Mathematics and numerics for data assimilation and state estimation – Lecture 19

UQ Mathematics for Uncertainty Quantification | RWTH AACHEN UNIVERSITY

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Overview

Summary lecture 18

- Stochastic processes, filtrations and Wiener processes.

- Plan for today: Itô integrals, theory and numerical integration of stochastic differential equations (SDE)

$$V_t = V_0 + \int_0^t b(V_s)ds + \int_0^t \sigma(V_s)dW_s$$

Overview

Wiener process and Itô integrals

We recall that the values of the Wiener process $\{W_{t_k}\}_{t_k}$ can be sampled exactly by

$$W_{t_{k+1}} = W_{t_k} + \underbrace{W_{t_{k+1}} - W_{t_k}}_{\sim N(0,\Delta t_k)}$$

with $W_0 = 0$.

And that given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with $\mathcal{F}_t = \mathcal{F}_t^W$, the Itô integral is defined by

$$\int_0^t \sigma(V_s)dW_s := \lim_{|\Delta|\to 0} \sum_k \sigma(V_{t_k})(W_{t_{k+1}} - W_{t_k})$$

where Δ denotes a mesh/subdivision of $[0, t]$ and one assumes that both V_t and W_t are $\mathcal{F}_t$-adapted.

It remains to describe what we mean by "$=$" in the above definition.

Integrals of simple and $\mathcal{F}_t$-adapted functions

Given a mesh $\{\tau_k\}_{k=0}^n$ over an interval $[S, T]$, we consider simple functions of the form

$$\phi_n(\omega, t) := \sum_{j=1}^{n-1} e_j(\omega) \mathbb{1}_{[\tau_j, \tau_{j+1})}(t)$$

with e_j being $\mathcal{F}_{\tau_j}$-measurable. This makes also ϕ_n $\mathcal{F}_t$-measurable.

The Itô integral is given by

$$\int_S^T \phi_n(t, \omega) dW_t := \lim_{|\Delta| \to 0} \sum_k \phi_n(t_k, \omega)(W_{t_{k+1}} - W_{t_k}) = \sum_{j=0}^{n-1} e_j(\omega)(W_{\tau_{j+1}} - W_{\tau_j})$$

Motivation: Summing over a finer mesh $\Delta \supset \{\tau_k\}_{k=0}^n$ leads to telescoping sums of Wiener increments over each $\tau - \textit{interval}$: if $[t_{k_1}, t_{k_2}) = [\tau_j, \tau_{j+1})$, then $\phi_n(\cdot, \omega)|_{[\tau_j, \tau_{j+1})} = \phi_n(\tau_j, \omega)$ and

$$\sum_{k=k_1}^{k_2-1} \phi_n(t_k, \omega)(W_{t_{k+1}} - W_{t_k}) = \phi_n(\tau_j, \omega) \sum_{k=k_1}^{k_2-1} (W_{t_{k+1}} - W_{t_k}) = e_j(\omega)(W_{\tau_{j+1}} - W_{\tau_j})$$

Properties of simple-function stochastic integrals

Since $e_j(\omega)$ is $\mathcal{F}_{\tau_j}$-measurable, it turns out that

$$e_j \perp \Delta W_k := W_{\tau_{k+1}} - W_{\tau_k} \quad \text{for any } k \geq j,$$

(since $\mathcal{F}_{\tau_j} \perp \sigma(\{W_s - W_{\tau_j}\}_{s \geq \tau_j})$).

Property 1: The Itô integral has mean zero:

$$\mathbb{E}\left[\int_S^T \phi_n(t,\cdot)dW_t\right] = \sum_{j=0}^{n-1} \mathbb{E}\left[\, e_j(\cdot)\Delta W_j\right] = \sum_{j=0}^{n-1} \mathbb{E}\left[\, e_j(\cdot)\right]\mathbb{E}\left[\,\Delta W_j\right] = 0$$

Property 2: Itô isometry:

$$\mathbb{E}\left[\left(\int_S^T \phi_n(t,\cdot)dW_t\right)^2\right] = \mathbb{E}\left[\int_S^T \phi_n^2(t,\cdot)dt\right]$$

