

Mathematics and numerics for data assimilation and state estimation – Lecture 4



Summer semester 2020

Summary of lecture 3

- Probability of G given H for events $G, H \in \mathcal{F}$:

$$\mathbb{P}(G \mid H) = \frac{\mathbb{P}(G \cap H)}{\mathbb{P}(H)}$$

where we use the division-by-zero convention $c/0 := 0$ whenever $\mathbb{P}(H) = 0$

- Probability of $X = a$ given Y for rv X, Y :

$$\mathbb{P}(X = a \mid Y)(\omega) = \mathbb{P}(X = a \mid \{Y = Y(\omega)\})$$

Summary of lecture 3

- Expectation of discrete rv $X : \Omega \rightarrow A$ given $H \in \mathcal{F}$:

$$\mathbb{E}[X | H] = \sum_{a \in A} a \mathbb{P}(X = a | H) = \frac{\mathbb{E}[X \mathbb{1}_H]}{\mathbb{P}(H)}$$

- Expectation of X given the rv Y :

$$\mathbb{E}[X | Y](\omega) = \mathbb{E}[X | \{Y = Y(\omega)\}]$$

- Optimal approximation property: Interesting property

$$\mathbb{E}[|X - \mathbb{E}[X | Y]|^2] = \mathbb{E}[|X - f(Y)|^2]$$

for any mapping $f(Y) \in \mathbb{R}^d$.

Last slides of lecture 3

For $X : \Omega \rightarrow A \subset \mathbb{R}^d$ and $Y : \Omega \rightarrow B$, the mapping

$$g(b) := \mathbb{E}[X \mid Y = b]$$

satisfies

$$g(Y(\omega)) := \mathbb{E}[X \mid Y = Y(\omega)].$$

Conclusion: $\mathbb{E}[X \mid Y]$ is an rv induced from the rv Y through the mapping g .

Question: Is $\mathbb{E}[X \mid Y]$ in some sense unique?

Question: Given a candidate mapping $g : B \rightarrow \mathbb{R}^d$, is there a way to verify whether $g(Y) = \mathbb{E}[Y \mid X]$?

Definition 1 (\mathbb{P} -almost surely equal)

Two rv X, Y are said to be \mathbb{P} -almost surely equal provided

$$\mathbb{P}(\{\omega \in \Omega \mid X(\omega) = Y(\omega)\}) = 1.$$

We write

$$X = Y \quad \mathbb{P} - a.s.$$

(or just “a.s.” whenever it is clear which probability measure \mathbb{P} is considered).

Motivation:

Example 2

$X : \Omega \rightarrow \{0, 1\}$ and $Y : \Omega \rightarrow \{0, 1, 2\}$ with

$$\mathbb{P}(X = Y) = 1 \quad \text{and} \quad \{Y = 2\} \neq \emptyset.$$

Then $X(\omega) \neq Y(\omega)$ for any $\omega \in \{Y = 2\}$, but $X = Y$ a.s.

Theorem 3

Consider discrete rv $X : \Omega \rightarrow A \subset \mathbb{R}^d$ and $Y : \Omega \rightarrow B$. If $g : \mathbb{R}^k \rightarrow \mathbb{R}^d$ is a mapping such that for every bounded mapping $f : \mathbb{R}^k \rightarrow \mathbb{R}$,

$$\mathbb{E}[f(Y)g(Y)] = \mathbb{E}[f(Y)X] \quad (1)$$

then

$$g(Y) = \mathbb{E}[X | Y] \quad \text{a.s.}$$

Interpretation: $\mathbb{E}[X | Y]$ is a a.s. unique rv of form $g(Y)$ satisfying (1).

