

# Mandatory Assignment 2 of 2

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## Problem 1

Let  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  be given by  $f(x) = |x|$ .

a) Show that the real Fourier series of  $f$  is

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos[(2n+1)x]}{(2n+1)^2} \quad (1)$$

As  $f(x) = f(-x)$  (even), we have  $b_n = 0$ . For we  $a_0$  calculate the average of the function on the interval  $[-\pi, \pi]$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{\pi}{2}$$

For the  $a_n$  terms we calculate

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx \\ a_n &= \frac{2}{\pi} \left[ \frac{\pi \sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2} - \frac{1}{n^2} \right] \\ a_n &= \frac{2}{\pi} \left[ \frac{\cos(n\pi)}{n^2} - \frac{1}{n^2} \right] \quad ; \quad (\sin(n\pi) = 0, \forall n \in \mathbb{N}) \end{aligned}$$

When  $n$  is even  $\cos(n\pi) = 1$  and when  $n$  is odd  $\cos(n\pi) = -1$ . Thus

$$a_n = \frac{2}{\pi} \left[ \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

We can see that for even  $n = 2k$  we get  $a_{2k} = 0$ . For odd  $n = 2k+1$  we have

$$a_{2k+1} = \frac{2}{\pi} \left[ \frac{(-1)^{2k+1}}{(2k+1)^2} - \frac{1}{(2k+1)^2} \right] = -\frac{4}{\pi} \frac{1}{(2k+1)^2}$$

Combining

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos[(2n+1)x]}{(2n+1)^2}$$

b) We shall later prove a theorem (Theorem 10.6.2) which implies that

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos [(2n+1)x]}{(2n+1)^2}$$

for all  $x \in [-\pi, \pi]$ . Use this to find the sum of the series  $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots$ .

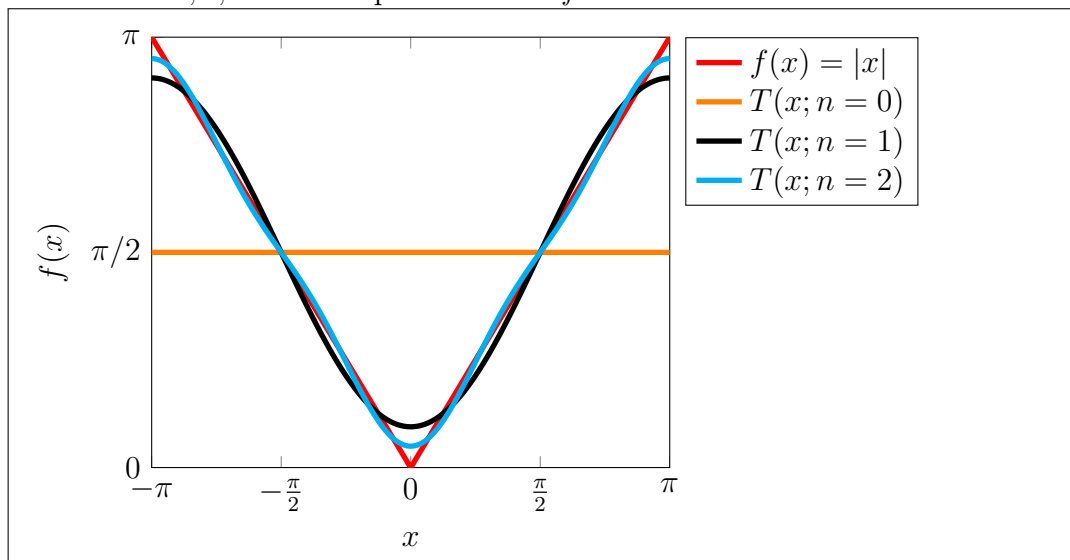
Evaluating the Taylor series of  $f(x) = |x|$  at  $x = 0$

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

Rearranging

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

c) Make plots of the finite approximations for  $N = 0, 1, 2$  and compare them to  $f$ .



## Problem 2

Assume that  $\{\mathbf{e}_n\}_{n \in \mathbb{N}}$  is an orthonormal set in an inner product space  $(V, \langle \cdot, \cdot \rangle)$ .

a) Show that if  $n \neq m$ , then  $\|\mathbf{e}_n - \mathbf{e}_m\| = \sqrt{2}$ .

