

TTK4130 - Cheat Sheet

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<https://github.com/haakonbaa/TTK4130-cheatsheet>

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1 Intro

1.2 State space methods

1.2.1 State space models

State Space Model is on the form

$$\dot{x} = f(x, u, t)$$

Linear time invariant system

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$y = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau) d\tau + Du(t)$$

1.2.2 Second order models of mechanical systems

Second order models on the form

$$M(q)\ddot{q} + f(q, \dot{q}) = u$$

Can be written as

$$\begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ M^{-1}(q)(-f(q, \dot{q}) + u) \end{pmatrix}$$

1.2.3 Linearization of state space models

Linearization of time varying systems

$$\dot{x} = f(x, u, t)$$

$$y = h(x, u, t)$$

Find two functions x_0 and u_0 begin solutions to the system

$$\dot{x}_0 = f(x_0(t), u_0(t), t)$$

Define perturbations

$$x(t) = x_0(t) + \Delta x(t)$$

$$u(t) = u_0(t) + \Delta u(t)$$

$$y(t) = y_0(t) + \Delta y(t)$$

Let $\mathcal{C} = \{x_0(t), u_0(t)\}$. The linearized system is

$$\Delta \dot{x} \approx \left. \frac{\partial f}{\partial x} \right|_{\mathcal{C}} \Delta x + \left. \frac{\partial f}{\partial u} \right|_{\mathcal{C}} \Delta u$$

$$\Delta \dot{y} \approx \left. \frac{\partial h}{\partial x} \right|_{\mathcal{C}} \Delta x + \left. \frac{\partial h}{\partial u} \right|_{\mathcal{C}} \Delta u$$

1.5 ODE's

General formulation

$$\varphi(y^{(m)}, \dots, y, u^{(m-1)}, \dots, u) = 0$$

Lipschitz continuous. A function

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

is said to be *Lipschitz continuous* if

$$\begin{aligned} \exists c > 0 \in \mathbb{R} & \quad \text{s.t.} \\ \|f(x) - f(y)\| < c\|x - y\| & \quad \forall x, y \in \mathbb{R} \end{aligned}$$

Theorem: existence of unique solution. Consider the ODE

$$\dot{x} = f(x)$$

If f is Lipschitz continuous then $x(t)$ exists and is unique for all t

Mean Value Theorem suppose f is continuous on $[a, b]$ and differentiable on (a, b) then

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

This can be used to show that if f is continuous and differentiable everywhere it is also Lipschitz.

Theorem 2: existence of unique solution. Consider the ODE

$$\dot{x} = f(x)$$

if f is continuously differentiable ($\frac{\partial f}{\partial x}$ exists and is continuous), then the solution to the ODE exists and is unique on some time interval.

2 Rotations

6.2 Vectors

The *skew-symmetric* matrix form of the coordinate vector \mathbf{u} is defined by

$$\mathbf{u}^x = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Notation: \mathbf{v}_{ab}^c means the vector from point a to point b (or often the origo of the reference frames a and b) described in the reference frame c

6.4 The Rotation Matrix

The coordinate transformation from frame b to frame a is given by

$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$

Properties of the rotation matrix

$$\begin{aligned} \mathbf{R}_a^b \mathbf{R}_b^a &= \mathbf{I} = \mathbf{R}_b^a \mathbf{R}_a^b \\ (\mathbf{R}_a^b)^{-1} &= (\mathbf{R}_a^b)^T = \mathbf{R}_b^a \\ \mathbf{R}_b^a &= (\mathbf{b}_1^a \quad \mathbf{b}_2^a \quad \mathbf{b}_3^a) \\ \det \mathbf{R}_a^b &= 1 \end{aligned}$$

\mathbf{R} is a rotation matrix if and only if it is an element of $SO(3)$

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3 \times 3} | \mathbf{R}^T \mathbf{R} = \mathbf{I} \wedge \det \mathbf{R} = 1\}$$

Rotation matrices in three dimensions

$$\begin{aligned} \mathbf{R}_x(\phi) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} \\ \mathbf{R}_y(\theta) &= \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \\ \mathbf{R}_z(\psi) &= \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Matrix transformations in different reference frames

$$\begin{aligned} \mathbf{D}^a &= \mathbf{R}_b^a \mathbf{D}^b \mathbf{R}_a^b \\ (\mathbf{u}^b)^\times &= \mathbf{R}_a^b (\mathbf{u}^a)^\times \mathbf{R}_b^a \end{aligned}$$