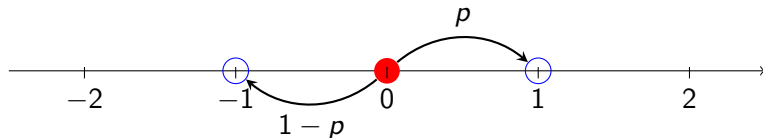
Usage: If a mapping $B \ni b \mapsto g(b) \in \mathbb{R}^d$ satisfies (1), i.e.,

$$\sum_{b \in B} f(b)g(b)P(Y = b) = \sum_{a \in A, b \in B} f(b)aP(X = a, Y = b) \quad \forall f : B \rightarrow \mathbb{R},$$

then $g(Y(\omega)) = \mathbb{E}[X|Y](\omega)$ for \mathbb{P} -almost all $\omega \in \Omega$.

Plan for this lecture

- Properties of Random walks (steps, symmetry, recurrence)



- Convergence of random variables

Random walks

- Are sequences of rv $\{X_n\}$ taking values on the lattice \mathbb{Z}^d for some $d \geq 1$.
- The subindex n can be associated to discrete time, and \mathbb{Z}^d to discrete space (really discrete state-space).

Definition 4 (Random walk (RW))

$X_n : \Omega \rightarrow \mathbb{Z}^d$ for $n = 0, 1, \dots$ is an RW if the sequence of steps $\Delta X_n := X_{n+1} - X_n$ is identically distributed and

$X_0, \Delta X_1, \Delta X_2, \dots$ are independent.

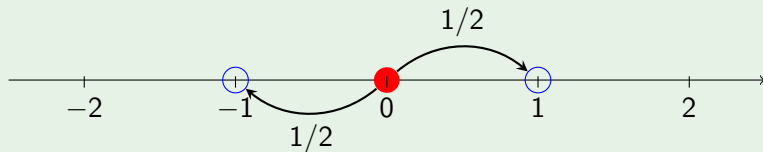
Random walk 2

Since $\{\Delta X_n\}$ are iid, an RW is defined by the two distributions:

- the initial state $\mathbb{P}_{X_0}(z) = \mathbb{P}(X_0 = z)$
- the step $\mathbb{P}_{\Delta X_0}(z) = \mathbb{P}(\Delta X_0 = z)$

Example 5 (Simple and symmetric RW on \mathbb{Z}^1)

Let $X_0 = 0$ and $\mathbb{P}(\Delta X_0 = \pm 1) = 1/2$, and **let us compute** $\mathbb{P}(X_n = k)$.



Solution:

Observe that the sequence $Y_k := \mathbb{1}_{\{\Delta X_k = 1\}} \sim \text{Bernoulli}(1/2)$ is iid and satisfies

$$\Delta X_k = 2Y_k - 1$$

Consequently,

$$X_n = X_0 + \sum_{k=0}^{n-1} \Delta X_k =$$

Symmetric random walks

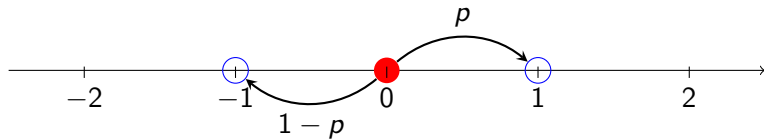
For rv X and Y , we introduce notation $X \stackrel{D}{=} Y$ to say that X and Y are identically distributed.

Definition 6 (Symmetric RW)

An RW on \mathbb{Z}^d is called symmetric if the step and the “reverse step” are identically distributed, meaning

$$X_1 - X_0 \stackrel{D}{=} X_0 - X_1.$$

Intuition: Equally likely to step in opposite directions.



The above RW symmetric if and only if $p = 1/2$.

