

Mandatory Assignment 1 of 2

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Problem 1

Let X be the set of all sequences $\{x_n\}_{n \in \mathbb{N}}$ of real numbers such that $\lim_{n \rightarrow \infty} x_n = 0$.

a) Use the definition of convergence to show that if $\{x_n\} \in X$, then there is a $K \in \mathbb{N}$ such that $|x_K| = \sup\{|x_n| : n \in \mathbb{N}\}$ (i.e. x_K is an element of maximal absolute value).

As all sequences $\{x_n\}_{n \in \mathbb{N}}$ in X converges to 0, $\lim_{n \rightarrow \infty} x_n = 0$, the definition of convergence states:

For every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$|x_n - 0| = |x_n| < \epsilon$$

As the series converges, one can choose an $\epsilon > 0$ such that there exists an $N \in \mathbb{N}$ where $|x_n|$ is no longer increasing. Therefore there must exist a $K \in \mathbb{N}$ such that $K \leq N$ where

$$|x_K| = \sup\{|x_n| : n \in \mathbb{N}\}$$

Note: This is a very wordy argument and I am not quite sure how to phrase it more rigorously.

b) Define $d : X \times X \rightarrow [0, \infty)$ by

$$d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}.$$

Show that d is a metric on X .

For (X, d) to be a metric space, it needs to satisfy the properties of postivity, symmetry and the triangle inequality:

- (Positivity) For all $x, y \in X$, we have $d(x, y) \geq 0$ with equality if and only if $x = y$.
- (Symmetry) For all $x, y \in X$ we have $d(x, y) = d(y, x)$.
- (Triangle Inequality) For all $x, y, z \in X$, we have

$$d(x, y) \leq d(x, z) + d(z, y)$$

The first two properties are fairly obvious as $|x_n - y_n| \geq 0$ (postivity) and $|x_n - y_n| = |y_n - x_n|$ (symmetry). To prove the triangle inequality, first assume there exists $\{x_n\}, \{y_n\}, \{z_n\} \in X$. Looking at the argument for the supreme function we have the triangle inequality

$$|x_n - y_n| \leq |x_n - z_n| + |y_n - z_n|$$

further evaluating the supremum

$$\begin{aligned} |x_n - z_n| + |y_n - z_n| &\leq \sup\{|x_n - z_n| + |y_n - z_n| : n \in \mathbb{N}\} \\ &\leq \sup\{|x_n - z_n| : n \in \mathbb{N}\} + \sup\{|z_n - y_n| : n \in \mathbb{N}\} \\ &= d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\}) \end{aligned}$$

where the fact that $\sup\{A + B\} = \sup\{A\} + \sup\{B\}$ is used.

This results in $d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\})$ being an upper bound ¹ for $|x_n - y_n|$ and we therefore get

$$d(\{x_n\}, \{y_n\}) \leq d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\})$$

c) Let Y be the set of all sequences $\{y_n\}_{n \in \mathbb{N}}$ of real numbers such that $\sum_{n=1}^{\infty} |y_n| < \infty$. Show that $Y \subseteq X$. Find a sequence $\{x_n\}$ that belongs to X but not to Y (you can use everything you know from calculus).

We have

$$X = \{\{x_n\} \mid \lim_{n \rightarrow \infty} x_n = 0, n \in \mathbb{N}\}$$

$$Y = \{\{y_n\} \mid \sum_{n=0}^{\infty} |y_n| < \infty, n \in \mathbb{N}\}$$

Define $S_N = \sum_{n=0}^N |y_n|$, which converges as it is smaller than infinity. We can then write

$$\lim_{n \rightarrow \infty} (S_N - S_{N-1}) = L - L = 0$$

but

$$\begin{aligned} S_N - S_{N-1} &= (|y_0| + |y_1| + \cdots + |y_N|) - (|y_0| + |y_1| + \cdots + |y_{N-1}|) \\ &= |y_N| \end{aligned}$$

So we have that $|y_N| = 0$ and therefore all sequences in Y converges to 0 and therefore $Y \subseteq X$.

