1

We have for Gram-Schmidt ortogonalization:

$$p_k = \phi_k - \sum_{j=0}^{k-1} \frac{(\phi_k, p_j)}{\|p_j\|^2} p_j .$$

a) Using Gram-Schmidt orthogonalization to construct p_2:

$$(\phi_2, p_0) = \int_{-1}^{1} \phi_2 \ p_0(x) \, dx = \int_{-1}^{1} x^2 \cdot 1 \ dx = \left[\tfrac{1}{3} x^3\right]_{-1}^{1} = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$

$$(\phi_2, p_1) = \int_{-1}^{1} \phi_2 \ p_1(x) \, dx = \int_{-1}^{1} x^2 \cdot x \ dx = \left[\tfrac{1}{4} x^4\right]_{-1}^{1} = \tfrac{1}{4} - \tfrac{1}{4} = 0$$

$$\|p_0\|^2 = \int_{-1}^{1} p_0(x)^2 \, dx = \int_{-1}^{1} 1 \, dx = 2$$

$$\|p_1\|^2 = \int_{-1}^{1} p_1(x)^2 \, dx = \int_{-1}^{1} (x)^2 \, dx = \tfrac{2}{3}$$

Now we can construct:

$$p_2 = \phi_2 - \sum_{j=0}^{1} \frac{(\phi_2, p_j)}{\|p_j\|^2} p_j .$$

$$= x^2 - \left(\frac{(\phi_2, p_0)}{\|p_0\|^2} p_0 + \frac{(\phi_2, p_1)}{\|p_1\|^2} p_1 \right)$$

$$= x^2 - \left(\frac{\frac{2}{3}}{2} + \frac{0}{\frac{4}{9}} \cdot x \right)$$

$$= x^2 - \frac{1}{3}$$

Now we repeat for p_3:

$$(\phi_3, p_0) = \int_{-1}^{1} \phi_3 \cdot p_0(x) \, dx = \int_{-1}^{1} x^3 \, dx = \left[\tfrac{1}{4} x^4\right]_{-1}^{1} = 0$$

$$(\phi_3, p_1) = \int_{-1}^{1} \phi_3 \cdot p_1(x) \, dx = \int_{-1}^{1} x^4 = \left[\tfrac{1}{5} x^5\right]_{-1}^{1} = \tfrac{2}{5}$$

$$(\emptyset_3, p_2) = \int_{-1}^{1} \emptyset_3 \cdot p_2(x)\, dx = \int_{-1}^{1} x^3 (x^2 - \tfrac{1}{3}) = \int_{-1}^{1} x^5 - \tfrac{1}{3}x^3 \, dx = \left[\tfrac{1}{6}x^6 - \tfrac{1}{12}x^4\right]_{-1}^{1} = 0$$

We already calculated that

$$\|p_1\|^2 = \frac{2}{3}$$

Now we can construct p_3:

$$p_3 = \emptyset_3 - \sum_{j=0}^{2} \frac{(\emptyset_3, p_j)}{\|p_j\|^2} p_j .$$

$$= x^3 - \left(\frac{(\emptyset_3, p_0)}{\|p_0\|^2} p_0 + \frac{(\emptyset_3, p_1)}{\|p_1\|^2} p_1 + \frac{(\emptyset_3, p_2)}{\|p_2\|^2} p_2 \right)$$

$$= x^3 - \left(\frac{0}{\|p_0\|^2} p_0 + \frac{3/5}{2/3} x + \frac{0}{\|p_2\|^2} p_2 \right)$$

$$\underline{\underline{= x^3 - \tfrac{3}{5}x}}$$

b)

Now we want to find the 3 roots of p_3

$$p_3 = x^3 - \tfrac{3}{5}x = 0$$

$$x^3 - \tfrac{3}{5}x = 0$$

$$x(x^2 - \tfrac{3}{5}) = 0$$

Now we know that $x = 0$ is one root, and to find the 2 other we need to solve

$$x^2 - \tfrac{3}{5} = 0$$
$$x^2 = \tfrac{3}{5}$$

$$\Rightarrow \quad x = \pm\sqrt{\tfrac{3}{5}}$$

Giving us the 3 roots:

$$\underline{x_1 = -\sqrt{\tfrac{3}{5}}}, \quad \underline{x_2 = 0}, \quad \underline{x_3 = \sqrt{\tfrac{3}{5}}}$$

