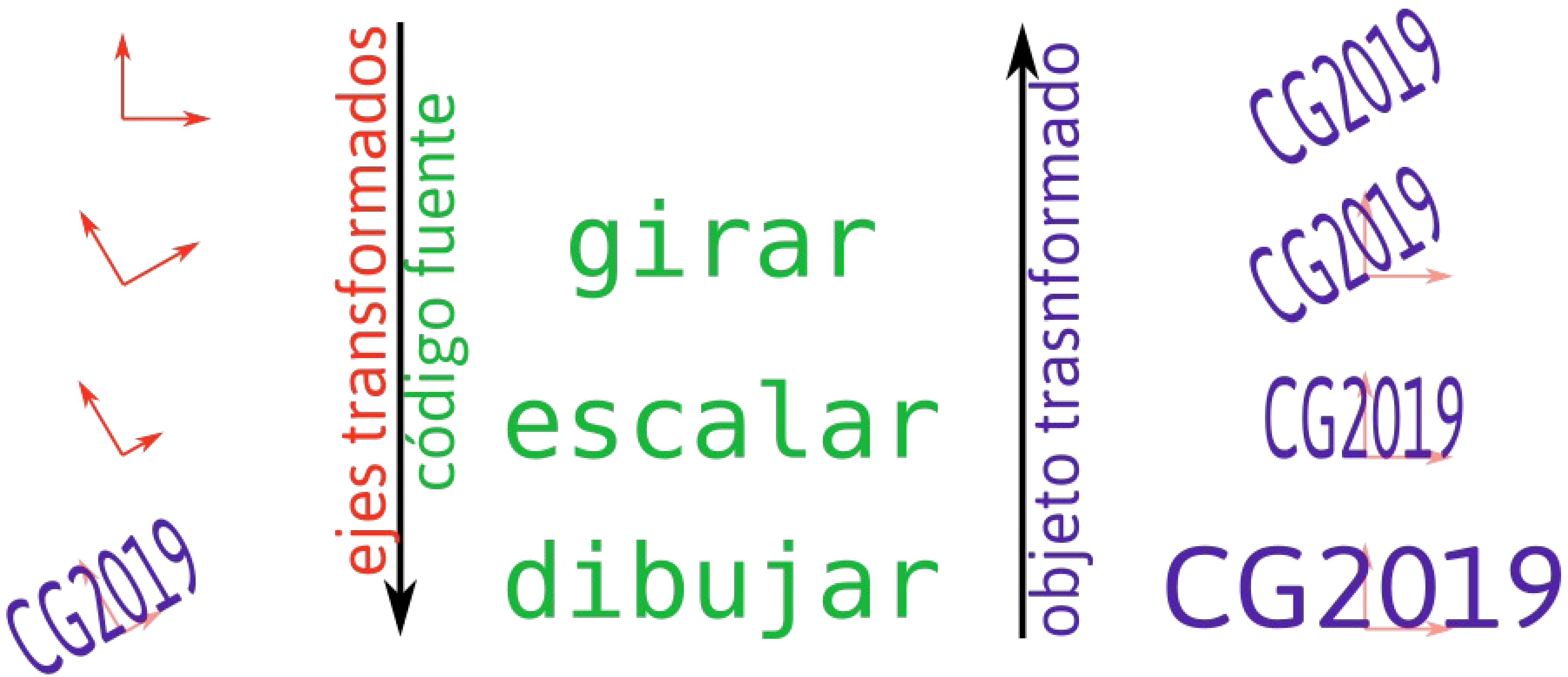
1. Desplazar ($\boldsymbol{T}^1$): $\underline{\hat{P}}^1 = \boldsymbol{T}^1(\underline{P})$
2. Escalar ($\boldsymbol{T}^2$): $\underline{\hat{P}}^2 = \boldsymbol{T}^2(\underline{\hat{P}}^1) = \boldsymbol{T}^2(\boldsymbol{T}^1(\underline{P}))$
3. Rotar ($\boldsymbol{T}^3$): $\underline{\hat{P}}^3 = \boldsymbol{T}^3(\underline{\hat{P}}^2) = \boldsymbol{T}^3(\boldsymbol{T}^2(\boldsymbol{T}^1(\underline{P})))$

Combinación: Utilizando las matrices asociadas:

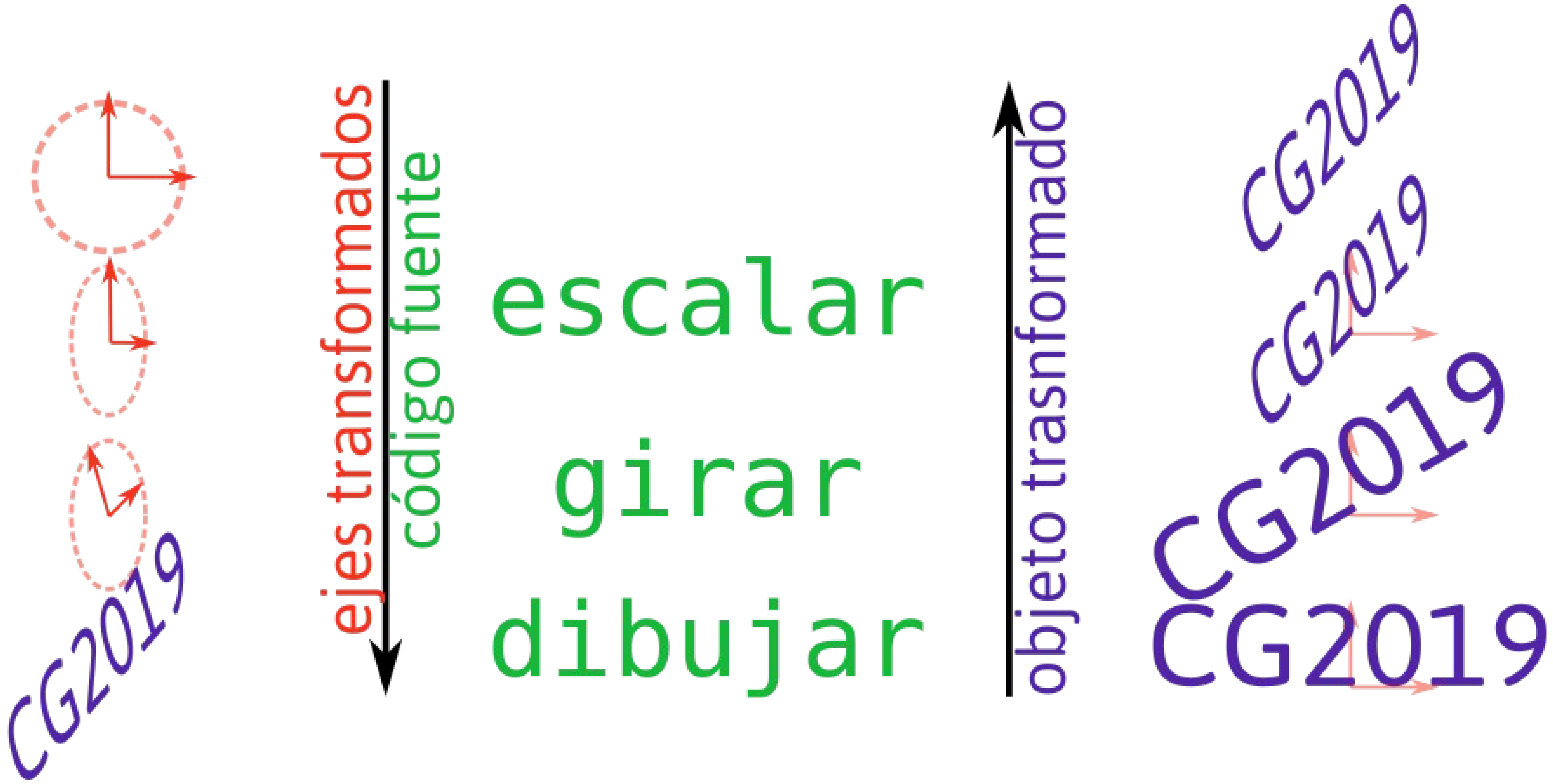
$$\underline{P}^3 = \boldsymbol{T}^3(\boldsymbol{T}^2(\boldsymbol{T}^1(\underline{P}))) =$$
$$= \underline{\underline{\boldsymbol{M}}}^3 \cdot (\underline{\underline{\boldsymbol{M}}}^2 \cdot (\underline{\underline{\boldsymbol{M}}}^1 \cdot \underline{P})) =$$
$$= \underbrace{\underline{\underline{\boldsymbol{M}}}^3 \cdot \underline{\underline{\boldsymbol{M}}}^2 \cdot \underline{\underline{\boldsymbol{M}}}^1}_{\underline{\underline{\boldsymbol{M}}}^*} \cdot \underline{P} = \underline{\underline{\boldsymbol{M}}}^*(\underline{P})$$

