TMA 4135

1)

$$f(x) = x^3 - x^2 - x + 1 = 0 \qquad f'(x) = 3x^2 - 2x - 1$$

$r_1 = -1 \qquad r_2 = +1$

a) $x_0 = 2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Iteration scheme

Iteration	Result
1	$x_1 = 1.5714$
2	$x_2 = 1.3143$
3	$x_3 = 1.1671$

b)

Based upon the theory developed in the lecture, we expect a convergence order of $p = 2$ (Quadratic convergence) for the iterations starting with $x_0 = 2$ and $x_0 = -2$ respectively.

c)

Using the code from "Preliminary.ipynb" we verify the convergence orders numerically:

```python
from math import sqrt, log
# Order of convergence for iterations
# Test problem: Newtons method for x^2-a=0
a = 4


def g(x):
    # return (x**2+a)/(2*x)   # g(x) = x-f(x)/f'(x)
    return (x-(x**3-x**2-x+1)/(3*x**2-2*x-1))   # g(x) = x-f(x)/f'(x)


x_exact = -1            # Exact solution
x = -2                       # Starting value
errors = [abs(x_exact-x)]          # Array to store errors
Nit = 10                     # Number of iterations

# Start the iterations
print('The Newton iterations:')
for k in range(Nit):
    x = g(x)                 # One iteration
    ek = abs(x_exact-x)      # Find the error
    print('k = {:2d},   x_k = {:10.8f},   e_k = {:8.2e}'.format(k, x, ek))
    if ek < 1.e-15:          # If the error is small, terminate.
        Nit = k+1
        break
    errors.append(ek)             # Append the new error to the array of errors

# Find the order and the error constant C
print('\nThe order p and the error constant C')
for k in range(Nit-2):
    p = log(errors[k+2]/errors[k+1])/log(errors[k+1]/errors[k])
    C = errors[k+2]/errors[k+1]**p
    print('k = {:2d},  p = {:4.2f},  C = {:6.4f}'.format(k, p, C))
```

```
The Newton iterations:
k =  0,   x_k = 1.57142857,   e_k = 5.71e-01
k =  1,   x_k = 1.31428571,   e_k = 3.14e-01
k =  2,   x_k = 1.16713460,   e_k = 1.67e-01
k =  3,   x_k = 1.08667011,   e_k = 8.67e-02
k =  4,   x_k = 1.04421671,   e_k = 4.42e-02
k =  5,   x_k = 1.02234490,   e_k = 2.23e-02
k =  6,   x_k = 1.01123383,   e_k = 1.12e-02
k =  7,   x_k = 1.00563256,   e_k = 5.63e-03
k =  8,   x_k = 1.00282023,   e_k = 2.82e-03
k =  9,   x_k = 1.00141111,   e_k = 1.41e-03

The order p and the error constant C
k =  0,  p = 1.07,  C = 0.5714
k =  1,  p = 1.06,  C = 0.5676
k =  2,  p = 1.04,  C = 0.5569
k =  3,  p = 1.02,  C = 0.5421
k =  4,  p = 1.01,  C = 0.5281
k =  5,  p = 1.01,  C = 0.5174
k =  6,  p = 1.00,  C = 0.5103
k =  7,  p = 1.00,  C = 0.5059
```

```
The Newton iterations:
k =  0,   x_k = -1.40000000,   e_k = 4.00e-01
k =  1,   x_k = -1.10000000,   e_k = 1.00e-01
k =  2,   x_k = -1.00869565,   e_k = 8.70e-03
k =  3,   x_k = -1.00007464,   e_k = 7.46e-05
k =  4,   x_k = -1.00000001,   e_k = 5.57e-09
k =  5,   x_k = -1.00000000,   e_k = 0.00e+00

The order p and the error constant C
k =  0,  p = 1.51,  C = 0.4000
k =  1,  p = 1.76,  C = 0.5024
k =  2,  p = 1.95,  C = 0.7716
k =  3,  p = 2.00,  C = 0.9746
```

2a)

	x_0	x_1	x_2	x_3
x_i	-2	-1	1	2
$f(x_i)$	-7	0	2	9

<u>Using lagrange interpolation</u>

The interpolation polynomial is:

$$L(x) := \sum_{j=0}^{k} y_j \, \ell_j(x)$$

where:

$$\ell_j(x) := \prod_{\substack{0 \le m \le k \\ m \ne j}} \frac{x - x_m}{x_j - x_m}$$

$$\ell_0(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)\left(\frac{x - x_3}{x_0 - x_3}\right) = \left(\frac{x+1}{-2+1}\right)\left(\frac{x-1}{-2-1}\right)\left(\frac{x-2}{-2-2}\right)$$

As $y_1 = 0$ we don't have to calculate $\ell_1(x)$.

$$\ell_2(x) = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right) = \left(\frac{x+2}{1+2}\right)\left(\frac{x+1}{1+1}\right)\left(\frac{x-2}{1-2}\right)$$

$$\ell_3(x) = \left(\frac{x - x_0}{x_3 - x_0}\right)\left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right) = \left(\frac{x+2}{2+2}\right)\left(\frac{x+1}{2+1}\right)\left(\frac{x-1}{2-1}\right)$$

