

# TTK4130 - Cheat Sheet

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May 21, 2023

<https://github.com/haakonbaa/TTK4130-cheatsheet>

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# 1 Intro

## 1.2 State space methods

### 1.2.1 State space models

State Space Model is on the form

$$\dot{x} = f(x, u, t)$$

Linear time invariant system

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$y = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau) d\tau + Du(t)$$

### 1.2.2 Second order models of mechanical systems

Second order models on the form

$$M(q)\ddot{q} + f(q, \dot{q}) = u$$

Can be written as

$$\begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ M^{-1}(q)(-f(q, \dot{q}) + u) \end{pmatrix}$$

### 1.2.3 Linearization of state space models

Linearization of time varying systems

$$\dot{x} = f(x, u, t)$$

$$y = h(x, u, t)$$

Find two functions  $x_0$  and  $u_0$  being solutions to the system

$$\dot{x}_0 = f(x_0(t), u_0(t), t)$$

Define perturbations

$$x(t) = x_0(t) + \Delta x(t)$$

$$u(t) = u_0(t) + \Delta u(t)$$

$$y(t) = y_0(t) + \Delta y(t)$$

Let  $\mathcal{C} = \{x_0(t), u_0(t)\}$ . The linearized system is

$$\Delta \dot{x} \approx \left. \frac{\partial f}{\partial x} \right|_{\mathcal{C}} \Delta x + \left. \frac{\partial f}{\partial u} \right|_{\mathcal{C}} \Delta u$$

$$\Delta \dot{y} \approx \left. \frac{\partial h}{\partial x} \right|_{\mathcal{C}} \Delta x + \left. \frac{\partial h}{\partial u} \right|_{\mathcal{C}} \Delta u$$

## 1.5 ODE's

General formulation

$$\varphi(y^{(m)}, \dots, y, u^{(m-1)}, \dots, u) = 0$$

Lipschitz continuous. A function

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

is said to be *Lipschitz continuous* if

$$\begin{aligned} \exists c > 0 \in \mathbb{R} & \quad \text{s.t.} \\ \|f(x) - f(y)\| < c\|x - y\| & \quad \forall x, y \in \mathbb{R} \end{aligned}$$

**Theorem: existence of unique solution.** Consider the ODE

$$\dot{x} = f(x)$$

If  $f$  is Lipschitz continuous then  $x(t)$  exists and is unique for all  $t$

**Mean Value Theorem** suppose  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  then

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

This can be used to show that if  $f$  is continuous and differentiable everywhere it is also Lipschitz.

**Theorem 2: existence of unique solution.** Consider the ODE

$$\dot{x} = f(x)$$

if  $f$  is continuously differentiable ( $\frac{\partial f}{\partial x}$  exists and is continuous), then the solution to the ODE exists and is unique on some time interval.

## 2 Rotations

### 6.2 Vectors

The *skew-symmetric* matrix form of the coordinate vector  $\mathbf{u}$  is defined by

$$\mathbf{u}^x = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Notation:  $\mathbf{v}_{ab}^c$  means the vector from point  $a$  to point  $b$  (or often the origin of the reference frames  $a$  and  $b$ ) described in the reference frame  $c$

### 6.4 The Rotation Matrix

The coordinate transformation from frame  $b$  to frame  $a$  is given by

$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$

Properties of the rotation matrix

$$\begin{aligned} \mathbf{R}_a^b \mathbf{R}_b^a &= \mathbf{I} = \mathbf{R}_b^a \mathbf{R}_a^b \\ (\mathbf{R}_a^b)^{-1} &= (\mathbf{R}_a^b)^T = \mathbf{R}_b^a \\ \mathbf{R}_b^a &= \begin{pmatrix} \mathbf{b}_1^a & \mathbf{b}_2^a & \mathbf{b}_3^a \end{pmatrix} \\ \det \mathbf{R}_a^b &= 1 \end{aligned}$$

$\mathbf{R}$  is a rotation matrix if and only if it is an element of  $SO(3)$

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3 \times 3} | \mathbf{R}^T \mathbf{R} = \mathbf{I} \wedge \det \mathbf{R} = 1\}$$

Rotation matrices in three dimensions

$$\begin{aligned} \mathbf{R}_x(\phi) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} \\ \mathbf{R}_y(\theta) &= \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \\ \mathbf{R}_z(\psi) &= \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Matrix transformations in different reference frames

$$\begin{aligned} \mathbf{D}^a &= \mathbf{R}_b^a \mathbf{D}^b \mathbf{R}_a^b \\ (\mathbf{u}^b)^\times &= \mathbf{R}_a^b (\mathbf{u}^a)^\times \mathbf{R}_b^a \end{aligned}$$