Notar que el orden altera el resultado

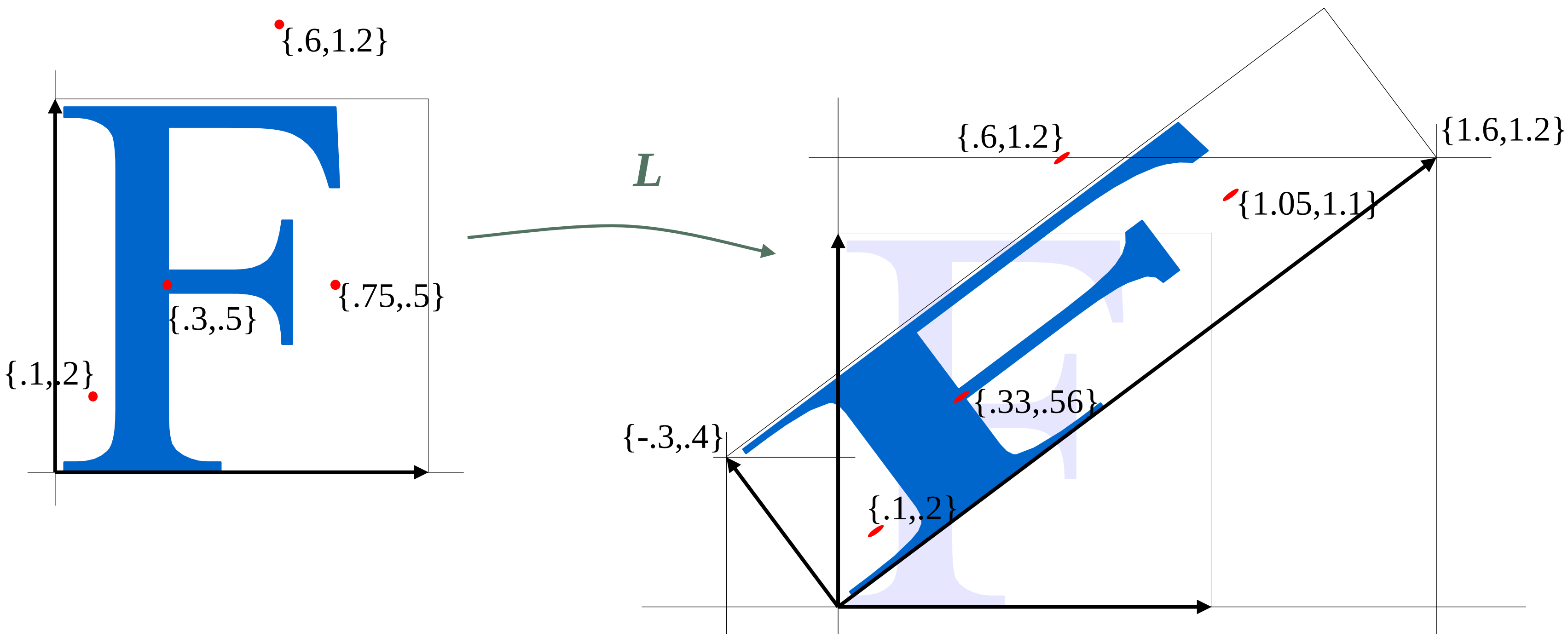
Orden de Interpretación



Orden de Interpretación



Ejemplo

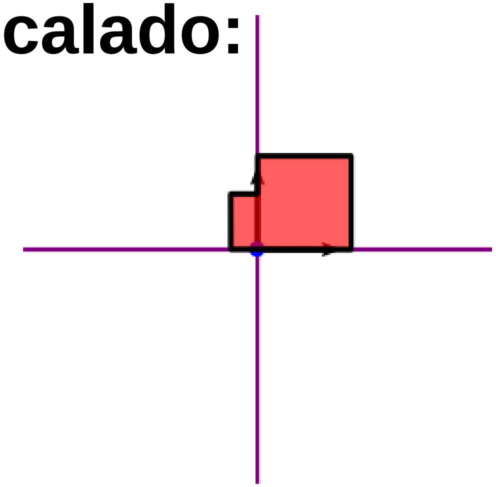


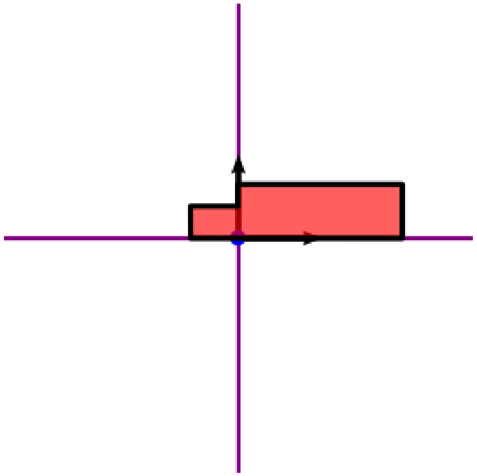
$$\begin{vmatrix} 0.1 \\ 0.2 \end{vmatrix} \begin{vmatrix} 0.3 \\ 0.5 \end{vmatrix} \begin{vmatrix} 0.6 \\ 1.2 \end{vmatrix} \begin{vmatrix} 0.75 \\ 0.5 \end{vmatrix} \Leftarrow \mathbf{v}$$

$$\begin{array}{cc} \underline{\mathbf{e}}_1 & \underline{\mathbf{e}}_2 \\ \Downarrow & \Downarrow \end{array}$$

$$\begin{vmatrix} 1.6 & -0.3 \\ 1.2 & 0.4 \end{vmatrix} \begin{vmatrix} 0.1 \\ 0.2 \end{vmatrix} \begin{vmatrix} 0.33 \\ 0.56 \end{vmatrix} \begin{vmatrix} 0.6 \\ 1.2 \end{vmatrix} \begin{vmatrix} 1.05 \\ 1.1 \end{vmatrix} \Leftarrow \underline{\mathbf{v}}$$

Escalado:

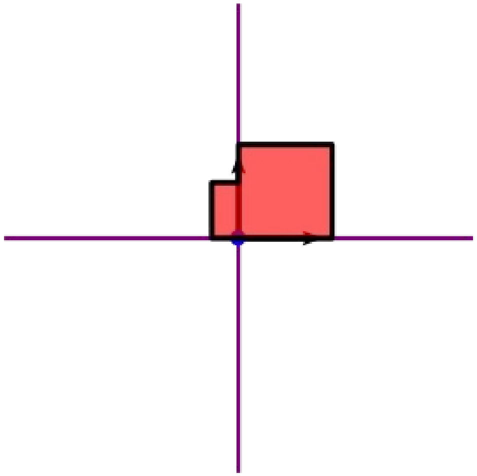


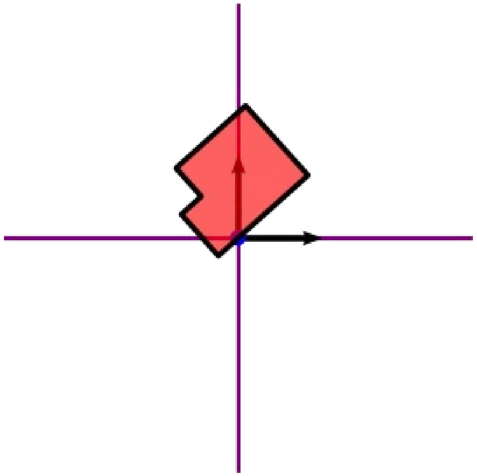


$$\underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

Rotación:

$$\underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{bmatrix}$$

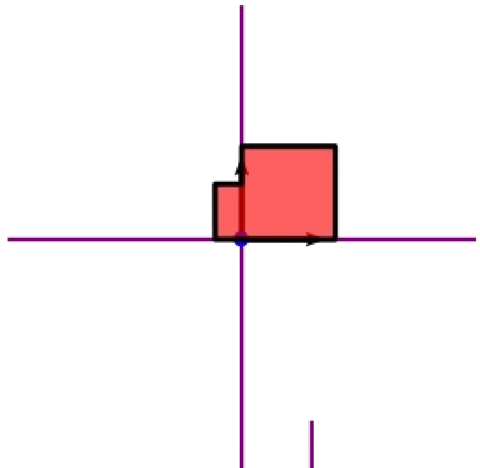


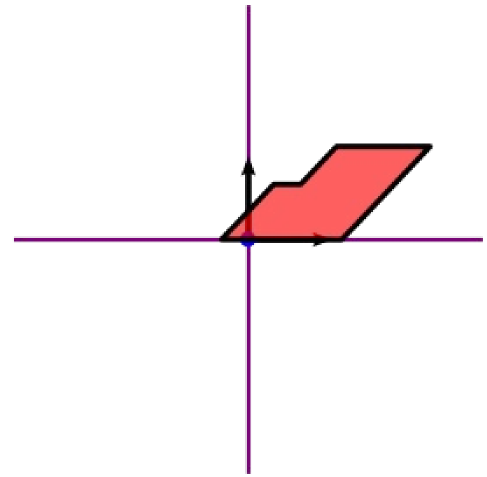


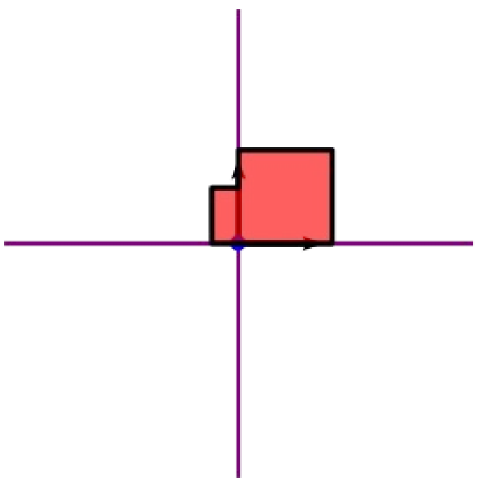
Transformaciones Lineales

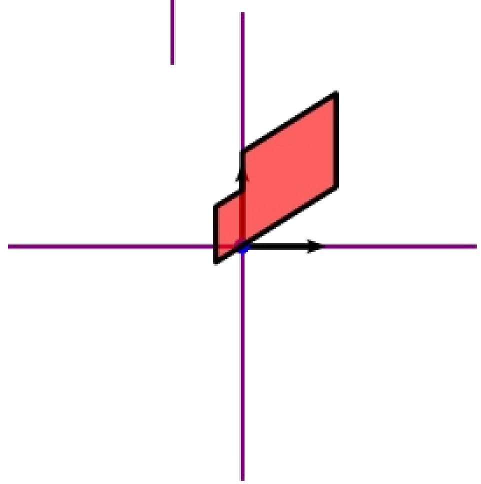
Shear/Deslizamiento:

$$\underline{\underline{M}} = \begin{bmatrix} 1 & S_x \\ 0 & 1 \end{bmatrix}$$



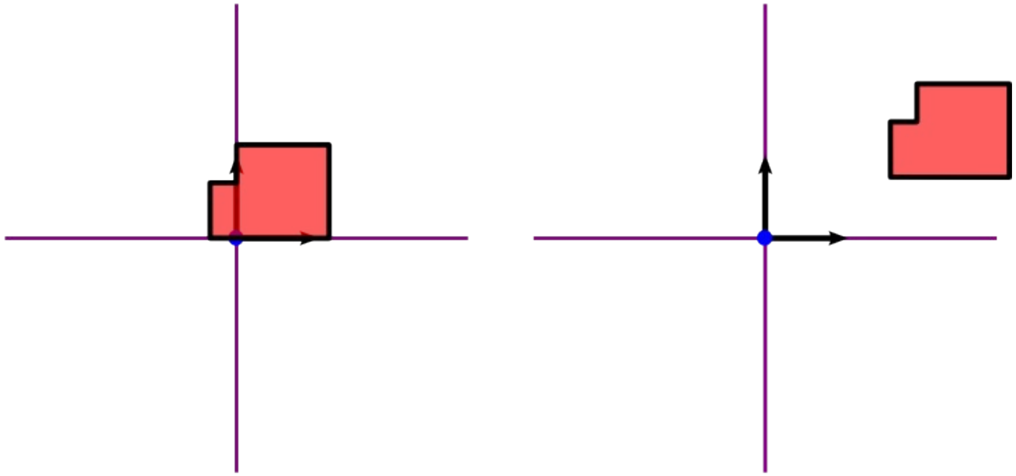






$$\underline{\underline{M}} = \begin{bmatrix} 1 & 0 \\ S_y & 1 \end{bmatrix}$$

Traslación/Desplazamiento:



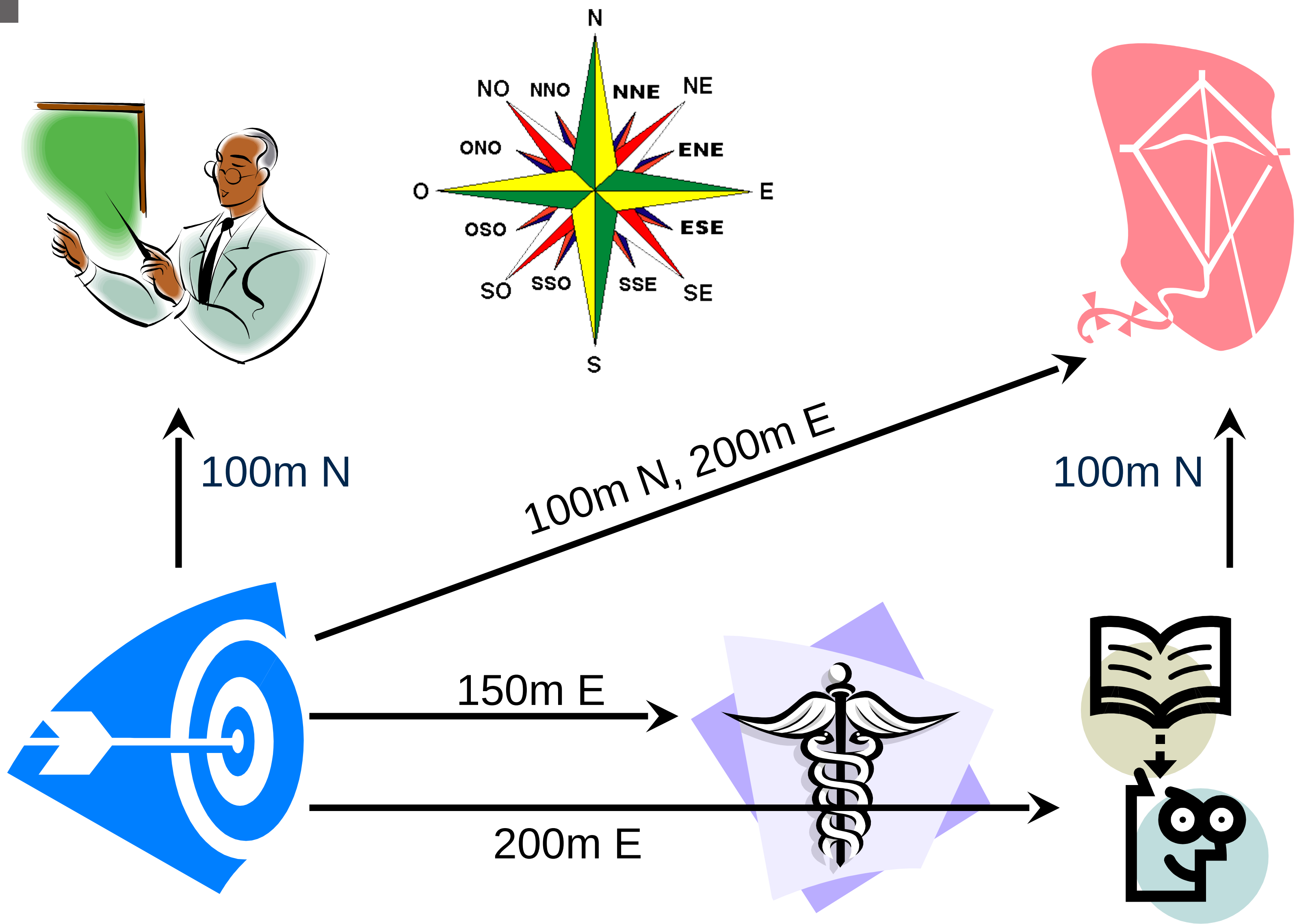
$$\hat{\underline{P}} = \underline{P} + \underline{D} \quad \Rightarrow \quad \underline{\underline{M}} = ? \quad (\text{NO es lineal en } R^N)$$

Peeero….

$$\underline{\underline{M}} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+t_x \\ y+t_y \\ 1 \end{bmatrix}$$

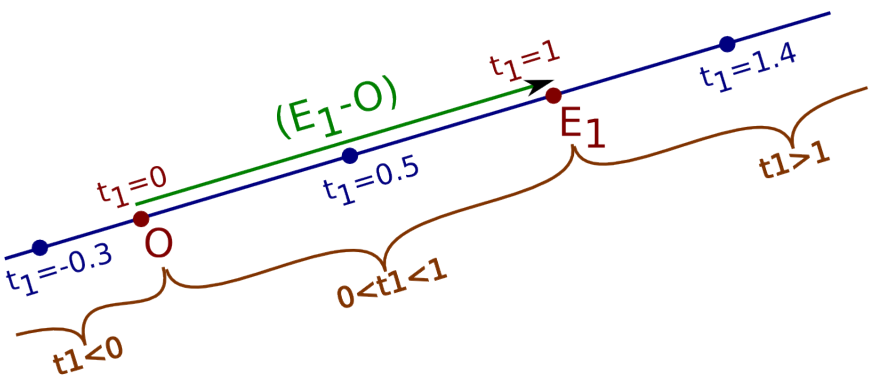
Esto no es Z!

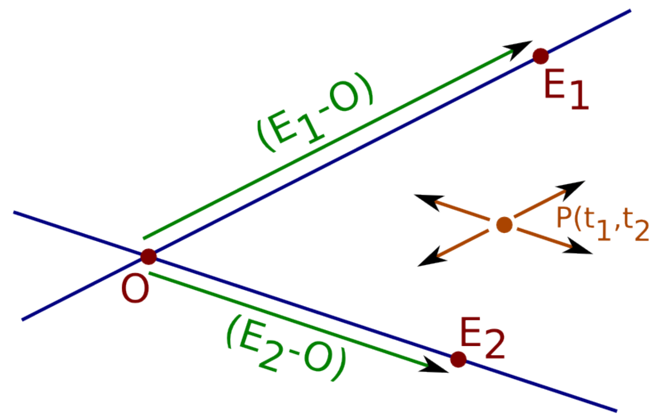
Espacio Afín: Puntos vs. Vectores



Expansión afín:

La expansión afín de N+1 puntos genera un espacio N-dimensional





Transformación afín:

transformación lineal + traslación del origen

Transformación Afín

Es **Afín** sii preserva la combinación afín:

$$\boldsymbol{T}\left(\alpha \underline{P}_1 + \beta \underline{P}_2\right) = \alpha \boldsymbol{T}\left(\underline{P}_1\right) + \beta \boldsymbol{T}\left(\underline{P}_2\right), \qquad \alpha + \beta = 1$$

Una **traslación** sí preserva la combinación afín

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A0 + B0 \\ C0 + D0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

No Puede representarse como matriz en R^N.

Transformación Afín

Es **Afín** sii preserva la combinación afín:

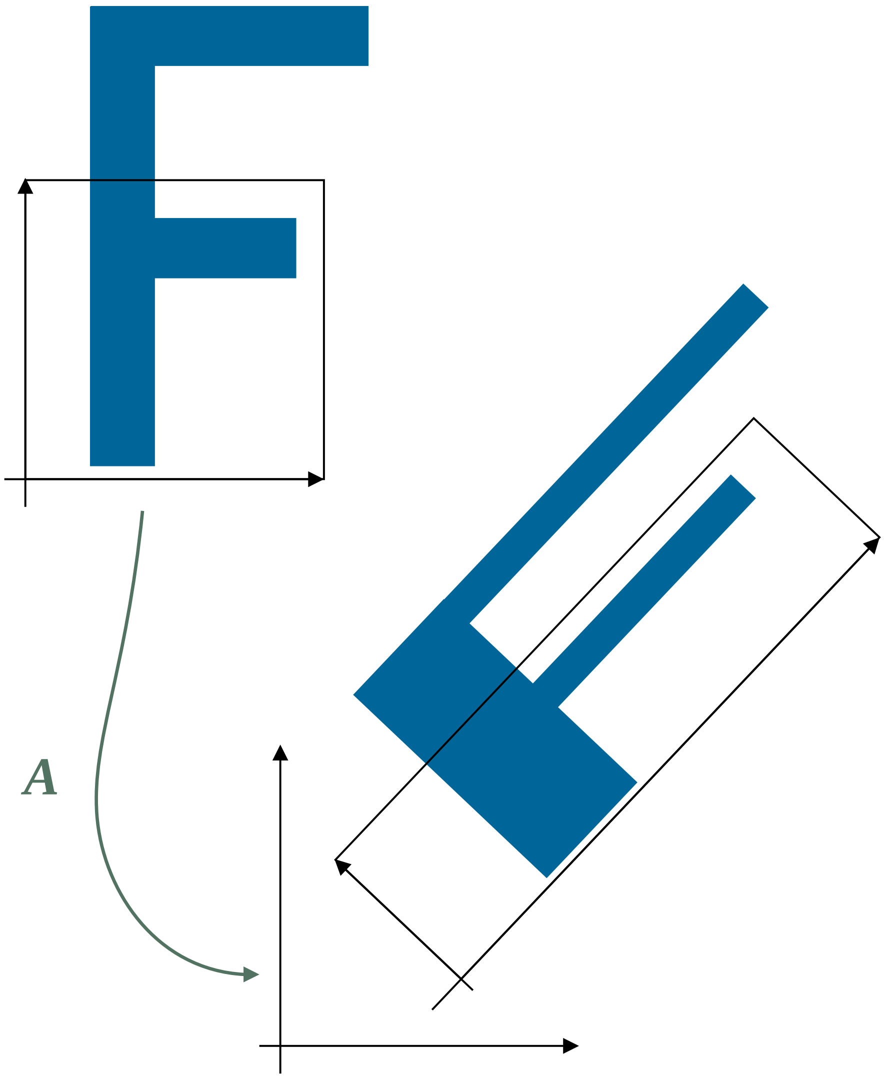
$$\boldsymbol{T}\left(\alpha \underline{P}_1+\beta \underline{P}_2\right)=\alpha \boldsymbol{T}\left(\underline{P}_1\right)+\beta \boldsymbol{T}\left(\underline{P}_2\right), \qquad \alpha+\beta=1$$

