Block Diagrams & Signal-Flow Graphs

Ref: Automatic Control Systems, 9th Edition
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Main Objectives

1. To study block diagrams, their components, and their underlying mathematics.
2. To obtain transfer function of systems through block diagram manipulation and reduction.
3. To introduce the signal-flow graphs.
4. To establish a parallel between block diagrams and signal-flow graphs.
5. To use Mason's gain formula for finding transfer function of systems.

Block Diagrams

- *Block diagrams*: the <u>composition</u> and <u>interconnection of the components</u> of a system
  ⇒ describe the cause-and-effect relationships throughout the system.

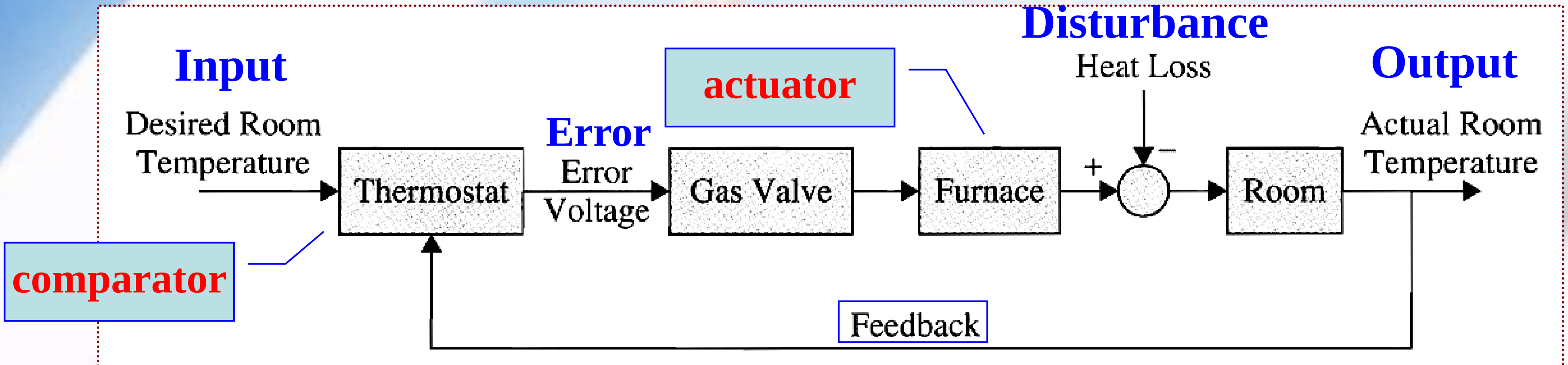
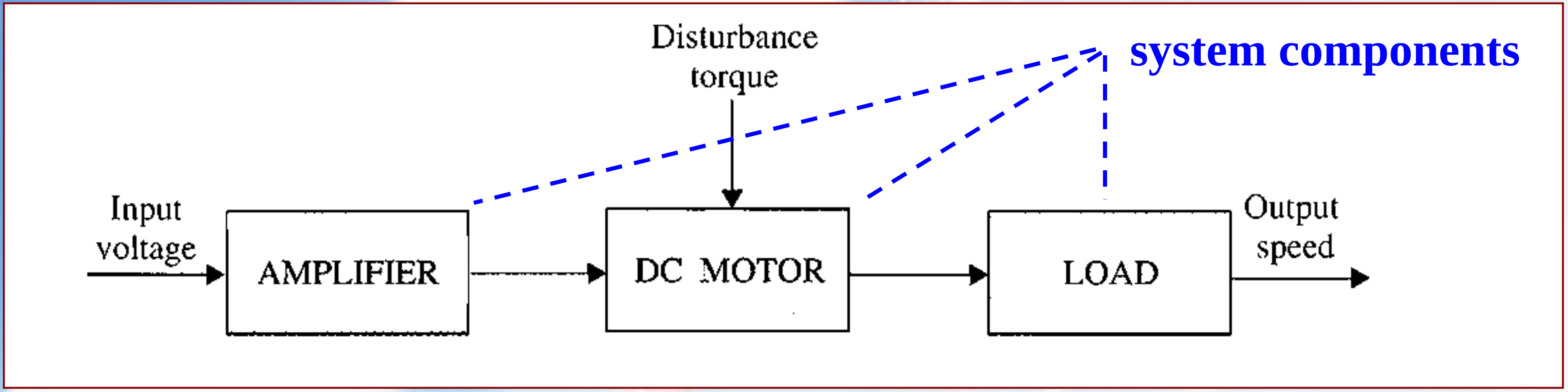


Figure 3-1 A simplified block diagram representation of a heating system.

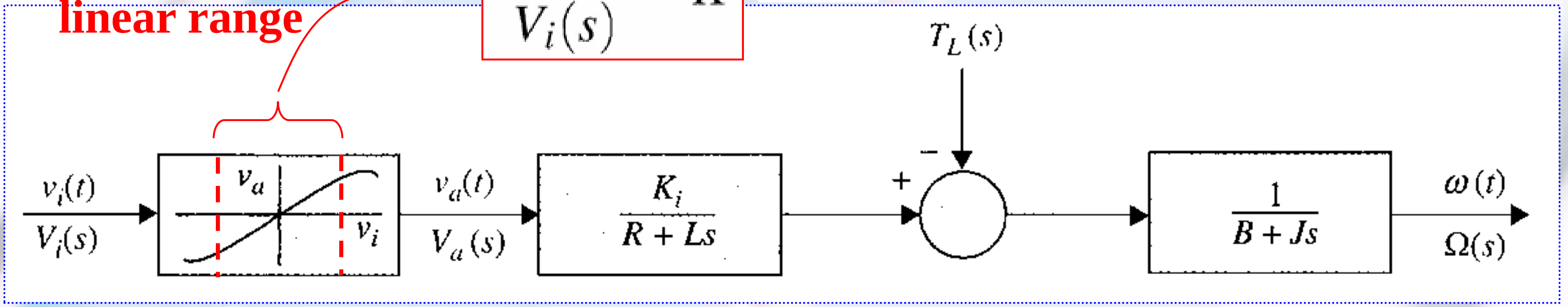
DC-Motor Control System



(a) Block diagram of a dc-motor control system.

$$\frac{V_a(s)}{V_i(s)} = K \quad (3\text{-}1)$$

linear range



(b) Block diagram with transfer function and amplifier characteristic.

