# TTK4130 - Cheat Sheet

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https://github.com/haakonbaa/TTK4130-cheatsheet

## Contents

# 1 Intro

## 1.2 State space methods

### 1.2.1 State space models

**State Space Model** is on the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$$

**Linear time invariant system**

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\
\boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \\
\boldsymbol{y} &= \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(t)\,d\tau + \boldsymbol{D}\boldsymbol{u}(t)
\end{aligned}$$

### 1.2.2 Second order models of mechanical systems

Second order models on the form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u}$$

Can be written as

$$\dot{\begin{pmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{pmatrix}} = \begin{pmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}^{-1}(\boldsymbol{q})(-\boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{u}) \end{pmatrix}$$

### 1.2.3 Linearization of state space models

**Linearization of time varying systems**

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \\
\boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}, t)
\end{aligned}$$

Find two functions $\boldsymbol{x}_0$ and $\boldsymbol{u}_0$ begin solutions to the sytem

$$\dot{\boldsymbol{x}}_0 = \boldsymbol{f}(\boldsymbol{x}_0(t), \boldsymbol{u}_0(t), t)$$

Define pertrubations

$$\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{x}_0(t) + \boldsymbol{\Delta x}(t) \\
\boldsymbol{u}(t) &= \boldsymbol{u}_0(t) + \boldsymbol{\Delta u}(t) \\
\boldsymbol{y}(t) &= \boldsymbol{y}_0(t) + \boldsymbol{\Delta y}(t)
\end{aligned}$$

Let $\mathcal{C} = \{\boldsymbol{x}_0(t), \boldsymbol{u}_0(t)\}$. The linearized system is

$$\begin{aligned}
\boldsymbol{\Delta}\dot{\boldsymbol{x}} &\approx \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\Big|_{\mathcal{C}} \boldsymbol{\Delta x} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\Big|_{\mathcal{C}} \boldsymbol{\Delta u} \\
\boldsymbol{\Delta}\dot{\boldsymbol{y}} &\approx \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\Big|_{\mathcal{C}} \boldsymbol{\Delta x} + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}\Big|_{\mathcal{C}} \boldsymbol{\Delta u}
\end{aligned}$$

## 1.5 ODE's

General formulation

$$\varphi(y^{(m)}, \cdots, y, u^{(m-1)}, \cdots, u) = 0$$

**Lipschitz continous**.A function

$$f : \mathbb{R} \to \mathbb{R}$$

is said to be *Lipschitz continous* if

$$\begin{aligned}
&\exists c > 0 \in \mathbb{R} && \text{s.t.} \\
&||f(x) - f(y)|| < c||x - y|| && \forall x, y \in \mathbb{R}
\end{aligned}$$

**Theorem: existence of unique solution**. Consider the ODE

$$\dot{x} = f(x)$$

If $f$ is Lipschitz continous then $x(t)$ exists and is unique for all $t$

**Mean Value Theorem** suppose $f$ is continous on $[a, b]$ and differnetiable on $(a, b)$ then

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

This can be sused to show that if $f$ is continous and differentiable everywhere it is also Lipschitz.

**Theorem 2: existence of unique solution**. Consider the ODE

$$\dot{x} = f(x)$$

if $f$ is continously differentiable ($\frac{\partial f}{\partial x}$ exists and is continous), then the solution to the ODE exists and is unique on some time interval.

# 2 Rotations

## 6.2 Vectors

The *skew-symetric* matrix form of the coordinate vector $\mathbf{u}$ is defined by

$$\mathbf{u}^x = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Notation: $\boldsymbol{v}_{ab}^c$ means the vector from point $a$ to point $b$ (or often the origo of the reference frames $a$ and $b$) described in the reference frame $c$

## 6.4 The Rotation Matrix

The coordinate transformation from frame $b$ to frame $a$ is given by

$$\boldsymbol{v}^a = \boldsymbol{R}_b^a \boldsymbol{v}^b$$

Properties of the rotation matrix

$$\begin{aligned} \boldsymbol{R}_a^b \boldsymbol{R}_b^a &= \boldsymbol{I} = \boldsymbol{R}_b^a \boldsymbol{R}_a^b \\ (\boldsymbol{R}_a^b)^{-1} &= (\boldsymbol{R}_a^b)^T = \boldsymbol{R}_b^a \\ \boldsymbol{R}_b^a &= \begin{pmatrix} \boldsymbol{b}_1^a & \boldsymbol{b}_2^a & \boldsymbol{b}_3^a \end{pmatrix} \\ \det \boldsymbol{R}_a^b &= 1 \end{aligned}$$

