TTK4130 - Cheat Sheet

Håkon Bårsaune

14. mai 2023

https://github.com/haakonbaa/TTK4130-cheatsheet

Innhold

1 Intro

1.1 What are models, what are simulations, notation

1.2 System dynamics and differential equations.

2 Rotations

6.2 Vectors

The *skew-symetric* matrix form of the coordinate vector $\mathbf{u}$ is defined by

$$\mathbf{u}^x = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Notation: $\boldsymbol{v}_{ab}^c$ means the vector from point a to point b (or often the origo of the reference frames a and b) described in the reference frame c

6.4 The Rotation Matrix

The coordinate transformation from frame b to frame a is given by

$$\boldsymbol{v}^a = \boldsymbol{R}_b^a \boldsymbol{v}^b$$

Properties of the rotation matrix

$$\begin{aligned} \boldsymbol{R}_a^b \boldsymbol{R}_b^a &= \boldsymbol{I} = \boldsymbol{R}_b^a \boldsymbol{R}_a^b \\ (\boldsymbol{R}_a^b)^{-1} &= (\boldsymbol{R}_a^b)^T = \boldsymbol{R}_b^a \\ \boldsymbol{R}_b^a &= \begin{pmatrix} \boldsymbol{b}_1^a & \boldsymbol{b}_2^a & \boldsymbol{b}_3^a \end{pmatrix} \\ \det \boldsymbol{R}_a^b &= 1 \end{aligned}$$

$\boldsymbol{R}$ is a rotation matrix if and only if it is an element of $SO(3)$

$$SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3\times 3} | \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I} \wedge \det \boldsymbol{R} = 1\}$$

Rotation matrices in three dimentions

$$\boldsymbol{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

$$\boldsymbol{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\boldsymbol{R}_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix transformations in different refrence frames

$$\begin{aligned} \boldsymbol{D}^a &= \boldsymbol{R}_b^a \boldsymbol{D}^b \boldsymbol{R}_a^b \\ (\boldsymbol{u}^b)^\times &= \boldsymbol{R}_a^b (\boldsymbol{u}^a)^\times \boldsymbol{R}_b^a \end{aligned}$$

The transformation of position and orientation from frame b to frame a is

$$\boldsymbol{T}_b^a = \begin{pmatrix} \boldsymbol{R}_b^a & \boldsymbol{r}_{ab}^a \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

$$\boldsymbol{T}_b^a \begin{pmatrix} \boldsymbol{v}^b \\ 1 \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{v}^a \\ 1 \end{pmatrix}$$

$$(\boldsymbol{T}_b^a)^{-1} = \boldsymbol{T}_a^b = \begin{pmatrix} \boldsymbol{R}_a^b & \boldsymbol{r}_{ba}^b \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

The Special Euclidean group is the set of all transformations from one reference frames to another

$$SE(3) = \left\{ \boldsymbol{T} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \middle| \boldsymbol{R} \in SO(3) \wedge \boldsymbol{r} \in \mathbb{R}^3 \right\}$$

6.5 Euler Angles

Roll-Pitch-Yaw Euler angles

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi)$$

Classical Euler angles. The orientation is described by a rotation bout the z axis, then the resulting y axis. And then again the resulting z axis.

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_z(\phi)$$

6.6 Angle Axis Description of rotation

6.6.5 Rotation Matrix

Angle-axis parameters All rotation matrices have an eigen vector with eigen value 1. A rotation can be uniquely described by the direction of this vector and an angle θ being the rotatoin about this vector.

$$\begin{aligned} &(\theta, \boldsymbol{k}) \text{ s.t. } ||\boldsymbol{k}|| = 1 \\ &\boldsymbol{R}_b^a = \cos\theta \boldsymbol{I} + \sin\theta(\boldsymbol{k}_a)^\times + (1-\cos\theta)\boldsymbol{k}_a\boldsymbol{k}_a^T \\ &\boldsymbol{R}_b^a = \exp\{\boldsymbol{k}^\times\theta\} \end{aligned}$$

6.7 Euler parameters

6.7.1 Definition

$$\begin{aligned} \eta &= \cos\frac{\theta}{2} \\ \boldsymbol{\epsilon} &= \boldsymbol{k}\sin\frac{\theta}{2} \\ \boldsymbol{R}_e(\eta, \boldsymbol{\epsilon}) &= \boldsymbol{I} + 2\eta\boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times \end{aligned}$$

6.7.3 Quaternions

The following can be treated as a unit quaternion

$$\boldsymbol{p} = \begin{pmatrix} \eta \\ \boldsymbol{\epsilon} \end{pmatrix}$$

A unit quaternion satisfies

$$\boldsymbol{p}^T\boldsymbol{p} = \eta^2 + \boldsymbol{\epsilon}^T\boldsymbol{\epsilon} = 1$$

Quaternion product

$$\begin{pmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 - \boldsymbol{\beta}_1^T\boldsymbol{\beta}_2 \\ \alpha_1\boldsymbol{\beta}_2 + \alpha_2\boldsymbol{\beta}_1 + \boldsymbol{\beta}_1^\times\boldsymbol{\beta}_2 \end{pmatrix}$$

6.7.6 Euler parameters from the rotation matrix

$$
\begin{aligned}
&\boldsymbol{R} = (r_{ij}) \\
&\boldsymbol{z} = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix}^T := 2 \begin{pmatrix} \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix}^T \\
&\boldsymbol{T} := r_{00} := \text{Trace}\boldsymbol{R}
\end{aligned}
$$

