

NATURAL METRICS ON TANGENT BUNDLES

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Abstract

In this Master's dissertation we study the geometry of tangent bundles TM of Riemannian manifolds (M, g) . We calculate their Lie bracket, introduce a natural class of Riemannian metrics \bar{g} on the tangent bundle TM transforming the natural projection from the tangent bundle TM onto (M, g) into a Riemannian submersion. As examples of natural metrics we discuss in detail the Sasaki and the Cheeger-Gromoll metrics. We calculate their Levi-Civita connections and various curvatures. This leads to some interesting connections between the geometry of the Riemannian manifold (M, g) and its tangent bundle TM equipped with these two natural metrics.

Throughout Chapters 2, 3 and 4 of this work it has been my firm intention to give reference to the stated results and credit the work of others. The results and proofs not marked with a number in brackets $[X]$ are the fruits of my efforts.

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Introduction

The main purpose of this Master's project is to write a survey on the geometry of tangent bundles. This is an area which has been developed over several decades and the authors have used a variety of approaches and very different notation. It is our main aim to write a unified presentation of some of the best known results in this field.

The research in this area of differential geometry began with the fundamental paper [16] of Sasaki published in 1958. He uses the metric g on a Riemannian manifold (M, g) and its Levi-Civita connection ∇ to construct a metric \hat{g} on the tangent bundle TM of M . Today this metric is a standard notion in Differential Geometry called the Sasaki metric.

In his famous article [6] from 1962, Dombrowski calculates the Lie bracket $[\cdot, \cdot]$ on the tangent bundle TM and discusses its natural almost complex structure J , constructed by Nagano in [14].

In 1971 Kowalski published [10] where he uses the Lie bracket on TM to get explicit formulae for the Levi-Civita connection $\hat{\nabla}$ of the tangent bundle (TM, \hat{g}) equipped with the Sasaki metric. Then he calculates the corresponding Riemann curvature tensor \hat{R} of (TM, \hat{g}) .

In 1981 Aso [1] and in 1988 Musso and Tricerri [13] use Kowalski's calculations of the curvature tensor to draw some very interesting conclusions on the connection between the geometry of (M, g) and that of (TM, \hat{g}) .

As early as 1972 Cheeger and Gromoll suggested a way of constructing a new natural Riemannian metric \tilde{g} on the tangent bundle TM of (M, g) . This heavily depends on Sasaki's idea of splitting the tangent bundle TTM of TM into a horizontal and a vertical part, using the Levi-Civita connection ∇ of (M, g) . The first explicit expression of the Cheeger-Gromoll metric \tilde{g} was given in the paper [13] of Musso and Tricerri published in 1988. Sekizawa uses this in [18] to calculate both, the Levi-Civita connection $\tilde{\nabla}$ and the curvature tensor \tilde{R} of (TM, \tilde{g}) . With this in hand he was able to give an interesting connection between the geometry of (M, g) and that of (TM, \tilde{g}) .

In Chapter 1 we lay the basis for this work. We introduce the concept of a differentiable manifold M and its tangent bundle TM , the Levi-Civita connection ∇ and the Riemann curvature tensor R . We discuss widely known facts, which are only proven in certain exceptional cases. They can be found in any book on Riemannian geometry, see for example [5] or [8].

In Chapter 2 we equip the tangent bundle with additional structures such as horizontal and vertical lifts of vector fields on M and an almost complex structure. We calculate the Lie bracket, define a class of natural metrics \bar{g} on TM and obtain formulae for its Levi-Civita connection $\bar{\nabla}$. In this chapter we mainly follow the work of Dombrowski [6].

Chapter 3 is devoted to the Sasaki metric as an example of a natural metric. We calculate its Levi-Civita connection and its Riemann curvature tensor and conclude with some geometric consequences. In this chapter we follow the work of Kowalski [10], Aso [1] and Musso-Tricerri [13].

Chapter 4 is devoted to the Cheeger-Gromoll metric. We calculate the Levi-Civita connection, the Riemann curvature tensor and obtain some geometric conclusions. Here we mainly follow the work of Musso-Tricerri [13] and Sekizawa [18], but without using the methods of moving frames, as in [13]. In the last section of this chapter we have found and corrected some mistakes in [18].

Throughout this work we have used Riemannian manifolds of constant sectional curvature as an example to illustrate the main results.

For later development in this field we recommend the reading of [11], [4] and [12].

The Tangent Bundle

In this chapter we introduce the main object under investigation in this work, namely the tangent bundle TM of a smooth manifold M . This is itself a smooth manifold of double the dimension of that of M . In order to set up the notation being used we give a very brief introduction to smooth manifolds following the presentation of [8]. The missing proofs are standard and can be found in most books on Riemannian geometry, see for example [5], [7] or [9].

1. Differentiable Manifolds

Definition 1.1. Let M be a topological Hausdorff space with a countable basis. M is called a *topological manifold*, if there exists an $m \in \mathbb{N}$ and for every point $p \in M$ an open neighborhood U_p of p such that U_p is homeomorphic to some open subset $V_p \subset \mathbb{R}^m$. The integer m is called the *dimension* of M . We write M^m to denote that M has dimension m .

Definition 1.2. Let M^m be a topological manifold, U an open and connected subset of M and $\varphi : U \rightarrow \mathbb{R}^m$ a continuous map homeomorphic onto its image $\varphi(U)$. Then (U, φ) is called a *local coordinate* on M . A collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ of local coordinates on M is called a C^r -atlas if

- i) $M = \bigcup_\alpha U_\alpha$, and
- ii) the corresponding *transition maps*

$$\varphi_\beta \circ \varphi_\alpha^{-1} \mid_{\varphi_\alpha(U_\alpha \cap U_\beta)} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \mathbb{R}^m$$

are C^r for all $\alpha, \beta \in I$.

If \mathcal{A} is a C^r -atlas on M then a local coordinate (U, φ) on M is said to be *compatible* with \mathcal{A} if $\mathcal{A} \cup \{(U, \varphi)\}$ is a C^r -atlas. A C^r -atlas $\hat{\mathcal{A}}$ is *maximal* if it contains all local coordinates compatible with it. It is also called a C^r -*structure* on M and the pair $(M, \hat{\mathcal{A}})$ is called a *differentiable C^r -manifold*. By *smooth* we mean C^∞ defined by $C^\infty = \bigcap_{k=1}^\infty C^k$.

We now are able to discuss maps between two differentiable manifolds. We are mainly interested in differentiable maps, so we will now define:

Definition 1.3. Let $(M^m, \hat{\mathcal{A}})$ and $(N^n, \hat{\mathcal{B}})$ be two C^r -manifolds. A map $\psi : M^m \rightarrow N^n$ is said to be a C^r -map if for all local coordinates $(U, \varphi) \in \hat{\mathcal{A}}$

and $(V, \chi) \in \hat{\mathcal{B}}$ the maps

$$\chi \circ \psi \circ \varphi^{-1} \big|_{\varphi(U \cap \psi^{-1}(V))} : \varphi(U \cap \psi^{-1}(V)) \rightarrow \mathbb{R}^n$$

are of class C^r . If ψ maps to \mathbb{R} it is called a C^r -function on M .

In what follows we shall assume that all manifolds and maps are smooth, i.e. in the C^∞ -category.

Proposition 1.4. Let $\varphi : (M, \hat{\mathcal{A}}) \rightarrow (N, \hat{\mathcal{B}})$ and $\psi : (N, \hat{\mathcal{B}}) \rightarrow (L, \hat{\mathcal{C}})$ be two C^∞ -maps, then the composition $\psi \circ \varphi : (M, \hat{\mathcal{A}}) \rightarrow (L, \hat{\mathcal{C}})$ is also a C^∞ -map.

Definition 1.5. Two C^r -manifolds $(M, \hat{\mathcal{A}})$ and $(N, \hat{\mathcal{B}})$ are said to be *diffeomorphic* if there exists a bijective C^∞ -map $\varphi : (M, \hat{\mathcal{A}}) \rightarrow (N, \hat{\mathcal{B}})$ such that its inverse also is C^∞ . The map φ is called a *diffeomorphism* between $(M, \hat{\mathcal{A}})$ and $(N, \hat{\mathcal{B}})$. If N is equal to M then the two C^∞ -structures $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are said to be *different*, if the identity map $id_M : (M, \hat{\mathcal{A}}) \rightarrow (M, \hat{\mathcal{B}})$ is not a diffeomorphism.

2. Tangent Spaces

In this section we introduce the tangent space $T_p M$ of M at a point p . Let M^m be a manifold and $p \in M$ a point of M and $\hat{\varepsilon}(p)$ be the set of all smooth functions on an open neighborhood around p , i.e

$$\hat{\varepsilon}(p) = \{f : U_f \rightarrow \mathbb{R} \mid U_f \subset M \text{ open and containing } p\}.$$

Now we define an equivalence relation on $\hat{\varepsilon}(p)$ by: $f \equiv g$ if and only if there exists an open neighborhood $V \subset U_f \cap U_g$ such that $f|_V = g|_V$. By $\varepsilon(p)$ we denote the set of equivalence classes of elements in $\hat{\varepsilon}(p)$. The elements \bar{f} of $\varepsilon(p)$ are called function germs at p . We can turn $\varepsilon(p)$ into an \mathbb{R} -algebra by the following operations $+$ and \cdot

$$\begin{aligned} i) \quad \bar{f} + \bar{g} &= \overline{f + g}, \\ ii) \quad \lambda \cdot \bar{f} &= \overline{\lambda \cdot f}, \\ iii) \quad \bar{f} \cdot \bar{g} &= \overline{f \cdot g} \end{aligned}$$

for all $f, g \in \hat{\varepsilon}(p)$ and $\lambda \in \mathbb{R}$. From now on we will denote the elements of $\varepsilon(p)$ by f instead of \bar{f} .

Definition 1.6. A *tangent vector* X_p at $p \in M$ is a map $X_p : \varepsilon(p) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} i) \quad X_p(\lambda \cdot f + \mu \cdot g) &= \lambda \cdot X_p(f) + \mu \cdot X_p(g), \\ ii) \quad X_p(f \cdot g) &= g(p) \cdot X_p(f) + f(p) \cdot X_p(g) \end{aligned}$$

for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in \varepsilon(p)$. The set of all tangent vectors X_p at $p \in M$ is denoted by $T_p M$ and is called the *tangent space* of M at p .

The tangent space $T_p M$ is turned into a real vector space by defining the operations $+$ and \cdot by

- i) $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$,
- ii) $(\lambda \cdot X_p)(f) = \lambda \cdot X_p(f)$

for all $\lambda \in \mathbb{R}$ and $X_p, Y_p \in T_p M$.

For $M = \mathbb{R}^m$ we denote ε_m by $\varepsilon(0)$ as the set of function germs at $0 \in \mathbb{R}$. It is well known from calculus, that for $v \in \mathbb{R}^m$ and $f \in \varepsilon_m$ the directional derivative of f at 0 in the direction of v is given by

$$\partial_v f = \lim_{t \rightarrow 0} \frac{f(tv) - f(0)}{t}.$$

For $v = (v_1, \dots, v_m)$ we get $\partial_v f = \sum_{i=1}^m v_i \frac{\partial f}{\partial x_i}(0)$ and we know that

- i) $\partial_v(\lambda \cdot f + \mu \cdot g) = \lambda \cdot \partial_v f + \mu \cdot \partial_v g$,
- ii) $\partial_v(f \cdot g) = g(0) \cdot \partial_v f + f(0) \cdot \partial_v g$,
- iii) $\partial_{\lambda \cdot v + \mu \cdot w} f = \lambda \cdot \partial_v f + \mu \cdot \partial_w g$

for all $\lambda, \mu \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $f, g \in \varepsilon_m$. Hence ∂_v is an element of $T_0 \mathbb{R}^m$.

Theorem 1.7. *The map $\Phi : \mathbb{R}^m \rightarrow T_0 \mathbb{R}^m$ given by $v \mapsto \partial_v$ is a vector space isomorphism.*

Corollary 1.8. *Let $\{e_k \mid k = 1, \dots, m\}$ be a basis for \mathbb{R}^m . Then*

$$\{\partial_{e_k} \mid k = 1, \dots, m\}$$

is a basis for the tangent space $T_0 \mathbb{R}^m$.

Definition 1.9. Let $\varphi : M \rightarrow N$ be a map between two manifolds. For a point $p \in M$ we define the map $d\varphi_p : T_p M \rightarrow T_{\varphi(p)} N$ by

$$(d\varphi_p)(X_p)(f) = X_p(f \circ \varphi)$$

for all $X_p \in T_p M$ and $f \in \varepsilon(\varphi(p))$. The map $d\varphi_p$ is called the *differential* of φ at $p \in M$.

Proposition 1.10. *Let $\varphi : M \rightarrow \tilde{M}$ and $\psi : \tilde{M} \rightarrow N$ be two maps between smooth manifolds, then*

- i) *the map $d\varphi_p : T_p M \rightarrow T_{\varphi(p)} \tilde{M}$ is linear,*
- ii) *if id_M is the identity map, then $d(id_M)_p = id_{T_p M}$,*
- iii) $d(\psi \circ \varphi)_p = d\psi_{\varphi(p)} \circ d\varphi_p$

for all $p \in M$. The equation iii) is called the Chain rule.

PROOF. The first two points follow directly from the definition, so we only have to prove the chain rule. If $X_p \in T_p M$ and $f \in \varepsilon(\psi \circ \varphi(p))$, then

$$\begin{aligned} (d\psi_{\varphi(p)} \circ d\varphi_p(X_p))(f) &= (d\varphi_p(X_p))(f \circ \psi) \\ &= X_p(f \circ \psi \circ \varphi) \\ &= d(\psi \circ \varphi)_p(X_p)(f). \end{aligned}$$

□

Corollary 1.11. Let $\varphi : M \rightarrow N$ be a diffeomorphism with inverse $\psi = \varphi^{-1} : N \rightarrow M$. Then the differential $d\varphi_p : T_p M \rightarrow T_{\varphi(p)} N$ at p is bijective and $(d\varphi_p)^{-1} = d\psi_{\varphi(p)}$.

As a direct consequence of Corollary 1.8 and Corollary 1.11 we obtain

Corollary 1.12. Let p be a point of a m -dimensional manifold M . Then the tangent space $T_p M$ at p is an m -dimensional real vector space.

Definition 1.13. Let M^m be a manifold, (U, x) be a local coordinate on M and $\{e_k \mid k = 1, \dots, m\}$ be the standard basis for \mathbb{R}^m . For $p \in M$ define $(\frac{\partial}{\partial x_k})_p \in T_p M$ by

$$(\frac{\partial}{\partial x_k})_p : f \mapsto \frac{\partial f}{\partial x_k}(p) = \partial_{e_k}(f \circ x^{-1})(x(p)).$$

Proposition 1.14. The set $\{(\frac{\partial}{\partial x_k})_p \mid k = 1, \dots, m\}$ is a basis for $T_p M$ for all $p \in U$.

PROOF. Because M is smooth, it follows that the inverse of x is smooth and therefore the differential of the inverse satisfies

$$(dx_{x(p)}^{-1})(\partial_{e_k})(f) = \partial_{e_k}(f \circ x^{-1})(x(p)) = ((\frac{\partial}{\partial x_k})_p)(f)$$

for all $f \in \varepsilon(p)$. □

The tangent space $T_p M$ may be viewed in an alternative way. For this we use the set $C(p)$ of all equivalence classes of locally defined C^1 -curves passing through the point $p \in M$. It is possible to identify $T_p M$ with $C(p)$ being the set of all tangents to curves going through the point p . Then a vector $v \in T_p M$ can be described by

$$v(f) = \frac{d}{dt}(f \circ \gamma(t))|_{t=0},$$

with $f : U \subset M \rightarrow \mathbb{R}$ a function defined on U containing p and $\gamma : I \rightarrow U$ an arbitrary curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

3. The Tangent Bundle

In this section we introduce the notion of a smooth vector bundle over a manifold. The tangent bundle turns out to be an important example and the main object in this work.

Definition 1.15. Let E and M be topological manifolds and $\pi : E \rightarrow M$ be a continuous surjective map. If

- i) for each $p \in M$ the fiber $E_p = \pi^{-1}(p)$ is an n -dimensional vector space and

- ii) for each $p \in M$ there exists a *bundle chart* $(\pi^{-1}(U), \psi)$ consisting of the pre-image of π of an open neighborhood U of p and a homeomorphism $\psi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ such that for all $q \in U$ the map $\psi_q = \psi|_{E_q} : E_q \rightarrow \{q\} \times \mathbb{R}^n$ is a vector space isomorphism,

then the triple (E, M, π) is called an n -dimensional *topological vector bundle* over M . It is said to be *trivial* if there exists a global bundle chart $\psi : E \rightarrow M \times \mathbb{R}^n$.

Definition 1.16. Let (E, M, π) be a topological vector bundle. A continuous map $\sigma : M \rightarrow E$ is called a *section* of the bundle if $\pi \circ \sigma(p) = p$ for each $p \in M$.

Definition 1.17. A collection

$$\mathcal{B} = \{(\pi^{-1}(U_\alpha), \psi_\alpha) \mid \alpha \in I\}$$

of bundle charts is called a *bundle atlas* for (E, M, π) if $M = \bigcup_\alpha U_\alpha$. For each pair (α, β) there exists a function $A_{\alpha, \beta} : U_\alpha \cap U_\beta \rightarrow GL(\mathbb{R}^n)$, into the general linear group $GL(\mathbb{R}^n)$ of \mathbb{R}^n , such that the corresponding continuous map

$$\psi_\alpha \circ \psi_\beta^{-1} \big|_{(U_\alpha \cap U_\beta) \times \mathbb{R}^n} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$(p, v) \mapsto (p, (A_{\alpha, \beta})(v)).$$

The elements of $\{A_{\alpha, \beta} \mid \alpha, \beta \in I\}$ are called the *transition maps* of the bundle atlas \mathcal{B} .

Remark 1.18. Since all the maps which we are using are smooth we call a topological vector bundle *smooth*, if \mathcal{B} is maximal. A smooth section of (E, M, π) is called a *vector field* and we denote the set of all vector fields of (E, M, π) by $C^\infty(E)$.

Definition 1.19. By the following operations we make $C^\infty(E)$ into a $C^\infty(M) = C^\infty(M, \mathbb{R})$ module

- i) $(v + w)_p = v_p + w_p$,
- ii) $(f \cdot v)_p = f(p) \cdot v_p$

for all $v, w \in C^\infty(E)$ and $f \in C^\infty(M) = C^\infty(M, \mathbb{R})$. In particular, $C^\infty(E)$ is a vector space over the real numbers.

Definition 1.20. Let M be a manifold and (E, M, π) be an n -dimensional vector bundle over M . A set $F = \{v_1, \dots, v_n\}$ of vector fields

$$v_1, \dots, v_n : U \subset M \rightarrow E$$

is called a *local frame* for E over U if for each $p \in U$ the set $\{(v_1)_p, \dots, (v_n)_p\}$ is a basis for the vector space E_p .

We can now define the tangent bundle TM on M . It can be thought of as the object you get by gluing the tangent space $T_p M$ at p on M for all $p \in M$. This way we obtain a $2m$ -dimensional topological vector bundle.

Definition 1.21. Let $(M, \hat{\mathcal{A}})$ be a manifold. The *tangent bundle* TM of M is given by

$$TM = \{(p, u) \mid p \in M, u \in T_p M\}.$$

The bundle map $\pi : TM \rightarrow M$ with $\pi : (p, u) \mapsto p$ is called the *natural projection* of TM .

Theorem 1.22. Let M be a topological manifold of dimension m with a C^∞ -atlas \mathcal{A} . Then the tangent bundle TM is a topological manifold of dimension $2m$ and \mathcal{A} induces a C^∞ -atlas \mathcal{A}^* on TM .

