TTK4130 - Cheat Sheet

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https://github.com/haakonbaa/TTK4130-cheatsheet

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1.1 What are models, what are simulations, notation

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6.2 Vectors

The *skew-symetric* matrix form of the coordinate vector $\mathbf{u}$ is defined by

$$\mathbf{u}^x = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Notation: $\boldsymbol{v}_{ab}^c$ means the vector from point a to point b (or often the origo of the reference frames a and b) described in the reference frame c

6.4 The Rotation Matrix

The coordinate transformation from frame b to frame a is given by

$$\boldsymbol{v}^a = \boldsymbol{R}_b^a \boldsymbol{v}^b$$

Properties of the rotation matrix

$$\begin{aligned} \boldsymbol{R}_a^b \boldsymbol{R}_b^a &= \boldsymbol{I} = \boldsymbol{R}_b^a \boldsymbol{R}_a^b \\ (\boldsymbol{R}_a^b)^{-1} &= (\boldsymbol{R}_a^b)^T = \boldsymbol{R}_b^a \\ \boldsymbol{R}_b^a &= \begin{pmatrix} \boldsymbol{b}_1^a & \boldsymbol{b}_2^a & \boldsymbol{b}_3^a \end{pmatrix} \\ \det \boldsymbol{R}_a^b &= 1 \end{aligned}$$

$\boldsymbol{R}$ is a rotation matrix if and only if it is an element of $SO(3)$

$$SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3\times3} | \boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I} \wedge \det \boldsymbol{R} = 1\}$$

Rotation matrices in three dimentions

$$\boldsymbol{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

$$\boldsymbol{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\boldsymbol{R}_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix transformations in different refrence frames

$$\begin{aligned} \boldsymbol{D}^a &= \boldsymbol{R}_b^a \boldsymbol{D}^b \boldsymbol{R}_a^b \\ (\boldsymbol{u}^b)^\times &= \boldsymbol{R}_a^b (\boldsymbol{u}^a)^\times \boldsymbol{R}_b^a \end{aligned}$$

The transformation of position and orientation from frame b to frame a is

$$\boldsymbol{T}_b^a = \begin{pmatrix} \boldsymbol{R}_b^a & \boldsymbol{r}_{ab}^a \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

$$\boldsymbol{T}_b^a \begin{pmatrix} \boldsymbol{v}^b \\ 1 \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{v}^a \\ 1 \end{pmatrix}$$

$$(\boldsymbol{T}_b^a)^{-1} = \boldsymbol{T}_a^b = \begin{pmatrix} \boldsymbol{R}_a^b & \boldsymbol{r}_{ba}^b \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

The Special Euclidean group is the set of all transformations from one reference frames to another

$$SE(3) = \left\{ \boldsymbol{T} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{3\times3} \middle| \boldsymbol{R} \in SO(3) \wedge \boldsymbol{r} \in \mathbb{R}^3 \right\}$$

6.5 Euler Angles

Roll-Pitch-Yaw Euler angles

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi) \boldsymbol{R}_y(\theta) \boldsymbol{R}_x(\phi)$$

Classical Euler angles. The orientation is described by a rotation bout the z axis, then the resulting y axis. And then again the resulting z axis.

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi) \boldsymbol{R}_y(\theta) \boldsymbol{R}_z(\phi)$$

6.6 Angle Axis Description of rotation

6.6.5 Rotation Matrix

Angle-axis parameters All rotation matrices have an eigen vector with eigen value 1. A rotation can be uniquely described by the direction of this vector and an angle θ being the rotatoin about this vector.

$$\begin{aligned} &(\theta, \boldsymbol{k}) \text{ s.t. } ||\boldsymbol{k}|| = 1 \\ &\boldsymbol{R}_b^a = \cos\theta \boldsymbol{I} + \sin\theta (\boldsymbol{k}_a)^\times + (1 - \cos\theta) \boldsymbol{k}_a \boldsymbol{k}_a^T \\ &\boldsymbol{R}_b^a = \exp\{\boldsymbol{k}^\times \theta\} \end{aligned}$$

