

## The Local Connectivity ATSP Problem

Fix some function  $\text{lb} : 2^V \rightarrow \mathbb{R}^+$  that obeys the following properties:

1.  $\text{lb}(S) = \sum_{v \in S} \text{lb}(\{v\})$  for all  $S \subseteq V$
2.  $\text{lb}(V) = \text{value}(I)$

One can think of  $\text{lb}$  as being defined by distributing the LP value among the vertices in some way.

### Problem Statement.

Input:  $I$  (the instance) and  $V_1, \dots, V_k$  (a partitioning of  $V$ ) such that  $V_i$  is strongly connected.

Output:  $F$ , an eulerian set of edges such that for every  $i$ ,  $F$  exits (and hence enters)  $V_i$  at least once.

**Definition 0.1.** An algorithm for local connectivity ATSP is called  **$\alpha$ -light** if it outputs  $F$  such that for each strongly connected component of  $F$ , say  $\tilde{G}$ ,  $w(\tilde{G}) \leq \alpha \cdot \text{lb}(\tilde{G})$ .

**Theorem 0.1** (Svensson '15). *An  $\alpha$ -light algorithm for local connectivity ATSP implies a  $O(\alpha)$ -approximation algorithm for ATSP.*

The paper describes an  $O(1)$ -light algorithm for local connectivity on vertebrate pairs, which gives us an  $O(1)$ -approximation algorithm for ATSP on vertebrate pairs, and hence an  $O(1)$ -approximation algorithm for general ATSP.

We will also assume that  $w(B) \leq O(1) \cdot \text{value}(I)$ , since in the previous reduction, this inequality is always true.

**Definition 0.2.** For this algorithm, we will define  $\text{lb}$  as

$$\text{lb}(v) := \begin{cases} \frac{\text{value}(I)}{|V(B)|} & \text{if } v \in B \\ 2 \cdot y_v & \text{otherwise} \end{cases}$$

Note that this definition does not guarantee that  $\text{lb}(V) = \text{value}(I)$ , but  $\text{lb}(V)$  is always a constant factor away from  $\text{value}(I)$ , so we can work with these  $\text{lb}$  values and suffer a constant factor loss in the approximation ratio.

To make corner cases easier, we add  $V$  to  $L_{\geq 2}$  for the rest of this write up.

The following is the main technical lemma of the paper.

**Lemma 0.1** (Lemma 7.3 in the paper). *Given  $U_1, \dots, U_l$  disjoint subsets of  $V \setminus V(B)$  that are strongly connected in  $G$ , such that for each  $S \in L_{\geq 2}$ , either  $U_i \in S$  or  $U_i$  is disjoint from  $S$ , we can find eulerian  $F \subseteq E$  such that:*

- (1)  $w(F) \leq 3 \cdot \text{value}(I)$
- (2)  $|\delta_F^-(U_i)| \geq 1$  for all  $i$
- (3)  $|\delta_F^-(v)| \leq 4$  for all  $v$  such that  $x(\delta^-(v)) = 1$
- (4) Any subtour in  $F$  that crosses a set in  $L_{\geq 2}$  visits a vertex of the backbone

We will use the lemma to prove the following theorem, and prove the lemma later on.

**Theorem 0.2.** *There is an  $O(1)$ -light algorithm for local connectivity ATSP on vertebrate pairs.*

*Proof.* We receive  $(I, B)$  and  $V_1, \dots, V_k$  as input. We output  $F^* = B \cup P \cup F$ , where  $P$  and  $F$  are defined as:

$P$ : If  $B$  exists entirely within some  $V_i$ , then we pick  $u \in B$  and  $v \notin V_i$ , and set  $P$  to be the shortest cycle containing  $u$  and  $v$ . Otherwise we keep  $P$  empty.

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$F$ : WLOG, let  $V_1, \dots, V_l$  be the partitions that are disjoint from  $B$ . Now let  $V'_i$  be the intersection of  $V_i$  with a minimal set in  $L_{\geq 2}$  that  $V_i$  intersects. Let  $U_i$  be the first strongly connected component of  $V'_i$  (in the topological order). Set  $F$  to be the eulerian set guaranteed by running the algorithm of Lemma 0.1 on  $U_1, \dots, U_l$ .