The transformation of position and orientation from frame  $b$  to frame  $a$  is

$$\begin{aligned} \mathbf{T}_b^a &= \begin{pmatrix} \mathbf{R}_b^a & \mathbf{r}_{ab}^a \\ \mathbf{0}^T & 1 \end{pmatrix} \\ \mathbf{T}_b^a \begin{pmatrix} \mathbf{v}^b \\ 1 \end{pmatrix}^T &= \begin{pmatrix} \mathbf{v}^a \\ 1 \end{pmatrix} \\ (\mathbf{T}_b^a)^{-1} &= \mathbf{T}_a^b = \begin{pmatrix} \mathbf{R}_a^b & \mathbf{r}_{ba}^b \\ \mathbf{0}^T & 1 \end{pmatrix} \end{aligned}$$

The Special Euclidean group is the set of all transformations from one reference frames to another

$$SE(3) = \left\{ \mathbf{T} = \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3} \middle| \mathbf{R} \in SO(3) \wedge \mathbf{r} \in \mathbb{R}^3 \right\}$$

### 6.5 Euler Angles

#### Roll-Pitch-Yaw Euler angles

$$\mathbf{R}_a^b = \mathbf{R}_z(\psi) \mathbf{R}_y(\theta) \mathbf{R}_x(\phi)$$

**Classical Euler angles.** The orientation is described by a rotation about the  $z$  axis, then the resulting  $y$  axis. And then again the resulting  $z$  axis.

$$\mathbf{R}_a^b = \mathbf{R}_z(\psi) \mathbf{R}_y(\theta) \mathbf{R}_z(\phi)$$

### 6.6 Angle Axis Description of rotation

#### 6.6.5 Rotation Matrix

**Angle-axis parameters** All rotation matrices have an eigen vector with eigen value 1. A rotation can be uniquely described by the direction of this vector and an angle  $\theta$  being the rotation about this vector.

$$\begin{aligned} (\theta, \mathbf{k}) \text{ s.t. } \|\mathbf{k}\| &= 1 \\ \mathbf{R}_b^a &= \cos \theta \mathbf{I} + \sin \theta (\mathbf{k}_a)^\times + (1 - \cos \theta) \mathbf{k}_a \mathbf{k}_a^T \\ \cos \theta &= \frac{\text{trace} \mathbf{R} - 1}{2} \\ \mathbf{R}_b^a &= \exp\{\mathbf{k}^\times \theta\} \end{aligned}$$

### 6.7 Euler parameters

#### 6.7.1 Definition

$$\begin{aligned} \eta &= \cos \frac{\theta}{2} \\ \boldsymbol{\epsilon} &= \mathbf{k} \sin \frac{\theta}{2} \\ \mathbf{R}_e(\eta, \boldsymbol{\epsilon}) &= \mathbf{I} + 2\eta \boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times \boldsymbol{\epsilon}^\times \\ \eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} &= 1 \end{aligned}$$

#### 6.7.3 Quaternions

The following can be treated as a unit quaternion

$$\mathbf{p} = \begin{pmatrix} \eta \\ \boldsymbol{\epsilon} \end{pmatrix}$$

A unit quaternion satisfies

$$\mathbf{p}^T \mathbf{p} = \eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = 1$$

Quaternion product

$$\begin{pmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 - \boldsymbol{\beta}_1^T \boldsymbol{\beta}_2 \\ \alpha_1 \boldsymbol{\beta}_2 + \alpha_2 \boldsymbol{\beta}_1 + \boldsymbol{\beta}_1^\times \boldsymbol{\beta}_2 \end{pmatrix}$$

### 6.7.6 Euler parameters from the rotation matrix

$$\begin{aligned}\mathbf{R} &= (r_{ij}) \\ \mathbf{z} &= (z_0 \ z_1 \ z_2 \ z_3)^T := 2 \begin{pmatrix} \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix}^T \\ T &:= r_{00} := \text{Trace} \mathbf{R}\end{aligned}$$

The algorithm from Shepperd (1978) goes like this:

- Let  $i = \arg \max_i \{r_{ii}\}$
- Compute  $|z_i| = \sqrt{1 + 2r_{ii} - T}$
- Determine sign of  $z_i$
- Determine the rest of  $\mathbf{z}$  from equations below

$$\begin{aligned}z_0 z_1 &= r_{32} - r_{23} & z_2 z_3 &= r_{32} + r_{23} \\ z_0 z_2 &= r_{13} - r_{31} & z_3 z_1 &= r_{13} + r_{31} \\ z_0 z_3 &= r_{21} - r_{12} & z_1 z_2 &= r_{21} + r_{12}\end{aligned}$$

## 6.8 Angular Velocity

Let  $R \in SO(3)$

$$\begin{aligned}0 &= \frac{d}{dt}(\mathbf{I}) = \frac{d}{dt}(\mathbf{R}\mathbf{R}^T) = \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}(\dot{\mathbf{R}})^T \\ &\Rightarrow \dot{\mathbf{R}}\mathbf{R}^T \text{ skew-symmetric}\end{aligned}$$

