TTK4130 - Cheat Sheet

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https://github.com/haakonbaa/TTK4130-cheatsheet

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1 Intro

1.2 State space methods

1.2.1 State space models

State Space Model is on the form

$$\dot{x} = f(x, u, t)$$

Linear time invariant system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$y = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(t)\,d\tau + Du(t)$$

1.2.2 Second order models of mechanical systems

Second order models on the form

$$M(q)\ddot{q} + f(q, \dot{q}) = u$$

Can be written as

$$\begin{pmatrix} \dot{q} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ M^{-1}(q)(-f(q, \dot{q}) + u) \end{pmatrix}$$

1.2.3 Linearization of state space models

Linearization of time varying systems

$$\dot{x} = f(x, u, t)$$
$$y = h(x, u, t)$$

Find two functions x_0 and u_0 being solutions to the system

$$\dot{x}_0 = f(x_0(t), u_0(t), t)$$

Define perturbations

$$x(t) = x_0(t) + \Delta x(t)$$
$$u(t) = u_0(t) + \Delta u(t)$$
$$y(t) = y_0(t) + \Delta y(t)$$

Let $\mathcal{C} = \{x_0(t), u_0(t)\}$. The linearized system is

$$\Delta\dot{x} \approx \frac{\partial f}{\partial x}\Big|_{\mathcal{C}}\Delta x + \frac{\partial f}{\partial u}\Big|_{\mathcal{C}}\Delta u$$
$$\Delta\dot{y} \approx \frac{\partial h}{\partial x}\Big|_{\mathcal{C}}\Delta x + \frac{\partial h}{\partial u}\Big|_{\mathcal{C}}\Delta u$$

1.5 ODE's

General formulation

$$\varphi(y^{(m)}, \cdots, y, u^{(m-1)}, \cdots, u) = 0$$

Lipschitz continuous. A function

$$f : \mathbb{R} \to \mathbb{R}$$

is said to be *Lipschitz continuous* if

$$\exists c > 0 \in \mathbb{R} \qquad\qquad \text{s.t.}$$
$$||f(x) - f(y)|| < c||x - y|| \qquad \forall x, y \in \mathbb{R}$$

Theorem: existence of unique solution. Consider the ODE

$$\dot{x} = f(x)$$

If f is Lipschitz continuous then $x(t)$ exists and is unique for all t

Mean Value Theorem suppose f is continuous on $[a, b]$ and differentiable on (a, b) then

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

This can be used to show that if f is continuous and differentiable everywhere it is also Lipschitz.

Theorem 2: existence of unique solution. Consider the ODE

$$\dot{x} = f(x)$$

if f is continuously differentiable ($\frac{\partial f}{\partial x}$ exists and is continuous), then the solution to the ODE exists and is unique on some time interval.

2   Rotations

6.2   Vectors

The *skew-symmetric* matrix form of the coordinate vector **u** is defined by

$$\mathbf{u}^x = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Notation: $\boldsymbol{v}_{ab}^c$ means the vector from point a to point b (or often the origin of the reference frames a and b) described in the reference frame c

6.4   The Rotation Matrix

The coordinate transformation from frame b to frame a is given by

$$\boldsymbol{v}^a = \boldsymbol{R}_b^a \boldsymbol{v}^b$$

Properties of the rotation matrix

$$\boldsymbol{R}_a^b \boldsymbol{R}_b^a = \boldsymbol{I} = \boldsymbol{R}_b^a \boldsymbol{R}_a^b$$
$$(\boldsymbol{R}_a^b)^{-1} = (\boldsymbol{R}_a^b)^T = \boldsymbol{R}_b^a$$
$$\boldsymbol{R}_b^a = \begin{pmatrix} \boldsymbol{b}_1^a & \boldsymbol{b}_2^a & \boldsymbol{b}_3^a \end{pmatrix}$$
$$\det \boldsymbol{R}_a^b = 1$$

$\boldsymbol{R}$ is a rotation matrix if and only if it is an element of $SO(3)$

$$SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3 \times 3} | \boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I} \wedge \det \boldsymbol{R} = 1\}$$