Indepencence of $\sigma-$algebras vs rv [cf. Durrett]

Given two rv on $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ and $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ defined on the same probability space, we recall that

$$X \perp Y \iff \mathbb{P}(X^{-1}(B_1) \cap Y^{-1}(B_2)) = \mathbb{P}(X^{-1}(B_1))\mathbb{P}(Y^{-1}(B_2)) \quad \forall B_1, B_2 \in \mathcal{B}.$$

The independence condition is equivalent to

$$\mathbb{P}(C_1 \cap C_2) = \mathbb{P}(C_1)\mathbb{P}(C_2) \qquad \forall C_1 \in \sigma(X) \text{ and } C_2 \in \sigma(Y),$$

since any $C_1 \in \sigma(X)$ can be written $C_1 = X^{-1}(B_1)$ for some $B_1 \in \mathcal{B}$ and any $C_2 \in \sigma(Y)$, $C_2 = Y^{-1}(B_2)$ for some $B_2 \in \mathcal{B}$.

Equivalence $\perp$ of rv and $\perp$ of σ-algebras: $X \perp Y \iff \sigma(X) \perp \sigma(Y)$.

This naturally extends to point evaluations etc of stochastic processes. E.g.,

$$e_j \perp \Delta W_j \iff \sigma(e_j) \perp \sigma(\Delta W_j)$$

And this holds since since $\sigma(e_j) \subset \mathcal{F}_{\tau_j} \perp \sigma(\{W_s - W_{\tau_j})\}_{s \geq \tau_j}) \supset \sigma(\Delta W_j)$.

Proof:

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_S^T \phi_n(t,\cdot)dW_t\right)^2\right] &= \mathbb{E}\left[\sum_{j,k} e_j e_k \Delta W_j \Delta W_k\right] \\
&= \sum_j \mathbb{E}\left[e_j^2 \Delta W_j^2\right] + 2\sum_{j<k} \mathbb{E}\left[\sum_{j,k} e_j e_k \Delta W_j \Delta W_k\right] \\
&= \sum_j \mathbb{E}\left[e_j^2\right] \mathbb{E}\left[\Delta W_j^2\right] + 2\sum_{j<k} \mathbb{E}\left[e_j e_k \Delta W_j\right] \mathbb{E}\left[\Delta W_k\right] \\
&= \sum_j \mathbb{E}\left[e_j^2\right](\tau_{j+1} - \tau_j) \\
&= \mathbb{E}\left[\int_S^T \phi_n^2(t,\cdot)dt\right]
\end{aligned}
$$

Where we used that $e_j \perp \Delta W_j$ and that for $k > j$, $e_j e_k \Delta W_j \perp \Delta W_k$ (since $\mathcal{F}_{\tau_k} \perp \sigma(\{W_s - W_{\tau_k}\}_{s \geq \tau_k})$.

We next extend the definition to more general integrands:

Definition 1

Let $\mathcal{V}[S, T]$ be the class of functions $f(t,\omega) \in \mathbb{R}$ that satisfying

- $f : [S, T] \times \Omega \to \mathbb{R}$ is $\mathcal{B} \times cF$-measurable (i.e., $f^{-1}(B) \in \mathcal{B} \times \mathcal{F}$ for any $B \in \mathbb{R}$)
- f is $\mathcal{F}_t$-adapted, (i.e., $f(t,\cdot)$ is $\mathcal{F}_t$-measurable for each $t \in [S, T]$)
- $f \in L^2(\Omega; L^2[S, T])$ meaning $\mathbb{E}^\omega\left[\int_S^T f^2(t,\omega)dt\right] < \infty$.

[ELV-E 7] For any $f \in \mathcal{V}[S, T]$ there exists a sequence of simple fcns $\{\phi_n\} \subset \mathcal{V}[S, T]$ such that

$$\|f - \phi_n\|^2_{L^2(\Omega;L^2[S,T])} = \mathbb{E}\left[\int_S^T (\phi_n(t,\cdot) - f(t,\cdot))^2 \, dt\right] \to 0 \quad \text{as } n \to \infty.$$

This implies that $\{\phi_n\}$ is Cauchy in the Banach space $L^2(\Omega; L^2[S, T])$.

