TTK4130 - Cheat Sheet

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https://github.com/haakonbaa/TTK4130-cheatsheet

Contents

1 Intro

1.2 State space methods

1.2.1 State space models

State Space Model is on the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$$

Linear time invariant system

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\
\boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \\
\boldsymbol{y} &= \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(t)\,d\tau + \boldsymbol{D}\boldsymbol{u}(t)
\end{aligned}$$

1.2.2 Second order models of mechanical systems

Second order models on the form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u}$$

Can be written as

$$\dot{\begin{pmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{pmatrix}} = \begin{pmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}^{-1}(\boldsymbol{q})(-\boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{u}) \end{pmatrix}$$

1.2.3 Linearization of state space models

Linearization of time varying systems

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \\
\boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}, t)
\end{aligned}$$

Find two functions $\boldsymbol{x}_0$ and $\boldsymbol{u}_0$ begin solutions to the sytem

$$\dot{\boldsymbol{x}}_0 = \boldsymbol{f}(\boldsymbol{x}_0(t), \boldsymbol{u}_0(t), t)$$

Define pertrubations

$$\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{x}_0(t) + \boldsymbol{\Delta x}(t) \\
\boldsymbol{u}(t) &= \boldsymbol{u}_0(t) + \boldsymbol{\Delta u}(t) \\
\boldsymbol{y}(t) &= \boldsymbol{y}_0(t) + \boldsymbol{\Delta y}(t)
\end{aligned}$$

Let $\mathcal{C} = \{\boldsymbol{x}_0(t), \boldsymbol{u}_0(t)\}$. The linearized system is

$$\begin{aligned}
\boldsymbol{\Delta}\dot{\boldsymbol{x}} &\approx \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\Big|_{\mathcal{C}} \boldsymbol{\Delta x} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\Big|_{\mathcal{C}} \boldsymbol{\Delta u} \\
\boldsymbol{\Delta}\dot{\boldsymbol{y}} &\approx \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\Big|_{\mathcal{C}} \boldsymbol{\Delta x} + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}\Big|_{\mathcal{C}} \boldsymbol{\Delta u}
\end{aligned}$$

1.5 ODE's

General formulation

$$\varphi(y^{(m)}, \cdots, y, u^{(m-1)}, \cdots, u) = 0$$

Lipschitz continous.A function

$$f : \mathbb{R} \to \mathbb{R}$$

is said to be *Lipschitz continous* if

$$\begin{aligned}
&\exists c > 0 \in \mathbb{R} && \text{s.t.} \\
&||f(x) - f(y)|| < c||x - y|| && \forall x, y \in \mathbb{R}
\end{aligned}$$

Theorem: existence of unique solution. Consider the ODE

$$\dot{x} = f(x)$$

If f is Lipschitz continous then $x(t)$ exists and is unique for all t

Mean Value Theorem suppose f is continous on $[a, b]$ and differnetiable on (a, b) then

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

This can be sused to show that if f is continous and differentiable everywhere it is also Lipschitz.

Theorem 2: existence of unique solution. Consider the ODE

$$\dot{x} = f(x)$$

if f is continously differentiable ($\frac{\partial f}{\partial x}$ exists and is continous), then the solution to the ODE exists and is unique on some time interval.

2 Rotations

6.2 Vectors

The *skew-symetric* matrix form of the coordinate vector $\mathbf{u}$ is defined by

$$\mathbf{u}^x = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Notation: $\boldsymbol{v}_{ab}^c$ means the vector from point a to point b (or often the origo of the reference frames a and b) described in the reference frame c

6.4 The Rotation Matrix

The coordinate transformation from frame b to frame a is given by

$$\boldsymbol{v}^a = \boldsymbol{R}_b^a \boldsymbol{v}^b$$

Properties of the rotation matrix

$$\begin{aligned} \boldsymbol{R}_a^b \boldsymbol{R}_b^a &= \boldsymbol{I} = \boldsymbol{R}_b^a \boldsymbol{R}_a^b \\ (\boldsymbol{R}_a^b)^{-1} &= (\boldsymbol{R}_a^b)^T = \boldsymbol{R}_b^a \\ \boldsymbol{R}_b^a &= \begin{pmatrix} \boldsymbol{b}_1^a & \boldsymbol{b}_2^a & \boldsymbol{b}_3^a \end{pmatrix} \\ \det \boldsymbol{R}_a^b &= 1 \end{aligned}$$