Typical Elements of Block Diagrams in Control Systems

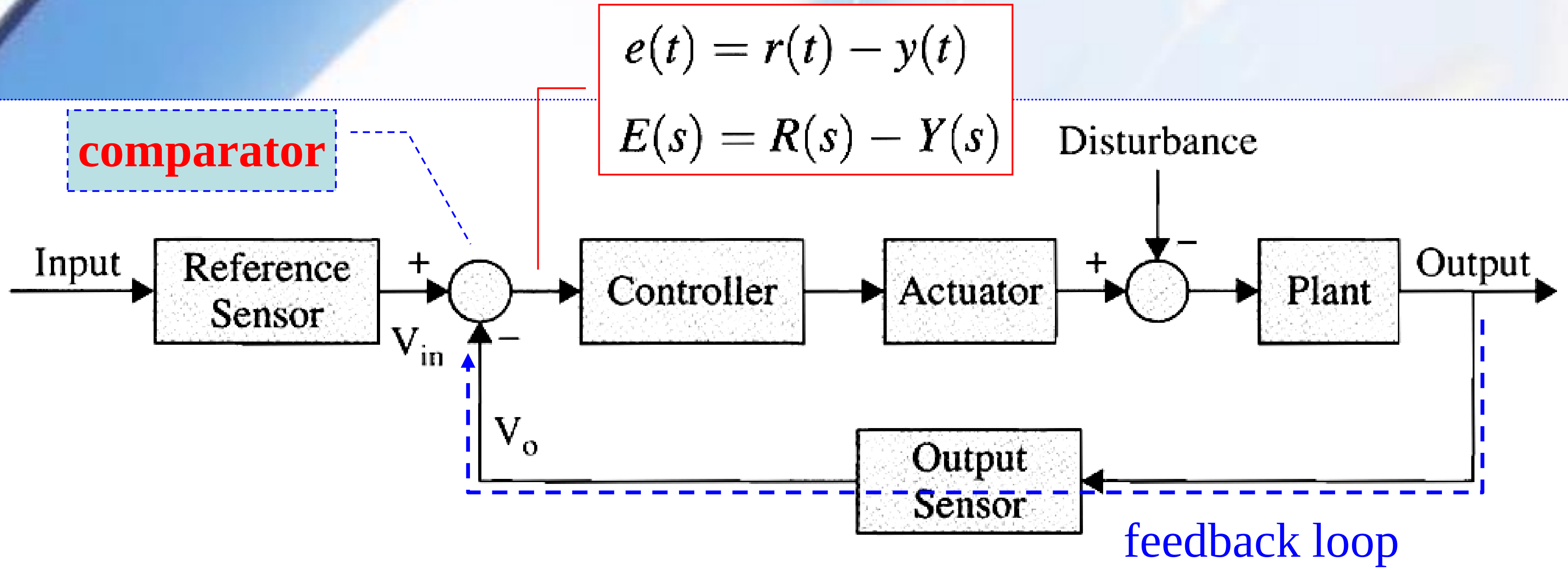
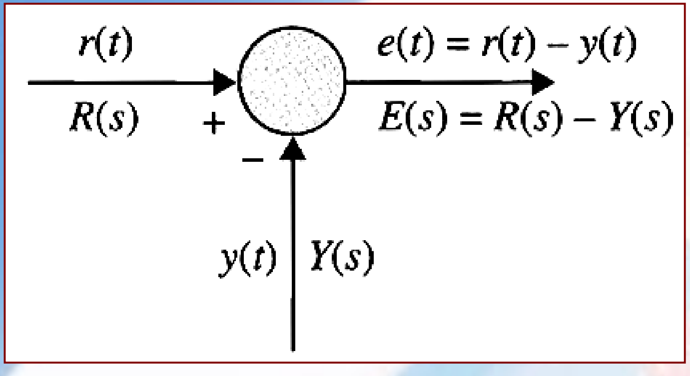


Figure 3-3 Block diagram representation of a general control system.

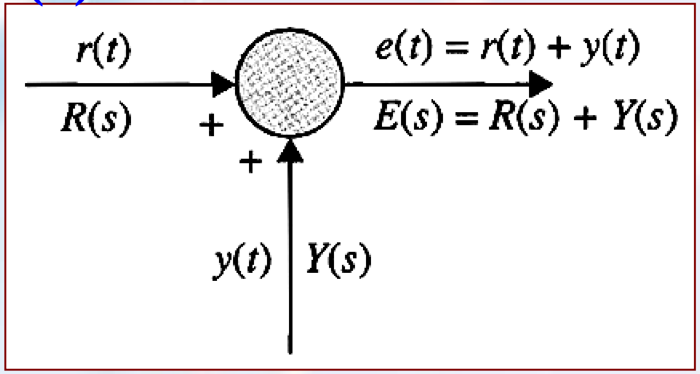
Typical block elements: <u>plant</u>, <u>controller</u>, <u>actuator</u>, and <u>sensor</u>

Block-Diagram Elements of Comparators

(a) Addition



(b) Subtraction



(c) Addition and subtraction

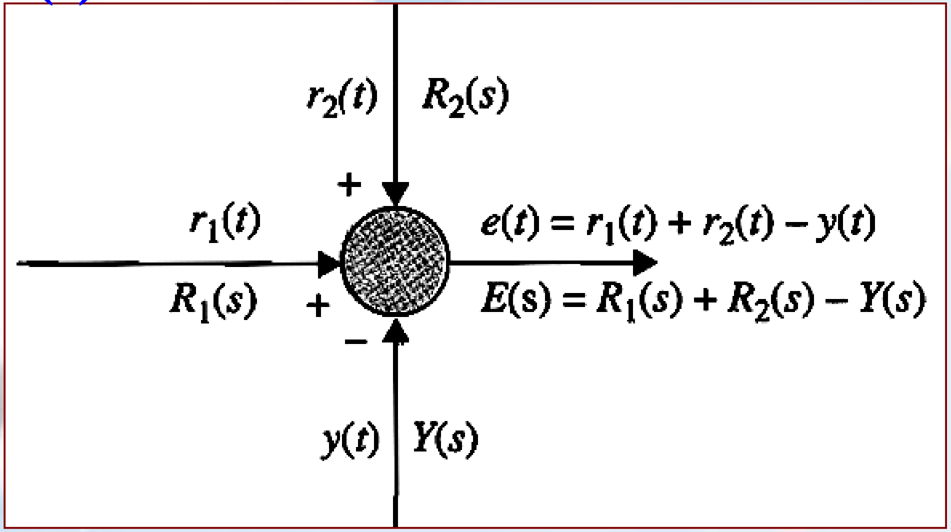


Fig. 3-4 Block diagram elements of typical sensing devices of control systems.

Time & Laplace Domain Block Diagrams

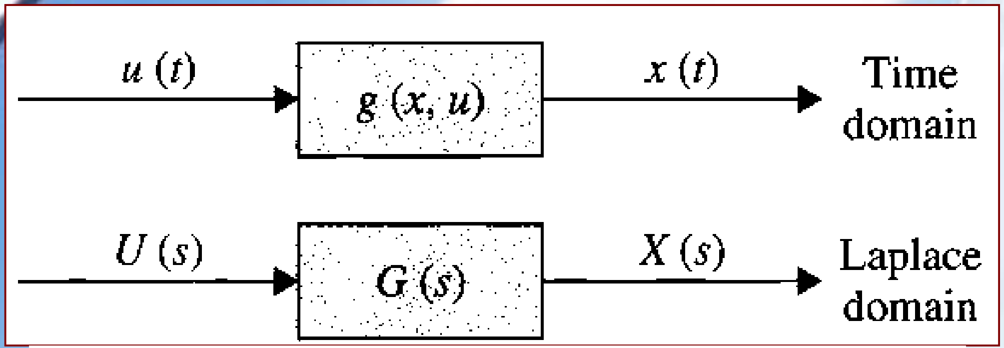


Figure 3-5 Time and Laplace domain block diagrams.

$$X(s) = G(s)\,U(s) \quad (3\text{-}4)$$

$$G(s) = \frac{X(s)}{U(s)} \quad (3\text{-}5)$$

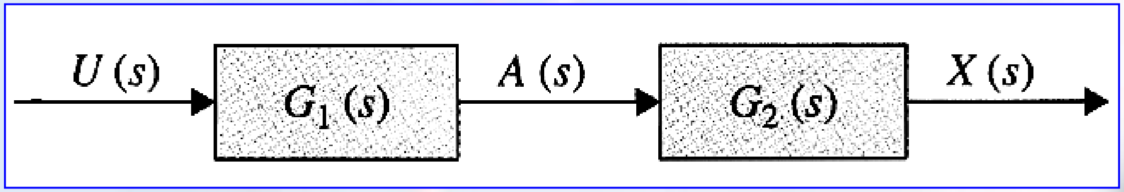


Figure 3-6 Block diagrams $G_1(s)$ and $G_2(s)$ connected in series.

