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Space Object Initial State Refinement Via Differential Correction

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Summary

This memo documents the exercise of deriving the mathematical relationships required to refine an epoch state resulting from an Initial Orbit Determination scheme via Differential Correction (Batch Processor).

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1. INTRODUCTION

There are times when *a priori* information is not available for a given space object's state. Based upon six independent parameters, a crude orbit estimate is achievable via several methods (e.g. Lagrange, Gauss, Gooding, etc.). Once a crude orbit estimate is made via Initial orbit Determination (IOD), a differential correction or batch process can be implemented to refine this crude IOD state estimate.

One issue that must be accounted for is the fact that the measured space object position is based upon the location of the spacecraft when the reflected light left the spacecraft. This is referred to as Spacecraft Event Time (SCET). The measurements are made at the image plane on Earth at Earth Receive Time (ERT). The difference between SCET and ERT is the One-Way-Light-Time (OWLT) from the spacecraft to Earth. Therefore the apparent location of the spacecraft at ERT is not the actual location of the spacecraft. To illustrate this Stellar Aberration, the space object-observer geometry is shown in Fig 1.

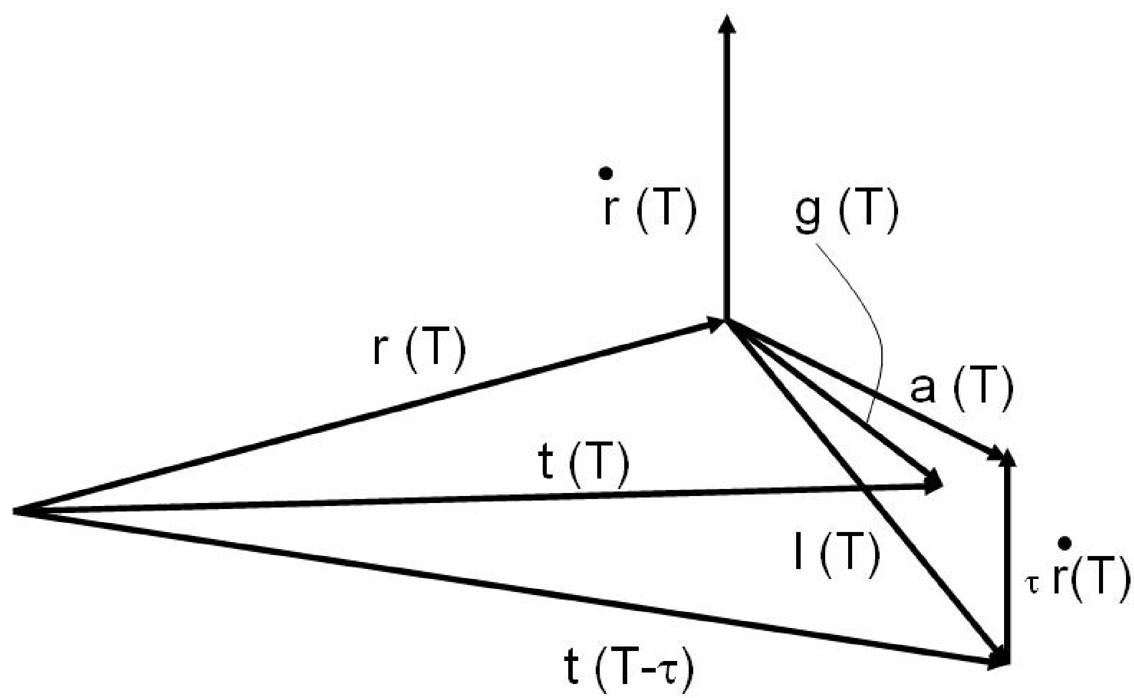

Fig. 1. Optical Measurement Inertial Frame Geometry. (T) is ERT, τ is the OWLT, and (T-τ) is SCET. r(T) is the geocentric inertial position of the observer at ERT. t(T) is the geocentric inertial position of the spacecraft at ERT. g(T) is the topocentric inertial position of the spacecraft at ERT. a(T) is the topocentric apparent inertial position of the spacecraft at ERT due to stellar aberration. t(T-τ) is the geocentric inertial position of the spacecraft at SCET. l(T) is the topocentric inertial position of the spacecraft at ERT when it was at SCET.