Definition of angular velocity

$$\begin{aligned}(\boldsymbol{\omega}_{ab}^a)^\times &= \dot{\mathbf{R}}_b^a (\mathbf{R}_b^a)^T \Rightarrow \\ \dot{\mathbf{R}}_b^a &= (\boldsymbol{\omega}_{ab}^a)^\times \mathbf{R}_b^a \\ \dot{\mathbf{R}}_b^a &= \mathbf{R}_b^a (\boldsymbol{\omega}_{ab}^b)^\times\end{aligned}$$

It can be shown that

$$\boldsymbol{\omega} = \dot{\theta} \mathbf{k}$$

Where  $\theta$  and  $\mathbf{k}$  are Angle Axis parameters.

$$\begin{aligned}\boldsymbol{\omega}_{ad}^a &= \boldsymbol{\omega}_{ab}^a + \boldsymbol{\omega}_{bc}^a + \boldsymbol{\omega}_{cd}^a \\ \dot{\mathbf{u}}^a &= \mathbf{R}_b^a (\dot{\mathbf{u}}^b + (\boldsymbol{\omega}_{ab}^b)^\times \mathbf{u}^b)\end{aligned}$$

## 6.9 Kinematic differential equations

### 6.9.4 Euler Angles

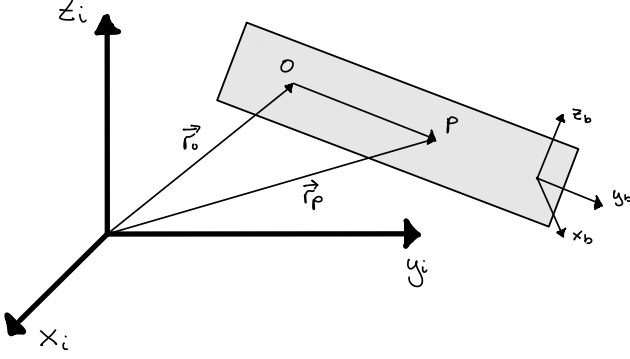
$$\begin{aligned}\boldsymbol{\omega}_{ad}^a &= \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \mathbf{R}_{z,\psi} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \mathbf{R}_{z,\psi} \mathbf{R}_{y,\theta} \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sin \psi \dot{\theta} + \cos \psi \cos \theta \dot{\phi} \\ \cos \psi \dot{\theta} + \sin \psi \cos \theta \dot{\phi} \\ \dot{\psi} - \sin \theta \dot{\phi} \end{pmatrix}\end{aligned}$$

### 3 Rigid Body Dynamics

#### 6.12 Kinematics of a rigid body

$\vec{\omega}_{io}$  is the angular velocity of the  $o$  frame with respect to the  $i$  frame.

${}^i \frac{d}{dt} \vec{r}_o$  is the derivative of  $\vec{r}_o$  in the  $i$  frame.



Velocity and Acceleration

$$\begin{aligned}\vec{v}_p &:= {}^i \frac{d}{dt} \vec{r}_p \\ &= \vec{v}_o + \frac{b}{dt} \vec{r} + \vec{\omega}_{ib} \times \vec{r} \\ \vec{a}_p &:= {}^i \frac{d^2}{dt^2} \vec{r}_p \\ &= \vec{a}_o + \frac{b}{dt^2} \vec{r} + 2\vec{\omega}_{ib} \times \frac{b}{dt} \vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})\end{aligned}$$

The last three terms are, respectively, the coriolis acceleration, Transversal acceleration and Centripetal acceleration. Note that

$$\vec{a}_o = \frac{i}{dt} \vec{v}_o = \frac{b}{dt} \vec{v}_o + \vec{\omega}_{ib} \times \vec{v}_o$$

#### 6.13 The center of mass

The center of mass of a rigid body  $\mathcal{C}$  is defined to be

$$\vec{r}_c := \frac{1}{m} \int_{\mathcal{C}} \vec{r}_p dm$$

It can be shown that

$$\vec{v}_c = \frac{1}{m} \int_{\mathcal{C}} \vec{v}_p dm \quad \vec{a}_c = \frac{1}{m} \int_{\mathcal{C}} \vec{a}_p dm$$

where  $c$  denotes *center*

#### 7.2 Forces and torques

**Moment.** The moment about a point  $P$  of the set  $S = \{F_j\}_{j \in [1, n_F]}$  for forces is

$$\vec{N}_{S/P} = \sum_{j=1}^{n_F} r_{P_j} \times \vec{F}_j$$

Where  $\vec{r}_{P_j}$  is an arbitrary point along the line of action of  $\vec{F}_j$

**Torque** is defined as the moment of the couple  $\mathcal{C}$ . A couple being a set of forces with  $\mathbf{0}$  resultant force.