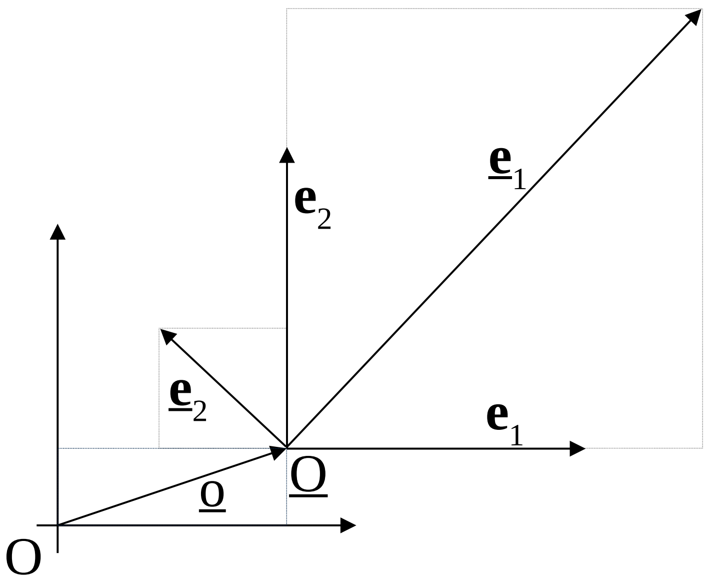
$$\Downarrow$$

$$\underline{\hat{P}}=\underline{\underline{\boldsymbol{M}}}\,\underline{P}$$

Sí puede representarse como matriz en "R^N+1".

$$\begin{bmatrix} A & B & T_x \\ C & D & T_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} Ax+By+1\,T_x \\ Cx+Dy+1\,T_y \\ 0\,x+0\,y+1 \end{bmatrix}$$

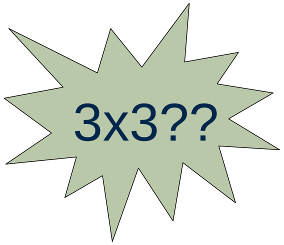




$$\begin{pmatrix} v^1 \\ v^2 \\ 1 \end{pmatrix}$$

$$\begin{array}{ccc} \underline{e}_1 & \underline{e}_2 & \underline{o} \\ \Downarrow & \Downarrow & \Downarrow \end{array}$$

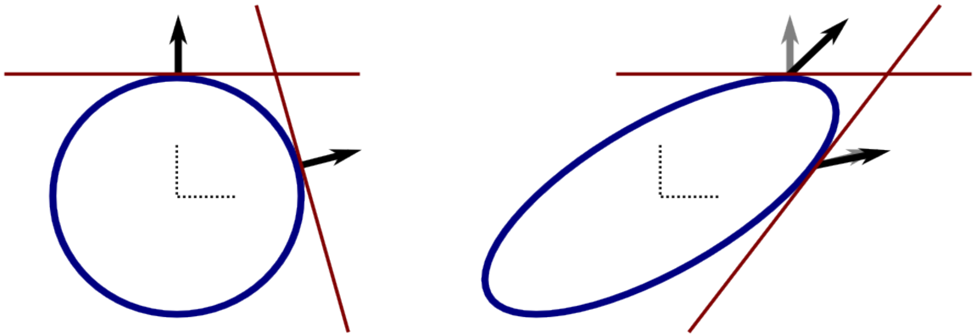
$$\begin{pmatrix} 1.3 & -0.5 & 0.6 \\ 1.5 & 0.4 & 0.1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{v}^1 \\ \underline{v}^2 \\ 1 \end{pmatrix}$$



Vector Normal: vector ortogonal a todos los vectores tangente

$$\underline{t}(\underline{P}) = \lim_{\underline{Q} \to \underline{P}} (\underline{Q} - \underline{P}) \qquad \underline{n} \cdot \underline{t} = 0$$

En una transformación afín, la transformación de la normal original no es igual a la normal de la curva transformada



Para una transformación afín ***A***:

$$\boldsymbol{n} \cdot \boldsymbol{t} = 0 \qquad \boldsymbol{n}^T \boldsymbol{t} = 0$$

$$\Rightarrow \quad \boldsymbol{n}^T \underbrace{\left(\boldsymbol{A}^{-1} \hat{\boldsymbol{t}}\right)}_{\boldsymbol{t}} = 0 \quad \Leftarrow$$

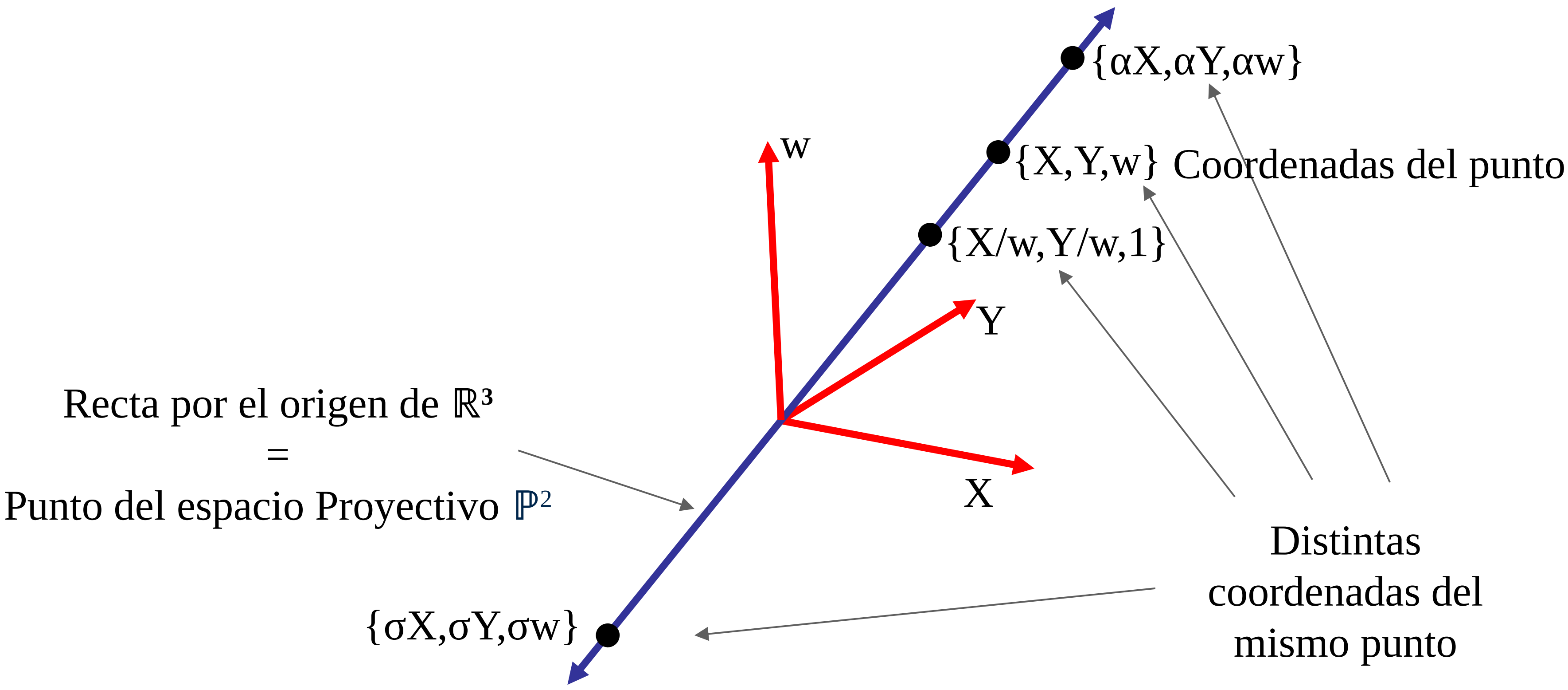
$$\hat{\boldsymbol{t}} = \boldsymbol{A}\,\boldsymbol{t} \qquad \boldsymbol{t} = \boldsymbol{A}^{-1} \hat{\boldsymbol{t}}$$

