The Local Connectivity ATSP Problem

Fix some function $\text{lb} : 2^V \to \mathbb{R}^+$ that obeys the following properties:

1. $\text{lb}(S) = \sum_{v \in S} \text{lb}(\{v\})$ for all $S \subseteq V$
2. $\text{lb}(V) = \text{value}(I)$

One can think of lb as being defined by distributing the LP value among the vertices in some way.

Problem Statement.
Input: I (the instance) and $V_1, \ldots, V_k$ (a partitioning of V) such that V_i is strongly connected.
Output: F, an eulerian set of edges such that for every i, F exits (and hence enters) V_i at least once.

Definition 0.1. An algorithm for local connectivity ATSP is called **$\boldsymbol{\alpha}$-light** if it outputs F such that for each strongly connected component of F, say $\tilde{G}$, $w(\tilde{G}) \leq \alpha \cdot \text{lb}(\tilde{G})$.

Theorem 0.1 (Svensson '15)**.** *An α-light algorithm for local connectivity ATSP implies a $O(\alpha)$-approximation algorithm for ATSP.*

The paper describes an $O(1)$-light algorithm for local connectivity on vertebrate pairs, which gives us an $O(1)$-approximation algorithm for ATSP on vertebrate pairs, and hence an $O(1)$-approximation algorithm for general ATSP.

We will also assume that $w(B) \leq O(1) \cdot \text{value}(I)$, since in the previous reduction, this inequality is always true.
Definition 0.2. For this algorithm, we will define lb as

$$\mathbf{lb(v)} := \begin{cases} \dfrac{\text{value}(I)}{|V(B)|} & \text{if } v \in B \\ 2 \cdot y_v & \text{otherwise} \end{cases}$$

Note that this definition does not guarantee that $\text{lb}(V) = \text{value}(I)$, but $\text{lb}(V)$ is always a constant factor away from $\text{value}(I)$, so we can work with these lb values and suffer a constant factor loss in the approximation ratio.

To make corner cases easier, we add V to $L_{\geq 2}$ for the rest of this write up.

The following is the main technical lemma of the paper.
Lemma 0.1 (Lemma 7.3 in the paper)**.** *Given $U_1, \ldots, U_l$ disjoint subsets of $V \setminus V(B)$ that are strongly connected in G, such that for each $S \in L_{\geq 2}$, either $U_i \in S$ or U_i is disjoint from S, we can find eulerian $F \subseteq E$ such that:*

(1) $w(F) \leq 3 \cdot \text{value}(I)$

(2) $|\delta_F^-(U_i)| \geq 1$ *for all* i

(3) $|\delta_F^-(v)| \leq 4$ *for all* v *such that* $x(\delta^-(v)) = 1$

(4) Any subtour in F that crosses a set in $L_{\geq 2}$ visits a vertex of the backbone

We will use the lemma to prove the following theorem, and prove the lemma later on.

Theorem 0.2. *There is an $O(1)$-light algorithm for local connectivity ATSP on vertebrate pairs.*

Proof. We receive (I, B) and $V_1, \ldots, V_k$ as input. We output $F^* = B \cup P \cup F$, where P and F are defined as:

P: If B exists entirely within some V_i, then we pick $u \in B$ and $v \notin V_i$, and set P to be the shortest cycle containing u and v. Otherwise we keep P empty.

- F: WLOG, let $V_1, \dots, V_l$ be the partitions that are disjoint from B. Now let V_i' be the intersection of V_i with a minimal set in $L_{\geq 2}$ that V_i intersects. Let U_i be the first strongly connected component of V_i' (in the topological order). Set F to be the eulerian set guaranteed by running the algorithm of Lemma 0.1 on $U_1, \dots, U_l$.

Now we need to verify that $B \cup P \cup F$ is feasible, and that it is $O(1)$-light.
For each V_i, there are three possibilities:

(a) V_i completely contains the backbone. The set P guarantees that F^* enters V_i at least once.

(b) V_i partially contains the backbone. In this case, B trivially crosses V_i, so F^* must enter V_i.

(c) V_i is disjoint from the backbone. In this case, (2) tells us that some edge (u, v) must enter U_i. $u \notin V_i'$ since U_i is the source component of V_i'. So either $u \in S \setminus V_i$ or $u \notin S$. If the former is true, then (u, v) enters V_i from outside. If the latter is true, then (u, v) crosses S, and guarantee (4) of Lemma 0.1 tells us that the subtour containing (u, v) visits the backbone, which is outside V_i.

In any case, F^* crosses each V_i, so it is indeed feasible.

