Mandatory Assignment 2 of 2

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Problem 1

Let $f : [-\pi, \pi] \to \mathbb{R}$ be given by $f(x) = |x|$.
a) Show that the real Fourier series of f is

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos\left[(2n+1)x\right]}{(2n+1)^2} \tag{1}$$

As $f(x) = f(-x)$ (even), we have $b_n = 0$. For we a_0 calculate the average of the function on the inteval $[-\pi, \pi]$

$$a_o = \frac{1}{2\pi} \int_0^{\pi} |x| dx = \frac{\pi}{2}$$

For the a_n terms we calculate

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx \\
a_n &= \frac{2}{\pi} \left[\frac{\pi \sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2} - \frac{1}{n^2} \right] \\
a_n &= \frac{2}{\pi} \left[\frac{\cos(n\pi)}{n^2} - \frac{1}{n^2} \right] \quad ; \quad (\sin(n\pi) = 0, \ \forall n \in \mathbb{N})
\end{aligned}$$

When n is even $\cos(n\pi) = 1$ and when n is odd $\cos(n\pi) = -1$. Thus

$$a_n = \frac{2}{\pi} \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

We can see that for even $n = 2k$ we get $a_{2k} = 0$. For odd $n = 2k+1$ we have

$$a_{2k+1} = \frac{2}{\pi} \left[\frac{(-1)^{2k+1}}{(2k+1)^2} - \frac{1}{(2k+1)^2} \right] = -\frac{4}{\pi} \frac{1}{(2k+1)^2}$$

Combining

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos\left[(2n+1)x\right]}{(2n+1)^2}$$

b) We shall later prove a theorem (Theorem 10.6.2) which implies that

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos\left[(2n+1)x\right]}{(2n+1)^2}$$

for all $x \in [-\pi, \pi]$. Use this to find the sum of the series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \frac{1}{(2n+1)^2} + \cdots$.

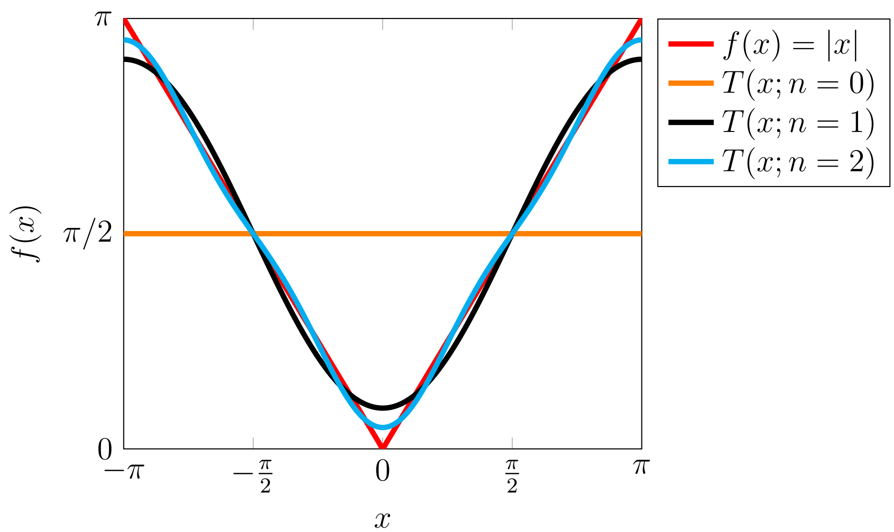
Evaluting the taylor series of $f(x) = |x|$ at $x = 0$

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

Rearranging

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

c) Make plots of the finite approximnations for $N = 0, 1, 2$ and compare them to f.



Problem 2

Assume that $\{\mathbf{e_n}\}_{\mathbf{n}\in\mathbb{N}}$ is an orthonormal set in an inner product space $(V, \langle\cdot,\cdot\rangle)$.
a) Show that if $n \neq m$, then $||\mathbf{e_n} - \mathbf{e_m}|| = \sqrt{\mathbf{2}}$.

