NAME ______________ *Solutions* ____________________

ES 442 Homework #2 Solutions

(**Spring 2019** – Due February 11, 2019)

Print out homework and do work on the printed pages.

Textbook: Samuel O. Agbo & Matthew O. Sadiku, ***Principles of Modern Communication Systems***, Cambridge University Press, United Kingdom, 2017.

Problem 1 Shannon Channel Capacity Nit Rate (10 points)

(a) Claude Shannon (then at Bell Telephone Laboratories) discovered an equation that gives the highest possible channel capacity of a communication system that can be achieved in the presence of noise (white Gaussian noise to be specific). The equation is known as the Shannon channel capacity relationship, namely

$$\text{Shannon channel capacity } C = B \cdot \log_2(1+SNR) \quad \text{(in bits/second)}$$

where B is the channel bandwidth (in Hz) and SNR is signal-to-noise as a power ratio. In the equation the SNR is a numerical value in the equation; but SNR is generally stated in decibels.

Consider the old landline telephone communication system with a bandwidth B = (3,400 – 300) Hz = 3,100 Hz. The telephone system is generally assumed that to require a SNR of at least 3162 to operate without unacceptable bit errors. Calculate the maximum channel transmission capacity C as predicted by the Shannon equation. [Refer to Section 1.5 of Agbo & Sadiku, page 10.]

[**Hint:** $SNR_{dB} = 10 \log_{10}(SNR)$ when stated in dB, so be careful about mixing the two logarithmic bases in this problem. The SNR quantity in the Shannon capacity equation is numerical (not dB) and the logarithm is base 2.]

Solution: We know that the bandwidth B = 3,100 Hz, and a SNR is numerically equal to 3,162. Plugging these numbers into Shannon's channel capacity equation gives

$C = 3{,}100 \times \log_2(1 + 3{,}162) = 3{,}100 \times \log_2(3{,}163)$
$= 3{,}100 \times [\log_{10}(3{,}163)/\log_{10}(2)] = 3{,}100 \times [3.5001/0.30103]$
$= 3{,}100 \times 11.6271 = 36{,}044$ bits/sec = 36.044 kbps $\Leftarrow$

Problem 2 Data Rate for CAT-5 Twisted Pair Cable (20 points)

Background: Category 5 cable, commonly referred to as CAT-5, is a twisted pair cable for computer networks. Since 2001, the variant commonly in use is the Category 5e specification (CAT-5e). The cable standard provides performance of up to 100 MHz and is suitable for most varieties of Ethernet over twisted pair up to 1000BASE-T (Gigabit Ethernet). https://en.wikipedia.org/wiki/Category_5_cable

We would like to transmit information over a CAT-5 twisted pair cable at 500 megabits per second (Mbps).

(a) Is a signal-to-noise ratio (SNR) of 30 dB adequate to transmit 500 Mbps? [Hint: You may want to refer to page 9 in Handout #2, "Useful Mathematical Relations."]

Solution: First we convert the SNR of 30 dB to a numerical value.

$$SNR_{dB} = 10 \times \log_2(SNR),$$
$$SNR = 10^{(SNR_{dB}/10)} = 10^{(30/10)} = 10^3 = 1000 \quad \Leftarrow$$

Next, we apply the Shannon Theorem,

$$C = B \times \log_2(1 + SNR)$$
$$C = 10^2 \text{ MHz} \times \log_2(1+1000) \times \left(\frac{1 \text{ Mbps}}{1 \text{ MHz}}\right)$$
$$C = 10^2 \text{ MHz} \times \left(\frac{\log_{10}(1001)}{\log_{10}(2)}\right) \times \left(\frac{1 \text{ Mbps}}{1 \text{ MHz}}\right)$$
$$C = 10^2 \times \left(\frac{\log_{10}(1001)}{\log_{10}(2)}\right) \text{Mbps} = 10^2 \times \left(\frac{3.0004}{0.30103}\right) \text{Mbps}$$
$$C = 10^2 \times (9.9672) \text{ Mbps} = 996.72 \text{ Mbps} \quad \Leftarrow$$

Clearly, this allows for reliable transmission at 500 Mbps. $\Leftarrow$

(b) What is the SNR required to just adequately transmit at 500 Mbps? Express your answer in decibels for the SNR (that is, SNR_{dB}).