#### 7.3 Newton-Euler Equations for rigid bodies

**Angular Momentum.** The angular momentum of the body  $b$  about the center of mass  $c$  is

$$\begin{aligned}\mathbf{h}_{b/c} &= \int_b \mathbf{r} \times \mathbf{v} dm \\ &= \mathbf{M}_{b/c} \boldsymbol{\omega}_{ib} \\ \mathbf{T}_{bc} &= \frac{d}{dt} \mathbf{h}_{b/c}\end{aligned}$$

**Rotational Inertia / The inertia dyadic.** The inertia matrix of the body  $b$  about the point  $c$  is

$$\begin{aligned}\mathbf{M}_{b/c} &= - \int_b \mathbf{r}^\times \mathbf{r}^\times dm \\ &= \int_b (\mathbf{r}^T \mathbf{r} \mathbb{I} - \mathbf{r} \mathbf{r}^T) dm \\ &= \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix}\end{aligned}$$

Where  $\mathbf{r}$  is the distance vector from the center of mass to the mass element being integrated

$$\begin{aligned}I_{xx} &= \int_b y^2 + z^2 dm & I_{xy} &= \int_b xy dm \\ I_{yy} &= \int_b x^2 + z^2 dm & I_{xz} &= \int_b xz dm \\ I_{zz} &= \int_b x^2 + y^2 dm & I_{yz} &= \int_b yz dm\end{aligned}$$

$$\mathbf{M}_{b/c}^i = \mathbf{R}_b^i \mathbf{M}_{b/c}^b \mathbf{R}_i^b$$

**Equations of motion.** Let  $b$  denote body,  $i$  an inertial frame,  $c$  the center of mass of  $b$ ,  $\mathbf{F}_{bc}$  a resultant force acting on  $b$  with line of action through  $c$  and  $\mathbf{T}_{bc}$  the torque about  $c$ . Then

$$\begin{aligned}\mathbf{F}_{bc} &= m \mathbf{a}_c \\ \mathbf{T}_{bc} &= \mathbf{M}_{b/c} \boldsymbol{\alpha}_{ib} + \boldsymbol{\omega}_{ib} \times (\mathbf{M}_{b/c} \boldsymbol{\omega}_{ib})\end{aligned}$$

On compact matrix form

$$\begin{aligned}\begin{pmatrix} m \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{b/c}^b \end{pmatrix} \begin{pmatrix} \mathbf{a}_c^b \\ \boldsymbol{\alpha}_{ib}^b \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b \end{pmatrix} &= \begin{pmatrix} \mathbf{F}_{bc}^b \\ \mathbf{T}_{bc}^b \end{pmatrix} \\ \begin{pmatrix} m \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{b/c}^b \end{pmatrix} \begin{pmatrix} \dot{\mathbf{v}}_c^b \\ \dot{\boldsymbol{\alpha}}_{ib}^b \end{pmatrix} + \begin{pmatrix} m (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{v}_c^b \\ (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b \end{pmatrix} &= \begin{pmatrix} \mathbf{F}_{bc}^b \\ \mathbf{T}_{bc}^b \end{pmatrix}\end{aligned}$$

**Kinetic energy.** The kinetic energy of the body  $b$  in an inertial reference frame  $i$  is

$$K = \frac{1}{2} m (\mathbf{v}_c^b)^T \mathbf{v}_c^b + \frac{1}{2} (\boldsymbol{\omega}_{ib}^b)^T \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b$$

**The parallel axes theorem.** The inertia matrix of  $b$  about  $o$  is related to the inertia matrix of  $b$  about  $c$  according to

$$\begin{aligned}\mathbf{M}_{b/o}^b &= \mathbf{M}_{b/c}^b - m (\mathbf{r}_g^b)^\times (\mathbf{r}_g^b)^\times \\ &= \mathbf{M}_{b/c}^b - m (\|\mathbf{r}_g^b\|^2 \mathbb{I} - \mathbf{r}_g^b (\mathbf{r}_g^b)^T)\end{aligned}$$

Where  $\mathbf{r}_g^b$  is the vector from  $c$  to  $o$

## 4 Lagrange Mechanics

### 8.2 Lagrange Mechanics

**The lagrangian.** Define a set of generalized coordinates  $\mathbf{q}$ . Let  $T(\mathbf{q}, \dot{\mathbf{q}}, t)$  be the kinetic energy and  $U(\mathbf{q}, \dot{\mathbf{q}}, t)$  the potential energy (Sometimes  $V$ ). Then the lagrangian is defined to be

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q})$$

The equation of motion is

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= \tau_i \\ \frac{d}{dt} (\nabla_{\dot{\mathbf{q}}} L) - \nabla_{\mathbf{q}} L &= \boldsymbol{\tau} \end{aligned}$$

Where  $\tau_i$  is a generalized actuator force

$$\tau_i = \sum_{k=1}^N \frac{\partial \mathbf{r}_k}{\partial q_i} \cdot \mathbf{F}_k$$

$\mathbf{r}_k(\mathbf{q})$  is the position of the point of application of the force  $\mathbf{F}_k$ . In general  $\tau_i$  is a force or a torque.