$$\left(\boldsymbol{n}^T \boldsymbol{A}^{-1}\right) \hat{\boldsymbol{t}} = 0$$

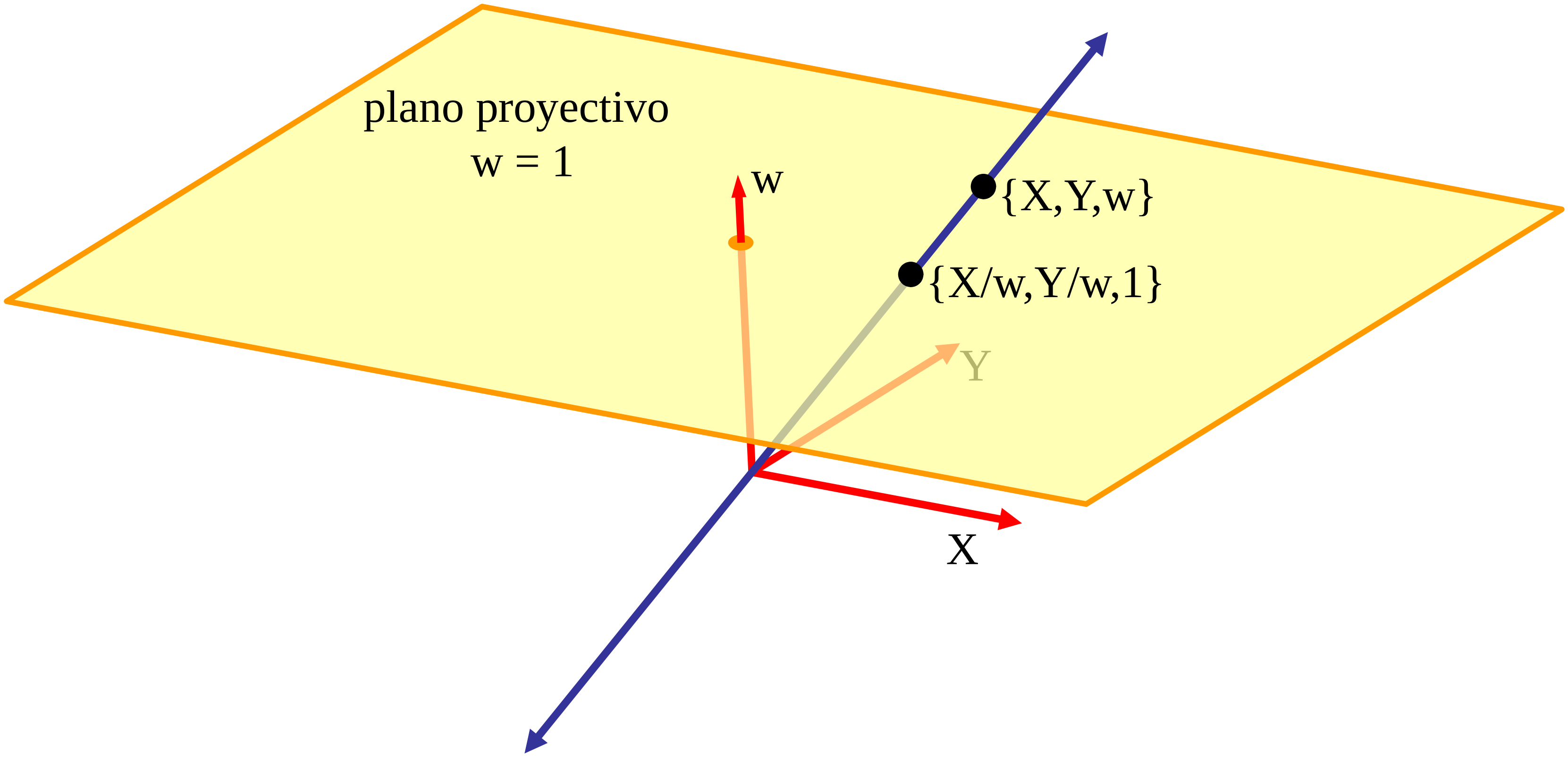
$$\underbrace{\left(\left(\boldsymbol{A}^{-1}\right)^T \boldsymbol{n}\right)^T}_{\hat{\boldsymbol{n}}^T} \hat{\boldsymbol{t}} = 0 \quad \Rightarrow \quad \boxed{\hat{\boldsymbol{n}} = \left(\boldsymbol{A}^{-1}\right)^T \boldsymbol{n}}$$

Espacio Proyectivo

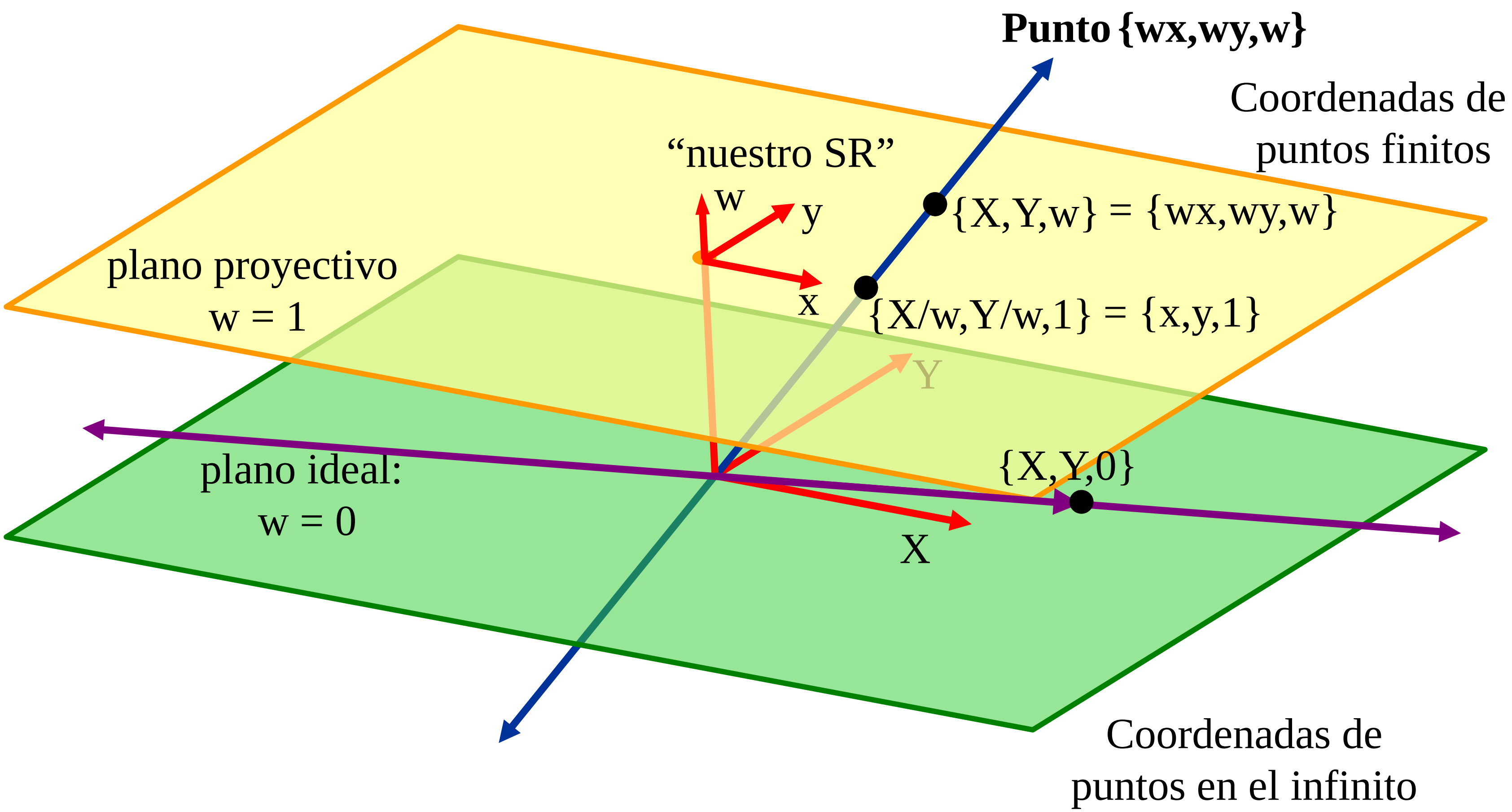
Espacio Proyectivo



Espacio Proyectivo

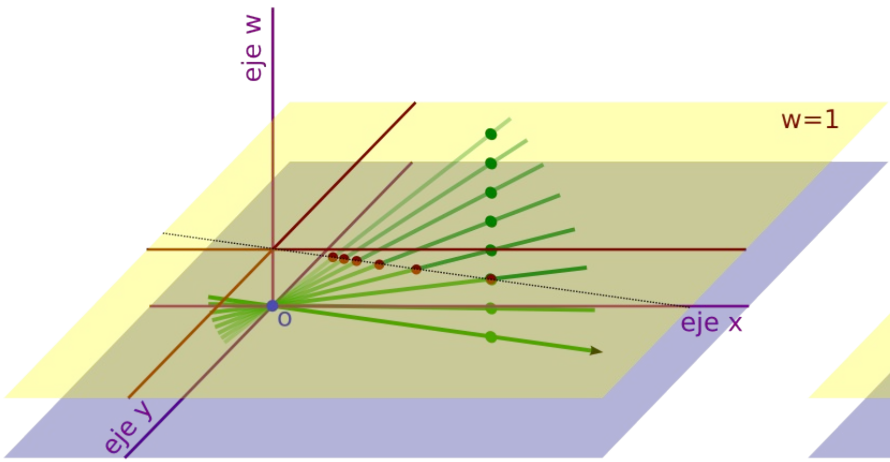


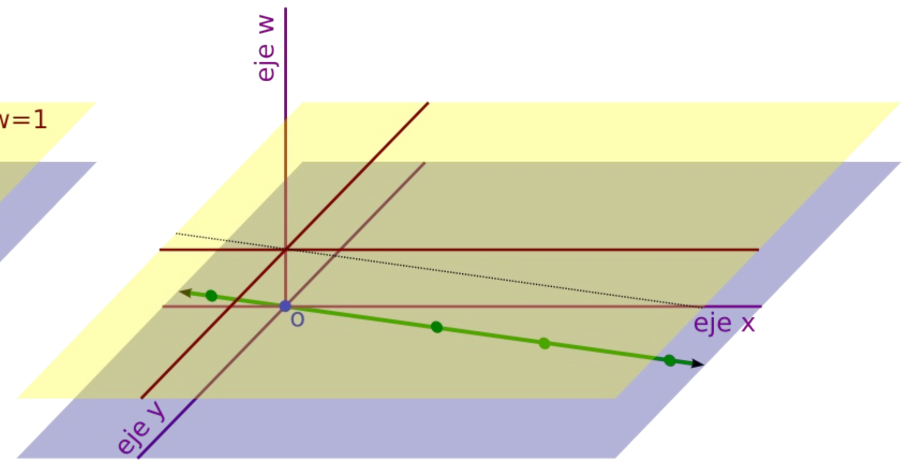
Espacio Proyectivo



Punto ideal:

- Si $w \neq 0$: la recta corta al plano en $(x/w, y/w, 1)$
- Si $w=0$: la recta es horizontal, y se corresponde con un punto ideal, en el infinito, definido por un “vector” dirección.