We know r(T) and its derivative, measure a(T), and are really interested in t(T-τ) so that we can compute a predicted a(T) that will be used to form a measurement residual that is minimized in a least-squared sense.

$$l(T) = t(T-\tau) - r(T) = R(T) - r(T) \quad (1)$$

$$\tau = \frac{l(T)}{c}$$

Where c is the speed of light

$$a(T) = l(T) + \tau \dot{r}(T) \quad (2)$$

Which can be re-written as

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} R_x - r_x + \tau \dot{r}_x \\ R_y - r_y + \tau \dot{r}_y \\ R_z - r_z + \tau \dot{r}_z \end{bmatrix} = \begin{bmatrix} R_x - r_x + \tau(\dot{R}_x - \dot{l}_x) \\ R_y - r_y + \tau(\dot{R}_y - \dot{l}_y) \\ R_z - r_z + \tau(\dot{R}_z - \dot{l}_z) \end{bmatrix} \quad (3)$$

As Eq. (1) suggests, the spacecraft inertial state at SCET needs to be computed in order to formulate l(T). Therefore, a sufficiently accurate estimate of OWLT (τ) is required.
The batch processor (differential correction) is commonly known as a least squares data fit. It is called a batch because all of the data are collected and accumulated prior to solving for the best estimate of the initial state and covariance. This best estimate will be the epoch state conditions (and parameter estimates) which minimize the sum of the squares of the observation residuals. The batch algorithm can be summarized as follows:

Given: P_{t-1} , $\hat{X}_{t-1}$, Y_t , $\Re_t$

If there is an *a priori* estimate, set $\Lambda = \overline{P}_0^{-1}$ and set $N = \overline{P}_0^{-1}\overline{x}_0$

If there is no *a priori* estimate, set $\Lambda = 0$ and set $N = 0$

(1) Integrate from t-1 to t,

$$\dot{\overline{X}} = F(\overline{X}, t)$$

$$\overline{X}(t) = \hat{X}_{t-1}$$

$$\dot{\Phi}(t, t-1) = A(t)\Phi(t, t-1) \quad (4)$$

$$\Phi(t-1, t-1) = I$$

(2) Compute the predicted observation(s) and associated partials

$$y_t = Y_t - G(\overline{X}_t, t) \quad (5)$$

$$\widetilde{H}_t = \partial G(\overline{X}_t, t) / \partial X_t \quad (6)$$

$$H_t = \widetilde{H}_t \ \Phi(t, t-1)$$

(3) Accumulate

$$\Lambda = \Lambda + H_i^T \Re_i^{-1} H_i$$

$$N = N + H_i^T \Re_i^{-1} y_i \tag{7}$$

(4) Replace t with t-1 and return to (1) until all the observations have been processed, then solve

$$\hat{x}_o = \Lambda^{-1} N$$

$$\hat{X}_o^* = \overline{X}_o^* + \hat{x}_o \tag{8}$$

(5) Iterate until the state deviation tends to zero (i.e. the initial state stops changing)

In the previous algorithm:

P is the state error covariance
X is the spacecraft state
x is the state deviation or error in the epoch state
ϕ is the state transition matrix
y is the observation measurement residual
Y is the actual observation (the measured quantity)
G is the predicted observation based on a model of the measurement
$\Re$ is the measurement noise covariance (assumed to be diagonal and white noise N[0,$\Re$])
H is the Jacobian matrix of observations with respect to the state
A is the Jacobian matrix of the state derivatives with respect to the state
I is the identity matrix

Overbars represent time-updated quantities and carrots represent measurement-updated quantities. For a more detailed description of the batch processor, please refer to Tapley, Schutz, and Born, 2004. For our purposes, the state (based upon Eq. 2) is defined as follows;

$$\overline{X} = \begin{bmatrix} R_x \\ R_y \\ R_z \\ \dot{R}_x \\ \dot{R}_y \\ \dot{R}_z \end{bmatrix}_{ECI} \tag{9}$$

For Earth orbiting satellites, the Earth-Centered-Inertial coordinate frame of J2000[1] (ECI) is an accepted standard.

[1] Let the J2000 frame be a body-centered inertial reference frame defined by the location of the mean equator and equinox on January 1, 2000 12:00:00.000 Terrestrial Time (TT). This is equivalent to January 1, 2000 11:58:55.816 UTC.