6.7 Euler parameters

6.7.1 Definition

$$\begin{aligned} \eta &= \cos\frac{\theta}{2} \\ \boldsymbol{\epsilon} &= \boldsymbol{k} \sin\frac{\theta}{2} \\ \boldsymbol{R}_e(\eta, \boldsymbol{\epsilon}) &= \boldsymbol{I} + 2\eta \boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times \boldsymbol{\epsilon}^\times \end{aligned}$$

6.7.3 Quaternions

The following can be treated as a unit quaternion

$$\boldsymbol{p} = \begin{pmatrix} \eta \\ \boldsymbol{\epsilon} \end{pmatrix}$$

A unit quaternion satisfies

$$\boldsymbol{p}^T \boldsymbol{p} = \eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = 1$$

Quaternion product

$$\begin{pmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 - \boldsymbol{\beta}_1^T \boldsymbol{\beta}_2 \\ \alpha_1 \boldsymbol{\beta}_2 + \alpha_2 \boldsymbol{\beta}_1 + \boldsymbol{\beta}_1^\times \boldsymbol{\beta}_2 \end{pmatrix}$$

6.7.6 Euler parameters from the rotation matrix

$$\boldsymbol{R} = (r_{ij})$$
$$\boldsymbol{z} = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix}^T := 2 \begin{pmatrix} \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix}^T$$
$$\boldsymbol{T} := r_{00} := \text{Trace}\boldsymbol{R}$$

The algorithm from Shepperd (1978) goes like this:

- Let $i = \arg\max_i \{r_{ii}\}$
- Compute $|z_i| = \sqrt{1 + 2r_{ii} - T}$
- Determine sign of z_i
- Determine the rest of $\boldsymbol{z}$ from equations below

$$\begin{aligned} z_0 z_1 &= r_{32} - r_{23} & z_2 z_3 &= r_{32} + r_{23} \\ z_0 z_2 &= r_{13} - r_{31} & z_3 z_1 &= r_{13} + r_{31} \\ z_0 z_3 &= r_{21} - r_{12} & z_1 z_2 &= r_{21} + r_{12} \end{aligned}$$

6.8 Angular Velocity

Let $R \in SO(3)$

$$0 = \frac{d}{dt}(\boldsymbol{I}) = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{R}^T) = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}(\dot{\boldsymbol{R}})^T$$
$$\Rightarrow \dot{\boldsymbol{R}}\boldsymbol{R}^T \textit{ skew-symmetric}$$

Definition of angluar velocity

$$\begin{aligned} (\boldsymbol{\omega}^a_{ab})^\times &= \dot{\boldsymbol{R}}^a_b (\boldsymbol{R}^a_b)^T \Rightarrow \\ \dot{\boldsymbol{R}}^a_b &= (\boldsymbol{\omega}^a_{ab})^\times \boldsymbol{R}^a_b \\ \dot{\boldsymbol{R}}^a_b &= \boldsymbol{R}^a_b (\boldsymbol{\omega}^b_{ab})^\times \end{aligned}$$

It can be shown that

$$\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}$$

Where θ and $\boldsymbol{k}$ are Angle Axis parameters.

$$\begin{aligned} \boldsymbol{\omega}^a_{ad} &= \boldsymbol{\omega}^a_{ab} + \boldsymbol{\omega}^a_{bc} + \boldsymbol{\omega}^a_{cd} \\ \dot{\boldsymbol{u}}^a &= \boldsymbol{R}^a_b (\dot{\boldsymbol{u}}^b + (\boldsymbol{\omega}^b_{ab})^\times \boldsymbol{u}^b) \end{aligned}$$

