

TTK4130 - Cheat Sheet

Håkon Bårsaune

May 15, 2023

<https://github.com/haakonbaa/TTK4130-cheatsheet>

Contents

1	Intro	2
1.2	State space methods	2
1.2.1	State space models	2
1.2.2	Second order models of mechanical systems	2
1.2.3	Linearization of state space models	2
1.5	ODE's	2
2	Rotations	2
6.2	Vectors	2
6.4	The Rotation Matrix	2
6.5	Euler Angles	3
6.6	Angle Axis Description of rotation	3
6.6.5	Rotation Matrix	3
6.7	Euler parameters	3
6.7.1	Definition	3
6.7.3	Quaternions	3
6.7.6	Euler parameters from the rotation matrix	3
6.8	Angular Velocity	3
6.9	Kinematic differential equations	3
6.9.4	Euler Angles	3
3	Rigid Body Dynamics	4
6.12	Kinematics of a rigid body	4
6.13	The center of mass	4
7.2	Forces and torques	4
7.3	Newton-Euler Equations for rigid bodies	4
8	Math	6

1 Intro

1.2 State space methods

1.2.1 State space models

State Space Model is on the form

$$\dot{x} = f(x, u, t)$$

Linear time invariant system

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$y = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau) d\tau + Du(t)$$

1.2.2 Second order models of mechanical systems

Second order models on the form

$$M(q)\ddot{q} + f(q, \dot{q}) = u$$

Can be written as

$$\begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ M^{-1}(q)(-f(q, \dot{q}) + u) \end{pmatrix}$$

1.2.3 Linearization of state space models

Linearization of time varying systems

$$\dot{x} = f(x, u, t)$$

$$y = h(x, u, t)$$

Find two functions x_0 and u_0 begin solutions to the system

$$\dot{x}_0 = f(x_0(t), u_0(t), t)$$

Define perturbations

$$\Delta x = x(t) - x_0(t)$$

$$\Delta u = u(t) - u_0(t)$$

$$\Delta y = y(t) - y_0(t)$$

Let $\mathcal{C} = \{x_0(t), u_0(t)\}$. The linearized system is

$$\Delta \dot{x} \approx \left. \frac{\partial f}{\partial x} \right|_{\mathcal{C}} \Delta x + \left. \frac{\partial f}{\partial u} \right|_{\mathcal{C}} \Delta u$$

$$\Delta \dot{y} \approx \left. \frac{\partial h}{\partial x} \right|_{\mathcal{C}} \Delta x + \left. \frac{\partial h}{\partial u} \right|_{\mathcal{C}} \Delta u$$

1.5 ODE's

General formulation

$$\varphi(y^{(m)}, \dots, y, u^{(m-1)}, \dots, u) = 0$$

Lipschitz continuous. A function

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

is said to be *Lipschitz continuous* if

$$\begin{aligned} \exists c > 0 \in \mathbb{R} & \quad \text{s.t.} \\ \|f(x) - f(y)\| & < c \|x - y\| \quad \forall x, y \in \mathbb{R} \end{aligned}$$

Theorem: existence of unique solution. Consider the ODE

$$\dot{x} = f(x)$$

If f is Lipschitz continuous then $x(t)$ exists and is unique for all t

Theorem 2: existence of unique solution. Consider the ODE

$$\dot{x} = f(x)$$

if f is continuously differentiable ($\frac{\partial f}{\partial x}$ exists and is continuous), then the solution to the ODE exists and is unique on some time interval.

2 Rotations

6.2 Vectors

The *skew-symmetric* matrix form of the coordinate vector u is defined by

$$u^x = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Notation: v_{ab}^c means the vector from point a to point b (or often the origin of the reference frames a and b) described in the reference frame c

6.4 The Rotation Matrix

The coordinate transformation from frame b to frame a is given by

$$v^a = R_b^a v^b$$

Properties of the rotation matrix

$$R_a^b R_b^a = I = R_b^a R_a^b$$

$$(R_a^b)^{-1} = (R_a^b)^T = R_b^a$$

$$R_b^a = (b_1^a \quad b_2^a \quad b_3^a)$$

$$\det R_a^b = 1$$

R is a rotation matrix if and only if it is an element of $SO(3)$

$$SO(3) = \{R \in \mathbb{R}^{3 \times 3} | R^T R = I \wedge \det R = 1\}$$

Rotation matrices in three dimensions

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

$$R_z(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix transformations in different reference frames

$$\begin{aligned} D^a &= R_b^a D^b R_a^b \\ (\mathbf{u}^b)^\times &= R_b^a (\mathbf{u}^a)^\times R_a^b \end{aligned}$$

