Explicit solution of extensible worm like chain model for low force regime

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DNA can extend to certain extent and act as a sping, an effect that becomes more pronounced if the applied force is very high. Still, an approximate formula for force regime also includes a correction term. For low force regime such as F < 10pN, the following approximation holds (`https://en.wikipedia.org/wiki/Worm-like_chain`):

$$\frac{FP}{k_BT} = \frac{1}{4}\left(1 - \frac{x}{L} + \frac{F}{S}\right)^{-2} - \frac{1}{4} + \frac{x}{L} - \frac{F}{S} \tag{1}$$

Here, F denotes the applied force, k_B is the Boltzmann constant, P is the persistence length, L is the contour length, S is the stretch modulus governing the extensibility. S is typically cited to be about 1200pN.

We define a new variable u satisfying:

$$\frac{1}{u} = 1 + \frac{F}{S} - \frac{x}{L} \tag{2}$$

using this variable, Eqn. 1 becomes

$$\begin{aligned} \frac{FP}{k_BT} &= \frac{1}{4}\left(\frac{1}{u}\right)^{-2} - \frac{1}{4} + 1 - \frac{1}{u} \\ 0 &= \frac{1}{4}u^2 + \frac{3}{4} - \frac{1}{u} - \frac{FP}{k_BT} \\ 0 &= u^3 + 4u\left(\frac{3}{4} - \frac{FP}{k_BT}\right) - 4 \end{aligned} \tag{3}$$

We note that this equation is of the form $u^3 + Au + B = 0$, which we can analytically solve by noting the Cordano's identity by making a binomial expansion of cubic power:

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \tag{4}$$

which can be rearranged to give

$$(a-b)^3 + 3ab(a-b) = a^3 - b^3 \tag{5}$$

We note that this is of the same form whose solution we are seeking, which we can achieve by assigning $u = a - b$, $A = 3ab$, $B = b^3 - a^3$. We can solve this latter system easily by the well-known quadratic formula following a simple subtitution $b = A/3a$:

$$a^6 + Ba^3 - A^3/27 = 0 \tag{6}$$

$$\begin{aligned} a^3 &= \frac{1}{2}\left(-B \pm \sqrt{B^2 + 4A^3/27}\right) \\ a &= \sqrt[3]{\frac{1}{2}\left(-B \pm \sqrt{B^2 + 4A^3/27}\right)} \\ b &= \sqrt[3]{\frac{1}{2}\left(B \pm \sqrt{B^2 + 4A^3/27}\right)} \end{aligned} \tag{7}$$

Using this, we now can find the analytical exprassion of the variable u:

$$u(F) = \sqrt[3]{\frac{1}{2}\left(-B \pm \sqrt{B^2 + 4A^3/27}\right)} - \sqrt[3]{\frac{1}{2}\left(B \pm \sqrt{B^2 + 4A^3/27}\right)} \tag{8}$$

which we can directly substitute into Eqn 2 to find the extension as an explicit function of F:

$$x(F) = L\left(1 + \frac{F}{S} - \left[\sqrt[3]{\frac{1}{2}\left(-B \pm \sqrt{B^2 + 4A^3/27}\right)} - \sqrt[3]{\frac{1}{2}\left(B \pm \sqrt{B^2 + 4A^3/27}\right)}\right]^{-1}\right) \tag{9}$$

Note that for our case $B = -4$ and $A = 4\left(3/4 - FP/k_BT\right)$. For computational purposes, you can also verify that Eqn 9 is guaranteed to be real and the $\pm$ sign choise is not of importance due to the symmetry.