PROOF. For every local coordinate $x : U \rightarrow \mathbb{R}^m$ in \mathcal{A} we define $x^* : \pi^{-1}(U) \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ by

$$x^* : (p, \sum_{k=1}^m u_k \frac{\partial}{\partial x_k} \Big|_p) \mapsto (x(p), (u_1, \dots, u_m)).$$

Then the collection

$$\{(x^*)^{-1}(W) \subset TM \mid (U, x) \in \hat{\mathcal{A}} \text{ and } W \subset x(U) \times \mathbb{R}^m \text{ open}\}$$

is a basis for a topology \mathcal{T}_{TM} on TM and $(\pi^{-1}(U), x^*)$ is a local coordinate on the $2m$ -dimensional topological manifold (TM, \mathcal{T}_{TM}) .

If (U, x) and (V, y) are two local coordinates in \mathcal{A} such that $p \in U \cap V$, then the transition map

$$(y^*) \circ (x^*)^{-1} : x^*(\pi^{-1}(U \cap V)) \rightarrow \mathbb{R}^m \times \mathbb{R}^m$$

is given by

$$(p, u) \mapsto \left(y \circ x^{-1}(x), \sum_{k=1}^m \frac{\partial y_1}{\partial x_k}(x^{-1}(p)) u_k, \dots, \sum_{k=1}^m \frac{\partial y_m}{\partial x_k}(x^{-1}(p)) u_k \right)$$

We are assuming that $y \circ x^{-1}$ is smooth, hence $(y^*) \circ (x^*)^{-1}$ is smooth and therefore $\mathcal{A}^* = \{(\pi^{-1}(U), x^*) \mid (U, \varphi) \in \hat{\mathcal{A}}\}$ is a C^∞ -atlas on TM and $(TM, \hat{\mathcal{A}}^*)$ is a smooth manifold. \square

Remark 1.23. For each point $p \in M$ the fiber $\pi^{-1}(p)$ of π is the tangent space $T_p M$ of M at p and hence an m -dimensional vector space. For a local coordinate $x : U \rightarrow \mathbb{R}^m \in \hat{\mathcal{A}}$ we define $\bar{x} : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m$ by

$$\bar{x} : (p, \sum_{k=1}^m u_k \frac{\partial}{\partial x_k} \Big|_p) \mapsto (p, (u_1, \dots, u_m)).$$

The restriction $\bar{x}_p = \bar{x}|_{T_p M} : T_p M \rightarrow \{p\} \times \mathbb{R}^m$ to $T_p M$ is given by

$$\bar{x}_p : \sum_{k=1}^m u_k \frac{\partial}{\partial x_k} \Big|_p \mapsto (u_1, \dots, u_m),$$

which obviously is a vector space isomorphism. Hence the $\bar{x} : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m$ is a bundle chart. This implies that

$$\mathcal{B} = \{(\pi^{-1}(U), \bar{x}) \mid (U, x) \in \hat{A}\}$$

is a bundle atlas transforming (TM, M, π) into an m -dimensional topological vector bundle. This implies that the vector bundle (TM, M, π) together with the maximal bundle atlas $\hat{\mathcal{B}}$ induced by \mathcal{B} is a smooth vector bundle.

4. The Lie Bracket

In this section we introduce the Lie bracket on a differentiable manifold (M, \mathcal{A}) .

Definition 1.24. Let M be a manifold and $X, Y \in C^\infty(TM)$ be vector fields on M . Then the *Lie bracket* $[X, Y]_p$ of X and Y at $p \in M$ is defined by

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)) \in \mathbb{R},$$

where $f \in C^\infty(M)$.

Lemma 1.25. Let M be a smooth manifold, $X, Y \in C^\infty(TM)$ be two vector fields on M , $f, g \in C^\infty(M)$ and $\lambda, \mu \in \mathbb{R}$. Then

- i) $[X, Y]_p(\lambda f + \mu g) = \lambda[X, Y]_p(f) + \mu[X, Y]_p(g)$,
- ii) $[X, Y]_p(f \cdot g) = f(p)[X, Y]_p(g) + g(p)[X, Y]_p(f)$.

Proposition 1.26. Let M be a manifold and $X, Y \in C^\infty(TM)$ be vector fields on M . Then

- i) $[X, Y]_p$ is an element of $T_p M$ for all $p \in M$,
- ii) the section $[X, Y] : p \mapsto [X, Y]_p$ is smooth.

Theorem 1.27. Let M be a smooth manifold. Then the vector space $C^\infty(TM)$ of smooth vector fields on M equipped with the Lie bracket is a Lie algebra over the real numbers i.e.

- i) $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z]$,
- ii) $[X, Y] = -[Y, X]$,
- iii) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$,

for all $X, Y, Z \in C^\infty(TM)$ and $\lambda, \mu \in \mathbb{R}$. The equation iii) is called the *Jacobi identity* for $C^\infty(TM)$.

Lemma 1.28. Let M be a smooth manifold. Then

- i) $[X, f \cdot Y] = X(f) \cdot Y + f \cdot [X, Y]$,
- ii) $[f \cdot X, Y] = f \cdot [X, Y] - Y(f) \cdot X$

for all $X, Y \in C^\infty(TM)$ and $f \in C^\infty(M)$.

Definition 1.29. Let M be a smooth manifold. Two vector fields $X, Y \in C^\infty(TM)$ are said to *commute* if $[X, Y] = 0$.

Lemma 1.30. Let $\varphi : M \rightarrow N$ be a smooth bijective map between two manifolds. If $X, Y \in C^\infty(TM)$ are vector fields on M , then

- i) $d\varphi(X) \in C^\infty(TN)$,
- ii) the map $d\varphi : C^\infty(TM) \rightarrow C^\infty(TN)$ is a Lie algebra homomorphism i.e. $[d\varphi(X), d\varphi(Y)] = d\varphi([X, Y])$.
In this case we say that X, Y and $d\varphi(X), d\varphi(Y)$ are φ -related.

PROOF. The first statement follows directly from the definition of the differential. Let $f : N \rightarrow \mathbb{R}$ be smooth, then

$$\begin{aligned}
[d\varphi(X), d\varphi(Y)](f) &= d\varphi(X)(d\varphi(Y)(f)) - d\varphi(Y)(d\varphi(X)(f)) \\
&= X(d\varphi(Y)(f) \circ \varphi) - Y(d\varphi(X)(f) \circ \varphi) \\
&= X(Y(f \circ \varphi)) - Y(X(f \circ \varphi)) \\
&= [X, Y](f \circ \varphi) = d\varphi([X, Y])(f).
\end{aligned}$$

□

5. Riemannian Metrics

In this section we will introduce the notion of a Riemannian manifold. It will be defined as a manifold with a certain symmetric $(2, 0)$ -tensor.

Definition 1.31. Let M be a smooth manifold, $C^\infty(M)$ the commutative ring of smooth functions on M and $C^\infty(TM)$ the module over $C^\infty(M)$ of smooth vector fields on M . Put $C_0^\infty(TM) = C^\infty(M)$ and for $k \in \mathbb{N}^+$ let

$$C_k^\infty(TM) = C^\infty(TM) \otimes \cdots \otimes C^\infty(TM),$$

be the k -fold tensor product of $C^\infty(TM)$. A *tensor field* B on M of type (r, s) is a map $B : C_r^\infty(TM) \rightarrow C_s^\infty(TM)$ satisfying

$$\begin{aligned}
&B(X_1 \otimes \cdots \otimes X_{l-1} \otimes (f \cdot X_l + g \cdot Y) \otimes X_{l+1} \otimes \cdots \otimes X_r) \\
&= f \cdot B(X_1 \otimes \cdots \otimes X_r) \\
&\quad + g \cdot B(X_1 \otimes \cdots \otimes X_{l-1} \otimes Y \otimes X_{l+1} \otimes \cdots \otimes X_r)
\end{aligned}$$

for all $X_1, \dots, X_r, Y \in C^\infty(TM), f, g \in C^\infty(M)$ and $l = 1, \dots, r$.

From now on we use the notation $B(X_1, \dots, X_r)$ for $B(X_1 \otimes \cdots \otimes X_r)$.

Proposition 1.32. Let B be a tensor field of type (r, s) , $p \in M$ a point of the manifold M and $X_1, \dots, X_r, Y_1, \dots, Y_r$ be smooth vector fields on M such that $(X_k)_p = (Y_k)_p$ for each $k = 1, \dots, r$. Then

$$B(X_1, \dots, X_r)(p) = B(Y_1, \dots, Y_r)(p).$$

By B_p we denote the restriction $B_p = B|_{\bigoplus_{l=1}^r T_p M}$ of B to the r -fold tensor product of $T_p M$ given by

$$B_p : ((X_1)_p, \dots, (X_r)_p) \mapsto B(X_1, \dots, X_r)(p).$$

Definition 1.33. The tensor field B is said to be *smooth* if for all $X_1, \dots, X_r \in C^\infty(TM)$ the map $B(X_1, \dots, X_r) : M \rightarrow C_s^\infty(TM)$ with

$$B(X_1, \dots, X_r) : p \mapsto B_p((X_1)_p, \dots, (X_r)_p)$$

is smooth.

Definition 1.34. Let (M, \mathcal{A}) be a smooth manifold. A smooth tensor field $g : C_2^\infty(TM) \rightarrow C_0^\infty(M)$ is called a *Riemannian metric* on M if for each $p \in M$ the restriction $g_p : T_p M \oplus T_p M \rightarrow \mathbb{R}$ with

$$g_p : (X_p, Y_p) \mapsto g(X, Y)(p)$$

is an inner product on $T_p M$. The pair (M, g) is called a *Riemannian manifold*.

Theorem 1.35. Let (M^m, \mathcal{A}) be a smooth manifold. Then there exists a Riemannian metric g on M .

Definition 1.36. Let (M, g) and (N, h) be Riemannian manifolds. A map $\varphi : M \rightarrow N$ is said to be *conformal* if there exists a positive function $\lambda : M \rightarrow \mathbb{R}^+$ such that

$$\lambda(p)g_p(X_p, Y_p) = h_{\varphi(p)}(d\varphi_p(X_p), d\varphi_p(Y_p)),$$

for all $X, Y \in C^\infty(TM)$ and $p \in M$. The function λ is called the *conformal factor* of φ . A conformal map with $\lambda \equiv 1$ is said to be *isometric*. An isometric diffeomorphism is called an *isometry*.

6. The Levi-Civita Connection

Later on we will discuss geometric properties of Riemannian manifolds. To be able to do this we must first explain how we can differentiate a vector field in the direction of another vector field. This will be done in this section, where we define the Levi-Civita connection ∇ on M , which is the unique metric and torsion-free connection on the manifold M .

Definition 1.37. Let (E, M, π) be a smooth vector bundle over M . A *connection* $\overset{*}{\nabla}$ on (E, M, π) is a map $\overset{*}{\nabla} : C^\infty(TM) \times C^\infty(E) \rightarrow C^\infty(E)$ satisfying

- i) $\overset{*}{\nabla}_X (\lambda \cdot v + \mu \cdot w) = \lambda \cdot \overset{*}{\nabla}_X v + \mu \cdot \overset{*}{\nabla}_X w$
- ii) $\overset{*}{\nabla}_X (f \cdot v) = X(f) \cdot v + f \cdot \overset{*}{\nabla}_X v$
- iii) $\overset{*}{\nabla}_{f \cdot X + g \cdot Y} v = f \cdot \overset{*}{\nabla}_X v + g \cdot \overset{*}{\nabla}_Y v$

for all $\lambda, \mu \in \mathbb{R}$, $X, Y \in C^\infty(TM)$, $v, w \in C^\infty(E)$ and $f, g \in C^\infty(M)$, such that the map $(\overset{*}{\nabla}_X v) : M \rightarrow E$ with $(\overset{*}{\nabla}_X v) : p \mapsto (\overset{*}{\nabla}_X v)_p$ is smooth. A section $v \in C^\infty(E)$ is said to be *parallel* with respect to $\overset{*}{\nabla}$ if $\overset{*}{\nabla}_X v = 0$ for all $X \in C^\infty(TM)$.

Definition 1.38. Let M be a smooth manifold and $\overset{*}{\nabla}$ be a connection on the tangent bundle (TM, M, π) . Then the *torsion* $T : C_2^\infty(TM) \rightarrow C^\infty(TM)$ of $\overset{*}{\nabla}$ is defined by

$$T(X, Y) = \overset{*}{\nabla}_X Y - \overset{*}{\nabla}_Y X - [X, Y].$$

The connection $\overset{*}{\nabla}$ is said to be *torsion-free* if the corresponding torsion T vanishes i.e.

$$T(X, Y) = 0$$

for all $X, Y \in C^\infty(TM)$.

Definition 1.39. Let (M, g) be a Riemannian manifold. Then $\overset{*}{\nabla}$ is said to be *compatible* with g (or *metric*) if

$$X(g(Y, Z)) = g(\overset{*}{\nabla}_X Y, Z) + g(Y, \overset{*}{\nabla}_X Z)$$

for all $X, Y, Z \in C^\infty(TM)$.

Theorem 1.40. Let (M, g) be a Riemannian manifold and let the map $\nabla : C^\infty(TM) \times C^\infty(TM) \rightarrow C^\infty(TM)$ be given by the so called Koszul formula

$$\begin{aligned} g(\nabla_X Y, Z) = & \frac{1}{2} (X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ & + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z])). \end{aligned}$$

Then ∇ is a connection on the tangent bundle (TM, M, π) .

PROOF. This theorem follows directly by the fact that g is a tensor field and by Definition 1.6, Theorem 2.11 and Lemma 1.28. \square

Definition 1.41. The connection ∇ on (M, g) defined in the Theorem 1.40 is called the *Levi-Civita connection* of g .

The next result is called The Fundamental Theorem of Riemannian Geometry.

Theorem 1.42. Let (M, g) be a Riemannian manifold. Then the Levi-Civita connection ∇ is the unique metric and torsion-free connection on (TM, M, π) .

Definition 1.43. Let (M, g, ∇) be a Riemannian manifold with its Levi-Civita connection. Further let (U, x) be a local coordinate on M and define $X_i = \frac{\partial}{\partial x_i} \in C^\infty(TU)$. Then $\{X_1, \dots, X_m\}$ is a local frame of TM on U . For (U, x) the *Christoffel symbols* $\Gamma_{ij}^k : U \rightarrow \mathbb{R}$ of ∇ with respect to (U, x) are given by

$$\sum_{k=1}^m \Gamma_{ij}^k X_k = \nabla_{X_i} X_j.$$

7. The Riemann Curvature Tensor

In this section we define the curvature tensor R , which is an important tool for understanding the geometry of the manifold (M, g) .

Definition 1.44. Let (M, g, ∇) be a Riemannian manifold with its Levi-Civita connection. For $i \in \{0, 1\}$ and a tensor field $A : C_r^\infty(TM) \rightarrow C_i^\infty(TM)$ we define its *covariant derivative* $\nabla A : C_{r+1}^\infty(TM) \rightarrow C_i^\infty(TM)$, by

$$\nabla A : (X, X_1, \dots, X_r) \mapsto (\nabla_X A)(X_1, \dots, X_r) = X(A(X_1, \dots, X_r)) - \sum_{i=1}^r A(X_1, \dots, X_{i-1}, \nabla_X X_i, X_{i+1}, \dots, X_r)$$

A tensor field A of type $(r, 0)$ or $(r, 1)$ is said to be *parallel* if $\nabla A \equiv 0$.

A vector field $Z \in C^\infty(TM)$ defines a smooth tensor field $\hat{Z} : C^\infty(TM) \rightarrow C^\infty(TM)$ given by $\hat{Z} : X \mapsto \nabla_X Z$. For two vector fields $X, Y \in C^\infty(TM)$ we define the *second covariant derivative* $\nabla_{X,Y}^2 : C^\infty(TM) \rightarrow C^\infty(TM)$ by

$$\nabla_{X,Y}^2 : Z \mapsto (\nabla_X \hat{Z})(Y).$$

Hence

$$\nabla_{X,Y}^2 Z = \nabla_X(\hat{Z}(Y)) - \hat{Z}(\nabla_X Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z.$$

Definition 1.45. Let (M, g, ∇) be a Riemannian manifold with the Levi-Civita connection. Let $R : C_3^\infty(TM) \rightarrow C_1^\infty(TM)$ be twice the skew-symmetric part of the second covariant derivative ∇^2 i.e.

$$R(X, Y)Z = \nabla_{X,Y}^2 Z - \nabla_{Y,X}^2 Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Then R is called the *Riemann curvature tensor* of (M, g) .

The next result shows that the curvature tensor has many nice symmetric properties.

Proposition 1.46. Let (M, g) a Riemannian manifold. Then following identities hold

- i) $R(X, Y)Z = -R(Y, X)Z$
- ii) $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$
- iii) $g(R(X, Y)Z, W) + g(R(Z, X)Y, W) + g(R(Y, Z)X, W) = 0$
- iv) $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$

The equation given in iii) is called the *1st Bianchi identity*.

For all $p \in M$ let $G_2(T_p M)$ denote the set of all 2-dimensional subspaces of $T_p M$.

Lemma 1.47. Let $X, Y, Z, W \in T_p M$ be vectors in the tangent space at the point p such that $\text{span}_{\mathbb{R}}\{X, Y\} = \text{span}_{\mathbb{R}}\{Z, W\}$. Then

$$\frac{g(X, Y)Y, X}{|X|^2|Y|^2 - g(X, Y)^2} = \frac{g(Z, W)W, Z}{|Z|^2|W|^2 - g(Z, W)^2}$$

Definition 1.48. For a point $p \in M$ the function $K_p : G_2(T_p M) \rightarrow \mathbb{R}$ given by

$$K_p : \text{span}_{\mathbb{R}}\{X, Y\} \mapsto \frac{g(R(X, Y)Y, X)}{|X|^2|Y|^2 - g(X, Y)^2}$$

is called the *sectional curvature* of the 2-plane spanned by X and Y at p . The Riemannian manifold (M, g) is said to be of *constant sectional curvature* if the function K_p is constant for all $p \in M$ and all $X, Y \in T_p M$. The Riemannian manifold (M, g) is said to be *flat* if its constant sectional curvature is zero.

In the case of constant sectional curvature the Riemann curvature tensor has a rather simple form.

Lemma 1.49. Let (M, g) be a Riemannian manifold of constant sectional curvature κ . Then the Riemann curvature tensor R of (M, g) is given by

$$R(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y).$$

Definition 1.50. Let (M, g) be a Riemannian manifold, $p \in M$ and $\{e_1, \dots, e_m\}$ be an orthonormal basis of $T_p M$. Then

i) the *Ricci tensor* at $p \in M$ $\text{Ric}_p(X)$ is defined by

$$\text{Ric}_p(X) = \sum_{i=1}^m R(X, e_i)e_i,$$

ii) the *Ricci curvature* at $p \in M$ $\text{Ric}_p(X, Y)$ by

$$\text{Ric}_p(X, Y) = \sum_{i=1}^m g(R(X, e_i)e_i, Y), \quad \text{and}$$

iii) the *scalar curvature* $\sigma(p)$ by

$$\sigma(p) = \sum_{j=1}^m \text{Ric}_p(e_j, e_j) = \sum_{j=1}^m \sum_{i=1}^m g(R(e_i, e_j)e_j, e_i).$$

Corollary 1.51. Let (M, g) be a Riemannian manifold of constant sectional curvature κ . Then the following holds

$$\sigma(p) = m \cdot (m - 1) \cdot \kappa.$$

PROOF. Let $\{e_1, \dots, e_m\}$ be an orthonormal basis, then Lemma 1.49 implies that

$$\begin{aligned}
Ric_p(e_j, e_j) &= \sum_{i=1}^m g(R(e_j, e_i)e_i, e_j) \\
&= \sum_{i=1}^m g(\kappa(g(e_i, e_i)e_j - g(e_j, e_i)e_i), e_j) \\
&= \kappa\left(\sum_{i=1}^m g(e_i, e_i)g(e_j, e_j) - \sum_{i=1}^m g(e_i, e_j)g(e_i, e_j)\right) \\
&= \kappa\left(\sum_{i=1}^m 1 - \sum_{i=1}^m \delta_{ij}\right) = (m-1) \cdot \kappa.
\end{aligned}$$

To obtain the formula for the scalar curvature σ we only have to multiply the constant Ricci curvature $Ric_p(e_j, e_j)$ by m . \square

The following result is called the 2nd Bianchi identity.