Now we need to verify that  $B \cup P \cup F$  is feasible, and that it is  $O(1)$ -light.

For each  $V_i$ , there are three possibilities:

- (a)  $V_i$  completely contains the backbone. The set  $P$  guarantees that  $F^*$  enters  $V_i$  at least once.
- (b)  $V_i$  partially contains the backbone. In this case,  $B$  trivially crosses  $V_i$ , so  $F^*$  must enter  $V_i$ .
- (c)  $V_i$  is disjoint from the backbone. In this case, (2) tells us that some edge  $(u, v)$  must enter  $U_i$ .  $u \notin V'_i$  since  $U_i$  is the source component of  $V'_i$ . So either  $u \in S \setminus V_i$  or  $u \notin S$ . If the former is true, then  $(u, v)$  enters  $V_i$  from outside. If the latter is true, then  $(u, v)$  crosses  $S$ , and guarantee (4) of Lemma 0.1 tells us that the subtour containing  $(u, v)$  visits the backbone, which is outside  $V_i$ .

In any case,  $F^*$  crosses each  $V_i$ , so it is indeed feasible.

Now consider any strongly connected component of  $F^*$ , say  $\tilde{G}$ . We would like to bound  $w(\tilde{G})$  by  $O(\text{lb}(\tilde{G}))$ . There are two possibilities for  $\tilde{G}$ :

- (a)  $\tilde{G}$  contains the backbone. So

$$w(\tilde{G}) \leq w(B) + w(P) + w(F) \leq O(\text{value}(I)) + \text{value}(I) + 3 \cdot \text{value}(I) = O(\text{value}(I))$$

$$\text{lb}(\tilde{G}) \geq \text{lb}(V(B)) = \text{value}(I)$$

And it follows that  $w(\tilde{G}) \leq O(\text{lb}(\tilde{G}))$ .

- (b)  $\tilde{G}$  is disjoint from the backbone. In this case, (4) guarantees that  $\tilde{G}$  did not cross any set in  $L_{\geq 2}$ , so the only sets it might have crossed are singletons. Additionally, (3) guarantees that it crosses each singleton at most four times.

$$w(\tilde{G}) \leq \sum_{v \in V(\tilde{G})} 4 \cdot 2 \cdot y_v = 4 \cdot \text{lb}(\tilde{G})$$

□

## Proof of Lemma 0.1: The final ingredient

Order the laminar sets in  $L_{\geq 2}$  by size so that  $|S_1| \leq |S_2| \leq \dots \leq |S_l| = |V|$ . For each  $v \in V$ , define  $\text{level}(v)$  to be the index of the smallest set it is contained in.

**Definition 0.3.** An edge  $e$  is called a **forward edge**, **backward edge**, or **neutral edge** if  $e \in E_f$ ,  $e \in E_b$ , or  $e \in E_n$  respectively, where  $E_f, E_b$ , and  $E_n$  are defined as:

- $E_f := \{(u, v) \in E \mid \text{level}(u) > \text{level}(v)\}$
- $E_b := \{(u, v) \in E \mid \text{level}(u) < \text{level}(v)\}$
- $E_n := E \setminus (E_f \cup E_b)$

**Definition 0.4.** Call  $G_{sp}$  the **split graph**, where we define it as follows: for each  $v \in V$ , create  $v^0$  and  $v^1$  in  $V(G_{sp})$ . Create the following edges along with weights  $w_{sp}$ :

- $(v^0, v^1)$  for all  $v \in V$  with weight 0. Call these edges 0 – 1 edges.
- $(v^1, v^0)$  for all  $v \in V(B)$  with weight 0. Call these edges 1 – 0 edges.
- $(u^1, v^1)$  for all  $(u, v) \in E_f \cup E_n$  with weight  $w(u, v)$ . Call these edges 1 – 1 edges.

–  $(u^0, v^0)$  for all  $(u, v) \in E_b \cup E_n$  with weight  $w(u, v)$ . Call these edges 0 – 0 edges.

Consider the lift of some subtour in  $G_{sp}$  that crosses at least one set in  $L_{\geq 2}$ . Let  $S$  be the smallest set in  $L_{\geq 2}$  that the subtour crosses. Then the edge of the subtour going into  $S$  must be a forward edge (i.e. a 1 – 1 edge) by definition, and the edge going out of  $S$  must be a backward edge (i.e. a 0 – 0 edge). This means at some point, the subtour must use a 1 – 0 edge, which it can only do if it crosses the backbone.

