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An approximation for Student's *t*-distribution

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SUMMARY

The tail area of the *t*-distribution is approximated by the reciprocal of a polynomial, the coefficients being smooth functions of the reciprocal of the degrees of freedom.

Sometimes it is convenient to have a numerical approximation to the tail probability of Student's *t*-distribution at various degrees of freedom. An obvious approach is to assume the form used by Hastings (1955) to approximate the tail probability of the normal distribution, viz. a polynomial raised to a large negative power, and to choose the coefficients to minimize the maximum error. If this is done, it is found

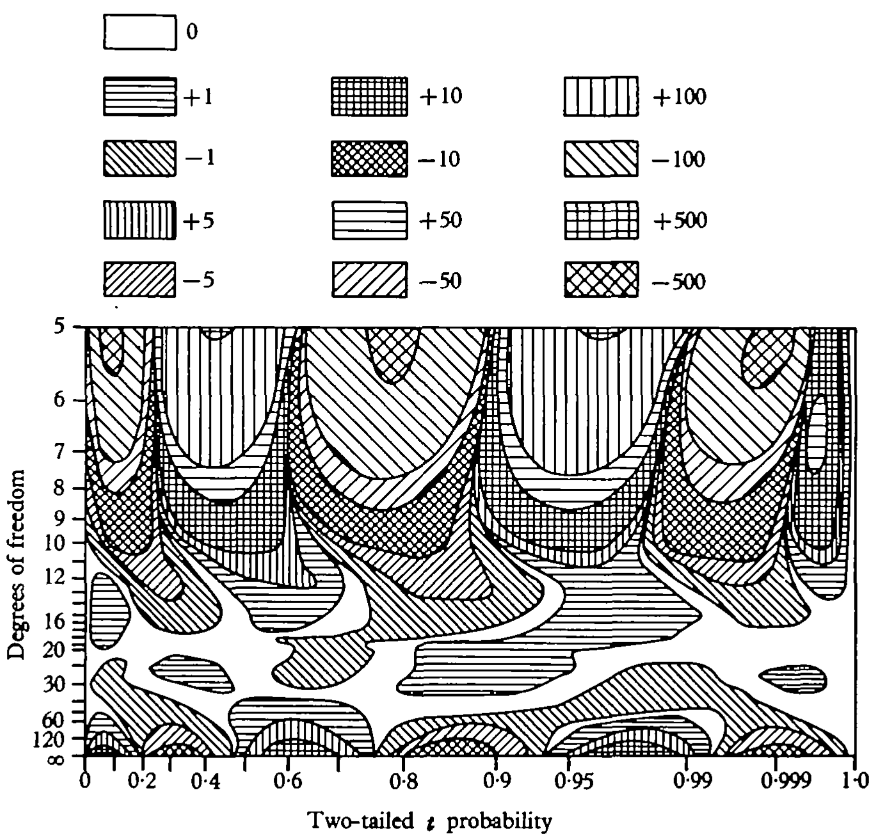

Fig. 1. Error contours for approximation to two-tailed Student's *t*-probability. Units are 10^{-6}.

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that the coefficients of these Chebyshev approximations are themselves smooth functions of reciprocal degrees of freedom and may be well approximated by polynomials or rational functions.

After some experimentation, we found the most satisfactory approximation to the tail probability for more than five degrees of freedom to be a fifth degree polynomial raised to the minus eighth power. The coefficients were fitted by the exchange algorithm using unpublished tables by Harter as known values for the function. These coefficients are not given here because it was found that the approximation to them (shown in Table 1), in the form of a quartic divided by a quadratic, gives errors of the same order and requires fewer tabled constants. The error contours corresponding to these coefficients are shown in Fig. 1. For more than 10 degrees of freedom the approximation gives an accuracy of 5 decimal places and at 5 degrees of freedom the approximation is still usable, although the error is now as big as 6×10^{-4}. Observe that for each value of ν, the ripples in the error contour are very nearly of equal size, indicating how close we are to the true minimum approximation.

Table 1. *Coefficients used in the approximation to the t-distribution*

i	a_{i0}	a_{i1}	a_{i2}	a_{i3}	a_{i4}	b_{i1}	b_{i2}
1	0·09979441	−0·5818210	1·390993	−1·222452	2·151185	−5·537409	11·42343
2	0·04431742	−0·2206018	−0·03317253	5·679969	−12·96519	−5·166733	13·49862
3	0·009694901	−0·1408854	1·889930	−12·75532	25·77532	−4·233736	14·39630
4	−0·00009187228	0·03789901	−1·280346	9·249528	−19·08115	−2·777816	16·46132
5	0·0005796020	−0·02763334	0·4517029	−2·657697	5·127212	−0·5657187	21·83269

We use
$$P(|t| < X) = 1-(c_5 X^5 + c_4 X^4 + c_3 X^3 + c_2 X^2 + c_1 X + 1)^{-8},$$
where $0 \leqslant X \leqslant \infty$, ν is the degrees of freedom and
$$c_i = \frac{a_{i4}\nu^{-4} + a_{i3}\nu^{-3} + a_{i2}\nu^{-2} + a_{i1}\nu^{-1} + a_{i0}}{b_{i2}\nu^{-2} + b_{i1}\nu^{-1} + 1}.$$

If we desire to approximate the inverse of this function, i.e. the fractiles of the t-distribution, we can use an inverse interpolation technique due to Tukey, which employs a good forward approximation to improve a crude inverse. Tukey's scheme is: if $F(x)$ is an accurate forward approximation to a function, but ϕ is only a poor approximation to the inverse function, we can obtain a better inverse
$$\phi^*(y) = \phi[2y - F\{\phi(y)\}].$$
This may be interpreted as suggesting that near y the error in ϕ can be regarded as being equivalent to a constant offset in y. Denoting $F\{\phi(y)\}$ by y', we can estimate this offset by $y' - y$ and correct for it by evaluating ϕ at $y - (y' - y)$. This technique can be applied recursively by considering the inverse function just generated as being itself a crude inverse, which could be improved.

Since the logarithm of the tail probability is a better quantity to interpolate in than the tail probability itself, we actually use
$$\phi_i(p) = \phi_{i-1}[p^2/F\{\phi_{i-1}(p)\}],$$
where $\phi_0(p)$ is Hastings's inverse for the normal distribution, corrected to have variance $\nu/(\nu-2)$, and where F is the approximation from the first part of this paper. Now ϕ_1, the first use of Tukey's scheme, provides 5 decimal places in the fractile from $p = 0·1$ to $0·001$ for 9 or more degrees of freedom, and ϕ_2, the first recursive usage, provides 5 decimal places over the same range for 5 or more degrees of freedom.

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