Solution: We start by writing the equation for Shannon's data rate,

$$C = B \times \log_2(1+SNR) \quad \Rightarrow \quad \frac{500}{100} = 5 = \log_2(1+SNR)$$
$$2^5 = 2^{\log_2(1+SNR)} = (1+SNR) \quad \Rightarrow \quad 32 - 1 = 31 = SNR$$
$$SNR_{dB} = 10 \times \log_{10}(SNR) = 10 \times \log_{10}(31) = 14.91 \text{ dB} \quad \Leftarrow$$

Problem 3 Nyquist Bit Rate (10 points)

Now consider the Nyquist formula which tells us the digital channel capacity as a function of the number of levels per symbol. The smallest number of levels per symbol is binary which is two levels. Other forms of coding can have more than two levels per symbol. Whereas Shannon's equation tells us the maximum data rate possible in the presence of noise, Nyquist's bit rate equation tells us the data rate *C* as a function of bandwidth *B* and the number of signal levels per symbol *M* we can achieve. Nyquist's equation is

$$\text{Nyquist channel capacity } C = 2B \times \log_2(M) \quad \text{(bits/second)}$$

where *M* is the number of signal levels per symbol. Calculate the Nyquist data rate given the same bandwidth as in Problem 1(namely, *B* = 3,100 Hz) above and assume 8 signal levels per symbol.

Solution: Substituting these values into the Nyquist equation gives

$$\text{Nyquist channel capacity } C = 2 \times 3{,}100 \times \log_2(8)$$

$$C = 6{,}200 \times \left(\frac{\log_{10}(8)}{\log_{10}(2)}\right) = 6{,}200 \times \left(\frac{0.90309}{0.30103}\right) = 6{,}200 \times 3 = 18{,}600 \text{ bps} \quad \Leftarrow$$

Problem 4 Nyquist Meets Shannon (10 points)

By combining the Nyquist channel capacity expression with the Shannon channel capacity equation, derive a relationship between the number of levels per symbol with the required signal-to-noise ratio needed to support each other with respect to bit rate.

Solution: Equating the two channel capacity equations we obtain,

$$2B \times \log_2(M) = B \times \log_2(1 + SNR)$$

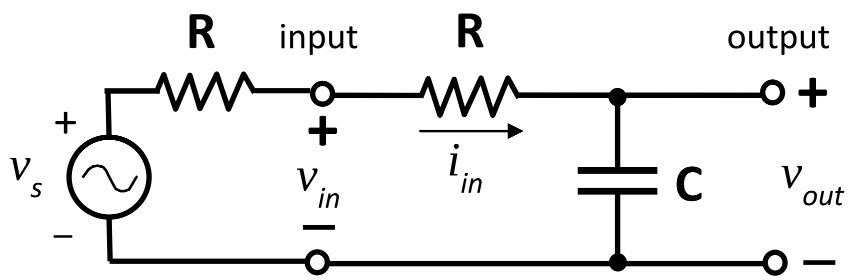
$$2B \times \log_2(M) = B \times \log_2(M^2) = B \times \log_2(1 + SNR)$$

$$2^{\log_2(M^2)} = 2^{\log_2(1+SNR)}$$

$$M^2 = 1 + SNR \quad or \quad M = \sqrt{1 + SNR} \quad \Leftarrow$$

Problem 5 RC Low-Pass Filter Problem (20 points)

Filters are a very important part of communication systems. Before we discuss filters in more detail in class, this problem is assigned to start thinking about filters; we select a very simple filter, namely, the RC low-pass filter. This is illustrated in the figure below:



Assume the two resistors (labeled **R**) are 1,000 ohms each.

(a) Write an expression for the input impedance $v_{out}(t)/i_{in}(t)$. Assume a sinusoidal steady-state situation with the source voltage generator being equal to $A_c \cdot \cos(\omega t)$.

Solution: The input impedance is found to be

$$v_{in}(\omega) = i_{in}(\omega)R + i_{in}(\omega)\frac{1}{j\omega C}$$

$$Z_{in}(\omega) = \frac{v_{in}(\omega)}{i_{in}(\omega)} = R + \frac{1}{j\omega C} \quad \Leftarrow$$

(b) What is the transfer function H(f) of this filter? Remember the transfer function is the ratio of If the -3 dB cutoff frequency of this low-pass filter is 3,400 Hz. What value of capacitance **C** gives this cutoff frequency?