Rotation matrices in three dimensions

$$\boldsymbol{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$
$$\boldsymbol{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$
$$\boldsymbol{R}_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix transformations in different reference frames

$$\boldsymbol{D}^a = \boldsymbol{R}_b^a \boldsymbol{D}^b \boldsymbol{R}_a^b$$
$$(\boldsymbol{u}^b)^\times = \boldsymbol{R}_a^b (\boldsymbol{u}^a)^\times \boldsymbol{R}_b^a$$

The transformation of position and orientation from frame b to frame a is

$$\boldsymbol{T}_b^a = \begin{pmatrix} \boldsymbol{R}_b^a & \boldsymbol{r}_{ab}^a \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$
$$\boldsymbol{T}_b^a \begin{pmatrix} \boldsymbol{v}^b \\ 1 \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{v}^a \\ 1 \end{pmatrix}$$
$$(\boldsymbol{T}_b^a)^{-1} = \boldsymbol{T}_a^b = \begin{pmatrix} \boldsymbol{R}_a^b & \boldsymbol{r}_{ba}^b \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

The Special Euclidean group is the set of all transformations from one reference frames to another

$$SE(3) = \left\{ \boldsymbol{T} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3} \middle| \boldsymbol{R} \in SO(3) \wedge \boldsymbol{r} \in \mathbb{R}^3 \right\}$$

6.5   Euler Angles

Roll-Pitch-Yaw Euler angles

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi) \boldsymbol{R}_y(\theta) \boldsymbol{R}_x(\phi)$$

Classical Euler angles. The orientation is described by a rotation bout the z axis, then the resulting y axis. And then again the resulting z axis.

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi) \boldsymbol{R}_y(\theta) \boldsymbol{R}_z(\phi)$$

6.6   Angle Axis Description of rotation

6.6.5   Rotation Matrix

Angle-axis parameters All rotation matrices have an eigen vector with eigen value 1. A rotation can be uniquely described by the direction of this vector and an angle θ being the rotation about this vector.

$$(\theta, \boldsymbol{k}) \text{ s.t. } ||\boldsymbol{k}|| = 1$$
$$\boldsymbol{R}_b^a = \cos\theta \boldsymbol{I} + \sin\theta (\boldsymbol{k}_a)^\times + (1 - \cos\theta) \boldsymbol{k}_a \boldsymbol{k}_a^T$$
$$\cos\theta = \frac{\text{trace}\boldsymbol{R} - 1}{2}$$
$$\boldsymbol{R}_b^a = \exp\{\boldsymbol{k}^\times \theta\}$$

6.7   Euler parameters

6.7.1   Definition

$$\eta = \cos\frac{\theta}{2}$$
$$\boldsymbol{\epsilon} = \boldsymbol{k}\sin\frac{\theta}{2}$$
$$\boldsymbol{R}_e(\eta, \boldsymbol{\epsilon}) = \boldsymbol{I} + 2\eta\boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times \boldsymbol{\epsilon}^\times$$
$$\eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = 1$$

6.7.3   Quaternions

The following can be treated as a unit quaternion

$$\boldsymbol{p} = \begin{pmatrix} \eta \\ \boldsymbol{\epsilon} \end{pmatrix}$$

A unit quaternion satisfies

$$\boldsymbol{p}^T \boldsymbol{p} = \eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = 1$$

Quaternion product

$$\begin{pmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 - \boldsymbol{\beta}_1^T \boldsymbol{\beta}_2 \\ \alpha_1 \boldsymbol{\beta}_2 + \alpha_2 \boldsymbol{\beta}_1 + \boldsymbol{\beta}_1^\times \boldsymbol{\beta}_2 \end{pmatrix}$$

6.7.6 Euler parameters from the rotation matrix

$$\boldsymbol{R} = (r_{ij})$$

$$\boldsymbol{z} = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix}^T := 2 \begin{pmatrix} \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix}^T$$

$$\boldsymbol{T} := r_{00} := \mathrm{Trace}\boldsymbol{R}$$

The algorithm from Shepperd (1978) goes like this:

- Let $i = \arg\max_i \{r_{ii}\}$

- Compute $|z_i| = \sqrt{1 + 2r_{ii} - T}$

- Determine sign of z_i

- Determine the rest of $\boldsymbol{z}$ from equations below

$$
\begin{aligned}
z_0 z_1 &= r_{32} - r_{23} & z_2 z_3 &= r_{32} + r_{23} \\
z_0 z_2 &= r_{13} - r_{31} & z_3 z_1 &= r_{13} + r_{31} \\
z_0 z_3 &= r_{21} - r_{12} & z_1 z_2 &= r_{21} + r_{12}
\end{aligned}
$$

6.8 Angular Velocity

Let $R \in SO(3)$

$$0 = \frac{d}{dt}(\boldsymbol{I}) = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{R}^T) = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}(\dot{\boldsymbol{R}})^T$$

$$\Rightarrow \dot{\boldsymbol{R}}\boldsymbol{R}^T \, \textit{skew-symmetric}$$

Definition of angular velocity

$$(\boldsymbol{\omega}_{ab}^a)^\times = \dot{\boldsymbol{R}}_b^a (\boldsymbol{R}_b^a)^T \Rightarrow$$

$$\dot{\boldsymbol{R}}_b^a = (\boldsymbol{\omega}_{ab}^a)^\times \boldsymbol{R}_b^a$$

$$\dot{\boldsymbol{R}}_b^a = \boldsymbol{R}_b^a (\boldsymbol{\omega}_{ab}^b)^\times$$

It can be shown that

$$\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}$$

Where θ and $\boldsymbol{k}$ are Angle Axis parameters.

$$\boldsymbol{\omega}_{ad}^a = \boldsymbol{\omega}_{ab}^a + \boldsymbol{\omega}_{bc}^a + \boldsymbol{\omega}_{cd}^a$$

$$\dot{\boldsymbol{u}}^a = \boldsymbol{R}_b^a(\dot{\boldsymbol{u}}^b + (\boldsymbol{\omega}_{ab}^b)^\times \boldsymbol{u}^b)$$

6.9 Kinematic differential equations

6.9.4 Euler Angles

$$
\boldsymbol{\omega}_{ad}^a = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \boldsymbol{R}_{z,\psi} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \boldsymbol{R}_{z,\psi}\boldsymbol{R}_{y,\theta} \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} -\sin\psi\dot{\theta} + \cos\psi\cos\theta\dot{\phi} \\ \cos\psi\dot{\theta} + \sin\psi\cos\theta\dot{\phi} \\ \dot{\psi} - \sin\theta\dot{\phi} \end{pmatrix}
$$

3 Rigid Body Dynamics

6.12 Kinematics of a rigid body

$\vec{\omega}_{io}$ is the angular velocity of the o frame with respect to the i frame.

${}^{i}\frac{d}{dt}\vec{r}_o$ is the derivative of $\vec{r}_o$ in the i frame.
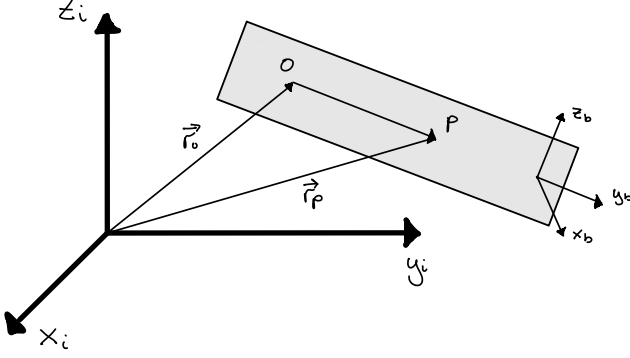
Velocity and Acceleration

$$\vec{v}_p := \frac{{}^{i}d}{dt}\vec{r}_p$$

$$= \vec{v}_o + \frac{{}^{b}d}{dt}\vec{r} + \vec{\omega}_{ib} \times \vec{r}$$

$$\vec{a}_p := \frac{{}^{i}d^2}{dt^2}\vec{r}_p$$

$$= \vec{a}_o + \frac{{}^{b}d^2}{dt^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^{b}d}{dt}\vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})$$

