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ON SOME CONVERGENCE PROPERTIES OF U-STATISTICS

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1. Introduction

In this paper, some covergence-properties of the U-statistics which are appreciably stronger than the usual property of consistency and which hold true even under less stringent regularity conditions, are studied. It is well known that the distribution of the standardized U-statistics is asymptotically normal (Hoeffding, 1948), whence follows the property of consistency. It has been shown here that even under less restrictive conditions, the U-statistic converges to its expectation, *with probability one.* A further convergence property termed the *structural convergence* (explained later, in this section) has also been studied and it leads to a consistent estimate of the variance of the U-statistic, which may not be readily obtained otherwise.

Let $X_1 \ldots\ldots, X_n$ be n independent sample units drawn from a population with a cumulative distribution function (c. d. f.) $\Phi(x, \theta)$ where θ indexes the distributions. Let now $g(\theta)$ be an *estimable parameter* of *degree* $m_0(\geqslant 1)$ and let $f_0(X_1, \ldots, X_m)$ be an *unbiased estimator* of it. Let $f(X_1, \ldots, X_m)$ denote the corresponding *symmetric function,* obtained by all possible (m !) permutations of the coordinantes $X_1, \ldots, X_m$. Then the corresponding U-statistic for a sample of size $n(\geqslant m \geqslant m_0)$ is defined by

$$U(X_1, \ldots, X_n) = \binom{n}{m}^{-1} \sum_{C} f(X_{\alpha_1}, \ldots, X_{\alpha_m}), \qquad \ldots(1.1)$$

where the summation C extends over $1 \leqslant \alpha_1 < \alpha_2 < \ldots\ldots \alpha_m \leqslant n$. Let us now write for $c = 1, \ldots, m$.

$$\left.\begin{array}{l} f_c(x_1, \ldots, x_c) = E_\theta\{f(x_1, \ldots, x_c, X_{c+1}, \ldots, X_m)\} \\ \quad \psi_c(x_1, \ldots, x_c) = f_c(x_1, \ldots, x_c) - g(\theta) \\ \text{and} \quad \zeta_c = E_\theta\{(\psi_c(X_1, \ldots, X_c)\}^2 \end{array}\right\} \qquad \ldots(1{\cdot}2)$$

$$\text{Then} \quad V_\theta\{U(X_1, \ldots, X_n)\} = \binom{n}{m}^{-1} \sum_{c=1}^{m} \binom{n}{m}\binom{n-m}{m-c}\zeta_c, \qquad \ldots(1{\cdot}3)$$

where E_θ and V_θ denote respectively the expectation and the variance of the required quantity. Now Hoeffding (1948) has shown that for $0<c<d\leqslant m$, $0<\zeta_c< \frac{c}{d}\zeta_d$ and $\zeta_0=0$. Again, it follows from the Central Limit Theorem that $Y_n = m.n^{-\frac{1}{2}}. \sum_{j=1}^{n} [f_1(X_j) - g(\theta)]$ has asymptotically a normal distribution. Hoeffding (1948) further showed that $\sqrt{n}[U_n - g(\theta)] \xrightarrow{P} Y_n$ (where by $\xrightarrow{P}$ we mean convergence in probabiliy) and hence the asymptotic normality of the U-statistics. Here, we would like to go a step further in showing that U_n may be decomposed into n components where the j-th component $\xrightarrow{P} f_1(X_j)$ for $j = 1, 2, \ldots$. This property has been termed the *Structural Convergence* and the property is utilized in having a consistent estimate of ζ_1 under the same set of regularity conditions. An important application of it is made in a class of non-parametric tests, where otherwise the tests are not valid and *distribution-free.*

Secondly, it has been shown here that for $n \geqslant n_0(\epsilon, \delta)$, depending on two arbitrarily small positive quantities ϵ and δ and for any arbitrary N,

$$P\left\{\bigcup_{j=1}^{N} [\,|\, U(X_1, \ldots, X_{n+j}) - g(\theta) \,|\, > \epsilon\,]\right\} < \delta, \qquad \ldots(1{\cdot}4)$$

and this can also be extented directly to any *regular function* of U-statistics. These results, however, do not follow from the classical Laws of Large Numbers, as here the components of the U-statistics are not all independent. Thus, we are to go through a translation of some of the Laws of Large Numbers from the case of independent random variables to our case of a mixture of *c-common* variables, with $c = 0, 1, \ldots, m-1$; where by 'c-common' we mean that there are c variables common between the two sets of m each, corresponding to two different $f(X_1, \ldots, X_m)$

2. Structural Convergence of U-Statistics

The following theorem will be proved first.

Theorem 1. *If $\zeta_m < \infty$, then $U(X_1, \ldots, X_n)$ may be decomposed into n identically distributed and asymptotically uncorrelated linear components, which converge in probability to a set of identically distributed, independent random variables, with the same mean and the same asymptotic variance.*

Proof : Let us define the j-the component of $U(X_1, \ldots, X_n)$ by

$$V_j = \binom{n-1}{m-1}^{-1} . \sum_{C_1} f(x_j, X_{\alpha_1}, \ldots, X_{\alpha_{m-1}}), \qquad \ldots(2{\cdot}1)$$

where the summation C_1 extends over $1 \leqslant \alpha_1 < \ldots < \alpha_{m-1} \leqslant n$ but $\alpha_i \neq j$ for all $i = 1, 2, \ldots, m-1$. It then follows combinatorially that

$$U(X_1, \ldots, X_n) = n^{-1} . \sum_{j=1}^{n} V_j. \qquad \ldots(2.2)$$

Since $X_1, \ldots, X_n$ have all the common c. d. f. $\Phi(x, \theta)$ and as the functional form of V_j remains the same for all $j = 1, 2, \ldots, n$, it follows directly that $V_1, \ldots, V_n$ have all the common c.d.f. $\Phi_1(v, \theta)$. It follows similarly that $f_1(X_j)$ $(j = 1, \ldots, n)$ have also the common c. d. f. $\Phi_2(f, \theta)$ and by definition $f_1(X_j) (j = 1, \ldots, n)$ are mutually independent. Hence, we require only to show that (*a*) $V_j \xrightarrow{P} f_1(X_j)$ for $j = 1, \ldots, n$ and (b) V_j and V_j' (with $j' \neq j = 1, \ldots, n$) are asymptotically uncorrelated.

As $\zeta_m < \infty$, for the first assertion, it is sufficient to show that for $j = 1, \ldots, n$, $E_\theta\{V_j - f_1(X_j)\}^2 \to 0$ as $n \to \infty$, as the rest will then follow from the Tshebysheff's lemma.

