1 Exercises of §2.2

Excercise 1.1 (The du Bois-Reymond Lemma)**.** Let f be a continuous function on $[a, b]$ such that

$$\int f\phi = 0$$

for any $\psi \in \mathscr{C}_0^\infty((a, b))$. Then f is identically zero.

Proof. Taking real and imaginary parts we may assume f real when ϕ is taken real. If $f(x_0) \neq 0$ we can find a non-negative $\psi \in \mathscr{C}_0^\infty$ with $\psi(x_0) \neq 0$ and support so close to x_0 that $f\psi$ has constant sign, which contradicts to the fact that its integral is 0. □

Excercise 1.2 (Leibniz's Formula, a Generalization)**.** The goal of this exercse is to prove the formula

$$P(x, \partial)(uv) = \sum_\alpha \frac{1}{\alpha!} \partial^\alpha u P^{(\alpha)}(x, \partial)v.$$

Let P be a linear differential operator with smooth coefficients. Prve that

(1) $P(x, \xi + \eta) = \sum_\alpha \frac{1}{\alpha!} \xi^\alpha P^{(\alpha)}(x, \eta)$;

(2) $P(x, \partial)(uv) = \sum_\alpha \partial u \cdot R_\alpha(x, \partial)v$, where R_α's are linear differential operators with smooth coefficients;

(3) $P(x, \partial)e^{\langle x,\eta\rangle} = P(x, \eta)e^{\langle x,\eta\rangle}$;

(4) $R_\alpha = \frac{1}{\alpha!} P^{(\alpha)}(x, \partial)$.

Proof. We write $P = \sum_\alpha a_\alpha(x)\partial^\alpha$. P defines a polynomial in ξ by $P(x, \xi) := e^{-\langle x,\xi\rangle} P e^{\langle x,\xi\rangle} = \sum_\alpha a_\alpha(x)\xi^\alpha$.

Since the polynomial $P(x, \xi)$ is analytic in ξ, we have by Taylor's theorem

$$P(x, \xi + \eta) = \sum_\alpha \frac{1}{\alpha!} \xi^\alpha P^{(\alpha)}(x, \eta).$$

The identity (2) comes after expansion and rewriting of the differentials.

Note that $\partial^\alpha e^{\langle x,\eta\rangle} = \eta^\alpha e^{\langle x,\eta\rangle}$, we have

$$\sum_\alpha a_\alpha(x)\partial^\alpha e^{\langle x,\eta\rangle} = \sum_\alpha a_\alpha(x)\eta^\alpha e^{\langle x,\eta\rangle}$$

and hence $P(x, \partial)e^{\langle x,\eta\rangle} = P(x, \eta)e^{\langle x,\eta\rangle}$.

For (4), note that $L : u \mapsto P(uv)$ is a linear differential operator. We have $L(x, \xi) = e^{-\langle x,\xi\rangle} L e^{\langle x,\xi\rangle} = e^{-\langle x,\xi\rangle} P(e^{\langle x,\xi\rangle} v)$. Note that

$$e^{-\langle x,\xi\rangle} \partial^\beta v \partial^\alpha e^{\langle x,\xi\rangle} = \xi^\alpha \partial^\beta v,$$

we have by Leibniz's formula

$$e^{-\langle x,\xi\rangle} \partial^\alpha (e^{\langle x,\xi\rangle} v) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \xi^\alpha \partial^\beta v = (\xi + \partial)^\alpha v$$

and therefore $L(x,\xi) = P(x,\xi+\partial)v = \sum_\alpha \frac{1}{\alpha!}\partial^\alpha v P^{(\alpha)}(x,\xi)$, which gives

$$P(uv) = Lu = L(x,\partial)u = \sum_\alpha \frac{1}{\alpha!}\partial^\alpha v P^{(\alpha)}(x,\partial)u.$$

$\square$

Excercise 1.3. Calculate $\Delta(uv)$ and $\Delta^2(uv)$, and prove that

$$e^{-\langle x,\xi\rangle} P(ue^{\langle x,\xi\rangle}) = P(x,\xi+\partial)u.$$

Proof. The identity is proven in the preceeding exercise.

$\Delta(uv) = 2\partial_i u \partial_i v + \partial_i^2 u \cdot v + u \cdot \partial_i^2 v$.

$\Delta^2(uv) = \Delta(\Delta(uv)) = 2\Delta u \Delta v + \Delta^2 u \cdot v + u \cdot \Delta^2 v + ...$ $\square$

Excercise 1.4. Let $f \in L^1$, prove that

$$\lim_{\epsilon\to 0} \|f_\epsilon - f\|_{L^1} = 0.$$

Proof. Note that

$$f_\epsilon - f = \int [f(x-\epsilon y) - f(x)]\phi(y)\,\mathrm{d}y.$$

Minkovskii's inequality gives

$$\|f_\epsilon - f\|_{L^1} \le \int \|f_{-\epsilon y} - f\|_{L^1} |\phi(y)|\,\mathrm{d}y,$$

where $f_{-\epsilon y}$ is the translation of f by ϵy to the right.

