INF5620 - Numerical methods for partial differential equations

Mandatory Assignment 4

Compute with a non-uniform mesh

Exercise 8

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Defining the linear system

In this exercise, our task is to derive the linear system for $-u''(x) = 2$ on the domain $[0, 1]$ with Dirichlet boundary conditions, using P1-elements and a non-uniform mesh. The cell number e has a length $h_e = x_{e+1} - x_e$.

We know the solution must be on the form

$$u(x) = B(x) + \sum_{j=0}^{N-1} c_j \psi_j \tag{1}$$

i.e., u can be written as a linear combination of different basis functions. We let these basis functions got to zero at the boundary and fix the boundary condition by adding the term $B(x)$.

We define the residual R, and require that the inner product of R and a test function v, should be 0, for any $v \in V$:

$$\int_\Omega \left(\frac{\partial^2 u}{\partial x^2} + 2 \right) v \, \mathrm{d}x = 0, \qquad \Omega \in [0, 1], \qquad \forall \, v \in V \tag{2}$$

In short-hand notation, we simply write the inner product as $(R, v) = 0$. When using P1-elements, we have discontinuous derivatives at the cell boundaries and therefore difficulties when computing the second-order derivative. To solve this, we can transform (or reduce) the second order derivative to first order, by doing integration by parts. Then we also end up with a first order derivative on the test function v, almost like we "moved" it from u to v. The boundary terms arising from the integration by parts, vanish and we are left with:

$$\begin{aligned}
(u'' + 2, v) = 0, & \qquad \forall \, v \in V \\
(u'', v) \quad = (-2, v), & \qquad \forall \, v \in V \\
(u', v') \quad = (2, v), & \qquad \forall \, v \in V
\end{aligned} \tag{3}$$

We say that the the basis functions span the vector space V, ($V = \mathrm{span}\{\psi_0, ..., \psi_{N-1}\}$), which means that V, contains all possible combinations of the basis functions. The condition in the above equations, that it must hold true for all $v \ \in V$, is equal to saying that $v = \psi_j, \forall \ j \in \{0, 1, .., N-1\}$, and thus v is used for ease of reading. Written as integrals over the domain we have:

$$\sum_{j=0}^{N-1} c_j \int_0^1 \psi_j' \psi_i' \, \mathrm{d}x = 2\int_0^1 \psi_i \, \mathrm{d}x - \int_0^1 B'(x)\psi_i' \, \mathrm{d}x \tag{4}$$

$$\int_0^1 \psi_j \psi_i \, \mathrm{d}x = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{5}$$

Since the basis is orthonormal, any integral over $\psi_j\psi_i$ is either 0 or 1 as described in equation 5. This means we pick out only the elements where $i = j$. We can now do the element-wise computations to find the matrix elements of A:

$$\mathbf{A}^{(e)}_{[0,0]} = \int_{x_e}^{x_{e+1}} \psi_e' \psi_e' dx \quad = \int_{x_e}^{x_{e+1}} \frac{1}{h_e^2} dx \quad = \frac{1}{h_e} \quad = \mathbf{A}^{(e)}_{[1,1]}$$

$$\mathbf{A}^{(e)}_{[1,0]} = \int_{x_e}^{x_{e+1}} \psi_{e+1}' \psi_e' dx = \int_{x_e}^{x_{e+1}} -\frac{1}{h_e^2} dx = -\frac{1}{h_e} = \mathbf{A}^{(e)}_{[0,1]}$$

We now have the diagonal and off-diagonal elements. We can now assemble the matrix $\mathbf{A}$ and see that it is positive definite, symmetric and tri-diagonal, which means it can be solved extremely fast with Thomas algorithm that run at $\mathcal{O}(6n)$ FLOPS.

$$\mathbf{A} = \begin{bmatrix} \frac{1}{h_0} & -\frac{1}{h_0} & 0 & \dots & 0 \\ -\frac{1}{h_0} & \frac{1}{h_0} + \frac{1}{h_1} & -\frac{1}{h_1} & \dots & 0 \\ 0 & -\frac{1}{h_1} & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & -\frac{1}{h_{N-1}} \\ 0 & 0 & 0 & -\frac{1}{h_{N-1}} & \frac{1}{h_{N-1}} \end{bmatrix} \tag{6}$$