The last three terms are, respectively, the coriolis acceleration, Transversal acceleration and Centripetal acceleration. Note that

$$\vec{a}_o = \frac{{}^{i}d}{dt}\vec{v}_o = \frac{{}^{b}d}{dt}\vec{v}_o + \vec{\omega}_{ib} \times \vec{v}_o$$

6.13 The center of mass

The center of mass of a rigid body $\mathcal{C}$ is defined to be

$$\vec{r}_c := \frac{1}{m}\int_{\mathcal{C}} \vec{r}_p \, dm$$

It can be shown that

$$\vec{v}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{v}_p \, dm \qquad \vec{a}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{a}_p \, dm$$

where c denotes *center*

7.2 Forces and torques

Moment. The moment about a point P of the set $S = \{F_j\}_{j \in [1, n_F]}$ for forces is

$$\vec{N}_{S/P} = \sum_{j=1}^{n_F} \vec{r_{Pj}} \times \vec{F}_j$$

Where $\vec{r}_{Pj}$ is an arbitrary point along the line of action of $\vec{F}_j$

Torque is defined as the moment of the couple $\mathcal{C}$. A couple being a set of forces with $\mathbf{0}$ resultant force.

7.3 Newton-Euler Equations for rigid bodies

Angular Momentum. The angular momentum of the body b about the center of mass c is

$$\boldsymbol{h}_{b/c} = \int_b \boldsymbol{r} \times \boldsymbol{v} \, dm$$

$$= \boldsymbol{M}_{b/c}\boldsymbol{\omega}_{ib}$$

$$\boldsymbol{T}_{bc} = \frac{d}{dt}\boldsymbol{h}_{b/c}$$

Rotational Inertia / The inertia dyadic. The inertia matrix of the body b about the point c is

$$\boldsymbol{M}_{b/c} = -\int_b \boldsymbol{r}^{\times}\boldsymbol{r}^{\times} \, dm$$

$$= \int_b (\boldsymbol{r}^T\boldsymbol{r}\mathbb{I} - \boldsymbol{r}\boldsymbol{r}^T) \, dm$$

$$= \begin{pmatrix} \boldsymbol{I}_{xx} & -\boldsymbol{I}_{xy} & -\boldsymbol{I}_{xz} \\ -\boldsymbol{I}_{xy} & \boldsymbol{I}_{yy} & -\boldsymbol{I}_{yz} \\ -\boldsymbol{I}_{xz} & -\boldsymbol{I}_{yz} & \boldsymbol{I}_{zz} \end{pmatrix}$$

Where $\boldsymbol{r}$ is the distance vector from the center of mass to the mass element being integrated

$$\boldsymbol{I}_{xx} = \int_b y^2 + z^2 \, dm \qquad \boldsymbol{I}_{xy} = \int_b xy \, dm$$

$$\boldsymbol{I}_{yy} = \int_b x^2 + z^2 \, dm \qquad \boldsymbol{I}_{xz} = \int_b xz \, dm$$

$$\boldsymbol{I}_{zz} = \int_b x^2 + y^2 \, dm \qquad \boldsymbol{I}_{yz} = \int_b yz \, dm$$

$$\boldsymbol{M}_{b/c}^i = \boldsymbol{R}_b^i \boldsymbol{M}_{b/c}^b \boldsymbol{R}_i^b$$

Equations of motion. Let b denote body, i an inertial frame, c the center of mass of b, $\boldsymbol{F}_{bc}$ a resultant force acting on b with line of action through c and $\boldsymbol{T}_{bc}$ the torque about c. Then

$$\boldsymbol{F}_{bc} = m\boldsymbol{a}_c$$

$$\boldsymbol{T}_{bc} = \boldsymbol{M}_{b/c}\boldsymbol{\alpha}_{ib} + \boldsymbol{\omega}_{ib} \times (\boldsymbol{M}_{b/c}\boldsymbol{\omega}_{ib})$$