$\boldsymbol{R}$ is a rotation matrix if and only if it is an element of $SO(3)$

$$SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3\times 3} | \boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I} \wedge \det \boldsymbol{R} = 1\}$$

Rotation matrices in three dimentions

$$\boldsymbol{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

$$\boldsymbol{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\boldsymbol{R}_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix transformations in different refrence frames

$$\begin{aligned} \boldsymbol{D}^a &= \boldsymbol{R}_b^a \boldsymbol{D}^b \boldsymbol{R}_a^b \\ (\boldsymbol{u}^b)^\times &= \boldsymbol{R}_a^b (\boldsymbol{u}^a)^\times \boldsymbol{R}_b^a \end{aligned}$$

The transformation of position and orientation from frame b to frame a is

$$\boldsymbol{T}_b^a = \begin{pmatrix} \boldsymbol{R}_b^a & \boldsymbol{r}_{ab}^a \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

$$\boldsymbol{T}_b^a \begin{pmatrix} \boldsymbol{v}^b \\ 1 \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{v}^a \\ 1 \end{pmatrix}$$

$$(\boldsymbol{T}_b^a)^{-1} = \boldsymbol{T}_a^b = \begin{pmatrix} \boldsymbol{R}_a^b & \boldsymbol{r}_{ba}^b \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

The Special Euclidean group is the set of all transformations from one reference frames to another

$$SE(3) = \left\{ \boldsymbol{T} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \middle| \boldsymbol{R} \in SO(3) \wedge \boldsymbol{r} \in \mathbb{R}^3 \right\}$$

6.5 Euler Angles

Roll-Pitch-Yaw Euler angles

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi)$$

Classical Euler angles. The orientation is described by a rotation bout the z axis, then the resulting y axis. And then again the resulting z axis.

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_z(\phi)$$

6.6 Angle Axis Description of rotation

6.6.5 Rotation Matrix

Angle-axis parameters All rotation matrices have an eigen vector with eigen value 1. A rotation can be uniquely described by the direction of this vector and an angle θ being the rotatoin about this vector.

$$\begin{aligned} &(\theta, \boldsymbol{k}) \text{ s.t. } ||\boldsymbol{k}|| = 1 \\ \boldsymbol{R}_b^a &= \cos\theta \boldsymbol{I} + \sin\theta(\boldsymbol{k}_a)^\times + (1-\cos\theta)\boldsymbol{k}_a\boldsymbol{k}_a^T \\ \boldsymbol{R}_b^a &= \exp\{\boldsymbol{k}^\times \theta\} \end{aligned}$$

6.7 Euler parameters

6.7.1 Definition

$$\begin{aligned} \eta &= \cos\frac{\theta}{2} \\ \boldsymbol{\epsilon} &= \boldsymbol{k}\sin\frac{\theta}{2} \\ \boldsymbol{R}_e(\eta, \boldsymbol{\epsilon}) &= \boldsymbol{I} + 2\eta\boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times \\ \eta^2 + \boldsymbol{\epsilon}^T\boldsymbol{\epsilon} &= 1 \end{aligned}$$

6.7.3 Quaternions

The following can be treated as a unit quaternion

$$\boldsymbol{p} = \begin{pmatrix} \eta \\ \boldsymbol{\epsilon} \end{pmatrix}$$

A unit quaternion satisfies

$$\boldsymbol{p}^T\boldsymbol{p} = \eta^2 + \boldsymbol{\epsilon}^T\boldsymbol{\epsilon} = 1$$

Quaternion product

$$\begin{pmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 - \boldsymbol{\beta}_1^T\boldsymbol{\beta}_2 \\ \alpha_1\boldsymbol{\beta}_2 + \alpha_2\boldsymbol{\beta}_1 + \boldsymbol{\beta}_1^\times\boldsymbol{\beta}_2 \end{pmatrix}$$

6.7.6 Euler parameters from the rotation matrix

$$\boldsymbol{R} = (r_{ij})$$
$$\boldsymbol{z} = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix}^T := 2\begin{pmatrix} \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix}^T$$
$$\boldsymbol{T} := r_{00} := \text{Trace}\boldsymbol{R}$$

