Mandatory Assignment 1 of 2

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Spring 2022

Problem 1

Let X be the set of all sequences $\{x_n\}_{n\in\mathbb{N}}$ of real numbers such that $\lim_{n\to\infty} x_n = 0$.

a) Use the definiton of convergence to show that if $\{x_n\} \in X$, then there is a $K \in \mathbb{N}$ such that $|x_K| = \sup\{|x_n| : n \in \mathbb{N}\}$ (i.e. x_K is an element of maximal absolute value).

As all sequences $\{x_n\}_{n\in\mathbb{N}}$ in X converges to 0, $lim_{n\to\infty}x_n = 0$, the defintion of convergence states:
For every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$|x_n - 0| = |x_n| < \epsilon$$

As the series converges, one can choose an $\epsilon > 0$ such that there exists an $N \in \mathbb{N}$ where $|x_n|$ is no longer increasing. Therefore there must exists a $K \in \mathbb{N}$ such that $K \leq N$ where

$$|x_K| = \sup\{|x_n| \ : |n \in \mathbb{N}\}$$

Note: This is a very wordy argument and I am not quite sure how to phrase it more rigorously.

b) Define $d : X \times X \to [0, \infty)$ by

$$d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}.$$

Show that d is a metric on X.

For (X, d) to be a metric space, it needs to satisfy the properties of postivity, symmetry and the triangle inequality:

- (Positivity) For all $x, y \in X$, we have $d(x, y) \geq 0$ with equalitiy if and only if $x = y$.
- (Symmetry) For all $x, y \in X$ we have $d(x, y) = d(y, x)$.
- (Triangle Inequality) For all $x, y, z \in X$, we have

$$d(x, y) \leq d(x, z) + d(z, y)$$

The first two properties are fairly obvious as $|x_n - y_n| \geq 0$ (postivity) and $|x_n - y_n| = |y_n - x_n|$ (symmetry). To prove the triangle inequality, first assume there exists $\{x_n\}, \{y_n\}, \{z_n\} \in X$. Looking at the argument for the supreme function we have the triangle inequality

$$|x_n - y_n| \leq |x_n - z_n| + |y_n - z_n|$$

further evaluating the supremum

$$\begin{aligned} |x_n - z_n| + |y_n - z_n| &\leq \sup\{|x_n - z_n| + |y_n - z_n| : n \in \mathbb{N}\} \\ &\leq \sup\{|x_n - z_n| : n \in \mathbb{N}\} + \sup\{|z_n - y_n| : n \in \mathbb{N}\} \\ &= d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\}) \end{aligned}$$

where the fact that $\sup\{A + B\} = \sup\{A\} + \sup\{B\}$ is used.
This results in $d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\})$ being an upper bound [1] for $|x_n - y_n|$ and we therefore get

$$d(\{x_n\}, \{y_n\}) \leq d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\})$$

c) Let Y be the set of all sequences $\{y_n\}_{n\in\mathbb{N}}$ of real numbers such that $\sum_{n=1}^{\infty}|y_n| < \infty$. Show that $Y \subseteq X$. Find a sequence $\{x_n\}$ that belongs to X but not to Y (you can use everything you know from calculus).

We have

$$X = \{\{x_n\} \mid \lim_{n\to\infty} x_n = 0, n \in \mathbb{N}\}$$
$$Y = \{\{y_n\} \mid \sum_{n=0}^{\infty} |y_n| < \infty, n \in \mathbb{N}\}$$

Define $S_N = \sum_{n=0}^{N} |y_n|$, which converges as it is smaller than infinity. We can then write

$$\lim_{n\to\infty} (S_N - S_{N-1}) = L - L = 0$$

but

$$\begin{aligned} S_N - S_{N-1} &= (|y_0| + |y_1| + \cdots |y_N|) - (|y_0| + |y_1| + \cdots |y_{N-1}|) \\ &= |y_n| \end{aligned}$$

So we have that $|y_N| = 0$ and therefore all sequences in Y converges to 0 and therefore $Y \subseteq X$.
A sequence $\{x_n\}$ that belongs to X but not to Y is the harmonic series, $\frac{1}{n}$. This series converges to zero as $\lim_{n\to\infty} 1/n = 0$, but the sum of the absolute value of the sequence is not smaller than infinity.

$$\sum_{n=1}^{\infty} \left|\frac{1}{n}\right| \not< \infty$$

d) Assume $\{x_n\} \in X \backslash Y$ and let $\epsilon > 0$. Show that the ball $B(\{x_n\};\epsilon)$ contains elements from Y. Explain why this shows that Y is not closed.