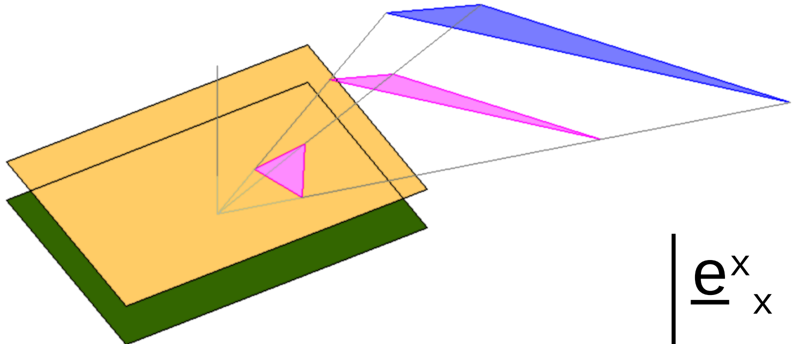
Transformaciones Proyectivas

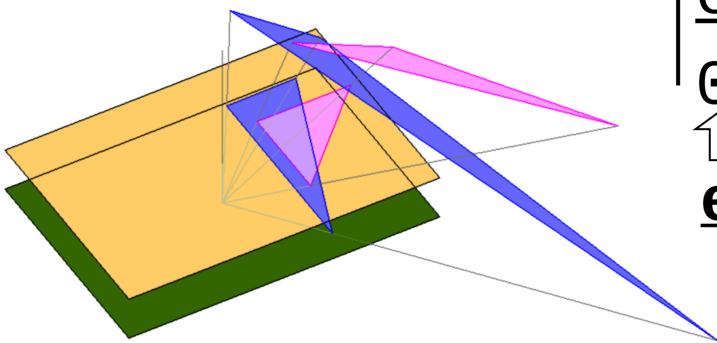
El efecto de una transformación lineal en $\mathbb{R}^3$ sobre los puntos de $\mathbb{P}^2$ se denomina transformación proyectiva.

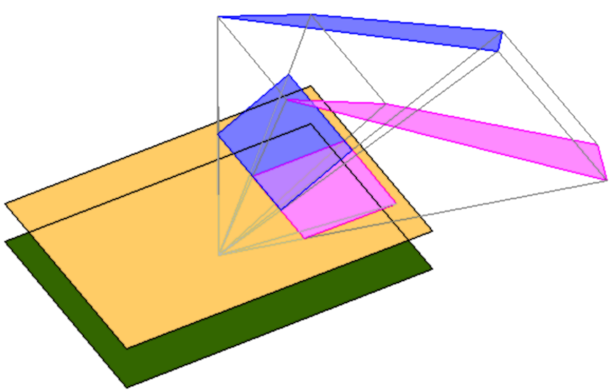
No proyecta

$$
\begin{array}{cc}
 & \left|\begin{matrix} p^x \\ p^y \\ p^w \end{matrix}\right| \\
\left|\begin{matrix} \underline{e}^x{}_x & \underline{e}^x{}_y & \underline{e}^x{}_w \\ \underline{e}^y{}_x & \underline{e}^y{}_y & \underline{e}^y{}_w \\ \underline{e}^w{}_x & \underline{e}^w{}_y & \underline{e}^w{}_w \end{matrix}\right| & \left|\begin{matrix} \underline{p}^x \\ \underline{p}^y \\ \underline{p}^w \end{matrix}\right| \\
\begin{matrix} \Uparrow & \Uparrow & \Uparrow \\ \underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_w \end{matrix} &
\end{array}
$$

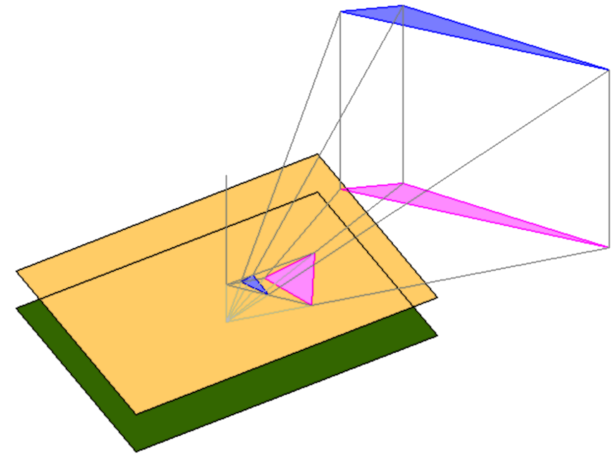
Transformaciones Afines en P^2

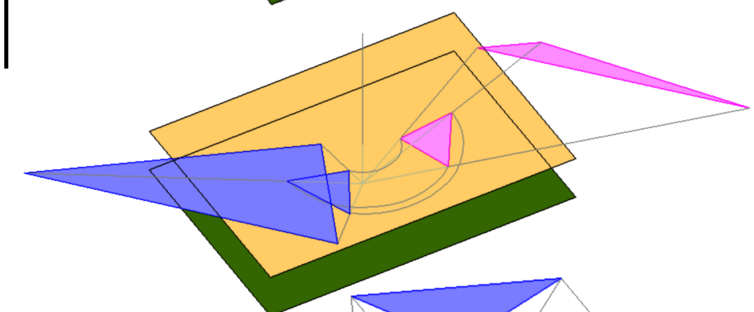


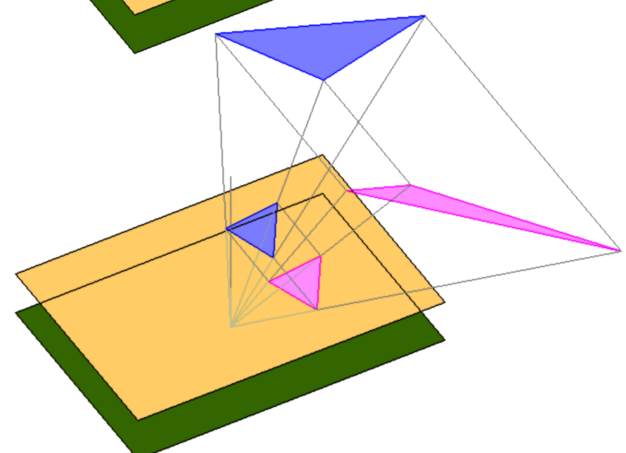




$$
\begin{array}{ccc}
\underline{e}^{x}{}_{x} & \underline{e}^{x}{}_{y} & \underline{O}^{x} \\
\underline{e}^{y}{}_{x} & \underline{e}^{y}{}_{y} & \underline{O}^{y} \\
0 & 0 & 1 \\
\Uparrow & \Uparrow & \Uparrow \\
\underline{\mathbf{e}}_{x} & \underline{\mathbf{e}}_{y} & \underline{\mathbf{e}}_{w}
\end{array}
$$