$\boldsymbol{R}$ is a rotation matrix if and only if it is an element of $SO(3)$

$$SO(3) = \{\boldsymbol{R} \in \mathbb{R}^{3\times 3} | \boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I} \wedge \det \boldsymbol{R} = 1\}$$

Rotation matrices in three dimentions

$$\boldsymbol{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

$$\boldsymbol{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\boldsymbol{R}_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix transformations in different refrence frames

$$\begin{aligned} \boldsymbol{D}^a &= \boldsymbol{R}_b^a \boldsymbol{D}^b \boldsymbol{R}_a^b \\ (\boldsymbol{u}^b)^\times &= \boldsymbol{R}_a^b (\boldsymbol{u}^a)^\times \boldsymbol{R}_b^a \end{aligned}$$

The transformation of position and orientation from frame $b$ to frame $a$ is

$$\boldsymbol{T}_b^a = \begin{pmatrix} \boldsymbol{R}_b^a & \boldsymbol{r}_{ab}^a \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

$$\boldsymbol{T}_b^a \begin{pmatrix} \boldsymbol{v}^b \\ 1 \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{v}^a \\ 1 \end{pmatrix}$$

$$(\boldsymbol{T}_b^a)^{-1} = \boldsymbol{T}_a^b = \begin{pmatrix} \boldsymbol{R}_a^b & \boldsymbol{r}_{ba}^b \\ \boldsymbol{0}^T & 1 \end{pmatrix}$$

The Special Euclidean group is the set of all transformations from one reference frames to another

$$SE(3) = \left\{ \boldsymbol{T} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \middle| \boldsymbol{R} \in SO(3) \wedge \boldsymbol{r} \in \mathbb{R}^3 \right\}$$

## 6.5 Euler Angles

**Roll-Pitch-Yaw Euler angles**

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi) \boldsymbol{R}_y(\theta) \boldsymbol{R}_x(\phi)$$

**Classical Euler angles**. The orientation is described by a rotation bout the $z$ axis, then the resulting $y$ axis. And then again the resulting $z$ axis.

$$\boldsymbol{R}_a^b = \boldsymbol{R}_z(\psi) \boldsymbol{R}_y(\theta) \boldsymbol{R}_z(\phi)$$

## 6.6 Angle Axis Description of rotation

### 6.6.5 Rotation Matrix

**Angle-axis parameters** All rotation matrices have an eigen vector with eigen value 1. A rotation can be uniquely described by the direction of this vector and an angle $\theta$ being the rotatoin about this vector.

$$\begin{aligned} &(\theta, \boldsymbol{k}) \text{ s.t. } ||\boldsymbol{k}|| = 1 \\ \boldsymbol{R}_b^a &= \cos\theta \boldsymbol{I} + \sin\theta (\boldsymbol{k}_a)^\times + (1 - \cos\theta) \boldsymbol{k}_a \boldsymbol{k}_a^T \\ \boldsymbol{R}_b^a &= \exp\{\boldsymbol{k}^\times \theta\} \end{aligned}$$

## 6.7 Euler parameters

### 6.7.1 Definition

$$\begin{aligned} \eta &= \cos\frac{\theta}{2} \\ \boldsymbol{\epsilon} &= \boldsymbol{k} \sin\frac{\theta}{2} \\ \boldsymbol{R}_e(\eta, \boldsymbol{\epsilon}) &= \boldsymbol{I} + 2\eta \boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times \boldsymbol{\epsilon}^\times \\ \eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} &= 1 \end{aligned}$$

### 6.7.3 Quaternions

The following can be treated as a unit quaternion

$$\boldsymbol{p} = \begin{pmatrix} \eta \\ \boldsymbol{\epsilon} \end{pmatrix}$$

A unit quaternion satisfies

$$\boldsymbol{p}^T \boldsymbol{p} = \eta^2 + \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} = 1$$