Assuming $\{x_n\} \in X \backslash Y$ means we are looking at the sets $\{x_n\}$ which have the following properties

$$\lim_{x\to\infty} \{x_n\} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |x_n| = \infty$$

We need to prove that the ball

$$B(\{x_n\};\epsilon) = \{\{z_n\} \in X \backslash Y :\ d(\{z_n\},\{x_n\}) < \epsilon\}$$

contains elements from Y: i.e. show that there exists a $\{z_n\}$ such that $|z_n - x_n| < \epsilon$ for $\sum_{n=1}^{\infty} |z_n| < 0$. We know that $\sum_{n=1}^{\infty} |z_n|$ converges if and only if $\sum_{n=N}^{\infty} |z_n|$ converges for all $N \in \mathbb{N}$.
Considering

$$\{\widetilde{z_n}\} = \{z_1, z_2, z_3, ..., z_n, 0, 0, ..., 0, ...\}$$

which also converges as $\sum_{n=N}^{\infty} |z_n|$ converges. Incorporating this we get

$$|\widetilde{z_n} - x_n| \leq |z_n - x_n| < \epsilon$$

Supposing N is where the entries in the sequence $\{\widetilde{z_n}\}$ become zero, choosing an $n > N$ we get

$$|\widetilde{z_n} - x_n| = |0 - x_n| \leq |z_n - x_n| < \epsilon$$
$$|x_n| < \epsilon$$

Which is true as $lim_{n\to\infty}|x_n| = 0$ converges and thus we can always find an N far enough out such that this holds.

We have proved that a ball $B(\{x_n\},\epsilon)$ centered around $\{x_n\} \in X \backslash Y$ contains atleast some of its boundary points. So since Y (the complement of $X \backslash Y$) does <u>not</u> contain <u>all</u> of its boundary points, it cannot be closed.

e) Assume $\{y_n\} \in Y$ and let $\epsilon > 0$. Show that $B(\{y_n\}; \epsilon)$ contains elements from $X \backslash Y$ Explain why this shows that Y is not open.

Assuming $\{y_n\} \in Y$ means we are looking at the sets $\{y_n\}$ which have the following properties

$$\lim_{x \to \infty} \{y_n\} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n| < \infty$$

We need to prove that the ball

$$B(\{y_n\}; \epsilon) = \{\{z_n\} \in Y : \ d(\{z_n\}, \{y_n\}) < \epsilon\}$$

contains elements from $X \backslash Y$: i.e. show that there exists a $\{z_n\}$ such that $|z_n - y_n| < \epsilon$ for $\sum_{n=1} |y_n| < \infty$ and $\sum_{n=1} |z_n| = \infty$.
Following from the previous argument, consider a

$$\{\widetilde{y_n}\} = \{y_1, y_2, y_3, ..., y_n, 0, 0, ..., 0, ...\}$$

which also converges as $\sum_{n=N} |y_n|$ convergers. By using the fact that $|z_n - y_n| = |y_n - z_n|$ we get

$$|y_n - z_n| < \epsilon$$

Supposing N is where the entries in the sequence $\{\widetilde{y_n}\}$ become zero, choosing an $n > N$ we get

$$|\widetilde{y_n} - z_n| = |0 - z_n| \leq |y_n - z_n| < \epsilon$$
$$|z_n| < \epsilon$$

Which is true as $lim_{n \to \infty} |z_n| = 0$ converges and thus we can always find an N far enough out such that this holds.
We have proved that a ball $B(\{y_n\}, \epsilon)$ centered around $\{y_n\} \in Y$ contains atleast some of its boundary points. So since Y (the complement of $X \backslash Y$) does contain <u>some</u> of its boundary points, it cannot be open.

