Self-Dual Cones in Euclidean Spaces

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ABSTRACT

Self-dual convex cones arise, for example, in the study of copositive matrices and copositive quadratic forms. We begin by giving necessary and sufficient conditions for a cone to be the orthogonal transform of the positive orthant. Next we give a technique for constructing self-dual cones in E^n which produces for all $n \geqslant 3$ many self-dual cones and even polyhedral self-dual cones which are not similar to the nonnegative orthant. We examine the structure of self-dual cones in E^n which contain an $n-1$ dimensional self-dual cone. Finally we show that if K is a cone which is contained in its dual, then there is a self-dual cone containing K.

A cone K in Euclidean n-space is called self-dual provided each linear functional f is non-negative on K if and only if there exists $y \in K$ such that $f(x) = (y,x)$, where (y,x) denotes the inner product. Self-dual cones arise in the study of copositive matrices and copositive quadratic forms (see [6]). In this paper we show that there are many such self-dual cones, provide a characterization of cones which are isometric to the non-negative orthant, and examine the structure of those self-dual cones in E^n which contain $n-1$ dimensional self-dual cones. The following definitions will be needed:

(1) A *cone* K in E^n is a set such that for all $x, y \in K$, $a, b \geqslant 0$, $ax + by \in K$.

Let K be a cone in E^n.

(2) The *partial order induced by* K on E^n is obtained by defining $x \leqslant y$ iff $y - x \in K$.

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(3) K is *closed* if it is topologically closed in the usual topology of E^n, *full* if $K^0 \neq \varnothing$ ($K^0 = K$ interior).

(4) K is *pointed* if $x \in K$ and $-x \in K$ imply $x = 0$.

(5) The *dual* of K is the set

$$K^* = \{ y : (y,x) \geqslant 0 \text{ for all } x \in K \}.$$

(6) K is *self-dual* if $K = K^*$.

For various properties of K and K^* see [4], [5], or [9]. In particular, if $K = K^*$, then K is closed, pointed, and full. The usual cones which arise in extensions of Perron-Frobenius theory [1] are closed, pointed, and full, but not self-dual. Examples of self-dual cones are the non-negative orthant, which consists of all $x \in E^n$ with non-negative components; the n-dimensional ice cream cone [7], and the cone of positive semi-definite matrices in the real space of all Hermitian matrices (cf. [4]).

In the study of cones certain subsets, called *faces*, have proven to be quite useful. For more results on faces see [2, 3, 8].

DEFINITION Let K be a closed, pointed cone in E^n. A subset F of K is a *face* if F is a cone and

$$0 \leqslant x \leqslant y \quad \text{and} \quad y \in F \qquad \text{implies} \qquad x \in F.$$

We write $F \trianglelefteq K$.

REMARK If $S \subset K$, then the intersection of all faces of K containing S is a face called the *face generated by* S. It is denoted $\phi(S)$. If $S = \{x\}$, we write $\phi(x)$ for $\phi(\{x\})$. As is well known, the space spanned by a face F is $F - F$, and the dimension of this space is called the dimension of F. If $\phi(x)$ is of dimension one, then x is called an *extremal* of K. If K has only finitely many extremals, it is called *polyhedral*; if K has exactly n extremals, it is called *simplicial*.

DEFINITION Let K be a closed, pointed cone in E^n.

(1) If $F \trianglelefteq K$, then we put

$$F^D = \{ y : y \in K^* \text{ and } \forall x \in F, \ (y,x) = 0 \}.$$

This is the *positive annihilator* of F (or just annihilator if the use is clear).

(2) We denote by F^V the dual of F in $\operatorname{span} F$.

p(3) The cone K is the *direct sum* of K_1 and K_2, and we write $K = K_1 \oplus K_2$ iff the following hold:

(i) $\forall x \in K \quad \exists x_i \in K_i, \qquad x = x_1 + x_2$

(ii) $\operatorname{span} K_1 \cap \operatorname{span} K_2 = \{0\}$.

REMARK (3) is just decomposability in the sense of Loewy and Schneider [7]. It follows that if $K = K_1 \oplus K_2$, then $K_i \trianglelefteq K$, $i = 1,2$, and the decomposition $x = x_1 + x_2$, $x_i \in K_i$, is unique.