Definition of Itô integral

We define
$$\int_S^T f(t,\omega)dW_t \overset{L^2(\Omega)}{:=} \lim_{n\to\infty}\int_S^T \phi_n(t,\omega)dW_t$$

This limit exists, since by Itô isometry,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_S^T \phi_n(t,\cdot)dW_t - \int_S^T \phi_m(t,\cdot)dW_t\right)^2\right] \\
&\quad = \mathbb{E}\left[\left(\int_S^T \phi_n(t,\cdot) - \phi_m(t,\cdot)dW_t\right)^2\right] \\
&\quad = \mathbb{E}\left[\int_S^T (\phi_n(t,\cdot) - \phi_m(t,\cdot))^2 dt\right] \\
&\quad = \|\phi_n - \phi_m\|^2_{L^2(\Omega;L^2[S,T])} \to 0 \quad \text{as } m,n\to\infty.
\end{aligned}$$

Properties of the Itô integral

For $f, g \in \mathcal{V}[S, T]$ and $u \in [S, T]$, the following integral properties extend from simple-function setting:

- Mean zero: $\mathbb{E}\left[\int_S^T f dW_t\right] = 0$,
- Itô isometry: $\mathbb{E}\left[\left(\int_S^T f dW_t\right)^2\right] = \mathbb{E}\left[\int_S^T f^2 dt\right]$,
- partition of integral: $\int_S^T f dW_t \overset{a.s.}{=} \int_S^u f dW_t + \int_u^T f dW_t$,
- for any scalar $c \in \mathbb{R}$, $\int_S^T f + cg dW_t \overset{a.s.}{=} \int_S^T f dW_t + c \int_S^T g dW_t$,
- $\int_S^T f dW_t$ is $\mathcal{F}_T$-measurable.

Overview

Definition 2 (1-D Itô process)

Given a Wiener process W_t defined on $(\Omega, \mathcal{F}, \mathbb{P})$, an Itô process over $[0, T]$ is defined by

$$X_t := X_0 + \int_0^t b(s,\omega)ds + \int_0^t \sigma(s,\omega)dW_s$$

where $\sigma \in \mathcal{V}[0, T]$ and $b : \Omega \times [0, T] \to \mathbb{R}$ is $\mathcal{F}_t$-adapted and $\int_0^T |b(t,\omega)|dt < \infty$ for a.a. ω. Or, equivalently,

$$dX_t := b(s,\omega)dt + \sigma(t,\omega)dW_t, \quad X_t|_{t=0} = X_0.$$

Question: For an Itô process X_t and $f \in C^2(\mathbb{R})$, what is the "Itô chain rule" for computing $df(X_t) =?$,

The classic chain rule yields: $$df(X_t) = f'(X_t)dX_t + \underbrace{\frac{1}{2}f''(X_t)dX_t^2 + \dots}_{h.o.t.}$$

but since X_t has less regularity than in classic settings, it turns out that some "classic h.o.t." needs to be reclassified as leading order.

Quadratic variation of the Wiener process

The quadtratic variation of W_t over $[0, T]$ can be shown to satisfy

$$[W, W]_t := \lim_{|\Delta|\downarrow 0} \sum_k (W_{t_{k+1}} - W_{t_k})^2$$

It can be shown that

$$[W, W]_t \overset{L^2(\Omega)}{=} t \quad \text{meaning} \quad \mathbb{E}\left[([W, W]_t - t)^2 \right] = 0.$$

We employ this property to motivate the following Itô integration:

$$\int_0^t W_s dW_s \approx \sum_j W_{t_j}(W_{t_{j+1}} - W_{t_j}) = \dots$$

$$= \frac{W_t^2}{2} - \frac{1}{2} \sum_j (W_{t_{j+1}} - W_{t_j})^2 \to \frac{W_t^2}{2} - \frac{t}{2}.$$

This corresponds to the differential equation

$$W_t dW_t = \frac{dW_t^2}{2} - \frac{dt}{2} \quad \text{or equivalently} \quad dW_t^2 = 2W_t dW_t + t$$

Note that this is different from the classic chain rule: $dW_t^2 = 2W_t dW_t$.

