Definitions

Definition 0.1. We say that a subtour B, is a backbone of the instance $\mathcal{I}$ if for every non-singleton set $S \in \mathcal{L}$. B crosses S i.e. $\delta(S) \cap B \neq \emptyset$.
The pair $(\mathcal{I}, B)$ is called the vertebrate pair.

Definition 0.2. For an instance $\mathcal{I}$ we say that a subtour B a quasi-backbone if

$$2 \sum_{S \in \mathcal{L}^*} y_S \leqslant (1-\delta)\,\mathrm{value}(\mathcal{I}) \tag{1}$$

where $\mathcal{L}^*$ is the set of all $S \in \mathcal{L}$ that are not crossed by B.

Here we note that a quasi-backbone may not be a backbone and vice-versa especially if majority of value($\mathcal{I}$) is concentrated on singleton laminar sets in $\mathcal{L}$.[1]

Finding a Quasi-Backbone

Lemma 0.1. *Given an irreducible instance $\mathcal{I}$, we can construct a quasi-backbone B such that it crosses all maximal non-singleton sets in $\mathcal{L}$ and $w(B) \leqslant (\alpha_{NW} + 3)\mathrm{value}(\mathcal{I})$.*

Proof. Let $\mathcal{L}_{\max}$ be the set of all maximal sets in $\mathcal{L}$ and $\mathcal{I}'$ be the instance obtained by contracting each set in $\mathcal{L}_{\max}$. The instance $\mathcal{I}'$ is node weighted and to find a subtour B' in $\mathcal{I}$ we find a tour T in $\mathcal{I}'$ using α_{NW} approx algorithm for Node-Weighted ATSP and lift it. From Lemma **??** and Fact **??** we know that $w_{\mathcal{I}}(B') \leqslant w_{\mathcal{I}'}(T) \leqslant \alpha_{NW}\mathrm{value}(\mathcal{I}') \leqslant \alpha_{NW}\mathrm{value}(\mathcal{I})$ and that B' crosses all maximal non-singleton laminar sets but, we still need to ensure that it crosses at least $\delta\mathrm{value}(\mathcal{I})$ of sets. For this we modify B' to get our quasi-backbone B as follows:

- Let u^S, v^S be the first entry and exit vertex for S in B' respectively and, $u^S_{\max}, v^S_{\max}$ be the vertices corresponding to $D_{\max}(S)$.
- Replace the $u^S - v^S$ path by the path P composed of the shortest paths[2] from u^S to $u^S_{\max}$ to $v^S_{\max}$ to v^S, from Lemma **??** $w(P) \leqslant 3\mathrm{value}_{\mathcal{I}}(S)$

As S is irreducible, with this modification we automatically satisfy our last requirement while only increasing the weight by at most $3\mathrm{value}(\mathcal{I})$. □

Obtaining a Vertebrate Pair

Now we would like to find a way to convert a quasi-backbone into a backbone and then use it, along with an approximation algorithm for ATSP on vertebrate pair to approximate ATSP on irreducible instances.

Theorem 0.1. *Given an irreducible instance $\mathcal{I}$, Algorithm 1 returns a Tour of $\mathcal{I}$ of cost no more than $\rho\mathrm{value}(\mathcal{I})$ where, $\rho = \frac{\kappa+\eta(\alpha_{NW}+3)}{1-2(1-\delta)}$.*

Proof.

$$w(F) \leqslant w(T') \leqslant \kappa\mathrm{value}(\mathcal{I}) + \eta(\alpha_{NW}+3)\mathrm{value}(\mathcal{I})$$
$$w(F) \leqslant (\kappa + \eta(\alpha_{NW}+3))\,\mathrm{value}(\mathcal{I})$$

[1] While the backbones we construct will also be quasi-backbones but our future reductions do not use this fact.
[2] These paths exists as $u^S, u^S_{\max} \in S_1$ and $v^S, v^S_{\max} \in S_l$.

Algorithm 1: Irreducible ATSP (A_{irr})

Input: Irreducible instance $\mathcal{I} = (G, \mathcal{L}, x, y)$
Algorithm A for ATSP on vertebrate pair that returns a tour of cost at most $\kappa\text{value}(\mathcal{I}') + \eta w(B)$.

Use Lemma 0.1 to obtain a quasi-backbone B.
$L^*_{\max} \leftarrow$ all non-singleton maximal sets in $\mathcal{L}^*$.
for *each* $S \in \mathcal{L}^*_{\max}$ **do**
  Find $T_S \leftarrow A_{irr}(\mathcal{I}[S])$.
  Use T_S to find F_S for which S is contractible.
end
$\mathcal{I}' =$ Instance obtained by contracting every $S \in \mathcal{L}^*_{\max}$.
$T' \leftarrow A(\mathcal{I}', B)$ and $T \leftarrow$ lift of T'.
return $T \cup \left(\cup_{S \in \mathcal{L}^*_{\max}} F_S\right)$

$$w(F_S) \leqslant w_{\mathcal{I}[S]}(T_S) \leqslant \rho\text{value}(\mathcal{I}[S])$$
$$w(F_S) \leqslant 2\rho\text{value}_{\mathcal{I}}(S)$$
$$\bigcup_{S \in \mathcal{L}^*_{\max}} w(F_S) \leqslant 2\rho\,(1-\delta)\,\text{value}(\mathcal{I}) \qquad - \text{ As all } S \text{ are disjoint and, from 1 as } B \text{ was a quasi-backbone.}$$

$$\begin{aligned} Cost(\mathcal{I}) &= w(F) + \cup_{S \in \mathcal{L}^*_{\max}} w(F_S) \\ &\leqslant (\kappa + \eta(\alpha_{NW} + 3))\,\text{value}(\mathcal{I}) + 2\rho\,(1-\delta)\,\text{value}(\mathcal{I}) \qquad \text{Substituting } \rho \text{ and } \delta = 0.75. \\ Cost(\mathcal{I}) &\leqslant \rho\text{value}(\mathcal{I}) \end{aligned}$$

$\square$