Lemma 1.52. *Let (M, g) be a Riemannian manifold and $R(X, Y)$ its Riemann curvature tensor, then*

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0,$$

for all $X, Y, Z \in C^\infty(TM)$.

Structures on Tangent Bundles

In this chapter we introduce necessary structures on the tangent bundle TM , some of which were introduced in Chapter 1 for a differentiable manifold M . We start with the vertical and the horizontal lifts of vector fields. These are then used to calculate the Lie bracket on TM . We then introduce the important class of natural metrics on the tangent bundle and calculate their Levi-Civita connections. For further interest we recommend [21].

1. The Vertical and Horizontal Lifts

Definition 2.1. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ , let TM denote the tangent bundle of M , and π be the natural projection of TM onto M . Then the differential $d\pi$ of π is a C^∞ -map from TTM onto TM . If $(p, u) \in TM$, then we denote the kernel of $d\pi$ at (p, u) by

$$\mathcal{V}_{(p,u)} = \text{Ker}(d\pi|_{(p,u)})$$

and call it the *vertical subspace* of $T_{(p,u)}TM$ at the point (p, u) .

Remark 2.2. If $p \in M$ and $i : T_pM \rightarrow TM$ is the inclusion map for the submanifold $T_pM = \pi^{-1}(\{p\})$ in TM with $i(u) = (p, u)$, then for every $u \in T_pM$ we have $\mathcal{V}_{(p,u)} = di((T_pM)_u)$.

In order to define the very important horizontal subspace we first need to introduce the connection map $K_{(p,u)}$ from $T_{(p,u)}TM$ to T_pM induced by the Levi-Civita connection on (M, g) .

Definition 2.3. Let V be a neighborhood of p in M such that the exponential map $\exp_p : T_pM \rightarrow M$ maps a neighborhood V' of 0 in T_pM diffeomorphically onto V . Furthermore let $\tau : \pi^{-1}(V) \rightarrow T_pM$ be the C^∞ -map into T_pM , which translates every $Y \in \pi^{-1}(V)$ in a parallel manner from $q = \pi(Y)$ to p along the unique geodesic arc in V between q and p . For $u \in T_pM$ let $R_{-u} : T_pM \rightarrow T_pM$ be the map given by $R_{-u}(X) := X - u$ for $X \in T_pM$. Then the *connection map*

$$K_{(p,u)} : T_{(p,u)}TM \rightarrow T_pM$$

of the Levi-Civita connection ∇ is defined as:

$$K(A) := d(\exp_p \circ R_{-u} \circ \tau)(A)$$

for all $A \in T_{(p,u)}TM$.

We now establish an important result for the connection map K .

Lemma 2.4. [6] *The connection map $K_{(p,u)} : T_{(p,u)}TM \rightarrow T_pM$ of ∇ satisfies*

$$K(dZ_p(X_p)) = (\nabla_X Z)_p,$$

where $Z \in C^\infty(TM)$ is viewed as a map $Z : M \rightarrow TM$ and $X_p \in T_pM$.

PROOF. Let Z be a vector field which takes the value (p, u) at p . Furthermore let γ be a geodesic curve such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$. Then the definition of the connection map implies

$$\begin{aligned} K_{(p,u)}(dZ_p(X_p)) &= d(\exp_p \circ R_{-u} \circ \tau)(dZ_p(X_p)) \\ &= \left. \frac{d}{dt} \right|_{t=0} \exp_p \circ R_{-u} \circ \tau(Z_{\gamma(t)}) \\ &= \left. \frac{d}{dt} \right|_{t=0} \exp_p(\tau(Z_{\gamma(t)}) - u) \\ &= d(\exp_p)_0 \left(\left. \frac{d}{dt} \right|_{t=0} \tau(Z_{\gamma(t)}) \right) \\ &= \left. \frac{d}{dt} \right|_{t=0} \tau(Z_{\gamma(t)}). \end{aligned}$$

In the second last step we use the fact that $\tau((p, u)) = u$. If $v \in T_pM$, then

$$g_p \left(\left. \frac{d}{dt} \right|_{t=0} \tau(Z_{\gamma(t)}), v \right) = \left. \frac{d}{dt} \right|_{t=0} g_p(\tau(Z_{\gamma(t)}), v).$$

Since τ is an isometry we have

$$\left. \frac{d}{dt} \right|_{t=0} g_{\gamma(t)}(Z_{\gamma(t)}, \tau_t^{-1} v) = \left. \frac{d}{dt} \right|_{t=0} g_{\gamma(t)}(Z_{\gamma(t)}, v_t).$$

The last equation follows from the fact that τ is a parallel transport of v along γ . The last term is equal to $g_p(\nabla_X Z, v)$, so

$$\left. \frac{d}{dt} \right|_{t=0} \tau(Z_{\gamma(t)}) = \nabla_X Z.$$

This completes the proof. \square

Let $\gamma : I \rightarrow M$ be a curve in M with $\gamma(0) = p$ and $\gamma'(0) = u$. Let $X : I \rightarrow TM$ be a vector field along γ , i.e. a curve in the tangent bundle through (p, u) with $\pi \circ X = \gamma$. In other words X maps every t to $(\gamma(t), U(t))$ with $U(t) \in T_{\gamma(t)}M$ and $U(0) = u$. Then the connection map K of ∇ satisfies

$$K_{(p,u)} : X'(t) \mapsto (\nabla_{\gamma'} U)(0).$$

Equipped with the connection map K of ∇ we can now define the horizontal subspace.

Definition 2.5. The *horizontal subspace* $\mathcal{H}_{(p,u)}$ of the tangent space $T_{(p,u)}TM$ at (p, u) of the tangent bundle TM is defined by

$$\mathcal{H}_{(p,u)} = \text{Ker}(K_{(p,u)}).$$

A curve $X : I \rightarrow TM$ is said to be *horizontal* if $X'(t) \in \mathcal{H}_{(\gamma(t), U(t))}$ for all $t \in I$ and *vertical* if $X'(t) \in \mathcal{V}_{(\gamma(t), U(t))}$ for all $t \in I$.

This means that horizontal curves in the tangent bundle TM correspond to parallel vector fields on the manifold (M, g, ∇) . This is the main motivation for defining the horizontal space as above.

Proposition 2.6. [16], [6] *The tangent space $T_{(p,u)}TM$ of the tangent bundle TM at the point (p, u) is the direct sum of its vertical and horizontal subspaces, i.e.*

$$T_{(p,u)}TM = \mathcal{H}_{(p,u)} \oplus \mathcal{V}_{(p,u)}.$$

PROOF. [6] The map $d(\exp_p \circ R_{-u})$ an isomorphism of $T_u T_p M$, i.e. the tangent space of $T_p M$ at $u \in T_p M$, and τ is defined as the parallel transport of π . Hence the image of K , which is the horizontal space $\mathcal{H}_{(p,u)}$ depends only on the image of π , but the vertical subspace $\mathcal{V}_{(p,u)}$ is just the kernel of π . Therefore, by the dimension theorem, the proposition holds. \square

With the horizontal and vertical subspaces we can now define the horizontal and the vertical lifts of tangent vectors on M .

Definition 2.7. Let $X \in T_p M$ be a tangent vector, then the *horizontal lift* of X at a point $(p, u) \in TM$ is the unique vector $X^h \in \mathcal{H}_{(p,u)}$ such that $d\pi(X^h) = X$. The *vertical lift* of X at (p, u) is the unique vector $X^v \in \mathcal{V}_{(p,u)}$ such that $X^v(df) = X(f)$ for all functions f on M . Here df is the function defined by $(df)(p, u) = u(f)$.

This can now be extended from tangent vectors to vector fields.

Definition 2.8. The *horizontal lift* of a vector field $X \in C^\infty(TM)$ on TM is the vector field $X^h \in C^\infty(TTM)$ whose value at a point (p, u) is the horizontal lift of X_p at (p, u) . The *vertical lift* of a vector field is defined in the same way. More precisely: If $X \in C^\infty(TM)$, then there is exactly one vector field $X^h \in C^\infty(TTM)$ on TM called the *horizontal lift* of X such that for all $Z \in TM$:

$$d\pi(X^h)_Z = X_{\pi(Z)} \quad \text{and} \quad KX_Z^h = 0_{\pi(Z)}.$$

The *vertical lift* X^v is the unique vector field satisfying

$$d\pi(X^v)_Z = 0_{\pi(Z)} \quad \text{and} \quad KX_Z^v = X_{\pi(Z)}.$$

Note that the maps $X \mapsto X^h$ and $X \mapsto X^v$ are isomorphisms between the vector space $T_p M$ and the subspaces $\mathcal{H}_{(p,u)}$ and $\mathcal{V}_{(p,u)}$, respectively. Each tangent vector $\hat{Z} \in T_{(p,u)}TM$ can then be written as

$$\hat{Z} = X^h + Y^v$$

where $X, Y \in T_p M$ are uniquely determined by $X = d\pi(\hat{Z})$ and $Y = K(\hat{Z})$.

It follows that if $\varphi : M \rightarrow \mathbb{R}$ is a smooth real valued function on M , then

$$X^h(\varphi \circ \pi) = (X\varphi) \circ \pi \text{ and } X^v(\varphi \circ \pi) = 0$$

for all $X \in C^\infty(TM)$.

2. The Lifts in Local Coordinates

In this section we shall express the vertical and the horizontal lifts of vector fields in terms of local coordinates on M .

Let M be a manifold of dimension m , $(x_1, \dots, x_m) : U \subset M \rightarrow \mathbb{R}^m$ be local coordinates on M and define the smooth functions $v_1, \dots, v_{2m} : TM \rightarrow \mathbb{R}$ in the following way:

$$\begin{aligned} v_i &= x_i \circ \pi, \\ v_{m+i}(Y) &= Y(x_i) = dx_i(Y) \end{aligned}$$

for $i = 1, \dots, m$ and $Y \in TM$. Then $(v_1, \dots, v_{2m}) : \pi^{-1}(U) \subset TM \rightarrow \mathbb{R}^{2m}$ are local coordinates on TM . Using Definition 2.8 we now see that

$$d\pi\left(\frac{\partial}{\partial v_i}\right)_Z = \left(\frac{\partial}{\partial x_i}\right)_{\pi(Z)} \quad \text{and} \quad d\pi\left(\frac{\partial}{\partial v_{m+i}}\right)_Z = 0$$

for all $Z \in TM$ and $i = 1, \dots, m$, since

$$d\pi\left(\frac{\partial}{\partial v_i}\right)(f) = \frac{\partial}{\partial v_i}(f \circ \pi) = \frac{\partial}{\partial x_i}(f)$$

and

$$d\pi\left(\frac{\partial}{\partial v_{m+i}}\right)(f) = \frac{\partial}{\partial v_{m+i}}(f \circ \pi) = \frac{\partial}{\partial v_{m+i}}(f(x)) = 0.$$

This leads to the following result for the horizontal and vertical lifts X^h, X^v of the vector field $X \in C^\infty(TM)$:

Lemma 2.9. [6] *Let (M, g) be a Riemannian manifold and $X \in C^\infty(TM)$ be a vector field on M which locally is represented by*

$$X = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}.$$

Then its vertical and horizontal lifts X^v and X^h are given by

$$X^v = \sum_{i=1}^m \xi_i \frac{\partial}{\partial v_{m+i}}$$

and

$$X^h = \sum_{i=1}^m \xi_i \frac{\partial}{\partial v_i} - \left(\sum_{i,j,k=1}^m \xi_j \eta_k \Gamma_{jk}^i \right) \frac{\partial}{\partial v_{m+i}},$$

where the coefficients Γ_{ab}^i are the Christoffel symbols of the Levi-Civita connection ∇ on (M, g) .

PROOF. The vertical part follows directly by Definition 2.8.

The fact that X^b and X are π -related, i.e. $d\pi(X_Z^b) = X_{\pi(Z)}$, follows directly from

$$d\pi\left(\left(\frac{\partial}{\partial v_{m+i}}\right)_Z\right) = 0 \quad \text{and} \quad d\pi\left(\left(\frac{\partial}{\partial v_i}\right)_Z\right) = \left(\frac{\partial}{\partial x_i}\right)_{\pi(Z)}.$$

Now let the vector field $Z \in C^\infty(TM)$ be represented by

$$Z = \sum_{i=1}^m \eta_i \frac{\partial}{\partial x_i} \in C^\infty(TM).$$

Then the map $Z : M \rightarrow TM$ is locally given by

$$Z : M \rightarrow TM, \quad Z : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, \eta_1, \dots, \eta_m).$$

Hence

$$\begin{aligned} dZ(X) &= dZ\left(\sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}\right) \\ &= \sum_{i=1}^m \xi_i \frac{\partial}{\partial v_i} + \sum_{i,k=1}^m \xi_i \frac{\partial \eta_k}{\partial x_i} \frac{\partial}{\partial v_{m+k}} \\ &= \sum_{i=1}^m \xi_i \frac{\partial}{\partial v_i} + \sum_{k=1}^m X(\eta_k) \frac{\partial}{\partial v_{m+k}} \\ &= \sum_{i=1}^m \xi_i \frac{\partial}{\partial v_i} + \sum_{i=1}^m X(\eta_i) \frac{\partial}{\partial v_{m+i}}. \end{aligned}$$

Put $X_i = \frac{\partial}{\partial x_i}$ and let Γ_{jm}^i be the Christoffel symbols of ∇ given by

$$(\nabla_{X_j} X_m) = \sum_{i=1}^m \Gamma_{jm}^i X_i.$$

Then we get the following expression for the Levi-Civita connection ∇ .

$$\begin{aligned} \nabla_X Z &= \sum_{i=1}^m \nabla_X (\eta_i X_i) \\ &= \sum_{i=1}^m (X(\eta_i) X_i + \eta_i \nabla_X X_i) \\ &= \sum_{i=1}^m X(\eta_i) X_i + \sum_{i,j=1}^m \eta_i \xi_j \nabla_{X_j} X_i \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j=1}^m X(\eta_i)X_i + \sum_{i,j,k=1}^m \xi_j \eta_i \Gamma_{ji}^k X_k \\
&= \sum_{i,j=1}^m X(\eta_i)X_i + \sum_{i,j,k=1}^m \xi_j \eta_k \Gamma_{jk}^i X_i \\
&= \sum_{i=1}^m (X(\eta_i) + \sum_{j,k=1}^m \xi_j \eta_k \Gamma_{jk}^i) X_i.
\end{aligned}$$

It is now a direct consequence of Definition 2.8 and Lemma 2.4 that

$$K(dZ(X)) = \nabla_X Z = \sum_{i=1}^m \left(\sum_{j=1}^m (\xi_j (\frac{\partial \eta_i}{\partial x_j} + \sum_{k=1}^m \Gamma_{jk}^i \eta_k)) \right) \frac{\partial}{\partial x_i}.$$

This implies that $K(dZ(X)) = 0$, if and only if

$$X(\eta_i) = - \sum_{j=1}^m \xi_j \sum_{k=1}^m \eta_k \Gamma_{jk}^i.$$

This means that $\nabla_X Z = 0$ if and only if $dZ(X)$ is in the kernel of K . This is equivalent to the fact that $dZ(X)$ takes of the form

$$dZ(X) = \sum_{i=1}^m \xi_i \frac{\partial}{\partial v_i} - \sum_{i=1}^m \left(\sum_{j,k=1}^m \xi_j \eta_k \Gamma_{jk}^i \right) \frac{\partial}{\partial v_{m+i}}.$$

Hence

$$X^h = \sum_{i=1}^m \xi_i \frac{\partial}{\partial v_i} - \sum_{i=1}^m \left(\sum_{j,k=1}^m \xi_j \eta_k \Gamma_{jk}^i \right) \frac{\partial}{\partial v_{m+i}}.$$

This proves the statement for the horizontal lift of X . \square

Corollary 2.10. [6] For $X_i = \frac{\partial}{\partial x_i}$ with (x_1, \dots, x_m) local coordinates of M and $U_i = \frac{\partial}{\partial v_i}$ with (v_1, \dots, v_{2m}) local coordinates of TM we obtain

$$(X_i)^v = U_{m+i} \quad \text{and} \quad (X_i)^h = U_i - \sum_{j,k=1}^m (\Gamma_{ik}^j \circ \pi) v_{m+k} U_{m+j}.$$

3. The Lie Bracket

In this section we use the vertical and horizontal lifts to calculate the Lie bracket on the tangent bundle.

Theorem 2.11. [6] Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemann curvature tensor of ∇ . Then the Lie bracket on the tangent bundle TM of M satisfies the following:

$$i) [X^v, Y^v] = 0,$$

- ii) $[X^b, Y^v] = (\nabla_X Y)^v$,
- iii) $[X^b, Y^b] = [X, Y]^b - (R(X, Y)u)^v$

for all $X, Y \in C^\infty(TM)$ and any point (p, u) in TM .

PROOF. [6] Using the inclusion map in Remark 2.2, we see that there exist vector fields $X', Y' \in C^\infty(T_u T_p M)$ which are i -related to X^v and Y^v , respectively i.e.

$$X_{(p,u)}^v = di(X'_u) \text{ and } Y_{(p,u)}^v = di(Y'_u)$$

for all $u \in T_p M$. Hence we get

$$[X^v, Y^v]_Z = di([X', Y']_u).$$

By Definition 2.8 we know that $K(X_{(p,u)}^v) = X_p$ for all $u \in T_p M$. Therefore X' and Y' are right-invariant vector fields on $T_p M$ in its capacity as a Lie group since $X_{(p,u)}^v, Y_{(p,u)}^v$ are in $\mathcal{V}_{(p,u)}$ and

$$K_u(di(X'_{u+w})) = K_{u+w}(di(CX'_{u+w}))$$

by the fact that $di \circ d\tau(A) = d\tau \circ di(A) = A$ for all $A \in \mathcal{V}_{(p,u+w)}$.

Remark 2.12. That X' is a *right-invariant vector field* means that $X'_{v+w} = (dR_w)_v(X'_v)$ for all $v, w \in T_p M$ with $R_w : T_p M \rightarrow T_p M$. The *right translation* by w defined by $R_w : v \mapsto w + v$.

Hence the right-hand side of the formula vanishes, since $T_p M$ is an abelian Lie group. This proves *i*).

According to Definition 2.8 we know that $d\pi(X_Z^b) = X_{\pi(Z)}$ and $d\pi(Y_Z^v) = 0$. Hence $d\pi([X^b, Y^v]) = [d\pi(X^b), d\pi(Y^v)] = 0$ and therefore by Definition 2.8, which tells us that $d\pi((\nabla_X Y)^v) = 0$, we get

$$d\pi([X^b, Y^v]) = d\pi((\nabla_X Y)^v),$$

and on the other hand

$$d\pi([X^b, Y^b]) = [X, Y].$$

So we only have to calculate the function K of the right-hand sides in the last two parts of the theorem. To calculate them we will again use our previous abbreviation $X_i = \frac{\partial}{\partial x_i}$, where (x_1, \dots, x_m) are local coordinates for M . It is sufficient to calculate both terms just for $X, Y \in \{X_1, \dots, X_m\}$, because all our functions are linear in every argument. By Corollary 2.10, including the abbreviations in this corollary, and using $[U_i, U_j] = 0$ and $U_{m+i}(v_{m+j}) = \delta_{ij}$ for all $i, j \in \{1, \dots, 2m\}$, with δ_{ij} the Kronecker symbol, we obtain

$$[(X_i)^b, (X_j)^v] = \sum_{k=1}^m (\Gamma_{ij}^k \circ \pi) U_{m+k}.$$

Using the formula for $\nabla_X Y$ in the proof of Lemma 2.9 and the Definition 2.8 we obtain

$$K([(X_i)^b, (X_j)^v]_{(p,u)}) = (\nabla_{X_i} X_j)_p.$$

This provides us with *ii*).