Solution:

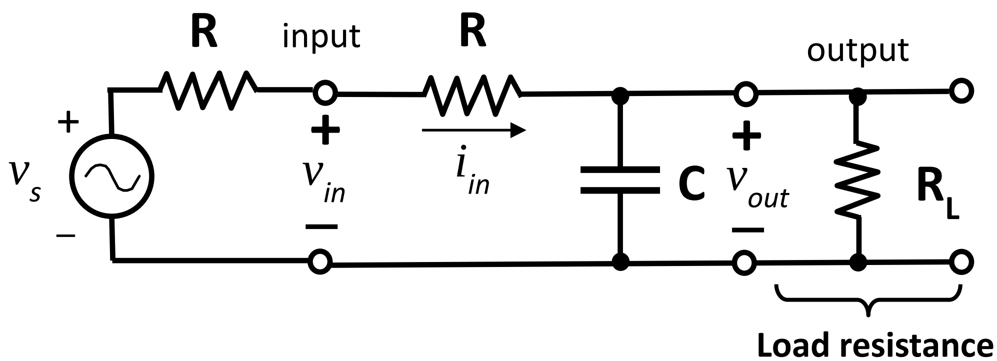
$$H(f) = \frac{(1/\,j2\pi fC)}{R + (1/\,j2\pi fC)} = \frac{1}{1 + j(2\pi fRC)} \quad \Leftarrow$$

and $f_{-3dB} = \dfrac{1}{2\pi RC}$ is the -3 dB bandwidth.

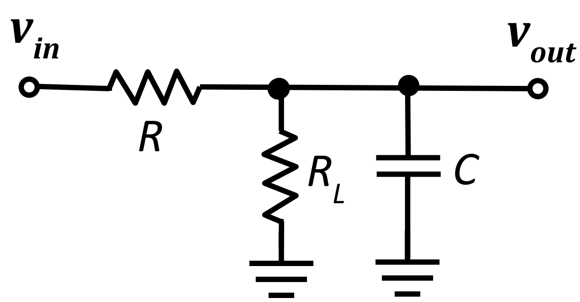
Hence, $C = \dfrac{1}{2\pi f_{-3dB} R} = \dfrac{1}{2\pi(3400)(1000)} = 4.68 \times 10^{-8}$ F $\Leftarrow$

or $C = 46.8$ nF

(c) When a filter is used in a communication system it is loaded with a load resistor **R_L** and such loading on the filter's output is important to account for. Suppose you use the filter you considered in sections (a) and (b) above, but when you measure its transfer function (and the cutoff frequency), you connect its output to an instrument with input resistance **R_L**. If **R_L** is 100 ohms, what is the approximate cutoff frequency you should measure? Use the capacitance value you found in section (b) above.



Solution: The problem with resistive loading of the simple RC low-pass filter, such as in cascading multiple *RC* filter sections, is that the added resistance modifies the cutoff frequency of the whole cascaded filter. That is, the *RC* product changes so that the -3 dB cutoff frequency is no longer just ($1/2\pi RC$). We can estimate the modified *RC* product by taking the circuit below where the capacitor C sees the parallel combination of resistors R and R_L.



Using the values of R = 1000 ohms, R_L = 100 ohms and C = 4.68 x 10-8 F, we approximate the -3 dB cutoff frequency from

$$f_{-3dB} \cong \frac{R+R_L}{2\pi CRR_L} = \frac{1{,}100}{2\pi(4.68\times10^{-8})(1{,}000)(100)}$$

$$f_{-3dB} \cong 37{,}400 \text{ Hz} \quad \Leftarrow$$

Problem 6 Free-Space Radio Propagation (30 points)

This problem involves free-space radio wave propagation using antennas between transmitter and receiver assuming line of sight transmission. In EE442 we don't study antennas analytically, but rather we will use a few simple results from antenna theory.