Theorem 3 (ELV-E 7.6)

Assume $f \in \mathcal{V}[0,T]$ is bounded and continuous for $t \in [0,T]$ for almost all ω. Then, in probability,

$$\lim_{|\Delta|\downarrow 0} \sum_j f(t_j^*, \omega)(W_{t_{j+1}} - W_{t_j})^2 = \int_0^T f(s,\omega)ds$$

for any choice $t_j^* \in [t_j, t_{j+1}]$

This motivates formally writing $dW_t^2 = dt$, and by introducing the additional formal h.o.t. rules

$$dt^2 = 0, \quad \text{and} \quad dtdW = dWdt = 0$$

we derive for the Itô process

$$dX_t = b(s,\omega)dt + \sigma(t,\omega)dW_t, \quad X_t|_{t=0} = X_0,$$

and $f \in C^2(\mathbb{R})$, the **1D Itô's formula**:

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)dX_t^2 = \left(f'(X_t)b + \frac{1}{2}f''(X)\sigma^2\right)dt + f'(X_t)dW_t. \tag{1}$$

Application of Itô's formula

To evaluate

$$X_t = \int_0^t W_s dW_s$$

consider the detour of introducing $f(x) = x^2/2$ and noting that

$$X_t = \int_0 f'(W_s) dW_s.$$

Next, apply Itô's formula to $Y_t = f(W_t)$:

$$dY_t = f'(W_t) dW_t + \frac{1}{2} f''(W_t)(dW_t)^2 = W_t dW_t + \frac{dt}{2}.$$

Integrating both sides yields,

$$W_t^2 = \int_0^t W_s dW_s + \frac{t}{2} \implies X_t = W_t^2 - \frac{t}{2}.$$

Itô integrals in higher dimensions

Multidimensional Itô integrals of the form

$$\int_0^T \sigma(t,\omega)dW_t$$

where

- each component of $\sigma : [0, T] \times \Omega \to \mathbb{R}^{d\times n}$ belongs to the function space $\mathcal{V}[0, T]$ and
- the components of $W_t : \Omega \times [0, T] \to \mathbb{R}^n$ are independent Wiener processes.

See [ELV-E 7.2] for more details on this and Itô's formula in higher dimensions.

Overview

Existence and uniqueness of Itô SDE

Theorem 4 (ELV-E 7.14)

For the Itô SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad \text{for} \quad t \in [0,T], \quad X_t|_{t=0} = X_0$$

with coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ *and* W *an* $n-$*dimensional Wiener process, assume that for some* $k > 0$ *that*

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq K|x-y|$$
$$|b(x)|^2 + |\sigma(x)|^2 \leq K(1+|x|^2)$$

for all $x, y \in \mathbb{R}^d$ *and that* $X_0 \in L^2(\Omega)$ *is independent from the history of the Wiener paths:* $\sigma(X_0) \perp \mathcal{F}^W_T$. *Then there exists a unique solution* $X \in L^2(\Omega; L^2[0,T])$ *satisfying* $X \in \mathcal{V}[0,T]$ *for each component.*

Remark: Unless X_0 is deterministic, the filtration must be augmented $\mathcal{F}_t = \mathcal{F}^W_t \vee \sigma(X_0) = \sigma(X_0, \{W_s\}_{s\leq t})$.

Proof ideas:

Existence: can be derived through a Picard iteration argument:

$$X_t^{(k+1)} = X_0 + \int_0^t b(X_s^{(k)})\, ds + \int_0^t \sigma(X_s^{(k)}) dW_s$$

and $X_t^{(0)} := X_0$.

