Øving 3 - Matematikk 40

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1) Try to verify the following computations

a) The Laplace transform of

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq a \\ 0 & \text{if } t > a \end{cases}$$

is

$$F(s) = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}$$

$$\mathcal{L}(f(t)) = \int_0^a e^{-st} \cdot t \, dt = \int_{t=0}^{t=a} e^u \cdot t\left(-\frac{1}{s}\right) du$$

Variabelskifte med
$u = -st \Rightarrow u' = -s$

$$= \int_{t=0}^{t=a} \frac{e^u u}{s^2} \, du$$

$$= \frac{1}{s^2} \int_{t=0}^{t=a} e^u u \, du$$

$$= \left[\frac{1}{s^2}\left(u e^u - e^u\right)\right]_{t=0}^{t=a}$$

$$= \left[\frac{1}{s^2}\left(-se^{-st}t - e^{-st}\right)\right]_0^a$$

$$= \left(\frac{1}{s^2}\left(-se^{-sa}a - e^{-sa}\right)\right) - \left(\frac{1}{s^2}\left(-se^{0} \cdot 0 - e^0\right)\right)$$

$$= \frac{1}{s^2} \cdot \left(-se^{-sa}a\right) - \frac{1}{s^2} \cdot e^{-sa} - \frac{1}{s^2} \cdot (-1)$$

$$= \frac{-e^{-sa}a}{s} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2} - \frac{e^{-sa}}{s^2} - a\frac{e^{-sa}}{s}$$

c)

The solution of $i(t)$ of

$$i'(t) + 2i(t) + \int_0^t i(\tau)\,d\tau = \delta(t-1), \quad i(0) = 0$$

is

$$i(t) = u(t-1)\left(e^{-(t-1)} - e^{-(t-1)}(t-1)\right)$$

$$\overset{\mathcal{L}}{\Rightarrow} sI - 0 + 2I + \tfrac{1}{s}I = e^{-s}$$

$$I\left(s + 2 + \tfrac{1}{s}\right) = e^{-s}$$

$$\Rightarrow I = \frac{e^{-s}}{s + 2 + \frac{1}{s}}$$

$$= e^{-s}\,\frac{s}{s^2 + 2s + 1}$$

$$i = u(t-1) \cdot \mathcal{L}^{-1}\left(\frac{s}{s^2+2s+1}\right) \qquad \text{der } t = t-1$$

$$\frac{s}{s^2+2s+1} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$s = As + A + B \Rightarrow A = 1 \;\wedge\; B = -1$$

$$\Rightarrow i(t) = u(t-1) \cdot \mathcal{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right) \qquad \text{der } t = t-1$$

$$= u(t-1)\left(e^{-(t-1)} - e^{-(t-1)}(t-1)\right) \quad \square$$

2) $y - y * t = t$

$\overset{\mathcal{L}}{\Rightarrow} \mathcal{L}(y - y * t) = \mathcal{L}(t)$

$\Rightarrow Y - Y \cdot \frac{1}{s^2} = \frac{1}{s^2}$

$\Rightarrow Y(1 - \frac{1}{s^2}) = Y(\frac{s^2-1}{s^2}) = \frac{1}{s^2}$

$Y = \dfrac{1/s^2}{\frac{s^2-1}{s^2}} = \frac{1}{s^2} \cdot \frac{s^2}{s^2-1} = \frac{1}{s^2-1}$

$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}(\frac{1}{s^2-1}) = \underline{\underline{-\sin(t)}}$

3) $\begin{cases} x' = 2x - y \\ y' = 3x - 2y \end{cases}$

$x(0) = 0 \quad y(0) = 1$

$\overset{\mathcal{L}}{\Rightarrow} sX - x(0) = 2X - Y \Rightarrow Y = X(2-s)$

$sY - y(0) = 3X - 2Y \Rightarrow 3X = Y(s+2) - 1$

$X = \frac{1}{3}(X(2-s))(s+2) - \frac{1}{3} \Leftrightarrow \frac{3X+1}{X} = 4 - s^2$

$\Rightarrow \frac{3X}{X} + \frac{1}{X} = 4 - s^2$

$\Rightarrow \frac{1}{X} = 1 - s^2$

$\Rightarrow X = \frac{-1}{s^2-1}$

$Y = X(2-s)$

$3X = Y(s+2) - 1 \qquad \Rightarrow X = \frac{-1}{s^2-1}$

$Y = \frac{-1}{s^2-1}(2-s) = \frac{s-2}{s^2-1} = \frac{s}{s^2-1} - \frac{2}{s^2-1}$

$= \frac{s}{s^2-1} + 2\frac{-1}{s^2-1}$

$x = \mathcal{L}^{-1}\left(\frac{-1}{s^2-1}\right) = \frac{1}{i}\sin(it)$

$y = \mathcal{L}^{-1}\left(\frac{s}{s^2+i^2}\right) + \frac{2}{i}\mathcal{L}^{-1}\left(\frac{i}{s^2+i^2}\right)$

$\underline{\underline{y = \cos(it) + \frac{2}{i}\sin(it)}}$

$\underline{\underline{x = \frac{1}{i}\sin(it)}}$

4)

a) $x = \sum_{n\neq 0} \frac{i(-1)^n}{n} e^{inx}$ when $-\pi < x < \pi$

$f(x) = x$

Kompleks Fourierserie av f

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}, \quad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

Finner C_n:

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left(\frac{-1}{in} x e^{-inx} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-inx}\,dx\right)$$

$$= \frac{1}{2\pi}\left[\left(\frac{-1}{in}x - \frac{1}{(in)^2}\right)e^{-inx}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{n^2} - \frac{x}{in}\right]_{-\pi}^{\pi}(-1)^n$$

$$= \frac{1}{2\pi}\left(\frac{1}{n^2} - \frac{\pi}{in} - \left(\frac{1}{n^2} + \frac{\pi}{in}\right)\right)(-1)^n$$

$$= -\frac{(-1)^n}{in}\cdot\frac{i}{i}$$

$$= \underline{\underline{\frac{i(-1)^n}{n}}}$$

$$\Rightarrow f(x) = x = \sum_{n=-\infty}^{\infty} C_n e^{inx} = \underline{\underline{\sum_{n\neq 0} \frac{i(-1)^n}{n} e^{inx}}} \qquad \square$$

b)

$$f(x) = x(2\pi - x)$$

$$f(x) = \sum C_n e^{inx}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(2\pi - x)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left(2\pi\int_{-\pi}^{\pi} xe^{-inx}\,dx - \int_{-\pi}^{\pi} x^2 e^{-inx}\,dx\right)$$

$$= \frac{1}{2\pi}\left(2\pi\,\frac{2\pi i(-1)^n}{n} + \frac{x^2}{in}e^{-inx} - \frac{2}{in}\cdot\frac{2\pi i(-1)^n}{n}\right]_{-\pi}^{\pi}$$

$$= \frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{n+1}}{n^2}$$

$$C_0 = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(2\pi - x)\,dx$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} 2\pi x\,dx - \int_{-\pi}^{\pi} x^2\,dx\right)$$

$$= \frac{1}{2\pi}\left(-\frac{1}{3}\pi^3 - \left(-\frac{1}{3}(-\pi)^3\right)\right)$$

$$= -\frac{\pi^2}{3}$$

$$\Rightarrow f(x) \approx -\frac{\pi^2}{3} + \sum_{n\neq 0}^{\infty}\left(\frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{n+1}}{n^2}\right)e^{inx} \qquad \text{for } -\pi < x < \pi$$

$\square$