The transformation of position and orientation from frame b to frame a is

$$\begin{aligned} T_b^a &= \begin{pmatrix} R_b^a & \mathbf{r}_{ab}^a \\ \mathbf{0}^T & 1 \end{pmatrix} \\ T_b^a \begin{pmatrix} \mathbf{v}^b \\ 1 \end{pmatrix}^T &= \begin{pmatrix} \mathbf{v}^a \\ 1 \end{pmatrix} \\ (T_b^a)^{-1} = T_a^b &= \begin{pmatrix} R_a^b & \mathbf{r}_{ba}^b \\ \mathbf{0}^T & 1 \end{pmatrix} \end{aligned}$$

The Special Euclidean group is the set of all transformations from one reference frames to another

$$SE(3) = \left\{ T = \begin{pmatrix} R & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3} \mid R \in SO(3) \wedge \mathbf{r} \in \mathbb{R}^3 \right\}$$

6.5 Euler Angles

Roll-Pitch-Yaw Euler angles

$$R_a^b = R_z(\psi) R_y(\theta) R_x(\phi)$$

Classical Euler angles. The orientation is described by a rotation about the z axis, then the resulting y axis. And then again the resulting z axis.

$$R_a^b = R_z(\psi) R_y(\theta) R_z(\phi)$$

6.6 Angle Axis Description of rotation

6.6.5 Rotation Matrix

Angle-axis parameters All rotation matrices have an eigen vector with eigen value 1. A rotation can be uniquely described by the direction of this vector and an angle θ being the rotation about this vector.

$$\begin{aligned} (\theta, \mathbf{k}) \text{ s.t. } \|\mathbf{k}\| &= 1 \\ R_b^a &= \cos \theta I + \sin \theta (\mathbf{k}_a)^\times + (1 - \cos \theta) \mathbf{k}_a \mathbf{k}_a^T \\ R_b^a &= \exp\{\mathbf{k}^\times \theta\} \end{aligned}$$

6.7 Euler parameters

6.7.1 Definition

$$\begin{aligned} \eta &= \cos \frac{\theta}{2} \\ \boldsymbol{\epsilon} &= \mathbf{k} \sin \frac{\theta}{2} \end{aligned}$$

$$R_e(\eta, \boldsymbol{\epsilon}) = I + 2\eta \boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times \boldsymbol{\epsilon}^\times$$

6.7.3 Quaternions

The following can be treated as a unit quaternion

$$\mathbf{p} = \begin{pmatrix} \eta \\ \boldsymbol{\epsilon} \end{pmatrix}$$

A unit quaternion satisfies

$$\mathbf{p}^T \mathbf{p} = \eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = 1$$

Quaternion product

$$\begin{pmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 - \boldsymbol{\beta}_1^T \boldsymbol{\beta}_2 \\ \alpha_1 \boldsymbol{\beta}_2 + \alpha_2 \boldsymbol{\beta}_1 + \boldsymbol{\beta}_1^\times \boldsymbol{\beta}_2 \end{pmatrix}$$

6.7.6 Euler parameters from the rotation matrix

$$\begin{aligned} \mathbf{R} &= (r_{ij}) \\ \mathbf{z} &= (z_0 \ z_1 \ z_2 \ z_3)^T := 2 \begin{pmatrix} \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix}^T \\ \mathbf{T} &:= r_{00} := \text{Trace} \mathbf{R} \end{aligned}$$

The algorithm from Shepperd (1978) goes like this:

- Let $i = \arg \max_i \{r_{ii}\}$
- Compute $|z_i| = \sqrt{1 + 2r_{ii} - T}$
- Determine sign of z_i
- Determine the rest of \mathbf{z} from equations below

$$\begin{aligned} z_0 z_1 &= r_{32} - r_{23} & z_2 z_3 &= r_{32} + r_{23} \\ z_0 z_2 &= r_{13} - r_{31} & z_3 z_1 &= r_{13} + r_{31} \\ z_0 z_3 &= r_{21} - r_{12} & z_1 z_2 &= r_{21} + r_{12} \end{aligned}$$

6.8 Angular Velocity

Let $R \in SO(3)$

$$\begin{aligned} 0 &= \frac{d}{dt}(I) = \frac{d}{dt}(R R^T) = \dot{R} R^T + R(\dot{R})^T \\ &\Rightarrow \dot{R} R^T \text{ skew-symmetric} \end{aligned}$$

Definition of angular velocity

$$\begin{aligned} (\boldsymbol{\omega}_{ab}^a)^\times &= \dot{R}_b^a (R_b^a)^T \Rightarrow \\ \dot{R}_b^a &= (\boldsymbol{\omega}_{ab}^a)^\times R_b^a \\ \dot{R}_b^a &= R_b^a (\boldsymbol{\omega}_{ab}^b)^\times \end{aligned}$$