Uniqueness in $L^2(\Omega; L^2[0,T])$: Given a pair of solutions $X, \hat{X}$, Itô isometry and the regularity of the coefficients yield

$$\begin{aligned}\mathbb{E}\left[|X_t - \hat{X}_t|^2\right] &\leq 2\mathbb{E}\left[\left(\int_0^t b(X_s) - b(\hat{X}_s) ds\right)^2\right] \\ &\quad + 2\mathbb{E}\left[\int_0^t (\sigma(X_s) - \sigma(\hat{X}_s))^2 ds\right] \\ &\leq 2K^2(1+t)\int_0^t \mathbb{E}\left[|X_s - \hat{X}_s|^2\right] ds\end{aligned}$$

By Grönwall's inequality, $X_t \overset{a.s.}{=} \hat{X}_t$ for all $t \in [0,T] \cap \mathbb{Q}$. Result follows by the (a.s.) continuity of solutions.

Example: Geometric Brownian Motion

$$dN_t = rN_t dt + \alpha N_t dW_t, \qquad N_t|_{t=0} = N_0$$

N_t the non-negative price of an asset, $r, \alpha >$ interest rate and volatility. Assuming $N_t > 0$ (once $N_t = 0$, it will remain 0-valued),

$$\frac{dN_t}{N_t} = rdt + \alpha dW_t,$$

Applying Ito's formula to $Y_t = \log(N_t)$ yields

$$\begin{aligned} d\log(N_t) &= \frac{1}{N_t} dN_t - \frac{1}{2N_t^2}(dN_t)^2 \\ &= \frac{rN_t dt + \alpha N_t dW_t}{N_t} - \frac{N_t^2 alpha^2 dt}{2N_t^2} \\ &= (r - \alpha^2/2)dt + \alpha dW_t \end{aligned}$$

and thus

$$N_t = N_0 e^{(r-\alpha^2/2)t + \alpha W_t}.$$

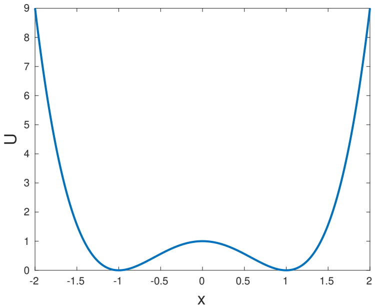
Langevin equation

$$
\begin{aligned}
dX_t &= V_t \\
mdV &= (-\gamma V_t - U'(X_t))dt + \sigma dW_t
\end{aligned}
$$

Particle-velocity system (X, V) in a force field potential $U : \mathbb{R} \to \mathbb{R}$.
Friction coefficient γ, σ - magnitude of noise force

This is a "stochastic version" the newtonian dynamics

$$
\begin{aligned}
\dot{x} &= v \\
m\dot{v} &= -U'(x)
\end{aligned}
$$



Potentials with local minima lead to pseudo-stable states for X_t.

Overview

The kernel density for SDE

Our plan is to study filtering problems

$$V_{j+1} = \Psi(V_j) := V_j + \int_0^1 b(V_{j+s})dt + \int_0^1 \sigma(V_{j+s})dW_s^{(j)}$$
$$Y_{j+1} = h(V_{j+1}) + \eta_{j+1}$$

where $W^{(j)}$ are independent Wiener processes.

The Bayes filter for this problem takes the form

$$\pi(v_{j+1}|y_{1:j+1}) \propto \pi(y_{j+1}|v_{j+1}) \int_{\mathbb{R}^d} \pi(v_{j+1}|v_j)\pi(v_j|y_{1:j})dv_j$$

with $\pi_{V_{j+1}|V_j}(x|y)$ equal to the kernel density for $t \in (0,1]$,

$$p(t,x|y) = \frac{\mathbb{P}(V_{j+t} \in dx|V_j \in dy)}{dx} = \frac{\mathbb{P}(V_t \in dx|V_0 \in dy)}{dx}$$

(due to the time-independent coefficients the SDE is stationary).

The density of an SDE

Consider the 1D SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW, \quad X_0 \sim p(0,x)$$

and assume that the density $p(t,x) = \mathbb{P}(X_t \in dx)/dx$ exists for any $t > 0$.