$$X(s) = A(s)G_2(s)$$

$$A(s) = U(s)G_1(s)$$

$$X(s) = G_1(s)G_2(s)U(s)$$

$$G(s) = \frac{X(s)}{U(s)}$$

$$G(s) = G_1(s)G_2(s) \quad (3\text{-}6)$$

Example

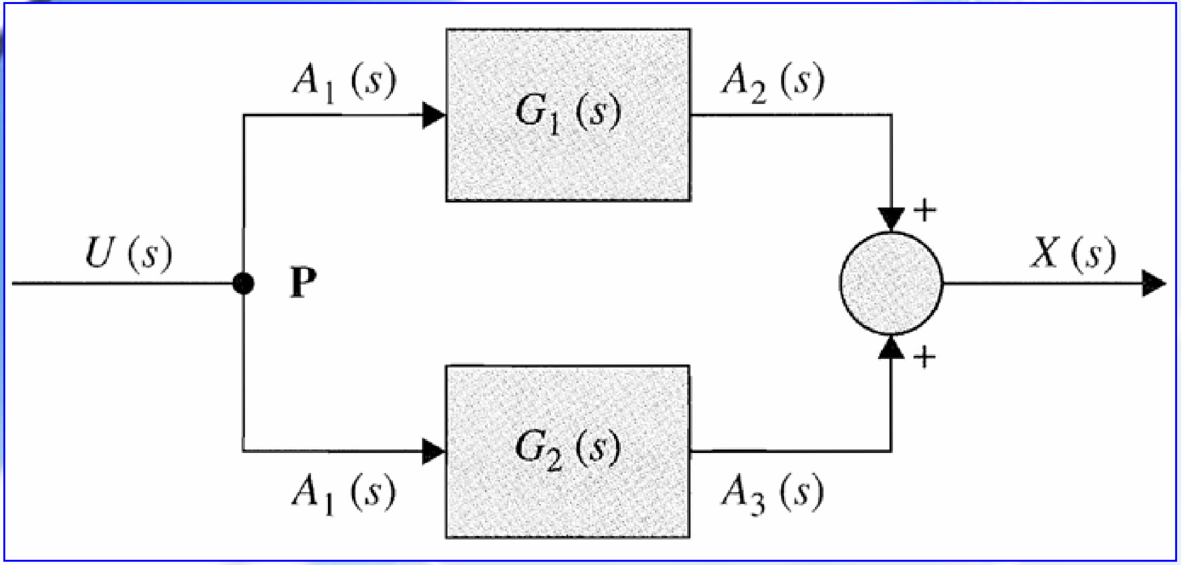


Figure 3-7 Block diagrams $G_1(s)$ and $G_2(s)$ connected in parallel.

$$A_1(s) = U(s)$$

$$A_2(s) = A_1(s)G_1(s)$$

$$A_3(s) = A_1(s)G_2(s)$$

$$X(s) = A_2(s) + A_3(s)$$

$$X(s) = U(s)(G_1(s) + G_2(s))$$

$$\Rightarrow G(s) = \frac{X(s)}{U(s)} \Rightarrow \boxed{G(s) = G_1(s) + G_2(s)} \tag{3-7}$$

Linear Feedback Control System

reference input (command)

actuating signal $U(s) = R(s) - B(s)$

output (controlled variable) $Y(s) = G(s)U(s)$

feedback signal $B(s) = H(s)Y(s)$

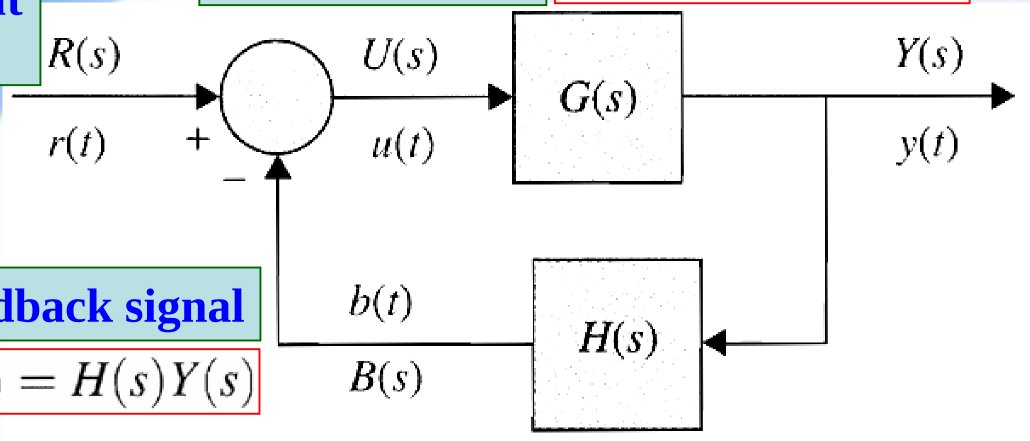


Figure 3-8 Basic block diagram of a feedback control system.

$$Y(s) = G(s)R(s) - G(s)H(s)Y(s) \quad (3\text{-}11)$$

Negative feedback:

$$\boxed{M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}} \quad (3\text{-}12)$$

Positive feedback:

$$\boxed{M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}} \quad (3\text{-}13)$$

Signal-Flow Graphs (SFGs)

- **Input-output (cause-and effect) relations:**

$$y_j = \sum_{k=1}^{N} a_{kj} y_k \qquad j = 1, 2, \ldots, N \qquad (3\text{-}34)$$

$$j\text{th effect} = \sum_{k=1}^{N} (\text{gain from } k \text{ to } j) \times (k\text{th cause}) \qquad (3\text{-}45)$$

$$\text{Output} = \sum (\text{gain}) \times (\text{input}) \qquad (3\text{-}46)$$

$$Y_j(s) = \sum_{k=1}^{N} G_{kj}(s) Y_k(s) \qquad j = 1, 2, \ldots, N \qquad (3\text{-}47)$$

- **Basic elements of an SFG:**

branch gain

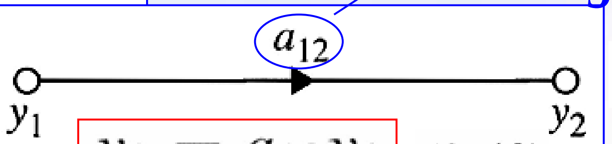


Figure 3-24

input $y_2 = a_{12} y_1$ (3-48) **output**

A signal can transmit through a branch only in the direction of the arrow.

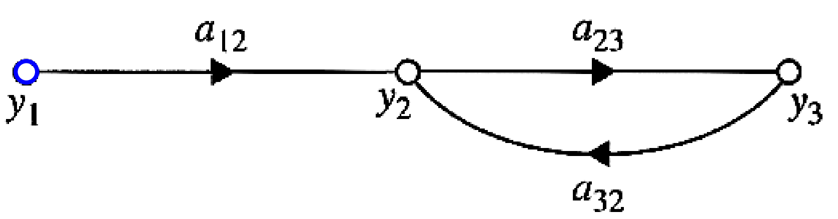
(3-48) $\Rightarrow$ $$y_1 = \frac{1}{a_{12}} y_2$$ (3-49) $\Leftarrow$ Figure 3-24

Definition of SFG Terms (1/3)

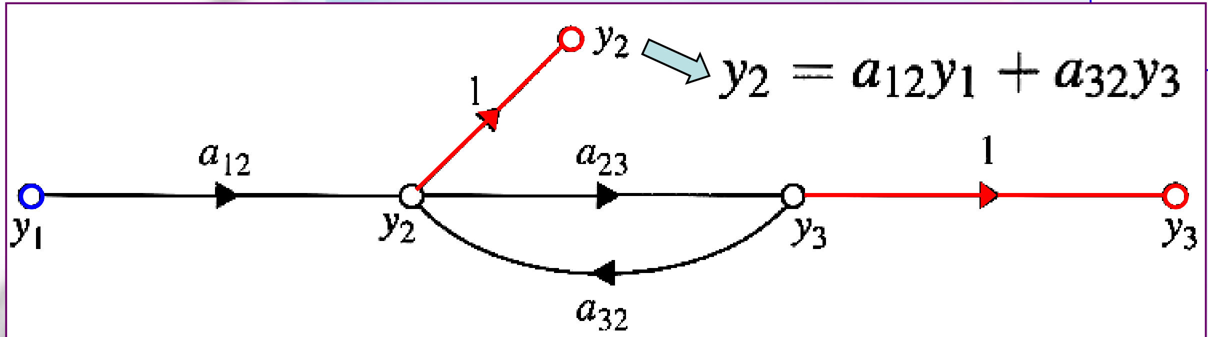
Input node (Source): only ***outgoing*** branches, e.g., ○ in Fig. 3-26

Output node (Sink): only ***incoming*** branches, e.g., ○ in Fig. 3-26

- We can make ***any*** noninput node of an SFG an *output*.
- We **cannot** convert a noninput node into an *input node*.