In the same way as above we will now, observing $U_i(v_{m+j}) = 0$ for all $i, j \in \{1, \dots, m\}$ and $U_i(\Gamma_{kl}^j \circ \pi) = X_i(\Gamma_{kl}^j) \circ \pi$, calculate

$$\begin{aligned} [(X_i)^b, (X_j)^b] &= \sum_{k,l,n=1}^m \{ U_j(\Gamma_{il}^k \circ \pi) - U_i(\Gamma_{jl}^k \circ \pi) + (\Gamma_{il}^n \circ \pi)(\Gamma_{jn}^k \circ \pi) \\ &\quad - (\Gamma_{jl}^n \circ \pi)(\Gamma_{in}^k \circ \pi) \} v_{m+l} U_{m+k} \\ &= - \sum_{k,l=1}^m (R_{lij}^k \circ \pi) v_{m+l} U_{m+k}. \end{aligned}$$

Therefore, by again using $K(U_{m+i}) = X_i$ we obtain for $Z = (p, u)$ as in Lemma 2.9

$$K([(X_i)^b, (X_j)^b]_Z) = - \sum_{k=1}^m \eta_k R(X_i, X_j) X_k = -R(X_i, X_j) Z.$$

This proves *iii*) and completes the proof. \square

4. The Natural Almost Complex Structure

We shall now see that the vertical and horizontal lifts of vector fields on the manifold (M, g) induce a natural almost complex structure J on the tangent bundle TM . It turns out that J is integrable if and only if (M, g) is flat. The natural almost complex structure on the tangent bundle TM was first studied by Nagano in [14].

Definition 2.13. Let $J : TTM \rightarrow TTM$ be the linear endomorphism of the tangent bundle of TM characterized by

$$d\pi(JA) = -K(A) \quad \text{and} \quad K(JA) = d\pi(A),$$

for all $A \in C^\infty(TTM)$. It follows from Definition 2.8 that

$$J(X^b) = X^v \quad \text{and} \quad J(X^v) = -X^b,$$

so the endomorphism $J : TTM \rightarrow TTM$ satisfies $J^2 = -Id_{TTM}$, hence it is an almost complex structure on TM . We call it the *natural almost complex structure* on TM with respect to g on M .

Definition 2.14. The *Nijenhuis tensor* N of an almost complex structure J is defined by

$$N(A, B) = [A, B] + J[JA, B] + J[A, JB] - [JA, JB].$$

Proposition 2.15. [19], [6] *Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita connection on M and R be the Riemann curvature tensor of ∇ . Furthermore let N be the Nijenhuis tensor of the almost complex structure J on TM . Then the following hold*

$$K(N(X^v, Y^v)_Z) = R(X, Y)Z, \quad \text{and} \quad d\pi(N(X^v, Y^v)_Z) = 0$$

for all $X, Y, Z \in C^\infty(TM)$.

PROOF. [6] From the Definitions 2.13 and 2.14 we obtain

$$N(X^v, Y^v) = [X^v, Y^v] - J([X^h, Y^v] + [X^v, Y^h]) - [X^h, Y^h].$$

The first term on the right hand side is zero, hence

$$\begin{aligned} K(N(X^v, Y^v)_Z) &= -K(J([X^h, Y^v] + [X^v, Y^h])_Z) - K([X^h, Y^h]_Z) \\ &= -d\pi([X^h, Y^v]_Z + [X^v, Y^h]_Z) + R(X, Y)Z \\ &= R(X, Y)Z. \end{aligned}$$

Furthermore

$$\begin{aligned} d\pi(N(X^v, Y^v)_Z) &= -d\pi(J([X^h, Y^v] + [X^v, Y^h])_Z) - d\pi([X^h, Y^h]_Z) \\ &= K([X^h, Y^v]_Z + [X^v, Y^h]_Z) - [X, Y] \\ &= \nabla_X Y - \nabla_Y X - [X, Y] \\ &= 0. \end{aligned}$$

□

Corollary 2.16. [19], [6] *The almost complex structure J is integrable if and only if M is flat, i.e. $R \equiv 0$.*

PROOF. [6] If J is integrable, then $N \equiv 0$ so it is clear that $R \equiv 0$. If now $R \equiv 0$ holds, then it follows that $N(X^v, Y^v) = 0$ for all $X, Y \in C^\infty(TM)$. The definition of N gives

$$N(A, B) = J(N(JA, B)) = J(N(A, JB)) = -N(JA, JB)$$

for all $A, B \in C^\infty(TTM)$. Hence

$$N(X^v, Y^v) = N(X^v, Y^h) = N(X^h, Y^v) = N(X^h, Y^h) = 0.$$

This implies $N \equiv 0$.

□

5. Natural Metrics

In this section we introduce the notion of a Riemannian submersion, defined by O'Neill in [15]. This leads to the class of natural metrics on the tangent bundle of a given Riemannian manifold (M, g) .

Definition 2.17. Let $\varphi : (N, h) \rightarrow (M, g)$ be a differentiable map between two Riemannian manifolds. Then φ is called a *submersion* if the differential $d\varphi_p : T_p N \rightarrow T_{\varphi(p)} M$ of φ is surjective at each point $p \in N$. Hence for each $q \in M$, $\varphi^{-1}(\{q\})$ is a submanifold of N of dimension $\dim N - \dim M$. We define \mathcal{V}_p as $\ker(d\varphi_p)$ i.e. the kernel of the differential of φ in the tangent space $T_p N$, and furthermore \mathcal{H}_p as \mathcal{V}_p^\perp i.e. the orthogonal complement of \mathcal{V}_p in $T_p N$ with respect to h .

This leads to the following:

Definition 2.18. A submersion is said to be *horizontally conformal* if there exists a positive function $\lambda : M \rightarrow \mathbb{R}^+$ such that

$$g(d\varphi(X), d\varphi(Y)) = \lambda^2 h(X, Y)$$

for all $X, Y \in \mathcal{H}$. If $\lambda \equiv 1$ then φ is said to be a *Riemannian submersion*.

The well known notions of a Riemannian submersion and vertical and horizontal spaces make the following definition very natural. It implies that the natural projection $\pi : (TM, \bar{g}) \rightarrow (M, g)$ is a Riemannian submersion and that horizontal and vertical vector fields on TM , introduced earlier in this chapter, are orthogonal.

Definition 2.19. Let (M, g) be a Riemannian manifold. A Riemannian metric \bar{g} on the tangent bundle TM of M is said to be *natural* with respect to g if:

- i) $\bar{g}_{(p,u)}(X^h, Y^h) = g_p(X, Y)$, and
- ii) $\bar{g}_{(p,u)}(X^h, Y^v) = 0$

for all vector fields $X, Y \in C^\infty(TM)$.

We can now use the Koszul formula in Theorem 1.40 to calculate the Levi-Civita connection $\bar{\nabla}$ for the tangent bundle (TM, \bar{g}) equipped with a natural metric \bar{g} with respect to g on M .

Lemma 2.20. Let (M, g) be a Riemannian manifold and TM be the tangent bundle of M . Then for each $(p, u) \in TM$ and every natural metric \bar{g} on TM the corresponding Levi-Civita connection $\bar{\nabla}$ satisfies

- i) $\bar{g}(\bar{\nabla}_{X^h} Y^h, Z^h) = g(\nabla_X Y, Z)$,
- ii) $\bar{g}(\bar{\nabla}_{X^h} Y^h, Z^v) = -\frac{1}{2} \bar{g}((R(X, Y)u)^v, Z^v)$,
- iii) $\bar{g}(\bar{\nabla}_{X^h} Y^v, Z^h) = -\frac{1}{2} \bar{g}(Y^v, (R(Z, X)u)^v)$,
- iv) $\bar{g}(\bar{\nabla}_{X^h} Y^v, Z^v) = \frac{1}{2} (X^h(\bar{g}(Y^v, Z^v)) + \bar{g}(Z^v, (\nabla_X Y)^v) - \bar{g}(Y^v, (\nabla_X Z)^v))$,

$$\begin{aligned}
v) \quad \bar{g}(\bar{\nabla}_{X^v} Y^b, Z^b) &= \frac{1}{2} \bar{g}(X^v, (R(Y, Z)u)^v), \\
vi) \quad \bar{g}(\bar{\nabla}_{X^v} Y^b, Z^v) &= \frac{1}{2} (Y^b(\bar{g}(Z^v, X^v)) - \bar{g}(Z^v, (\nabla_Y X)^v) \\
&\quad - \bar{g}(X^v, (\nabla_Y Z)^v)), \\
vii) \quad \bar{g}(\bar{\nabla}_{X^v} Y^v, Z^b) &= \frac{1}{2} (-Z^b(\bar{g}(X^v, Y^v)) + \bar{g}(Y^v, (\nabla_Z X)^v) \\
&\quad + \bar{g}(X^v, (\nabla_Z Y)^v)), \\
viii) \quad \bar{g}(\bar{\nabla}_{X^v} Y^v, Z^v) &= \frac{1}{2} (X^v(\bar{g}(Y^v, Z^v)) + Y^v(\bar{g}(Z^v, X^v)) \\
&\quad - Z^v(\bar{g}(X^v, Y^v))),
\end{aligned}$$

for all $X, Y, Z \in C^\infty(TM)$.

PROOF. For any vector fields $X, Y, Z \in C^\infty(TM)$ and $i, j, k \in \{h, v\}$

$$\begin{aligned}
\bar{g}(\bar{\nabla}_{X^i} Y^j, Z^k) &= \frac{1}{2} (X^i(\bar{g}(Y^j, Z^k)) + Y^j(\bar{g}(Z^k, X^i)) - Z^k(\bar{g}(X^i, Y^j))) \\
&\quad - \bar{g}(X^i, [Y^j, Z^k]) + \bar{g}(Y^j, [Z^k, X^i]) + \bar{g}(Z^k, [X^i, Y^j]).
\end{aligned}$$

i) This is a consequence of Theorem 2.11, Definition 2.19 and the following calculations

$$\begin{aligned}
2\bar{g}(\bar{\nabla}_{X^h} Y^h, Z^h) &= (X^h(\bar{g}(Y^h, Z^h)) + Y^h(\bar{g}(Z^h, X^h)) \\
&\quad - Z^h(\bar{g}(X^h, Y^h))) - \bar{g}(X^h, [Y^h, Z^h]) \\
&\quad + \bar{g}(Y^h, [Z^h, X^h]) + \bar{g}(Z^h, [X^h, Y^h]) \\
&= X(g(Y, Z)) + Y(g(Z, Y)) - Z(g(X, Y)) \\
&\quad - \bar{g}(X^h, [Y, Z]^h) + \bar{g}(Y^h, [Z, X]^h) + \bar{g}(Z^h, [X, Y]^h) \\
&\quad + \bar{g}(X^h, (R(Y, Z)u)^v) - \bar{g}(Y^h, (R(Z, X)u)^v) \\
&\quad - \bar{g}(Z^h, (R(X, Y)u)^v) \\
&= 2g(\nabla_X Y, Z).
\end{aligned}$$

ii) The second part of the lemma is obtained as follows

$$\begin{aligned}
2\bar{g}(\bar{\nabla}_{X^h} Y^h, Z^v) &= X^h(\bar{g}(Y^h, Z^v)) + Y^h(\bar{g}(Z^v, X^h)) \\
&\quad - Z^v(\bar{g}(X^h, Y^h)) - \bar{g}(X^h, [Y^h, Z^v]) \\
&\quad + \bar{g}(Y^h, [Z^v, X^h]) + \bar{g}(Z^v, [X^h, Y^h]) \\
&= -Z^v g(X, Y) - \bar{g}(X^h, [Y^h, Z^v]) \\
&\quad + \bar{g}(Y^h, [Z^v, X^h]) + \bar{g}(Z^v, [X^h, Y^h]).
\end{aligned}$$

The first term vanishes, because differentiating a horizontal vector field in a vertical direction gives zero. The second and third terms also vanish, because the Lie bracket of a horizontal vector field and a vertical vector field is vertical, hence

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_{X^h} Y^h, Z^h) &= \bar{g}(Z^v, [X^h, Y^h]) \\ &= \bar{g}(Z^v, [X, Y]^h - (R(X, Y)u)^v) \\ &= -\bar{g}(Z^v, (R(X, Y)u)^v). \end{aligned}$$

iii) This is analogous to the proof of part **ii**).

iv) The Koszul formula gives

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_{X^h} Y^v, Z^v) &= X^h(\bar{g}(Y^v, Z^v)) + Y^v(\bar{g}(Z^v, X^h)) \\ &\quad - Z^v(\bar{g}(X^h, Y^v)) - \bar{g}(X^h, [Y^v, Z^v]) \\ &\quad + \bar{g}(Y^v, [Z^v, X^h]) + \bar{g}(Z^v, [X^h, Y^v]) \\ &= X^h(\bar{g}(Y^v, Z^v)) - \bar{g}(X^h, [Y^v, Z^v]) \\ &\quad + \bar{g}(Y^v, [Z^v, X^h]) + \bar{g}(Z^v, [X^h, Y^v]), \end{aligned}$$

but the Lie bracket of two vertical vector fields is equal to zero and hence the result is proven.

v) This is analogous to the proof of part **ii**)

vi) and **vii)** are analogous to **iv)**

viii) This is a direct consequence of the fact that the Lie bracket of two vertical vector fields vanishes. \square

Corollary 2.21. *Let (M, g) be a Riemannian manifold and \bar{g} be a natural metric on the tangent bundle TM of M . Then the corresponding Levi-Civita connection satisfies*

$$(\bar{\nabla}_{X^h} Y^h)_{(p,u)} = (\nabla_X Y)^h_{(p,u)} - \frac{1}{2}(R_p(X, Y)u)^v$$

for all $X, Y \in C^\infty(TM)$.

Corollary 2.22. *Let \bar{K} be the sectional curvature of the tangent bundle (TM, \bar{g}) equipped with a natural metric \bar{g} . Then holds*

$$\bar{K}(X^h, Y^h) = K(X, Y) - \frac{3}{4} \| (R(X, Y)u)^v \|^2,$$

for any orthonormal vector fields X and $Y \in C^\infty(TM)$.

PROOF. This follows directly by the formula of O'Neill, provided in [15], page 465 and Corollary 2.21. \square

For later purposes we need the following definition and lemma.

Definition 2.23. Let (M, g) be a Riemannian manifold and let $\bar{\nabla}$ be the Levi-Civita connection on the tangent bundle (TM, \bar{g}) , equipped with a natural metric \bar{g} . Let $F : TM \rightarrow TM$ be a differentiable map preserving the fibers and linear on each of them. Then we define the *vertical* and *horizontal lifts* F^v and F^h by

$$F(\eta)^v = \sum_{i=1}^m \eta_i F(\partial_i)^v \quad \text{and} \quad F(\eta)^h = \sum_{i=1}^m \eta_i F(\partial_i)^h,$$

where $\eta = \sum_{i=1}^m \eta_i \partial_i \in \pi^{-1}(V)$ is a local representation of $\eta \in C^\infty(TM)$.

Lemma 2.24. [10] *For any vector field $X \in C^\infty(TM)$, $\xi = (p, u) \in TM$ and $\eta = \sum_{i=1}^m \eta_i \partial_i \in \pi^{-1}(V)$, we have*

- i) $(\hat{\nabla}_{X^v} F^v)_\xi = F(X_p)_\xi^v + \sum_{i=1}^m \eta_i \bar{\nabla}_{X^v} F(\partial_i)^v,$
- ii) $(\hat{\nabla}_{X^v} F^h)_\xi = F(X_p)_\xi^h + \sum_{i=1}^m \eta_i \bar{\nabla}_{X^v} F(\partial_i)^h,$
- iii) $(\hat{\nabla}_{X^h} F^v)_\xi = (\hat{\nabla}_{X^h} (F(u)^v))_\xi,$
- iv) $(\hat{\nabla}_{X^h} F^h)_\xi = (\hat{\nabla}_{X^h} (F(u)^h))_\xi.$

PROOF. Let (x_1, \dots, x_m) be a local coordinate of M in a neighborhood V of p . Then, using the known abbreviation X_i for $\frac{\partial}{\partial x_i}$, we have $X^v \cdot dx_i = dx_i(X)$ for $i \in \{1, \dots, m\}$. Hence we get

$$\begin{aligned} \hat{\nabla}_{X^v} F(\eta)^v &= \sum_{i=1}^m \nabla_{X^v} (\eta_i F(\partial_i)^v) \\ &= \sum_{i=1}^m X^v(\eta_i) F(\partial_i)^v + \eta_i \hat{\nabla}_{X^v} F(\partial_i)^v \\ &= \sum_{i=1}^m \eta_i (X) F(\partial_i)^v + \eta_i \hat{\nabla}_{X^v} F(\partial_i)^v \\ &= F(X_p)_\xi^v + \sum_{i=1}^m \eta_i \hat{\nabla}_{X^v} F(\partial_i)^v. \end{aligned}$$

The second step follows by the product rule. Similarly we calculate:

$$\begin{aligned} \hat{\nabla}_{X^v} F(\eta)^h &= \sum_{i=1}^m \nabla_{X^v} (\eta_i F(\partial_i)^h) \\ &= \sum_{i=1}^m X^v(\eta_i) F(\partial_i)^h + \eta_i \hat{\nabla}_{X^v} F(\partial_i)^h \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^m \eta_i(X) F(\partial_i)^b + \eta_i \hat{\nabla}_{X^v} F(\partial_i)^b \\
&= F(X_p)_\xi^b + \sum_{i=1}^m \eta_i \hat{\nabla}_{X^v} F(\partial_i)^b.
\end{aligned}$$

For the two last equations of the lemma we use a differentiable curve $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$ to get a differentiable curve $U \circ \gamma : [0, 1] \rightarrow TM$ such that $U \circ \gamma(0) = \xi$ and $(U \circ \gamma)'(0) = X_\xi^b$. By the definition of our functions F^v and F^b we get $F^v|_{U \circ \gamma} = (F \circ U)^v|_{U \circ \gamma}$ and $F^b|_{U \circ \gamma} = (F \circ U)^b|_{U \circ \gamma}$. This proves part iii) and iv) of the lemma. \square

The Sasaki Metric

This chapter is devoted to the geometry of the tangent bundle (TM, \hat{g}) of a Riemannian manifold (M, g) equipped with the Sasaki metric \hat{g} introduced in [16]. This has interested many mathematicians in the last decades, see for example Dombrowski [6], Kowalski [10], Aso [1] or Musso and Tricerri [13].

1. The Levi-Civita Connection

Definition 3.1. Let (M, g) be a Riemannian manifold. Then the *Sasaki metric* \hat{g} on the tangent bundle TM is the natural metric given by

- i) $\hat{g}_{(x,u)}(X^h, Y^h) = g_p(X, Y),$
- ii) $\hat{g}_{(x,u)}(X^v, Y^h) = 0,$
- iii) $\hat{g}_{(x,u)}(X^v, Y^v) = g_p(X, Y),$

for all vector fields $X, Y \in C^\infty(TM)$.

We can now calculate the Levi-Civita connection of the tangent bundle with respect to \hat{g} .

Proposition 3.2. [10] *Let $\hat{\nabla}$ be the Levi-Civita connection of (TM, \hat{g}) equipped with the Sasaki metric \hat{g} . Then*

- i) $(\hat{\nabla}_{X^h} Y^h)_{(p,u)} = (\nabla_X Y)_{(p,u)}^h - \frac{1}{2}(R_p(X, Y)u)^v,$
- ii) $(\hat{\nabla}_{X^h} Y^v)_{(p,u)} = (\nabla_X Y)_{(p,u)}^v + \frac{1}{2}(R_p(u, Y)X)^h,$
- iii) $(\hat{\nabla}_{X^v} Y^h)_{(p,u)} = \frac{1}{2}(R_p(u, X)Y)^h,$
- iv) $(\hat{\nabla}_{X^v} Y^v)_{(p,u)} = 0$

for all vector fields $X, Y \in C^\infty(TM)$.

