

**IV 11.4500 – UBUNG 4**

U4.1 a) Let  $X, Y \in L^2(\Omega)$  be scalar-valued rv. Show that if  $X \perp Y$ , then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

b) Let  $(X, Y) \sim U(0, 1)^2$ . Verify that  $X \perp Y$ .

c) Let  $X \sim N(\mu_X, \sigma_X^2)$  and  $Y \sim N(\mu_Y, \sigma_Y^2)$  be scalar-valued rv. Show that if  $(X, Y)$  is multivariate normal and

$$\text{Cov}[X, Y] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = 0,$$

then  $X \perp Y$ . Hint: joint pdf.

d) Let  $X \sim U(-1, 1)$  and  $Y \sim U(-1, 1)$  with  $X \perp Y$  and  $Z = X + Y$ . Compute the pdf  $\pi_Z(z)$ .

U4.2 Let  $(X, Y) \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$  with  $\rho \in (-1, 1)$ . Compute  $\pi_{X|Y}(x|y)$ .

U4.3 Let  $X, Y$  and  $Z$  be rv defined on the same probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and assume that  $X \in L^2(\Omega)$ . Show that

$$(i) \quad \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X],$$

$$(ii) \quad \mathbb{E}[X|\mathcal{V}] = X \text{ a.s., for any sigma-algebra } \mathcal{V} \text{ satisfying } \sigma(X) \subset \mathcal{V} \subset \mathcal{F},$$

(iii) if  $\sigma(Z) \subset \sigma(Y) \subset \mathcal{F}$  then

$$\mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[\mathbb{E}[X|Z]|Y] = \mathbb{E}[X|Z] \quad \text{a.s..}$$

U4.4 a) Verify that the Hellinger distance is a metric.

b) Consider an Bayesian inverse problem  $Y = G(U) + \eta$  with prior density with compact support:  $\sup\{u \in \mathbb{R}^d \mid \pi_U(u) > 0\} \leq C_U$ , and with  $\eta \sim N(0, 1)$  and an observation  $Y = y$ . Assume that  $G_\delta$  is a perturbation of the forward model  $G$  giving rise to a perturbed posterior density  $\pi^\delta(u|y)$ . State constraints on mappings  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  ensuring that

$$|\mathbb{E}^{\pi(\cdot|y)}[f] - \mathbb{E}^{\pi^\delta(\cdot|y)}[f]| \leq d_{TV}(\pi(\cdot|y), \pi^\delta(\cdot|y)),$$

and state constraints on  $G_\delta$  such that

$$d_{TV}(\pi(\cdot|y), \pi^\delta(\cdot|y)) \leq C\delta$$

for some  $C > 0$ .

c) Consider the Bayesian inverse problem

$$Y = U + \eta$$

with  $U, \eta \sim U(0, 1)$  and  $U \perp \eta$ . For  $Y = 0$ , where we assume that  $\pi_Y(y) > 0$ , compute the posterior  $\pi_{U|Y}(u|0)$ . Explain why this fails in producing a posterior density.

Hint: Check the consistency of the assumptions.

U4.5 Consider the fair coin example in Lecture 10. Let  $y = (y_1, y_2, \dots)$  be a sequence with cumulative sums  $\bar{y}_n = \sum_{k=1}^n y_k$  given by

$$\bar{y}_{10} = 4, \quad \bar{y}_{100} = 48, \quad \bar{y}_{1000} = 532, \quad \bar{y}_{10000} = 5267$$

Explore numerically the probability that  $U$  is a fair coin given these measurements.

Hint: implemented naively, you may not be able to normalize your density properly.

U4.6 a) State sufficient conditions for a density  $\pi$  to ensure that its posterior mean equals its maximum posterior, i.e.,

$$(1) \quad u_{PM}[\pi] = u_{MAP}[\pi].$$

b) Verify that (1) holds for any Gaussian pdf.