(a) Original signal-flow graph



$$y_2 = a_{12}y_1 + a_{32}y_3$$

(b) Modified signal-flow graph Figure 3-26

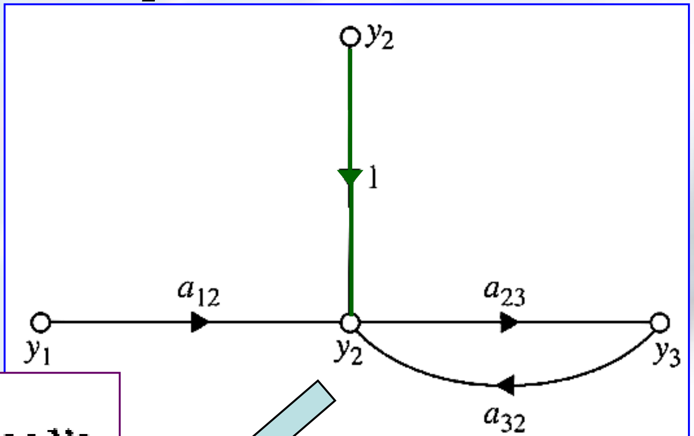


Figure 3-27

$$y_2 = y_2 + a_{12}y_1 + a_{32}y_3 \quad (3\text{-}51)$$

Definition of SFG Terms (2/3)

- **Path:** any connection of a ***continuous succession branches*** traversed in the same direction, e.g., $y_1 - y_2 - y_3$ or $y_2 - y_3 - y_2$.
- **Forward Path:** a path that ***starts** at an input node* and ***ends** at an output node* and along which *no node is traversed more than once*.
- **Path Gain**: ***the product of the branch gains*** encountered in traversing a path,
  e.g., path = $y_1 - y_2 - y_3 - y_4 \Rightarrow$ path gain = $a_{12}a_{23}a_{34}$
- **Forward-Path Gain**: the path gain for a forward path.

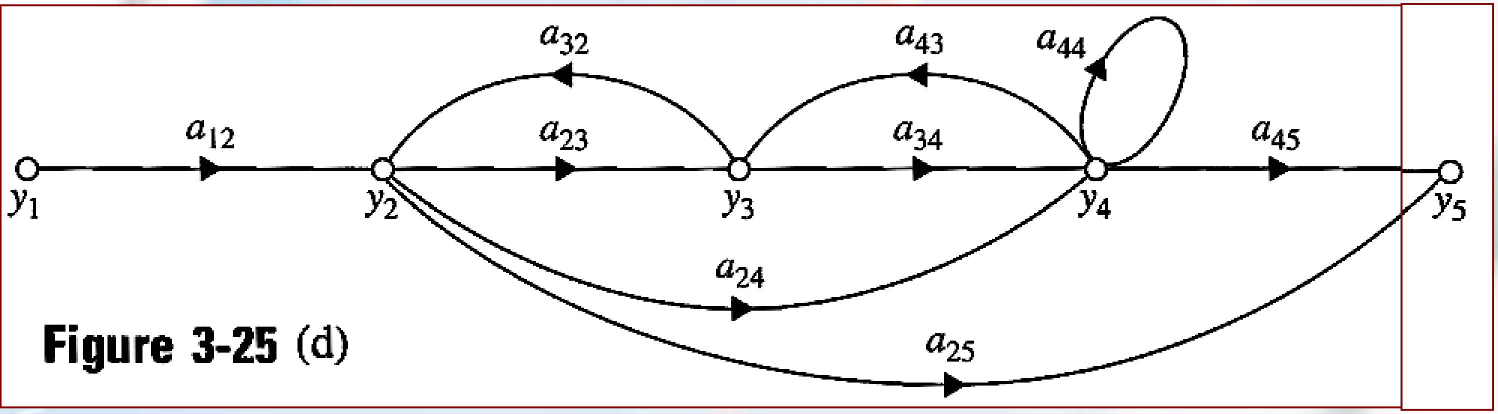


Figure 3-25 (d)

Definition of SFG Terms (3/3)

- **Loop:** a path that ***originates*** and ***terminates*** on the same node and along which no other node is encountered more than once.
- **Loop Gain:** the path gain of a loop.
- **Nontouching Loops:** they do ***not*** share a common node.

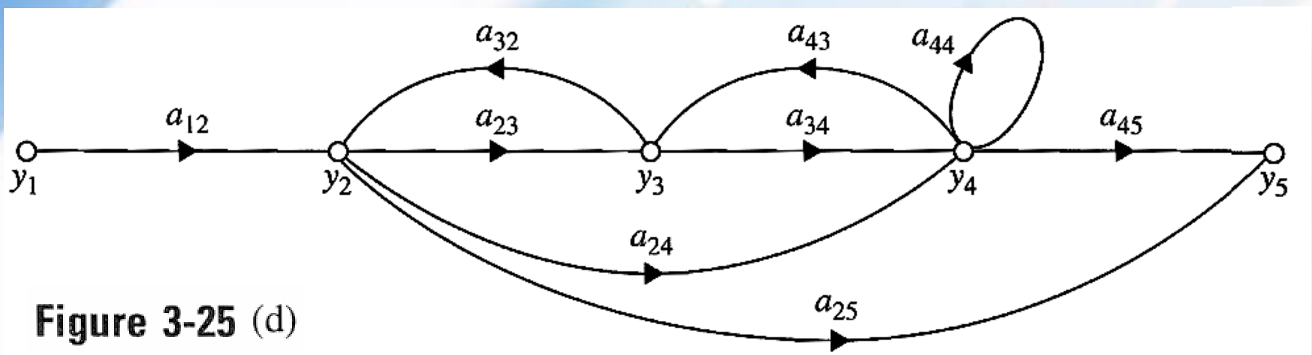


Figure 3-25 (d)

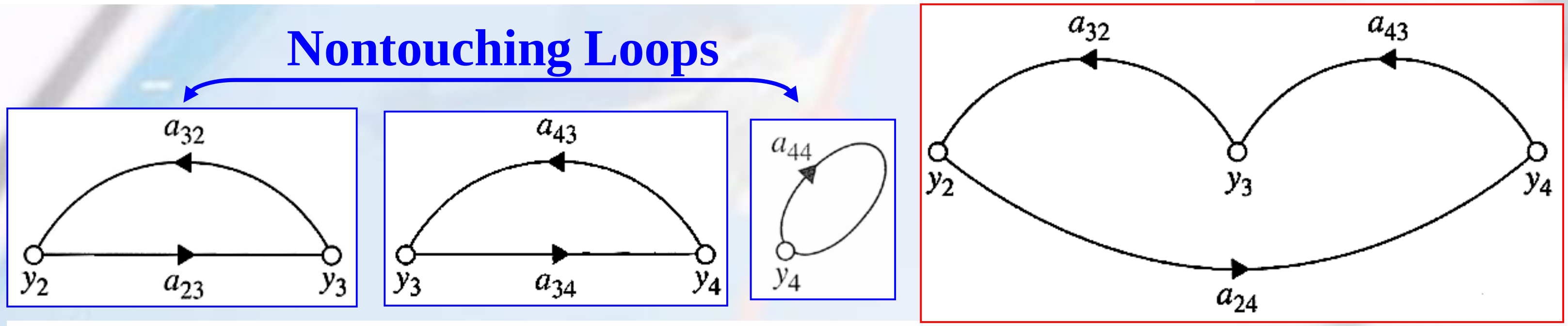


Figure 3-28 Four loops in the signal-flow graph of Fig. 3-25(d).