2. METHODS AND PROCEDURES

The batch processor requires that several matrices be computed. One of them is the matrix containing the partial derivatives of the observations with respect to the state variables. Since we are interested in the relationship between our observations and our state, we need an appropriate Jacobian matrix defined as follows:

$$\widetilde{H} = \begin{bmatrix} \frac{\partial \alpha_t}{\partial R_x} & \frac{\partial \alpha_t}{\partial R_y} & \frac{\partial \alpha_t}{\partial R_z} & \frac{\partial \alpha_t}{\partial \dot{R}_x} & \frac{\partial \alpha_t}{\partial \dot{R}_y} & \frac{\partial \alpha_t}{\partial \dot{R}_z} \\ \frac{\partial \delta_t}{\partial R_x} & \frac{\partial \delta_t}{\partial R_y} & \frac{\partial \delta_t}{\partial R_z} & \frac{\partial \delta_t}{\partial \dot{R}_x} & \frac{\partial \delta_t}{\partial \dot{R}_y} & \frac{\partial \delta_t}{\partial \dot{R}_z} \end{bmatrix} \tag{10}$$

Considering that:

$$\sin(\delta_t) = \frac{a_z}{a} \tag{11}$$

Where

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{12}$$

If $\sqrt{a_x^2 + a_y^2} \neq 0$

$$\sin(\alpha_t) = \frac{a_y}{\sqrt{a_x^2 + a_y^2}} \qquad \cos(\alpha_t) = \frac{a_x}{\sqrt{a_x^2 + a_y^2}} \tag{13}$$

Else

$$\sin(\alpha_t) = \frac{\dot{a}_y}{\sqrt{\dot{a}_x^2 + \dot{a}_y^2}} \qquad \cos(\alpha_t) = \frac{\dot{a}_x}{\sqrt{\dot{a}_x^2 + \dot{a}_y^2}} \tag{14}$$

Thus

$$\tan(\alpha_t) = \frac{\sin(\alpha_t)}{\cos(\alpha_t)} \tag{15}$$

$(0^\circ \leq \alpha_t \leq 360^\circ)$

Knowing that

$$\frac{\partial}{\partial x} \tan^{-1}(u) = \frac{1}{1+u^2} \frac{\partial u}{\partial x} \qquad \frac{\partial}{\partial x} \sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \frac{\partial u}{\partial x} \tag{16}$$

Then, following differentiation and algebraic manipulation we have our results:

$$
\begin{aligned}
\frac{\partial \alpha_t}{\partial R_x} &= -\frac{\tan(\alpha_t)}{a_x + a_y \tan(\alpha_t)} \\
\frac{\partial \alpha_t}{\partial R_y} &= \frac{1}{a_x + a_y \tan(\alpha_t)} \\
\frac{\partial \alpha_t}{\partial R_z} &= 0 \\
\frac{\partial \alpha_t}{\partial \dot{R}_x} &= -\frac{\tau \tan(\alpha_t)}{a_x + a_y \tan(\alpha_t)} \\
\frac{\partial \alpha_t}{\partial \dot{R}_y} &= \frac{\tau}{a_x + a_y \tan(\alpha_t)} \\
\frac{\partial \alpha_t}{\partial \dot{R}_z} &= 0 \\
\frac{\partial \delta_t}{\partial R_x} &= -\frac{a_x \tan(\delta_t)}{a^2} \\
\frac{\partial \delta_t}{\partial R_y} &= -\frac{a_y \tan(\delta_t)}{a^2} \\
\frac{\partial \delta_t}{\partial R_z} &= \frac{\sqrt{1-\sin^2(\delta_t)}}{a} \\
\frac{\partial \delta_t}{\partial \dot{R}_x} &= -\frac{\tau a_x \tan(\delta_t)}{a^2} \\
\frac{\partial \delta_t}{\partial \dot{R}_y} &= -\frac{\tau a_y \tan(\delta_t)}{a^2} \\
\frac{\partial \delta_t}{\partial \dot{R}_z} &= \frac{\tau \sqrt{1-\sin^2(\delta_t)}}{a}
\end{aligned}
\tag{17}
$$