**Constrained Lagrange.** Having the constraints

$$\mathbf{c}(\mathbf{q}) = \mathbf{0}$$

The system can be described by

$$\begin{aligned} L(\mathbf{q}, \dot{\mathbf{q}}, t) &= T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}) - \mathbf{z}^T \mathbf{c}(\mathbf{q}) \\ \frac{d}{dt} (\nabla_{\dot{\mathbf{q}}} L) - \nabla_{\mathbf{q}} L &= \boldsymbol{\tau} \\ \mathbf{c}(\mathbf{q}) &= \mathbf{0} \end{aligned}$$

**Baumgarte stabilization.** Instead of imposing

$$\ddot{\mathbf{c}}(\mathbf{q}) = \mathbf{0}$$

Impose

$$\ddot{\mathbf{c}} + 2\alpha\dot{\mathbf{c}} + \alpha^2\mathbf{c} = \mathbf{0}$$

As to reduce drifts in the constraints resulting from

$$\ddot{\mathbf{c}} = \mathbf{0}$$

not begin satisfied exactly when doing numeric computations.

## 5 Differential Algebraic Equations

### 14.2 Preliminaries

#### 5.1 Differential Algebraic Equations

**Definition of DEA.** The differential equation defined by

$$F(\dot{x}, x, u, t) = 0$$

Es a DAE if

$$\frac{\partial F}{\partial \dot{x}}$$

is rank deficient

#### Fully-explicit DAE

$$F(\dot{x}, x, z, u) = 0$$

$$\det \left| \frac{\partial F}{\partial \dot{x}} \right| = 0$$

Can be rewritten as

$$\dot{x} = v$$

$$0 = F(v, x, z, u)$$

#### Semi-explicit DAE

$$\dot{x} = f(x, z, u)$$

$$0 = g(x, z, u)$$

This can be rewritten as

$$F(\dot{x}, x, z, u) = \begin{pmatrix} \dot{x} - f(x, z, u) \\ g(x, z, u) \end{pmatrix} = 0$$

**Tikhonov Theorem.** Consider the ordinary differential equation

$$\dot{x} = f(x, z)$$

$$\varepsilon \dot{z} = g(x, z)$$

If

- dynamics of  $\dot{z} = g(x, z)$  stable  $\forall x$
- $\frac{\partial g}{\partial z}$  is full rank

then

$$\lim_{\varepsilon \rightarrow 0} x(t) = x_0(t)$$

$$\lim_{\varepsilon \rightarrow 0} z(t) = z_0(t)$$

where  $z_0(t)$  and  $x_0(t)$  is the solution to the ODE above modified to a DAE where  $\varepsilon = 0$

**Theorem: Solvability of DAE.** A fully implicit DAE with smooth

$$F(\dot{x}, x, z, u) = 0$$

Can be readily solved (solved for  $\dot{x}$  and  $z$ ) if

$$\begin{pmatrix} \frac{\partial F}{\partial \dot{x}} & \frac{\partial F}{\partial z} \end{pmatrix}$$

is full rank on all trajectories  $\dot{x}$ ,  $z$ ,  $x$  and  $u$ . Note that all **Index 1** DAEs fullfil these requirements. The theorem implies that

$$\dot{x} = f(x, z, u)$$

$$0 = g(x, z, u)$$

with smooth  $f$  can be solved if

$$\frac{\partial g}{\partial z}$$

is full rank on all trajectories  $z$ ,  $x$  and  $u$

**Definition: Differential index of a DAE** is the number of times the differentiation operator  $\frac{d}{dt}$  must be applied to the equations in order to convert the DAE into an ODE.

## 6 Simulation methods

### Butcher tableau

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

$$K_n = f(t_k + \Delta t c_n, x_n + \Delta t(A_n)K)$$

$$x_{k+1} = x_k + \Delta t \mathbf{b}^T K$$

The method is **consistent** if

$$c_n = \Sigma_i A_{ni}$$

$$\Sigma_i \mathbf{b}_i = 1$$

The **stages** of a Runge Kutta method is the number of elements in  $\mathbf{c}$ . The Butcher Tableau defines an explicit integrator if and only if the diagonal elements and the upper-diagonal elements are zero.

**Stability function** For the Butcher Tableau

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

The stability function is

$$R(z) = \frac{\det[\mathbb{I} - z(\mathbf{A} - \mathbf{1}\mathbf{b}^T)]}{\det[\mathbb{I} - z\mathbf{A}]}$$

**Explicit Euler**

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \cdot \mathbf{f}(\mathbf{x}, \mathbf{u})$$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Global error  $\|\mathbf{x}_N - \mathbf{x}(T)\| = \mathcal{O}(\Delta t)$

Stability function  $R(z) = 1 + z$

**Explicit mid-point rule**

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1/2 \end{array}$$

Global error  $\|\mathbf{x}_N - \mathbf{x}(T)\| = \mathcal{O}(\Delta t^2)$

**Ralston's RK2**

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 1/3 \end{array}$$

**Heun's RK2**

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

**Generic second-order method**

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \alpha & \alpha & 0 \\ \hline & 1 - \frac{1}{2\alpha} & \frac{1}{2\alpha} \end{array}$$