The transformation of position and orientation from frame b to frame a is

$$\begin{aligned} \mathbf{T}_b^a &= \begin{pmatrix} \mathbf{R}_b^a & \mathbf{r}_{ab}^a \\ \mathbf{0}^T & 1 \end{pmatrix} \\ \mathbf{T}_b^a \begin{pmatrix} \mathbf{v}^b \\ 1 \end{pmatrix}^T &= \begin{pmatrix} \mathbf{v}^a \\ 1 \end{pmatrix} \\ (\mathbf{T}_b^a)^{-1} &= \mathbf{T}_a^b = \begin{pmatrix} \mathbf{R}_a^b & \mathbf{r}_{ba}^b \\ \mathbf{0}^T & 1 \end{pmatrix} \end{aligned}$$

The Special Euclidean group is the set of all transformations from one reference frames to another

$$SE(3) = \left\{ \mathbf{T} = \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3} \middle| \mathbf{R} \in SO(3) \wedge \mathbf{r} \in \mathbb{R}^3 \right\}$$

6.5 Euler Angles

Roll-Pitch-Yaw Euler angles

$$\mathbf{R}_a^b = \mathbf{R}_z(\psi) \mathbf{R}_y(\theta) \mathbf{R}_x(\phi)$$

Classical Euler angles. The orientation is described by a rotation about the z axis, then the resulting y axis. And then again the resulting z axis.

$$\mathbf{R}_a^b = \mathbf{R}_z(\psi) \mathbf{R}_y(\theta) \mathbf{R}_z(\phi)$$

6.6 Angle Axis Description of rotation

6.6.5 Rotation Matrix

Angle-axis parameters All rotation matrices have an eigen vector with eigen value 1. A rotation can be uniquely described by the direction of this vector and an angle θ being the rotation about this vector.

$$(\theta, \mathbf{k}) \text{ s.t. } \|\mathbf{k}\| = 1$$

$$\mathbf{R}_b^a = \cos \theta \mathbf{I} + \sin \theta (\mathbf{k}_a)^\times + (1 - \cos \theta) \mathbf{k}_a \mathbf{k}_a^T$$

$$\mathbf{R}_b^a = \exp\{\mathbf{k}^\times \theta\}$$

6.7 Euler parameters

6.7.1 Definition

$$\eta = \cos \frac{\theta}{2}$$

$$\boldsymbol{\epsilon} = \mathbf{k} \sin \frac{\theta}{2}$$

$$\mathbf{R}_e(\eta, \boldsymbol{\epsilon}) = \mathbf{I} + 2\eta \boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times \boldsymbol{\epsilon}^\times$$

$$\eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = 1$$

6.7.3 Quaternions

The following can be treated as a unit quaternion

$$\mathbf{p} = \begin{pmatrix} \eta \\ \boldsymbol{\epsilon} \end{pmatrix}$$

A unit quaternion satisfies

$$\mathbf{p}^T \mathbf{p} = \eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = 1$$

Quaternion product

$$\begin{pmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 - \boldsymbol{\beta}_1^T \boldsymbol{\beta}_2 \\ \alpha_1 \boldsymbol{\beta}_2 + \alpha_2 \boldsymbol{\beta}_1 + \boldsymbol{\beta}_1^\times \boldsymbol{\beta}_2 \end{pmatrix}$$

6.7.6 Euler parameters from the rotation matrix

$$\begin{aligned}\mathbf{R} &= (r_{ij}) \\ \mathbf{z} &= (z_0 \ z_1 \ z_2 \ z_3)^T := 2 \begin{pmatrix} \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix}^T \\ T &:= r_{00} := \text{Trace} \mathbf{R}\end{aligned}$$

The algorithm from Shepperd (1978) goes like this:

- Let $i = \arg \max_i \{r_{ii}\}$
- Compute $|z_i| = \sqrt{1 + 2r_{ii} - T}$
- Determine sign of z_i
- Determine the rest of \mathbf{z} from equations below

$$\begin{aligned}z_0 z_1 &= r_{32} - r_{23} & z_2 z_3 &= r_{32} + r_{23} \\ z_0 z_2 &= r_{13} - r_{31} & z_3 z_1 &= r_{13} + r_{31} \\ z_0 z_3 &= r_{21} - r_{12} & z_1 z_2 &= r_{21} + r_{12}\end{aligned}$$