Using (1.2), we get directly that

$$\left.\begin{aligned} &E_\theta\{f_1(X_j) - g(\theta)\}^2 = \zeta_1 \\ &E_\theta\{V_j - g(\theta)\}^2 = \binom{n-1}{m-1}^{-1} \sum_{c=1}^{m} \binom{m-1}{c-1}\binom{n-m}{m-c}\zeta_c \\ \text{and } &E_\theta[\{f_1(X_j) - g(\theta)\}\{V_j - g(\theta)\}] = \zeta_1 \end{aligned}\right\} \qquad \ldots(2.3)$$

If therefore follows from (2.3), by simple algebra that

$$\begin{aligned} E_\theta\{f_1(X_j) - V_j\}^2 &= \binom{n-1}{m-1}^{-1} \sum_{c=1}^{m} \binom{m-1}{c-1}\binom{n-m}{m-c}(\zeta_c - \zeta_1) \\ &\leqslant \binom{n-1}{m-1}^{-1} \sum_{c=2}^{m} \binom{m-1}{c-1}\binom{n-m}{m-c} \frac{c}{m} \zeta_m \\ &\leqslant \frac{2(m-1)^2}{m(n-1)} \zeta_m, \end{aligned} \qquad \ldots(2{\cdot}4)$$

and hence converges to zero as $n \to \infty$ uniformly in $j = 1, \ldots, n$. Thus (a) is proved.

Also, we can write for any $j' \neq j$ that

$$\binom{n-1}{m-1} V_j = \sum_{C'_1} f\,(x_j, x_{j'}, X_{\alpha_1}, \ldots, X_{\alpha_{m-2}}) + \sum_{C''_1} f(x_j, X_{\alpha_1}, \ldots X_{\alpha_{m-1}}) \qquad \ldots(2.5)$$

where the summation C'_1 extends over the $\binom{n-2}{m-2}$ terms with $1 \leqslant \alpha_1 < \ldots < \alpha_{m-2} \leqslant n$ but $\alpha_i \neq j \neq j'$ for $i = 1, \ldots, m-2$; and the summation C_1'' extends over the $\binom{n-2}{m-1}$ terms with $1 \leqslant \alpha_1 < \ldots < \alpha_{m-1} \leqslant n$ but $\alpha_i \neq j \neq j'$ for $i = 1, \ldots, m-1$. And a similar decomposition of V_j' yields directly that

$$E_\theta[\{V_j - g(\theta)\}\{V_j' - g(\theta)\}] = \left[\sum_{c=1}^{m-1} \binom{m-1}{c}\binom{n-m-1}{m-c-1} \zeta_c\right]\binom{n-2}{m-1} \Big/ \binom{n-1}{m-1}^2 +$$

$$\frac{m-1}{n-1}\left[\sum_{c=2}^{m} \binom{m-2}{c-2}\binom{n-m}{m-c}\zeta_c + 2\sum_{c=1}^{m-1} \binom{m-2}{c-1}\binom{n-m}{m-c}\zeta_c\right]\binom{n-1}{m-1}^{-1}.$$

$$- \frac{m^2-1}{n-1}\zeta_1 \qquad \ldots(2{\cdot}6)$$

Now it follows from (2·3) that $E_\theta\{V_j - g(\theta)\}^2$ is finite and is asymptotically equal to ζ_1 and hence from (2.6), we get that V_j and $V_j'(j \neq j')$ are asymptotically uncorrelated, uniformly in $j \neq j' = 1, \ldots, n$. Hence, the theorem.

We are now in a position to interpret statistically the variable $f_1(X_j)$ for $j = 1, \ldots, n$. Apart from their usual definitions as some partially expected random variables, they may also be interpretod as the limiting values of the components V_j for $j = 1, 2, \ldots$; i.e. for $n \geqslant n_0(\epsilon, \delta)$, depending on two arbitrarily small positive quantities ϵ and δ

$$P\{V_j - \epsilon \leqslant f_1(X_j) \leqslant V_j + \epsilon\} \geqslant 1 - \delta \qquad \ldots(2{\cdot}7)$$

Remark one : Let us now consider an extended decomposition of the U-statistics and for any $k > 0$, we define for a k-th order decomposition, a typical term of the $\binom{n}{k}$ possible ones by

$$V_{j_1, \ldots; j_k} = \binom{n-k}{m-k}^{-1} \sum_{C_k} f(X_{j_1}, \ldots, X_{j_k}, X_{\alpha_1}, \ldots, X_{\alpha_{m-k}}), \qquad \ldots(2.3)$$

where the summation C_k extends over $1 \leqslant \alpha_1 < \ldots < \alpha_{m-k} \leqslant n$ with

$\alpha_i \neq j_1 \neq \ldots\ldots \neq j_k$ for $i=1, \ldots, m-k$, so that, $U(X_1, \ldots, X_n)$ may be written as

$$U(X_1, \ldots, X_n) = \binom{n}{k}^{-1} \sum_{S_k} V_{j_1, \ldots\ldots, j_k}, \qquad \ldots(2\cdot9)$$

where S_k extends over all possible $1 \leqslant j_1 < \ldots\ldots < j_k \leqslant n$. It then follows similarly that for any $0<k\leqslant m$, and $\zeta_k \geqslant 0$, we will have

$$P\{V_{j_1, \ldots, j_k} - \epsilon \leqslant f_k(X_{j_1}, \ldots, X_{j_k}) \leqslant V_{j_1, \ldots, j_k} + \epsilon\} \geqslant 1-\delta, \qquad \ldots(2.10)$$

for $n \geqslant n_0(\epsilon, \delta)$ and uniformly in $(j_1, \ldots, j_k) \in (1, \ldots, n)$.

Remark Two: If we now define by $V_j^{(1)} = \binom{j-1}{m-1}^{-1} \Sigma_C f(X_j, X_{\alpha_1}, \ldots\ldots, X_{\alpha_{m-1}})$ where C extends over $1 \leqslant \alpha_1 < \ldots\ldots < \alpha_{m-1} \leqslant j-1$, it then follows similarly that for $n \geqslant n_0(\epsilon, \delta)$ and for any r, however large a positive integer it may be, $V_{n+r}^{(1)}$ will converge in probability to $f_1(X_{n+r})$ This property will be helpful in sequential tests with U-statistics.

With these properties of the U-statistics, we will prove the following prepositions with their applications.

Proposition 1. *Let* $s_v^2 = \frac{1}{n-1} \cdot \sum_{j=1}^{n} [V_j - U(X_1, \ldots, X_n)]^2, \qquad \ldots(2.11)$

then (i) $E_\theta(s_v^2) \to \zeta_1$, *as* $n \to \infty$
and (ii) for $n \geqslant n_0(\epsilon, \delta)$, $P\{|s_v^2 - \zeta_1| > \epsilon\} < \delta$.