It can be shown that

$$\boldsymbol{\omega} = \dot{\theta} \mathbf{k}$$

Where θ and \mathbf{k} are Angle Axis parameters.

$$\begin{aligned} \boldsymbol{\omega}_{ad}^a &= \boldsymbol{\omega}_{ab}^a + \boldsymbol{\omega}_{bc}^a + \boldsymbol{\omega}_{cd}^a \\ \dot{\mathbf{u}}^a &= R_b^a (\dot{\mathbf{u}}^b + (\boldsymbol{\omega}_{ab}^b)^\times \mathbf{u}^b) \end{aligned}$$

6.9 Kinematic differential equations

6.9.4 Euler Angles

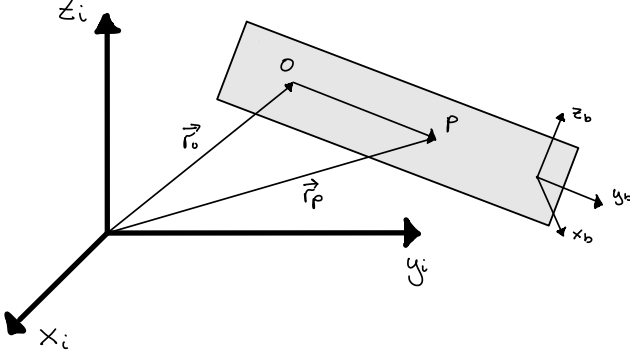
$$\begin{aligned} \boldsymbol{\omega}_{ad}^a &= \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + R_{z,\psi} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + R_{z,\psi} R_{y,\theta} \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sin \psi \dot{\theta} + \cos \psi \cos \theta \dot{\phi} \\ \cos \psi \dot{\theta} + \sin \psi \cos \theta \dot{\phi} \\ \dot{\psi} - \sin \theta \dot{\phi} \end{pmatrix} \end{aligned}$$

3 Rigid Body Dynamics

6.12 Kinematics of a rigid body

$\vec{\omega}_{io}$ is the angular velocity of the o frame with respect to the i frame.

${}^i \frac{d}{dt} \vec{r}_o$ is the derivative of \vec{r}_o in the i frame.



Velocity and Acceleration

$$\begin{aligned}\vec{v}_p &:= \frac{{}^i d}{dt} \vec{r}_p \\ &= \vec{v}_o + \frac{{}^b d}{dt} \vec{r} + \vec{\omega}_{ib} \times \vec{r} \\ \vec{a}_p &:= \frac{{}^i d^2}{dt^2} \vec{r}_p \\ &= \vec{a}_o + \frac{{}^b d^2}{dt^2} \vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^b d}{dt} \vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})\end{aligned}$$

The last three terms are, respectively, the coriolis acceleration, Transversal acceleration and Centripetal acceleration. Note that

$$\vec{a}_o = \frac{{}^i d}{dt} \vec{v}_o = \frac{{}^b d}{dt} \vec{v}_o + \vec{\omega}_{ib} \times \vec{v}_o$$

6.13 The center of mass

The center of mass of a rigid body \mathcal{C} is defined to be

$$\vec{r}_c := \frac{1}{m} \int_{\mathcal{C}} \vec{r}_p dm$$

It can be shown that

$$\vec{v}_c = \frac{1}{m} \int_{\mathcal{C}} \vec{v}_p dm \quad \vec{a}_c = \frac{1}{m} \int_{\mathcal{C}} \vec{a}_p dm$$

where c denotes *center*

7.2 Forces and torques

Moment. The moment about a point P of the set $S = \{F_j\}_{j \in [1, n_F]}$ for forces is

$$\vec{N}_{S/P} = \sum_{j=1}^{n_F} r_{Pj} \times \vec{F}_j$$

Where \vec{r}_{Pj} is an arbitrary point along the line of action of \vec{F}_j

Torque is defined as the moment of the couple \mathcal{C} . A couple being a set of forces with $\mathbf{0}$ resultant force.