6.9 Kinematic differential equations

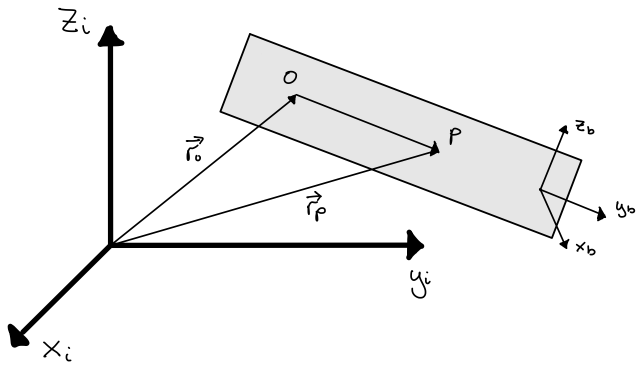
6.9.4 Euler Angles

$$\begin{aligned} \boldsymbol{\omega}^a_{ad} &= \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \boldsymbol{R}_{z,\psi} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \boldsymbol{R}_{z,\psi}\boldsymbol{R}_{y,\theta} \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sin\psi\dot{\theta} + \cos\psi\cos\theta\dot{\phi} \\ \cos\psi\dot{\theta} + \sin\psi\cos\theta\dot{\phi} \\ \dot{\psi} - \sin\theta\dot{\phi} \end{pmatrix} \end{aligned}$$

3 Rigid Body Dynamics

6.12 Kinematics of a rigid body

$\vec{\omega}_{io}$ is the angular velocity of the o frame with respect to the i frame.
${}^i\frac{d}{dt}\vec{r}_o$ is the derivative of $\vec{r}_o$ in the i frame.



Velocity and Acceleration

$$\begin{aligned} \vec{v}_p &:= \frac{{}^id}{dt}\vec{r}_p \\ &= \vec{v}_o + \frac{{}^bd}{dt}\vec{r} + \vec{\omega}_{ib} \times \vec{r} \\ \vec{a}_p &:= \frac{{}^id^2}{dt^2}\vec{r}_p \\ &= \vec{a}_o + \frac{{}^bd^2}{dt^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^bd}{dt}\vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}) \end{aligned}$$

The last three terms are, respectively, the coriolis acceleration, Transveral acceleration and Centripetal acceleration. Note that

$$\vec{a}_o = \frac{{}^id}{dt}\vec{v}_o = \frac{{}^bd}{dt}\vec{v}_o + \vec{\omega}_{ib} \times \vec{v}_o$$

6.13 The center of mass

The center of mass of a rigid body $\mathcal{C}$ is defined to be

$$\vec{r}_c := \frac{1}{m}\int_{\mathcal{C}} \vec{r}_p \, dm$$

It can be shown that

$$\vec{v}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{v}_p \, dm \qquad \vec{a}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{a}_p \, dm$$

where c denotes *center*

7.2 Forces and torques

Moment. The moment about a point P of the set $S = \{F_j\}_{j\in[1,n_F]}$ for forces is

$$\vec{N}_{S/P} = \sum_{j=1}^{n_F} r\vec{_P}_j \times \vec{F}_j$$

Where $\vec{r}_{Pj}$ is an arbitrary point along the line of action of $\vec{F}_j$

Torque is defined as the moment of the couple $\mathcal{C}$. A couple being a set of forces with $\mathbf{0}$ resultant force.

7.3 Newton-Euler Equations for rigid bodies

Angular Momentum. The angular momentum of the body b about the center of mass c is

$$\begin{aligned}\boldsymbol{h}_{b/c} &= \int_b \boldsymbol{r} \times \boldsymbol{v}\, dm \\ &= \boldsymbol{M}_{b/c}\boldsymbol{\omega}_{ib} \\ \boldsymbol{T}_{bc} &= \frac{d}{dt}\boldsymbol{h}_{b/c}\end{aligned}$$

