# Mandatory Assignment 1 of 2

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# Problem 1

Let $X$ be the set of all sequences $\{x_n\}_{n\in\mathbb{N}}$ of real numbers such that $lim_{n\to\infty}x_n = 0$.

a) Use the definiton of converge to show that if $x_n \in X$, then there is a $K \in \mathbb{N}$ such that $|x_K| = \sup\{|x_n| : n \in \mathbb{N}\}$ (i.e. $x_K$ is an element of maximal absolute value).

Definition of convergence:
A sequence $\{x_n\}$ of real numbers converges to $a \in \mathbb{R}$ if for every $\epsilon > 0$ (no matter how small), there is an $N \in \mathbb{N}$ such that $|x_n - a| < \epsilon$ for all $n \leq N$. We write $lim_{n\to\infty}x_n = a$.

(Or in laymans terms: You can always find a point $x_n$ which is arbitrarily close, $\epsilon > 0$, to $a$ by going deeper and deeper into the sequence $\{x_n\}$, i.e. choosing larger and larger $N$).

b) Define $d : X \times X \to [0, \infty)$ by

$$d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}.$$

Show that d is a metric on $X$.

Definition of a metric:
A metric space $(X, d)$ consits of a nonempty set $X$ and a function $d : X \times X \to [0, \infty)$ such that:

- (Positivity) For all $x, y \in X$, we have $d(x, y) \geq 0$ with equalitiy if and only if $x = y$.
- (Symmetry) For all $x, y \in X$ we have $d(x, y) = d(y, x)$.
- (Triangle Inequality) For all $x, y, z \in X$, we have

$$d(x, y) \leq d(x, z) + d(z, y)$$

c) Let $Y$ be the set of all sequences $\{y_n\}_{n\in\mathbb{N}}$ of real numbers such that

$\sum_{n=1}^{\infty} |y_n| < \infty$. Show that $Y \subseteq X$. Find a sequence $\{x_n\}$ that belongs to $X$ but not to $Y$ (you can use everything you know from calculus).

d) Assume $\{x_n\} \in X \setminus Y$ and let $\epsilon > 0$. Show that the ball $B(\{x_n\}; \epsilon)$ contains elements from $Y$. Explain why this shows that $Y$ is not closed.

e) Assume $\{y_n\} \in Y$ and let $\epsilon > 0$. Show that $B(\{y_n\}; \epsilon)$ contains elements from $X \setminus Y$ Explain why this shows that $Y$ is not open.

# Problem 2

A metric space $(X, d$ is called disconnected if there are two non-empty, open subsets $O_1, O_2$ such that $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \emptyset$.

a) Let $X = [0, 1] \cup [2, 3]$ have the usual metric $d(x, y) = |x - y|$.
The metric space is disconnected as there are two non-empty, open subsets $O_1 = [0, 1]$ and $O_2 = [2, 3]$ which have the properties

$$O_1 \cup O_2 = [0, 1] \cup [2, 3] = X$$
$$O_1 \cap O_2 = [0, 1] \cap [2, 3] = \emptyset$$

b) Show that $\mathbb{Q}$ with the usual metric $d(x, y) = |x - y|$ is disconnected.
(Hint: Consider $O_1 = \{x \in \mathbb{Q} : x^2 > 2\}$ and $O_1 = \{x \in \mathbb{Q} : x^2 > 2\}$.)
c) Assume that $(X, d)$ is a connected (i.e. not disconnected) metric space and that $f : X \to \mathbb{R}$ is a continous function such that there are two points $a, b \in X$ with $f(a) < 0 < f(b)$. Show that there is a point $c \in X$ such that $f(c) = 0$.(This is an abstract version of the Intermediate Value Theorem.)