Øving 6 - Matte 4

Different from the convolution in Laplace transform, the convolution in Fourier transform is defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(u)\, g(x-u)\, du \qquad (1)$$

We have the following Fourier Convolution formula

$$\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g) \qquad (2)$$

Where $\mathcal{F}(f)$ denotes the Fourier transform of f.

1)

$$\int_{-\infty}^{\infty} f(p) \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-p)^2}{2}}\, dp = \int_{\mathbb{R}} f(u)\, g(x-u)\, du$$

$$= (f * g)(x)$$

Fourier transform:

$$\Rightarrow \quad \mathcal{F}(g)(\omega) = \frac{1}{\sqrt{2\pi}} \mathcal{F}(e^{-\frac{x^2}{2}})(\omega)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}}$$

$$\mathcal{F}(e^{-\frac{x^2}{4}})(\omega) = \sqrt{2}\, e^{-\omega^2}$$

Which gives us:

$$\tilde{\hat{f}}(\omega) e^{-\frac{\omega^2}{2}} = \mathcal{F}(f * g)(\omega)$$

$$= \mathcal{F}(e^{-\frac{x^2}{4}})(\omega)$$

$$= \sqrt{2}\, e^{-\omega^2}$$

$$\Rightarrow \quad \tilde{\hat{f}}(\omega) = \sqrt{2}\, e^{-\frac{\omega^2}{2}}$$

$$\overset{inv}{\Rightarrow} \quad \underline{\underline{f(x) = \sqrt{2}\, e^{-\frac{x^2}{2}}}}$$

2)

Solve the following wave equation:

$$u_{tt} = u_{xx}$$

with boundary conditions:

$$u(t,0) = u(t,\pi) = 0 \quad , \quad \forall t \geq 0$$

and initial conditions:

$$u(0,x) = \sin x , \quad u_t(0,x) = \sin 3x \quad , \quad \forall \; 0 \leq x \leq \pi$$

Solutions of the form $u(t,x) = G(t)F(x)$ wher $G \neq 0 \wedge F \neq 0$

in the wave equation, gives:

$$G''(t)F(x) = G(t)F''(x)$$

$$\Rightarrow \frac{G''(t)}{G(t)} = \frac{F''(x)}{F(x)} = k \qquad \text{where} \quad k \in \mathbb{R}$$

We know from the bounding conditions that

$$G(t)F(0) = 0$$

and

$$G(t)F(\pi) = 0$$

For $k = 0$ we have $F'' = 0 \Rightarrow F(x) = ax + b$
this is not possible because our bounding constraints.

For $k < 0$:

$$\Rightarrow k = -\mu^2$$

$$\Rightarrow F''(x) + \mu^2 F(x) = 0$$

With the general solution:

$$F(x) = C\cos \mu x + \tilde{C}\sin \mu x$$

Now since our boundary conditions give us that

$$F(0) = 0 \quad \text{and} \quad F(\pi) = 0$$

we know that $C = 0$ and $\sin \mu\pi = 0$

Thus we know that

$$F_n(x) = A_n \sin nx \qquad \text{for } n \in \mathbb{N}$$

Now the second diff. equation

$$G'' + n^2 G = 0$$

gives us

$$G_n = B_n \cos nt + C_n \sin nt$$

Given this we know that

$$u_n(t,x) = (B_n \cos nt + C_n \sin nt)\sin nx \qquad \text{for } n \in \mathbb{N}$$

satisfies the wave equation, along with any linear combination

Pluging in the initial conditions:

$$u(0,x) = \sin x \implies \sum_{n=1}^{\infty} B_n \sin nx = \sin nx \implies \begin{matrix} B_n = 0 \\ B_1 = 1 \end{matrix}$$

$$u_t(0,x) = \sin 3x \implies \sum_{n=1}^{\infty} nC_n \sin nx = \sin 3x \implies \begin{matrix} C_n = 0 \\ C_3 = 1/3 \end{matrix}$$

Adding all of this up, we get:

$$u(t,x) = \cos(t)\sin(x) + \frac{1}{3}\sin(3t)\sin(3x)$$

4)

$$u(t,x) = \frac{1}{\sqrt{2\pi t}}\left(f * e^{-\frac{x^2}{2t}}\right) = \int_{-\infty}^{\infty} f(p)\,\frac{1}{\sqrt{2\pi t}}\, e^{\frac{-(x-p)^2}{2t}}\,dp$$

a)

b)

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}}\, e^{\frac{-(x-p)^2}{2t}}\,dp = \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-p)^2}{2t}}\,dp$$

Substituting $\sqrt{ts} = (p-x) \Rightarrow$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{s^2}{2}}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\sqrt{2\pi}$$

$$= 1 \quad \square$$