The algorithm from Shepperd (1978) goes like this:

- Let $i = \arg\max_i\{r_{ii}\}$
- Compute $|z_i| = \sqrt{1 + 2r_{ii} - T}$
- Determine sign of z_i
- Determine the rest of $\boldsymbol{z}$ from equations below

$$\begin{aligned} z_0 z_1 &= r_{32} - r_{23} & z_2 z_3 &= r_{32} + r_{23} \\ z_0 z_2 &= r_{13} - r_{31} & z_3 z_1 &= r_{13} + r_{31} \\ z_0 z_3 &= r_{21} - r_{12} & z_1 z_2 &= r_{21} + r_{12} \end{aligned}$$

6.8 Angular Velocity

Let $R \in SO(3)$

$$\begin{aligned} &0 = \frac{d}{dt}(\boldsymbol{I}) = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{R}^T) = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}(\dot{\boldsymbol{R}})^T \\ &\Rightarrow \dot{\boldsymbol{R}}\boldsymbol{R}^T \, \textit{skew-symmetric} \end{aligned}$$

Definition of angluar velocity

$$\begin{aligned} (\boldsymbol{\omega}^a_{ab})^\times &= \dot{\boldsymbol{R}}^a_b(\boldsymbol{R}^a_b)^T \Rightarrow \\ \dot{\boldsymbol{R}}^a_b &= (\boldsymbol{\omega}^a_{ab})^\times \boldsymbol{R}^a_b \\ \dot{\boldsymbol{R}}^a_b &= \boldsymbol{R}^a_b(\boldsymbol{\omega}^b_{ab})^\times \end{aligned}$$

It can be shown that

$$\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}$$

Where θ and $\boldsymbol{k}$ are Angle Axis parameters.

$$\begin{aligned} \boldsymbol{\omega}^a_{ad} &= \boldsymbol{\omega}^a_{ab} + \boldsymbol{\omega}^a_{bc} + \boldsymbol{\omega}^a_{cd} \\ \dot{\boldsymbol{u}}^a &= \boldsymbol{R}^a_b(\dot{\boldsymbol{u}}^b + (\boldsymbol{\omega}^b_{ab})^\times \boldsymbol{u}^b) \end{aligned}$$

6.9 Kinematic differential equations

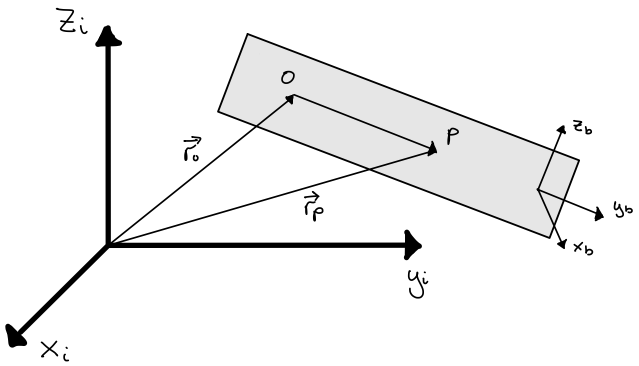
6.9.4 Euler Angles

$$\begin{aligned} \boldsymbol{\omega}^a_{ad} &= \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \boldsymbol{R}_{z,\psi}\begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \boldsymbol{R}_{z,\psi}\boldsymbol{R}_{y,\theta}\begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sin\psi\dot{\theta} + \cos\psi\cos\theta\dot{\phi} \\ \cos\psi\dot{\theta} + \sin\psi\cos\theta\dot{\phi} \\ \dot{\psi} - \sin\theta\dot{\phi} \end{pmatrix} \end{aligned}$$

3 Rigid Body Dynamics

6.12 Kinematics of a rigid body

$\vec{\omega}_{io}$ is the angular velocity of the o frame with respect to the i frame.
${}^i\frac{d}{dt}\vec{r}_o$ is the derivative of $\vec{r}_o$ in the i frame.