Simple RW

Definition 7 (Simple RW)

An RW on \mathbb{Z}^d is called **simple** if the values of the step ΔX_0 belong to the set $\{e_k\}_{k=1}^d$ of canonical basis vectors in \mathbb{R}^d . In other words,

$$\{X_n\} \text{ is simple} \iff \mathbb{P}(|\Delta X_0| = 1) = 1.$$

Furthermore, an RW is called **simple symmetric** if

$$\mathbb{P}(\Delta X_0 = e_k) = \mathbb{P}(\Delta X_0 = -e_k) = \frac{1}{2d}, \quad k = 1, 2, \dots, d.$$

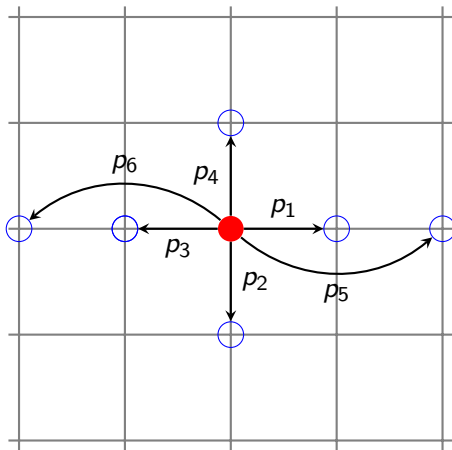
Example

Consider RW with steps satisfying

$$\sum_{i=1}^6 p_i = 1.$$

Constraints for the RW being

- symmetric?
- simple?
- simple symmetric?



Matlab implementation of simple symmetric RW on \mathbb{Z}^2

Core idea $X_{n+1} = X_n + \Delta X_n$ where

$$\mathbb{P}(\Delta X_n = \pm e_1) = \mathbb{P}(\Delta X_n = \pm e_2) = 1/4.$$

Use **randi(4)** in matlab to draw random integer in $[1, 4]$, all integers with same probability, and assign walk direction from drawn integer.

```
% Four step directions
Step = [-1 0; 1 0; 0 1; 0 -1];

for n = 1:200
    direction = randi(4);
    dX = Step(direction, :);
    plot([X(1) X(1)+dX(1)], [X(2), X(2)+dX(2)])
    X = X+dX;
end
```

See **randWalk2d.m** for more details.

Recurrence and transience

Definition 8

An RW on \mathbb{Z}^d is **recurrent** if it (over its whole path $\{X_n\}_{n \in \mathbb{N}}$) visits its initial state infinitely often \mathbb{P} -almost surely, and **transient** otherwise (i.e., if it visits its initial state only a finite number of times \mathbb{P} -almost surely).

- Description of a quasi-stable property: assume you are gambling, you win with probability $\mathbb{P}(\Delta X_n = 1) = p$ lose with $\mathbb{P}(\Delta X_n = -1) = 1 - p$. Unless $p = 1/2$, $\{X_n\}$ is transient.
- Recurrence is a form of quasi-periodic behavior. In some settings (but not for RW) it connects spatial distribution of limit processes and time-averages over path realizations

$$\mathbb{P}(X_\infty = y) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^N \mathbb{1}_{X_n=y}.$$

Theorem 9

Consider an RW on \mathbb{Z}^d with $X_0 = 0$ and let

$$T := \inf\{n \geq 1 \mid X_n = 0\}$$

with the convention that $\inf \emptyset := \infty$ and

$$N := \sum_{n \in \mathbb{N}} \mathbb{1}_{X_n=0} \quad (\text{total visits of origin})$$

Then $\{X_n\}$ is recurrent if and only if $\lambda := \mathbb{P}(T < \infty) = 1$ and for $j \in \mathbb{N} \cup \{\infty\}$,

$$\mathbb{P}(N = j) = \begin{cases} (1 - \lambda)\lambda^{j-1} & \text{if } \lambda < 1 \\ \mathbb{1}_{j=\infty} & \text{if } \lambda = 1 \end{cases}$$

Note that $N : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$.

Proof of Theorem 9

Define $\tau_0 = 0$, and

$$\tau_{k+1} = \inf\{n > \tau_k \mid X_n = 0\} \quad \text{for } k = 0, 1, \dots$$

Note that $\Delta\tau_k = \tau_{k+1} - \tau_k$ is a sequence of independent and T -distributed rv.