Transformaciones Afines en P^3

Parte lineal 3D

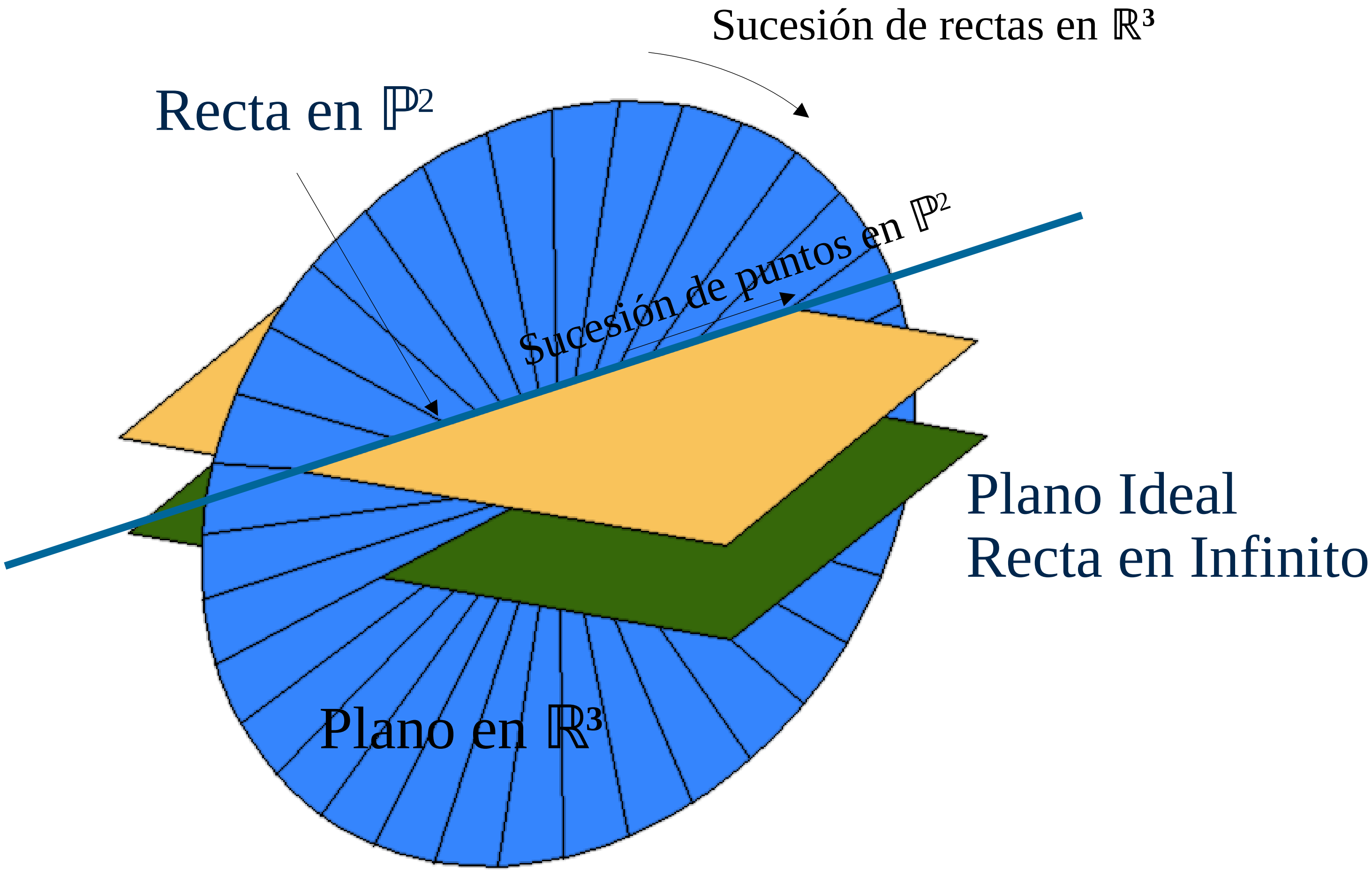
Traslación

$$\begin{array}{c}
\begin{vmatrix}
\underline{e}_x{}^x & \underline{e}_y{}^x & \underline{e}_z{}^x & \underline{o}^x \\
\underline{e}_x{}^y & \underline{e}_y{}^y & \underline{e}_z{}^y & \underline{o}^y \\
\underline{e}_x{}^z & \underline{e}_y{}^z & \underline{e}_z{}^z & \underline{o}^z \\
0 & 0 & 0 & 1
\end{vmatrix} \\
\begin{array}{cccc}
\Uparrow & \Uparrow & \Uparrow & \Uparrow \\
\underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_z & \underline{\mathbf{e}}_w\texttt{(origen)}
\end{array}
\end{array}$$

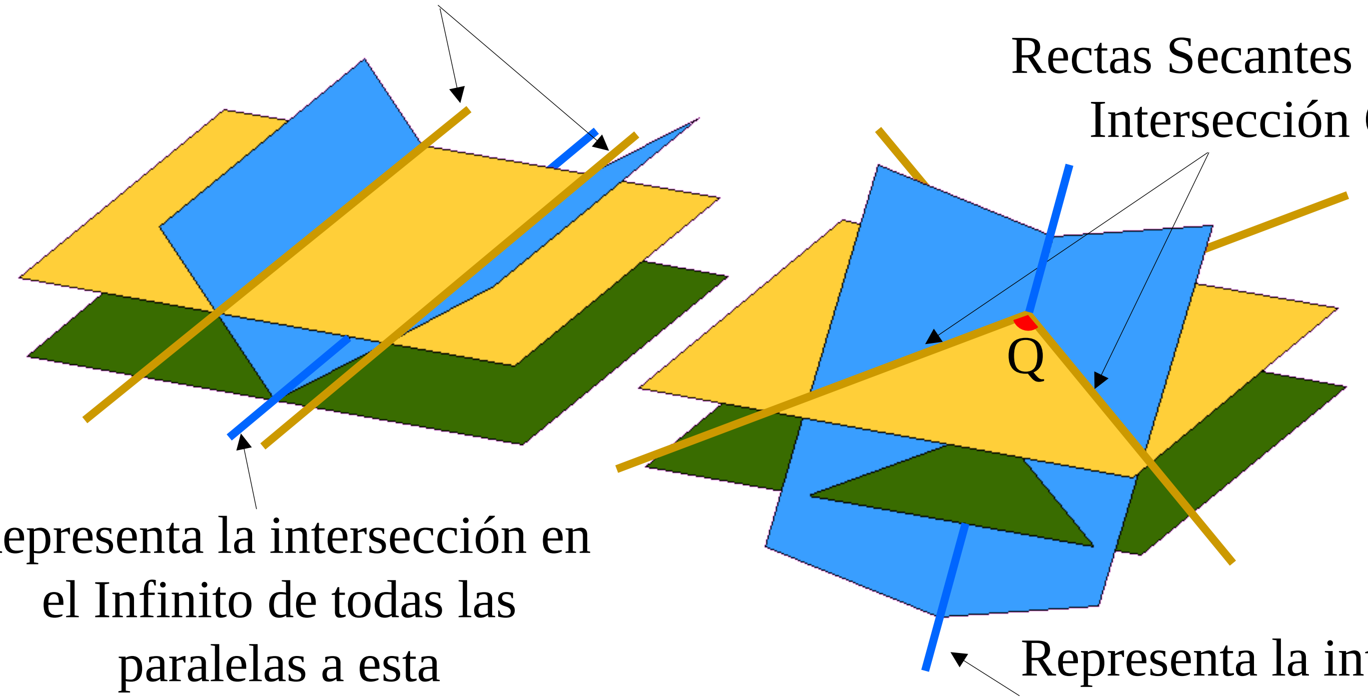
El plano ideal se mapea sobre si mismo

(puede girar o deformarse, pero no puede inclinarse)

Planos por el Origen en P² = Rectas en R²



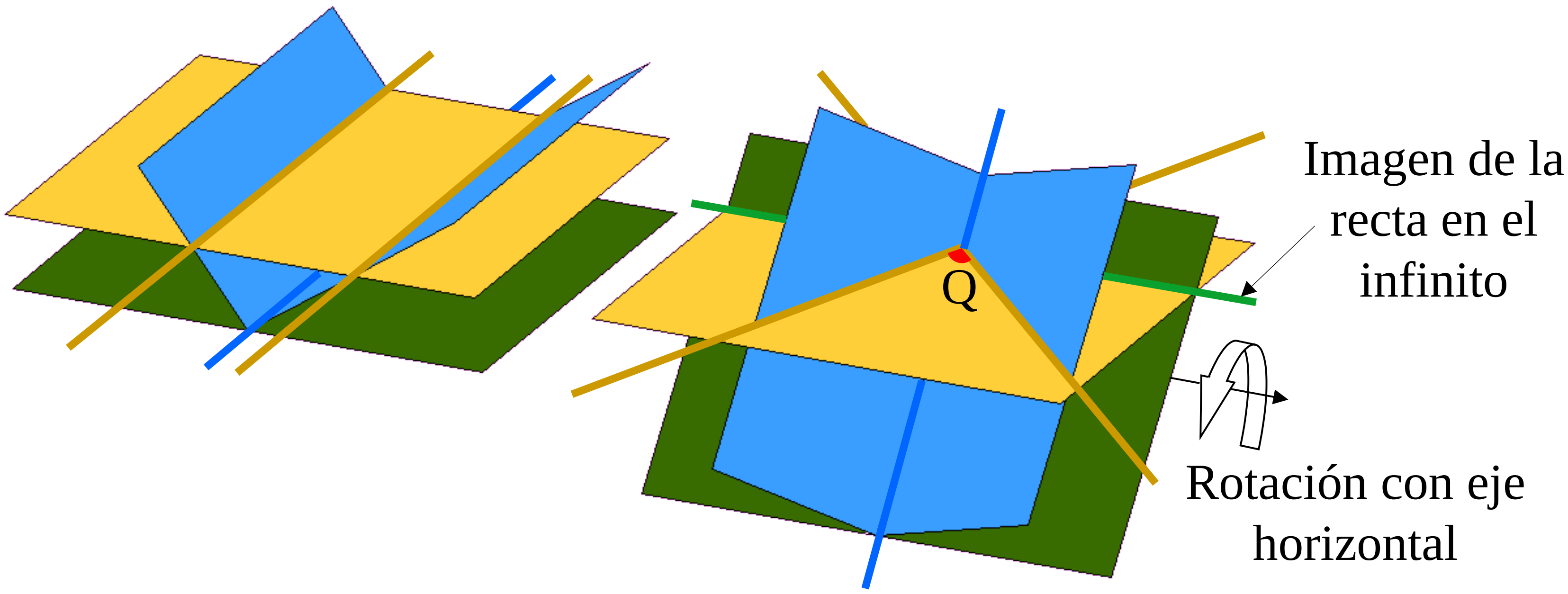
Planos por el Origen en P^2 = Rectas en R^2



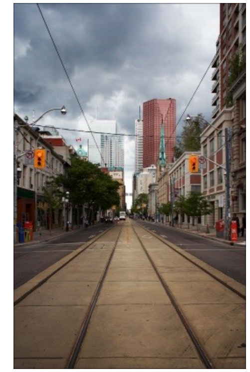
En el espacio proyectivo todo par de rectas tiene un punto común.