L_1

2

Considering the initial value problem:

$$y' - xy^2 = 0 \quad , \quad y(0) = 1$$

a)

$$y' - xy^2 = 0$$

$$\frac{1}{y^2} y' = x$$

$$-\frac{1}{y} = \frac{x^2}{2} + C_1$$

Inserting $y(0) = 1$:

$$\Rightarrow -\frac{1}{1} = \frac{0^2}{2} + C_1$$

$$\Rightarrow -1 = C_1$$

Inserting $C_1 = -1$ back into the equation:

$$-\frac{1}{y} = \frac{x^2}{2} + C_1$$

$$-\frac{1}{y} = \frac{x^2}{2} - 1$$

$$\underline{\underline{y = -\frac{2}{x^2 - 2}}}$$

Now computing for $x = 0.3$:

$$y(0.3) = -\frac{2}{0.3^2 - 2} = \frac{200}{191} \approx 1.04712$$

b) Eulers method with $h=0.1$:

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \qquad y' = f(x,y) = xy^2$$

For $n=0$:

$$y_1 = y_0 + h\, f(x_0, y_0)$$

We have

$$y(0) = 1 \Rightarrow y_0 = 1, \ x_0 = 0$$

which gives us

$$y_1 = 1 + 0.1 \cdot xy^2 = 1$$

Now for $n=1$:

$$y_2 = y_1 + h \cdot x_1 y_1^2$$
$$= 1 + 0.1 \cdot (0.1)\, 1^2$$
$$= 1.01$$

and $n=2$:

$$y_3 = y_2 + 0.1 \cdot x_2 y_2^2$$
$$= 1.01 + 0.1(0.2)(1.01)^2$$
$$= \underline{\underline{1.0304}}$$

after 3 iterations of Eulers Method.

The error at the last step:

$$e_3 = |y_3 - y(0.3)| = |1.0304 - 1.04712| = \underline{\underline{0.01672}}$$

c)

Heuns method with $h = 0.15$, $y_0 = 1$, $x_0 = 0$

$$u_{n+1} = y_n + hf(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, u_{n+1})\right) \qquad \text{wher} \qquad f(x,y) = y' = xy^2$$

Now for $n=0$ we have

$$u_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.15 \cdot 0 \cdot 1^2$$
$$= 1$$

$$y_1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, u_1)\right)$$
$$= 1 + \frac{0.15}{2}\left(0 \cdot 1^2 + 1 \cdot 1^2\right)$$
$$= 1.075$$

for $n = 1$:

$$u_2 = y_1 + hf(x_1, x_2)$$
$$= 1.075 + 0.15\,(0.15 \cdot 1.075^2)$$
$$= 1.101$$

$$y_2 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, u_2)\right)$$
$$= 1.075 + \frac{0.15}{2}\left(0.15 \cdot 1.075^2 + 0.30 \cdot 1.101^2\right)$$
$$\underline{\underline{= 1.11528}}$$

After 2 iterations of Heuns method

Error at the last step:

$$e_2 = |y_2 - y(0.3)| = |1.11528 - 1.04712| = \underline{\underline{0.06816}}$$

d)

Runge-Kutta method with $h = 0.3$, $y_0 = 1$, $x_0 = 0$

$$k_1 = 0 \cdot 1^2 = 0$$

$$k_2 = \left(0 + \frac{0.3}{2}\right)\left(1 + \frac{0.3}{2} \cdot 0\right)^2 = 0.15$$

$$k_3 = \left(0 + \frac{0.3}{2}\right)\left(1 + \frac{0.3}{2} \cdot 0.15\right)^2 = 0.15683$$

$$k_4 = (0 + 0.3)(1 + 0.3 \cdot 0.15683)^2 = 0.32889$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{0.3}{6}(0 + 2 \cdot 0.15 + 2 \cdot 0.15683 + 0.32889)$$

$$= \underline{\underline{1.04713}}$$

The error:

$$e_1 = |y_1 - y(0.3)| = |1.04713 - 1.04712| = \underline{\underline{0.00001}}$$

In my case the Runge-Kutta method performed best, by far!