Introducing the rv

$$\bar{k} = \sup\{k \geq 0 \mid \tau_k < \infty\},$$

we can write

$$N = \sum_{n=0}^{\infty} \mathbb{1}_{X_n=0} = \sum_{k=0}^{\bar{k}} \mathbb{1}_{X_{\tau_k}=0} = \bar{k} + 1.$$

Observe that

$$\mathbb{P}(\bar{k} = j) =$$

Which RW are recurrent?

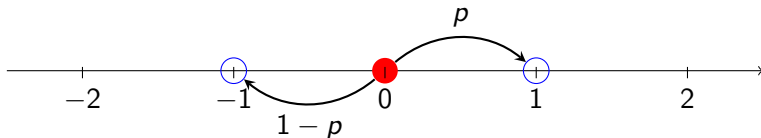
- **(Related to FJK 2.1.13)** Symmetric and simple RW on \mathbb{Z}^d are recurrent if $d \leq 2$ and transient otherwise.



A drunk man will eventually find his way home, but a drunk bird may get lost forever

Shizuo Kakutani

- **(Related to FJK 2.1.14)** Non-symmetric RW are always transient.



Always transient when $p \neq 1/2$.

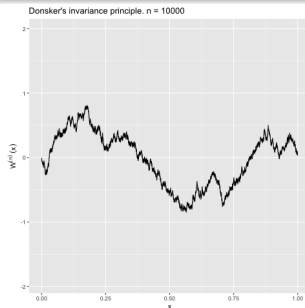
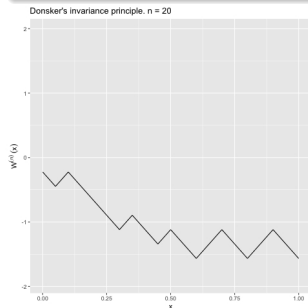
Scaling property of RW

Theorem 10 (Random walk case of Donsker's theorem)

Let $\{X_n\}$ be a simple symmetric RW on \mathbb{Z} with $X_0 = 0$ and consider

$$W^{(n)}(t) := \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \quad t \in [0, 1],$$

where $\lfloor x \rfloor := \max\{k \in \mathbb{Z} \mid k \leq x\}$. Then $\{W^{(n)}(t)\}_{t \in [0,1]}$ converges in distribution to a standard Brownian motion $\{W(t)\}_{t \in [0,1]}$.



Convergence of random variables

Assume you can draw iid samples $X_k \sim \mathbb{P}_X$ and that you approximate $\mu = \mathbb{E}[X]$ by the sample average

$$\bar{X}_M := \frac{1}{M} \sum_{k=1}^M X_k. \quad (2)$$

Questions:

- Will $\bar{X}_M \rightarrow \mu$ as $M \rightarrow \infty$, and, if so, in what sense?
- Is there a convergence rate of the form

$$\|\bar{X}_M - \mu\| \leq \frac{C}{M^\beta}$$

for some norm $\|\cdot\|$ and some rate $\beta > 0$?

Mean-square convergence

- For rv $Y, Z : \Omega \rightarrow \mathbb{R}^d$ we introduce the scalar product

$$\langle Y, Z \rangle_{L^2(\Omega)} := \mathbb{E}[Y \cdot Z]$$

- the function space

$$L^2(\Omega) := \{ \mathcal{F} - \text{measurable mappings } Y : \Omega \rightarrow \mathbb{R}^d \mid \mathbb{E}[|Y|^2] < \infty \}$$

with norm

$$\|Y\|_{L^2(\Omega)} := \sqrt{\mathbb{E}[|Y|^2]},$$

is a Hilbert space.

- The notation is shorthand for $L^2(\Omega) = L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$.