Because of the need to map computed quantities to the epoch, the state transition matrix must be calculated. In order to do this, a matrix of partial derivatives of the state derivatives with respect to the state is required. Assume that all the A matrix elements not defined are zero and that each defined matrix element is a scalar.

$$A = \frac{\partial \overline{F}}{\partial \overline{X}} = \frac{\partial \dot{\overline{X}}}{\partial \overline{X}} = \begin{bmatrix} \frac{\partial \dot{R}_x}{\partial R_x} & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\partial \dot{R}_x}{\partial \dot{R}_z} \\ \cdot & \cdot & & & & & \cdot \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ \frac{\partial \ddot{R}_z}{\partial R_x} & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\partial \ddot{R}_z}{\partial \dot{R}_z} \end{bmatrix} \tag{18}$$

$$A(1,4) = A(2,5) = A(3,6) = 1$$

For the partials of the gravitational accelerations with respect to the state, we have the following. Higher order gravity terms are not included because their effect is negligible within the integration time for the state transition matrix, as compared to the dominant zonal term:

$$A(4,1) = \frac{\partial \ddot{R}_x}{\partial R_x} = -\frac{\mu}{R^3}\left[1 - \frac{3}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(5\left(\frac{R_z}{R}\right)^2 - 1\right)\right] + 3\frac{\mu R_x^2}{R^5}\left[1 - \frac{5}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(7\left(\frac{R_z}{R}\right)^2 - 1\right)\right]$$

$$A(4,2) = \frac{\partial \ddot{R}_x}{\partial R_y} = 3\frac{\mu R_x R_y}{R^5}\left[1 - \frac{5}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(7\left(\frac{R_z}{R}\right)^2 - 1\right)\right]$$

$$A(4,3) = \frac{\partial \ddot{R}_x}{\partial R_z} = 3\frac{\mu R_x R_z}{R^5}\left[1 - \frac{5}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(7\left(\frac{R_z}{R}\right)^2 - 3\right)\right]$$

$$A(5,1) = \frac{\partial \ddot{R}_y}{\partial R_x} = 3\frac{\mu R_x R_y}{R^5}\left[1 - \frac{5}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(7\left(\frac{R_z}{R}\right)^2 - 1\right)\right]$$

$$A(5,2) = \frac{\partial \ddot{R}_y}{\partial R_y} = -\frac{\mu}{R^3}\left[1 - \frac{3}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(5\left(\frac{R_z}{R}\right)^2 - 1\right)\right] + 3\frac{\mu R_y^2}{R^5}\left[1 - \frac{5}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(7\left(\frac{R_z}{R}\right)^2 - 1\right)\right]$$

$$A(5,3) = \frac{\partial \ddot{R}_y}{\partial R_z} = 3\frac{\mu R_y R_z}{R^5}\left[1 - \frac{5}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(7\left(\frac{R_z}{R}\right)^2 - 3\right)\right]$$

$$A(6,1) = \frac{\partial \ddot{R}_z}{\partial R_x} = 3\frac{\mu R_x R_z}{R^5}\left[1 - \frac{5}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(7\left(\frac{R_z}{R}\right)^2 - 3\right)\right]$$

$$A(6,2) = \frac{\partial \ddot{R}_z}{\partial R_y} = 3\frac{\mu R_y R_z}{R^5}\left[1 - \frac{5}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(7\left(\frac{R_z}{R}\right)^2 - 3\right)\right]$$

$$A(6,3) = \frac{\partial \ddot{R}_z}{\partial R_z} = -\frac{\mu}{R^3}\left[1 - \frac{3}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(5\left(\frac{R_z}{R}\right)^2 - 3\right)\right] + 3\frac{\mu R_z^2}{R^5}\left[1 - \frac{5}{2}\mathbf{J}_2\left(\frac{R_\oplus}{R}\right)^2\left(7\left(\frac{R_z}{R}\right)^2 - 5\right)\right].$$

3. SUMMARY AND CONCLUSIONS

This work presents the process of formulating the batch processor or differential correction algorithm in order to refine an epoch state, based upon the reduction of angles data.

4. REFERENCES

Vallado, D. (1997). *Fundamentals of Astrodynamics and Applications.* McGraw Hill Companies, Inc., New York, NY.

Tapley, B, Schutz, B., Born, G. (2004). *Statistical Orbit Determination*. Elsevier Academic press, Burlington, MA.