On compact matrix form

$$\begin{pmatrix} m\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}_{b/c}^b \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_c^b \\ \boldsymbol{\alpha}_{ib}^b \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ (\boldsymbol{\omega}_{ib}^b)^{\times}\boldsymbol{M}_{b/c}^b\boldsymbol{\omega}_{ib}^b \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}_{bc}^b \\ \boldsymbol{T}_{b/c}^b \end{pmatrix}$$

$$\begin{pmatrix} m\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}_{b/c}^b \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}_c^b \\ \boldsymbol{\alpha}_{ib}^b \end{pmatrix} + \begin{pmatrix} m(\boldsymbol{\omega}_{ib}^b)^{\times}\boldsymbol{v}_c^b \\ (\boldsymbol{\omega}_{ib}^b)^{\times}\boldsymbol{M}_{b/c}^b\boldsymbol{\omega}_{ib}^b \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}_{bc}^b \\ \boldsymbol{T}_{b/c}^b \end{pmatrix}$$

Kinetic energy. The kinetic energy of the body b in an inertial reference frame i is

$$K = \frac{1}{2}m(\boldsymbol{v}_c^b)^T\boldsymbol{v}_c^b + \frac{1}{2}(\boldsymbol{\omega}_{ib}^b)^T\boldsymbol{M}_{b/c}^b\boldsymbol{\omega}_{ib}^b$$

The parallel axes theorem. The inertia matrix of b about o is related to the inertia matrix of b about c according to

$$\boldsymbol{M}_{b/o}^b = \boldsymbol{M}_{b/c}^b - m(\boldsymbol{r}_g^b)^\times(\boldsymbol{r}_g^b)^\times$$
$$= \boldsymbol{M}_{b/c}^b - m(||\boldsymbol{r}_g^b||^2\mathbb{I} - \boldsymbol{r}_g^b(\boldsymbol{r}_g^b)^T)$$

Where $\boldsymbol{r}_g^b$ is the vector from c to o

4 Lagrange Mechanics

8.2 Lagrange Mechanics

The lagrangian. Define a set of generalized coordinates $\boldsymbol{q}$. Let $T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ be the kinetic energy and $U(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ the potential energy (Sometimes V). Then the lagrangian is defined to be

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - U(\boldsymbol{q})$$

The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$$

$$\frac{d}{dt}\left(\nabla_{\dot{\boldsymbol{q}}} L\right) - \nabla_{\boldsymbol{q}} L = \boldsymbol{\tau}$$

Where τ_i is a generalized actuator force

$$\tau_i = \sum_{k=1}^{N} \frac{\partial \boldsymbol{r_k}}{\partial q_i} \cdot \boldsymbol{F}_k$$

$\boldsymbol{r}_k(\boldsymbol{q})$ is the position of the point of application of the force $\boldsymbol{F}_k$. In general τ_i is a force or a torque.

Constrained Lagrange. Having the constraints

$$\boldsymbol{c}(\boldsymbol{q}) = \boldsymbol{0}$$

The system can be described by

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - U(\boldsymbol{q}) - \boldsymbol{z}^T \boldsymbol{c}(\boldsymbol{q})$$

$$\frac{d}{dt}\left(\nabla_{\dot{\boldsymbol{q}}} L\right) - \nabla_{\boldsymbol{q}} L = \boldsymbol{\tau}$$

$$\boldsymbol{c}(\boldsymbol{q}) = \boldsymbol{0}$$

Baumgarte stabilization. Instead of imposing

$$\ddot{\boldsymbol{c}}(\boldsymbol{q}) = \boldsymbol{0}$$

Impose

$$\ddot{\boldsymbol{c}} + 2\alpha\dot{\boldsymbol{c}} + \alpha^2 \boldsymbol{c} = \boldsymbol{0}$$

As to reduce drifts in the constraints resulting from

$$\ddot{\boldsymbol{c}} = \boldsymbol{0}$$

not begin satisfied exactly when doing numeric computations.