Proof: Writing equivalently,

$$s_v^2 = (n-1)^{-1} \cdot \left\{ \sum_{j=1}^{n} [V_j - g(\theta)]^2 - n[U(X_1, \ldots, X_n) - g(\theta)]^2 \right\}$$

it follows directly from (1.3) and (2·3) that $E_\theta(s_v^2) \to \zeta_1$ as $n \to \infty$. To prove *(ii)*, let us first prove the following simple lemma.

Lemma 1: *If* $\{a_i\}$ *and* $\{b_i\}$ *be two sequences of random variables, such that (a)* $n^{-1} \sum_{i=1}^{n} a_i^2 \xrightarrow{P} A < \infty$ *and (b)* $n^{-1} \sum_{i=1}^{n} (a_i - b_i)^2 \xrightarrow{P} 0$, *then*

$$n^{-1} \cdot \sum_{i=1}^{n} b_i^2 \xrightarrow{P} A.$$

Proof : We have $\left|\frac{1}{n}\sum_{i=1}^{n}(b_i^2-a_i^2)-\frac{1}{n}\sum_{i=1}^{n}(a_i-b_i)^2\right|$

$$=\left|\frac{2}{n}\sum_{i=j}^{n}a_i(a_i-b_i)\right|$$

$$\leqslant \frac{2}{n}\sum_{i=1}^{n}|a_i|\ |a_i-b_i|$$

$$\leqslant 2\left\{\frac{1}{n}\sum_{i=1}^{n}a_i^2\right\}^{\frac{1}{2}}.\left\{\frac{1}{n}\sum_{i=1}^{n}(a_i-b_i)^2\right\}^{\frac{1}{2}} \qquad \text{...(2'12)}$$

Since, the right hand side of (2.12) $\rightarrow 0$ as $n\rightarrow\infty$, so also

$$n^{-1}\sum_{i=1}^{n}(a_i-b_i)^2 \xrightarrow{P} 0,$$

on the left hand side, it follows directly that

$$\frac{1}{n}\sum_{i=1}^{n}b_i^2 \xrightarrow{P} \frac{1}{n}\sum_{i=1}^{n}a_i^2 \xrightarrow{P} A.$$

Hence, the lemma.

Now in our case, we take $a_i=f_1(X_i)-g(\theta)$ and $b_i=V_i-g(\theta)$ for $i=1, ..., n$. It then follows from Kintchine's Law of Large Numbers that $n^{-1}\sum_{i=1}^{n}a_i^2 \xrightarrow{P} E_\theta\{f_1(X_1)-g(\theta)\}^2=\zeta_1$. Also, it follows from (2.4) that $E_\theta\left\{n^{-1}\sum_{i=1}^{n}(a_i-b_i)^2\right\}=E_\theta(a_1-b_1)^2 \leqslant \frac{2(m-1)^2}{m(n-1)}\zeta_m$, and hence an application of Tshebysheff's inequality yields that $n^{-1}\sum_{i=1}^{n}(a_i-b_i)^2 \xrightarrow{P} 0$.

Thus from lemma 1 and (1'3), we get that

$$\left.\begin{aligned} &n^{-1}\sum_{i=1}^{n}[V_i-g(\theta)]^2 \xrightarrow{P} n^{-1}\sum_{i=1}^{n}[f_1(X_i)-g(\theta)]^2 \xrightarrow{P} \zeta_1 \\ &\text{and } \frac{n}{n-1}\Big[V(X_1, ..., X_n)-g(\theta)\Big]^2 \xrightarrow{P} 0 \end{aligned}\right\} \qquad \text{...(2'13)}$$

Hence, it follows from (2.11) through (2.13) that $s_v^2 \xrightarrow{P} \zeta_1$, which completes the proof.

Proposition 2 : *The distribution of* $t=\sqrt{n}[U_n - g(\theta)]/(m/s_v)$ *is asymptotically normal with zero mean and unit variance.*

The proof follows directly by writing
$t=\{\sqrt{n}[U_n - g(\theta)]/(m\sqrt{\zeta_1})\}/\{\sqrt{s_v^2/\zeta_1}\}$ and noting that the numerator has asymptotically a normal distribution (vide Hoefding, 1948) with zero mean and unit variance and finally by applying Proposition I through a convergence theorem of Cramer (1946, p. 254).

These concepts can readily be extended to the case of generalized U-statistics, as defined by Lehmann (1951). By way of summary we recapitulate it as follows :

Corresponding to a symmetric unbiased estimator of an estimable parameter $g(\theta)$, the generalized U-statistic based on two independent samples of sizes n_1 and n_2 respectively, is defined by

$$U(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}) = \binom{n_1}{m_1}^{-1} \cdot \binom{n_2}{m_2}^{-1} \cdot \sum_{C} f(X_{\alpha_1}, \ldots, X_{\alpha m_1}, Y_{\beta_1}, \ldots, Y_{\beta m_2}) \quad \ldots(2.14)$$

where C extends over $1 \leqslant \alpha_1 < \ldots < \alpha m_1 \leqslant n_1$; $1 \leqslant \beta_1 < \ldots < \beta m_2 \leqslant n_2$.

$$\left.\begin{aligned} &\text{Let } f_{10}(x_j) = E_\theta\{f(x_j, X_{\alpha_1}, \ldots, X_{\alpha m_1 - 1}, Y_{\beta_1}, \ldots, Y_{\beta m_2})\} \\ &V_{j(1)} = \binom{n_1 - 1}{m_1 - 1}^{-1} \binom{n_2}{m_2}^{-1} \cdot \sum_{C_{10}} f(x_j, X_{\alpha_1}, \ldots, X_{\alpha m_1 - 1}, Y_{\beta_1}, \ldots, Y_{\beta m_2}). \end{aligned}\right\} \quad \ldots(2.15)$$

where C_{10} extends over $1 \leqslant \alpha_1 < \ldots < \alpha m_1 - 1 \leqslant n_1$; $\alpha_i \neq j$ for all i and $1 \leqslant \beta_1 < \ldots < \beta m_2 \leqslant n_2$.

$$\left.\begin{aligned} &\text{Also } f_{01}(y_j) = E_\theta\{f(X_{\alpha_1}, \ldots, X_{\alpha m_1}, y_j, X_{\beta_1}, \ldots, Y_{\beta m_2 - 1})\} \\ &V_j(2) = \binom{n_1}{m_1}^{-1} \binom{n_2 - 1}{m_2 - 1}^{-1} \sum_{Co_1} f(X_{\alpha_1}, \ldots, X_{\alpha m_1}, y_j, Y_{\beta_1}, \ldots, Y_{\beta m_2 - 1}) \end{aligned}\right\} \quad \ldots(2.16)$$

where C_{01} extends over $1 \leqslant \alpha_1 < \ldots < \alpha m_1 \leqslant n_1$; $1 \leqslant \beta_1 < \ldots \ldots < \beta m_2 - 1 \leqslant n_2$ but $\beta_i \neq j$ for all i.

$$\left.\begin{aligned} &\text{Let further } \zeta_{10} = E_\theta\{f_{10}(X_j) - g(\theta)\}^2 \text{ and } \zeta_{01} = E_\theta\{f_{01}(Y_j) - g(\theta)\}^2 \\ &\text{and } s_{v(i)}^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} [V_{j(i)} - U(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})]^2 \\ &\qquad \text{for } i = 1, 2 \end{aligned}\right\} \quad \ldots(2.17)$$

Then, the following propositions are true.

