Game Theory and Control - Cheat Sheet

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https://github.com/haakonbaa/TTK4130-cheatsheet

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1 Static Games

Game representation
A game contains two players

$$P_1 \qquad\qquad P_2$$

with a finite number of actions

$$\Gamma = \{\gamma_1, \cdots, \gamma_n\} \qquad \Sigma = \{\sigma_1, \cdots, \sigma_m\}$$

Each possible combination of action has an outcome for each player

$$J_1(\gamma_i, \sigma_j) \qquad\qquad J_2(\gamma_i, \sigma_j)$$

this can be represented in a matrix

$$\begin{aligned} A \in \mathbb{R}^{n\times m}, \quad A_{ij} = J_1(\gamma_i, \sigma_j) \\ B \in \mathbb{R}^{n\times m}, \quad B_{ij} = J_2(\gamma_i, \sigma_j) \end{aligned}$$

Action dominance
We say that γ_i dominates γ_k for the minimizing player P_1 if

$$J_1(\gamma_i, \sigma_j) \leq J_1(\gamma_k, \sigma_j) \quad \forall \sigma_j \in \Sigma$$

Security level
The security level of the minimizing player P_1 is defined by

$$\bar{J}_1 = \min_{\gamma_i \in \Gamma} \max_{\sigma_j \in \Sigma} J_1(\gamma_i, \sigma_j)$$

Security policy
The security policy of the minimizing player P_1 is defined by

$$\operatorname*{argmin}_{\gamma_i \in \Gamma} \max_{\sigma_j \in \Sigma} J_1(\gamma_i, \sigma_j)$$

Nash equilibrium
Given a static two-player game described by the two payoff functions $J_1(\gamma, \sigma)$ and $J_2(\gamma, \sigma)$, a Nash equilibrium (for minimizing players) is a pair of actions

$$(\gamma^*, \sigma^*) \in \Gamma \times \Sigma$$

such that

$$\begin{aligned} J_1\gamma^*, \sigma^* \leq J_1\gamma, \sigma^* \qquad & \forall \gamma \in \Gamma \\ J_2(\gamma^*, \sigma^*) \leq J_2(\gamma^*, \sigma) \qquad & \forall \sigma \in \Sigma \end{aligned}$$

Admissible Nash equilibria
A Nash equilibrium (γ^*, σ^*) is admissible if there is no other Nash equilibrium $(\tilde{\gamma}, \tilde{\sigma})$ such that

$$\begin{aligned} J_1(\tilde{\gamma}, \tilde{\sigma}) \leq J_1(\gamma^*, \sigma^*) \\ J_2(\tilde{\gamma}, \tilde{\sigma}) \leq J_2(\gamma^*, \sigma^*) \end{aligned}$$

with at least one strict inequality.

Mixed strategies
A mixed strategy is a stochastic strategy. Each action is played with a certain probability. Let γ be the chosen strategy for player P_1 and σ the chosen strategy for player P_2. Define

$$\begin{aligned} P(\gamma = \gamma_i) = y_i \qquad & \boldsymbol{y} = (y_i)_i \\ P(\sigma = \sigma_j) = z_j \qquad & \boldsymbol{z} = (z_j)_j \end{aligned}$$

Expected outcome
The expected outcome of a game is

$$\begin{aligned} J_1(\boldsymbol{y}, \boldsymbol{z}) = \sum_{i=1}^{n} \sum_{j=1}^{m} y_i z_j A_{ij} = \boldsymbol{y^T A z} \\ J_2(\boldsymbol{y}, \boldsymbol{z}) = \sum_{i=1}^{n} \sum_{j=1}^{m} y_i z_j B_{ij} = \boldsymbol{y^T B z} \end{aligned}$$

Mixes security level The mixed security level of the minimizing player P_1 is defined by

$$\bar{J}_1^m = \min_{\boldsymbol{y} \in \mathcal{Y}} \max_{\boldsymbol{z} \in \mathcal{Z}} y^T A z$$

Mixed security strategy

$$\operatorname*{argmin}_{\boldsymbol{y} \in \mathcal{Y}} \max_{\boldsymbol{z} \in \mathcal{Z}} \boldsymbol{y^T A z}$$

Mixed Nash equilibria Given a static two-player game described by the two payoff matrices A and B, we say that the pair of mixed strategies

$$(\boldsymbol{y}^*, \boldsymbol{z}^*) \in \mathcal{Y} \times \mathcal{Z}$$

are a mixed Nash Equilibrium for the two minimizing players if

$$\begin{aligned} (\boldsymbol{y*})^T \boldsymbol{A z^*} \leq (\boldsymbol{y})^T \boldsymbol{A z^*} \quad & \forall \boldsymbol{y} \in \mathcal{Y} \\ (\boldsymbol{y*})^T \boldsymbol{B z^*} \leq (\boldsymbol{y*})^T \boldsymbol{B z} \quad & \forall \boldsymbol{z} \in \mathcal{Z} \end{aligned}$$

In general terms

$$\begin{aligned} J_1(\boldsymbol{y^*}, \boldsymbol{z^*}) \leq J_1(\boldsymbol{y}, \boldsymbol{z^*}) \quad & \forall \boldsymbol{y} \in \mathcal{Y} \\ J_2(\boldsymbol{y^*}, \boldsymbol{z^*}) \leq J_2(\boldsymbol{y^*}, \boldsymbol{z}) \quad & \forall \boldsymbol{z} \in \mathcal{Z} \end{aligned}$$