Real Analaysis Notes

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De Morgan's Laws

These laws state that the complement of

$$\begin{array}{rl} \text{(i)} & (A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap ... A_n^c \\ \text{(ii)} & (A_1 \cap A_2 \cap ... \cap A_n)^c = A_1^c \cup A_2^c \cup ... \cup ... A_n^c \end{array}$$

Families of sets

A collection of sets, think set of sets, is usually called a family. An example is the family

$$\mathbb{A} = \{[a,b] | a, b \in \mathbb{R}\}$$

of all closed and bounded intervals on the real line.

Functions

If $\mathbf{A}$ is a subset of X, the set $f(A) \subset Y$ defined by

$$f(A) = v\{f(a)|a \in A\}$$

is called the *images of A under f*.

If B is a subset of Y, the set $f^{-1}(B) \subset X$ is defined by

$$f^{-1}(B) = \{x|x \in B\}$$

Note that the inverse function only is defined when the funciton is bijective. However the inverse images $f^{-1}(B)$ that is studied above are defined for all functions f.

Relations and partitions

Relations are an abstract way of relating something to each other. Tangible examples of this can be the difference in magnitude (denoted by less than or greater than signs), angle(s) between vectors, similiar matricies and properties thereof. Below is an abstract defintions of such relations.

Defintion: By a relation on a set X, we mean a subset R of the carteisan product $X \times X$. We usually write xRy instead of $(x, y) \in \mathbb{R}$ to denote that x and y are related. The symbols $\sim$ and $\equiv$ [1] are often used to denote relations, and we then write $x \sim y$ and $x \equiv y$.

Example: Equality (denoted by the symbol =) and less than (<) are relations on $\mathbb{R}$. To see that they fit into the formal definition above, note that they can be defines as

$$R = \{(x, y) \in \mathbb{R}^2|x = y\}$$
$$S = \{(x, y) \in \mathbb{R}^2|x < y\}$$

Partion is a division of sets into nonoverlapping pieces. More preciesly, if X is a set, a partition $\mathbb{P}$ of X is a family of nonempty subsets of X such

[1] the LaTeX symbol for these signs are "sim" and "equiv"

that each element in x belongs to exactly one set $P \in \mathbb{P}$. These sets $P \in \mathbb{P}$ are called parition classes of $\mathbb{P}$.

Example: Given a parition of X, we may introduce a relation $\sim$ on X by

$$x \sim y \Leftrightarrow x \text{ and } y \text{ belong to the same set } P \in \mathbb{P}..$$

Equivalence relations - Used to partion sets into subsets.

Defintion: An equvialnce relation on X is a relation $\sim$ satisfying the follonw conditions:

- Reflexivity: $x \sim x$ for all $x \in X$.
- Symmetry: If $x \sim y$, then $y \sim x$.
- Transistivity: If $x \sim y$ and $y \sim z$ then $x \sim z$.

Completeness

Some key defintions for central terms are

- **Bounded − above**- A is bounded above if there is a number $b \in \mathbf{R}$ such that $b \geq a$ for all $a \in A$.
- **Bounded − above** -
- **Bounded − below** - A is bounded below if there is a number $c \in \mathbf{R}$ such that $c \leq a$ for all $a \in A$.
- **Bounded − above** -

Intermediate Value Theorem

The Bolzano-Weierstrass Theorem

The Extreme Value Theorem

The Mean Value Theorem

Assume that $f : [a, b] \to \mathbf{R}$ is continous in all of $[a, b]$ and differentiable at all inner points $x \in (a, b)$. Then there is a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Include graph for clarity's sake.