So if we restrict ourself to lifts of subtours of  $G_{sp}$  (instead of all subtours of  $G$ ), guarantee (4) will always be true.

**Lemma 0.2.** *We can find  $x_{sp}$ , an eulerian vector on  $G_{sp}$  that the image of  $x_{sp}$  in  $G$  is  $x$ .*

If we want the image of  $x_{sp}$  to be  $x$ , we must necessarily have the following conditions:

- (a)  $x_{sp}(u^0, v^0) = 0$ , for all  $(u, v) \in E_f$ , since there is no  $(u^0, v^0)$  edge in  $G_{sp}$ .
- (b)  $x_{sp}(u^1, v^1) = 0$ , for all  $(u, v) \in E_b$ , since there is no  $(u^1, v^1)$  edge in  $G_{sp}$ .
- (c)  $x_{sp}(u^0, v^0) + x_{sp}(u^1, v^1) = x(u, v)$  for all  $(u, v) \in E$ , since the image of  $x_{sp}$  in  $G$  must be  $x$ .

After defining some  $x_{sp}$  values on the 0 – 0 and 1 – 1 edges, there will be some “potential” on each vertex (i.e. the net outgoing flow), and since  $x$  was eulerian, the potential of the 0-copy of a vertex, say  $v$ , is exactly the negative potential of the 1-copy. If the potential of the 1-copy is positive,  $v^0$  and  $v^1$  can be brought down to zero potential by routing that amount of flow through  $(v^0, v^1)$ . But if it is negative, this can only be equalized using a  $(v^1, v^0)$  edge, which is impossible unless  $v \in V(B)$ . In other words, if we want  $x_{sp}$  to be eulerian, it must also fulfil the following condition:

- (d)  $x_{sp}(\delta^-(v_1) \setminus (v^0, v^1)) \geq x_{sp}(\delta^+(v_1) \setminus (v^0, v^1))$  for all  $v \notin V(B)$ .

It can easily be seen that (a), (b), (c), and (d) are also sufficient for some nonnegative  $x_{sp}$  to fulfil the guarantees of Lemma 0.2. This gives rise to the following LP and its dual where  $f$  can be interpreted as  $x_{sp}$  restricted to 1 – 1 edges:

$$\begin{array}{ll|ll}
 \max & \sum_{e \in E_f} f(e) & \min & \sum_{e \in E_f \cup E_n} x(e)z(e) \\
 s.t. & f(\delta^+(v)) \geq f(\delta^-(v)) & & \pi_v - \pi_u + z(u, v) \geq 1 \quad (u, v) \in E_f \\
 & f(e) = 0 & & \pi_v - \pi_u + z(u, v) \geq 0 \quad (u, v) \in E_n \\
 & 0 \leq f(e) \leq x(e) & & \pi_v = 0 \quad v \in V(B) \\
 & & & \pi, z \geq 0
 \end{array}$$

Once we have some primal feasible  $f$ , we can extend it by defining  $x_{sp}$  as  $f$  on 1 – 1 edges,  $x - f$  on 0 – 0 edges, and eliminating potential differences between 0 and 1 copies using 0 – 1 and 1 – 0 edges. The first inequality of the LP guarantees (d) by definition. The second inequality guarantees (b) by definition. And the way we extend  $f$  to 0 – 0 edges guarantees (c). Now if we know that the primal optimal value is equal to  $\sum_{e \in E_f} x_e$ , then it guarantees (a) as well.

**Claim 0.1.** *The primal optimal value is equal to  $\sum_{e \in E_f} x_e$ .*

*Proof.* Note that if there were a dual optimal solution where  $\pi = 0$ , then the dual optimal (and hence primal optimal) value would be precisely  $\sum_{e \in E_f} 1 \cdot x_e$ . So consider any dual optimal solution, say  $(z, \pi)$ . Our goal is to convert it into a dual optimal solution whose  $\pi$  has strictly smaller support. Running this procedure repeatedly proves that there is a dual optimal where  $\pi = 0$ .

Let  $T$  be the set of vertices in  $\text{support}(\pi)$  with the smallest level, and let  $S$  be the set corresponding to that level.

□