It is time to define our boundary term, $B(x) = \psi_{N-1}$, where $\psi_{N-1} = 1$ on x_{N-1}. We can now find the right-hand side of the linear equation $\mathbf{Ac} = \mathbf{b}$ (where the c's are the weights in eq. 1):

$$b_0^{(e)} = 2\int_{x_e}^{x_{e+1}} \psi_e \, \mathrm{d}x = \frac{1}{h_e} = b_1^{(e)} \tag{7}$$

For the very last term $b^{(e)}$, $e = N-1$, we get a special relation to take care of the boundary conditions $u(x=1) = 1$:

$$b^{(e)} = h_e - \frac{1}{h_e}, \quad \text{where} \quad e = N-1 \tag{8}$$

The full linear system, $\mathbf{Ac} = \mathbf{b}$ may now be assembled:

$$\begin{bmatrix} \frac{1}{h_0} & -\frac{1}{h_0} & 0 & ... & 0 \\ -\frac{1}{h_0} & \frac{1}{h_0}+\frac{1}{h_1} & -\frac{1}{h_1} & ... & 0 \\ 0 & -\frac{1}{h_1} & ... & ... & 0 \\ 0 & 0 & ... & ... & -\frac{1}{h_{N-1}} \\ 0 & 0 & 0 & -\frac{1}{h_{N-1}} & \frac{1}{h_{N-1}} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ ... \\ c_{N-2} \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} h_0 \\ h_0 + h_1 \\ ... \\ h_{N-2} + h_{N-1} \\ h_{N-1} - \frac{1}{h_{N-1}} \end{bmatrix} \tag{9}$$

We can write this system of linear equations on vectorized form to get a formula for any row j: (special care must be taken for the first, and last row)

$$\begin{aligned} c_j\frac{1}{h_j} - c_{j+1}\frac{1}{h_j} &= h_j, && \text{where} \quad j = 0 \\ -c_{j-1}\frac{1}{h_{j-1}} + c_j\left(\frac{1}{h_{j-1}} + \frac{1}{h_j}\right) - c_{j+1}\frac{1}{h_j} &= h_{j-1} + h_j, && \text{where} \quad j \in 1, 2, .., N-2 \\ -c_{j-1}\frac{1}{h_{j-1}} + c_j\frac{1}{h_{j-1}} &= h_j - \frac{1}{h_j}, && \text{where} \quad j = N-1 \end{aligned} \tag{10}$$

An interesting observation here is that, if the mesh was uniform (and not non-uniform as in this task), then spacing between cells, h_e would be the same for any cell e. This means we could have written the entire system as:

$$\begin{bmatrix} 1 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & ... & 0 \\ 0 & -1 & 2 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ ... \\ c_{N-2} \\ c_{N-1} \end{bmatrix} = h^2 \begin{bmatrix} 1 \\ 2 \\ ... \\ 2 \\ 1 - h^{-2} \end{bmatrix} \tag{11}$$

Very clean, huh?

Comparison with a central finite difference scheme

Let us start by deriving the (central) finite difference scheme of $[D_x D_x u]_i = -2$:

$$[D_x D_x u]_i \approx D_x \left(\frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \right) \tag{12}$$

$$\approx \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i} - \frac{u_i - u_{i-1}}{x_i - x_{i-1}} \right) \tag{13}$$

We may now replace the "in-between-values" $x_{i\,\pm\frac{1}{2}}$ with the simple arithmetic mean, and we end up with the following (after cancelling some terms):

$$x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \frac{1}{2}\left(x_{i+1} - x_{i-1}\right) \tag{14}$$

If we now use the relationship that $h_i = x_{i+1} - x_i$, we can replace this in the above equation and get:

$$u_{xx} = -2$$

$$\frac{1}{h_i + h_{i-1}} \left(\frac{u_{i+1} - u_i}{h_i} - \frac{u_i - u_{i-1}}{h_{i-1}} \right) = 1 \tag{15}$$

With a little fantasy (and rearrangement), we can see that equation 10 gives the same equation when changing the weights c_j with u_i.