SFG Algebra

- $y_1 = a_{21}y_2 + a_{31}y_3 + a_{41}y_4 + a_{51}y_5$ (3-52)

  the sum of all signals entering the node

- $y_6 = a_{16}y_1$

  $y_7 = a_{17}y_1$ (3-53)

  $y_8 = a_{18}y_1$

  transmit through all branches leaving the node

- **Parallel branches**:

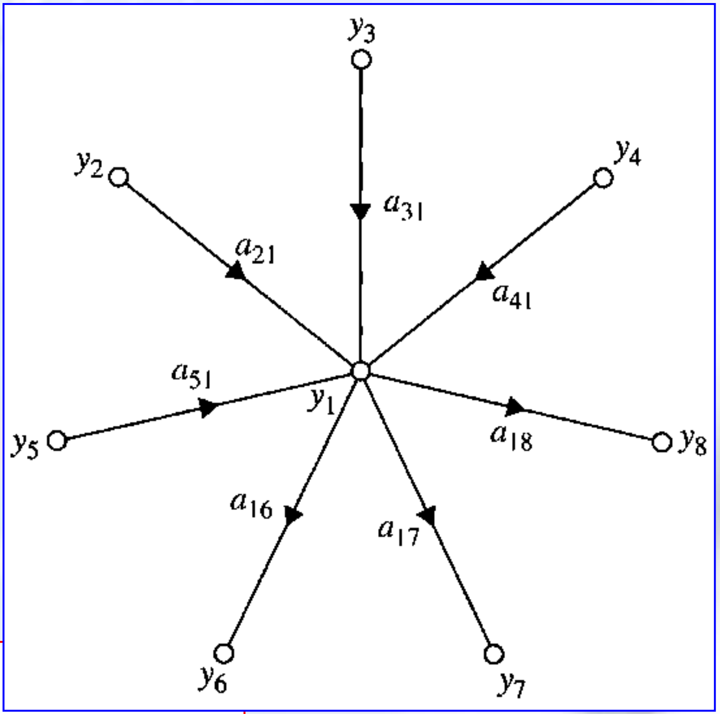


Figure 3-29

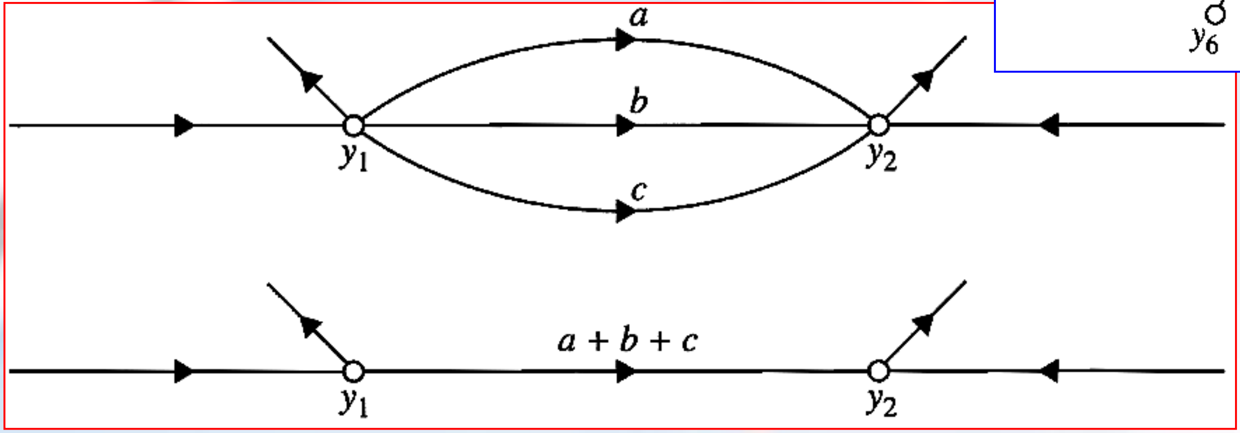


Figure 3-30

SFG Algebra & Feedback Control

- **Series connection**:

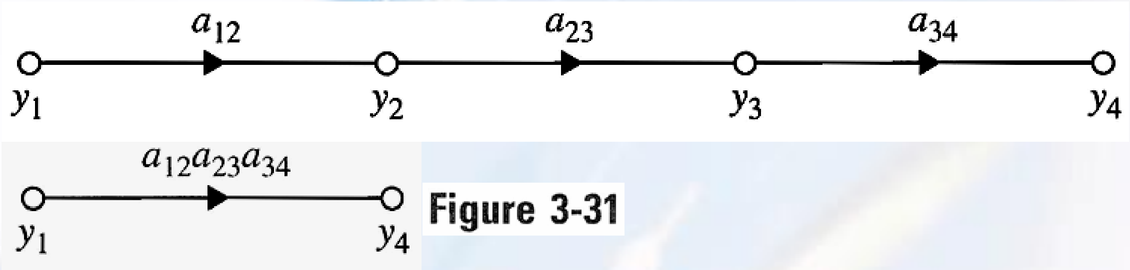


Figure 3-31

- **SFG of a Feedback Control System**:

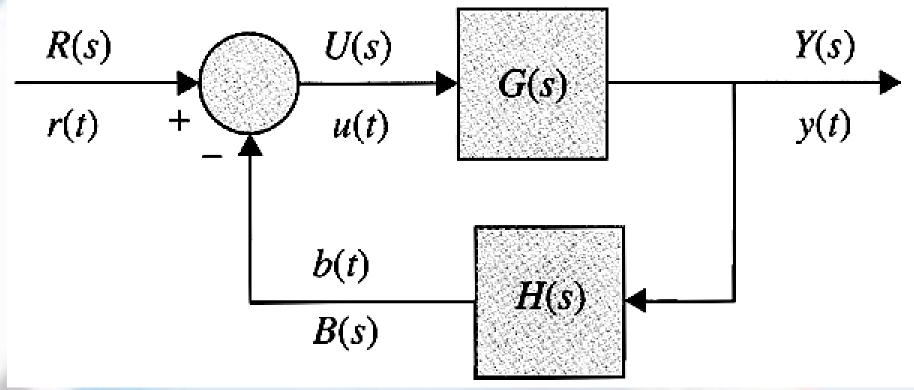


Figure 3-8

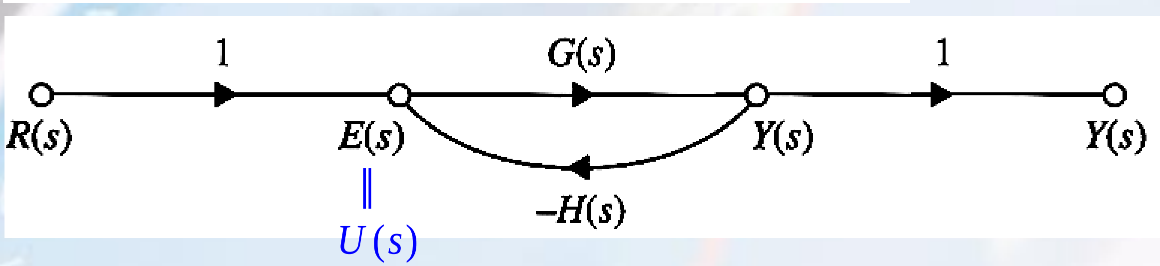
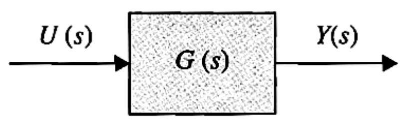
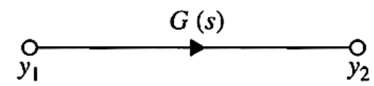
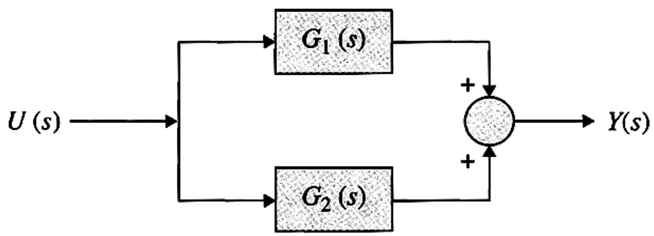
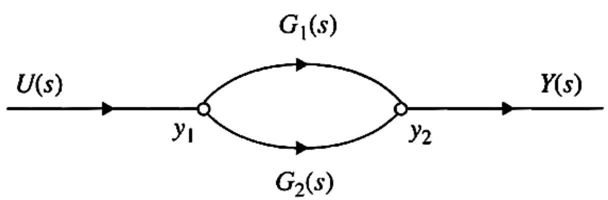
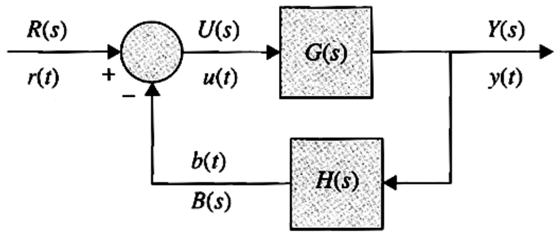
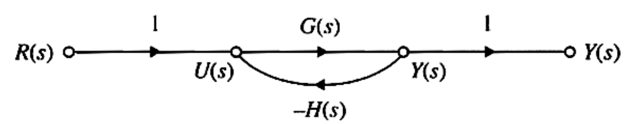