5 Differential Algebraic Equations

14.2 Preliminaries

5.1 Differential Algebraic Equations

Definition of DEA. The differential equation defined by

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{u}, t) = \boldsymbol{0}$$

Es a DAE if

$$\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$$

is rank deficient

Fully-explicit DAE

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \boldsymbol{0}$$

$$\det \left| \frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}} \right| = 0$$

Can be rewritten as

$$\dot{\boldsymbol{x}} = \boldsymbol{v}$$

$$\boldsymbol{0} = \boldsymbol{F}(\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u})$$

Semi-explicit DAE

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u})$$

$$\boldsymbol{0} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u})$$

This can be rewritten as

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \begin{pmatrix} \dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \\ \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{pmatrix} = \boldsymbol{0}$$

Tikhonov Theorem. Consider the ordinary differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z})$$

$$\varepsilon \dot{\boldsymbol{z}} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z})$$

If

- dynamics of $\dot{\boldsymbol{z}} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z})$ stable $\forall \boldsymbol{x}$

- $\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}}$ is full rank

then

$$\lim_{\epsilon \to 0} \boldsymbol{x}(t) = \boldsymbol{x}_0(t)$$

$$\lim_{\epsilon \to 0} \boldsymbol{z}(t) = \boldsymbol{z}_0(t)$$

where $\boldsymbol{z}_0(t)$ and $\boldsymbol{x}_0(t)$ is the solution to the ODE above modified to a DAE where $\varepsilon = 0$

Theorem: Solvability of DAE. A fully implicit DAE with smooth

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \boldsymbol{0}$$

Can be readily solved (solved for $\dot{\boldsymbol{x}}$ and $\boldsymbol{z}$) if

$$\begin{pmatrix} \frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}} & \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{z}} \end{pmatrix}$$

is full rank on all trajectories $\dot{\boldsymbol{x}}$, $\boldsymbol{z}$, $\boldsymbol{x}$ and $\boldsymbol{u}$. Note that all **Index 1** DAEs fullfil these requirements. The theorem implies that

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u})$$

$$\boldsymbol{0} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u})$$

with smooth $\boldsymbol{f}$ can be solved if

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}}$$

is full rank on all trajectories $\boldsymbol{z}$, $\boldsymbol{x}$ and $\boldsymbol{u}$

Definition: Differential index of a DAE is the number of times the differentiation operator $\frac{d}{dt}$ must be applied to the equations in order to convert the DAE into an ODE.

6 Simulation methods

Butcher tableau

$$\frac{\boldsymbol{c} \ \Big| \ A}{\boldsymbol{b}^T}$$

$$K_n = f(t_k + \Delta t c_n, x_n + \Delta t (A_n) K)$$
$$x_{k+1} = x_k + \Delta t \boldsymbol{b}^T K$$

The method is **consistent** if

$$c_n = \Sigma_i A_{ni}$$
$$\Sigma_i \boldsymbol{b}_i = 1$$

The **stages** of a Runge Kutta method is the number of elements in $\boldsymbol{c}$. The Butcher Tableu defines an explicit integrator if and only if the diagonal elements and the upper-diagonal elements are zero.

Stability function For the Butcher Tableu

$$\frac{\boldsymbol{c} \ \Big| \ \boldsymbol{A}}{\boldsymbol{b}^T}$$

The stability function is

$$R(z) = \frac{\det |\mathbb{I} - z(\boldsymbol{A} - \boldsymbol{1}\boldsymbol{b}^T)|}{\det |\mathbb{I} - z\boldsymbol{A}|}$$

Explicit Euler

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t \cdot \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \qquad \frac{0 \ \big| \ 0}{1}$$

Global error $||\boldsymbol{x}_N - \boldsymbol{x}(T)|| = \mathcal{O}(\Delta t)$
Stability function $R(z) = 1 + z$

Explicit mid-point rule

0	0	0
1/2	1/2	0
	0	1/2

Global error $||\boldsymbol{x}_N - \boldsymbol{x}(T)|| = \mathcal{O}(\Delta t^2)$

Ralston's RK2

0	0	0
2/3	2/3	0
	1/4	1/3

Heun's RK2

0	0	0
1	1	0
	1/2	1/2

Generic second-order method

0	0	0
α	α	0
	$1 - \frac{1}{2\alpha}$	$\frac{1}{2\alpha}$

Generic third-order method

0	0	0	0
α	α	0	0
1	$1 + \frac{1-\alpha}{\alpha(3\alpha-2)}$	$-\frac{1-\alpha}{\alpha(3\alpha-2)}$	0
	$\frac{1}{2} - \frac{1}{6\alpha}$	$\frac{1}{6\alpha(1-\alpha)}$	$\frac{2-3\alpha}{6(1-\alpha)}$