Problem 2

A metric space (X, d) is called disconnected if there are two non-empty, open subsets O_1, O_2 such that $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \emptyset$.

a) Let $X = [0, 1] \cup [2, 3]$ have the usual metric $d(x, y) = |x - y|$. Show that (X, d) is disconnected.

The metric space (X, d) is composed of two closed sets.
By carefully splitting X into $O_1 = [0, \frac{1}{2}) \cup (\frac{3}{2}, 3]$ and $O_2 = (\frac{1}{2}, 1] \cup [2, \frac{3}{2})$, the "inside" and the "outside" of X, we now have two non-empty, open subsets which satisfy the properties of a disconnected metric space.

$$O_1 \cup O_2 = \left([0, \frac{1}{2}) \cup (\frac{3}{2}, 3]\right) \cup \left((\frac{1}{2}, 1] \cup [2, \frac{3}{2})\right) = X$$
$$O_1 \cap O_2 = \left([0, \frac{1}{2}) \cup (\frac{3}{2}, 3]\right) \cap \left((\frac{1}{2}, 1] \cup [2, \frac{3}{2})\right) = \emptyset$$

b) Show that $\mathbb{Q}$ with the usual metric $d(x, y) = |x - y|$ is disconnected.
(Hint: Consider $O_1 = \{x \in \mathbb{Q} : x^2 > 2\}$ and $O_2 = \{x \in \mathbb{Q} : x^2 > 2\}$.)

Considering $O_1 = \{x \in \mathbb{Q} : x^2 > 2\}$ and $O_2 = \{x \in \mathbb{Q} : x^2 < 2\}$, which are open, non-empty subsets of $\mathbb{Q}$. The metric space is disconnected as the subset are disjointed

$$O_1 \cup O_2 = \{x \in \mathbb{Q} : x^2 > 2\} \cap \{x \in \mathbb{Q} : x^2 < 2\} = \mathbb{Q}$$
$$O_1 \cap O_2 = \{x \in \mathbb{Q} : x^2 > 2\} \cup \{x \in \mathbb{Q} : x^2 < 2\} = \emptyset$$

c) Assume that (X, d) is a connected (i.e. not disconnected) metric space and that $f : X \to \mathbb{R}$ is a continous function such that there are two points $a, b \in X$ with $f(a) < 0 < f(b)$. Show that there is a point $c \in X$ such that $f(c) = 0$.(This is an abstract version of the Intermediate Value Theorem.)

To prove there exists a $c \in \mathbb{X}$ such that $f(c) = 0$, we need to prove that given (X, d) is connected we have $f(X)$ to be connected. Because if $f(X)$ is connected, the two sets $(a, c]$ and $[c, b)$ contradicts the defintion of disconnected sets

$$O_1 \cap O_2 = (a, c] \cap [c, b) \neq \emptyset$$

which means there must be a $c \in X$ such that $f(c) = 0$. Under is the proof that given (X, d) is connected and $f(X)$ is continous, then $f(X)$ is connected:
Proof by the contrapositive:
Suppose $f : X \to F(X)$ is a surjective (?), continous function and that $f(X)$ is disconnected. Since $f(X)$ is disconnected, there must be two open subsets U_1 and U_2 in $f(X)$ such that

$$U_1 \cup U_2 = f(X)$$
$$U_1 \cap U_2 = \emptyset$$

Let $O_1 = f^{-1}(U_1)$ and $O_2 = f^{-1}(U_2)$, and as U_1, U_2 are open sets so are O_1, O_2. For every $x \in X$, the union of the two subsets are

$$O_1 \cup O_2 = X$$

because $f(x)$ is either in U_1 or U_2. Since $f(x)$ cannot be in both U_1 and U_2 since they are disconnected, we have that

$$O_1 \cap O_2 = \emptyset$$

Thus $f(X)$ is disconnected.

Sketch of continous $f(X)$ when (X, d) is disconnected.