Proposition 3 : $P\{\,|\,V_{j(1)} - f_{10}(x_j)\,| > \epsilon\,\} < \delta$ for $j = 1, \ldots, n_1$ and $P\{\,|\,V_{j(2)} - f_{01}(Y_j)\,| > \epsilon\,\} < \delta$ for $j = 1, \ldots, n_2$

for all $n_1 \geqslant n_0(\epsilon, \delta)$ and $\lim_{n_1 \to \infty} n_2/n_1 = \rho < \infty$.

Proposition 4 : For $n_1 \geqslant n_0(\epsilon, \delta)$ and $\lim_{n_1 \to \infty} n_2/n_1 = \rho < \infty$ we will have $P\{\,|\,s^2_{v(1)} - \zeta_{10}\,| > \epsilon\,\} < \delta$ and $P\{\,|\,s^2_{v(2)} - \zeta_{01}\,| > \epsilon\,\} < \delta$

Proposition 5 : The distribution cf

$$t = \sqrt{n_1}\,[U_{n_1} - g(\theta)]/\sqrt{m_1^2\, s^2_{v(1)} + \rho m_2^2\, s^2_{v(2)}}$$

is asymptotically normal with zero mean and unit variance.

Proposition 5 is of great value in some two-sample non-parametric tests, where otherwise the distribution of the test criterion is not distribution-free, even asymptotically and even under the null hypothesis.. As an example, we may refer the two-sample scale test by Lehmann (1951, p. 170). The test is not distribution-free and the modifications necessary to make such tests distribution free were considered by Sukhatme (1958). It has been shown by him that only under some restrictive regularity conditions, one may expect some modified tests to be distribution-free but not all the available ones. But if modified in the light of Proposition 5, we require no assumption such as the symmetry and boundedness of the density function as in Sukhatme (1958) and all the available tests will enjoy this property. Hence, the importance of Proposition 5.

Extension to the c-sample case (with $c > 2$) is obvious and hence is omitted.

3. Strong Convergence of U-Statistics

The following theorem, under Hoeffding's regularity conditions, will be proved first, while subsequently, the regularity conditions will be relaxed to some extent.

Theorem 2 : *If $\zeta_m < \infty$, then $U(X_1, \ldots, X_n)$ converges with probability one to $g(\theta)$ i. e. for any two arbitrarily small positive quantities ϵ and δ, there exists a value of n, say $n_0(\epsilon, \delta)$, such that for all $n \geqslant n_0(\epsilon, \delta)$*

$$P\left\{ \bigcup_{j=1}^{N} [\,|\,U(X_1, \ldots, X_{n+j}) - g(\theta)\,| > \epsilon] \right\} < \delta \qquad \ldots(3.1)$$

uniformly in N.

Proof : To start with, we may take without any loss of generality (*i*) $g(\theta)=0$ and (*ii*) $m \geqslant 2$, as in the case $m=1$, Kolmogorov's Law of Large Numbers will apply.

Let us now denote by $U^{(1)}(Z_1, \ldots, Z_n) = \frac{m}{n} \sum_{j=1}^{n} Z_j$, ...(3.2)

where $Z_i = E_\theta\{f(x_j, X_{\alpha_1}, \ldots, X_{\alpha_{m-1}})\} = f_1(x_j)$ for $j=1, \ldots, n$. Then, we can write $U(X_1, \ldots, X_n) = \binom{n}{m}^{-1} \sum_C f(X_{\alpha_1}, \ldots, X_{\alpha_m})$

$$= U^{(1)}(Z_1, \ldots, Z_n) + U^{(2)}(R_n), \qquad \ldots(3.3)$$

where $U^{(2)}(R_n)$, is the remainder component. Let us further denote by

$$\left.\begin{aligned} P^{(1)}_{n.N} &= P\left\{\bigcup_{j=1}^{N} [\,|\,U^{(1)}(Z_1, \ldots, Z_{n+j})\,| < \epsilon_1]\right\} \\ \text{and} \quad P^{(2)}_{n.N} &= P\left\{\bigcup_{j=1}^{N} [\,|\,U^{(2)}(R_{n+j})\,| > \epsilon_2]\right\} \end{aligned}\right\} \qquad \ldots(3.4)$$

where ϵ_1 and ϵ_2 are arbitrarily small. Now, if we can show that there exists two arbitrarily small positive quantities $\epsilon\ (=\epsilon_1+\epsilon_2)$ and $\delta(=\delta_1+\delta_2)$ such that for all $n \geqslant n_0(\epsilon, \delta)$ and any N

$$P^{(1)}_{n.N} < \delta_1 \quad \text{and} \quad P^{(2)}_{n.N} < \delta_2 \qquad \ldots(3.5)$$

then (3.1) would follow through (3.3), (3.4) and (3.5).

Now $\{Z_j\}$ is a sequence of independent random variables with mean zero and a finite variance $\zeta_1 < \frac{1}{m}\zeta_m < \infty$. Hence, an application of Kolmogorov's Law of Large Numbers yields that for $n \geqslant n_0\ (\epsilon, \delta)$, $P^{(1)}_{n.N} < \delta_1$. Also it follows by direct computation, using (1.2) and (1.3) that

$$E_\theta\{U^{(2)}(R_{n+j})\}^2 = E_\theta\{U(X_1, \ldots, X_{n+j}) - U(Z_1, \ldots, Z_{n+j})\}^2$$

$$= \frac{m}{n+j}\left\{\binom{n+j-1}{m-1}^{-1} \sum_{c=1}^{m} \binom{m}{c}\binom{n+j-m}{m-c}\zeta_c - m\zeta_1\right\}$$

$$\leqslant \frac{m}{n+j}\left\{\binom{n+j-1}{m-1}^{-1} \sum_{c=2}^{m} \binom{m}{c}\binom{n+j-m}{m-c}\zeta_c\right\}$$

$$\leqslant \frac{m}{n+j}\binom{n+j-1}{m-1}^{-1} \sum_{c=2}^{m} \frac{m}{c} \cdot \binom{m-1}{c-1}\binom{n+j-m}{m-c}\frac{c}{m}\,\zeta_m.$$