6.8 Angular Velocity

Let $R \in SO(3)$

$$\begin{aligned}0 &= \frac{d}{dt}(\mathbf{I}) = \frac{d}{dt}(\mathbf{R}\mathbf{R}^T) = \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}(\dot{\mathbf{R}})^T \\ &\Rightarrow \dot{\mathbf{R}}\mathbf{R}^T \text{ skew-symmetric}\end{aligned}$$

Definition of angular velocity

$$\begin{aligned}(\boldsymbol{\omega}_{ab}^a)^\times &= \dot{\mathbf{R}}_b^a (\mathbf{R}_b^a)^T \Rightarrow \\ \dot{\mathbf{R}}_b^a &= (\boldsymbol{\omega}_{ab}^a)^\times \mathbf{R}_b^a \\ \dot{\mathbf{R}}_b^a &= \mathbf{R}_b^a (\boldsymbol{\omega}_{ab}^b)^\times\end{aligned}$$

It can be shown that

$$\boldsymbol{\omega} = \dot{\theta} \mathbf{k}$$

Where θ and \mathbf{k} are Angle Axis parameters.

$$\begin{aligned}\boldsymbol{\omega}_{ad}^a &= \boldsymbol{\omega}_{ab}^a + \boldsymbol{\omega}_{bc}^a + \boldsymbol{\omega}_{cd}^a \\ \dot{\mathbf{u}}^a &= \mathbf{R}_b^a (\dot{\mathbf{u}}^b + (\boldsymbol{\omega}_{ab}^b)^\times \mathbf{u}^b)\end{aligned}$$

6.9 Kinematic differential equations

6.9.4 Euler Angles

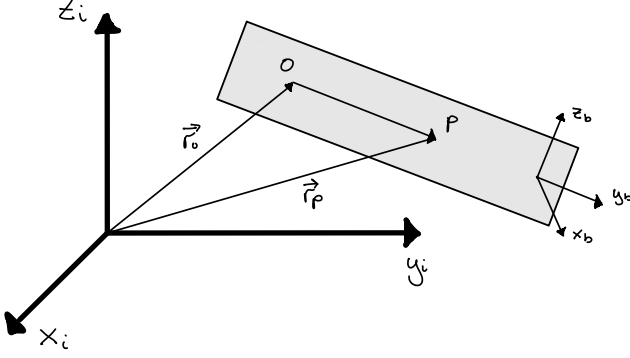
$$\begin{aligned}\boldsymbol{\omega}_{ad}^a &= \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \mathbf{R}_{z,\psi} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \mathbf{R}_{z,\psi} \mathbf{R}_{y,\theta} \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sin \psi \dot{\theta} + \cos \psi \cos \theta \dot{\phi} \\ \cos \psi \dot{\theta} + \sin \psi \cos \theta \dot{\phi} \\ \dot{\psi} - \sin \theta \dot{\phi} \end{pmatrix}\end{aligned}$$

3 Rigid Body Dynamics

6.12 Kinematics of a rigid body

$\vec{\omega}_{io}$ is the angular velocity of the o frame with respect to the i frame.

${}^i \frac{d}{dt} \vec{r}_o$ is the derivative of \vec{r}_o in the i frame.



Velocity and Acceleration

$$\begin{aligned}\vec{v}_p &:= \frac{{}^i d}{dt} \vec{r}_p \\ &= \vec{v}_o + \frac{{}^b d}{dt} \vec{r} + \vec{\omega}_{ib} \times \vec{r} \\ \vec{a}_p &:= \frac{{}^i d^2}{dt^2} \vec{r}_p \\ &= \vec{a}_o + \frac{{}^b d^2}{dt^2} \vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^b d}{dt} \vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})\end{aligned}$$

The last three terms are, respectively, the coriolis acceleration, Transversal acceleration and Centripetal acceleration. Note that

$$\vec{a}_o = \frac{{}^i d}{dt} \vec{v}_o = \frac{{}^b d}{dt} \vec{v}_o + \vec{\omega}_{ib} \times \vec{v}_o$$