Quaternion product

$$\begin{pmatrix} \alpha_1 \\ \boldsymbol{\beta}_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \boldsymbol{\beta}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 - \boldsymbol{\beta}_1^T \boldsymbol{\beta}_2 \\ \alpha_1 \boldsymbol{\beta}_2 + \alpha_2 \boldsymbol{\beta}_1 + \boldsymbol{\beta}_1^\times \boldsymbol{\beta}_2 \end{pmatrix}$$

### 6.7.6 Euler parameters from the rotation matrix

$$\begin{aligned}
&\boldsymbol{R} = (r_{ij}) \\
&\boldsymbol{z} = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix}^T := 2\begin{pmatrix} \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix}^T \\
&\boldsymbol{T} := r_{00} := \text{Trace}\boldsymbol{R}
\end{aligned}$$

The algorithm from Shepperd (1978) goes like this:

- Let $i = \arg\max_i\{r_{ii}\}$
- Compute $|z_i| = \sqrt{1 + 2r_{ii} - T}$
- Determine sign of $z_i$
- Determine the rest of $\boldsymbol{z}$ from equations below

$$\begin{aligned}
z_0z_1 &= r_{32} - r_{23} & z_2z_3 &= r_{32} + r_{23} \\
z_0z_2 &= r_{13} - r_{31} & z_3z_1 &= r_{13} + r_{31} \\
z_0z_3 &= r_{21} - r_{12} & z_1z_2 &= r_{21} + r_{12}
\end{aligned}$$

## 6.8 Angular Velocity

Let $R \in SO(3)$

$$\begin{aligned}
&0 = \frac{d}{dt}(\boldsymbol{I}) = \frac{d}{dt}(\boldsymbol{R}\boldsymbol{R}^T) = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}(\dot{\boldsymbol{R}})^T \\
&\Rightarrow \dot{\boldsymbol{R}}\boldsymbol{R}^T \, \textit{skew-symmetric}
\end{aligned}$$

Definition of angluar velocity

$$\begin{aligned}
(\boldsymbol{\omega}^a_{ab})^\times &= \dot{\boldsymbol{R}}^a_b(\boldsymbol{R}^a_b)^T \Rightarrow \\
\dot{\boldsymbol{R}}^a_b &= (\boldsymbol{\omega}^a_{ab})^\times \boldsymbol{R}^a_b \\
\dot{\boldsymbol{R}}^a_b &= \boldsymbol{R}^a_b(\boldsymbol{\omega}^b_{ab})^\times
\end{aligned}$$

It can be shown that

$$\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}$$

Where $\theta$ and $\boldsymbol{k}$ are Angle Axis parameters.

$$\begin{aligned}
\boldsymbol{\omega}^a_{ad} &= \boldsymbol{\omega}^a_{ab} + \boldsymbol{\omega}^a_{bc} + \boldsymbol{\omega}^a_{cd} \\
\dot{\boldsymbol{u}}^a &= \boldsymbol{R}^a_b(\dot{\boldsymbol{u}}^b + (\boldsymbol{\omega}^b_{ab})^\times \boldsymbol{u}^b)
\end{aligned}$$

## 6.9 Kinematic differential equations

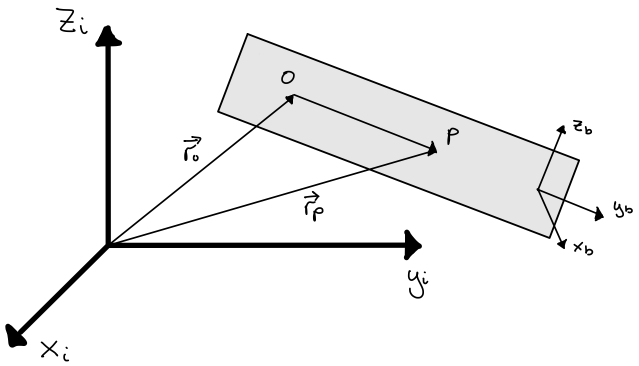
### 6.9.4 Euler Angles

$$\begin{aligned}
\boldsymbol{\omega}^a_{ad} &= \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \boldsymbol{R}_{z,\psi}\begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \boldsymbol{R}_{z,\psi}\boldsymbol{R}_{y,\theta}\begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sin\psi\dot{\theta} + \cos\psi\cos\theta\dot{\phi} \\ \cos\psi\dot{\theta} + \sin\psi\cos\theta\dot{\phi} \\ \dot{\psi} - \sin\theta\dot{\phi} \end{pmatrix}
\end{aligned}$$

# 3 Rigid Body Dynamics

## 6.12 Kinematics of a rigid body

$\vec{\omega}_{io}$ is the angular velocity of the $o$ frame with respect to the $i$ frame.
${}^i\frac{d}{dt}\vec{r}_o$ is the derivative of $\vec{r}_o$ in the $i$ frame.