$$\leqslant \frac{m(m-1)^2}{(n+j)(n+j-1)}\,\zeta_m. \qquad \ldots(3.6)$$

Also, from Tshebysheff's inequality, we get

$$p_{n+j}^{(2)} = P\{ \mid U^{(2)}(R_{n+j}) \mid > \epsilon_2\} < E_\theta\{U^{(2)}(R_{n+j})\}^2/\epsilon_2^2$$

and finally on applying Poincare's theorem on total probability,

$$\text{we get } P_{n}{}^{(2)}_{.N} \leqslant \sum_{j=1}^{N} p_{n+j}^{(2)} < \frac{m(m-1)^2\, \zeta_m}{\epsilon_2^2} \cdot \sum_{j=1}^{N} \frac{1}{(n+j)(n+j-1)}$$

$$< \frac{m(m-1)^2 \zeta_m}{n\epsilon_2^2} \qquad \ldots(3.7)$$

Since, $1/(n\epsilon_2^2)$ can be made arbitrarily small by a proper choice of n it follows that for $n \geqslant n_0(\epsilon, \delta)$, $P_{n.N}^{(2)} < \delta_2$. Hence, the theorem.

As for this theorem, the condition $\zeta_m < \infty$, appears to be a sufficient one and thereby attempts may be made to replace it by less stringent ones. Deduction of the theorem, under Kintchine's condition (*i.e.* $\mid E\{f(X_1, \ldots, X_m)\} \mid < \infty$) does not appear to be feasible for $m > 1$ while we may formulate the following condition (3.8), which corresponds to Markoff's condition (cf. Uspensky, 1937, p. 191) in the case $m = 1$.

Theorem 3. *If, for any symmetric unbiased estimator $f(X_1, \ldots, X_m)$ of an estimable function $g(\theta)$, we have*

$$E_\theta\{ \mid f(X_1, \ldots, X_m) \mid^{1+\delta}\} < \infty, \text{ for some } \delta > 1 - 1/m, \qquad \ldots(3.8)$$

then the corresponding U-statistic converges with probability one to $g(\theta)$.

Proof: In this case also we take (*i*) $g(\theta) = 0$, (ii) $m \geqslant 2$ and $\delta < 1$, as in the case $\delta \geqslant 1$, theorem 2 will apply.

Now there are $\binom{n}{m}$ terms in $U(X_1, \ldots, X_n)$. Let us designate the general term by f_s and write

$$U_n = U(X_1, \ldots, X_n) = \binom{n}{m}^{-1} \cdot \sum_{s=1}^{\binom{n}{m}} f_s,$$

where the suffixes "s" are attached in the following manner. For $f(X_1, \ldots, X_m)$ we take $s = \binom{m}{m} = 1$ and for any $j(>m)$ we take, for the $\binom{j-1}{m-1}$ possible terms of the type $f(X_{\alpha_1}, \ldots, X_{\alpha_{m-1}}, X_j)$ (where $1 \leqslant \alpha_1 < \ldots < \alpha_{m-1} \leqslant j$ in any arbitrary way, $s = \binom{j-1}{m} + 1, \ldots, \binom{j}{m}$ for $j = m, m+1, \ldots n, \ldots$.

Let us now adopt Markoff's truncation principle and for this a new stochastic variable y_s is introduced and defined by

$$\left.\begin{aligned} y_s &= f_s \text{ if } \mid f_s \mid \leqslant \tbinom{n}{m} \text{ for } s = \tbinom{n-1}{m} + 1, \ldots, \tbinom{n}{m} \\ &= 0, \text{ otherwise} \end{aligned}\right\} \qquad \ldots(3.9)$$

It then follows from (3.8) that

$$\left\{E_\theta\left[\,|\,f(X_1,\,\ldots,\,X_m)\,|^{\,1+\delta}\right]\right\}^{1/(1+\delta)} \geqslant E_\theta\{\,|\,f(X_1,\,\ldots,\,X_m)\,|\,\}$$

$$\geqslant \sum_{s=0}^{\infty} sP\{s < |f| \leqslant s+1\}$$

$$\geqslant \sum_{n=m}^{\infty} \tbinom{n}{m} P\{\tbinom{n}{m} < |f| \leqslant \tbinom{n+1}{m}\}. \qquad \ldots(3.10)$$

$$\text{Then, } \sum_{n=m}^{\infty} \sum_{s=\binom{n-1}{m}+1}^{\binom{n}{m}} P\{y_s \neq f_s\}$$

$$= \sum_{n=m}^{\infty} \sum_{s=\binom{n-1}{m}+1}^{\binom{n}{m}} P\{\,|f_s| > \tbinom{n}{m}\}$$

$$= \sum_{n=m}^{\infty} \sum_{s=\binom{n-1}{m}+1}^{\binom{n}{m}} \sum_{j=n}^{\infty} P\{\tbinom{j}{m} < |f| \leqslant \tbinom{j+1}{m}\}$$

$$= \sum_{n=m}^{\infty} \tbinom{n}{m} P\{\tbinom{n}{m} < |f| \leqslant \tbinom{n+1}{m}\}$$

$$\leqslant E_\theta\{f(X_1,\,\ldots,\,X_m)\} < \infty. \qquad \ldots(3{\cdot}11)$$

It therefore follows from an "equivalence lemma" of Loeve (1955, p. 233) that $U_n^{(1)} = \binom{n}{m}^{-1} \sum_{s=1}^{\binom{n}{m}} y_s$ and $U(X_1, \ldots, X_n)$ are convergence-equivalent. Consequently, we are only to show that $U_n^{(1)}$ converges to zero with probability one. Since, $|\,E_\theta\,[U_n^{(1)}]\,| \to 0$ as $n \to \infty$, it follows that on writing $\bar{y}_s = y_s - E_\theta(y_s)$ for all s and $U_n^{(2)} = \binom{n}{m}^{-1} \sum_{s=1}^{\binom{n}{m}} \bar{y}_s$; that it is sufficient to show that $U_n^{(2)}$ converges to zero with probability one.

Let us now denote by $Z_j = E_\theta\left\{\bar{y}(x_j,\,X_{\alpha_1},\,\ldots,\,X_{\alpha_{m-1}})\right\}$; $1 \leqslant \alpha_1 < \ldots < \alpha_{m-1} \leqslant j-1$ and $j = m,\,\ldots,\,n,\,\ldots$; and $\zeta_{c\cdot kk'} = E_\theta\left\{\bar{y}_s\,.\,\bar{y}_{s'} \Big/ \binom{k-1}{m} < s \leqslant \binom{k}{m},\ \binom{k'-1}{m} < s' \leqslant \binom{k'}{m}\right\}$, where among the two sets of

m variables each, there are c variables common for $c=0, 1, \ldots m$ and for $k, k'=m, \ldots, n, \ldots$. It then follows as in Hoeffding (1948) that $\zeta_{c\cdot kk'}=0$ and $0<\zeta_{c\cdot kk}<\frac{c}{d}\ \zeta_{d\cdot kk}$ for $1\leqslant c<d\leqslant m$. We will next show that for $k'\geqslant k$, $0\leqslant |\zeta_{c\cdot kk'}| <A_1\binom{k}{m}^{1-\delta}$, where $A_1<\infty$, independently of $k,k'=m, \ldots, n, \ldots$. For this let us first prove the following simple lemma.