Recall that for any $f \in C_C^2(\mathbb{R})$ (mapping with compact support),

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 = (f'b + \sigma^2/2f'')dt + f'\sigma dW_t$$

By integration,

$$f(X_t) - f(X_0) = \int_0^t (bf' + \frac{\sigma^2}{2}f'')(X_s)ds + \int_0^t (f'\sigma)(X_s)dW_s.$$

Taking the expectation, and recalling that Itô integrals are mean-zero,

$$\mathbb{E}\left[f(X_t) - f(X_0)\right] = \int_0^t \mathbb{E}\left[(bf' + \frac{\sigma^2}{2}f'')(X_s)\right] ds$$

Note: expectation is wrt the density $p(s,x)$

Fokker-Planck equation

$$\int_{\mathbb{R}} f(x)(p(t,x)-p(0,x))dx = \int_0^t \int_{\mathbb{R}} \left[b(x)f'(x)+\sigma^2(x)\frac{f''(x)}{2}\right]p(s,x)dxds$$

Integration by parts, using the compact support of f (and its derivatives), we obtain

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}} f(x)p_t(s,x)dxds \\ &= \int_0^t \int_{\mathbb{R}} f(x)\Big[-\partial_x\Big(b(s)p(s,x)\Big) + \partial_{xx}\Big(\frac{\sigma^2(x)}{2}p(s,x)\Big)\Big]dxds \quad \forall f \in C_C^2(\mathbb{R})\end{aligned}$$

Conclusion: The density $p(t,x) = \mathbb{P}(X_t \in dx)/dx$ must satisfy the **Fokker-Planck** PDE

$$\begin{aligned} p_t &= \partial_x(-bp) + \partial_{xx}(\frac{\sigma^2}{2}p) \quad (t,x) \in [0,T] \times \mathbb{R} \\ p(t,x)|_{t=0} &= p(0,x). \end{aligned} \tag{2}$$

If the SDE coefficients are sufficiently smooth and $\sigma > 0$, then (3) is well-posed and a classical solution exists for all $t > 0$.

Fokker-Planck for kernel densities

The PDE extends to kernel densities $p(t,x|y) = \mathbb{P}(X_t \in dx | y \in dy)/dx$:

$$\begin{aligned} & p_t(\cdot,\cdot|y) = \partial_x(-bp(\cdot,\cdot|y)) + \partial_{xx}(\frac{\sigma^2}{2}p(\cdot,\cdot|y)) \quad (t,x) \in [0,T] \times \mathbb{R} \\ & p(0,x|y) = \delta_y(x). \end{aligned} \tag{3}$$

Remarks: The operator

$$(\mathcal{L}^* p)(x) := \partial_x(-bp)(x) + \partial_{xx}(\frac{\sigma^2}{2}p)(x)$$

may be associated to the transition function of Markov chains (here denoted P):

$$p(t+\Delta t, x) \approx p(t,x) + \Delta t(\mathcal{L}^* p)(x),$$

vs

$$\pi_i^{n+1} = \sum_{j=1}^{N} P_{ji}\pi_j^n = \pi_i^n + \Big((P-I)^T \pi^n)\Big)_i$$

And just like Markov chains, SDE may have stationary distributions:

$$\mathcal{L}^* p = 0 \iff p \text{ stationary }, \qquad (P-I)^T\pi = 0 \iff \pi \text{ stationary}.$$

Application in filtering

Returning to the filtering problem

$$V_{j+1} = \Psi(V_j) := V_j + \int_0^1 b(V_{j+s})dt + \int_0^1 \sigma(V_{j+s})dW_s^{(j)}$$
$$Y_{j+1} = h(V_{j+1}) + \eta_{j+1}$$

the iterative Bayes filter equation

$$\pi(v_{j+1}|y_{1:j+1}) \propto \pi(y_{j+1}|v_{j+1})\pi(v_{j+1}|y_{1:j})$$

can be written

$$\pi(v_{j+1}|y_{1:j+1}) \propto \pi(y_{j+1}|v_{j+1})p(1, v_{j+1})$$

where p solves

$$p_t = \mathcal{L}^* p \qquad (t, x) \in [0, T] \times \mathbb{R}$$
$$p(t, x)|_{t=0} = \pi_{V_j|Y_{1:j}}(x|y_{1:j})$$