Now consider any strongly connected component of F^*, say $\tilde{G}$. We would like to bound $w(\tilde{(G)})$ by $O(\text{lb}(\tilde{G}))$. There are two possibilities for $\tilde{G}$:

(a) $\tilde{G}$ contains the backbone. So

$$w(\tilde{G}) \leq w(B) + w(P) + w(F) \leq O(\text{value}(I)) + \text{value}(I) + 3 \cdot \text{value}(I) = O(\text{value}(I))$$

$$\text{lb}(\tilde{G}) \geq \text{lb}(V(B)) = \text{value}(I)$$

And it follows that $w(\tilde{G}) \leq O(\text{lb}(\tilde{G}))$.

(b) $\tilde{G}$ is disjoint from the backbone. In this case, (4) guarantees that $\tilde{G}$ did not cross any set in $L_{\geq 2}$, so the only sets it might have crossed are singletons. Additionally, (3) guarantees that it crosses each singleton at most four times.

$$w(\tilde{G}) \leq \sum_{v \in V(\tilde{G})} 4 \cdot 2 \cdot y_v = 4 \cdot \text{lb}(\tilde{G})$$

□

Proof of Lemma 0.1: The final ingredient

Order the laminar sets in $L_{\geq 2}$ by size so that $|S_1| \leq |S_2| \leq \cdots \leq |S_l| = |V|$. For each $v \in V$, define $\textit{level}(v)$ to be the index of the smallest set it is contained in.

Definition 0.3. An edge e is called a **forward edge**, **backward edge**, or **neutral edge** if $e \in E_f$, $e \in E_b$, or $e \in E_n$ respectively, where E_f, E_b, and E_n are defined as:

- $\boldsymbol{E_f} := \{(u, v) \in E \mid \textit{level}(u) > \textit{level}(v)\}$
- $\boldsymbol{E_b} := \{u, v) \in E \mid \textit{level}(u) < \textit{level}(v)\}$
- $\boldsymbol{E_n} := E \setminus (E_f \cup E_b)$

Definition 0.4. Call $\boldsymbol{G_{sp}}$ the **split graph**, where we define it as follows: for each $v \in V$, create v^0 and v^1 in $V(G_{sp})$. Create the following edges along with weights $\boldsymbol{w_{sp}}$:

- (v^0, v^1) for all $v \in V$ with weight 0. Call these edges $0 - 1$ edges.
- (v^1, v^0) for all $v \in V(B)$ with weight 0. Call these edges $1 - 0$ edges.
- (u^1, v^1) for all $(u, v) \in E_f \cup E_n$ with weight $w(u, v)$. Call these edges $1 - 1$ edges.

– (u^0, v^0) for all $(u,v) \in E_b \cup E_n$ with weight $w(u,v)$. Call these edges $0-0$ edges.

Consider the lift of some subtour in G_{sp} that crosses at least one set in $L_{\geq 2}$. Let S be the smallest set in $L_{\geq 2}$ that the subtour crosses. Then the edge of the subtour going into S must be a forward edge (i.e. a $1-1$ edge) by definition, and the edge going out of S must be a backward edge (i.e. a $0-0$ edge). This means at some point, the subtour must use a $1-0$ edge, which it can only do if it crosses the backbone.

So if we restrict ourself to lifts of subtours of G_{sp} (instead of all subtours of G), guarantee (4) will always be true.

Lemma 0.2. *We can find x_{sp}, an eulerian vector on G_{sp} that the image of x_{sp} in G is x.*

Proof. If we want the image of x_{sp} to be x, we must necessarily have the following conditions:

(a) $x_{sp}(u^0, v^0) = 0$, for all $(u,v) \in E_f$, since there is no (u^0, v^0) edge in G_{sp}.

(b) $x_{sp}(u^1, v^1) = 0$, for all $(u,v) \in E_b$, since there is no (u^1, v^1) edge in G_{sp}.

(c) $x_{sp}(u^0, v^0) + x_{sp}(u^1, v^1) = x(u,v)$ for all $(u,v) \in E$, since the image of x_{sp} in G must be x.

After defining some x_{sp} values on the $0-0$ and $1-1$ edges, there will be some "potential" on each vertex (i.e. the net outgoing flow), and since x was eulerian, the potential of the 0-copy of a vertex, say v, is exactly the negative potential of the 1-copy. If the potential of the 1-copy is positive, v^0 and v^1 can be brought down to zero potential by routing that amount of flow through (v^0, v^1). But if it is negative, this can only be equalized using a (v^1, v^0) edge, which is impossible unless $v \in V(B)$. In other words, if we want x_{sp} to be eulerian, it must also fulfil the following condition:

(d) $x_{sp}(\delta^-(v_1) \setminus (v^0, v^1)) \geq x_{sp}(\delta^+(v_1) \setminus (v^0, v^1))$ for all $v \notin V(B)$.