PROOF. i) This is nothing but Corollary 2.21.

ii) The part iii) of Lemma 2.20 and the symmetry rules in Proposition 1.46 give us

$$\begin{aligned} 2\hat{g}(\hat{\nabla}_{X^h} Y^v, Z^h) &= -\hat{g}((R(Z, X)u)^v, Y^v) \\ &= -g(R(u, Y)Z, X) \\ &= g(R(u, Y)X, Z). \end{aligned}$$

Part iv) of Lemma 2.20 implies

$$\begin{aligned}
2\hat{g}(\hat{\nabla}_{X^b}Y^v, Z^v) &= X^b(\hat{g}(Y^v, Z^v)) + \hat{g}(Z^v, (\nabla_X Y)^v) \\
&\quad - \hat{g}(Y^v, (\nabla_X Z)^v) \\
&= X(g(Y, Z)) + g(Z, \nabla_X Y) - g(Y, \nabla_X Z) \\
&= g(Z, \nabla_X Y) + g(Y, \nabla_X Z) + g(Z, \nabla_X Y) - g(Y, \nabla_X Z) \\
&= 2g(Z, \nabla_X Y).
\end{aligned}$$

The last important step follows by the definition of a metric connection.

iii) As above we use part v) of Lemma 2.20 and Proposition 1.46 and get

$$\begin{aligned}
2\hat{g}(\hat{\nabla}_{X^v}Y^b, Z^b) &= \hat{g}(X^v, (R(Y, Z)u)^v) \\
&= g(X, R(Y, Z)u) \\
&= g(R(u, X)Y, Z).
\end{aligned}$$

Part vi) of Lemma 2.20 gives us further

$$\begin{aligned}
2\hat{g}(\hat{\nabla}_{X^v}Y^b, Z^v) &= (Y^b(\hat{g}(Z^v, X^v)) \\
&\quad - \hat{g}(Z^v, (\nabla_Y X)^v) - \hat{g}(X^v, (\nabla_Y Z)^v)) \\
&= Y(g(Z, X) - g(Z, \nabla_Y X) - g(X, \nabla_Y Z)) \\
&= g(Z, \nabla_Y X) + g(X, \nabla_Y Z) \\
&\quad - g(Z, \nabla_Y X) - g(X, \nabla_Y Z) \\
&= 0.
\end{aligned}$$

iv) As usual we use Lemma 2.20 to get

$$\begin{aligned}
2\hat{g}(\hat{\nabla}_{X^v}Y^v, Z^b) &= (-Z^b(\hat{g}(X^v, Y^v)) \\
&\quad + \hat{g}(Y^v, (\nabla_Z X)^v) + \hat{g}(X^v, (\nabla_Z Y)^v)) \\
&= -Z(g(X, Y)) + g(Y, \nabla_Z X) + g(X, \nabla_Z Y) \\
&\quad - g(Y, \nabla_Z X) - g(X, \nabla_Z Y) + g(Y, \nabla_Z X) \\
&\quad + g(X, \nabla_Z Y) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
2\hat{g}(\hat{\nabla}_{X^v}Y^v, Z^v) &= X^v(\hat{g}(Y^v, Z^v)) + Y^v(\hat{g}(Z^v, X^v)) \\
&\quad - Z^v(\hat{g}(X^v, Y^v)) \\
&= X^v(g(Y, Z)) + Y^v(g(Z, X)) - Z^v(g(X, Y)) \\
&= 0.
\end{aligned}$$

The last equation we have, because differentiating a horizontal vector field in the direction of a vertical one gives zero and by the definition of the metric holds $g(X, Y) = \hat{g}(X^h, Y^h)$. This completes the proof. \square

2. The Curvature Tensor

For calculating the Riemann curvature tensor we need the following result, which is a direct consequence of Lemma 2.24.

Corollary 3.3. [10] *Let (M, g) be a Riemannian manifold and let $\hat{\nabla}$ be the Levi-Civita connection on the tangent bundle (TM, \hat{g}) , equipped with the Sasaki metric. Let $F : TM \rightarrow TM$ be a differentiable map preserving the fibers and linear on each of them. Then for any $x \in M$ and $\eta \in C^\infty(TTM)$ we have*

$$\hat{\nabla}_{X^v} F(\eta)^v = F(X)^v$$

$$\hat{\nabla}_{X^v} F(\eta)^h = F(X)^h + \frac{1}{2}(R(u, X)F(\eta))^h.$$

Proposition 3.4. [10] *Let (M, g) be a Riemannian manifold and \hat{R} be the Riemann curvature tensor of the tangent bundle (TM, \hat{g}) equipped with the induced Sasaki metric. Then we have the following formulae:*

$$\begin{aligned} i) \quad \hat{R}_{(p,u)}(X^v, Y^v)Z^v &= 0, \\ ii) \quad \hat{R}_{(p,u)}(X^v, Y^v)Z^h &= (R(X, Y)Z + \frac{1}{4}R(u, X)(R(u, Y)Z) \\ &\quad - \frac{1}{4}R(u, Y)(R(u, X)Z))_p^h, \\ iii) \quad \hat{R}_{(p,u)}(X^h, Y^v)Z^v &= -(\frac{1}{2}R(Y, Z)X + \frac{1}{4}R(u, Y)(R(u, Z)X))_p^h, \\ iv) \quad \hat{R}_{(p,u)}(X^h, Y^v)Z^h &= (\frac{1}{4}R(R(u, Y)Z, X)u + \frac{1}{2}R(X, Z)Y)_p^v \\ &\quad + \frac{1}{2}((\nabla_X R)(u, Y)Z)_p^h, \\ v) \quad \hat{R}_{(p,u)}(X^h, Y^h)Z^v &= (R(X, Y)Z + \frac{1}{4}R(R(u, Z)Y, X)u \\ &\quad - \frac{1}{4}R(R(u, Z)X, Y)u)_p^v \\ &\quad + \frac{1}{2}((\nabla_X R)(u, Z)Y - (\nabla_Y R)(u, Z)X)_p^h, \\ vi) \quad \hat{R}_{(p,u)}(X^h, Y^h)Z^h &= \frac{1}{2}((\nabla_Z R)(X, Y)u)_p^v \\ &\quad + (R(X, Y)Z + \frac{1}{4}R(u, R(Z, Y)u)X \end{aligned}$$

$$+\frac{1}{4}R(u, R(X, Z)u)Y + \frac{1}{2}R(u, R(X, Y)u)Z_p^b$$

for $X, Y, Z \in T_p M$.

PROOF. [10] **i)** The first part of the proposition follows directly by the last part of Proposition 3.2 and the fact that the Lie bracket of two vertical vector fields vanishes.

iii) The last part of Proposition 3.2 and the fact that

$$[X^b, Y^v] = (\nabla_X Y)^v,$$

by Theorem 2.11, provide us with

$$\begin{aligned} \hat{R}(X^b, Y^v)Z^v &= \hat{\nabla}_{X^b} \hat{\nabla}_{Y^v} Z^v - \hat{\nabla}_{Y^v} \hat{\nabla}_{X^b} Z^v - \hat{\nabla}_{[X^b, Y^v]} Z^v \\ &= -\hat{\nabla}_{Y^v} \hat{\nabla}_{X^b} Z^v \\ &= -\hat{\nabla}_{Y^v} ((\nabla_X Z)^v + F(u)^b), \end{aligned}$$

where $F : TM \rightarrow TM$ is the linear fiber preserving map

$$F : u \mapsto \frac{1}{2}R(u, Z_p)X_p$$

for any $(p, u) \in TM$. By the last part of Proposition 3.2 we know that $\hat{\nabla}_{Y^v} (\nabla_X Z)^v = 0$ and according to Corollary 3.3 we have

$$\hat{\nabla}_{Y^v} F(u)^b = F(Y)^b + \frac{1}{2}(R(u, Y)F(u))^b.$$

Therefore we get

$$\begin{aligned} \hat{R}(X^b, Y^v)Z^v &= -\hat{\nabla}_{Y^v} \hat{\nabla}_{X^b} Z^v \\ &= -\hat{\nabla}_{Y^v} ((\nabla_X Z)^v + F(u)^b) \\ &= -\hat{\nabla}_{Y^v} F(u)^b \\ &= -F(Y)^b - \frac{1}{2}(R(u, Y)F(u))^b \\ &= -(\frac{1}{2}R(Y, Z)X + \frac{1}{4}R(u, Y)(R(u, Z)X))^b. \end{aligned}$$

Hence the third part of the proposition is proven.

ii) Using the 1st Bianchi identity in Proposition 1.46

$$\hat{R}(X^v, Y^v)Z^b = \hat{R}(Z^b, Y^v)X^v - \hat{R}(Z^b, X^v)Y^v,$$

we get by using part iii)

$$\begin{aligned}\hat{R}(X^v, Y^v)Z^b &= (-\frac{1}{2}R(Y, X)Z - \frac{1}{4}R(u, Y)(R(u, X)Z))^b \\ &\quad + (\frac{1}{2}R(X, Y)Z + \frac{1}{4}R(u, X)(R(u, Y)Z))^b.\end{aligned}$$

From which the statement follows.

iv) As above we now introduce two mappings $F_1 : TM \rightarrow TM$ and $F_2 : TM \rightarrow TM$ by

$$F_1(u) \mapsto \frac{1}{2}R(u, Y_p)Z_p$$

and

$$F_2(u) \mapsto -\frac{1}{2}R(X_p, Z_p)u$$

and can now write the third part of Proposition 3.2 in the form

$$\hat{\nabla}_{Y^v}Z^b = F_1(u)^b.$$

By the definition of the Riemann curvature tensor we obtain

$$\begin{aligned}\hat{R}(X^b, Y^v)Z^b &= \hat{\nabla}_{X^b}\hat{\nabla}_{Y^v}Z^b - \hat{\nabla}_{Y^v}\hat{\nabla}_{X^b}Z^b - \hat{\nabla}_{[X^b, Y^v]}Z^b \\ &= \hat{\nabla}_{X^b}F_1(u)^b - \hat{\nabla}_{Y^v}((\nabla_X Z)^b + F_2(u)^v) - \hat{\nabla}_{(\nabla_X Y)^v}Z^b \\ &= (\nabla_X(F_1(u)))^b - \frac{1}{2}(R(X, F_1(u))u)^v \\ &\quad - \frac{1}{2}(R(u, Y)\nabla_X Z)^b - F_2(Y)^v - \frac{1}{2}(R(u, \nabla_X Y)Z)^b \\ &= (\frac{1}{4}R(R(u, Y)Z, X)u + \frac{1}{2}R(X, Z)Y)^v \\ &\quad + \frac{1}{2}((\nabla_X R)(u, Y)Z)^b.\end{aligned}$$

The last step is only inserting the mappings F_1, F_2 and the definition of a covariant derivative. The middle step uses Proposition 3.2 and Corollary 3.3.

v) Using part iv) and the 1st Bianchi identity

$$\hat{R}(X^b, Y^b)Z^v = \hat{R}(X^b, Z^v)Y^b - \hat{R}(Y^b, Z^v)X^b$$

and we see that

$$\begin{aligned}\hat{R}(X^b, Y^b)Z^v &= (\frac{1}{4}R(R(u, Z)Y, X)u)^v + \frac{1}{2}((\nabla_X R)(u, Z)Y)^b \\ &\quad - (\frac{1}{4}R(R(u, Z)X, Y)u)^v - \frac{1}{2}((\nabla_Y R)(u, Z)X)^b\end{aligned}$$

$$+\frac{1}{2}(R(X, Y)Z - R(Y, X)Z)^v.$$

Which implies the result.

vi) For the last part we have the following standard calculations

$$\begin{aligned}\hat{R}(X^b, Y^b)Z^b &= \hat{\nabla}_{X^b}\hat{\nabla}_{Y^b}Z^b - \hat{\nabla}_{Y^b}\hat{\nabla}_{X^b}Z^b - \hat{\nabla}_{[X^b, Y^b]}Z^b \\ &= \hat{\nabla}_{X^b}((\nabla_Y Z)^b - \frac{1}{2}(R(Y, Z)u)^v) \\ &\quad - \hat{\nabla}_{Y^b}((\nabla_X Z)^b - \frac{1}{2}(R(X, Z)u)^v) \\ &\quad - \hat{\nabla}_{[X, Y]^b}Z^b + \hat{\nabla}_{(R(X, Y)u)^v}Z^b \\ &= (\nabla_X \nabla_Y Z)^b - \frac{1}{2}(R(X, \nabla_Y Z)u)^v \\ &\quad - \frac{1}{2}(\nabla_X R(Y, Z)u)^v - \frac{1}{4}(R(u, R(Y, Z)u)X)^b \\ &\quad - (\nabla_Y \nabla_X Z)^b + \frac{1}{2}(R(Y, \nabla_X Z)u)^v \\ &\quad + \frac{1}{2}(\nabla_Y R(X, Z)u)^v + \frac{1}{4}(R(u, R(X, Z)u)Y)^b \\ &\quad - (\nabla_{[X, Y]}Z)^b + \frac{1}{2}(R([X, Y], Z)u)^v \\ &\quad + \frac{1}{2}(R(u, R(X, Y)u)Z)^b \\ &= \frac{1}{2}((\nabla_Z R)(X, Y)u)^v + (R(X, Y)Z)^b \\ &\quad + \frac{1}{4}(R(u, R(Z, Y)u)X)^b + \frac{1}{4}(R(u, R(X, Z)u)Y)^b \\ &\quad + \frac{1}{2}(R(u, R(X, Y)u)Z)^b.\end{aligned}$$

The last step of these calculations follows by the 2nd Bianchi identity, which tells us that

$$(\nabla_X R)(Y, Z)u + (\nabla_Y R)(Z, X)u + (\nabla_Z R)(X, Y)u = 0.$$

□

3. Geometric Consequences

Theorem 3.5. [10], [1] *(TM, \hat{g}) is flat if and only if (M, g) is flat.*

PROOF. [1] It is a direct consequence of Proposition 3.4 that if $R \equiv 0$ then $\hat{R} \equiv 0$. We now assume that $\hat{R} = 0$ and calculate the Riemann curvature

tensor for three horizontal vector fields at $(p, 0)$:

$$R_p(X, Y)Z = \hat{R}_{(p,0)}(X^h, Y^h)Z^h = 0.$$

Therefore (M, g) is flat. \square

For the sectional curvatures of (TM, \hat{g}) we have

Lemma 3.6. [1] *Let $(p, u) \in TM$ and $X, Y \in C^\infty(TM)$ be two orthonormal tangent vectors at p . Let $\hat{K}(X^i, Y^j)$ denote the sectional curvature of the plane spanned by X^i and Y^j , $i, j \in \{h, v\}$. Then we have:*

- i) $\hat{K}(X^v, Y^v) = 0$,
- ii) $\hat{K}(X^h, Y^v) = \frac{1}{4}|R(Y, u)X|^2$,
- iii) $\hat{K}(X^h, Y^h) = K(X, Y) - \frac{3}{4}|R(X, Y)u|^2$,

where $|\cdot|$ is the norm induced by g .

PROOF. **i)** It follows directly from Proposition 3.4 that for two vertical vectors the sectional curvature vanishes.

ii) If one of the vectors is horizontal and the other is vertical, then Proposition 3.4 implies

$$\begin{aligned} \hat{g}(\hat{R}(X^h, Y^v)Y^v, X^h) &= g(-\frac{1}{2}R(Y, Y)X, X) \\ &\quad + g(-\frac{1}{4}R(u, Y)(R(u, Y)X), X). \end{aligned}$$

But the first term vanishes and the result follows by using Proposition 1.46.

iii) If both vectors are horizontal, then Proposition 3.4 and the fact that the scalar product of a vertical and a horizontal vector vanishes gives

$$\begin{aligned} \hat{g}(\hat{R}(X, Y)Y, X) &= g(R(X, Y)Y, X) + \frac{3}{4}g(R(u, R(X, Y)u)Y, X) \\ &\quad + \frac{1}{4}g(R(u, R(X, Y)u)X, X). \end{aligned}$$

The last term vanishes and hence the lemma is proven. \square

We can also proof this lemma by the formulae of O'Neill, see page 465 of [15].

PROOF. We first observe that in our case O'Neill's tensor T equals zero. Hence the first part is proven, because the sectional curvature on the fibers is zero.

For the second part we observe, that his formula reduces to:

$$\begin{aligned} \hat{K}(X^h, Y^v) &= \|A_{X^h}Y^v\|^2 \\ &= \|(\hat{\nabla}_{X^h}Y^v)^h\|^2 \end{aligned}$$

$$= -\frac{1}{2}|R(u, Y)X|^2.$$

The last part is just Corollary 2.22 and the definition of the Sasaki metric. \square

Corollary 3.7. [1] *If the sectional curvatures of (TM, \hat{g}) are bounded, then (TM, \hat{g}) is flat.*

PROOF. [1] Let us assume, that TM is not flat. Then it follows by Theorem 3.5 that M is not flat. Hence there exist a point $p \in M$ and a pair of orthonormal vectors $V, W \in T_p M$ such that $R(V, W)u \neq 0$ for some $u \in T_p M$. Then we have

$$\hat{K}(V^h, W^h) = K(V, W) - \frac{3}{4}|R(X, Y)u|^2.$$

Since the set of u satisfying this condition is unbounded, $\hat{K}(V^h, W^h)$ is unbounded from below. Similarly we show that $\hat{K}(V^h, W^h)$ is unbounded above, by using the second formula of Lemma 3.6. \square

Corollary 3.8. *Let (M, g) be a Riemannian manifold of constant sectional curvature κ . Then*

- i) $\hat{K}(X^v, Y^v) = 0,$
- i) $\hat{K}(X^h, Y^v) = \frac{1}{4}\kappa^2 g(u, X)^2,$
- i) $\hat{K}(X^h, Y^h) = \kappa - \frac{3}{4}\kappa^2(g(u, X)^2 + g(u, Y)^2),$

for any orthonormal vector fields $X, Y \in C^\infty(TM)$.

Using Lemma 3.6 we obtain the following result for the scalar curvature $\hat{\sigma}$ of (TM, \hat{g})

Lemma 3.9. [13] *Let σ be the scalar curvature of g and $\hat{\sigma}$ be the scalar curvature of \hat{g} . Then the following equation holds*

$$\hat{\sigma} = \sigma - \frac{1}{4} \sum_{i,j=1}^m |R(X_i, X_j)u|^2,$$

where $\{X_1, \dots, X_m\}$ are a orthonormal basis for M .