Velocity and Acceleration

$$\begin{aligned}
\vec{v}_p &:= \frac{{}^id}{dt}\vec{r}_p \\
&= \vec{v}_o + \frac{{}^bd}{dt}\vec{r} + \vec{\omega}_{ib} \times \vec{r} \\
\vec{a}_p &:= \frac{{}^id^2}{dt^2}\vec{r}_p \\
&= \vec{a}_o + \frac{{}^bd^2}{dt^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^bd}{dt}\vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})
\end{aligned}$$

The last three terms are, respectively, the coriolis acceleration, Transveral acceleration and Centripetal acceleration. Note that

$$\vec{a}_o = \frac{{}^id}{dt}\vec{v}_o = \frac{{}^bd}{dt}\vec{v}_o + \vec{\omega}_{ib} \times \vec{v}_o$$

## 6.13 The center of mass

The center of mass of a rigid body $\mathcal{C}$ is defined to be

$$\vec{r}_c := \frac{1}{m}\int_{\mathcal{C}} \vec{r}_p \, dm$$

It can be shown that

$$\vec{v}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{v}_p \, dm \qquad \vec{a}_c = \frac{1}{m}\int_{\mathcal{C}} \vec{a}_p \, dm$$

where $c$ denotes *center*

## 7.2 Forces and torques

**Moment**. The moment about a point $P$ of the set $S = \{F_j\}_{j\in[1,n_F]}$ for forces is

$$\vec{N}_{S/P} = \sum_{j=1}^{n_F} r\vec{_{P}}_j \times \vec{F}_j$$

Where $\vec{r}_{Pj}$ is an arbitrary point along the line of action of $\vec{F}_j$

**Torque** is defined as the moment of the couple $\mathcal{C}$. A couple being a set of forces with **0** resultant force.

## 7.3 Newton-Euler Equations for rigid bodies

**Angular Momentum**. The angular momentum of the body $b$ about the center of mass $c$ is

$$\begin{aligned}
\boldsymbol{h}_{b/c} &= \int_b \boldsymbol{r} \times \boldsymbol{v} \, dm \\
&= \boldsymbol{M}_{b/c}\boldsymbol{\omega}_{ib} \\
\boldsymbol{T}_{bc} &= \frac{d}{dt}\boldsymbol{h}_{b/c}
\end{aligned}$$

**Rotational Inertia / The intertia dyadic**. The inertia matrix of the body $b$ about the point $c$ is

$$\begin{aligned}
\boldsymbol{M}_{b/c} &= -\int_b \boldsymbol{r}^\times \boldsymbol{r}^\times \, dm \\
&= \int_b (\boldsymbol{r}^T\boldsymbol{r}\mathbb{I} - \boldsymbol{r}\boldsymbol{r}^T) \, dm \\
&= \begin{pmatrix} \boldsymbol{I}_{xx} & -\boldsymbol{I}_{xy} & -\boldsymbol{I}_{xz} \\ -\boldsymbol{I}_{xy} & \boldsymbol{I}_{yy} & -\boldsymbol{I}_{yz} \\ -\boldsymbol{I}_{xz} & -\boldsymbol{I}_{yz} & \boldsymbol{I}_{zz} \end{pmatrix}
\end{aligned}$$

Where $\boldsymbol{r}$ is the distance vector from the center of mass to the mass element being integrated

$$\begin{aligned}
\boldsymbol{I}_{xx} &= \int_b y^2 + z^2 \, dm & \boldsymbol{I}_{xy} &= \int_b xy \, dm \\
\boldsymbol{I}_{yy} &= \int_b x^2 + z^2 \, dm & \boldsymbol{I}_{xz} &= \int_b xz \, dm \\
\boldsymbol{I}_{zz} &= \int_b x^2 + y^2 \, dm & \boldsymbol{I}_{yz} &= \int_b yz \, dm
\end{aligned}$$

$$\boldsymbol{M}^i_{b/c} = \boldsymbol{R}^i_b \boldsymbol{M}^b_{b/c} \boldsymbol{R}^b_i$$