Figure 3-32

Relation between Block Diagram & SFGs

TABLE 3-1 Block diagrams and their SFG equivalent representations

	Block Diagram	Signal Flow Diagram
Simple Transfer Function $\frac{Y(s)}{U(s)} = G(s)$		
Parallel Feedback		
$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$		

Gain Formula for SFG

- ***Mason's gain formula***:

$$M = \frac{y_{\text{out}}}{y_{\text{in}}} = \sum_{k=1}^{N} \frac{M_k \Delta_k}{\Delta} \quad (3\text{-}54)$$

y_{in} = input-node variable

y_{out} = output-node variable

M = gain between y_{in} and y_{out}

N = total number of forward paths between y_{in} and y_{out}

M_k = gain of the kth forward paths between y_{in} and y_{out}

$$\Delta = 1 - \sum_i L_{i1} + \sum_j L_{j2} - \sum_k L_{k3} + \ldots \quad (3\text{-}55)$$

L_{mr}= gain product of the *m*th possible combination of *r* nontouching loops

Δ = 1 – (sum of the gains of **all individual** loops) + (sum of products of gains of all possible combinations of **two** nontouching loops) – (sum of products of gains of all possible combinations of **three** nontouching loops) + – + – …

Δ_k = the Δ for that part of the SFG that is nontouching with the *k*th forward path

Example

- $Y(s) \,/\, R(s) = ?$

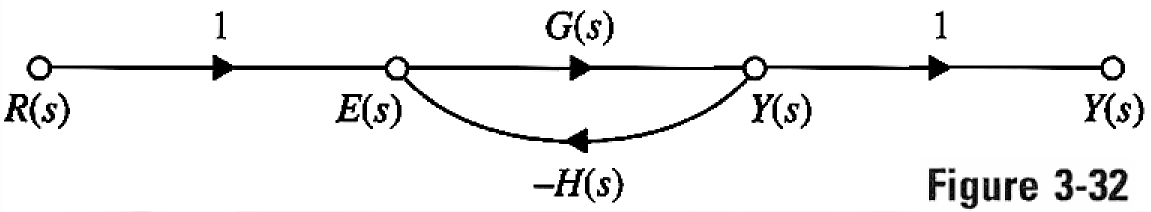


Figure 3-32

1. There is only one forward path between $R(s)$ and $Y(s)$ ⇨ $M_1 = G(s)$ (3-56)

2. There is only one loop ⇨ $L_{11} = -G(s)H(s)$ (3-57)

3. There are no nontouching loops ⇨ $\Delta_1 = 1$,

$$\Delta = 1 - L_{11} = 1 + G(s)H(s) \quad (3\text{-}58)$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{M_1\Delta_1}{\Delta} = \frac{G(s)}{1 + G(s)H(s)}} \quad (3\text{-}59)$$

Example

$y_5 / y_1 = ?$

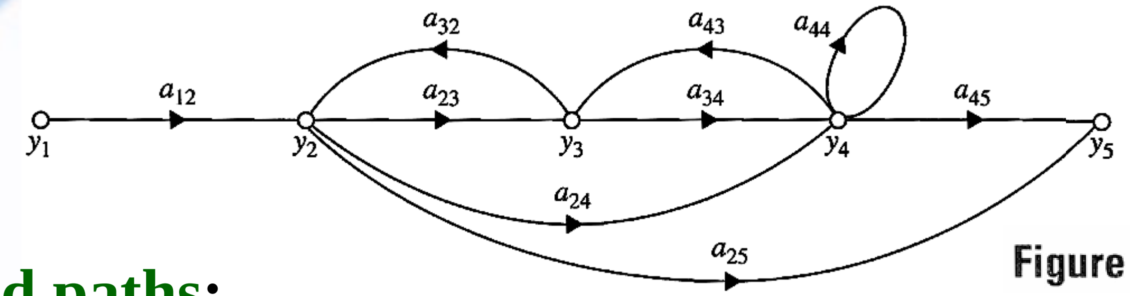


Figure 3-25 (d)

- ***Three* forward paths:**

Forward path: $y_1 - y_2 - y_3 - y_4 - y_5$
Forward path: $y_1 - y_2 - y_5$
Forward path: $y_1 - y_2 - y_4 - y_5$

$$M_1 = a_{12}a_{23}a_{34}a_{45}$$
$$M_2 = a_{12}a_{25}$$
$$M_3 = a_{12}a_{24}a_{45}$$

- ***Four* loops:** $L_{11} = a_{23}a_{32} \quad L_{21} = a_{34}a_{43} \quad L_{31} = a_{24}a_{43}a_{32} \quad L_{41} = a_{44}$

- ***One pair* of nontouching loops:** $y_2 - y_3 - y_2$ and $y_4 - y_4$

$$L_{12} = a_{23}a_{32}a_{44} \quad (3\text{-}60)$$

- $\Delta_1 = \Delta_3 = 1.$ $\Delta_2 = 1 - a_{34}a_{43} - a_{44}$ (3-61)

$$\begin{aligned}\Delta &= 1 - (L_{11} + L_{21} + L_{31} + L_{41}) + L_{12} \\ &= 1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{32}a_{43} + a_{44}) + a_{23}a_{32}a_{44}\end{aligned} \quad (3\text{-}63)$$

Example 3-2-3 (cont.)

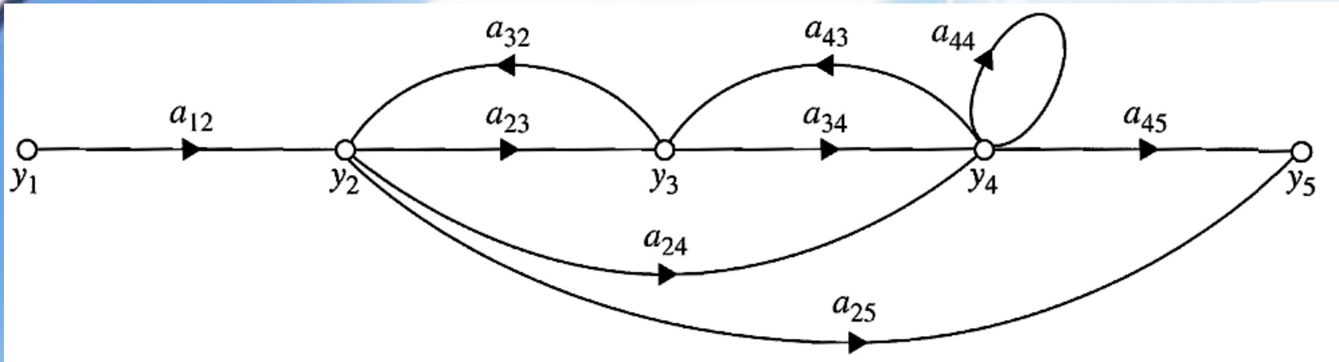


Figure 3-25 (d)

$$\frac{y_5}{y_1} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}$$

$$= \frac{(a_{12}a_{23}a_{34}a_{45}) + (a_{12}a_{25})(1 - a_{34}a_{43} - a_{44}) + a_{12}a_{24}a_{45}}{1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{32}a_{43} + a_{44}) + a_{23}a_{32}a_{44}} \quad (3\text{-}62)$$

$$\frac{y_2}{y_1} = \frac{a_{12}(1 - a_{34}a_{43} - a_{44})}{\Delta} \quad (3\text{-}64)$$