PROOF. Lemma 3.6 provides us with a formula of the sectional curvatures. We only have to add them for an orthonormal basis. This means that for $\{Y_1, \dots, Y_{2m}\}$, an orthonormal basis on TM with $X_i^h = Y_i$ and $X_i^v = Y_{m+i}$ we get

$$\hat{\sigma} = \sum_{i,j=1}^{2m} \hat{K}(Y_i, Y_j)$$

$$\begin{aligned}
&= \sum_{i,j=1}^m (\hat{K}(X_i^b, X_j^b) + 2\hat{K}(X_i^b, X_j^v) + \hat{K}(X_i^v, X_j^v)) \\
&= \sum_{i,j=1}^m (K(X_i, X_j) - \frac{3}{4}|R(X_i, X_j)u|^2) + 2 \sum_{i,j=1}^m \frac{1}{4}|R(X_j, u)X_i|^2.
\end{aligned}$$

Therefore we just have to prove that the two sums are equal. But with $u = \sum_{i=1}^m u_i X_i$ we get

$$\begin{aligned}
&\sum_{i,j=1}^m |R(X_j, u)X_i|^2 \\
&= \sum_{i,j,k,l=1}^m u_k u_l g(R(X_j, X_k)X_i, R(X_j, X_l)X_i) \\
&= \sum_{i,j,k,l,s=1}^m u_k u_l g(R(X_j, X_k)X_i, X_s) g(R(X_j, X_l)X_i, X_s) \\
&= \sum_{i,j,k,l,s=1}^m u_k u_l g(R(X_s, X_i)X_k, X_j) g(R(X_s, X_i)X_l, X_j) \\
&= \sum_{i,j,k,l=1}^m u_k u_l g(R(X_j, X_i)X_k, R(X_j, X_i)X_l) \\
&= \sum_{i,j=1}^m |R(X_j, X_i)u|^2.
\end{aligned}$$

In the middle step we use three parts of Proposition 1.46. \square

Corollary 3.10. *Let (M, g) be a Riemannian manifold with constant sectional curvature κ . Then*

$$\hat{\sigma} = (m-1)\kappa(m - \frac{1}{2}\kappa \|u\|^2).$$

PROOF. By Corollary 1.51 we know that $\sigma = m(m-1)\kappa$. By Lemma 1.49 we therefore obtain

$$\begin{aligned}
\hat{\sigma} &= \sigma - \frac{1}{4} \sum_{i,j=1}^m |R(X_i, X_j)u|^2 \\
&= \sigma - \frac{1}{4} \kappa^2 \sum_{i,j=1}^m |g(X_j, u)X_i - g(X_i, u)X_j|^2
\end{aligned}$$

$$\begin{aligned}
&= \sigma - \frac{1}{4}\chi^2 \sum_{i,j=1}^m (g(X_j, u)^2 + g(X_i, u)^2 - 2(g(X_i, u)g(X_j, u)g(X_i, X_j))) \\
&= m(m-1)\chi - \frac{1}{2}\chi^2(m-1) \|u\|^2.
\end{aligned}$$

In the last step we just split the sum and use the formula for σ . \square

As a direct consequence we get

Theorem 3.11. [13] *(TM, \hat{g}) has constant scalar curvature if and only if (M, g) is flat.*

Hence by Theorem 3.5 follows

Corollary 3.12. *(TM, \hat{g}) has constant scalar curvature if and only if the scalar curvature is zero.*

By Theorem 3.11 and by section 1.B of [20] we obtain directly

Corollary 3.13. [13] *(TM, \hat{g}) is locally homogeneous if and only if (TM, \hat{g}) is flat.*

Where a manifold (M, g) is said to be *locally homogeneous* if for each $p, q \in M$, there exists a neighbourhood V of p , a neighbourhood W of q and a local isometry $\varphi : V \rightarrow W$ such that $\varphi(p) = q$.

In particular, by Theorem 3, page 303 in [9] volume I,

Corollary 3.14. [13] *(TM, \hat{g}) is locally symmetric if and only if (TM, \hat{g}) is flat.*

And finally follows by the proof of Lemma 3.9 we have

Corollary 3.15. [13] *(TM, \hat{g}) is Einstein if and only if (TM, \hat{g}) is flat i.e. that (TM, \hat{g}) has constant Ricci curvature if and only if (TM, \hat{g}) is flat.*

For a deeper look on Einstein manifolds we refer to [2].

The Cheeger-Gromoll Metric

The Cheeger-Gromoll metric on the tangent bundle TM of a Riemannian manifold M has also been of great interest to mathematicians in the last decades. This metric was introduced by J. Cheeger and D. Gromoll [3] in 1972, but an explicit expression was first given by E. Musso and F. Tricerri [13] in 1988.

1. The Levi-Civita Connection

Definition 4.1. Let (M, g) be a Riemannian manifold. Then the *Cheeger-Gromoll metric* \tilde{g} is the natural Riemannian metric on the tangent bundle TM such that

$$\begin{aligned} i) \quad & \tilde{g}_{(p,u)}(X^h, Y^h) = g_p(X, Y), \\ ii) \quad & \tilde{g}_{(p,u)}(X^h, Y^v) = 0, \\ iii) \quad & \tilde{g}_{(p,u)}(X^v, Y^v) = \frac{1}{1+r^2}(g_p(X, Y) + g_p(X, u)g_p(Y, u)) \end{aligned}$$

for all vector fields $X, Y \in C^\infty(TM)$. Here r denotes $|u| = \sqrt{g(u, u)}$.

From now on we set $\alpha = 1 + r^2$ to simplify the notion.

Proposition 4.2. [18] *Let $\tilde{\nabla}$ be the Levi-Civita connection of TM with Cheeger-Gromoll metric \tilde{g} . If $X, Y \in C^\infty(TM)$, then for all $(p, u) \in TM$*

$$\begin{aligned} \text{i)} \quad & (\tilde{\nabla}_X Y^h) = (\nabla_X Y)^h - \frac{1}{2}(R(X, Y)u)^v, \\ \text{ii)} \quad & (\tilde{\nabla}_X Y^v) = \frac{1}{2\alpha}(R(u, Y)X)^h + (\nabla_X Y)^v, \\ \text{iii)} \quad & (\tilde{\nabla}_{X^v} Y^h) = \frac{1}{2\alpha}(R(u, X)Y)^h, \\ \text{iv)} \quad & (\tilde{\nabla}_{X^v} Y^v) = -\frac{1}{\alpha}(\tilde{g}(X^v, U)Y^v + \tilde{g}(Y^v, U)X^v) \\ & + \frac{1+\alpha}{\alpha}\tilde{g}(X^v, Y^v)U - \frac{1}{\alpha}\tilde{g}(X^v, U)\tilde{g}(Y^v, U)U, \end{aligned}$$

where $U \in C^\infty(TM)$ is the canonical vertical vector at (p, u) defined by

$$U = \sum_{i=1}^m v_{m+i} \left(\frac{\partial}{\partial v_{m+1}} \right)_{(p,u)},$$

with $u = (v_{m+1}, \dots, v_{2m})$.

PROOF. [18] **i)** The first part of this proposition is just the Corollary 2.21.

ii) For the rest we first claim that

$$\tilde{g}(Y^v, (R(Z, X)u)^v) = -\frac{1}{\alpha}\tilde{g}((R(u, Y)X)^b, Z^b).$$

In fact, by earlier definitions and the skew-symmetry of R , we get

$$\begin{aligned}\tilde{g}(Y^v, (R(Z, X)u)^v) &= \frac{1}{\alpha}(g(Y, R(Z, X)u) \\ &\quad + g(Y, u)g(R(Z, X)u, u)) \\ &= -\frac{1}{\alpha}g(R(u, Y)X, Z) \\ &= -\frac{1}{\alpha}\tilde{g}((R(u, Y)X)^b, Z^b).\end{aligned}$$

Hence by part iii) of Lemma 2.20

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_X^b Y^v, Z^b) &= -\frac{1}{2}\tilde{g}(Y^v, (R(Z, X)u)^v) \\ &= \frac{1}{2\alpha}\tilde{g}((R(u, Y)X)^b, Z^b).\end{aligned}$$

By Definition 2.8 and Lemma 2.9

$$X^b\left(\frac{1}{\alpha}\right) = 0 \quad \text{and} \quad X^b(g(Y, u) \circ \pi) = g(\nabla_X Y, u) \circ \pi.$$

Hence $X^b(\tilde{g}(Y^v, Z^v)) = \tilde{g}((\nabla_X Y)^v, Z^v) + \tilde{g}(Y^v, (\nabla_X Z)^v)$ by using the definition of the metric and so we get by part iv) of Lemma 2.20

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_X^b Y^v, Z^v) &= \frac{1}{2}(X^b(\tilde{g}(Y^v, Z^v)) + \tilde{g}(Z^v, (\nabla_X Y)^v) - \tilde{g}(Y^v, (\nabla_X Z)^v) \\ &= \tilde{g}((\nabla_X Y)^v, Z^v).\end{aligned}$$

These two equations and Lemma 2.20 complete the proof of the second part.

iii) Similar calculations give us

$$\tilde{g}(X^v, (R(Y, Z)u)^v) = \frac{1}{\alpha}\tilde{g}((R(u, X)Y)^b, Z^b)$$

and so, by part v) of Lemma 2.20, we get

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_X^v Y^b, Z^b) &= \frac{1}{2}\tilde{g}(X^v, (R(Y, Z)u)^v) \\ &= \frac{1}{2\alpha}\tilde{g}((R(u, X)Y)^b, Z^b).\end{aligned}$$

By part vi) of Lemma 2.20

$$2\tilde{g}(\tilde{\nabla}_X^v Y^b, Z^v) = Y^b(\tilde{g}(Z^v, X^v)) - \tilde{g}(Z^v, (\nabla_Y X)^v) - \tilde{g}(X^v, (\nabla_Y Z)^v)$$

$$\begin{aligned}
&= \tilde{g}(Z^v, (\nabla_Y X)^v) + \tilde{g}(X^v, (\nabla_Y Z)^v) \\
&\quad - \tilde{g}(Z^v, (\nabla_Y X)^v) - \tilde{g}(X^v, (\nabla_Y Z)^v) \\
&= 0.
\end{aligned}$$

which proves the third part of the proposition.

iv) With similar calculations as above and part vii) of Lemma 2.20 we get that

$$\begin{aligned}
\tilde{g}(\tilde{\nabla}_{X^v} Y^v, Z^v) &= -Z^v(\tilde{g}(X^v, Y^v)) + \tilde{g}(Y^v, (\nabla_Z X)^v) + \tilde{g}(X^v, (\nabla_Z Y)^v) \\
&= -\tilde{g}(Y^v, (\nabla_Z X)^v) - \tilde{g}(X^v, (\nabla_Z Y)^v) \\
&\quad + \tilde{g}(Y^v, (\nabla_Z X)^v) + \tilde{g}(X^v, (\nabla_Z Y)^v) \\
&= 0.
\end{aligned}$$

By using the equation $X^v(f(r^2)) = 2f'(r^2)g(X, u)$ we see that

$$\begin{aligned}
X^v(\tilde{g}(Y^v, Z^v)) &= -\frac{2}{\alpha^2}g(X, u)(g(Y, Z) + g(Y, u)g(Z, u)) \\
&\quad + \frac{1}{\alpha}(g(X, Y)g(Z, u) + g(X, Z)g(Y, u)),
\end{aligned}$$

because α is a function of r^2 . The equation $\tilde{g}(X^v, U) = g(X, u)$ holds, because the canonical vertical vector U is equal to u at the point (p, u) and by the definition of the Cheeger-Gromoll metric

$$\tilde{g}(X^v, U) = \frac{1}{\alpha}(g(X, u) + g(X, u)g(u, u)) = \frac{\alpha}{\alpha}g(X, u).$$

This fact and part viii) of Lemma 2.20 give

$$\begin{aligned}
\alpha^2 \tilde{g}(\tilde{\nabla}_{X^v} Y^v, Z^v) &= \frac{\alpha^2}{2}(X^v(\tilde{g}(Y^v, Z^v)) + Y^v(\tilde{g}(Z^v, X^v)) \\
&\quad - Z^v(\tilde{g}(X^v, Y^v))) \\
&= -g(X, u)(g(Y, Z) + g(Y, u)g(Z, u)) \\
&\quad + \frac{\alpha}{2}(g(X, Y)g(Z, u) + g(X, Z)g(Y, u)) \\
&\quad - g(Y, u)(g(Z, X) + g(Y, u)g(X, u)) \\
&\quad + \frac{\alpha}{2}(g(Y, Z)g(X, u) + g(Y, X)g(Z, u)) \\
&\quad + g(Z, u)(g(X, Y) - g(X, u)g(Y, u)) \\
&\quad - \frac{\alpha}{2}(g(Z, X)g(Y, u) + g(Z, Y)g(X, u)) \\
&= -(g(X, u)g(Y, Z) + g(Y, u)g(Z, X)) \\
&\quad + (1 + \alpha)g(X, Y)g(Z, u) \\
&\quad - g(X, u)g(Y, u)g(Z, u).
\end{aligned}$$

By using the definition of the metric we see that this is equal to what is stated in the proposition. \square

2. The Curvature Tensor

Equipped with the Levi-Civita connection we can now relatively easy calculate the Riemann curvature tensor of TM . But first we state the following direct consequence of Lemma 2.24.

Corollary 4.3. *Let (M, g) be a Riemannian manifold and let $\tilde{\nabla}$ be the Levi-Civita connection on the tangent bundle (TM, \tilde{g}) , equipped with the Cheeger-Gromoll metric. Let $F : TM \rightarrow TM$ be a differentiable map preserving the fibers and linear on each of them, then for any $x \in M$ and any $\eta \in C^\infty(TTM)$ we have*

$$\begin{aligned} \tilde{\nabla}_{X^v} F(\eta)^v &= F(X)^v \\ &\quad - \frac{1}{\alpha} (\tilde{g}(X^v, U) F(\eta)^v + \tilde{g}(F(\eta)^v, U) X^v \\ &\quad - (1 + \alpha) \tilde{g}(F(\eta)^v, X^v) U + \tilde{g}(X^v, U) \tilde{g}(F(\eta)^v, U) U) \end{aligned}$$

and

$$\tilde{\nabla}_{X^v} F(\eta)^b = F(X)^b + \frac{1}{2\alpha} (R(u, X) F(\eta))^b.$$

Proposition 4.4. [18] *Let \tilde{R} be the Riemann curvature tensor of TM equipped with the Cheeger-Gromoll metric \tilde{g} . If $X, Y, Z \in T_p M$, then*

$$\begin{aligned} \text{i)} \quad \tilde{R}(X^b, Y^b) Z^b &= (R(X, Y) Z)^b \\ &\quad - \frac{1}{4\alpha} (R(u, R(Y, Z)u) X - R(u, R(X, Z)u) Y \\ &\quad - 2R(u, R(X, Y)u) Z)^b \\ &\quad + \frac{1}{2} ((\nabla_Z R)(X, Y)u)^v, \\ \text{ii)} \quad \tilde{R}(X^b, Y^b) Z^v &= (R(X, Y) Z)^v \\ &\quad + \frac{1}{2\alpha} ((\nabla_X R)(u, Z) Y - (\nabla_Y R)(u, Z) X)^b \\ &\quad - \frac{1}{4\alpha} (R(X, R(u, Z) Y)u - R(Y, R(u, Z) X)u)^v \\ &\quad - 4\tilde{g}(Z^v, U) (R(X, Y)u)^v \\ &\quad + \frac{1 + \alpha}{\alpha} \tilde{g}((R(X, Y)u)^v, Z^v) U, \\ \text{iii)} \quad \tilde{R}(X^b, Y^v) Z^b &= \frac{1}{2\alpha} ((\nabla_X R)(u, Y) Z)^b + \frac{1}{2} (R(X, Z) Y)^v \\ &\quad - \frac{1}{4\alpha} ((R(X, R(u, Y) Z)u)^v \end{aligned}$$

$$\begin{aligned}
& -2g(Y, u)(R(X, Z)u)^v \\
& + \frac{1+\alpha}{2\alpha} \tilde{g}((R(X, Z)u)^v, Y^v) U, \\
\text{iv)} \tilde{R}(X^b, Y^v)Z^v &= -\frac{1}{2\alpha} (R(Y, Z)X)^b \\
& + \frac{1}{2\alpha^2} (g(Y, u)(R(u, Z)X)^b \\
& \quad - g(Z, u)(R(u, Y)X)^b) \\
& - \frac{1}{4\alpha^2} (R(u, Y)R(u, Z)X)^b, \\
\text{v)} \tilde{R}(X^v, Y^v)Z^b &= \frac{1}{\alpha} (R(X, Y)Z)^b \\
& + \frac{1}{4\alpha^2} (R(u, X)R(u, Y)Z - R(u, Y)R(u, X)Z)^b \\
& + \frac{1}{\alpha^2} (g(Y, u)(R(u, X)Z)^b - g(X, u)(R(u, Y)Z)^b), \\
\text{vi)} \tilde{R}(X^v, Y^v)Z^v &= \frac{\alpha+2}{\alpha^2} (\tilde{g}(X^v, Z^v)g(Y, u)U - \tilde{g}(Y^v, Z^v)g(X, u)U) \\
& + \frac{1+\alpha+\alpha^2}{\alpha^2} (\tilde{g}(Y^v, Z^v)X^v - \tilde{g}(X^v, Z^v)Y^v) \\
& + \frac{\alpha+2}{\alpha^2} (g(X, u)g(Z, u)Y^v - g(Y, u)g(Z, u)X^v).
\end{aligned}$$

PROOF. **i)** With standard calculations we get:

$$\begin{aligned}
\tilde{R}(X^b, Y^b)Z^b &= \tilde{\nabla}_{X^b} \tilde{\nabla}_{Y^b} Z^b - \tilde{\nabla}_{Y^b} \tilde{\nabla}_{X^b} Z^b - \tilde{\nabla}_{[X^b, Y^b]} Z^b \\
&= \tilde{\nabla}_{X^b} (\nabla_Y Z)^b - \frac{1}{2} (R(Y, Z)u)^v \\
&\quad - \tilde{\nabla}_{Y^b} (\nabla_X Z)^b - \frac{1}{2} (R(X, Z)u)^v \\
&\quad - \tilde{\nabla}_{[X, Y]^b} - (R(X, Y)u)^v Z^b \\
&= (\nabla_X \nabla_Y Z)^b - \frac{1}{2} (R(X, \nabla_Y Z)u)^v \\
&\quad - \frac{1}{4\alpha} (R(u, R(Y, Z)u)X)^b - \frac{1}{2} (\nabla_X R(Y, Z)u)^v \\
&\quad - (\nabla_Y \nabla_X Z)^b + \frac{1}{2} (R(Y, \nabla_X Z)u)^v \\
&\quad + \frac{1}{4\alpha} (R(u, R(X, Z)u)Y)^b + \frac{1}{2} (\nabla_Y R(X, Z)u)^v \\
&\quad - (\nabla_{[X, Y]} Z)^b + \frac{1}{2} (R([X, Y], Z)u)^v
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2\alpha}(R(u, R(X, Y)u)Z)^b \\
= & (R(X, Y)Z)^b + \frac{1}{2}((\nabla_Z R)(X, Y)u)^v \\
& - \frac{1}{4\alpha}(R(u, R(Y, Z)u)X - R(u, R(X, Z)u)Y \\
& - 2R(u, R(X, Y)u)Z)^b.
\end{aligned}$$

The last step follows by the 2nd Bianchy identity.