**Equations of motion**. Let $b$ denote body, $i$ an intertial frame, $c$ the center of mass of $b$, $\boldsymbol{F}_{bc}$ a resultant force acting on $b$ with line of action throgh $c$ and $\boldsymbol{T}_{bc}$ the torque about $c$. Then

$$\begin{aligned}
\boldsymbol{F}_{bc} &= m\boldsymbol{a}_c \\
\boldsymbol{T}_{bc} &= \boldsymbol{M}_{b/c}\boldsymbol{\alpha}_{ib} + \boldsymbol{\omega}_{ib} \times (\boldsymbol{M}_{b/c}\boldsymbol{\omega}_{ib})
\end{aligned}$$

On compact matrix form

$$\begin{pmatrix} m\mathbb{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}^b_{b/c} \end{pmatrix} \begin{pmatrix} \boldsymbol{a}^b_c \\ \boldsymbol{\alpha}^b_{ib} \end{pmatrix} + \begin{pmatrix} \boldsymbol{0} \\ (\boldsymbol{\omega}^b_{ib})^\times \boldsymbol{M}^b_{b/c}\boldsymbol{\omega}^b_{ib} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}^b_{bc} \\ \boldsymbol{T}^b_{b/c} \end{pmatrix}$$
$$\begin{pmatrix} m\mathbb{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}^b_{b/c} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}^b_c \\ \boldsymbol{\alpha}^b_{ib} \end{pmatrix} + \begin{pmatrix} m(\boldsymbol{\omega}^b_{ib})^\times \boldsymbol{v}^b_c \\ (\boldsymbol{\omega}^b_{ib})^\times \boldsymbol{M}^b_{b/c}\boldsymbol{\omega}^b_{ib} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}^b_{bc} \\ \boldsymbol{T}^b_{b/c} \end{pmatrix}$$

**Kinetic energy**. The kinetic energy of the body $b$ in an inertial refrence frame $i$ is

$$K = \frac{1}{2}m(\boldsymbol{v}^b_c)^T\boldsymbol{v}^b_c + \frac{1}{2}(\boldsymbol{\omega}^b_{ib})^T\boldsymbol{M}^b_{b/c}\boldsymbol{\omega}^b_{ib}$$

**The parallel axes theorem**. The inertia matrix of $b$ about $o$ is related to the inertia matrix of $b$ about $c$ according to

$$\begin{aligned}\boldsymbol{M}^b_{b/o} &= \boldsymbol{M}^b_{b/c} - m(\boldsymbol{r}^b_g)^\times(\boldsymbol{r}^b_g)^\times \\ &= \boldsymbol{M}^b_{b/c} - m(||\boldsymbol{r}^b_g||^2\mathbb{I} - \boldsymbol{r}^b_g(\boldsymbol{r}^b_g)^T)\end{aligned}$$

Where $\boldsymbol{r}^b_g$ is the vector from $c$ to $o$

# 4 Lagrange Mechanics

## 8.2 Lagrange Mechanics

**The lagrangian**. Define a set of generalized coordinates $\boldsymbol{q}$. Let $T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ be the kinetic energy and $U(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ the potential energy (Sometimes $V$). Then the lagrangian is defined to be

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - U(\boldsymbol{q})$$

The equation of motion is

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i \\ &\frac{d}{dt}\left(\nabla_{\dot{\boldsymbol{q}}} L\right) - \nabla_{\boldsymbol{q}} L = \boldsymbol{\tau} \end{aligned}$$

Where $\tau_i$ is a generalized actuator force

$$\tau_i = \sum_{k=1}^{N} \frac{\partial \boldsymbol{r_k}}{\partial q_i} \cdot \boldsymbol{F}_k$$

$\boldsymbol{r}_k(\boldsymbol{q})$ is the position of the point of application of the force $\boldsymbol{F}_k$. In general $\tau_i$ is a force or a torque.