Velocity and Acceleration

$$\begin{aligned}
\vec{v}_p &:= \frac{{}^i d}{dt}\vec{r}_p \\
&= \vec{v}_o + \frac{{}^b d}{dt}\vec{r} + \vec{\omega}_{ib} \times \vec{r} \\
\vec{a}_p &:= \frac{{}^i d^2}{dt^2}\vec{r}_p \\
&= \vec{a}_o + \frac{{}^b d^2}{dt^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^b d}{dt}\vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})
\end{aligned}$$

The last three terms are, respectively, the coriolis acceleration, Transveral acceleration and Centripetal acceleration. Note that

$$\vec{a}_o = \frac{{}^i d}{dt}\vec{v}_o = \frac{{}^b d}{dt}\vec{v}_o + \vec{\omega}_{ib} \times \vec{v}_o$$

6.13 The center of mass

The center of mass of a rigid body $\mathcal{C}$ is defined to be

$$\vec{r}_c := \frac{1}{m}\int_{\mathcal{C}} \vec{r}_p \, dm$$

It can be shown that

$$\vec{v}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{v}_p \, dm \qquad \vec{a}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{a}_p \, dm$$

where c denotes *center*

7.2 Forces and torques

Moment. The moment about a point P of the set $S = \{F_j\}_{j\in[1,n_F]}$ for forces is

$$\vec{N}_{S/P} = \sum_{j=1}^{n_F} r\vec{r}_{Pj} \times \vec{F}_j$$

Where $\vec{r}_{Pj}$ is an arbitrary point along the line of action of $\vec{F}_j$

Torque is defined as the moment of the couple $\mathcal{C}$. A couple being a set of forces with $\mathbf{0}$ resultant force.

7.3 Newton-Euler Equations for rigid bodies

Angular Momentum. The angular momentum of the body b about the center of mass c is

$$\begin{aligned}
\boldsymbol{h}_{b/c} &= \int_b \boldsymbol{r} \times \boldsymbol{v} \, dm \\
&= \boldsymbol{M}_{b/c}\boldsymbol{\omega}_{ib} \\
\boldsymbol{T}_{bc} &= \frac{d}{dt}\boldsymbol{h}_{b/c}
\end{aligned}$$

Rotational Inertia / The intertia dyadic. The inertia matrix of the body b about the point c is

$$\begin{aligned}
\boldsymbol{M}_{b/c} &= -\int_b \boldsymbol{r}^\times \boldsymbol{r}^\times \, dm \\
&= \int_b (\boldsymbol{r}^T\boldsymbol{r}\mathbb{I} - \boldsymbol{r}\boldsymbol{r}^T) \, dm \\
&= \begin{pmatrix} \boldsymbol{I}_{xx} & -\boldsymbol{I}_{xy} & -\boldsymbol{I}_{xz} \\ -\boldsymbol{I}_{xy} & \boldsymbol{I}_{yy} & -\boldsymbol{I}_{yz} \\ -\boldsymbol{I}_{xz} & -\boldsymbol{I}_{yz} & \boldsymbol{I}_{zz} \end{pmatrix}
\end{aligned}$$

Where $\boldsymbol{r}$ is the distance vector from the center of mass to the mass element being integrated

$$\begin{aligned}
\boldsymbol{I}_{xx} &= \int_b y^2 + z^2 \, dm & \boldsymbol{I}_{xy} &= \int_b xy \, dm \\
\boldsymbol{I}_{yy} &= \int_b x^2 + z^2 \, dm & \boldsymbol{I}_{xz} &= \int_b xz \, dm \\
\boldsymbol{I}_{zz} &= \int_b x^2 + y^2 \, dm & \boldsymbol{I}_{yz} &= \int_b yz \, dm
\end{aligned}$$

$$\boldsymbol{M}^i_{b/c} = \boldsymbol{R}^i_b \boldsymbol{M}^b_{b/c} \boldsymbol{R}^b_i$$

Equations of motion. Let b denote body, i an intertial frame, c the center of mass of b, $\boldsymbol{F}_{bc}$ a resultant force acting on b with line of action throgh c and $\boldsymbol{T}_{bc}$ the torque about c. Then

$$\begin{aligned}
\boldsymbol{F}_{bc} &= m\boldsymbol{a}_c \\
\boldsymbol{T}_{bc} &= \boldsymbol{M}_{b/c}\boldsymbol{\alpha}_{ib} + \boldsymbol{\omega}_{ib} \times (\boldsymbol{M}_{b/c}\boldsymbol{\omega}_{ib})
\end{aligned}$$