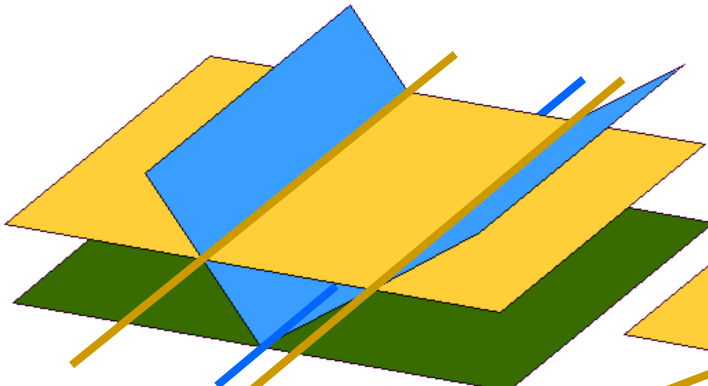
Transformación Proyectiva General

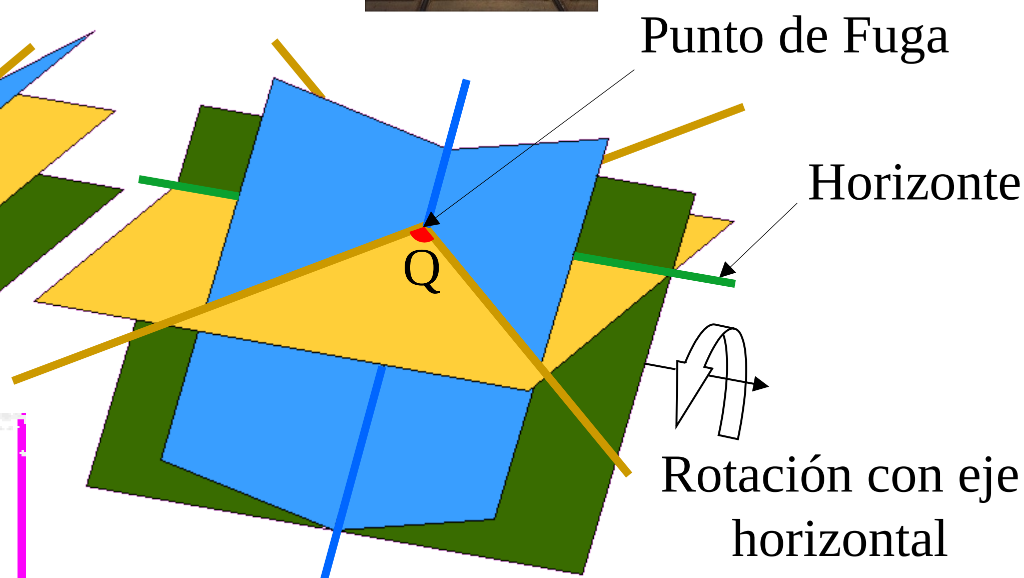
El plano ideal puede inclinarse

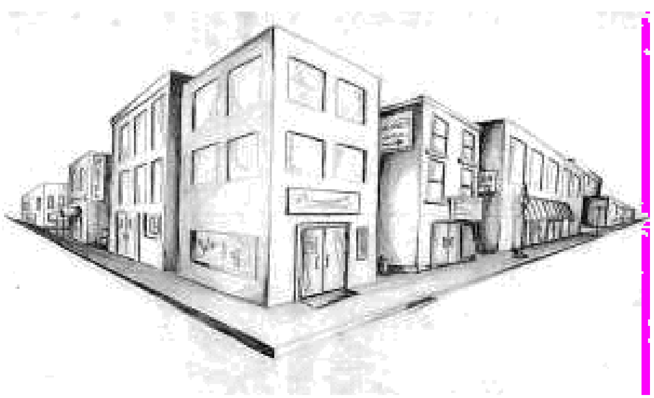


Transformación Proyectiva General

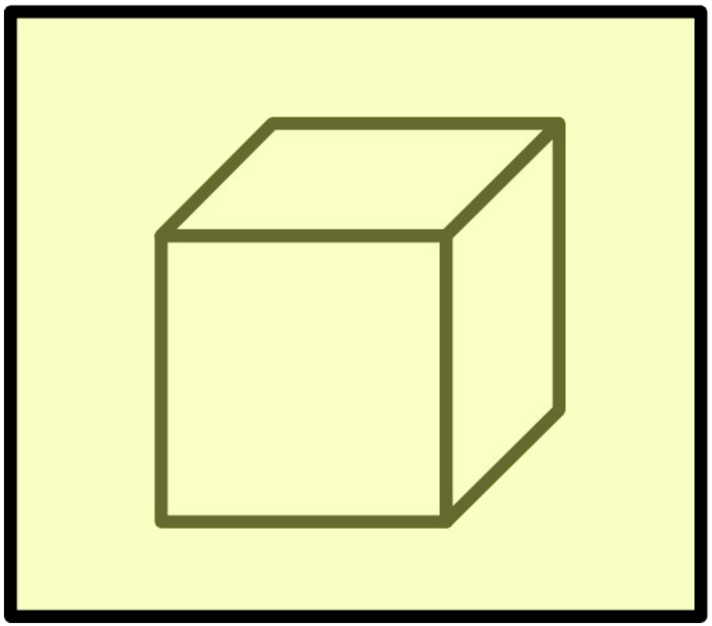




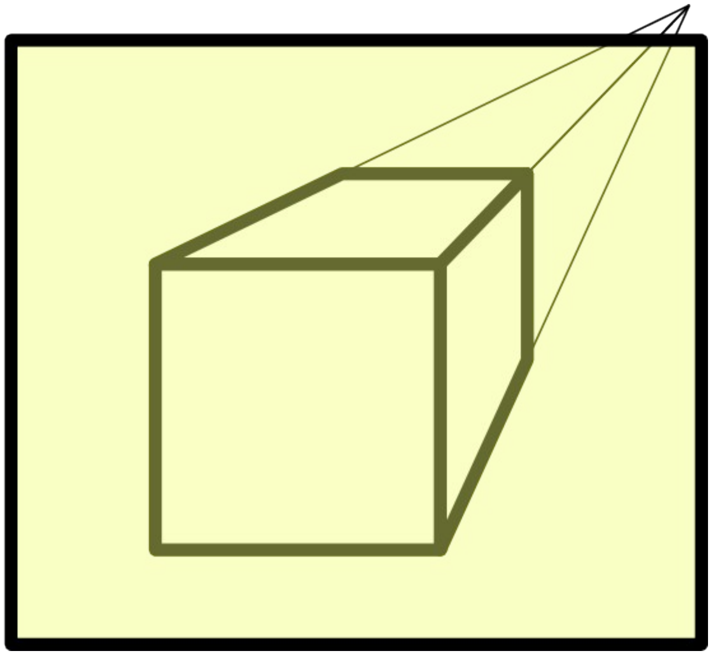




Transformación no invertibles, de rango incompleto.



Ortogonal



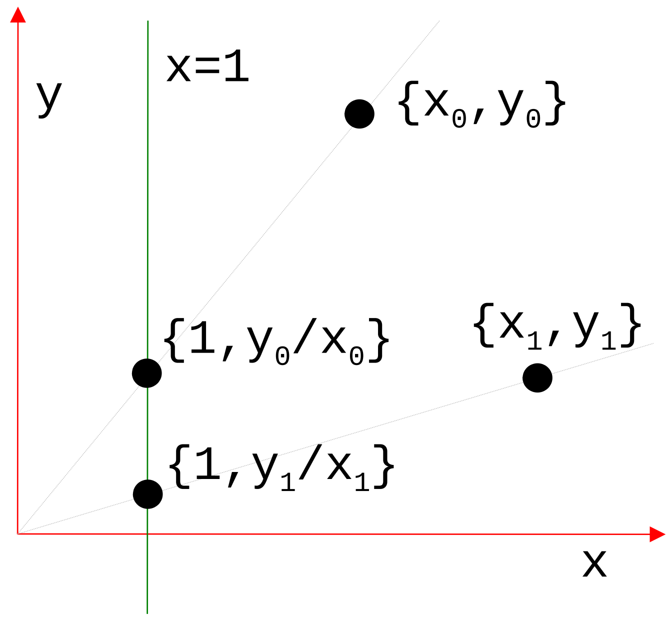
Perspectiva

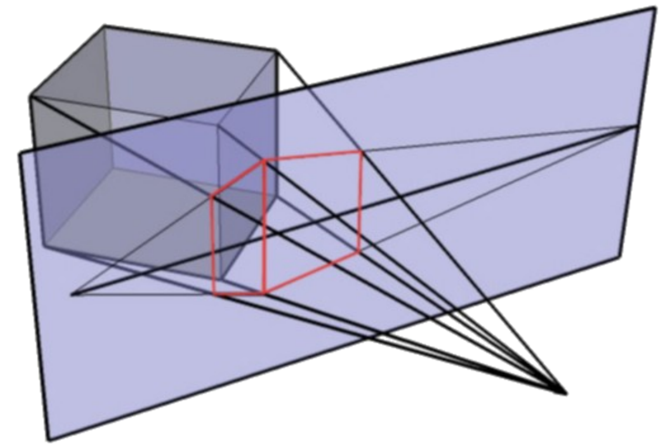
Transformación Proyectiva General 3D

$$\begin{array}{cc} \mathbf{P} & \mathbf{v} \\ \Downarrow & \Downarrow \\ \begin{vmatrix} wx \\ wy \\ wz \\ w \end{vmatrix} & \begin{vmatrix} x \\ y \\ z \\ 0 \end{vmatrix} \end{array}$$

$$\begin{array}{ccc} \begin{vmatrix} \underline{e}_x{}^x & \underline{e}_y{}^x & \underline{e}_z{}^x & \underline{e}_w{}^x \\ \underline{e}_x{}^y & \underline{e}_y{}^y & \underline{e}_z{}^y & \underline{e}_w{}^y \\ \underline{e}_x{}^z & \underline{e}_y{}^z & \underline{e}_z{}^z & \underline{e}_w{}^z \\ \underline{e}_x{}^w & \underline{e}_y{}^w & \underline{e}_z{}^w & \underline{e}_w{}^w \end{vmatrix} & \begin{vmatrix} \underline{wx} \\ \underline{wy} \\ \underline{wz} \\ \underline{w} \end{vmatrix} & \begin{vmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \\ 0 \end{vmatrix} \\ \begin{matrix} \Uparrow & \Uparrow & \Uparrow & \Uparrow \end{matrix} & \Uparrow & \Uparrow \\ \begin{matrix} \underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_z & \underline{\mathbf{e}}_w \end{matrix} & \underline{\mathbf{P}} & \underline{\mathbf{v}} \end{array}$$

Ejemplo: Perspectiva Central





$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1/x & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & 1/x \end{bmatrix} \begin{bmatrix} 1 \\ y/x \\ z/x \end{bmatrix}$$

$$\begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} wx \\ wy \\ wz \\ wx \end{bmatrix}$$

¿Preguntas?