**Generic third-order method**

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \alpha & \alpha & 0 & 0 \\ 1 & 1 + \frac{1-\alpha}{\alpha(3\alpha-2)} & -\frac{1-\alpha}{\alpha(3\alpha-2)} & 0 \\ \hline & \frac{1}{2} - \frac{1}{6\alpha} & \frac{1}{6\alpha(1-\alpha)} & \frac{2-3\alpha}{6(1-\alpha)} \end{array}$$

**"The" RK4 method**

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Global error  $\|\mathbf{x}_N - \mathbf{x}(T)\| = \mathcal{O}(\Delta t^4)$

**One-Step error**

$$\|\mathbf{x}_{k+1} - \mathbf{x}(t_{k+1})\|$$

Taylor expand  $\mathbf{x}(t)$  around  $t_k$ . Do the same for  $\mathbf{x}_{k+1}$ . Subtract and get the results

**Global error** Get error on time step  $T$  by multiplying local error with  $N = \frac{T}{\Delta t}$

**stability of integration method:** Consider the ODE

$$\dot{x} = \lambda x \quad x(0) = x_0$$

Perform one step of the RK method to find

$$\mathbf{x}_{n+1} = R(z)\mathbf{x}_n \quad z = \Delta t \lambda$$

The integration method is unstable if

$$|R(z)| > 1$$

**A-stability** A method is A-stable if the region of stability is the entire left-half plane.

$$|R(\lambda \Delta t)| \leq 1 \forall \lambda \in \mathbb{C} \text{ s.t. } \Re(\lambda) \leq 0$$

**L-stability.** A Runge Kutta method is L-stable if it is A-stable and

$$\lim_{\omega \rightarrow \pm\infty} |R(i\omega h)| = 0$$

**Implicit methods**

- Can achieve high and systematic orders
- Can be stable regardless of the step size
- Can handle the simulation of DAEs (must be index 1)
- Can give huge speed-up when system is stiff
- Can achieve order  $o = 2s$  for any stage  $s$ . Explicit methods can only achieve  $o = s$  and only for  $s \leq 4$



### Implicit Euler

$$\frac{1}{1} \bigg| \frac{1}{1}$$

$$\begin{aligned} k_1 &= f(x_n + \Delta t k_1) \\ x_{n+1} &= x_n + \Delta t k_1 \\ &= x_n + \Delta t f(x_{n+1}) \end{aligned}$$

### Trapezoidal method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

**Gauss-Legendre collocation method** Generates order  $o = 2s$  A-stable IRK methods. Let  $s$  be the number of stages. Find the roots  $(\tau_i)$  of

$$P_s(\tau) = \frac{1}{s!} \frac{d^s}{d\tau^s} ((\tau^2 - \tau)^s)$$

Build polynomials

$$\ell_i(\tau) = \prod_{j \neq i} \frac{\tau - \tau_j}{\tau_i - \tau_j}$$

Integrate them

$$L_i(\tau) = \int_0^\tau \ell_i(\xi) d\xi$$

Calculate  $A$ ,  $b$  and  $c$

$$A_{ji} = L_i(\tau_j) \quad b_i = L_i(1) \quad c_j = \tau_j$$

**Adaptive integrator.** Use two methods:  $x_{n+1}$  and  $\hat{x}_{n+1}$ , of different order:  $p$  and  $p+1$ . The local error is

$$\begin{aligned} \epsilon_{n+1} &:= |x_{n+1} - \hat{x}_{n+1}| \\ &\approx \mathcal{O}(\Delta t^{p+1}) - \mathcal{O}(\Delta t^{p+2}) \\ &\approx C \Delta t^{p+1} \\ e_{\text{tol}} &= C \Delta t_{\text{new}}^{p+1} \Rightarrow \\ \Delta t_{\text{new}} &= \Delta t \left( \frac{e_{\text{tol}}}{\epsilon_{n+1}} \right)^{\frac{1}{p+1}} \end{aligned}$$

### Newtons method

$$\begin{aligned} f(x + \Delta x) &\approx f(x) + \frac{\partial f}{\partial x}(x) \Delta x = 0 \\ \Delta x &= - \left( \frac{\partial f}{\partial x}(x) \right)^{-1} f(x) \\ x_{n+1} &= x_n + \Delta x \end{aligned}$$

- May not work if jacobian is singular
- Has quadratic convergence when it converges
- May diverge

**RK method on implicit DAEs/ODEs** Given the implicit ODE / DAE

$$F(\dot{x}, x, z, u, t) = 0$$

The  $K = (K_1, \dots, K_s)^T$  vector / matrix can be found by solving

$$\begin{aligned} r(K, x_k, z, u(\cdot), t_k) &= \\ &\begin{pmatrix} F(K_1, x_n + \Delta t a_1^T K, z_1, u(t_k + \Delta t c_1), t_k + \Delta t c_1) \\ \vdots \\ F(K_s, x_n + \Delta t a_s^T K, z_s, u(t_k + \Delta t c_s), t_k + \Delta t c_s) \end{pmatrix} \\ &= 0 \end{aligned}$$

for  $K$  and  $z$

**Computational cost of ERK methods.** Let  $T_{\text{ol}}$  be the global tolerance.

$$\begin{aligned} \|\mathbf{x}_n - \mathbf{x}(T)\| &\leq c \Delta t^o \leq T_{\text{ol}} \\ \Delta t &\leq \left( \frac{T_{\text{ol}}}{c} \right)^{(1/o)} \end{aligned}$$