Lemma 2 : *If $E|X|<\infty$ and $E|Y|<\infty$, then for any real* α : $0<\alpha<1$, *we have* $E\{|X|^{\alpha}. |Y|^{1-\alpha}\}<\infty$.

Proof : For any a, $b>0$ and any two positive integers p and q we have

$$(pa+qb)/(p+q)\geqslant a^{p/(p+q)}.\ b^{q/(p+q)}, \qquad \ldots(3\cdot12)$$

this being consequent of the elementary inequalities between the arithmetic mean and the geometric mean. Thus, if α is rational, we can choose p and q such that $\alpha=p/(p+q)$ and hence on taking expectation on both sides of (3. 12), we get on replacing a by $|X|$ and b by $|Y|$ that

$$E\{|X|^{\alpha}. |Y|^{1-\alpha}\}\leqslant\alpha E|X| +(1-\alpha)E|Y|<\infty \qquad \ldots(3.13).$$

Passing to any real number through a sequence of rational numbers the lemma follows.

$$\text{Now}\quad \underset{k'\geqslant k}{|\zeta_{c\cdot kk'}|}\leqslant E_\theta\left\{|\bar{y}_s|.|\bar{y}_{s'}| \Big/ \binom{k-1}{m}<s\leqslant\binom{k}{m},\ \binom{k'-1}{m}<s'\leqslant\binom{k'}{m}\right\}$$

$$\leqslant 2\binom{k}{m}^{1-\delta}.\ E_\theta\left\{|\bar{y}_s|^{\delta}|\bar{y}_{s'}| \Big/ \binom{k-1}{m}<s\leqslant\binom{k}{m},\ \binom{k'-1}{m}<s'\leqslant\binom{k'}{m}\right\}$$

as for $0<\delta<1$, $|\bar{y}_s|^{1-\delta}\leqslant|y_s|^{1-\delta}+|E(y_s)|^{1-\delta}\leqslant2\binom{k}{m}^{1-\delta}$.

Hence, an application of Lemma 2 and the condition (3. 8) directly yields that

$$\underset{k'\geqslant k}{|\zeta_{c\cdot kk'}|}\leqslant A_1\binom{k}{m}^{1-\delta} \qquad \ldots(3\cdot14)$$

where $A_1<\infty$ independently of (k, k').

Let us now write $U_n^{(2)} = U_n^{(2)}(Z) + U_n^{(2)}(R)$

$$\left.\begin{aligned} &\text{where} \quad U_n^{(2)}(Z) = \binom{m}{n}^{-1} \sum_{j=m}^{n} \binom{j-1}{m-1} Z_j \end{aligned}\right\} \qquad \ldots(3\cdot15)$$

and $U_n^{(2)}(R)$ is the residual component. Then, it is sufficient to show that both $U_n^{(2)}(Z)$ and $U_n^{(2)}(R)$ strongly converge to zero. It now follows from Toeplitz's lemma (Loeve, 1955, p. 238) that $U_n^{(2)}$ will converge to zero with probability one, if the same is true of $n^{-1}(Z_1 + \ldots + Z_n)$. Now $\{Z_j\}$ is a sequence of independent random variables (though not identically distributed) with means zero and variance $\{\zeta_{1\cdot jj}\}$ and thus, it is sufficient to show that $\sum_{j=m}^{\infty} \frac{\zeta_{1\cdot jj}}{j^2} < \infty$.

Since $\delta > 1 - \frac{1}{m}$ we get on writing $\delta = 1 - \frac{1}{m} + \alpha$ that $\alpha > 0$ and hence from (3. 14) that $\zeta_{1\cdot jj} \leqslant \frac{1}{m}\, \zeta_{m\cdot jj} \leqslant \frac{A_1}{m}\, \left(\frac{j}{m}\right)^{1-\delta} < \frac{A_1}{m}\, j^{m(1-\delta)} = \frac{A_1}{m}\, j^{1-m\alpha}$

and therefore $\sum_{j=m}^{\infty} \frac{\zeta_{1\cdot jj}}{j^2} \leqslant \frac{A_1}{m} \sum_{j=m}^{\infty} \frac{1}{j^{1+m\alpha}} < \infty$, as $\sum_{1}^{\infty} \frac{1}{n^{1+\delta}}$

converges for any $\delta > 0$. Consequently, Kolmogorov's Law of Large Numbers leads to the convergence of $n^{-1}(Z_1 + \ldots + Z_n)$ to zero with probability one. Thus $U_n^{(2)}(Z)$ strongly converges to zero. Also, we have as in (1.2) that

$$E_\theta\,\{U_n^{(2)}\}^2 = \binom{n}{m}^{-2} \left\{ \sum_{j=m}^{n} \binom{j-1}{m-1} \sum_{c=1}^{m} \binom{m-1}{c-1} \binom{j-m}{m-c} \zeta_{c\cdot jj} \right.$$

$$\left. + 2 \sum_{j'>j=m}^{n} \binom{j-1}{m-1} \sum_{c=1}^{m-1} \binom{m}{c} \binom{j'-1-m}{m-1-c} \zeta_{c\cdot jj'} \right\} \qquad \ldots(3\cdot16)$$

Also $E_\theta\,\{U_n^{(2)}.\ U_n^{(2)}(Z)\}$

$$\left.\begin{aligned} &= \binom{n}{m}^{-2} \left\{ \sum_{j=m}^{n} \binom{j-1}{m-1}^2 \zeta_{1\cdot jj} \right. \\ &\qquad \left. + 2 \sum_{j'>j=m}^{n} \binom{j-1}{m-1} \binom{j'-2}{m-2} \zeta_{1\cdot jj'} \right\} \\ &\text{and} \quad E_\theta\,\{U_n^{(2)}(Z)\}^2 = \binom{n}{m}^{-2} \left\{ \sum_{j=m}^{n} \binom{j-1}{m-1}^2 \zeta_{1\cdot jj} \right\} \end{aligned}\right\} \qquad \ldots(3\cdot17)$$