6.13 The center of mass

The center of mass of a rigid body \mathcal{C} is defined to be

$$\vec{r}_c := \frac{1}{m} \int_{\mathcal{C}} \vec{r}_p dm$$

It can be shown that

$$\vec{v}_c = \frac{1}{m} \int_{\mathcal{C}} \vec{v}_p dm \quad \vec{a}_c = \frac{1}{m} \int_{\mathcal{C}} \vec{a}_p dm$$

where c denotes *center*

7.2 Forces and torques

Moment. The moment about a point P of the set $S = \{F_j\}_{j \in [1, n_F]}$ for forces is

$$\vec{N}_{S/P} = \sum_{j=1}^{n_F} r_{Pj} \times \vec{F}_j$$

Where \vec{r}_{Pj} is an arbitrary point along the line of action of \vec{F}_j

Torque is defined as the moment of the couple \mathcal{C} . A couple being a set of forces with $\mathbf{0}$ resultant force.

7.3 Newton-Euler Equations for rigid bodies

Angular Momentum. The angular momentum of the body b about the center of mass c is

$$\begin{aligned}h_{b/c} &= \int_b \mathbf{r} \times \mathbf{v} dm \\ &= \mathbf{M}_{b/c} \boldsymbol{\omega}_{ib} \\ \mathbf{T}_{bc} &= \frac{d}{dt} h_{b/c}\end{aligned}$$

Rotational Inertia / The inertia dyadic. The inertia matrix of the body b about the point c is

$$\begin{aligned}\mathbf{M}_{b/c} &= - \int_b \mathbf{r}^\times \mathbf{r}^\times dm \\ &= \int_b (\mathbf{r}^T \mathbf{r} \mathbb{I} - \mathbf{r} \mathbf{r}^T) dm \\ &= \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix}\end{aligned}$$

Where \mathbf{r} is the distance vector from the center of mass to the mass element being integrated

$$\begin{aligned}I_{xx} &= \int_b y^2 + z^2 dm & I_{xy} &= \int_b xy dm \\ I_{yy} &= \int_b x^2 + z^2 dm & I_{xz} &= \int_b xz dm \\ I_{zz} &= \int_b x^2 + y^2 dm & I_{yz} &= \int_b yz dm\end{aligned}$$

$$\mathbf{M}_{b/c}^i = \mathbf{R}_b^i \mathbf{M}_{b/c}^b \mathbf{R}_i^b$$

Equations of motion. Let b denote body, i an inertial frame, c the center of mass of b , \mathbf{F}_{bc} a resultant force acting on b with line of action through c and \mathbf{T}_{bc} the torque about c . Then

$$\begin{aligned}\mathbf{F}_{bc} &= m \mathbf{a}_c \\ \mathbf{T}_{bc} &= \mathbf{M}_{b/c} \boldsymbol{\alpha}_{ib} + \boldsymbol{\omega}_{ib} \times (\mathbf{M}_{b/c} \boldsymbol{\omega}_{ib})\end{aligned}$$

On compact matrix form

$$\begin{pmatrix} m \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{b/c}^b \end{pmatrix} \begin{pmatrix} \mathbf{a}_c^b \\ \boldsymbol{\alpha}_{ib}^b \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{bc}^b \\ \mathbf{T}_{bc}^b \end{pmatrix}$$

$$\begin{pmatrix} m \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{b/c}^b \end{pmatrix} \begin{pmatrix} \dot{\mathbf{v}}_c^b \\ \dot{\boldsymbol{\alpha}}_{ib}^b \end{pmatrix} + \begin{pmatrix} m (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{v}_c^b \\ (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{bc}^b \\ \mathbf{T}_{bc}^b \end{pmatrix}$$

Kinetic energy. The kinetic energy of the body b in an inertial reference frame i is

$$K = \frac{1}{2} m (\mathbf{v}_c^b)^T \mathbf{v}_c^b + \frac{1}{2} (\boldsymbol{\omega}_{ib}^b)^T \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b$$

The parallel axes theorem. The inertia matrix of b about o is related to the inertia matrix of b about c according to

$$\begin{aligned} \mathbf{M}_{b/o}^b &= \mathbf{M}_{b/c}^b - m(\mathbf{r}_g^b)^\times (\mathbf{r}_g^b)^\times \\ &= \mathbf{M}_{b/c}^b - m(\|\mathbf{r}_g^b\|^2 \mathbb{I} - \mathbf{r}_g^b (\mathbf{r}_g^b)^T) \end{aligned}$$