**Constrained Lagrange**. Having the constraints

$$\boldsymbol{c}(\boldsymbol{q}) = \boldsymbol{0}$$

The system can be described by

$$\begin{aligned} &L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - U(\boldsymbol{q}) - \boldsymbol{z}^T \boldsymbol{c}(\boldsymbol{q}) \\ &\frac{d}{dt}\left(\nabla_{\dot{\boldsymbol{q}}} L\right) - \nabla_{\boldsymbol{q}} L = \boldsymbol{\tau} \\ &\boldsymbol{c}(\boldsymbol{q}) = \boldsymbol{0} \end{aligned}$$

**Baumgarte stabilization**. Instead of imposing

$$\ddot{\boldsymbol{c}}(\boldsymbol{q}) = \boldsymbol{0}$$

Impose

$$\ddot{\boldsymbol{c}} + 2\alpha\dot{\boldsymbol{c}} + \alpha^2 \boldsymbol{c} = \boldsymbol{0}$$

As to reduce drifts in the constraints resulting from

$$\ddot{\boldsymbol{c}} = \boldsymbol{0}$$

not begin satisfied exactly when doing numeric computations.

# 5 Differential Algebraic Equations

## 14.2 Preliminaries

## 5.1 Differential Algebraic Equations

**Definition of DEA**. The differential equation defined by

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{u}, t) = \boldsymbol{0}$$

Es a DAE if

$$\frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}}$$

is rank defficient

**Fully-explicit DAE**

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \boldsymbol{0}$$
$$\det \left| \frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}} \right| = 0$$

Can be rewritten as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{v} \\ \boldsymbol{0} &= \boldsymbol{F}(\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{aligned}$$

**Semi-explicit DAE**

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{aligned}$$

This can be rewritten as

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \begin{pmatrix} \dot{\boldsymbol{x}} - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \\ \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{pmatrix} = \boldsymbol{0}$$

**Tikhonov Theorem**. Consider the ordinary differential equation

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}) \\ \varepsilon \dot{\boldsymbol{z}} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}) \end{aligned}$$

If

- dynamics of $\dot{\boldsymbol{z}} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z})$ stable $\forall \boldsymbol{x}$
- $\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}}$ is full rank

then

$$\begin{aligned} \lim_{\epsilon \to 0} \boldsymbol{x}(t) &= \boldsymbol{x}_0(t) \\ \lim_{\epsilon \to 0} \boldsymbol{z}(t) &= \boldsymbol{z}_0(t) \end{aligned}$$

where $\boldsymbol{z}_0(t)$ and $\boldsymbol{x}_0(t)$ is the solution to the ODE above modified to a DAE where $\varepsilon = 0$

**Theorem: Solvability of DAE**. A fully implicit DAE with smooth

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \boldsymbol{0}$$

Can be readily solved (solved for $\dot{\boldsymbol{x}}$ and $\boldsymbol{z}$) if

$$\begin{pmatrix} \frac{\partial \boldsymbol{F}}{\partial \dot{\boldsymbol{x}}} & \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{z}} \end{pmatrix}$$

is full rank on all trajectories $\dot{\boldsymbol{x}}$, $\boldsymbol{z}$, $\boldsymbol{x}$ and $\boldsymbol{u}$. Note that all **Index 1** DAEs fullfil these requirements. The theorem implies that

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{aligned}$$

with smooth $\boldsymbol{f}$ can be solved if

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}}$$

is full rank on all trajectories $\boldsymbol{z}$, $\boldsymbol{x}$ and $\boldsymbol{u}$

**Definition: Differential index of a DAE** is the numer of times the differentiation operator $\frac{d}{dt}$ must be applied to the equations in order to convert the DAE into an ODE.

# 6 Simulation methods

**Butcher tableau**

$$\begin{array}{c|c} \boldsymbol{c} & A \\ \hline & \boldsymbol{b}^T \end{array}$$

$$K_n = f(t_k + \Delta t c_n, x_n + \Delta t (A_n) K)$$
$$x_{k+1} = x_k + \Delta t \boldsymbol{b}^T K$$

The method is **consistent** if

$$c_n = \Sigma_i A_{ni}$$
$$\Sigma_i \boldsymbol{b}_i = 1$$

The **stages** of a Runge Kutta method is the number of elements in $\boldsymbol{c}$. The Butcher tableu defines an explicit integrator if and only if the diagonal elements and the upper-diagonal elements are zero.