A sequence $\{x_n\}$ that belongs to X but not to Y is the harmonic series, $\frac{1}{n}$. This series converges to zero as $\lim_{n \rightarrow \infty} 1/n = 0$, but the sum of the absolute value of the sequence is not smaller than infinity.

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \right| \not< \infty$$

d) Assume $\{x_n\} \in X \setminus Y$ and let $\epsilon > 0$. Show that the ball $B(\{x_n\}; \epsilon)$ contains elements from Y . Explain why this shows that Y is not closed.

Assuming $\{x_n\} \in X \setminus Y$ means we are looking at the sets $\{x_n\}$ which have the following properties

$$\lim_{n \rightarrow \infty} \{x_n\} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |x_n| = \infty$$

We need to prove that the ball

$$B(\{x_n\}; \epsilon) = \{\{z_n\} \in X \setminus Y : d(\{z_n\}, \{x_n\}) < \epsilon\}$$

contains elements from Y : i.e. show that there exists a $\{z_n\}$ such that

$$|z_n - x_n| < \epsilon$$

for $\sum_{n=1}^{\infty} |z_n| < \infty$. We know that $\sum_{n=1}^{\infty} |z_n|$ converges if and only if $\sum_{n=N}^{\infty} |z_n|$ converges for all $N \in \mathbb{N}$. Considering

$$\{\tilde{z}_n\} = \{z_1, z_2, z_3, \dots, z_n, 0, 0, \dots, 0, \dots\}$$

which also converges as $\sum_{n=N}^{\infty} |z_n|$ converges. Incorporating this we get

$$|\tilde{z}_n - x_n| \leq |z_n - x_n| < \epsilon$$

Supposing N is where the entries in the sequence $\{\tilde{z}_n\}$ become zero, choosing an $n > N$ we get

$$\begin{aligned} |\tilde{z}_n - x_n| &= |0 - x_n| \leq |z_n - x_n| < \epsilon \\ |x_n| &< \epsilon \end{aligned}$$

Which is true as $\lim_{n \rightarrow \infty} |x_n| = 0$ converges and thus we can always find an N far enough out such that this holds.

Since we have chosen the open ball, this means that Y is not closed as [...].

e) Assume $\{y_n\} \in Y$ and let $\epsilon > 0$. Show that $B(\{y_n\}; \epsilon)$ contains elements from $X \setminus Y$. Explain why this shows that Y is not open.

Assuming $\{y_n\} \in Y$ means we are looking at the sets $\{y_n\}$ which have the following properties

$$\lim_{n \rightarrow \infty} y_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n| < \infty$$

We need to prove that the ball

$$B(\{y_n\}; \epsilon) = \{\{z_n\} \in Y : d(\{z_n\}, \{y_n\}) < \epsilon\}$$

contains elements from $X \setminus Y$: i.e. show that there exists a $\{z_n\}$ such that

$$|z_n - y_n| < \epsilon$$

for $\sum_{n=1}^{\infty} |y_n| < \infty$ and $\sum_{n=1}^{\infty} |z_n| = \infty$.

Following from the previous argument, consider a

$$\{\tilde{y}_n\} = \{y_1, y_2, y_3, \dots, y_n, 0, 0, \dots, 0, \dots\}$$

which also converges as $\sum_{n=N}^{\infty} |y_n|$ converges. By using the fact that $|z_n - y_n| = |y_n - z_n|$ we get

$$|y_n - z_n| < \epsilon$$

Supposing N is where the entries in the sequence $\{\tilde{y}_n\}$ become zero, choosing an $n > N$ we get

$$\begin{aligned} |\tilde{y}_n - z_n| &= |0 - z_n| \leq |y_n - z_n| < \epsilon \\ |z_n| &< \epsilon \end{aligned}$$

Which is true as $\lim_{n \rightarrow \infty} |z_n| = 0$ converges and thus we can always find an N far enough out such that this holds.

Since we have chosen the open ball, this means that Y is not open as [...].