Where \mathbf{r}_g^b is the vector from c to o

4 Lagrange Mechanics

8.2 Lagrange Mechanics

The lagrangian. Define a set of generalized coordinates \mathbf{q} . Let $T(\mathbf{q}, \dot{\mathbf{q}}, t)$ be the kinetic energy and $U(\mathbf{q}, \dot{\mathbf{q}}, t)$ the potential energy (Sometimes V). Then the lagrangian is defined to be

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q})$$

The equation of motion is

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= \tau_i \\ \frac{d}{dt} (\nabla_{\dot{\mathbf{q}}} L) - \nabla_{\mathbf{q}} L &= \boldsymbol{\tau} \end{aligned}$$

Where τ_i is a generalized actuator force

$$\tau_i = \sum_{k=1}^N \frac{\partial \mathbf{r}_k}{\partial q_i} \cdot \mathbf{F}_k$$

$\mathbf{r}_k(\mathbf{q})$ is the position of the point of application of the force \mathbf{F}_k . In general τ_i is a force or a torque.

Constrained Lagrange. Having the constraints

$$\mathbf{c}(\mathbf{q}) = \mathbf{0}$$

The system can be described by

$$\begin{aligned} L(\mathbf{q}, \dot{\mathbf{q}}, t) &= T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}) - \mathbf{z}^T \mathbf{c}(\mathbf{q}) \\ \frac{d}{dt} (\nabla_{\dot{\mathbf{q}}} L) - \nabla_{\mathbf{q}} L &= \boldsymbol{\tau} \\ \mathbf{c}(\mathbf{q}) &= \mathbf{0} \end{aligned}$$

Baumgarte stabilization. Instead of imposing

$$\ddot{\mathbf{c}}(\mathbf{q}) = \mathbf{0}$$

Impose

$$\ddot{\mathbf{c}} + 2\alpha\dot{\mathbf{c}} + \alpha^2\mathbf{c} = \mathbf{0}$$

As to reduce drifts in the constraints resulting from

$$\ddot{\mathbf{c}} = \mathbf{0}$$

not begin satisfied exactly when doing numeric computations.

5 Differential Algebraic Equations

14.2 Preliminaries

5.1 Differential Algebraic Equations

Definition of DEA. The differential equation defined by

$$F(\dot{x}, x, u, t) = 0$$

Es a DAE if

$$\frac{\partial F}{\partial \dot{x}}$$

is rank deficient

Fully-explicit DAE

$$\begin{aligned} F(\dot{x}, x, z, u) &= 0 \\ \det \left| \frac{\partial F}{\partial \dot{x}} \right| &= 0 \end{aligned}$$

Can be rewritten as

$$\begin{aligned} \dot{x} &= v \\ 0 &= F(v, x, z, u) \end{aligned}$$

Semi-explicit DAE

$$\begin{aligned} \dot{x} &= f(x, z, u) \\ 0 &= g(x, z, u) \end{aligned}$$

This can be rewritten as

$$F(\dot{x}, x, z, u) = \begin{pmatrix} \dot{x} - f(x, z, u) \\ g(x, z, u) \end{pmatrix} = 0$$

Tikhonov Theorem. Consider the ordinary differential equation

$$\begin{aligned} \dot{x} &= f(x, z) \\ \varepsilon \dot{z} &= g(x, z) \end{aligned}$$

If

- dynamics of $\dot{z} = g(x, z)$ stable $\forall x$
- $\frac{\partial g}{\partial z}$ is full rank

then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} x(t) &= x_0(t) \\ \lim_{\varepsilon \rightarrow 0} z(t) &= z_0(t) \end{aligned}$$

where $z_0(t)$ and $x_0(t)$ is the solution to the ODE above modified to a DAE where $\varepsilon = 0$

Theorem: Solvability of DAE. A fully implicit DAE with smooth

$$F(\dot{x}, x, z, u) = 0$$

Can be readily solved (solved for \dot{x} and z) if

$$\begin{pmatrix} \frac{\partial F}{\partial \dot{x}} & \frac{\partial F}{\partial z} \end{pmatrix}$$

is full rank on all trajectories \dot{x} , z , x and u . Note that all **Index 1** DAEs fullfil these requirements. The theorem implies that

$$\begin{aligned} \dot{x} &= f(x, z, u) \\ 0 &= g(x, z, u) \end{aligned}$$

with smooth f can be solved if

$$\frac{\partial g}{\partial z}$$

is full rank on all trajectories z , x and u

Definition: Differential index of a DAE is the numner of times the differentiation operator $\frac{d}{dt}$ must be applied to the equations in order to convert the DAE into an ODE.