**Stability function** For the Butcher Tableu

$$\begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{b}^T \end{array}$$

The stability function is

$$R(z) = \frac{\det |\mathbb{I} - z(\boldsymbol{A} - \boldsymbol{1}\boldsymbol{b}^T)|}{\det |\mathbb{I} - z\boldsymbol{A}|}$$

**Explicit Euler**

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta t \cdot \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \qquad \begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Global error $||\boldsymbol{x}_N - \boldsymbol{x}(T)|| = \mathcal{O}(\Delta t)$
Stability function $R(z) = 1 + z$

**Explicit mid-point rule**

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1/2 \end{array}$$

Global error $||\boldsymbol{x}_N - \boldsymbol{x}(T)|| = \mathcal{O}(\Delta t^2)$

**Ralston's RK2**

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 1/3 \end{array}$$

**Heun's RK2**

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

**Generic second-order method**

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \alpha & \alpha & 0 \\ \hline & 1 - \frac{1}{2\alpha} & \frac{1}{2\alpha} \end{array}$$

**Generic third-order method**

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \alpha & \alpha & 0 & 0 \\ 1 & 1 + \frac{1-\alpha}{\alpha(3\alpha-2)} & -\frac{1-\alpha}{\alpha(3\alpha-2)} & 0 \\ \hline & \frac{1}{2} - \frac{1}{6\alpha} & \frac{1}{6\alpha(1-\alpha)} & \frac{2-3\alpha}{6(1-\alpha)} \end{array}$$

**"The" RK4 method**

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Global error $||\boldsymbol{x}_N - \boldsymbol{x}(T)|| = \mathcal{O}(\Delta t^4)$

**One-Step error**

$$||\boldsymbol{x}_{k+1} - \boldsymbol{x}(t_{k+1})||$$

Taylor expand $\boldsymbol{x}(t)$ around $t_k$. Do the same for $\boldsymbol{x}_{k+1}$. Subtract and get the results

**Global error** Get error on timestep $T$ by multiplying local error with $N = \frac{T}{\Delta t}$

**stability of integration method**: Consdier the ODE

$$\dot{x} = \lambda x \qquad x(0) = x_0$$

Perform one step of the RK method to find

$$x_{n+1} = R(z) x_n \qquad z = \Delta t \lambda$$

The integration method is unstable if

$$|R(z)| > 1$$

**A-stability** A method is A-stable if the region of stability is the entire left-half plane.

$$|R(\lambda \Delta t)| \leq 1 \forall \lambda \in \mathbb{C} \text{ s.t. } \mathcal{R}(\lambda) \leq 0$$

**L-stability**. A Runge kutta method is L-stable if it is A-stable and

$$\lim_{\omega \to \pm\infty} |R(i\omega h)| = 0$$

**Implicit methods**

- Can chaieve high and systematic orders
- Can be stable regardless of the step size
- Can handle the simulation of DAEs (must be index 1)
- Can give huge speed-up when system is stiff
- Can achieve order $o = 2s$ for any stage $s$. Explicit methods can only achieve $o = s$ and only for $s \leq 4$

**Implicit Euler**

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

$$\begin{aligned} k_1 &= f(x_n + \Delta t k_1) \\ x_{n+1} &= x_n + \Delta t k_1 \\ &= x_n + \Delta t f(x_{n+1}) \end{aligned}$$

**Gauss-Legendre collocation method** Generates order $o = 2s$ A-stable IRK methods. Let $s$ be the number of stages. Find the roots $(\tau_i)$ of

$$P_s(\tau) = \frac{1}{s!}\frac{d^s}{d\tau^s}\left((\tau^2 - \tau)^s\right)$$

Build polynomials

$$\ell_i(\tau) = \prod_{j \neq i} \frac{\tau - \tau_j}{\tau_i - \tau_j}$$

Integrate them

$$L_i(\tau) = \int_0^\tau \ell_i(\xi)\, d\xi$$

Calculate $A$ ,$\boldsymbol{b}$ and $\boldsymbol{c}$

$$A_{ji} = L_i(\tau_j) \qquad b_i = L_i(1) \qquad c_j = \tau_j$$

**Adaptive integrator**. Use two methods: $x_{n+1}$ and $\hat{x}_{n+1}$, of different order: $p$ and $p+1$. The local error is

$$\begin{aligned} \epsilon_{n+1} &:= |x_{n+1} - \hat{x}_{n+1}| \\ &\approx \mathcal{O}(\Delta t^{p+1}) - \mathcal{O}(\Delta t^{p+2}) \\ &\approx C\Delta t^{p+1} \\ e_{\text{tol}} &= C\Delta t_{\text{new}}^{p+1} \Rightarrow \\ \Delta t_{\text{new}} &= \Delta t \left(\frac{e_{\text{tol}}}{\epsilon_{n+1}}\right)^{\frac{1}{p+1}} \end{aligned}$$