Problem 2

A metric space (X, d) is called disconnected if there are two non-empty, open subsets O_1, O_2 such that $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \emptyset$.

a) Let $X = [0, 1] \cup [2, 3]$ have the usual metric $d(x, y) = |x - y|$. Show that (X, d) is disconnected.

The metric space (X, d) is composed of two closed sets.

By carefully splitting X into $O_1 = [0, \frac{1}{2}) \cup (\frac{3}{2}, 3]$ and $O_2 = (\frac{1}{2}, 1] \cup [2, \frac{3}{2})$, the "inside" and the "outside" of X , we now have two non-empty, open subsets which satisfy the properties of a disconnected metric space.

$$\begin{aligned} O_1 \cup O_2 &= \left([0, \frac{1}{2}) \cup (\frac{3}{2}, 3] \right) \cup \left((\frac{1}{2}, 1] \cup [2, \frac{3}{2}) \right) = X \\ O_1 \cap O_2 &= \left([0, \frac{1}{2}) \cup (\frac{3}{2}, 3] \right) \cap \left((\frac{1}{2}, 1] \cup [2, \frac{3}{2}) \right) = \emptyset \end{aligned}$$

b) Show that \mathbb{Q} with the usual metric $d(x, y) = |x - y|$ is disconnected. (Hint: Consider $O_1 = \{x \in \mathbb{Q} : x^2 > 2\}$ and $O_2 = \{x \in \mathbb{Q} : x^2 < 2\}$.)

Considering $O_1 = \{x \in \mathbb{Q} : x^2 > 2\}$ and $O_2 = \{x \in \mathbb{Q} : x^2 < 2\}$, which are open, non-empty subsets of \mathbb{Q} . The metric space is disconnected as the subset are disjoint

$$\begin{aligned} O_1 \cup O_2 &= \{x \in \mathbb{Q} : x^2 > 2\} \cup \{x \in \mathbb{Q} : x^2 < 2\} = \mathbb{Q} \\ O_1 \cap O_2 &= \{x \in \mathbb{Q} : x^2 > 2\} \cap \{x \in \mathbb{Q} : x^2 < 2\} = \emptyset \end{aligned}$$

c) Assume that (X, d) is a connected (i.e. not disconnected) metric space and that $f : X \rightarrow \mathbb{R}$ is a continuous function such that there are two points $a, b \in X$ with $f(a) < 0 < f(b)$. Show that there is a point $c \in X$ such that $f(c) = 0$. (This is an abstract version of the Intermediate Value Theorem.)

To prove there exists a $c \in \mathbb{X}$ such that $f(c) = 0$, we need to prove that given (X, d) is connected we have $f(X)$ to be connected. Because if $f(X)$ is connected, the two sets $(a, c]$ and $[c, b)$ contradicts the definition of disconnected sets

$$O_1 \cap O_2 = (a, c] \cap [c, b) \neq \emptyset$$

which means there must be a $c \in X$ such that $f(c) = 0$. Under is the proof that given (X, d) is connected and $f(X)$ is continuous, then $f(X)$ is connected:

Proof by the contrapositive:

Suppose $f : X \rightarrow F(X)$ is a surjective (?), continuous function and that $f(X)$ is disconnected. Since $f(X)$ is disconnected, there must be two open subsets U_1 and U_2 in $f(X)$ such that

$$U_1 \cup U_2 = f(X)$$

$$U_1 \cap U_2 = \emptyset$$

Let $O_1 = f^{-1}(U_1)$ and $O_2 = f^{-1}(U_2)$, and as U_1, U_2 are open sets so are O_1, O_2 . For every $x \in X$, the union of the two subsets are

$$O_1 \cup O_2 = X$$

because $f(x)$ is either in U_1 or U_2 . Since $f(x)$ cannot be in both U_1 and U_2 since they are disconnected, we have that

$$O_1 \cap O_2 = \emptyset$$

Thus $f(X)$ is disconnected.

Sketch of continuous $f(X)$ when (X, d) is disconnected.