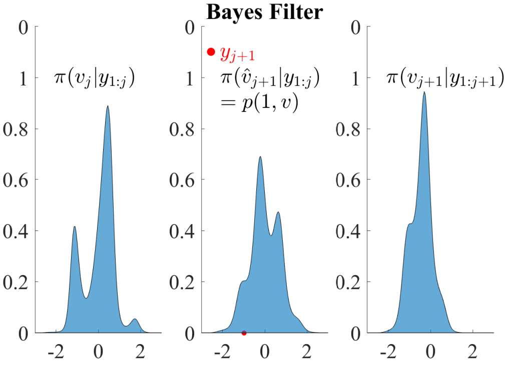
Conclusion: In principle we can solve these filtering problems exactly!

Example

Filtering problem:

$$V_{j+1} = V_j + \int_0^1 U'(V_{j+s})dt + \int_0^1 dW_s^{(j)}$$
$$Y_{j+1} = V_{j+1} + \eta_{j+1}$$

with $U(x) = x^2/2 + 0.15\sin(2\pi x)$ and for some j, suppose $\pi(v_j|y_{1:j}) \propto \exp\big(-2U(v_j) + \sin(4v_j)\big)$.



Overview

Euler–Maruyama scheme

For the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \quad t \in [0, T], \qquad X_t|_{t=0} = X_0,$$

the Euler-Maruyama scheme, computing a numerical solution $\bar{X}_j \approx X_{j\Delta t}$, is defined by

$$\bar{X}_{j+1} = b(\bar{X}_j)\Delta t + \sigma(\bar{X}_j)\Delta W_j$$

where $\Delta W_j = W_{(j+1)\Delta t} - W_{j\Delta t}$ and $\bar{X}_0 = X_0$.

Motivation:

$$\begin{aligned} X_{(j+1)\Delta t} - X_{j\Delta t} &= \int_{j\Delta t}^{(j+1)\Delta t} b(X_t)dt + \int_{j\Delta t}^{(j+1)\Delta t} \sigma(X_t)dW_t \\ &\approx \int_{j\Delta t}^{(j+1)\Delta t} b(X_{j\Delta t})dt + \int_{j\Delta t}^{(j+1)\Delta t} \sigma(X_{j\Delta t})dW_t \end{aligned}$$

Let $\bar{X}_t := LinInterp(t; \{(j\Delta t, \bar{X}_j)\}_{j=0}^{T/\Delta t})$.

Strong and weak convergence of numerical methods

Under the coefficient assumptions in Theorem 4, the Euler–Maruyama satisfies

$$\sqrt{\max_{t\in[0,T]} \mathbb{E}\left[\,|\bar{X}_t - X_t|^2\right]} \leq C\Delta t^{1/2}$$

for some $C > 0$. We say that Euler-Maruyama converges strongly with rate $1/2$.

Under more restrictive regularity conditions, it also holds that

$$\max_{t\in[0,T]} \left|\mathbb{E}\left[\, f(\bar{X}_t) - f(X_t)\right]\right| \leq C_f \Delta t$$

for any mapping $f \in C_b^\infty(\mathbb{R})$ with $C_f > 0$ depending on f. We say that Euler–Maruyama converges weakly with rate 1.

Remark: See [ELV-E 7, 8] for extensions of results to higher-dimensional state space, and higher order numerical methods.

Summary

- Have introduced stochastic integrals and differential equations.
- The density of SDE is described by the Fokker-Planck equation.
- SDE extend the previously studied dynamics $\Psi(V_j) + \xi_j$ in many ways:
  1. the dynamics may now be nonlinear in both the drift and the diffusion coefficient,
  2. the noise enters in a more general way (not only as additive noise) through the diffusion coefficient,
  3. the dynamics is now continuous ... so one may generalize observation frequency as well.
- Next time: Filtering problems with SDE dynamics.