From (3.16) and (3.17), we get after little combinatorial adjustments that

$$0 < E_\theta\{U_n^{(2)}(R)\}^2 = \binom{n}{m}^{-2}\left\{\sum_{j=m}^{n}\binom{j-1}{m-1}\left[\binom{j-m}{m-1}-\binom{j-1}{m-1}\right]\zeta_{1\cdot jj}\right.$$

$$+\sum_{j=m}^{n}\binom{j-1}{m-1}\sum_{c=2}^{m}\binom{m-1}{c-1}\binom{j-m}{m-c}\zeta_{c\cdot jj}$$

$$+2\sum_{j'>j=m}^{n}\binom{j-1}{1}\left[\binom{j'-1-m}{m-2}-\binom{j'-2}{m-2}\right]\zeta_{1\cdot jj'}$$

$$\left.+2\sum_{j'>j=m}\binom{j-1}{m-1}\sum_{c=2}^{m-1}\binom{m}{c}\binom{j'-1-m}{m-1-c}\zeta_{c\cdot jj'}\right\}$$

$$=\frac{m^2}{n^2}\{S_1+S_2+S_3+S_4\} \qquad \text{(say)} \qquad \ldots(3\cdot18)$$

Let us find an upper bound for the right hand side of (3.18). Since, the first summation S_1 has all the terms negative, we may delete it and consequently, the equality will have to be replaced by a "less than or equal to" sign. Also, it follows from (3.14) and using simple combinatorial properties that the 2nd and the 4th summation (S_2 and S_4) being added will be less than (in absolute value)

$$3A_1\frac{(m+1)(m-1)^2}{(n-1)}\left[\sum_{j=m}^{n}\binom{j-m}{m-1}\zeta_{m\cdot nn}\right]\binom{n-1}{m-1}^{-1}$$

$$\leqslant 3A_1(m+1)(m-1)^2 \quad \zeta_{m\cdot nn}\left\{\frac{1}{n-1}\sum_{j=m}^{n}\frac{(j-1)^{m-1}}{(n-m)^{m-1}}\right\}$$

$$\leqslant c_1\,\zeta_{m\cdot nn} \qquad (\text{with } c_1<\infty) \qquad \ldots(3.19)$$

as $\displaystyle\frac{1}{n-1}\sum_{j=m}^{n}\frac{(j-1)^{m-1}}{(n-m)^{m-1}}<\infty$ for all $n>m$ and tends to $\frac{1}{m}$ as $n\to\infty$.

Again, it follows after some little algebraic readjustments that the absolute value of the third summation is less than

$$2\binom{n-1}{m-1}^{-2}\sum_{m=j}^{n}\binom{j-1}{m-1}\left[\binom{j-1}{m-1}-\binom{j-m}{m-1}\right]\zeta_{m\cdot nn} \qquad \ldots(3.20)$$

And, it can be shown readily that for any $j\leqslant n$, $m\geqslant 1$

$$\left[\binom{j-1}{\ }\ \binom{j-m}{\ }\right]\binom{n-1}{\ }^{-1} \ldots \qquad (3\cdot21)$$

Thus, from (3.21), we get that (3.20) will always be less than $c_3 \zeta_{m \cdot nn}$ where $c_3 < \infty$. Hence, from (3.18) through (3.21) we get

$$E_\theta\{U^{(2)}_n(R)\}^2 < \frac{m^2}{n^2}\left[A_1 \binom{n}{m}^{1-\delta}\right](c_1+c_3)$$

$$< A_2/n^{1+m\alpha} \qquad \ldots(3.22)$$

where $A_2 < \infty$, independently of n and where $\alpha > 0$.
Writing now $p_{n+j}^{(2)} = P\{\,|\,U_{n+j}^{(2)}(R)\,| > \epsilon\}$ for $j = 1, \ldots, N$
we get from (3.22) by Tshebysheff's lemma that

$$p_{n+j}^{(2)} < A_2\,\epsilon^{-2}/(n+j)^{1+m\alpha} \qquad \text{for } j = 1, \ldots, N.$$

Thus, on denoting by $P^{(2)}_{n \cdot N} = P\left\{\begin{matrix}\text{atleast one}\\ j=1,\ldots,N\end{matrix}\,|\,U^{(2)}_{n+j}(R)\,| > \epsilon\right\}$

we get from Poincare's theorem on total probability that

$$P^{(2)}_{n \cdot N} \leqslant \sum_{j=1}^{N} p^{(2)}_{n+j} < A_2\,\epsilon^{-2}\sum_{j=1}^{N}\frac{1}{(n+j)^{1+m\alpha}} \qquad \ldots(3.23)$$

Since, for any $\lambda > 0$, $\sum_{j=1}^{\infty}\frac{1}{(n+j)^{1+m\alpha\lambda}}$ converges to zero, as $n \to \infty$,

we get on letting $\epsilon = kn^{-\frac{1}{2}m\alpha(1-\lambda)}$ with $0 < \lambda < 1$ that $\epsilon \to 0$ as $n \to \infty$

$$\text{and} \quad P^{(2)}_{n \cdot N} < \frac{A_2}{k^2}\cdot\sum_{j=1}^{N}\frac{1}{(n+j)^{1+m\alpha\lambda}} < \delta \qquad \ldots(3.24)$$

where $\delta = 0\left(n^{-\frac{m\alpha\lambda}{2}}\right)$ and hence converges to zero as $n \to \infty$. As, (3.24) holds true for any N, the theorem follows directly.

Let us now consider an example where theorem 3 applies but neither theorem 2 nor Hoeffding's consistency theorem. Let a sample of n units be drawn from a Cauchy distribution with the specification

$$\phi(x, \theta)dx = d\Phi(x) = \frac{1}{\pi}\quad\frac{dx}{1+(x-\theta)^2}, \quad -\infty < x < \infty.$$

Let us now consider a subset of 3 variables X_{α_1}, X_{α_2}, and X_{α_3} and let $f(X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_3}) = x^*(\alpha_1, \alpha_2, \alpha_3)$ denote the median of these three values. This is obviously a symmetric function. It now follows from a theorem on the moments of the sample quantiles (cf. Sen, 1959) that $x^*(\alpha_1, \alpha_2, \alpha_3)$ has a finite moment of the order $1+\delta$ for all $\delta < 1$ and the unbiasedness of $x^*(\alpha_1, \alpha_2, \alpha_3)$ follows directly from the symmetry of $\phi(x, \theta)$. The corresponding U-statistics based on a sample of size n, is given by

$$U(X_1, \ldots, X_n) = \binom{n}{3}^{-1} \sum_{1 \leqslant \alpha_1 < \alpha_2 < \alpha_3 \leqslant n} x^*(\alpha_1, \alpha_2, \alpha_3)$$

$$= \binom{n}{3}^{-1} \sum_{j=2}^{n-1} (j-1)(n-j)\, x_{(j)} = T_n \quad \text{(say)} \qquad \ldots(3.25)$$

where $x_{(1)} < \ldots < x_{(n)}$ are the sample ordered variables. Since, for all $\delta < 1$, $E_\theta\{\, | x^*(\alpha_1, \alpha_2, \alpha_3) |^{1+\delta}\} < \infty$, it follows that by taking $\delta > \frac{2}{3}$ we can apply our theorem 3. Hence the consistency of T_n follows. To show that the other two theorems will not hold true, it is sufficient to show that $\zeta_3 = E_\theta\{\, | x^*(\alpha_1 \alpha_2, \alpha_3) |^2\} \not< \infty$.