Example

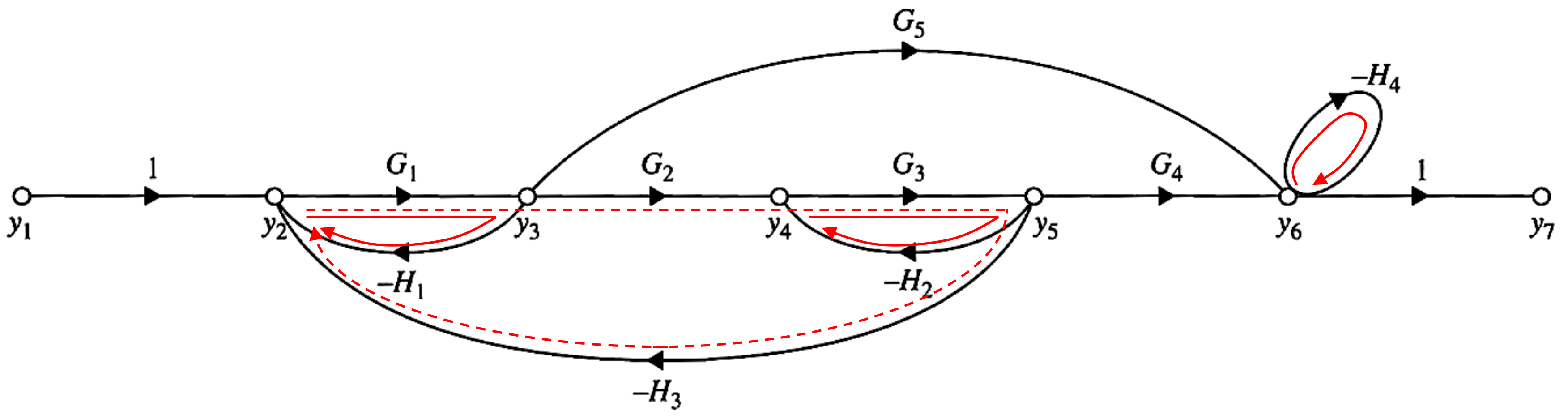


Figure 3-33 Signal-flow graph for Example 3-2-4.

$$\frac{y_2}{y_1} = \frac{1 + G_3H_2 + H_4 + G_3H_2H_4}{\Delta} \quad (3\text{-}65)$$

$$\frac{y_4}{y_1} = \frac{G_1G_2(1 + H_4)}{\Delta} \quad (3\text{-}66)$$

$$\frac{y_6}{y_1} = \frac{y_7}{y_1} = \frac{G_1G_2G_3G_4 + G_1G_5(1 + G_3H_2)}{\Delta} \quad (3\text{-}67)$$

$$\Delta = 1 + \underbrace{G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + H_4}_{} + G_1G_3H_1H_2 + G_1H_1H_4 + G_3H_2H_4 + G_1G_2G_3H_3H_4 + G_1G_3H_1H_2H_4 \quad (3\text{-}68)$$

Gain Formula between Output Node and Noninput Nodes & Example

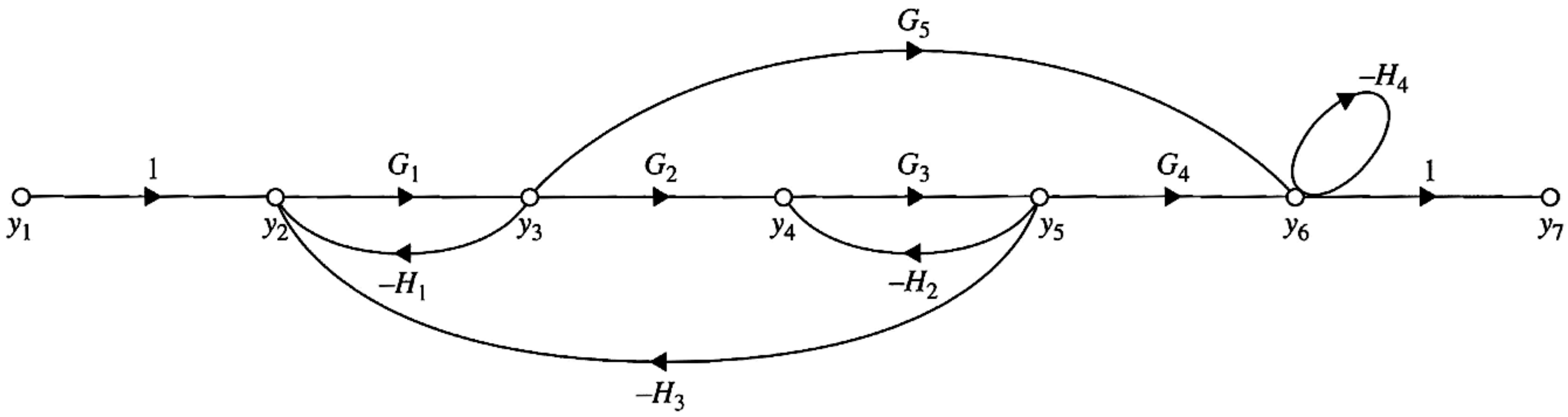


Figure 3-33 Signal-flow graph for Example 3-2-4.

$$\frac{y_{out}}{y_2} = \frac{\dfrac{y_{out}}{y_{in}}}{\dfrac{y_2}{y_{in}}} = \frac{\dfrac{\Sigma M_k \Delta_k|_{\text{from } y_{in} \text{ to } y_{out}}}{\Delta}}{\dfrac{\Sigma M_k \Delta_k|_{\text{from } y_{in} \text{ to } y_2}}{\Delta}} = \frac{\Sigma M_k \Delta_k|_{\text{from } y_{in} \text{ to } y_{out}}}{\Sigma M_k \Delta_k|_{\text{from } y_{in} \text{ to } y_2}} \quad \begin{matrix}(3\text{-}69)\\(3\text{-}70)\end{matrix}$$

Example :

$$\frac{y_7}{y_2} = \frac{y_7/y_1}{y_2/y_1} = \frac{G_1G_2G_3G_4 + G_1G_5(1 + G_3H_2)}{1 + G_3H_2 + H_4 + G_3H_2H_4} \quad (3\text{-}71)$$