Path loss *PL*(dB) is the attenuation of a signal traveling from transmitter to a receiver along a free path. Let power P_t be the power emitted by the transmitter's antenna and power P_r be the signal power received by the receiver's antenna. Then the path loss is

$$PL(\text{dB}) = 10 \cdot \log_{10}\left(\frac{P_t}{P_r}\right) = -10 \cdot \log_{10}\left(\frac{\lambda^2}{(4\pi)^2 d^2}\right) \quad \text{(note the minus sign)}$$

where λ = wavelength of signal and d = distance between the transmittter and the receiver antennas.

This equation comes from the Friis free-space model and we have assumed the antennas gain are both unity (this is not important unless you know antenna theory). Your cell phone transmits about ½ watt of power when it is communicating with the base station in your cellular network. Assume the frequency of transmission in the uplink band to be 915 MHz.

(a) What is the transmit power of the cell phone in dBm?

Solution:

$$P_t[\text{dBm}] = 10 \cdot \log_{10}\left(\frac{P_t[\text{mW}]}{1\text{ mW}}\right) = 10 \cdot \log_{10}\left(\frac{500\text{ mW}}{1\text{mW}}\right) = 27\text{ dBm} \quad \Leftarrow$$

(b) Suppose the distance d between your cell phone and the base station is one kilometer (1 km = 1000 m = 3280 feet). What is the received power in dBm?

Solution:

First we find the wavelength, $\lambda = \dfrac{c}{f} = \dfrac{3 \times 10^8 \text{ m/sec}}{9.15 \times 10^8 \text{ 1/sec}} = 0.328\text{ m}.$

From the Friis equation for free-space propagation we can write

$$10 \cdot \log_{10}\left(\frac{P_t}{P_r}\right) = -10 \cdot \log_{10}\left(\frac{P_r}{P_t}\right); \quad \textit{Thus,} \ \frac{P_r}{P_t} = \frac{\lambda^2}{(4\pi)^2 d^2} \ \text{ so } \ P_r = \frac{P_t \lambda^2}{(4\pi)^2 d^2}$$

$$P_r = \frac{(500\text{ mW})(0.328\text{ m})^2}{(4\pi)^2(1000\text{ m})^2} = \frac{500 \times 0.1075}{(12.57)^2 \times 10^6}\text{ mW} = 3.404 \times 10^{-7}\text{ mW} \quad \Leftarrow$$

In dBm we get $P_r[\text{dBm}] = 10 \cdot \log_{10}\left(\dfrac{3.404 \times 10^{-7}\text{ mW}}{1\text{ mW}}\right) = -64.7\text{ dBm} \quad \Leftarrow$

(c) Suppose the weakest signal the base station can still pick up is -84 dBm. What is the greatest distance d that your cell phone can be from the base station and still function?

Solution:

$$-84\,\text{dBm} = 10 \cdot \log_{10}\left(\frac{P_r\,[\text{mW}]}{1\,\text{mW}}\right) \quad \Rightarrow \quad 10^{-8.1} = 10^{Log(P_r)}$$

$$P_r = 3.9811 \times 10^{-9}\ \text{mW} = \frac{500 \times 0.1075}{(12.57)^2 d^2}\,\text{mW}; \quad \text{Solve for } d$$

$$d^2 = \frac{500 \times 0.1075}{(12.57)^2 P_r}\,\text{m}^2 = \frac{500 \times 0.1075}{(157.9)(7.943 \times 10^{-9})}\,\text{m}^2$$

$$d = 9{,}247\ m \quad \Leftarrow \quad \text{(Note: This is about } 5\,{}^3\!/_4 \text{ miles.)}$$

Note: There are approximately 1,609 meters in one mile.

EXTRA CREDIT QUESTION (up to 10 points)

Going back to Problem 5 above ,suppose a student suggests that we could obtain a much shaper roll off (or sharper filter skirt) by cascading multiple RC sections of the *RC* filter presented in Problem 5, section (a). Give two reasons why this would be a poor selection for a low-pass filter with a specified cutoff frequency.

ANSWER:

(1) The -3 dB frequency, or corner frequency, for each *RC* filter section would experience changes because of the changes in resistance presented to each capacitor. That would require significant changes in the component values to compensate for corner frequency changes.

(2) The series resistor in each *RC* section adds signal transmission loss. With several cascaded *RC* filters, the total loss can become quite large. Generally, high filter losses are not acceptable in communication systems.