On compact matrix form

$$\begin{pmatrix} m\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}^b_{b/c} \end{pmatrix} \begin{pmatrix} \boldsymbol{a}^b_c \\ \boldsymbol{\alpha}^b_{ib} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ (\boldsymbol{\omega}^b_{ib})^\times \boldsymbol{M}^b_{b/c}\boldsymbol{\omega}^b_{ib} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}^b_{bc} \\ \boldsymbol{T}^b_{b/c} \end{pmatrix}$$

$$\begin{pmatrix} m\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}^b_{b/c} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}^b_c \\ \boldsymbol{\alpha}^b_{ib} \end{pmatrix} + \begin{pmatrix} m(\boldsymbol{\omega}^b_{ib})^\times \boldsymbol{v}^b_c \\ (\boldsymbol{\omega}^b_{ib})^\times \boldsymbol{M}^b_{b/c}\boldsymbol{\omega}^b_{ib} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}^b_{bc} \\ \boldsymbol{T}^b_{b/c} \end{pmatrix}$$

Kinetic energy. The kinetic energy of the body b in an inertial refrence frame i is

$$K = \frac{1}{2}m(\boldsymbol{v}^b_c)^T\boldsymbol{v}^b_c + \frac{1}{2}(\boldsymbol{\omega}^b_{ib})^T\boldsymbol{M}^b_{b/c}\boldsymbol{\omega}^b_{ib}$$

The parallel axes theorem. The inertia matrix of b about o is related to the inertia matrix of b about c according to

$$\begin{aligned}\boldsymbol{M}^b_{b/o} &= \boldsymbol{M}^b_{b/c} - m(\boldsymbol{r}^b_g)^\times(\boldsymbol{r}^b_g)^\times \\ &= \boldsymbol{M}^b_{b/c} - m(||\boldsymbol{r}^b_g||^2\mathbb{I} - \boldsymbol{r}^b_g(\boldsymbol{r}^b_g)^T)\end{aligned}$$

Where $\boldsymbol{r}^b_g$ is the vector from c to o

4 Lagrange Mechanics

8.2 Lagrange Mechanics

The lagrangian. Define a set of generalized coordinates $\boldsymbol{q}$. Let $T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ be the kinetic energy and $U(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ the potential energy (Sometimes V). Then the lagrangian is defined to be

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - U(\boldsymbol{q})$$

The equation of motion is

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i \\ &\frac{d}{dt}\left(\nabla_{\dot{\boldsymbol{q}}} L\right) - \nabla_{\boldsymbol{q}} L = \boldsymbol{\tau} \end{aligned}$$

Where τ_i is a generalized actuator force

$$\tau_i = \sum_{k=1}^{N} \frac{\partial \boldsymbol{r_k}}{\partial q_i} \cdot \boldsymbol{F}_k$$

$\boldsymbol{r}_k(\boldsymbol{q})$ is the position of the point of application of the force $\boldsymbol{F}_k$. In general τ_i is a force or a torque.

Constrained Lagrange. Having the constraints

$$\boldsymbol{c}(\boldsymbol{q}) = \boldsymbol{0}$$

The system can be described by

$$\begin{aligned} &L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - U(\boldsymbol{q}) - \boldsymbol{z}^T \boldsymbol{c}(\boldsymbol{q}) \\ &\frac{d}{dt}\left(\nabla_{\dot{\boldsymbol{q}}} L\right) - \nabla_{\boldsymbol{q}} L = \boldsymbol{\tau} \\ &\boldsymbol{c}(\boldsymbol{q}) = \boldsymbol{0} \end{aligned}$$

Baumgarte stabilization. Instead of imposing

$$\ddot{\boldsymbol{c}}(\boldsymbol{q}) = \boldsymbol{0}$$

Impose

$$\ddot{\boldsymbol{c}} + 2\alpha\dot{\boldsymbol{c}} + \alpha^2 \boldsymbol{c} = \boldsymbol{0}$$

As to reduce drifts in the constraints resulting from

$$\ddot{\boldsymbol{c}} = \boldsymbol{0}$$

not begin satisfied exactly when doing numeric computations.

