Gaffke (2005) proposes the following test of

$H_0: \mu \leq 1$

$H_A: \mu > 1$

with $\mu := \mathbb{E}X$ and $X \geq 0$:

Given a sample $X_1, \dots, X_n$, reject the null at nominal level α when $K(X_1, \dots, X_n) < \alpha$, where $K(x_1, \dots, x_n)$ is given (eqn (10)) by

(1) $$K(x_1, \dots, x_n) = \mathbb{P}\left(\sum_{j=1}^{n} (x_j - 1) Z_j \leq Z_0 \right),$$ and

$Z_0, \dots, Z_n$ are iid exponential, ie, $F_{Z_j}(x) = 1 - e^{-x}$.

So, $$K(x_1, \dots, x_n) = \mathbb{E}\left\{ \exp\left(- \sum_{j=1}^{n} (x_j - 1) Z_j \right) \right\}$$
$$= \prod_{j=1}^{n} \mathbb{E}\left\{ \exp\left(-(x_j - 1) Z_j \right) \right\}$$
$$= \prod_{j=1}^{n} \int_0^\infty \exp\left(-(x_j - 1) z \right) e^{-z} \, dz$$
$$= \prod_{j=1}^{n} \int_0^\infty e^{-x_j z} \, dz$$
$$= \prod_{j=1}^{n} \frac{1}{x_j}.$$

$(H_0: \mathbb{E}X_j \leq 1)$

The false positive rate is $\mathbb{P}_{H_0}(K(X_1, \dots, X_n) < \alpha)$,

Gaffke says it is unknown whether

(*) $$FPR = \mathbb{P}(K(X_1, \dots, X_n) < \alpha) \leq \alpha$$ (Gaffke eqn (14))

except for distributions supported on 2 points (where it holds – his Lemma 2.2).

Suppose (*) does not hold, ie, there exists $\alpha_0 \in (0,1)$ such that

$$\alpha_0 < \mathbb{P}(K(X_1, \dots, X_n) < \alpha_0)$$
$$= \mathbb{P}\left(\prod_{j=1}^{n} X_j^{-1} < \alpha_0 \right)$$
$$= \mathbb{P}\left(\prod_{j=1}^{n} X_j > 1/\alpha_0 \right).$$

Then $$\mathbb{E}\left(\prod_{j=1}^{n} X_j \right) = \int_0^\infty \mathbb{P}\left(\prod X_j > u\right) du$$
$$\geq \int_0^{1/\alpha_0} \mathbb{P}\left(\prod X_j > u\right) du$$
$$> \int_0^{1/\alpha_0} \alpha_0 = 1.$$

Just assuming $X_1, \dots, X_n$ are uncorrelated (not necessarily iid), $\mathbb{E}(\prod X_j) = \prod(\mathbb{E}X_j)$, so $\mathbb{E}(\prod X_j) > 1$ implies $\mathbb{E}X_j > 1$ for at least 1 of the X_j. So the null doesn't hold.

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eqn (1): $$K(x_1, \dots, x_n) = \mathbb{P}\left(\sum_{j=1}^{n} (x_j - 1) Z_j < Z_0 \right)$$

$$= \mathbb{E}\left\{ \mathbb{P}\left(\sum_{j=1}^{n} (x_j - 1) Z_j < Z_0 \,\middle|\, \sum_{j=1}^{n} (x_j - 1) Z_j \right) \right\}$$ law of total probability

(2) $$= \mathbb{E}\left\{ \mathbb{P}(u < Z_0) \Big|_{u = \sum_{j=1}^{n} (x_j - 1) Z_j} \right\}$$

$$= \mathbb{E}\left\{ 1 - F_{Z_0}(u) \Big|_{u = \sum (x_j - 1) Z_j} \right\}$$ $F_{Z_0}(u) = 1 - e^{-u}$ is the CDF of Z_0

$$= \mathbb{E}\left(e^{-\sum (x_j - 1) Z_j} \right)$$

Equation (2) is this rule: If X and Y are independent, then for any (integrable) f,

$$\mathbb{E}(f(X, Y) \mid Y) = \mathbb{E}(f(X, y)) \Big|_{y = Y}$$ ie,

compute $\mathbb{E}(f(X, y))$ treating y as a constant, then substitute Y for y (eg, Durrett Example 4.1.7). In our setting, $X = Z_0$, $Y = \sum_{j=1}^{n} (x_j - 1) Z_j$, and $f: (u, v) \mapsto \mathbb{1}\{v < u\}$. Also I used the notation "$\mathbb{P}$" instead of "$\mathbb{E}$" as in the rule since the argument f is an indicator.