"The" RK4 method

0	0	0	0	0
1/2	1/2	0	0	0
1/2	0	1/2	0	0
1	0	0	1	0
	1/6	1/3	1/3	1/6

Global error $||\boldsymbol{x}_N - \boldsymbol{x}(T)|| = \mathcal{O}(\Delta t^4)$

One-Step error

$$||\boldsymbol{x}_{k+1} - \boldsymbol{x}(t_{k+1})||$$

Taylor expand $\boldsymbol{x}(t)$ around t_k. Do the same for $\boldsymbol{x}_{k+1}$. Subtract and get the results

Global error Get error on time step T by multiplying local error with $N = \frac{T}{\Delta t}$

stability of integration method: Consider the ODE

$$\dot{x} = \lambda x \qquad\qquad x(0) = x_0$$

Perform one step of the RK method to find

$$x_{n+1} = R(z)x_n \qquad\qquad z = \Delta t \lambda$$

The integration method is unstable if

$$|R(z)| > 1$$

A-stability A method is A-stable if the region of stability is the entire left-half plane.

$$|R(\lambda \Delta t)| \le 1 \forall \lambda \in \mathbb{C} \text{ s.t. } \mathcal{R}(\lambda) \le 0$$

L-stability. A Runge Kutta method is L-stable if it is A-stable and

$$\lim_{\omega \to \pm\infty} |R(i\omega h)| = 0$$

Implicit methods

- Can achieve high and systematic orders

- Can be stable regardless of the step size

- Can handle the simulation of DAEs (must be index 1)

- Can give huge speed-up when system is stiff

- Can achieve order $o = 2s$ for any stage s. Explicit methods can only achieve $o = s$ and only for $s \le 4$

Implicit Euler

$$\frac{1}{\quad} \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

$$k_1 = f(x_n + \Delta t k_1)$$
$$x_{n+1} = x_n + \Delta t k_1$$
$$= x_n + \Delta t f(x_{n+1})$$

Gauss-Legendre collocation method Generates order $o = 2s$ A-stable IRK methods. Let s be the number of stages. Find the roots (τ_i) of

$$P_s(\tau) = \frac{1}{s!} \frac{d^s}{d\tau^s} \left((\tau^2 - \tau)^s \right)$$

Build polynomials

$$\ell_i(\tau) = \prod_{j \neq i} \frac{\tau - \tau_j}{\tau_i - \tau_j}$$

Integrate them

$$L_i(\tau) = \int_0^\tau \ell_i(\xi) \, d\xi$$

Calculate A, $\boldsymbol{b}$ and $\boldsymbol{c}$

$$A_{ji} = L_i(\tau_j) \qquad b_i = L_i(1) \qquad c_j = \tau_j$$

Adaptive integrator. Use two methods: x_{n+1} and $\hat{x}_{n+1}$, of different order: p and $p+1$. The local error is

$$\epsilon_{n+1} := |x_{n+1} - \hat{x}_{n+1}|$$
$$\approx \mathcal{O}(\Delta t^{p+1}) - \mathcal{O}(\Delta t^{p+2})$$
$$\approx C \Delta t^{p+1}$$
$$e_{\text{tol}} = C \Delta t_{\text{new}}^{p+1} \Rightarrow$$
$$\Delta t_{\text{new}} = \Delta t \left(\frac{e_{\text{tol}}}{\epsilon_{n+1}} \right)^{\frac{1}{p+1}}$$

Newtons method

$$\boldsymbol{f}(\boldsymbol{x} + \Delta\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x})\Delta\boldsymbol{x} = \boldsymbol{0}$$
$$\Delta\boldsymbol{x} = -\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}) \right)^{-1} \boldsymbol{f}(\boldsymbol{x})$$
$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \alpha \Delta\boldsymbol{x}$$

- May not work if jacobian is singular
- Has quadratic convergence when it converges
- May diverge