ii) Note for the second formula that, since $\tilde{g}_{(p,u)}(X^v, U) = g_p(X, u)$, we get $\tilde{g}_{(p,u)}((R(X, Y)u)^v, U) = g_p(R(X, Y)u, u) = 0$. Hence

$$\begin{aligned}
\alpha\tilde{R}(X^b, Y^b)Z^v &= \alpha\tilde{\nabla}_{X^b}\tilde{\nabla}_{Y^b}Z^v - \alpha\tilde{\nabla}_{Y^b}\tilde{\nabla}_{X^b}Z^v - \alpha\tilde{\nabla}_{[X^b, Y^b]}Z^v \\
&= \tilde{\nabla}_{X^b}(\frac{1}{2}(R(u, Z)Y)^b + \alpha(\nabla_Y Z)^v) \\
&\quad - \tilde{\nabla}_{Y^b}(\frac{1}{2}(R(u, Z)X)^b + \alpha(\nabla_X Z)^v) \\
&\quad - \alpha\tilde{\nabla}_{[X, Y]^b} - (R(X, Y)u)^v Z^v \\
&= \frac{1}{2}(\nabla_X R(u, Z)Y)^b - \frac{1}{4}(R(X, R(u, Z)Y)u)^v \\
&\quad + \frac{1}{2}(R(u, \nabla_Y Z)X)^b + \alpha(\nabla_X \nabla_Y Z)^v \\
&\quad - \frac{1}{2}(\nabla_Y R(u, Z)X)^b + \frac{1}{4}(R(Y, R(u, Z)X)u)^v \\
&\quad - \frac{1}{2}(R(u, \nabla_X Z)Y)^b - \alpha(\nabla_Y \nabla_X Z)^v \\
&\quad - \frac{1}{2}(R(u, Z)[X, Y])^b - \alpha(\nabla_{[X, Y]}Z)^v \\
&\quad - (\tilde{g}((R(X, Y)u)^v, U)Z^v + \tilde{g}(Z^v, U)(R(X, Y)u)^v) \\
&\quad + (1 + \alpha)\tilde{g}((R(X, Y)u)^v, Z^v)U \\
&\quad - \tilde{g}((R(X, Y)u)^v, U)\tilde{g}(Z^v, U)U \\
&= \alpha(R(X, Y)Z)^v \\
&\quad + \frac{1}{2}((\nabla_X R)(u, Z)Y - (\nabla_Y R)(u, Z)X)^b \\
&\quad - \frac{1}{4}((R(X, R(u, Z)Y)u)^v - (R(Y, R(u, Z)X)u)^v) \\
&\quad + -g(Z, u)(R(X, Y)u)^v \\
&\quad + (1 + \alpha)\tilde{g}((R(X, Y)u)^v, Z^v)U.
\end{aligned}$$

iii) Calculations similar to those above produce the third formula:

$$\begin{aligned}
\tilde{R}(X^b, Y^v)Z^b &= \tilde{\nabla}_{X^b}\tilde{\nabla}_{Y^v}Z^b - \tilde{\nabla}_{Y^v}\tilde{\nabla}_{X^b}Z^b - \tilde{\nabla}_{[X^b, Y^v]}Z^b \\
&= \frac{1}{2\alpha}\tilde{\nabla}_{X^b}(R(u, Y)Z)^b - \tilde{\nabla}_{(\nabla_X Y)^v}Z^b \\
&\quad - \tilde{\nabla}_{Y^v}((\nabla_X Z)^b - \frac{1}{2}(R(X, Z)u)^v) \\
&= \frac{1}{2\alpha}((\nabla_X R(u, Y)Z)^b - \frac{1}{2}(R(X, R(u, Y)Z)u)^v) \\
&\quad - \frac{1}{2\alpha}(R(u, \nabla_X Y)Z)^b - \frac{1}{2\alpha}(R(u, Y)\nabla_X Z)^b \\
&\quad + \frac{1}{2}(R(X, Z)Y)^v - \frac{1}{2\alpha}\tilde{g}(Y^v, U)(R(X, Z)u)^v \\
&\quad - \frac{1}{2\alpha}\tilde{g}((R(X, Z)u)^v, U)Y^v \\
&\quad + \frac{1+\alpha}{2\alpha}\tilde{g}((R(X, Z)u)^v, Y^v)U \\
&\quad - \frac{1}{2\alpha}\tilde{g}(Y^v, U)\tilde{g}((R(X, Z)u)^v, U)U \\
&= \frac{1}{2\alpha}((\nabla_X R)(u, Y)Z)^b + \frac{1}{2}(R(X, Z)Y)^v \\
&\quad - \frac{1}{4\alpha}(R(X, R(u, Y)Z)u)^v \\
&\quad - \frac{1}{2\alpha}\tilde{g}(Y^v, U)(R(X, Z)u)^v \\
&\quad + \frac{1+\alpha}{2\alpha}\tilde{g}((R(X, Z)u)^v, Y^v)U.
\end{aligned}$$

Here we used Corollary 4.3 to calculate $\tilde{\nabla}_{Y^v}(R(X, Z)u)^v$.

iv) For the fourth formula note that $(\tilde{\nabla}_{X^b}U)_{(p,u)} = 0$ and $X_{(p,u)}^v(f(r^2)) = 2f'(r^2)g_p(X, u)$.

$$\begin{aligned}
2\alpha\tilde{R}(X^b, Y^v)Z^v &= 2\alpha(\tilde{\nabla}_{X^b}\tilde{\nabla}_{Y^v}Z^v - \tilde{\nabla}_{Y^v}\tilde{\nabla}_{X^b}Z^v \\
&\quad - \tilde{\nabla}_{[X^b, Y^v]}Z^v) \\
&= -2\tilde{\nabla}_{X^b}(\tilde{g}(Y^v, U)Z^v - (1+\alpha)\tilde{g}(Y^v, Z^v)U \\
&\quad + \tilde{g}(Z^v, U)Y^v + \tilde{g}(Y^v, U)\tilde{g}(Z^v, U)U) \\
&\quad - \alpha\tilde{\nabla}_{Y^v}\frac{1}{\alpha}(R(u, Z)X)^b \\
&\quad - 2\alpha(\tilde{\nabla}_{Y^v}(\nabla_X Z)^v - \tilde{\nabla}_{(\nabla_X Y)^v}Z^v)
\end{aligned}$$

$$\begin{aligned}
&= -g(Y, u)\left(\frac{1}{\alpha}(R(u, Z)X)^b + 2(\nabla_X Z)^v\right) \\
&\quad -g(Z, u)\left(\frac{1}{\alpha}(R(u, Y)X)^b + 2(\nabla_X Y)^v\right) \\
&\quad + \frac{2}{\alpha}g(Y, u)(R(u, Z)X)^b \\
&\quad - \frac{1}{2\alpha}(R(u, Y)R(u, Z)X)^b - (R(Y, Z)X)^b \\
&\quad + 2(g(Y, u)(\nabla_X Z)^v + g(\nabla_X Z, u)Y^v \\
&\quad \quad - (1 + \alpha)\tilde{g}(Y^v, (\nabla_X Z)^v)U \\
&\quad \quad + g(Y, u)g(\nabla_X Z, u)U \\
&\quad \quad + g(\nabla_X Y, u)Z^v + g(Z, u)(\nabla_X Y)^v \\
&\quad \quad - (1 + \alpha)\tilde{g}((\nabla_X Y)^v, Z^v)U + g(\nabla_X Y, u)g(Z, u)U \\
&= -(R(Y, Z)X)^b - \frac{1}{2\alpha}(R(u, Y)R(u, Z)X)^b \\
&\quad + \frac{1}{\alpha}g(Y, u)(R(u, Z)X)^b - \frac{1}{\alpha}g(Z, u)(R(u, Y)X)^b.
\end{aligned}$$

To calculate $\tilde{\nabla}_{Y^v}(R(X, Z)u)^b$ we used Corollary 4.3. For the last equation we have to show that all the terms not containing the Riemannian tensor R vanish.

But

$$\tilde{g}(Y^v, (\nabla_X Z)^v)U = \frac{1}{\alpha}(g(Y, \nabla_X Z) + g(Y, u)g(\nabla_X Z))U$$

and therefore the rest becomes

$$-\frac{2}{\alpha}(g(Y, \nabla_X Z) + g(Y, u)g(\nabla_X Z, u) + g(Z, \nabla_X Y) + g(Z, u)g(\nabla_X Y, u))U.$$

This is now equal to

$$-\frac{2}{\alpha}X^h(\tilde{g}(Y^v, Z^v) + \tilde{g}(Y^v, U)\tilde{g}(Z^v, U))U$$

because the Levi-Civita connection is a metric one. The last terms are equal to zero, because we are differentiating vertical vector fields in the direction of a horizontal vector fields.

▼) Since $[X^v, Y^v] = 0$, we only have to calculate $\tilde{\nabla}_{X^v}\tilde{\nabla}_{Y^v}Z^b$.

$$\begin{aligned}
\tilde{\nabla}_{X^v}\tilde{\nabla}_{Y^v}Z^b &= \tilde{\nabla}_{X^v}\frac{1}{2\alpha}(R(u, Y)Z)^b \\
&= -\frac{1}{\alpha^2}g(X, u)(R(u, Y)Z)^b \\
&\quad + \frac{1}{4\alpha^2}(R(u, X)R(u, Y)Z)^b + \frac{1}{2\alpha}(R(X, Y)Z)^b.
\end{aligned}$$

If we now subtract the analog equation for $\tilde{\nabla}_{Y^v}\tilde{\nabla}_{X^v}Z^b$, we get

$$\begin{aligned}\tilde{R}(X^v, Y^v)Z^b &= -\frac{1}{\alpha^2}g(X, u)(R(u, Y)Z)^b + \frac{1}{4\alpha^2}(R(u, X)R(u, Y)Z)^b \\ &\quad + \frac{1}{2\alpha}(R(X, Y)Z)^b + \frac{1}{\alpha^2}g(Y, u)(R(u, X)Z)^b \\ &\quad - \frac{1}{4\alpha^2}(R(u, Y)R(u, X)Z)^b - \frac{1}{2\alpha}(R(Y, X)Z)^b.\end{aligned}$$

vi) Note that $\tilde{g}(U, U) = r^2$ and $\tilde{\nabla}_{X^v}U = \frac{1}{\alpha}(X^v + \tilde{g}(X^v, U)U)$. The last statement is true, because

$$\begin{aligned}\tilde{\nabla}_{X^v}U &= \tilde{\nabla}_{X^v}\sum_{i=1}^m v_{m+i}\left(\frac{\partial}{\partial v_{m+i}}\right) \\ &= \sum_{i=1}^m X^v(v_{m+i})\frac{\partial}{\partial v_{m+i}} \\ &\quad + \sum_{i=1}^m v_{m+i}\tilde{\nabla}_{X^v}\left(\frac{\partial}{\partial v_{m+i}}\right).\end{aligned}$$

By Lemma 2.9 the first sum is equal to X^v and the last sum is, by part iv) of Proposition 4.2, equal to $\frac{1}{\alpha}((1 - \alpha)X^v + \tilde{g}(X^v, U)U)$. Then

$$\begin{aligned}\alpha^2\tilde{\nabla}_{X^v}\tilde{\nabla}_{Y^v}Z^v &= \alpha^2\tilde{\nabla}_{X^v}\frac{1}{\alpha}(-g(Y, u)Z^v - g(Z, u)Y^v \\ &\quad - g(Y, u)g(Z, u)U + (1 + \alpha)\tilde{g}(Y^v, Z^v)U) \\ &= -\alpha^2X^v\left(\frac{1}{\alpha}\right)(g(Y, u)Z^v + g(Z, u)Y^v \\ &\quad + g(Y, u)g(Z, u)U - (1 + \alpha)\tilde{g}(Y^v, Z^v)U) \\ &\quad - \alpha(g(X, Y)Z^v + g(Y, u)\tilde{\nabla}_{X^v}Z^v + g(X, Z)Y^v \\ &\quad + g(Z, u)\tilde{\nabla}_{X^v}Y^v + g(X, Y)g(Z, u)U \\ &\quad + g(X, Z)g(Y, u)U + g(Y, u)g(Z, u)\tilde{\nabla}_{X^v}U \\ &\quad - X^v(\alpha + 1)\tilde{g}(Y^v, Z^v)U - (1 + \alpha)X^v(\tilde{g}(Y^v, Z^v))U \\ &\quad - (1 + \alpha)\tilde{g}(Y^v, Z^v)\tilde{\nabla}_{X^v}U) \\ &= 2g(X, u)g(Y, u)Z^v + 2g(X, u)g(Z, u)Y^v \\ &\quad + 2g(X, u)g(Y, u)g(Z, u)U - 2(1 + \alpha)g(X, u)\tilde{g}(Y^v, Z^v)U \\ &\quad - \alpha g(X, Y)Z^v + g(Y, u)g(X, u)Z^v + g(Y, u)g(Z, u)X^v \\ &\quad + g(Y, u)g(X, u)g(Z, u)U - (1 + \alpha)g(Y, u)\tilde{g}(X^v, Z^v)U \\ &\quad - \alpha g(X, Z)Y^v + g(Z, u)g(X, u)Y^v + g(Z, u)g(Y, u)X^v \\ &\quad + g(Z, u)g(X, u)g(Y, u)U - (1 + \alpha)g(Z, u)\tilde{g}(X^v, Y^v)U \\ &\quad - \alpha g(X, Y)g(Z, u)U - \alpha g(X, Z)g(Y, u)U\end{aligned}$$

$$\begin{aligned}
& -g(Y, u)g(Z, u)X^v - g(Y, u)g(Z, u)g(X, u)U \\
& + 2\alpha g(X, u)\tilde{g}(Y^v, Z^v)U \\
& + (1 + \alpha)\left(-\frac{2}{\alpha}g(X, u)g(Y, Z)U\right. \\
& \quad \left.- \frac{2}{\alpha}g(X, u)g(Y, u)g(Z, u)U\right. \\
& \quad + g(X, Y)g(Z, u)U + g(X, Z)g(Y, u)U \\
& \quad \left. + \tilde{g}(Y^v, Z^v)X^v + \tilde{g}(Y^v, Z^v)g(X, u)U\right) \\
= & g(Y, u)g(Z, u)X^v + (1 + \alpha)\tilde{g}(Y^v, Z^v)X^v \\
& + 3g(X, u)g(Z, u)Y^v - \alpha g(X, Z)Y^v \\
& + 3g(X, u)g(Y, u)Z^v - \alpha g(X, Y)Z^v \\
& + g(X, u)g(Y, u)g(Z, u)U - (\alpha + 3)g(X, u)\tilde{g}(Y^v, Z^v)U \\
& - g(Y, u)\tilde{g}(X, Z)U - g(Z, u)\tilde{g}(X, Y)U.
\end{aligned}$$

Therefore we get by taking the difference of $\tilde{\nabla}_{X^v}\tilde{\nabla}_{Y^u}Z^v$ and $\tilde{\nabla}_{Y^v}\tilde{\nabla}_{X^u}Z^v$:

$$\begin{aligned}
\alpha^2 \tilde{R}(X^v, Y^v)Z^v &= g(Y, u)g(Z, u)X^v + (1 + \alpha)\tilde{g}(Y^v, Z^v)X^v \\
&+ 3g(X, u)g(Z, u)Y^v - \alpha g(X, Z)Y^v \\
&+ 3g(X, u)g(Y, u)Z^v - \alpha g(X, Y)Z^v \\
&+ g(X, u)g(Y, u)g(Z, u)U - (\alpha + 3)g(X, u)\tilde{g}(Y^v, Z^v)U \\
&- g(Y, u)\tilde{g}(X, Z)U - g(Z, u)\tilde{g}(X, Y)U \\
&- g(X, u)g(Z, u)Y^v - (1 + \alpha)\tilde{g}(X^v, Z^v)Y^v \\
&- 3g(Y, u)g(Z, u)X^v + \alpha g(Y, Z)X^v \\
&- 3g(Y, u)g(X, u)Z^v + \alpha g(Y, X)Z^v \\
&- g(Y, u)g(X, u)g(Z, u)U + (\alpha + 3)g(Y, u)\tilde{g}(X^v, Z^v)U \\
&+ g(X, u)\tilde{g}(Y, Z)U + g(Z, u)\tilde{g}(Y, X)U \\
= & (\alpha^2 + \alpha + 1)(\tilde{g}(Y^v, Z^v)X^v - \tilde{g}(X^v, Z^v)Y^v) \\
&- (\alpha + 2)(g(Y, u)g(Z, u)X^v - g(X, u)g(Z, u)Y^v) \\
&- (\alpha + 2)(\tilde{g}(Y^v, Z^v)g(X, u)U - \tilde{g}(X^v, Z^v)g(Y, u)U).
\end{aligned}$$

This finishes the proof of this proposition. \square

3. Geometric Consequences

In the following let $\|\cdot\|$ denote the norm with respect to \tilde{g} and let

$$\tilde{Q}(V, W) = \|V\|^2 \|W\|^2 - \tilde{g}(V, W)^2$$

be the *square of the area of the parallelogram with sides V and W* for $V, W \in C^\infty(TTM)$.

Lemma 4.5. [18] *Let $X, Y \in C^\infty(TM)$ be two orthonormal vector fields, then*

- i) $\tilde{Q}(X^h, Y^h) = 1,$
- ii) $\tilde{Q}(X^h, Y^v) = \frac{1}{\alpha}(1 + g(Y, u)^2),$
- iii) $\tilde{Q}(X^v, Y^v) = \frac{1}{\alpha^2}(1 + g(Y, u)^2 + g(X, u)^2).$

PROOF. **i)** The first part follows directly from the first part of the definition of the Cheeger-Gromoll metric.

ii) The second part is proved by calculations similar to those in the first part

$$\begin{aligned}\tilde{Q}(X^h, Y^v) &= \tilde{g}(X^h, X^h)\tilde{g}(Y^v, Y^v) - \tilde{g}(X^h, Y^v)^2 \\ &= \frac{1}{\alpha}(1 + g(Y, u)^2).\end{aligned}$$

iii) The last part follows from

$$\begin{aligned}\tilde{Q}(X^v, Y^v) &= \tilde{g}(X^v, X^v)\tilde{g}(Y^v, Y^v) - \tilde{g}(X^v, Y^v)^2 \\ &= \frac{1}{\alpha}(1 + g(X, u)^2)\frac{1}{\alpha}(1 + g(Y, u)^2) \\ &\quad - \left(\frac{1}{\alpha}(g(X, Y) + g(X, u)g(Y, u))\right)^2 \\ &= \frac{1}{\alpha^2}(1 + g(Y, u)^2 + g(X, u)^2).\end{aligned}$$

□

Now let \tilde{G} be the $(2, 0)$ -tensor on the tangent bundle TM with

$$\tilde{G} : (V, W) \mapsto \tilde{g}(\tilde{R}(V, W)W, V)$$

for $V, W \in C^\infty(TTM)$.

In [18] Sekizawa made a mistake when calculating the term $\tilde{G}(X^v, Y^v)$. This has been corrected in Lemma 4.6 below. This means that some of the results which follow are different from those Sekizawa stated in [18]. Those which are identical to his are marked with the symbol [18] and the rest are not.