The number of integrator steps is

$$N = T / \Delta t$$

The system dynamics must be evaluated  $s$  times for each integration step

$$\begin{aligned} n = Ns &= \frac{sT}{\Delta t} \geq sT \left( \frac{T_{\text{ol}}}{c} \right)^{-1/o} \\ \frac{n}{T} &\geq s \left( \frac{T_{\text{ol}}}{c} \right)^{-1/o} \end{aligned}$$

$n$  is the number of function evaluations,  $T$  is the simulation time,  $s = \#$  stages.  $T_{\text{ol}}$  is the global tolerance and  $o$  is the order. Using the correlation between maximum order and number of stages we get that the optimum order is between 3 and 6.

## 7 System Modeling and Actuators

### 3.2 Electrical Motors

**Basic equations.** Let  $\omega_m$  be the angular velocity of the motor shaft,  $T$  the motor torque,  $T_L$  the torque of the load,  $J_m$  the rotational inertia in the motor,  $P_m$  the power delivered from the motor the shaft and  $P_L$  the mechanical power delivered to the load, then

$$J_m \dot{\omega}_m = T - T_L$$

$$P_m = T \omega_m$$

$$P_L = T_L \omega_m$$

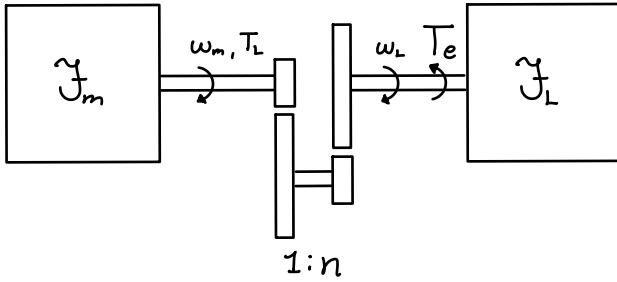
#### Gears

$$\omega_{out} = n \omega_{in}$$

$$T_{out} = \frac{1}{n} T_{in}$$

$$P_{in} = T_{in} \omega_{in} = T_{out} \omega_{out} = P_{out}$$

#### Motor and gear



$$(J_m + n^2 J_L) \dot{\omega}_m = T - n T_e$$

**Torque characteristics** The motor torque  $T$  and the torque of the load  $T_L$  are often functions of the motor speed  $\omega_m$ .

$$T_L(\omega_m)$$

$$T(\omega_m)$$

Stability can be investigated in a torque-speed diagram

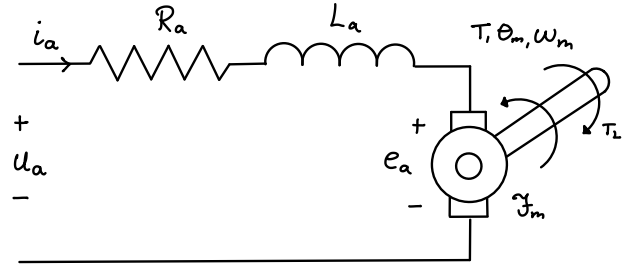
$$k = \left( \frac{\partial T}{\partial \omega_m} - \frac{\partial T_L}{\partial \omega_m} \right) \Big|_{\omega_m}$$

The system is stable in an equilibrium point  $T_L(\omega_m) = T(\omega_m)$  if  $k \leq 0$

**The four quadrants of the motor** A motor can both deliver and receive power, it can act as both a motor and a generator.

$$P = T \omega$$

### 3.3 The DC motor with constant field



$$L_a \frac{d}{dt} i_a = -R_a i_a - K_E \omega_m + u_a$$

$$J_m \dot{\omega}_m = K_T i_a - T_L$$

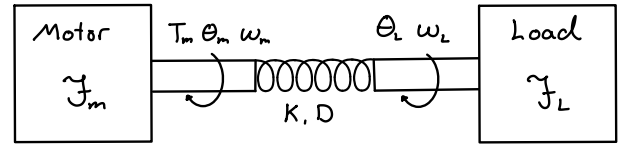
$$\dot{\theta}_m = \omega_m$$

$K_T$  and  $K_E$  defined by

$$T = K_T i_a$$

$$K_E = K_T$$

### 3.5 Motor and load with elastic transmission



Spring is damped

$$\theta_e = \theta_L - \theta_m$$

$$T_L = -K \theta_e - D \dot{\theta}_e$$

The equations are

$$\ddot{\theta}_e + \frac{D}{J_e} \dot{\theta}_e + \frac{K}{J_e} \theta_e = -\frac{1}{J_m} T_m$$

$$\ddot{\theta}_r = \frac{T_m}{J_m}$$

where

$$J_e = \frac{J_m J_L}{J_m + J_L}$$

$$\theta_r = \theta_m + \frac{J_L}{J_m} \theta_L$$

### 4.2 Valves

#### Pressure conversion rules

- 1 Pa = 1 N/m<sup>2</sup>
- 1 bar = 10<sup>5</sup> Pa
- 1 atm = 1.01325 · 10<sup>5</sup> Pa
- 1 psi = 6897 Pa