RK method on implicit DAEs/ODEs Given the implicit ODE / DAE

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}, t) = \boldsymbol{0}$$

The $\boldsymbol{K} = (\boldsymbol{K}_1, \cdots, \boldsymbol{K}_s)^T$ vector / matrix can be found by solving

$$\boldsymbol{r}(\boldsymbol{K}, \boldsymbol{x}_k, \boldsymbol{z}, \boldsymbol{u}(\cdot), t_k) =$$
$$\begin{pmatrix} F(K_1, x_n + \Delta t a_1^T K, z_1, u(t_k + \Delta t c_1), t_k + \Delta t c_i) \\ \vdots \\ F(K_s, x_n + \Delta t a_s^T K, z_s, u(t_k + \Delta t c_s), t_k + \Delta t c_i) \end{pmatrix}$$
$$= \boldsymbol{0}$$

for $\boldsymbol{K}$ and $\boldsymbol{z}$

Computational cost of ERK methods. Let T_{ol} be the global tolerance.

$$||\boldsymbol{x}_n - \boldsymbol{x}(T)|| \leq c\Delta t^o \leq T_{\text{ol}}$$
$$\Delta t \leq \left(\frac{T_{\text{ol}}}{c} \right)^{(1/o)}$$

The number of integrator steps is

$$N = T/\Delta t$$

The system dynamics must be evaluated s times for each integration step

$$n = Ns = \frac{sT}{\Delta t} \geq sT \left(\frac{T_{\text{ol}}}{c} \right)^{-1/o}$$
$$\frac{n}{T} \geq s \left(\frac{T_{\text{ol}}}{c} \right)^{-1/o}$$

n is the number of function evaluations, T is the simulation time, $s = \#$ stages. T_{ol} is the global tolerance and o is the order. Using the correlation between maximum order and number of stages we get that the optimum order is between 3 and 6.

7 Math

Inverse of 2×2 Matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Multivariable derivative rules

$$\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{a}^T \boldsymbol{x} = \boldsymbol{a}^T \qquad \nabla_{\boldsymbol{x}} \boldsymbol{a}^T \boldsymbol{x} = \boldsymbol{a}$$

$$\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{A} \boldsymbol{x} = \boldsymbol{A} \qquad \nabla_{\boldsymbol{x}} \boldsymbol{A} \boldsymbol{x} = \boldsymbol{A}^T$$

$$\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{x}^T \boldsymbol{A} = \boldsymbol{A}^T \qquad \nabla_{\boldsymbol{x}} \boldsymbol{x}^T \boldsymbol{A} = \boldsymbol{A}$$

Second order terms

$$\frac{\partial}{\partial \boldsymbol{x}} \frac{1}{2} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = \frac{1}{2} \boldsymbol{x}^T (\boldsymbol{A}^T + \boldsymbol{A})$$

$$\nabla_{\boldsymbol{x}} \frac{1}{2} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = \frac{1}{2} (\boldsymbol{A}^T + \boldsymbol{A}) \boldsymbol{x}$$

Multivariable Chain rule

$$\frac{\partial \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{g}} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}$$

$$\frac{\partial \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{h}(\boldsymbol{x}))}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{g}} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{h}} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}$$

Some derivatives

$$\frac{d}{dt} \sinh(t) = \cosh(t)$$

$$\frac{d}{dt} \cosh(t) = \sinh(t)$$

$$\frac{d}{dt} \tanh(t) = \frac{d}{dt} \frac{\sinh(t)}{\cosh(t)} = \frac{1}{\cosh^2 t} = 1 - \tanh^2(x)$$

$$\frac{d}{dt} \arctan(t) = \frac{1}{1 + t^2}$$

Taylor's theorem
Let $k \geq 1$ be an integer and let the function $f : \mathbb{R} \to \mathbb{R}$ be $k + 1$ times differentiable at the point $a \in \mathbb{R}$. Then

$$f(x) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(a)(x - a)^n + \frac{1}{(1+k)!} f^{(k+1)}(\xi)(x - a)^{k+1}$$

for some $\xi \in [x, a]$

Common Taylor expansions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Integration in polar coordinates

$$dA = r \, dr \, d\theta$$

Integration in spherical coordinates

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$