Rotational Inertia / The intertia dyadic. The inertia matrix of the body b about the point c is

$$\begin{aligned}\boldsymbol{M}_{b/c} &= -\int_b \boldsymbol{r}^\times \boldsymbol{r}^\times\, dm \\ &= \int_b (\boldsymbol{r}^T\boldsymbol{r}\mathbb{I} - \boldsymbol{r}\boldsymbol{r}^T)\, dm \\ &= \begin{pmatrix} \boldsymbol{I}_{xx} & -\boldsymbol{I}_{xy} & -\boldsymbol{I}_{xz} \\ -\boldsymbol{I}_{xy} & \boldsymbol{I}_{yy} & -\boldsymbol{I}_{yz} \\ -\boldsymbol{I}_{xz} & -\boldsymbol{I}_{yz} & \boldsymbol{I}_{zz} \end{pmatrix}\end{aligned}$$

Where $\boldsymbol{r}$ is the distance vector from the center of mass to the mass element being integrated

$$\begin{aligned}\boldsymbol{I}_{xx} &= \int_b y^2 + z^2\, dm & \boldsymbol{I}_{xy} &= \int_b xy\, dm \\ \boldsymbol{I}_{yy} &= \int_b x^2 + z^2\, dm & \boldsymbol{I}_{xz} &= \int_b xz\, dm \\ \boldsymbol{I}_{zz} &= \int_b x^2 + y^2\, dm & \boldsymbol{I}_{yz} &= \int_b yz\, dm\end{aligned}$$

$$\boldsymbol{M}^i_{b/c} = \boldsymbol{R}^i_b \boldsymbol{M}^b_{b/c} \boldsymbol{R}^b_i$$

Equations of motion. Let b denote body, i an intertial frame, c the center of mass of b, $\boldsymbol{F}_{bc}$ a resultant force acting on b with line of action throgh c and $\boldsymbol{T}_{bc}$ the torque about c. Then

$$\begin{aligned}\boldsymbol{F}_{bc} &= m\boldsymbol{a}_c \\ \boldsymbol{T}_{bc} &= \boldsymbol{M}_{b/c}\boldsymbol{\alpha}_{ib} + \boldsymbol{\omega}_{ib} \times (\boldsymbol{M}_{b/c}\boldsymbol{\omega}_{ib})\end{aligned}$$

On compact matrix form

$$\begin{pmatrix} m\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}^b_{b/c} \end{pmatrix} \begin{pmatrix} \boldsymbol{a}^b_c \\ \boldsymbol{\alpha}^b_{ib} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ (\boldsymbol{\omega}^b_{ib})^\times \boldsymbol{M}^b_{b/c}\boldsymbol{\omega}^b_{ib} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}^b_{bc} \\ \boldsymbol{T}^b_{b/c} \end{pmatrix}$$
$$\begin{pmatrix} m\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}^b_{b/c} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}^b_c \\ \boldsymbol{\alpha}^b_{ib} \end{pmatrix} + \begin{pmatrix} m(\boldsymbol{\omega}^b_{ib})^\times \boldsymbol{v}^b_c \\ (\boldsymbol{\omega}^b_{ib})^\times \boldsymbol{M}^b_{b/c}\boldsymbol{\omega}^b_{ib} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}^b_{bc} \\ \boldsymbol{T}^b_{b/c} \end{pmatrix}$$

Kinetic energy. The kinetic energy of the body b in an inertial refrence frame i is

$$K = \frac{1}{2}m(\boldsymbol{v}^b_c)^T\boldsymbol{v}^b_c + \frac{1}{2}(\boldsymbol{\omega}^b_{ib})^T\boldsymbol{M}^b_{b/c}\boldsymbol{\omega}^b_{ib}$$

The parallel axes theorem. The inertia matrix of b about o is related to the inertia matrix of b about c according to

$$\boldsymbol{M}^b_{b/o} = \boldsymbol{M}^b_{b/c} - m(\boldsymbol{r}^b_g)^\times(\boldsymbol{r}^b_g)^\times$$

Where $\boldsymbol{r}^b_g$ is the vector from c to o