5 Differential Algebraic Equations

14.2 Preliminaries

5.1 Differential Algebraic Equations

Definition of DEA. The differential equation defined by

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{u}, t) = \boldsymbol{0}$$

Es a DAE if

$$\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$$

is rank defficient

Fully-explicit DAE

$$\begin{aligned} \boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) &= \boldsymbol{0} \\ \det \left| \frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}} \right| &= 0 \end{aligned}$$

Can be rewritten as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{v} \\ \boldsymbol{0} &= \boldsymbol{F}(\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{aligned}$$

Semi-explicit DAE

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{aligned}$$

This can be rewritten as

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \begin{pmatrix} \dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \\ \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{pmatrix} = \boldsymbol{0}$$

Tikhonov Theorem. Consider the ordinary differential equation

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}) \\ \varepsilon \dot{\boldsymbol{z}} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}) \end{aligned}$$

If

- dynamics of $\dot{\boldsymbol{z}} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z})$ stable $\forall \boldsymbol{x}$
- $\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}}$ is full rank

then

$$\begin{aligned} \lim_{\epsilon \to 0} \boldsymbol{x}(t) &= \boldsymbol{x}_0(t) \\ \lim_{\epsilon \to 0} \boldsymbol{z}(t) &= \boldsymbol{z}_0(t) \end{aligned}$$

where $\boldsymbol{z}_0(t)$ and $\boldsymbol{x}_0(t)$ is the solution to the ODE above modified to a DAE where $\varepsilon = 0$

Theorem: Solvability of DAE. A fully implicit DAE with smooth

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \boldsymbol{0}$$

Can be readily solved (solved for $\dot{\boldsymbol{x}}$ and $\boldsymbol{z}$) if

$$\begin{pmatrix} \frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}} & \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{z}} \end{pmatrix}$$

is full rank on all trajectories $\dot{\boldsymbol{x}}$, $\boldsymbol{z}$, $\boldsymbol{x}$ and $\boldsymbol{u}$. Note that all **Index 1** DAEs fullfil these requirements. The theorem implies that

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{aligned}$$

with smooth $\boldsymbol{f}$ can be solved if

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}}$$

is full rank on all trajectories $\boldsymbol{z}$, $\boldsymbol{x}$ and $\boldsymbol{u}$

Definition: Differential index of a DAE is the numer of times the differentiation operator $\frac{d}{dt}$ must be applied to the equations in order to convert the DAE into an ODE.

6 Simulation methods

Butcher tableau

$$\begin{array}{c|c} \boldsymbol{c} & A \\ \hline & \boldsymbol{b}^T \end{array}$$

$$K_n = f(t_k + \Delta t c_n, x_n + \Delta t(A_n)K)$$
$$x_{k+1} = x_k + \Delta t \boldsymbol{b}^T K$$

The method is valid if

$$c_n = \Sigma_i A_{ni}$$
$$\Sigma_i \boldsymbol{b}_i = 1$$

The **stages** of a Runge Kutta method is the number of elements in $\boldsymbol{c}$. The Butcher tableu defines an explicit integrator if and only if the diagonal elements and the upper-diagonal elements are zero.

Explicit Euler

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t \cdot \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \qquad \begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Global error $||\boldsymbol{x}_N - \boldsymbol{x}(T)|| = \mathcal{O}(\Delta t)$
Region of stability $R(z) = 1 + z$

Explicit mid-point rule

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1/2 \end{array}$$

Global error $||\boldsymbol{x}_N - \boldsymbol{x}(T)|| = \mathcal{O}(\Delta t^2)$

Ralston's RK2

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 1/3 \end{array}$$

Heun's RK2

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

Generic second-order method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \alpha & \alpha & 0 \\ \hline & 1 - \frac{1}{2\alpha} & \frac{1}{2\alpha} \end{array}$$

Generic third-order method

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \alpha & \alpha & 0 & 0 \\ 1 & 1 + \frac{1-\alpha}{\alpha(3\alpha-2)} & -\frac{1-\alpha}{\alpha(3\alpha-2)} & 0 \\ \hline & \frac{1}{2} - \frac{1}{6\alpha} & \frac{1}{6\alpha(1-\alpha)} & \frac{2-3\alpha}{6(1-\alpha)} \end{array}$$