We have the inner product given as

$$\|\mathbf{e}_n - \mathbf{e}_m\| = \sqrt{\langle \mathbf{e}_n - \mathbf{e}_m, \mathbf{e}_n - \mathbf{e}_m \rangle}$$

Using the distributive property of  $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$  :

$$\begin{aligned} &= \sqrt{\langle \mathbf{e}_n, \mathbf{e}_n \rangle - 2\langle \mathbf{e}_n, \mathbf{e}_m \rangle + \langle \mathbf{e}_m, \mathbf{e}_m \rangle} \\ &= \sqrt{1^2 - 2 \cdot 0 + 1^2} = \sqrt{2} \end{aligned}$$

where  $\langle \mathbf{e}_n, \mathbf{e}_m \rangle = 0$  for  $m \neq n$ .

b) Let

$$S = \{\mathbf{v} \in V : \|\mathbf{v}\| = 1\}$$

be the unit sphere in  $V$ . Show that  $S$  is not compact.

<https://wiki.math.ntnu.no/linearmethods/basicspaces/ballsandspheres>

### Problem 3

Recall the Gram-Schmidt process from linear algebra: If  $\{\mathbf{v}_n\}_{n=0}^\infty$  is a linearly independent sequence in an inner product space  $(V, \langle \cdot, \cdot \rangle)$ , define a new sequence  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  by

Then the new sequence is orthogonal (i.e.  $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$  for  $i \neq j$ ) and  $\text{Span}(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_n) = \text{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n)$  for all  $n$ . We get an orthonormal sequence  $\{\mathbf{e}_n\}$  by putting  $\mathbf{e}_n = \frac{\mathbf{u}_n}{\|\mathbf{u}_n\|}$ .

a) Let  $V = C([0, 1], \mathbb{R})$  and define an inner product on  $V$  by

$$\langle u, v \rangle = \int_0^1 u(x)v(x)dx$$

Assume that we perform the Gram-Schmidt process on the polynomials  $v_0(x) = 1, v_1(x) = x, v_2(x) = x^2, \dots, v_n(x) = x^n, \dots$  and get an orthonormal sequence  $e_0(x), e_1(x), e_2(x), \dots, e_n(x), \dots$ . Find  $e_0(x)$  and  $e_1(x)$ .

To find  $e_0(x), e_1(x)$  we have

$$\begin{aligned} e_0(x) &= v_0(x) = 1 \\ e_1(x) &= v_1(x) - \frac{\langle v_0(x), v_1(x) \rangle}{\langle v_0(x), v_0(x) \rangle} v_0(x) \end{aligned}$$

Calculating the inner products

$$\begin{aligned} \langle v_0(x), v_1(x) \rangle &= \int_0^1 1 \cdot x dx = \frac{1}{2} \\ \langle v_0(x), v_0(x) \rangle &= \int_0^1 1 \cdot 1 dx = 1 \end{aligned}$$

Dividing by the absolute value we get the orthonormal series. Absolute value of  $e_1(x)$  is

$$\begin{aligned} \|e_1(x)\|^2 &= \langle e_1(x), e_1(x) \rangle \\ &= \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} u^2 du \\ &= \frac{1}{3} \left[ \left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3 \right] \\ &= \frac{1}{12} \end{aligned}$$

And then we get the new, orthogonal vectors

$$\begin{aligned} e_0(x) &= 1 \\ e_1(x) &= \frac{x}{2\sqrt{3}} - \frac{1}{4\sqrt{3}} \end{aligned}$$

b) Let  $h \in V$  and assume that  $\langle h, e_n \rangle = 0$  for  $n = 0, 1, 2, \dots$ . Show that  $\langle h, p \rangle = 0$  for all polynomials  $p$ .

Note:  $h$  is "orthogonal" to every orthogonal vector. Therefore it must be orthogonal to every polynomial: Write this argument but in "math". <https://math.stackexchange.com/questions/400331/show-that-a-vector-that-is-orthogonal-to-every-other-vector-is-the-zero-vector>

c) Show that  $\langle h, h \rangle = 0$ , and conclude that  $h = 0$ .

d) Let  $f \in V$  and put  $\alpha_n = \langle f, e_n \rangle$ . Assume that  $g(x) = \sum_{n=0}^{\infty} \alpha_n e_n(x)$  is continuous (the sum here is with respect to the norm  $\|\cdot\|$  generated by  $\langle \cdot, \cdot \rangle$ , hence  $\lim_{N \rightarrow \infty} \|g(x) - \sum_{n=0}^N \alpha_n e_n(x)\| = 0$ ). Show that  $g = f$ .