Returning to the approximation

$$\bar{X}_M = \frac{1}{M} \sum_{k=1}^M X_k$$

- Since $\mathbb{E}[X_k] = \mu$, it holds that

$$\bar{X}_M - \mu = \sum_{k=1}^M \frac{X_k - \mu}{M}$$

- Since $\{X_k - \mu\}$ is a mean-zero and independent sequence of rv, it holds for $j \neq k$ that

$$\begin{aligned} \langle X_k - \mu, X_j - \mu \rangle_{L^2(\Omega)} &= \mathbb{E}[(X_k - \mu) \cdot (X_j - \mu)] \\ &= \sum_{(x_k, x_j) \in A \times A} (x_k - \mu) \cdot (x_j - \mu) \underbrace{\mathbb{P}(X_k = x_k, X_j = x_j)}_{=\mathbb{P}(X_k=x_k)\mathbb{P}(X_j=x_j)} \\ &= \mathbb{E}[(X_k - \mu)] \cdot \mathbb{E}[(X_j - \mu)] = 0 \end{aligned}$$

(Here we assumed discrete rv $X_k : \Omega \rightarrow A$, but it also holds for continuous rv.)

This yields

$$\begin{aligned}\|\bar{X}_M - \mu\|_{L^2(\Omega)}^2 &= \left\langle \sum_{k=1}^M \frac{X_k - \mu}{M}, \sum_{k=1}^M \frac{X_k - \mu}{M} \right\rangle \\ &= \end{aligned}$$

Conclusion: For a sequence of d -dimensional discrete independent rv $X_i \sim \mathbb{P}_X$,

$$\|\bar{X}_M - \mu\|_{L^2(\Omega)} = \frac{\|X - \mu\|_{L^2(\Omega)}}{\sqrt{M}}, \quad (3)$$

i.e., the mean-square convergence rate is $1/2$.

Weaker form of convergence

Definition 11 (Convergence in probability)

A sequence of rv $\{\bar{Y}_k\}$ converges in probability towards the rv Y if for all $\epsilon > 0$,

$$\lim_{k \rightarrow \infty} \mathbb{P}(|Y_k - Y| > \epsilon) = 0.$$

Theorem 12 (Weak law of large numbers (Durrett 2.2.14))

For a sequence of d -dimensional independent rv $X_i \sim \mathbb{P}_X$ with $\mathbb{E}[|X_i|] < \infty$ it holds that

$$\bar{X}_M \rightarrow \mu \quad \text{in probability.}$$

Chebychev's inequality

To prove the theorem, we will apply **Chebychev's inequality**: for any rv Y with $\bar{\mu} = \mathbb{E}[Y]$

$$\mathbb{P}(|Y - \bar{\mu}| > \epsilon) \leq \mathbb{E} \left[\frac{|Y - \bar{\mu}|^2}{\epsilon^2} \right]$$

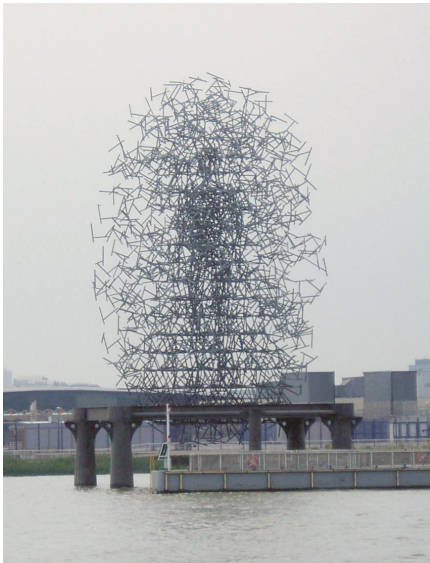
Verification:

Proof of Theorem 12

$$\mathbb{P}(|\bar{X}_M - \mu| > \epsilon) \leq$$

Next time

Discrete time and space Markov Chains



Caption: Quantum Cloud, designed by Antony Gormley. Random walk algorithm starting from points on the surface of an enlarged figure based on Gormley's body.