6 Simulation methods

Butcher tableau

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

$$K_n = f(t_k + \Delta t c_n, x_n + \Delta t (A_n) K)$$

$$x_{k+1} = x_k + \Delta t \mathbf{b}^T K$$

The method is **consistent** if

$$c_n = \Sigma_i A_{ni}$$

$$\Sigma_i \mathbf{b}_i = 1$$

The **stages** of a Runge Kutta method is the number of elements in \mathbf{c} . The Butcher tableau defines an explicit integrator if and only if the diagonal elements and the upper-diagonal elements are zero.

Stability function For the Butcher Tableau

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

The stability function is

$$R(z) = \frac{\det |\mathbb{I} - z(\mathbf{A} - \mathbf{1}\mathbf{b}^T)|}{\det |\mathbb{I} - z\mathbf{A}|}$$

Explicit Euler

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \cdot \mathbf{f}(\mathbf{x}, \mathbf{u})$$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Global error $\|\mathbf{x}_N - \mathbf{x}(T)\| = \mathcal{O}(\Delta t)$

Stability function $R(z) = 1 + z$

Explicit mid-point rule

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1/2 \end{array}$$

Global error $\|\mathbf{x}_N - \mathbf{x}(T)\| = \mathcal{O}(\Delta t^2)$

Ralston's RK2

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 1/3 \end{array}$$

Heun's RK2

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

Generic second-order method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \alpha & \alpha & 0 \\ \hline & 1 - \frac{1}{2\alpha} & \frac{1}{2\alpha} \end{array}$$

Generic third-order method

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \alpha & \alpha & 0 & 0 \\ 1 & 1 + \frac{1-\alpha}{\alpha(3\alpha-2)} & -\frac{1-\alpha}{\alpha(3\alpha-2)} & 0 \\ \hline & \frac{1}{2} - \frac{1}{6\alpha} & \frac{1}{6\alpha(1-\alpha)} & \frac{2-3\alpha}{6(1-\alpha)} \end{array}$$

"The" RK4 method

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Global error $\|\mathbf{x}_N - \mathbf{x}(T)\| = \mathcal{O}(\Delta t^4)$

One-Step error

$$\|\mathbf{x}_{k+1} - \mathbf{x}(t_{k+1})\|$$

Taylor expand $\mathbf{x}(t)$ around t_k . Do the same for \mathbf{x}_{k+1} . Subtract and get the results

Global error Get error on timestep T by multiplying local error with $N = \frac{T}{\Delta t}$

stability of integration method: Consider the ODE

$$\dot{x} = \lambda x \quad x(0) = x_0$$

Perform one step of the RK method to find

$$x_{n+1} = R(z)x_n \quad z = \Delta t \lambda$$

The integration method is unstable if

$$|R(z)| > 1$$

A-stability A method is A-stable if the region of stability is the entire left-half plane.

$$|R(\lambda \Delta t)| \leq 1 \forall \lambda \in \mathbb{C} \text{ s.t. } \Re(\lambda) \leq 0$$

L-stability. A Runge kutta method is L-stable if it is A-stable and

$$\lim_{\omega \rightarrow \pm\infty} |R(i\omega h)| = 0$$

Implicit methods

- Can achieve high and systematic orders
- Can be stable regardless of the step size
- Can handle the simulation of DAEs (must be index 1)
- Can give huge speed-up when system is stiff
- Can achieve order $o = 2s$ for any stage s . Explicit methods can only achieve $o = s$ and only for $s \leq 4$

Implicit Euler

$$\frac{1}{1} \bigg| \frac{1}{1}$$

$$\begin{aligned} k_1 &= f(x_n + \Delta t k_1) \\ x_{n+1} &= x_n + \Delta t k_1 \\ &= x_n + \Delta t f(x_{n+1}) \end{aligned}$$