**Flow through restriction** The flow  $q$  [kg/m<sup>3</sup>] is related to the density of the fluid  $\rho$ , the change in pressure  $\Delta p$ , the cross sectional area of the orifice  $A$  and the discharge coefficient  $C_d$

$$q \approx C_d A \sqrt{\frac{2}{\rho} \Delta p}$$

$C_d = 1$  when no energy is lost, in practice  $C_d \in [0.60, 0.65]$  for orifices with sharpe edges and  $C_d \in [0.8, 0.9]$  with rounded edges.

**Reynolds number for flow through restrictions**

$$R_e = \frac{D}{A\nu} q$$

$D$  is diameter of the restriction,  $A$  is the cross sectional area of the flow,  $\nu$  is the kinematic viscosity and is approximately  $30 \cdot 10^{-6}$  m<sup>2</sup>/s for hydraulic oil. When  $R_e > 1000$  the flow will be turbulent and the forula for

the flow  $q$  is as stated above. When  $R_e < 10$  the flow will be laminar and

$$q = C_l \Delta p$$

### 4.3 Motor models

#### 7.1 system modeling

**Generalization of concepts accross fields**

- **$e$  - Effort**
- **$f$  - Flow**
- **$I$  - Intertance.**
- **$P$  - Power.**  $P = e \cdot f$
- **$p$  - Momentum.**  $p = I \cdot f$
- **$q$  - Displacement.**  $q = \int f_1 - f_2$

	Effort	Flow	Inertance	Momentum	Displacement
Linear Mechanics	$F$	$v$	$m$	$p$	$x$
Angular Mechanics	$T$	$\omega$	$J$	$h$	$\theta$
Hydraulics	$p$	$q$		$\Gamma$	$V$
Electrical	$U$	$i$	$L$	$\lambda$	$Q$

## 8 Mathematics

### Inverse of $2 \times 2$ Matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

### Multivariable derivative rules

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \mathbf{a}^T \mathbf{x} &= \mathbf{a}^T & \nabla_{\mathbf{x}} \mathbf{a}^T \mathbf{x} &= \mathbf{a} \\ \frac{\partial}{\partial \mathbf{x}} \mathbf{A} \mathbf{x} &= \mathbf{A} & \nabla_{\mathbf{x}} \mathbf{A} \mathbf{x} &= \mathbf{A}^T \\ \frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{A} &= \mathbf{A}^T & \nabla_{\mathbf{x}} \mathbf{x}^T \mathbf{A} &= \mathbf{A} \end{aligned}$$

### Second order terms

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} &= \frac{1}{2} \mathbf{x}^T (\mathbf{A}^T + \mathbf{A}) \\ \nabla_{\mathbf{x}} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} &= \frac{1}{2} (\mathbf{A}^T + \mathbf{A}) \mathbf{x} \end{aligned}$$

### Multivariable Chain rule

$$\begin{aligned} \frac{\partial \mathbf{f}(\mathbf{g}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial \mathbf{f}}{\partial \mathbf{g}} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \\ \frac{\partial \mathbf{f}(\mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial \mathbf{f}}{\partial \mathbf{g}} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{h}} \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \end{aligned}$$

### Some derivatives

$$\begin{aligned} \frac{d}{dt} \sinh(t) &= \cosh(t) \\ \frac{d}{dt} \cosh(t) &= \sinh(t) \\ \frac{d}{dt} \tanh(t) &= \frac{d}{dt} \frac{\sinh(t)}{\cosh(t)} = \frac{1}{\cosh^2 t} = 1 - \tanh^2(t) \\ \frac{d}{dt} \arctan(t) &= \frac{1}{1+t^2} \end{aligned}$$

### Taylor's theorem

Let  $k \geq 1$  be an integer and let the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  be  $k+1$  times differentiable at the point  $a \in \mathbb{R}$ . Then

$$f(x) = \sum_{n=0}^k \frac{1}{n!} f^{(n)}(a)(x-a)^n + \frac{1}{(k+1)!} f^{(k+1)}(\xi)(x-a)^{k+1}$$

for some  $\xi \in [x, a]$

### Common Taylor expansions

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \sin(x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \\ \cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$

### Integration in polar coordinates

$$dA = r \, dr \, d\theta$$

### Integration in spherical coordinates

$$\begin{aligned} x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta \\ z &= \rho \cos \phi \\ dV &= \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{aligned}$$