Application of Gain Formula to Block Diagram

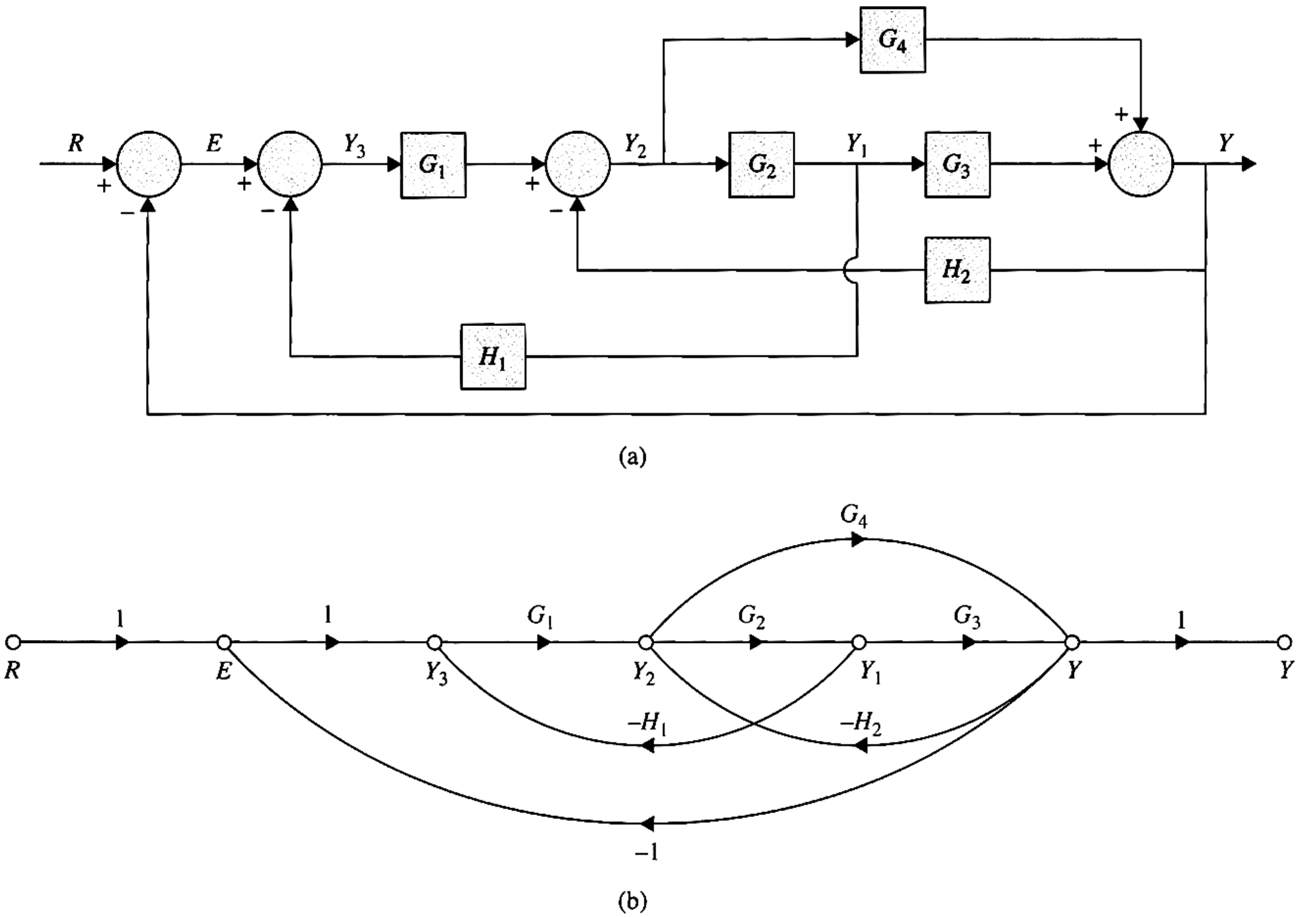
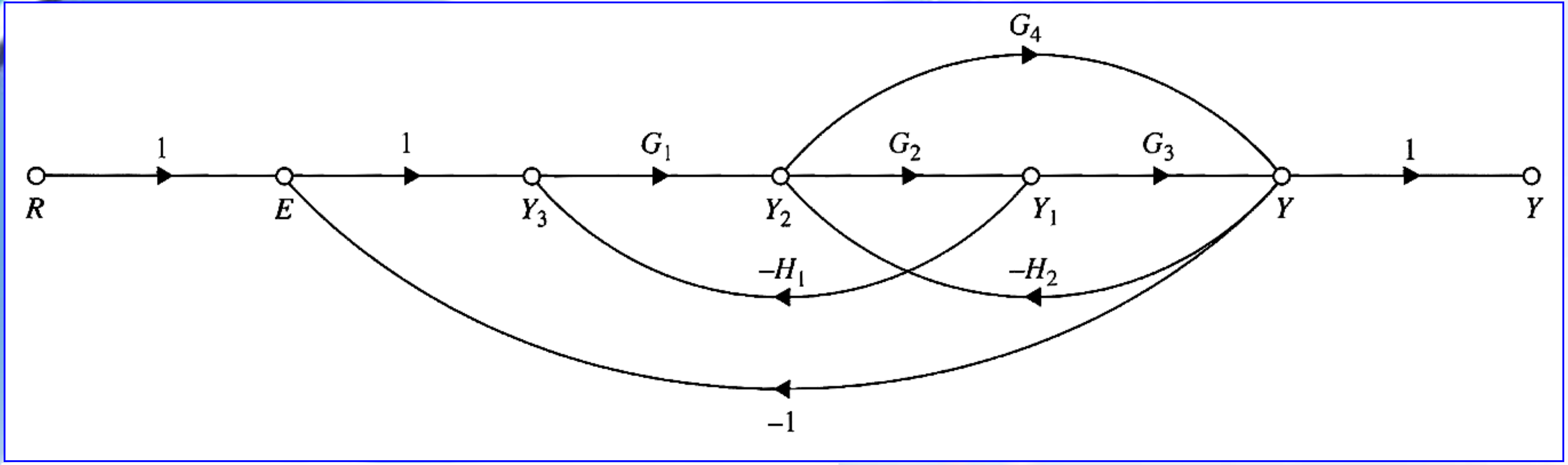


Figure 3-34 (a) Block diagram of a control system. (b) Equivalent signal-flow graph.

Example



Forward Path Gains: 1. $G_1G_2G_3$; 2. G_1G_4

Loop Gains: 1. $-G_1G_2H_1$; 2. $-G_2G_3H_2$; 3. $-G_1G_2G_3$; 4. $-G_4H_2$; 5. $-G_1G_4$

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{\Delta} \quad (3\text{-}72) \qquad \frac{E(s)}{R(s)} = \frac{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}{\Delta} \quad (3\text{-}74)$$

$$\frac{Y(s)}{E(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2} \quad (3\text{-}75)$$

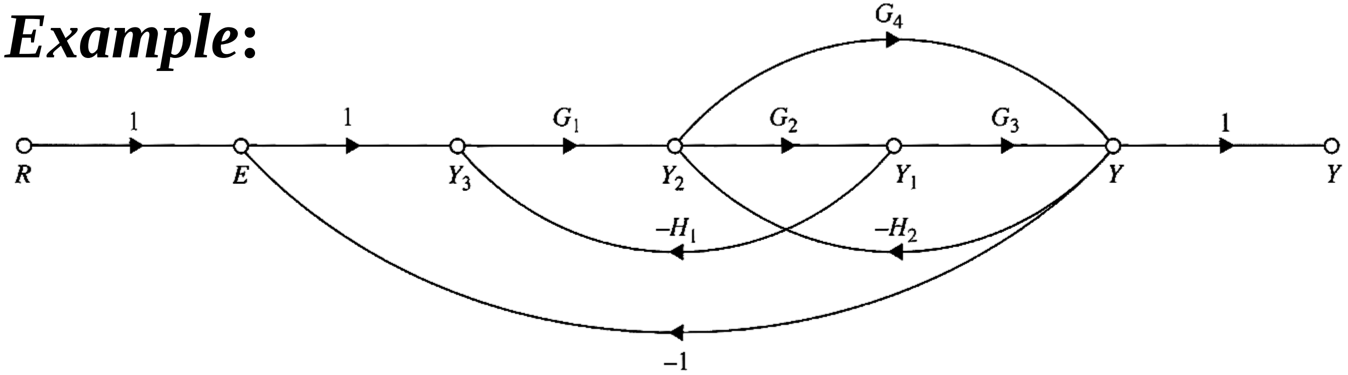
$$\Delta = 1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4H_2 + G_1G_4 \quad (3\text{-}73)$$

Simplified Gain Formula

- **All loops and forward paths are touching:**

$$M = \frac{y_{\text{out}}}{y_{\text{in}}} = \sum \frac{Forward\ Path\ Gains}{1 - Loop\ Gains} \qquad (3\text{-}76)$$

- ***Example:***



Forward Path Gains: 1. $G_1G_2G_3$; 2. G_1G_4

Loop Gains: 1. $-G_1G_2H_1$; 2. $-G_2G_3H_2$; 3. $-G_1G_2G_3$; 4. $-G_4H_2$; 5. $-G_1G_4$

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{\Delta}$$

$$\Delta = 1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4H_2 + G_1G_4$$