Lemma 4.6. *Let X, Y be two orthonormal vector fields in $C^\infty(TM)$, then*

- i) $\tilde{G}(X^h, Y^h) = K(X, Y) - \frac{3}{4\alpha}|R(X, Y)u|^2,$
- ii) $\tilde{G}(X^h, Y^v) = \frac{1}{4\alpha^2}|R(u, Y)X|^2,$
- iii) $\tilde{G}(X^v, Y^v) = \frac{1-\alpha}{\alpha^4}(1 + g(X, u)^2 + g(Y, u)^2) + \frac{\alpha+2}{\alpha^3}.$

PROOF. **i)** The first part is proved as follows

$$\begin{aligned}\alpha\tilde{G}(X^h, Y^h) &= \alpha\tilde{g}(\tilde{R}(X^h, Y^h)Y^h, X^h) \\ &= \alpha\tilde{g}((R(X, Y)Y)^h, X^h)\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}(\tilde{g}((R(u, R(Y, Y)u)X)^b, X^b) \\
& \quad -\tilde{g}((R(u, R(X, Y)u)Y)^b, X^b) \\
& \quad -\tilde{g}((2R(u, R(X, Y)u)Y)^b, X^b) \\
& +\frac{\alpha}{2}\tilde{g}((\nabla_Z R(X, Y)u)^v, X^b) \\
& = \alpha g(R(X, Y)Y, X) \\
& \quad +\frac{3}{4}g(R(u, R(X, Y)u)Y, X).
\end{aligned}$$

The last step of this calculation can be done in this way, because a natural metric of a vertical and a horizontal vector field vanishes and the fact that the Riemann curvature tensor is skew-symmetric. Finally we get by the rules for the Riemann curvature tensor $g(R(u, R(X, Y)u)Y, X) = |R(X, Y)u|^2$. Hence we have proved this part.

ii) The proof of the second part follows along the same line of calculations.

$$\begin{aligned}
\alpha^2 \tilde{G}(X^b, Y^v) &= \alpha^2 \tilde{g}(\tilde{R}(X^b, Y^v)Y^v, X^b) \\
&= -\frac{\alpha}{2}\tilde{g}((R(Y, Y)X)^b, X^b) \\
&\quad +\frac{2\alpha-1}{2\alpha}(\tilde{g}(Y^v, U)\tilde{g}(R(u, Y)X)^b, X^b) \\
&\quad -\frac{1}{2\alpha}\tilde{g}(Z^v, U)\tilde{g}((R(u, Y)X)^b, X^b) \\
&\quad -\frac{1}{4}\tilde{g}((R(u, Y)R(u, Y)X)^b, X^b) \\
&= -\frac{1}{4}g(R(u, Y)R(u, Y)X, X) \\
&= \frac{1}{4}|R(u, Y)X|^2.
\end{aligned}$$

Here the last two steps follow from the skew-symmetry of the Riemann curvature tensor.

iii) In the last case we calculate

$$\begin{aligned}
\alpha^2 \tilde{G}(X^v, Y^v) &= \alpha^2 \tilde{g}(\tilde{R}(X^v, Y^v)Y^v, X^v) \\
&= (1 + \alpha + \alpha^2)(\tilde{g}(Y^v, Y^v)\tilde{g}(X^v, X^v) - \tilde{g}(X^v, Y^v)^2) \\
&\quad +(\alpha + 2)(\tilde{g}(X^v, Y^v)g(Y, u)g(X, u) - \tilde{g}(Y^v, Y^v)g(X, u)^2) \\
&\quad +(\alpha + 2)(g(X, u)g(Y, u)\tilde{g}(X^v, Y^v) - g(Y, u)^2\tilde{g}(X^v, X^v)) \\
&= \frac{1 + \alpha + \alpha^2}{\alpha^2}((1 + g(X, u)^2)(1 + g(Y, u)^2)
\end{aligned}$$

$$\begin{aligned}
& -g(X, u)^2 g(Y, u)^2 \\
& + \frac{\alpha+2}{\alpha} (g(Y, u)^2 g(X, u)^2 - (1 + g(Y, u)^2) g(X, u)^2) \\
& + \frac{\alpha+2}{\alpha} (g(Y, u)^2 g(X, u)^2 - g(Y, u)^2 (1 + g(X, u)^2)) \\
& = \frac{1-\alpha}{\alpha^2} (1 + g(X, u)^2 + g(Y, u)^2) + \frac{\alpha+2}{\alpha}.
\end{aligned}$$

□

We are now able to calculate the sectional curvatures of TM .

Proposition 4.7. *Let \tilde{K} be the sectional curvature function of the tangent bundle TM equipped with the Cheeger-Gromoll metric \tilde{g} and X, Y be two orthonormal vector fields in $C^\infty(TM)$. Then we get*

$$\begin{aligned}
i) \quad & \tilde{K}(X^h, Y^h) = K(X, Y) - \frac{3}{4\alpha} |R(X, Y)u|^2, \\
ii) \quad & \tilde{K}(X^h, Y^v) = \frac{1}{4\alpha} \frac{|R(u, Y)X|^2}{1+g(Y, u)^2}, \\
iii) \quad & \tilde{K}(X^v, Y^v) = \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha} \frac{1}{1+g(X, u)^2+g(Y, u)^2}.
\end{aligned}$$

PROOF. The statements follow directly by dividing $\tilde{G}(X^i, Y^j)$ by the $\tilde{Q}(X^i, Y^j)$ for $i, j \in \{h, v\}$. □

As for the Sasaki metric we can prove formula *i)* and *ii)* of the above results for the sectional curvatures with O'Neill's formulae.

PROOF. **i)** The first equation follows by Corollary 2.22 and the definition of the Cheeger-Gromoll metric.

ii) The second equation reduces to:

$$\begin{aligned}
\tilde{K}(X^h, Y^h) \cdot \tilde{g}(Y^v, Y^v) &= \|A_{X^h} Y^v\|^2 \\
&= \|(\tilde{\nabla}_{X^h} Y^v)^h\|^2 \\
&= \frac{1}{4\alpha^2} |R(u, Y)X|^2
\end{aligned}$$

□

Corollary 4.8. *Let (M, g) be a manifold of constant sectional curvature κ . Then*

$$\begin{aligned}
i) \quad & \tilde{K}(X^h, Y^h) = \kappa - \frac{3\kappa^2}{4\alpha} (g(u, X)^2 + g(u, Y)^2), \\
ii) \quad & \tilde{K}(X^h, Y^v) = \frac{\kappa^2 g(X, u)^2}{4\alpha(1+g(Y, u)^2)}, \\
iii) \quad & \tilde{K}(X^v, Y^v) = \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha} \frac{1}{1+g(X, u)^2+g(Y, u)^2},
\end{aligned}$$

for any orthonormal vector fields $X, Y \in C^\infty(TM)$.

The next theorem differs from Sekizawa [18] in the parts *ii)* and *iii)*. He has stated, that that these parts are only non-negative, if κ is non-negative, but

it is obvious that κ has nothing to do with part *iii*) and in part *ii*) it appears only as a square.

Theorem 4.9. *Let (M, g) be a Riemannian manifold with constant sectional curvature κ . Let \tilde{K} be the sectional curvature of the tangent bundle TM equipped with Cheeger-Gromoll metric \tilde{g} and $X, Y \in C^\infty(TM)$ be two orthonormal vector fields on M . Then*

- i) $\tilde{K}(X^h, Y^h)$ is non-negative if $0 \leq \kappa \leq \frac{4}{3}$,*
- ii) $\tilde{K}(X^h, Y^v)$ is non-negative,*
- iii) $\tilde{K}(X^v, Y^v)$ is positive.*

PROOF. If we use an orthonormal basis $\{e_1, \dots, e_m\}$ of the tangent space $T_p M$ with two vectors of the basis equal to X and Y , then we get

$$g(Y, u)^2 + g(X, u)^2 \leq \sum_{i=1}^m g(e_i, u)^2 = |u|^2 = \alpha - 1 \leq \alpha.$$

Hence the theorem follows directly by Corollary 4.8. \square

Corollary 4.10. [18] *If the Riemannian manifold (M, g) is flat, then the Cheeger-Gromoll metric \tilde{g} of the tangent bundle TM has non-negative sectional curvatures, which are nowhere constant.*

We now introduce an orthonormal basis for the tangent space $T_{(p,u)} TM$ of the tangent bundle TM at the point (p, u) . Let $\{e_1, \dots, e_m\}$ be an orthonormal basis for $T_p M$, the tangent space of M at the point p , with $e_1 = \frac{u}{r}$, where r again is the norm of u with respect to the Cheeger-Gromoll metric. Then we get an orthonormal basis $\{f_1, \dots, f_{2m}\}$ for $T_{(p,u)} TM$ by setting

$$f_i = e_i^h, \quad f_{m+1} = e_1^v \quad \text{and} \quad f_{m+j} = \sqrt{\alpha} e_j^v,$$

for $i \in \{1, \dots, m\}$ and $j \in \{2, \dots, m\}$.

The following lemma is a direct consequence of Proposition 4.7.

Lemma 4.11. *Let $\{f_1, \dots, f_{2m}\}$ be an orthonormal basis for the tangent space $T_{(p,u)} TM$ of the tangent bundle TM at the point (p, u) as above and let \tilde{K} be the sectional curvatures of TM . Then the following hold*

$$\begin{aligned} \tilde{K}(f_i, f_j) &= K(e_i, e_j) - \frac{3}{4\alpha} |R(e_i, e_j)u|^2, \\ \tilde{K}(f_i, f_{m+1}) &= 0, \\ \tilde{K}(f_i, f_{m+k}) &= \frac{1}{4} |R(u, e_k)e_i|^2, \\ \tilde{K}(f_{m+1}, f_{m+k}) &= \frac{3}{\alpha^2}, \\ \tilde{K}(f_{m+k}, f_{m+l}) &= \frac{\alpha^2 + \alpha + 1}{\alpha^2}, \end{aligned}$$

for $i, j \in \{1, \dots, m\}$ and $k, l \in \{2, \dots, m\}$.

Proposition 4.12. *Let $\{f_1, \dots, f_{2m}\}$ be an orthonormal basis for the tangent space $T_{(p,u)}TM$ of the tangent bundle TM at the point (p, u) as above and let $\tilde{\sigma}$ be the scalar curvature of TM . Then*

$$\begin{aligned}\tilde{\sigma} &= \sigma + \frac{2\alpha - 3}{2\alpha} \sum_{i < j} |R(e_i, e_j)u|^2 \\ &\quad + \frac{m-1}{\alpha^2} (6 + (m-2)(\alpha^2 + \alpha + 1)).\end{aligned}$$

PROOF. By the definition of the scalar curvature we know that

$$\begin{aligned}\tilde{\sigma} &= \sum_{k,l=1}^{2m} \tilde{K}(f_k, f_l) \\ &= \sum_{i,j=1}^m \tilde{K}(f_i, f_j) + 2 \sum_{i,j=1}^m \tilde{K}(f_i, f_{m+j}) + \sum_{i,j=1}^m \tilde{K}(f_{m+i}, f_{m+j}) \\ &= \sum_{i,j=1}^m (K(e_i, e_j) - \frac{3}{4\alpha} |R(e_i, e_j)u|^2) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^m |R(u, e_j)e_i|^2 \\ &\quad + 2 \sum_{i=2}^m \frac{3}{\alpha^2} + \sum_{\substack{i,j=2 \\ i \neq j}}^m \frac{\alpha^2 + \alpha + 1}{\alpha^2} \\ &= \sigma + \frac{2\alpha - 3}{2\alpha} \sum_{i < j} |R(e_i, e_j)u|^2 \\ &\quad + \frac{m-1}{\alpha^2} (6 + (m-2)(\alpha^2 + \alpha + 1)).\end{aligned}$$

For the third step we only use the results of Lemma 4.11. For the fact that

$$\sum_{i,j=1}^m |R(e_i, e_j)u|^2 = \sum_{i,j=1}^m |R(u, e_j)e_i|^2$$

see the proof of Lemma 3.9. □

Corollary 4.13. *Let (M, g) be a Riemann manifold of constant sectional curvature κ . Let the tangent bundle TM be equipped with the Cheeger-Gromoll metric \tilde{g} . Then the scalar curvature $\tilde{\sigma}$ of TM satisfies*

$$\tilde{\sigma} = \frac{m-1}{2\alpha^2} (2\alpha^2 m\kappa + (2\alpha - 3)(\alpha - 1)\kappa^2 \alpha)$$

$$+2(6 + (m - 2)(\alpha^2 + \alpha + 1))).$$

PROOF. By Lemma 1.49 and for $i \neq j$ we get

$$\begin{aligned} |R(e_i, e_j)u|^2 &= g(\chi(g(e_j, u)e_i - g(e_i, u)e_j), \chi(g(e_j, u)e_i - g(e_i, u)e_j)) \\ &= \chi^2(\alpha - 1)(\delta_{i1} + \delta_{j1}). \end{aligned}$$

□

Corollary 4.14. [18] *If the base manifold M has constant sectional curvature χ , then its tangent bundle TM with Cheeger-Gromoll metric \tilde{g} is not (curvature) homogeneous.*

PROOF. [18] Corollary 4.13 implies that $\tilde{\sigma}$ is never constant if χ is constant. □

For a given $m \geq 2$, we are now interested in determining the sign of the scalar curvature $\tilde{S}_m(\alpha, \chi)$ as a function of $(\alpha, \chi) \in D = [1, \infty) \times \mathbb{R}$. The contour $D_0 = \{(\alpha, \chi) \in D \mid \tilde{S}_m(\alpha, \chi) = 0\}$ in the (α, χ) plane is determined by the equation

$$f(\alpha, \chi) = 0,$$

with

$$f(\alpha, \chi) = \alpha(\alpha - 1)(2\alpha - 3)\chi^2 + 2m\alpha^2\chi + 2(6 + (m - 2)(1 + \alpha + \alpha^2)).$$

We obtain first, that for any $(\alpha, \chi) \in D$ the value of $f(\alpha, \chi)$ is increasing if m is increasing and that $f(\alpha, 0)$ is positive.

If $\alpha \neq 1$ and $\alpha \neq 3/2$ and the discriminant of $f(\alpha, \chi) = 0$ is non-negative, we get the solutions

$$\chi_{\pm} = \frac{-m\alpha^2 \pm \sqrt{m^2\alpha^4 - \alpha(\alpha - 1)(2\alpha - 3)2(6 + (m - 2)(1 + \alpha + \alpha^2))}}{\alpha(\alpha - 1)(2\alpha - 3)}.$$

Which can be used to plot the contour D_0 .

When removing D_0 from D the rest falls into three connected components. The scalar curvature is positive in the component D_+ containing the point $(1, 0)$ and negative in the other two.

In Figure 1 we have plotted the contour D_0 in the (α, χ) plane for the case when $m = 3$.

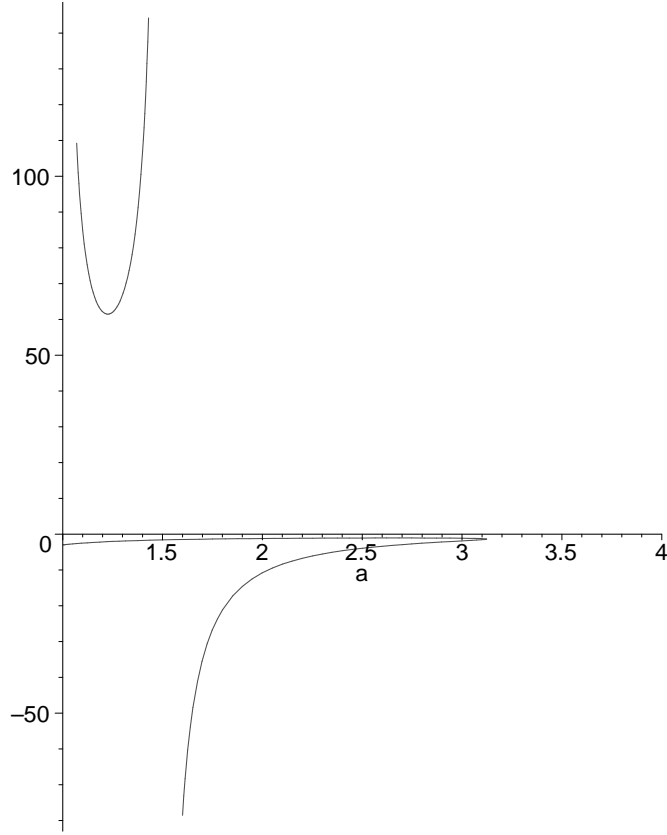


Figure 1.

We are interested in determining those $\kappa \in \mathbb{R}$ such that $S_3(\alpha, \kappa)$ is positive (non-negative) for all $\alpha \in [1, \infty)$ i.e. which are the horizontal half-lines which are completely contained in the component D_+ . The connected component of D_0 which is contained in the upper halfspace ($\kappa > 0$) is a graph of the solution κ_- given above. It has exactly one minimum $C_3 > 0$. The graph of the other solution κ_+ , where defined, has exactly one maximum $c_3 < 0$. The family of horizontal lines that we are looking for are then parametrized by $\kappa \in (c_3, C_3)$

It is easy to see that for $m > 3$ we get exactly the same qualitative behaviour of the two solutions κ_- and κ_+ as for $m = 3$. This provides us with the following result:

Theorem 4.15. *Let (M, g) be a Riemannian manifold of dimension $m \geq 3$ and of constant sectional curvature κ . Then there exist real numbers $c_m < \dots < c_3 < 0$ and $60 < C_3 < \dots < C_m$ such that the tangent bundle (TM, \tilde{g})*

- i) *has positive scalar curvature if and only if $\kappa \in (c_m, C_m)$,*
- ii) *has non-negative scalar curvature if and only if $\kappa \in [c_m, C_m]$,*

When $m = 2$ the discriminant of $f(\alpha, \kappa) = 0$ is positive everywhere, and the solutions κ_{\pm} are given by

$$\kappa_{\pm} = \frac{-2\alpha^2 \pm 2\sqrt{\alpha^4 - 3\alpha(\alpha - 1)(2\alpha - 3)}}{\alpha(\alpha - 1)(2\alpha - 3)}.$$

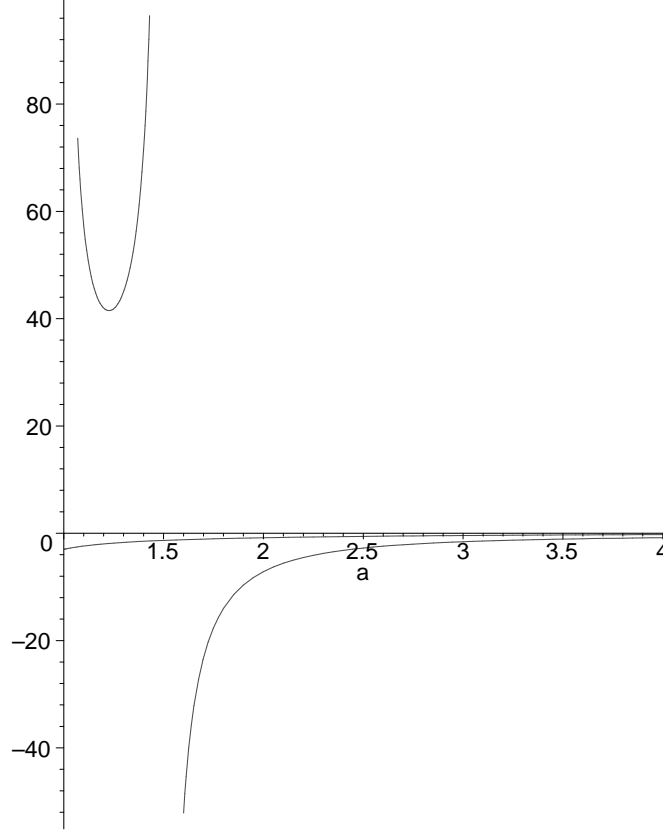


Figure 2.

In Figure 2 we have plotted the contour D_0 in the (α, κ) for $m = 2$. When this is removed from D the rest falls into four connected components. The scalar curvature is positive in the component D_+ containing the point $(2, \pm 30)$ and negative in the other two. When $\alpha > 3/2$ both the solutions κ_{\pm} are negative, approaching 0 as $\alpha \rightarrow \infty$.

This leads to the following result

Theorem 4.16. *Let (M, g) be a surface of constant sectional curvature κ . Then there exists a real number $C_2 \geq 41$ such that the tangent bundle (TM, \tilde{g})*

- i. *has positive scalar curvature if and only if $\kappa \in [0, C_2)$,*
- ii. *has non-negative scalar curvature if and only if $\kappa \in [0, C_2]$.*

Corollary 4.17. *Let (M, g) be a Riemannian manifold of dimension m and of constant sectional curvature κ . Then for large m the real number C_m is approximative $(10 + 4\sqrt{6})m$.*

PROOF. For large m the term under the root in the formula for κ_- is approximative $m^2 \alpha^4$. Hence we get

$$\kappa_- \approx \frac{-2m\alpha}{(\alpha - 1)(2\alpha - 3)}.$$

The function $\frac{\alpha}{(\alpha-1)(2\alpha-3)}$ has for $\alpha \geq 1$ its minimum at $\frac{\sqrt{6}}{2}$. If we put $\alpha = \frac{\sqrt{6}}{2}$ in the equation above, we get the result of the corollary. \square

Remark 4.18. If we compare the formula of Proposition 4.12 for the scalar curvature of the tangent bundle (TM, \tilde{g}) equipped with the Cheeger-Gromoll metric with the formula of Lemma 3.9 for the scalar curvature of (TM, \hat{g}) equipped with the Sasaki metric, we can see, that they do not differ a lot. The question in the end of this master thesis is, if in general the scalar curvature of (TM, \bar{g}) equipped with any natural metric can be written as

$$\bar{\sigma} = \sigma + A \sum_{i,j=1}^m |R(e_i, e_j)u|^2 + B,$$

where A and B are functions on the norm of u .

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