"The" RK4 method

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Global error $||\boldsymbol{x}_N - \boldsymbol{x}(T)|| = \mathcal{O}(\Delta t^4)$

One-Step error

$$||\boldsymbol{x}_{k+1} - \boldsymbol{x}(t_{k+1})||$$

Taylor expand $\boldsymbol{x}(t)$ around t_k. Do the same for $\boldsymbol{x}_{k+1}$. Subtract and get the results

Global error Get error on timestep T by multiplying local error with $N = \frac{T}{\Delta t}$

stability of integration method: Consdier the ODE

$$\dot{x} = \lambda x \qquad x(0) = x_0$$

Perform one step of the RK method to find

$$x_{n+1} = R(z)x_n \qquad z = \Delta t \lambda$$

The integration method is unstable if

$$|R(z)| > 1$$

A-stability A method is a-stable if the region of stability is the entire left-half plane.

Implicit methods

- Can chaieve high and systematic orders
- Can be stable regardless of the step size
- Can handle the simulation of DAEs (must be index 1)
- Can achieve order $o = 2s$ for any stage s. Explicit methods can only achieve $o = s$ and only for $s \leq 4$

Implicit Euler

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

$$\begin{aligned} k_1 &= f(x_n + \Delta t k_1) \\ x_{n+1} &= x_n + \Delta t k_1 \\ &= x_n + \Delta t f(x_{n+1}) \end{aligned}$$

Gauss-Legendre collocation method Let s be the number of stages. Find the roots of

$$P_s(\tau) = \frac{1}{s!}\frac{d^s}{d\tau^s}\left((\tau^2 - \tau)^s\right)$$

Build polynomials

$$\ell_i(\tau) = \prod_{j \neq i} \frac{\tau - \tau_j}{\tau_i - \tau_j}$$

Integrate them

$$L_i(\tau) = \int_0^\tau \ell_i(\xi)\, d\xi$$

Calculate A ,$\boldsymbol{b}$ and $\boldsymbol{c}$

$$A_{ji} = L_i(\tau_j) \qquad b_i = L_i(1) \qquad c_j = \tau_j$$

7 Math

Inverse of 2×2 Matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Multivariable derivative rules

$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{a}^T\boldsymbol{x} = \boldsymbol{a}^T \qquad \nabla_{\boldsymbol{x}}\boldsymbol{a}^T\boldsymbol{x} = \boldsymbol{a}$$

$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A} \qquad \nabla_{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T$$

$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x}^T\boldsymbol{A} = \boldsymbol{A}^T \qquad \nabla_{\boldsymbol{x}}\boldsymbol{x}^T\boldsymbol{A} = \boldsymbol{A}$$

Second order terms

$$\frac{\partial}{\partial \boldsymbol{x}}\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \frac{1}{2}\boldsymbol{x}^T(\boldsymbol{A}^T+\boldsymbol{A})$$

$$\nabla_{\boldsymbol{x}}\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \frac{1}{2}(\boldsymbol{A}^T+\boldsymbol{A})\boldsymbol{x}$$

Multivariable Chain rule

$$\frac{\partial \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{g}}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}$$

$$\frac{\partial \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}),\boldsymbol{h}(\boldsymbol{x}))}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{g}}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{h}}\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}$$

Some derivatives

$$\begin{aligned}
\frac{d}{dt}\sinh(t) &= \cosh(t) \\
\frac{d}{dt}\cosh(t) &= \sinh(t) \\
\frac{d}{dt}\tanh(t) = \frac{d}{dt}\frac{\sinh(t)}{\cosh(t)} &= \frac{1}{\cosh^2 t} = 1-\tanh^2(x) \\
\frac{d}{dt}\arctan(t) &= \frac{1}{1+t^2}
\end{aligned}$$

Taylor's theorem
Let $k \geq 1$ be an integer and let the function $f : \mathbb{R} \to \mathbb{R}$ be $k+1$ times differentiable at the point $a \in \mathbb{R}$. Then

$$f(x) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(a)(x-a)^n + \frac{1}{(1+k)!} f^{(k+1)}(\xi)(x-a)^{k+1}$$

for some $\xi \in [x,a]$
Common Taylor expansions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$