31st March, 2022

MAT2400

Mandatory assignment 2 of 2

Submission deadline

Thursday 28th April 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course, and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. You only have one attempt at each assignment, and you need to have both assignments approved in order to take the exam. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

There is a new regime for the mandatory assignments this year. In the new regime you only have one attempt at each assignment and not two as in earlier years. As the purpose of the new regime is to handle the assignments in a more efficient and pedagogical way and not to fail more students, we shall put more emphasis on effort *in the grading this year: As long as you have documented that you have made a serious attempt at the majority of the problems, we will pass you. The best way to document that you have tried, is, of course, to solve the problems, but you can also do it by telling us what you have tried and why it failed. We encourage you to discuss, collaborate, and help each other. Do not hesitate to contact the teachers (preferably well in advance of the deadline) if you have problems.*

If there is a problem you cannot solve, you can still use the result freely in any subsequent problem.

Problem 1. Let $f\colon [-\pi,\pi] \to \mathbb{R}$ be given by $f(x) = |x|$.

a) Show that the real Fourier series of f is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos[(2n+1)x]}{(2n+1)^2}\,.$$

b) We shall later prove a theorem (Theorem 10.6.2) which impliess that

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos[(2n+1)x]}{(2n+1)^2}$$

for all $x \in [-\pi,\pi]$. Use this to find the sum of the series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots$.

c) Make plots of the finite approximations

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{N} \frac{\cos[(2n+1)x]}{(2n+1)^2}$$

for $N = 0, 1, 2$ and compare them to f.

Problem 2. Assume that $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ is an orthonormal set in an inner product space $(V, \langle\cdot,\cdot\rangle)$.

a) Show that if $n \neq m$, then $\|\mathbf{e}_n - \mathbf{e}_m\| = \sqrt{2}$.

b) Let

$$S = \{\mathbf{v} \in V : \|\mathbf{v}\| = 1\}$$

be the unit sphere in V. Show that S is not compact.

Problem 3. Recall the Gram–Schmidt process from linear algebra: If $\{\mathbf{v}_n\}_{n=0}^{\infty}$ is a linearly independent sequence in an inner product space $(V, \langle \cdot, \cdot \rangle)$, define a new sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ by

$$\begin{aligned}
\mathbf{u}_0 &= \mathbf{v}_0 \\
\mathbf{u}_1 &= \mathbf{v}_1 - \frac{\langle \mathbf{v}_1, \mathbf{u}_0 \rangle}{\|\mathbf{u}_0\|^2}\mathbf{u}_0 \\
\vdots \quad & \quad \vdots \qquad \vdots \\
\mathbf{u}_n &= \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{u}_0 \rangle}{\|\mathbf{u}_0\|^2}\mathbf{u}_0 - \frac{\langle \mathbf{v}_n, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{u}_{n-1} \rangle}{\|\mathbf{u}_{n-1}\|^2}\mathbf{u}_{n-1} \\
\vdots \quad & \quad \vdots \qquad \vdots
\end{aligned}$$

Then the new sequence is orthogonal (i.e. $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for $i \neq j$) and $\mathrm{Span}(\mathbf{u}_0, \mathbf{u}_1, \dots \mathbf{u}_n) = \mathrm{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots \mathbf{v}_n)$ for all n. We get an orthonormal sequence $\{\mathbf{e}_n\}$ by putting $\mathbf{e}_n = \frac{\mathbf{u}_n}{\|\mathbf{u}_n\|}$.

a) Let $V = C([0,1], \mathbb{R})$ and define an inner product on V by

$$\langle u, v \rangle = \int_0^1 u(x)v(x)\,dx\,.$$

Assume that we perform the Gram–Schmidt process on the polynomials $v_0(x) = 1, v_1(x) = x, v_2(x) = x^2, \dots, v_n(x) = x^n, \dots$ and get an orthonormal sequence $e_0(x), e_1(x), e_2(x), \dots, e_n(x), \dots$ Find $e_0(x)$ and $e_1(x)$.

b) Let $h \in V$ and assume that $\langle h, e_n \rangle = 0$ for $n = 0, 1, 2, \dots$ Show that $\langle h, p \rangle = 0$ for all polynomials p.

c) Show that $\langle h, h \rangle = 0$, and conclude that $h = 0$.

d) Let $f \in V$ and put $\alpha_n = \langle f, e_n \rangle$. Assume that $g(x) = \sum_{n=0}^{\infty} \alpha_n e_n(x)$ is continuous (the sum here is with respect to the norm $\|\cdot\|$ generated by $\langle \cdot, \cdot \rangle$, hence $\lim_{N\to\infty} \|g(x) - \sum_{n=0}^{N} \alpha_n e_n(x)\| = 0$). Show that $g = f$.

Good Luck!