7.3 Newton-Euler Equations for rigid bodies

Angular Momentum. The angular momentum of the body b about the center of mass c is

$$\begin{aligned}h_{b/c} &= \int_b \mathbf{r} \times \mathbf{v} dm \\ &= \mathbf{M}_{b/c} \boldsymbol{\omega}_{ib} \\ \mathbf{T}_{bc} &= \frac{d}{dt} h_{b/c}\end{aligned}$$

Rotational Inertia / The inertia dyadic. The inertia matrix of the body b about the point c is

$$\begin{aligned}\mathbf{M}_{b/c} &= - \int_b \mathbf{r}^\times \mathbf{r}^\times dm \\ &= \int_b (\mathbf{r}^T \mathbf{r} \mathbb{I} - \mathbf{r} \mathbf{r}^T) dm \\ &= \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix}\end{aligned}$$

Where \mathbf{r} is the distance vector from the center of mass to the mass element being integrated

$$\begin{aligned}I_{xx} &= \int_b y^2 + z^2 dm & I_{xy} &= \int_b xy dm \\ I_{yy} &= \int_b x^2 + z^2 dm & I_{xz} &= \int_b xz dm \\ I_{zz} &= \int_b x^2 + y^2 dm & I_{yz} &= \int_b yz dm\end{aligned}$$

$$\mathbf{M}_{b/c}^i = \mathbf{R}_b^i \mathbf{M}_{b/c}^b \mathbf{R}_i^b$$

Equations of motion. Let b denote body, i an inertial frame, c the center of mass of b , \mathbf{F}_{bc} a resultant force acting on b with line of action through c and \mathbf{T}_{bc} the torque about c . Then

$$\begin{aligned}\mathbf{F}_{bc} &= m \mathbf{a}_c \\ \mathbf{T}_{bc} &= \mathbf{M}_{b/c} \boldsymbol{\alpha}_{ib} + \boldsymbol{\omega}_{ib} \times (\mathbf{M}_{b/c} \boldsymbol{\omega}_{ib})\end{aligned}$$

On compact matrix form

$$\begin{pmatrix} m \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{b/c}^b \end{pmatrix} \begin{pmatrix} \mathbf{a}_c^b \\ \boldsymbol{\alpha}_{ib}^b \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{bc}^b \\ \mathbf{T}_{bc}^b \end{pmatrix}$$

$$\begin{pmatrix} m \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{b/c}^b \end{pmatrix} \begin{pmatrix} \dot{\mathbf{v}}_c^b \\ \dot{\boldsymbol{\alpha}}_{ib}^b \end{pmatrix} + \begin{pmatrix} m (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{v}_c^b \\ (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{bc}^b \\ \mathbf{T}_{bc}^b \end{pmatrix}$$

Kinetic energy. The kinetic energy of the body b in an inertial reference frame i is

$$K = \frac{1}{2} m (\mathbf{v}_c^b)^T \mathbf{v}_c^b + \frac{1}{2} (\boldsymbol{\omega}_{ib}^b)^T \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b$$

The parallel axes theorem. The inertia matrix of b about o is related to the inertia matrix of b about c according to

$$\mathbf{M}_{b/o}^b = \mathbf{M}_{b/c}^b - m(\mathbf{r}_g^b)^\times (\mathbf{r}_g^b)^\times$$

Where \mathbf{r}_g^b is the vector from c to o

8 Math

Inverse of 2×2 Matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Multivariable derivative rules

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \mathbf{a}^T \mathbf{x} &= \mathbf{a}^T & \nabla_{\mathbf{x}} \mathbf{a}^T \mathbf{x} &= \mathbf{a} \\ \frac{\partial}{\partial \mathbf{x}} \mathbf{A} \mathbf{x} &= \mathbf{A} & \nabla_{\mathbf{x}} \mathbf{A} \mathbf{x} &= \mathbf{A}^T \\ \frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{A} &= \mathbf{A}^T & \nabla_{\mathbf{x}} \mathbf{x}^T \mathbf{A} &= \mathbf{A} \end{aligned}$$

Second order terms

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} &= \frac{1}{2} \mathbf{x}^T (\mathbf{A}^T + \mathbf{A}) \\ \nabla_{\mathbf{x}} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} &= \frac{1}{2} (\mathbf{A}^T + \mathbf{A}) \mathbf{x} \end{aligned}$$

Multivariable Chain rule

$$\begin{aligned} \frac{\partial f(\mathbf{g}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial f}{\partial \mathbf{g}} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \\ \frac{\partial f(\mathbf{g}(\mathbf{x}), \mathbf{h}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial f}{\partial \mathbf{g}} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} + \frac{\partial f}{\partial \mathbf{h}} \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \end{aligned}$$

Some derivatives

$$\begin{aligned} \frac{d}{dt} \sinh(t) &= \cosh(t) \\ \frac{d}{dt} \cosh(t) &= \sinh(t) \\ \frac{d}{dt} \tanh(t) &= \frac{d \sinh(t)}{dt \cosh(t)} = \frac{1}{\cosh^2 t} = 1 - \tanh^2(x) \\ \frac{d}{dt} \arctan(t) &= \frac{1}{1+t^2} \end{aligned}$$