We have the inner product given as

$$||\mathbf{e_n} - \mathbf{e_m}|| = \sqrt{\langle \mathbf{e_n} - \mathbf{e_m}, \mathbf{e_n} - \mathbf{e_m} \rangle}$$

Using the distributive property of $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$:

$$\begin{aligned} &= \sqrt{\langle \mathbf{e_n}, \mathbf{e_n} \rangle - \mathbf{2}\langle \mathbf{e_n}, \mathbf{e_m} \rangle + \langle \mathbf{e_m}, \mathbf{e_m} \rangle} \\ &= \sqrt{1^2 - 2 \cdot 0 + 1^2} = \sqrt{2} \end{aligned}$$

where $\langle \mathbf{e_n}, \mathbf{e_m} \rangle = \mathbf{0}$ for $m \neq n$.

b) Let

$$S = \{\mathbf{v} \in \mathbf{V} : ||\mathbf{v} = \mathbf{1}\}$$

be the unit sphere in V. Show that S is not compact.

In the unit sphere of V we have $\mathbf{e_i} \in \mathbf{S}$ for $i = 0, 1, 2, ...$ Constructing a subsquence of the unit sphere $S_k = \{e_{n_k}\}_{k=0}^{N} \subset \{e_n\}_{n=0}^{N}$, we can check for convergence by

$$||S_n - S|| < \epsilon \quad \text{for } n > N$$

However from a) we have that

$$||e_n - e_m|| = \sqrt{2} \quad \text{for } n \neq m$$

And so there is no convergent subsequence of S. Therefore S is not compact.

Problem 3

Recall the Gram-Schmidt process from linear algebra: If $\{\mathbf{v_n}\}_{\mathbf{n=0}}^{\infty}$ is a linearly independent sequence in an inner product space $(V, \langle\cdot,\cdot\rangle)$, define a new sequence $\{\mathbf{u}\}_{\mathbf{n}\in\mathbb{N}}$ by

$$\begin{aligned}
\mathbf{u}_0 &= \mathbf{v}_0 \\
\mathbf{u}_1 &= \mathbf{v}_1 - \frac{\langle \mathbf{v}_1, \mathbf{u}_0\rangle}{||\mathbf{u}_0||^2}\mathbf{u}_0 \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\mathbf{u}_n &= \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{u}_0\rangle}{||\mathbf{u}_0||^2}\mathbf{u}_0 - \frac{\langle \mathbf{v}_n, \mathbf{u}_1\rangle}{||\mathbf{u}_1||^2}\mathbf{u}_1 - \cdots \frac{\langle \mathbf{v}_n, \mathbf{u}_{n-1}\rangle}{||\mathbf{u}_{n-1}||^2}\mathbf{u}_{n-1}
\end{aligned}$$

Then the new sequence is orthogonal (i.e. $\langle \mathbf{u_i}, \mathbf{u_j}\rangle = \mathbf{0}$ for $i \neq j$) and $\text{Span}(\mathbf{u_0}, \mathbf{u_1}, ..., \mathbf{u_n}) = \text{Span}(\mathbf{v_0}, \mathbf{v_1}, ..., \mathbf{v_n})$ for all n. We get an orthonormal sequence $\{\mathbf{e_n}\}$ by putting $\mathbf{e_n} = \frac{\mathbf{u_n}}{||\mathbf{u_n}||}$.

a) Let $V = C([0,1], \mathbb{R})$ and define an inner product on V by

$$\langle u, v \rangle = \int_0^1 u(x)v(x)dx$$

Assume that we perform the Gram-Schmidt process on the polynomials $v_0(x) = 1, v_1(x) = x, v_2(x) = x^2, ..., v_n(x) = x^n, ...$ and get an orthonormal sequence $e_o(x), e_1(x), e_2(x), ..., e_n(x), ...$ Find $e_0(x)$ and $e_1(x)$.

