Mandatory Assignment 1 of 2

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Problem 1

Let X be the set of all sequences $\{x_n\}_{n\in\mathbb{N}}$ of real numbers such that $lim_{n\to\infty}x_n = 0$.

a) Use the definiton of convergence to show that if $\{x_n\} \in X$, then there is a $K \in \mathbb{N}$ such that $|x_K| = \sup\{|x_n| : n \in \mathbb{N}\}$ (i.e. x_K is an element of maximal absolute value).

As all sequences $\{x_n\}_{n\in\mathbb{N}}$ in X converges to 0, $lim_{n\to\infty}x_n = 0$, we have that for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $|\{x_n\} - 0| = |\{x_n\}| < \epsilon$.

Definition of convergence:
"A sequence $\{x_n\}$ of real numbers converges to $a \in \mathbb{R}$ if for every $\epsilon > 0$ (no matter how small), there is an $N \in \mathbb{N}$ such that $|x_n - a| <$ "ϵ for all $n \leq N$. We write $lim_{n\to\infty}x_n = a$."

$$|x_K| = \sup\{|x_n| \ : |n \in \mathbb{N}\}$$

b) Define $d : X \times X \to [0, \infty)$ by

$$d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}.$$

Show that d is a metric on X.

For (X, d) to be a metric, it ness to satisfy the properties of postivity, symmetry and the triangle inequality:

- (Positivity) For all $x, y \in X$, we have $d(x, y) \geq 0$ with equalitiy if and only if $x = y$.
- (Symmetry) For all $x, y \in X$ we have $d(x, y) = d(y, x)$.
- (Triangle Inequality) For all $x, y, z \in X$, we have

$$d(x, y) \leq d(x, z) + d(z, y)$$

The first two properties are fairly obvious as $|x_n - y_n| \geq 0$ (postivity) and $|x_n - y_n| = |y_n - x_n|$ (symmetry). To prove the triangle inequality, first assume there exists $\{x_n\}, \{y_n\}, \{z_n\} \in X$. Looking at the argument for the supreme function we have the triangle inequality

$$|x_n - y_n| \leq |x_n - z_n| + |y_n - z_n|$$

further evaluating the supremum

$$\begin{aligned} |x_n - z_n| + |y_n - z_n| &\leq \sup\{|x_n - z_n| + |y_n - z_n| : n \in \mathbb{N}\} \\ &\leq \sup\{|x_n - z_n| : n \in \mathbb{N}\} + \sup\{|z_n - y_n| : n \in \mathbb{N}\} \\ &= d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\}) \end{aligned}$$

where the fact that $\sup\{A + B\} = \sup\{A\} + \sup\{B\}$ is used.
This results in $d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\})$ being an upper bound [1] for $|x_n - y_n|$ and we therefore get

$$d(\{x_n\}, \{y_n\}) \leq d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\})$$

c) Let Y be the set of all sequences $\{y_n\}_{n\in\mathbb{N}}$ of real numbers such that $\sum_{n=1}^{\infty} |y_n| < \infty$. Show that $Y \subseteq X$. Find a sequence $\{x_n\}$ that belongs to X but not to Y (you can use everything you know from calculus).

We have

$$X = \{\{x_n\} \mid \lim_{n\to\infty} x_n = 0, n \in \mathbb{N}\}$$
$$Y = \{\{y_n\} \mid \sum_{n=0}^{\infty} |y_n| < \infty, n \in \mathbb{N}\}$$

Define $S_N = \sum_{n=0}^{N} |y_n|$, which converges as it is smaller than infinity. We can then write

$$\lim_{n\to\infty} (S_N - S_{N-1}) = L - L = 0$$

but

$$\begin{aligned} S_N - S_{N-1} &= (|y_0| + |y_1| + \cdots |y_N|) - (|y_0| + |y_1| + \cdots |y_{N-1}|) \\ &= |y_n| \end{aligned}$$

So we have that $|y_N| = 0$ and therefore all sequences in Y converges to 0 and therefore $Y \subseteq X$.
A sequence $\{x_n\}$ that belongs to X but not to Y is the harmonic series, $\frac{1}{n}$. This series converges to zero as $\lim_{n\to\infty} 1/n = 0$, but the sum of the absolute value of the sequence is not smaller than infinity.

$$\sum_{n=1}^{\infty} \left|\frac{1}{n}\right| \not< \infty$$

d) Assume $\{x_n\} \in X \backslash Y$ and let $\epsilon > 0$. Show that the ball $B(\{x_n\}; \epsilon)$ contains elements from Y. Explain why this shows that Y is not closed.