For each y, $\|f_{-\epsilon y} - f\|_{L^1}$ tends to zero as $\epsilon \to 0$[1] and is bounded by $2\|f\|_{L^1}$, the desired result then follows from the dominated convergence theorem. $\square$

Excercise 1.5. Let $\phi \in \mathscr{C}_0^\infty$ with $\int \phi = 1$, and v be continuous. Define

$$u(x,t) = \int v(x-ty)\phi(y)\,\mathrm{d}y.$$

Prove that

(1) When $t > 0$

$$\partial_{x_i}(t^k u(x,t)) = t^{k-1}\int v(x-ty)\partial_{y_i}\phi(y)\,\mathrm{d}y;$$

$$\partial_t(t^k u(x,t)) = t^{k-1}\int v(x-ty)[(k-n)\phi(y) - \sum_i y_i \partial_{y_i}\phi(y)]\,\mathrm{d}y;$$

(2) As $t \to 0^+$,

$$\partial_t^j(t^k u(x,t)) \to 0, j < k;$$

$$\partial_t^k(t^k u(x,t)) \to k! v(x).$$

[1]The case where f is uniformly continuous is simple and the general case follows from approximating f with uniformly continuous functions.

Proof. By a change of variable we have

$$t^k u(x,t) = t^k |t|^{-n} \int v(y)\phi(\frac{x-y}{t})\,\mathrm{d}y.$$

Also notice that the integrand is bounded, hence differentiation commutes with integration. We have by direct calculation

$$\partial_{x_i}(t^k u(x,t)) = t^{k-1}\int v(x-ty)\partial_{y_i}\phi(y)\,\mathrm{d}y;$$

$$\partial_t(t^k u(x,t)) = t^{k-1}\int v(x-ty)[(k-n)\phi(y) - \sum_i y_i\partial_{y_i}\phi(y)]\,\mathrm{d}y;$$

Since $u(x,t) \to v(x)$ as $t \to 0$, (2) follows from Leibniz's formula. □

Excercise 1.6. Let $\{\phi_\epsilon\}_\epsilon$ be an approximation to the identity and $f_\epsilon(x) = \int \phi_\epsilon(x-y)\,\mathrm{d}y$. Prove that

$$|\partial^\alpha f_\epsilon| \le C(\alpha, n)\epsilon^{-|\alpha|}.$$

Proof. Notice that

$$f_\epsilon(x) = \int \phi_\epsilon(x-y)\,\mathrm{d}y = \epsilon^{-n}\int \phi(\frac{x-y}{\epsilon})\,\mathrm{d}y,$$

and therefore $\partial^\alpha f_\epsilon = \epsilon^{-n}\epsilon^{-|\alpha|}\int \partial^\alpha\phi(\frac{x-y}{\epsilon})\,\mathrm{d}y$.

The desired inequality holds since ϕ is a fixed function and $\int \partial^\alpha\phi(\frac{x-y}{\epsilon})\,\mathrm{d}y$ is always bounded for each α. □

Excercise 1.7. We define $H_a = a^{-1}1_{[0,a]}$. Prove that $u * H_a \in \mathscr{C}^{k+1}$ if $u \in \mathscr{C}^k$.

Proof. We have by definition

$$u * H_a(x) = \frac{1}{a}\int u(x-y)1_{[0,a]}(y)\,\mathrm{d}y = \frac{1}{a}\int_0^a u(x-y)\,\mathrm{d}y = a^{-1}\int_{x-a}^x u.$$

And the assertion follows from the fundamental theorem of integration. □

Excercise 1.8. The goal of this exercise is to give another proof of the theorem for partition of the unity.

(1) Let $\{X_\alpha\}$ be an open covering of the compact set U. There exist a finite number of $K_i \subset\subset X_i$ such that $\{K_i\}_i$ is an open coveing of U.

(2) Let $\phi \in \mathscr{C}_0^\infty$. There is for each i a $\phi_i \in \mathscr{C}_0^\infty$ such that $0 \le \phi_i \le 1$, $\phi_i \equiv 1$ on K_i and $\phi_i \equiv 0$ outside X_i.

Proof. For (1), let X_i^ϵ be the set of points in X_i with distance from $\mathbb{R}^n \setminus X_i$ larger than ϵ. Therefore $X_i^\epsilon \subset\subset X_i$.

We claim $U \subset \bigcup_1^N X_i^\epsilon$ if ϵ is small enough while $X_1, ..., X_N$ s a finite subcovering of U. Assume otherwise, for each $\epsilon > 0$ there exists $x_\epsilon \in U \setminus \bigcup_1^N X_i^\epsilon$. $\{x_\epsilon\}$ has a limit point x as $\epsilon \to 0$ since U is compact. However $x \in \bigcup_1^N X_i$ then, which contradicts to our assumption that $X_1, ..., X_N$ covers U.

For (2), we choose $K_1, ..., K_N$ relatively compact in X_i so that $\operatorname{Supp}\phi \subset \bigcup_1^N K_i$. Then we choose $\psi_i \in \mathscr{C}_0^\infty(X_i)$ with $0 \le \psi_i \le 1$ and $\psi_i \equiv 1$ on K_i.

Let $\phi_1 = \phi\psi_1, \phi_2 = \phi\psi_2(1-\psi_1), ..., \psi_N = \phi\psi_N(1-\psi_1)\cdot ... \cdot(1-\psi_{N-1})$.

Then the functions ϕ_i satisfy the desired properties since $\sum_1^N \phi_i - \phi = -\phi\prod_1^N(1-\psi_i) = 0$. □

2 Exercises of §2.3