This gives us the interpolation polynomial:

$$L(x) = -7\left(\frac{x+1}{-2+1}\right)\left(\frac{x-1}{-2-1}\right)\left(\frac{x-2}{-2-2}\right) + 0 \cdot \ell_1(x)$$
$$+ 2\left(\frac{x+2}{1+2}\right)\left(\frac{x+1}{1+1}\right)\left(\frac{x-2}{1-2}\right) + 9\left(\frac{x+2}{2+2}\right)\left(\frac{x+1}{2+1}\right)\left(\frac{x-1}{2-1}\right)$$

$$= -7\left(\frac{x^3 - 2x^2 - x + 2}{-12}\right) + 2\left(\frac{x^3 + x^2 - 4x - 4}{-6}\right) + 9\left(\frac{x^3 + 2x^2 - x - 2}{12}\right)$$

$$= \left(\frac{7}{12}x^3 - \frac{7}{6}x^2 - \frac{7}{12}x + \frac{7}{6}\right) + \left(-\frac{1}{3}x^3 - \frac{1}{3}x^2 + \frac{4}{3}x + \frac{4}{3}\right) + \left(\frac{3}{4}x^3 + \frac{3}{2}x^2 - \frac{3}{4}x - \frac{3}{2}\right)$$

$$= \underline{\underline{x^3 + 1}}$$

b) Our results in (a) gives us this approximation of $f(0)$:

$$f(x) \approx x^3 + 1 \Rightarrow f(0) \approx 0^3 + 1 = \underline{\underline{1}}$$

3) a)

	x_0	x_1	x_2
x_i	-1	0	1
y_i	2	0	0

$$f(x_0) = 2 \Rightarrow c_0 = 2$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{0-2}{0--1} = -2 \quad \Rightarrow c_1 = -2$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0}{1} = 0$$

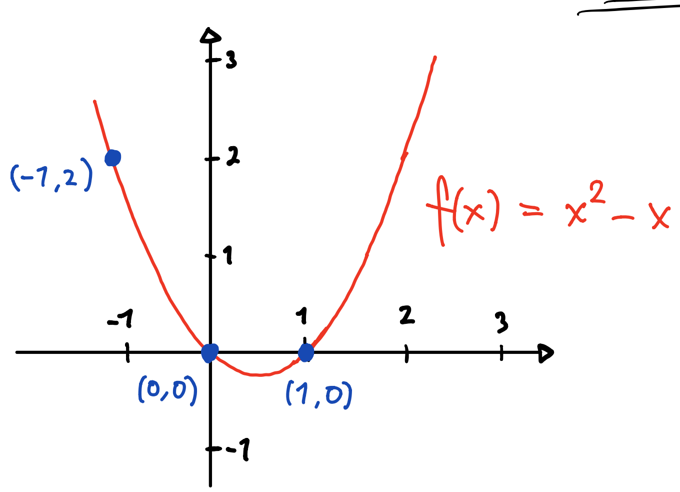
$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{0-(-2)}{1-(-1)} = \frac{2}{2} = 1 \quad \Rightarrow c_2 = 1$$

Which gives us this table of divided differences:

$$\begin{array}{c|cccc} -1 & 2 & & \\ & & -2 & \\ 0 & 0 & & 1 \\ & & 0 & \\ 1 & 0 & & \end{array}$$

And the second order interpolation polynomial in the Newton form:

$$\begin{aligned} p_2(x) &= c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) \\ &= 2 + (-2)(x+1) + (x+1)(x) \\ &= x^2 + x - 2x - 2 + 2 \\ &= x^2 - x \end{aligned}$$



b)

	x_0	x_1	x_2	x_3
x_i	-1	0	1	2
y_i	2	0	0	-4

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{-4-(0)}{2-1} = -4$$

$$f[x_1, x_2, x_3] = \frac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1} = \frac{-4-0}{2-0} = -\frac{4}{2} = -2$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0} = \frac{-2-1}{2-(-1)} = \frac{-3}{3} = -1 \quad \Rightarrow \quad C_3 = -1$$

Which gives us this table of divided differences:

$$\begin{array}{r|lllll}
-1 & 2 & & & \\
 & & -2 & & \\
0 & 0 & & 1 & \\
 & & 0 & & -1 \\
1 & 0 & & -2 & \\
 & & -4 & & \\
2 & -4 & & &
\end{array}$$

And the interpolation polynomial:

$$p_3(x) = C_0 + C_1(x-x_0) + C_2(x-x_0)(x-x_1) + C_3(x-x_0)(x-x_1)(x-x_2)$$

$$= 2 + (-2)(x+1) + (x+1)(x) - (x+1)(x)(x-1)$$

$$= 2 - 2x - 2 + x^2 + x - (x^2 + x)(x-1)$$

$$= 2 - 2x - 2 + x^2 + x - x^3 + x^2 - x^2 + x$$

$$\underline{\underline{= -x^3 + x^2}}$$