$$\text{Now} \quad \zeta_3 = 6 \int_{-\infty}^{\infty} z^2\, \Phi(z)[1 - \Phi(z)]\, \phi(z) dz \qquad \left(\text{where } \phi(z) = \frac{1}{\pi} \cdot \frac{1}{1+z^2} \right)$$

Therefore for any $X_1 > X_0 \geqslant 1$ we get

$$\int_{X_0}^{X_1} z^2\, \Phi(z)[1 - \Phi(z)]\, \phi(z) dz \geqslant \frac{1}{4\pi} \int_{X_0}^{X_1} [1 - \Phi(z)]\, dz$$

$$= \frac{1}{4\pi^2} \left[\log \frac{X_1}{X_0} + \sum_{s=1}^{\infty} \frac{(-1)^s}{2s(2s+1)} \left[\frac{1}{X_0^{2s}} - \frac{1}{X_1^{2s}} \right] \right] \ldots(3{\cdot}26)$$

as $z^2\, \phi(z) \geqslant \frac{1}{2\pi}$ for all $z \geqslant 1$ and $\Phi(z) \geqslant \frac{1}{2}$ for $z \geqslant 0$.

Therefore, proceeding to the limit $X_1 \to \infty$ we have $\log X_1 / X_0 \to \infty$ and hence (3.26) does not converge to any finite limit. Similarly, the divergence at $-\infty$ follows. Hencc $\zeta_3 \not< \infty$ and this completes the illustration.

In the theorem following, we have briefly sketched the case of any regular function of U-statistics.

Theorem 4 : *Let $U_n = (U_n^{(1)}, \ldots, U_n^{(k)})$ denote a vector of $k (\geqslant 1)$ U-statistics based on a sample of size n and let $G_\theta = (g_1(\theta), \ldots, g_k(\theta))$ denote the population vector and let U_n satisfy the regularity conditions of Theorem 3. If now $H(U_n)$ denotes a continuons function of U_n with continuous first order partial derivatives with respect to $U_n^{(i)}$ $(i = 1, \ldots, k)$ for all U_n in a non-degenerate region containing G_θ as an inner point, then $H(U_n)$ converges to $H(G_\theta)$ with probability one.*

Proof : Let $h_n^{(i)} = U_n^{(i)} - g_i(\theta)$ for $i=1, \ldots, k$ and $h_n = (h_n^{(1)} \ldots, h_n^{(k)})$ and let further $(\lambda h_n) = (\lambda_1 . h_n^{(1)}, \ldots, \lambda_k . h_n^{(k)})$ with $0 < \lambda_1, \ldots, \lambda_k < 1$. Then, by the first mean value theorem, we get

$$H(U_n) = H(G_\theta) + \sum_{i=1}^{k} h_n^{(i)} . \left. \frac{\partial H_n}{\partial U_n^{(i)}} \right|_{U_n = G_\theta + (\lambda h_n)} \qquad \ldots(3{\cdot}27)$$

Also, by a direct extension of theorem 3, we have for $n \geqslant n_0$ (ϵ, δ)

$$P\left\{ \bigcup_{j=1}^{N} \bigcup_{i=1}^{K} \left[\left| U_{n+j}^{(i)} - g_i(\theta) \right| > \epsilon \right] \right\} < \delta \qquad \ldots(3{\cdot}28)$$

uniformly in N. Also it follows from the regularity conditions on H (U_n) that for all $n \geqslant n_0$ (ϵ, δ) and $|h_{n+j}^{(i)}| < \epsilon$ for $i=1, \ldots, k$ and $j=1, \ldots, N$ we have

$$\left| \left. \frac{\partial H_{n+j}}{\partial U_{n+j}^{(i)}} \right|_{U_{n+j} = G_\theta + (\lambda h_{n+j})} \right| < c_i \ (i=1, \ldots, k) \qquad \ldots(3{\cdot}29)$$

where $c_1, \ldots, c_k$ are all finite. Thus, on writing $\eta = \epsilon \left(\sum_{i=1}^{k} c_i \right)$ and noting that η will also be arbitrarily small (depending on ϵ), we get from (3.27) through (3.29) that for $n \geqslant n_0$ (ϵ, δ) and any N,

$$P\left\{ \bigcup_{j=1}^{N} \left[\left| H(U_{n+j}) - H(G_\theta) \right| > \eta \right] \right\} < \delta \qquad \ldots(3{\cdot}30)$$

Hence, the theorem.

In our discussion so far, we have considered the case $\zeta_1 > 0$. Situations may arise, as indicated by Von Mises (1947) and Hoeffding (1948), in which for some $c \geqslant 1$, $\zeta_0 = \ldots = \zeta_c = 0$ and $\zeta_{c+1} > 0$. We are then to use a $(c+1)$-th order decomposition (as has been indicated in Remark 1 of Theorem 1) with f_{c+1} $(X_{\alpha_1}, \ldots, X_{\alpha_{c+1}})$, which will then be uncorrelated for all possible sets. The rest of ourproofs will remain essentially the same, while the condition (3.8) can further be relaxed to $\delta > 1 - (c+1)/m$.

Incidentally, as for the consistency of U-statistics, we require only

$$\text{that} \quad E_\theta \{ |f(X_1, \ldots, X_m)|^{1+\delta} \} < \infty \qquad \ldots(3{\cdot}31)$$

$$\left. \begin{array}{ll} \text{for some } \delta \geqslant 0 & \text{for } m=1 \\ \quad > 1 - 2/m & \text{for } m > 1 \end{array} \right\} \qquad \ldots(3{\cdot}32)$$

The proof of the statement is as follows. Following the lines of theorem 3, we may decompose $U_n^{(2)}$ as in (3.15). Now, it follows from Markoff's Law of Large Numbers that $U_n^{(2)}$ (Z) converges in probability to $g(\theta)$ for any $\delta>0$. While proceeding as in (3.22), we note that for any $\delta>1-2/m$ the quantity $E_\theta\{U_n^{(2)}(R)\}^2\to 0$ as $n\to\infty$, and hence an application of Tshebysheff's lemma yields that $U_n^{(2)}(R)$ also converges in probability to zero. In the case, $m=1$, the Kintchine's Law of Large Numbers applies and hence $\delta=0$ is sufficient. Hence, the assertion.

Finally, it should be noted that all these concepts can directly be extended to the case of generalized U-statistics, as defined by Lehmann (1951), but for the close analogy in approach, they are not reproduced here.

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