Gauss-Legendre collocation method Generates order $o = 2s$ A-stable IRK methods. Let s be the number of stages. Find the roots (τ_i) of

$$P_s(\tau) = \frac{1}{s!} \frac{d^s}{d\tau^s} ((\tau^2 - \tau)^s)$$

Build polynomials

$$\ell_i(\tau) = \prod_{j \neq i} \frac{\tau - \tau_j}{\tau_i - \tau_j}$$

Integrate them

$$L_i(\tau) = \int_0^\tau \ell_i(\xi) d\xi$$

Calculate A , \mathbf{b} and \mathbf{c}

$$A_{ji} = L_i(\tau_j) \quad b_i = L_i(1) \quad c_j = \tau_j$$

Adaptive integrator. Use two methods: x_{n+1} and \hat{x}_{n+1} , of different order: p and $p+1$. The local error is

$$\begin{aligned} \epsilon_{n+1} &:= |x_{n+1} - \hat{x}_{n+1}| \\ &\approx \mathcal{O}(\Delta t^{p+1}) - \mathcal{O}(\Delta t^{p+2}) \\ &\approx C \Delta t^{p+1} \\ e_{\text{tol}} &= C \Delta t_{\text{new}}^{p+1} \Rightarrow \\ \Delta t_{\text{new}} &= \Delta t \left(\frac{e_{\text{tol}}}{\epsilon_{n+1}} \right)^{\frac{1}{p+1}} \end{aligned}$$

Newtons method

$$\begin{aligned} \mathbf{f}(\mathbf{x} + \Delta \mathbf{x}) &\approx \mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \Delta \mathbf{x} = \mathbf{0} \\ \Delta \mathbf{x} &= - \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) \right)^{-1} \mathbf{f}(\mathbf{x}) \\ \mathbf{x}_{n+1} &= \mathbf{x}_n + \alpha \Delta \mathbf{x} \end{aligned}$$

- May not work if jacobian is singular
- Has quadratic convergence when it converges
- May diverge

7 Math

Inverse of 2×2 Matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Multivariable derivative rules

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \mathbf{a}^T \mathbf{x} &= \mathbf{a}^T & \nabla_{\mathbf{x}} \mathbf{a}^T \mathbf{x} &= \mathbf{a} \\ \frac{\partial}{\partial \mathbf{x}} \mathbf{A} \mathbf{x} &= \mathbf{A} & \nabla_{\mathbf{x}} \mathbf{A} \mathbf{x} &= \mathbf{A}^T \\ \frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{A} &= \mathbf{A}^T & \nabla_{\mathbf{x}} \mathbf{x}^T \mathbf{A} &= \mathbf{A} \end{aligned}$$

Second order terms

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} &= \frac{1}{2} \mathbf{x}^T (\mathbf{A}^T + \mathbf{A}) \\ \nabla_{\mathbf{x}} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} &= \frac{1}{2} (\mathbf{A}^T + \mathbf{A}) \mathbf{x} \end{aligned}$$

Multivariable Chain rule

$$\begin{aligned} \frac{\partial \mathbf{f}(\mathbf{g}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial \mathbf{f}}{\partial \mathbf{g}} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \\ \frac{\partial \mathbf{f}(\mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial \mathbf{f}}{\partial \mathbf{g}} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{h}} \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \end{aligned}$$

Some derivatives

$$\begin{aligned} \frac{d}{dt} \sinh(t) &= \cosh(t) \\ \frac{d}{dt} \cosh(t) &= \sinh(t) \\ \frac{d}{dt} \tanh(t) &= \frac{d \sinh(t)}{dt \cosh(t)} = \frac{1}{\cosh^2 t} = 1 - \tanh^2(x) \\ \frac{d}{dt} \arctan(t) &= \frac{1}{1 + t^2} \end{aligned}$$

Taylor's theorem

Let $k \geq 1$ be an integer and let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ be $k + 1$ times differentiable at the point $a \in \mathbb{R}$. Then

$$f(x) = \sum_{n=0}^k \frac{1}{n!} f^{(n)}(a)(x-a)^n + \frac{1}{(1+k)!} f^{(k+1)}(\xi)(x-a)^{k+1}$$

for some $\xi \in [x, a]$

Common Taylor expansions

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \sin(x) &= \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$