To find $e_0(x), e_1(x)$ we have

$$\begin{aligned} u_0(x) &= v_0(x) = 1 \\ u_1(x) &= v_1(x) - \frac{\langle v_0(x), v_1(x)\rangle}{\langle v_0(x), v_0(x)\rangle} v_0(x) \end{aligned}$$

Calculating the inner products

$$\begin{aligned} \langle v_0(x), v_1(x)\rangle &= \int_0^1 1 \cdot x \; dx = \frac{1}{2} \\ \langle v_0(x), v_0(x)\rangle &= \int_0^1 1 \cdot 1 \; dx = 1 \end{aligned}$$

Normalizing orthogonal vectors by their norms:

$$\begin{aligned} ||u_1(x)||^2 &= \langle u_1(x), u_1(x)\rangle \\ &= \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \int_{-1/2}^{1/2} u^2 du \\ &= \frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right] = \frac{1}{12} \end{aligned}$$

And then we get the new, orthogonal vectors

$$\begin{aligned} e_0(x) &= 1 \\ e_1(x) &= \frac{x}{2\sqrt{3}} - \frac{1}{4\sqrt{3}} \end{aligned}$$

b) Let $h \in V$ and assume that $\langle h, e_n \rangle = 0$ for $n = 0, 1, 2, ...$ Show that $\langle h, p \rangle = 0$ for all polynomials p.

Rewrite the polynomial p to a linear combination

$$p = p_0 e_0(x) + p_1 e_1(x) + ... + p_n e_n(x)$$

Taking the inner product with h:

$$\begin{aligned}
\langle h, p \rangle &= \langle h, p_0 e_0(x) + p_1 e_1(x) + ... + p_n e_n(x) \rangle \\
&= \langle h, p_0 e_0(x) \rangle + \langle h, p_1 e_1(x) \rangle + ... + \langle h, p_n e_n(x) \rangle \\
&= p_0 \langle h, e_0(x) \rangle + p_1 \langle h, e_1(x) \rangle + ... + p_n \langle h, e_n(x) \rangle \\
&= p_0 \cdot 0 + p_1 \cdot 0 + ... + p_n \cdot 0 = 0
\end{aligned}$$

With the assumption $\langle h, e_n \rangle = 0,\ \forall n \in \mathbb{N}$.

c) Show that $\langle h, h \rangle = 0$, and conclude that $h = 0$.

Writing $h \in V$ into a linear combination of its orthonormal vectors $e_n(x)$ we get

$$h = h_0 e_0(x) + h_1 e_1(x) + ... h_n e_n(x)$$

Taking the inner product of h on itself

$$\begin{aligned}
\langle h, h \rangle &= \langle h, h_0 e_0(x) + h_1 e_1(x) + ... h_n e_n(x) \rangle \\
&= \langle h, h_0 e_0(x) \rangle + \langle h, h_1 e_1(x) \rangle + ... + \langle h, h_n e_n(x) \rangle \\
&= h_0 \langle h, e_0(x) \rangle + \langle h, e_1(x) \rangle + ... + \langle h, e_n(x) \rangle \\
&= h_0 \cdot 0 + h_1 \cdot 0 + ... + h_n \cdot 0 = 0
\end{aligned}$$

As $\langle h, h \rangle = 0$ we conclude that $h = 0$ from definition 5.3.1 (iv) of inner products.

d) Let $f \in V$ and put $\alpha_n = \langle f, e_n \rangle$. Assume that $g(x) = \sum_{n=0}^{\infty} \alpha_n e_n(x)$ is continous (the sum here is with respect to the norm $||\cdot||$ generated by $\langle \cdot, \cdot \rangle$, hence $\lim_{N\to\infty} ||g(x) - \sum_{n=0}^{N} \alpha_n e_n(x)|| = 0$). Show that $g = f$.

We have $f \in V$ which can be written as a linear combination as

$$f = \sum_{n=0}^{\infty} f_n e_n = f_0 e_0 + f_1 e_1 + ... + f_n e_n + ...$$

and we have defined α_n as $\alpha_n = \langle f, e_n \rangle$. Calculating the inner product

$$\begin{aligned} \alpha_n = \langle f, e_n \rangle &= \langle f_0 e_0 + f_1 e_1 + ... + f_n e_n + ... \ , \ e_n \rangle \\ &= f_n \end{aligned}$$

Since we have

$$g(x) = \sum_{n=0}^{\infty} \alpha_n e_n(x) = \sum_{n=0}^{\infty} f_n e_n(x) = f(x)$$

We therefore conclude that $g = f$.