Real Analaysis Notes

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De Morgan's Laws

These laws state that the complement of

$$\begin{aligned} &\text{(i)} \quad (A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap ...A_n^c \\ &\text{(ii)} \quad (A_1 \cap A_2 \cap ... \cap A_n)^c = A_1^c \cup A_2^c \cup ... \cup ...A_n^c \end{aligned}$$

Families of sets

A collection of sets, think set of sets, is usually called a family. An example is the family

$$\mathbb{A} = \{[a, b] | a, b \in \mathbb{R}\}$$

of all closed and bounded intervals on the real line.

Functions

If **A** is a subset of X, the set $f(A) \subset Y$ defined by

$$f(A) = \{f(a)|a \in A\}$$

is called the *images of* A *under* f.

If B is a subset of Y, the set $f^{-1}(B) \subset X$ is defined by

$$f^{-1}(B) = \{x|x \in B\}$$

Note that the inverse function only is defined when the funciton is bijective. However the inverse images $f^{-1}(B)$ that is studied above are defined for all functions f.

Relations and partitions

Relations are an abstract way of relating something to each other. Tangible examples of this can be the difference in magnitude (denoted by less than or greater than signs), angle(s) between vectors, similiar matricies and properties thereof. Below is an abstract defintions of such relations.

Defintion: By a relation on a set X, we mean a subset R of the carteisan product $X \times X$. We usually write xRy instead of $(x, y) \in \mathbb{R}$ to denote that x and y are related. The symbols $\sim$ and $\equiv$ [1] are often used to denote relations, and we then write $x \sim y$ and $x \equiv y$.

Example: Equality (denoted by the symbol =) and less than (<) are relations on $\mathbb{R}$. To see that they fit into the formal definition above, note that they can be defines as

$$R = \{(x, y) \in \mathbb{R}^2 | x = y\}$$
$$S = \{(x, y) \in \mathbb{R}^2 | x < y\}$$

Partion is a division of sets into nonoverlapping pieces. More preciesly, if X is a set, a partition $\mathbb{P}$ of X is a family of nonempty subsets of X such

[1]the LaTeX symbol for these signs are "sim" and "equiv"

that each element in x belongs to exactly one set $P \in \mathbb{P}$. These sets $P \in \mathbb{P}$ are called parition classes of $\mathbb{P}$.

Example: Given a parition of X, we may introduce a relation $\sim$ on X by

$$x \sim y \Leftrightarrow x \text{ and } y \text{ belong to the same set } P \in \mathbb{P}..$$

Equivalence relations - Used to partion sets into subsets.

Defintion: An equvialnce relation on X is a relation $\sim$ satisfying the follonw conditions:

- Reflexivity: $x \sim x$ for all $x \in X$.
- Symmetry: If $x \sim y$, then $y \sim x$.
- Transistivity: If $x \sim y$ and $y \sim z$ then $x \sim z$.

Countability

A set A is called countable if it is possible to make a list $a_1, a_2, ..., a_n, ...$ which contains all elements of A. If this isn't possible, the set is called countable. The infinte countable sets are the smalles infintes sets, and such this is a way to give a "magnitude" to infinity. Note that $\mathbb{R}$ is too large to be uncountable. Finites sets are obivously countable and so are the subsets of finite sets.
The set of natural numbers $\mathbb{N}$ is also countable, listed as $1, 2, 3, ...$, and such is the set of integers $\mathbb{Z}$ aswell, however this is less obvious.

Propostiion If the sets A, B are countable, so is the cartesian product $A \times B$.

Completeness

Some key defintions for central terms are

- **Bounded** - This just means there is a limit how big (or how small, if one is looking at bounded below) a number in a sequence can get.
- **Supremum** - This is the smallest possible upper bound.There are many bounds one can set, but this is the closest one can get to the upper bound.

- **Infinitum(?)** - This is the largest possible lower bound. There are many bounds one can set, but this is the closest one can get to the lower bound
- **Bounded – above**- A is bounded above if there is a number $b \in \mathbf{R}$ such that $b \geq a$ for all $a \in A$.
- **Bounded – above** -
- **Bounded – below** - A is bounded below if there is a number $c \in \mathbf{R}$ such that $c \leq a$ for all $a \in A$.
- **Bounded – above** -

Cauchy Sequences

Definition: A sequence $\{x_n\}$ of poitns in $\mathbb{R}^d$ is called a Cauche-sequence if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ (or viewed as $N(\epsilon)$, such that for all $n, m \geq M$we have $||\vec{x}_n - \vec{x}_m|| < \epsilon$.

Lemma: All convergent sequences are Cauchy sequencess.

Proof: Since $\{\vec{x}_n\}$ converges to a point $\vec{x}$, these are for any $\epsilon > 0$ and $N \in \mathbb{N}$ such that if $n \geq N$, then $||\vec{x}_n - \vec{m}|| < \frac{\epsilon}{2}$. Hence if $n, m \geq N$, then

$$||\vec{x}_n - \vec{x}_m|| = ||(\vec{x}_n - \vec{x}_m) + (\vec{x} - \vec{x}_m)|| \leq ||(\vec{x}_n - \vec{x})|| + ||(\vec{x} - \vec{x}_m)|| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Intermediate Value Theorem

The Bolzano-Weierstrass Theorem

A Preview of Compactness

Given a sequence $\{\vec{x_m}\}$ is in $\mathbb{R}^d$.

$$\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_5, \vec{x}_6, \vec{x}_7, \vec{x}_8, \vec{x}_9, ...$$

Subsequence: $\vec{x}_2, \vec{x}_5, \vec{x}_7, \vec{x}_9, ...$

Formally: Let $n_1 < n_2 < n_3 < ....$ be a strictly increasing sequences of natural numbers. The sequence

$$\vec{x}_{n_1}, \vec{x}_{n_2}, \vec{x}_{n_3}...$$

is called a subsequence of $\{\vec{x}_n\}$.

Bolzano-Weierstrass Theorem: All bounded sequences in $\mathbb{R}^d$ have convergent subsequences. Proof for $d = 2$ (2-dimensional case).

The Extreme Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continous defined on a closed and bounded interval, then f has maximum and minimum points on $[a, b]$.

graph here

Proof: Let

$$A = \{f(x) : x \in [a, b]\}$$

and put $M = \sup A$ (with $M = \infty$ if A is upward bounded).

Pick a sequence $\{x_n\}$ from $[a, b)$ such that $f(x_n) \to M$.

By Bolzano-Weierstrass Theorem, $\{x_n\}$ has a convergent subsuequence $\{x_{n_k}\}$, i.e. $C = \lim x_{u_k}$.

Since $[a, b]$ is closed $c \in [a, b]$. Since f is continous $x_{u_k} \to C$ implies $f(x_{u_k}) \to f(c)$.

But we also $f(x_{u_k}) \to M$.

$$f(x_n) \to M$$

The Mean Value Theorem

Assume that $f : [a, b] \to \mathbf{R}$ is continous in all of $[a, b]$ and differentiable at all inner points $x \in (a, b)$. Then there is a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Include graph for clarity's sake.