It can easily be seen that (a), (b), (c), and (d) are also sufficient for some nonnegative x_{sp} to fulfil the guarantees of Lemma 0.2. This gives rise to the following LP and its dual where f can be interpreted as x_{sp} restricted to $1-1$ edges:

$$
\begin{aligned}
\max \quad & \sum_{e \in E_f} f(e) \\
s.t. \quad & f(\delta^+(v)) \geq f(\delta^-(v)) && v \notin V(B) \\
& f(e) = 0 && e \in E_b \\
& 0 \leq f(e) \leq x(e) && e \in E
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \sum_{e \in E_f \cup E_n} x(e)z(e) \\
s.t. \quad & \pi_v - \pi_u + z(u,v) \geq 1 && (u,v) \in E_f \\
& \pi_v - \pi_u + z(u,v) \geq 0 && (u,v) \in E_n \\
& \pi_v = 0 && v \in V(B) \\
& \pi, z \geq 0
\end{aligned}
$$

Once we have some primal feasible f, we can extend it by defining x_{sp} as f on $1-1$ edges, $x-f$ on $0-0$ edges, and eliminating potential differences between 0 and 1 copies using $0-1$ and $1-0$ edges. The first inequality of the LP guarantees (d) by definition. The second inequality guarantees (b) by definition. And the way we extend f to $0-0$ edges guarantees (c). Now if we know that the primal optimal value is equal to $\sum_{e \in E_f} x_e$, then it guarantees (a) as well. The following claim proves this, so (a), (b), (c), and (d) must all hold. $\square$

Claim 0.1. *The primal optimal value is equal to $\sum_{e \in E_f} x_e$.*

Proof. Note that if there were a dual optimal solution where $\pi = 0$, then the dual optimal (and hence primal optimal) value would be precisely $\sum_{e \in E_F} 1 \cdot x_e$. So consider any dual optimal solution, say (z, π). Our goal is to convert it into a dual optimal solution whose π has strictly smaller support. Running this procedure repeatedly proves that there is a dual optimal where $\pi = 0$.

Let T be the set of vertices in support(π) with the smallest level, and let S be the set corresponding to that level. Also, let $F = \delta(V \setminus S, T)$, and $F' = \delta(T, S \setminus T)$.

To start off, let's decrement the π-values of all vertices in T by the smallest π-value of any vertex in T, say ε. Now π has a smaller support, but it may not be feasible. In order to correct this, we increase $z(e)$ by ε for all $e \in F$. Now the solution might not be optimal. In order to correct this, we decrease $z(e)$ by ε for all $e \in F'$. In order to see why the new dual is feasible, note that an edge (u, v) can be of the following types:

Case 1: $v \notin T$. Since we only decreased the π-value of vertices in T (and left all other vertices unchanged), the left hand side of each of the first two dual inequalities can only increase. So these inequalities are still satisfied.

Case 2: $u, v \in T$. In this case, the decrease in π_v is matched by the increase in $-\pi_u$, so the inequalities are still satisfied.

Case 3: $u \notin S, v \in T$. In this case, the decrease in π_v is matched by the increase in $z(u, v)$, so the inequalities are still satisfied.

Case 4: $u \in S \setminus T, v \in T$. In this case, (u, v) cannot be a forward edge, since if it were, then T would intersect a set in $L_{\geq 2}$ which was smaller than S. But we defined S to be the smallest such set. So we only consider the second inequality, which must be satisfied, since π_u was already 0 and π_v is nonnegative.

The inequality $\pi \geq 0$ must be satisfied simply because we reduced π_v for $v \in T$ by $\min_{u \in T} \pi_u$. Also, since all edges (u, v) in F' are either neutral or forward, we originally satisfied $\pi_v - \pi_u + z(u, v) \geq 0$. Since $u \in T$ and $v \in S \setminus T$, we must have that $0 - \varepsilon + z(u, v) \geq 0$, that is, $z(u, v) \geq \varepsilon$. So the inequality $z \geq 0$ must be satisfied since we only reduced z-value in F', and for all edges in F', we already had ε z-value to begin with.

Now we will see that the new dual value is still optimal. The decrease in dual objective value was precisely $\varepsilon(x(F') - x(F \cap (E_f \cup E_n)) \geq \varepsilon(x(F') - x(F))$. If we can show that this is positive, then the new dual value is still optimal.

$\square$