The algorithm from Shepperd (1978) goes like this:

- Let $i = \arg\max_i \{r_{ii}\}$
- Compute $|z_i| = \sqrt{1 + 2r_{ii} - T}$
- Determine sign of z_i
- Determine the rest of $\boldsymbol{z}$ from equations below

$$
\begin{aligned}
z_0 z_1 &= r_{32} - r_{23} & \qquad z_2 z_3 &= r_{32} + r_{23} \\
z_0 z_2 &= r_{13} - r_{31} & \qquad z_3 z_1 &= r_{13} + r_{31} \\
z_0 z_3 &= r_{21} - r_{12} & \qquad z_1 z_2 &= r_{21} + r_{12}
\end{aligned}
$$

6.8 Angular Velocity

Let $R \in SO(3)$

$$
\begin{aligned}
0 &= \frac{d}{dt}(\boldsymbol{I}) = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{R}^T) = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}(\dot{\boldsymbol{R}})^T \\
&\Rightarrow \dot{\boldsymbol{R}}\boldsymbol{R}^T \, \textit{skew-symmetric}
\end{aligned}
$$

Definition of angluar velocity

$$
\begin{aligned}
(\boldsymbol{\omega}^a_{ab})^\times &= \dot{\boldsymbol{R}}^a_b (\boldsymbol{R}^a_b)^T \Rightarrow \\
\dot{\boldsymbol{R}}^a_b &= (\boldsymbol{\omega}^a_{ab})^\times \boldsymbol{R}^a_b \\
\dot{\boldsymbol{R}}^a_b &= \boldsymbol{R}^a_b (\boldsymbol{\omega}^b_{ab})^\times
\end{aligned}
$$

It can be shown that

$$\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}$$

Where θ and $\boldsymbol{k}$ are Angle Axis parameters.

$$
\begin{aligned}
\boldsymbol{\omega}^a_{ad} &= \boldsymbol{\omega}^a_{ab} + \boldsymbol{\omega}^a_{bc} + \boldsymbol{\omega}^a_{cd} \\
\dot{\boldsymbol{u}}^a &= \boldsymbol{R}^a_b (\dot{\boldsymbol{u}}^b + (\boldsymbol{\omega}^b_{ab})^\times \boldsymbol{u}^b)
\end{aligned}
$$

6.9 Kinematic differential equations

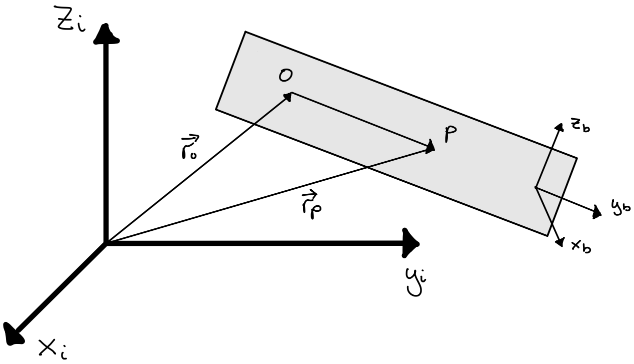
6.9.4 Euler Angles

$$
\begin{aligned}
\boldsymbol{\omega}^a_{ad} &= \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \boldsymbol{R}_{z,\psi} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \boldsymbol{R}_{z,\psi}\boldsymbol{R}_{y,\theta} \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sin\psi\dot{\theta} + \cos\psi\cos\theta\dot{\phi} \\ \cos\psi\dot{\theta} + \sin\psi\cos\theta\dot{\phi} \\ \dot{\psi} - \sin\theta\dot{\phi} \end{pmatrix}
\end{aligned}
$$

3 Rigid Body Dynamics

6.12 Kinematics of a rigid body

$\vec{\omega}_{io}$ is the angular velocity of the o frame with respect to the i frame.
${}^i\frac{d}{dt}\vec{r}_o$ is the derivative of $\vec{r}_o$ in the i frame.



Velocity and Acceleration

$$
\begin{aligned}
\vec{v}_p &:= \frac{{}^id}{dt}\vec{r}_p \\
&= \vec{v}_o + \frac{{}^bd}{dt}\vec{r} + \vec{\omega}_{ib} \times \vec{r} \\
\vec{a}_p &:= \frac{{}^id^2}{dt^2}\vec{r}_p \\
&= \vec{a}_o + \frac{{}^bd^2}{dt^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^bd}{dt}\vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})
\end{aligned}
$$

The last three terms are, respectively, the coriolis acceleration, Transveral acceleration and Centripetal acceleration. Note that

$$\vec{a}_o = \frac{{}^id}{dt}\vec{v}_o = \frac{{}^bd}{dt}\vec{v}_o + \vec{\omega}_{ib} \times \vec{v}_o$$

6.13 The center of mass

The center of mass of a rigid body $\mathcal{C}$ is defined to be

$$\vec{r}_c := \frac{1}{m}\int_{\mathcal{C}} \vec{r}_p \, dm$$

It can be shown that

$$\vec{v}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{v}_p \, dm \qquad \vec{a}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{a}_p \, dm$$

where c denotes *center*

7.2 Forces and torques

7.3 Newton-Euler Equations for rigid bodies