**Newtons method**

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x} + \Delta\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x})\Delta\boldsymbol{x} = \boldsymbol{0} \\ \Delta\boldsymbol{x} = -\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x})\right)^{-1} \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \alpha\Delta\boldsymbol{x} \end{aligned}$$

- May not work if jacobian is singular
- Has quadratic convergence when it converges
- May diverge

**RK method on implicit DAEs/ODEs** Given the implicit ODE / DAE

$$\boldsymbol{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}, t) = \boldsymbol{0}$$

The $\boldsymbol{K} = (\boldsymbol{K}_1, \cdots, \boldsymbol{K}_s)^T$ vector / matrix can be found by solving

$$\boldsymbol{r}(\boldsymbol{K}, \boldsymbol{x}_k, \boldsymbol{z}, \boldsymbol{u}(\cdot), t_k) = \\ \begin{pmatrix} \boldsymbol{F(K_1, x_n + \Delta t a_1^T K, z, u(t_k + \Delta t c_1), t_k + \Delta t c_i)} \\ \vdots \\ \boldsymbol{F(K_s, x_n + \Delta t a_s^T K, z, u(t_k + \Delta t c_s), t_k + \Delta t c_i)} \end{pmatrix} \\ = \boldsymbol{0}$$

for $\boldsymbol{K}$ and $\boldsymbol{z}$

## 7 Math

**Inverse of $2 \times 2$ Matrix**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

**Multivariable derivative rules**

$$\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{a}^T \boldsymbol{x} = \boldsymbol{a}^T \qquad \nabla_{\boldsymbol{x}} \boldsymbol{a}^T \boldsymbol{x} = \boldsymbol{a}$$
$$\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{A} \boldsymbol{x} = \boldsymbol{A} \qquad \nabla_{\boldsymbol{x}} \boldsymbol{A} \boldsymbol{x} = \boldsymbol{A}^T$$
$$\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{x}^T \boldsymbol{A} = \boldsymbol{A}^T \qquad \nabla_{\boldsymbol{x}} \boldsymbol{x}^T \boldsymbol{A} = \boldsymbol{A}$$

Second order terms

$$\frac{\partial}{\partial \boldsymbol{x}} \frac{1}{2} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = \frac{1}{2} \boldsymbol{x}^T (\boldsymbol{A}^T + \boldsymbol{A})$$
$$\nabla_{\boldsymbol{x}} \frac{1}{2} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = \frac{1}{2} (\boldsymbol{A}^T + \boldsymbol{A}) \boldsymbol{x}$$

**Multivariable Chain rule**

$$\frac{\partial \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{g}} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}$$
$$\frac{\partial \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{h}(\boldsymbol{x}))}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{g}} \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{h}} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}$$

**Some derivatives**

$$\begin{aligned}
\frac{d}{dt} \sinh(t) &= \cosh(t) \\
\frac{d}{dt} \cosh(t) &= \sinh(t) \\
\frac{d}{dt} \tanh(t) = \frac{d}{dt} \frac{\sinh(t)}{\cosh(t)} &= \frac{1}{\cosh^2 t} = 1 - \tanh^2(x) \\
\frac{d}{dt} \arctan(t) &= \frac{1}{1 + t^2}
\end{aligned}$$

**Taylor's theorem**
Let $k \geq 1$ be an integer and let the function $f : \mathbb{R} \to \mathbb{R}$ be $k + 1$ times differentiable at the point $a \in \mathbb{R}$. Then

$$f(x) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(a)(x - a)^n + \frac{1}{(1 + k)!} f^{(k+1)}(\xi)(x - a)^{k+1}$$

for some $\xi \in [x, a]$
**Common Taylor expansions**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n + 1)!}$$
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$