If the ball $B(\{x_n\};\epsilon)$ contains elements, either has an interior- or boundary point in Y then Y is not closed. We can check this by firstly examining if there exists an exterior point of Y.

If $B(\{x_n\};\epsilon) \subset Y^c$ where $\{x_n\} \in X\backslash Y$.

- an interior point of Y is there if $r > 0$ such that $B(\{x_n\};\epsilon) \subset Y$
- an exterior point of Y is there if $r > 0$ such that $B(\{x_n\};\epsilon) \subset Y^c$
- a boundary point if of A is there if all $r > 0$ we have

$$B(\{x_n\},\epsilon) \cap A \neq \emptyset \quad and B(\{x_n\},\epsilon) \cap A^c \neq \emptyset$$

Note: The ball is defined as such:

Let a be a point in a metric space (X, d), and assume that r is a positive, real number. The (open) ball centered at a with radius r is set

$$\mathrm{B}(a;r) = \{x \in X \ : \ d(x,a) < r\}$$

And the closed ball centered at a with radius r is the set

$$\overline{\mathrm{B}}(a;r) = \{x \in X \ : \ d(x,a) \leq r\}$$

e) Assume $\{y_n\} \in Y$ and let $\epsilon > 0$. Show that $B(\{y_n\};\epsilon)$ contains elements from $X\backslash Y$ Explain why this shows that Y is not open.

$\square$

Problem 2

A metric space (X, d) is called disconnected if there are two non-empty, open subsets O_1, O_2 such that $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \emptyset$.

a) Let $X = [0, 1] \cup [2, 3]$ have the usual metric $d(x, y) = |x - y|$. Show that (X, d) is disconnected.

The metric space (X, d) is composed of two closed sets.
By carefully splitting X into $O_1 = [0, \frac{1}{2}) \cup (\frac{3}{2}, 3]$ and $O_2 = (\frac{1}{2}, 1] \cup [2, \frac{3}{2})$, the "inside" and the "outside" of X, we now have two non-empty, open subsets which satisfy the properties of a disconnected metric space.

$$O_1 \cup O_2 = \left([0, \frac{1}{2}) \cup (\frac{3}{2}, 3]\right) \cup \left((\frac{1}{2}, 1] \cup [2, \frac{3}{2})\right) = X$$
$$O_1 \cap O_2 = \left([0, \frac{1}{2}) \cup (\frac{3}{2}, 3]\right) \cap \left((\frac{1}{2}, 1] \cup [2, \frac{3}{2})\right) = \emptyset$$

b) Show that $\mathbb{Q}$ with the usual metric $d(x, y) = |x - y|$ is disconnected.
(Hint: Consider $O_1 = \{x \in \mathbb{Q} : x^2 > 2\}$ and $O_2 = \{x \in \mathbb{Q} : x^2 > 2\}$.)

Considering $O_1 = \{x \in \mathbb{Q} : x^2 > 2\}$ and $O_2 = \{x \in \mathbb{Q} : x^2 < 2\}$, which are open, non-empty subsets of $\mathbb{Q}$. The metric space is disconnected as the subset are disjointed

$$O_1 \cup O_2 = \{x \in \mathbb{Q} : x^2 > 2\} \cap \{x \in \mathbb{Q} : x^2 < 2\} = \mathbb{Q}$$
$$O_1 \cap O_2 = \{x \in \mathbb{Q} : x^2 > 2\} \cup \{x \in \mathbb{Q} : x^2 < 2\} = \emptyset$$

c) Assume that (X, d) is a connected (i.e. not disconnected) metric space and that $f : X \to \mathbb{R}$ is a continous function such that there are two points $a, b \in X$ with $f(a) < 0 < f(b)$. Show that there is a point $c \in X$ such that $f(c) = 0$.(This is an abstract version of the Intermediate Value Theorem.)

The outline of the proof should look like this:

- Since (X, d) is connected and f is continous, $f(X)$ is connected.
- Since f is a function from X to $\mathbb{R}$, $f(X) \subseteq \mathbb{R}$
- Since $f(a)$ and $f(b)$ are in $f(X)$, $f(X)$ is nonempty
- Since all nonempty, connected subsets of $\mathbb{R}$ are intervals, $f(X)$ is an interval
- Intervals are characterized by the property that any point lying between two points of an interval is also in the interval.
- Therefore, since $f(a) < 0 < f(b)$ and $f(a)$ and $f(b)$ are points of the interval $f(X)$, we conclude that $0 \in f(X)$.
- I.e.: there is a point $c \in X$ such that $f(c) = 0$.