We now prove a theorem which gives necessary and sufficient conditions for a self-dual cone to be isometric to an orthant. The following lemma is needed for this purpose.

LEMMA *Let K be a closed, full, pointed cone. If $K = K_1 \oplus K_2$ and if $x \in K_1$, $y \in K_2$ implies $(x,y) = 0$, then $(K_1 \oplus K_2)^* = K_1{}^V \oplus K_2{}^V$.*

Proof. This lemma is well known (cf. [4, p. 5]). We include a short proof for completeness. Under the hypotheses it is clear that $K_1{}^V \oplus K_2{}^V \subset (K_1 \oplus K_2)^*$. If $z \in (K_1 \oplus K_2)^*$, $x \in K_1$, and $y \in K_2$, put $z_1(x+y) = (z,x)$ and $z_2(x+y) = (z,y)$. This determines vectors z_1 and z_2. Then $z_i \in K_i{}^V$, $i = 1,2$, and for any $x = x_1 + x_2 \in K$,

$$(z,x) = (z,x_1) + (z,x_2) = (z_1,x_1) + (z_2,x_2)$$

$$= (z_1,x) + (z_2,x) = (z_1 + z_2, x).$$

Since K is full, $z = z_1 + z_2$ and the lemma is proved. ∎

COROLLARY *Under the conditions of the lemma, if K is self-dual, then $K_i^V = K_i$.*

Proof. If $x \in K_1 \subset K = K^*$, then for any $y \in K_1 \subset K$, $(x,y) \geqslant 0$. So $x \in K_1^V$. If $y \in K_1^V \subset K^* = K$, then for any $x \in K_1^V \subset K^*$, $(x,y) \geqslant 0$, so $y \in (K_1^V)^V = K_1$. This latter follows because K is self-dual and so every face is closed, pointed, and full in its span. $K_2^V = K_2$ is the same. ∎

THEOREM 1. *If K is a self-dual polyhedral cone such that every proper maximal face is also self-dual, then K is the image of the non-negative orthant under an orthogonal transformation.*

REMARK The converse is obvious.

Proof. It is easy to show that a self-dual simplicial cone is the image of the non-negative orthant under an orthogonal transformation. Thus it is enough to show that K is simplicial. We show first by induction that every face of K is self-dual in its span. For $\dim K = 2$ this is obvious. Suppose that for every self-dual polyhedral cone of dimension $\leqslant n-1$, if every proper maximal face is self-dual, then every face is self-dual (in its span). Let $\dim K = n$. Let x be an extremal of K, and put $F = [\phi(x)]^D$. Then F is a maximal proper face of K, $\dim F = n-1$, so $F^V = F$. Let G be a maximal face of F. We claim that $G \oplus \phi(x)$ is a face of K. Suppose that $0 \leqslant y \leqslant g + ax$. Note that $F \oplus \phi(x)$ satisfies the hypotheses of the lemma. Hence

$$K = K^* \subset [F + \phi(x)]^* = F^V \oplus \phi(x)^V = F + \phi(x) \subset K.$$

Therefore, $K = F \oplus \phi(x)$ and $y = f_1 + b_1 x$ with $f_1 \in F$, $b_1 \geqslant 0$, and there is a $y_1 = f_2 + b_2 x$, $f_2 \in F$, $b_2 \geqslant 0$, such that

$$0 \leqslant y + y_1 = g + ax = (f_1 + f_2) + (b_1 + b_2)x.$$

But because the representation is unique we have

$$g = f_1 + f_2 \geqslant f_1 \geqslant 0,$$

whence $f_1 \in G$, since $G \trianglelefteq F$. The claim is established. Since $\dim[G \oplus \phi(x)] = n-1$, the face is maximal; hence $G \oplus \phi(x)$ is self-dual. Since $G \subset [\phi(x)]^D$, the conditions of the lemma are satisfied, and $G = G^V$. By the induction hypothesis every face of F is self-dual. Since every face of K is contained in a maximal face, every face of K is self-dual.

We now show, again by induction, that K is simplicial. For $\dim K = 2$ the result is trivial. So suppose that whenever $\dim K \leqslant n-1$ and every maximal face is self-dual, K is simplicial. Let $\dim K = n$. Since every maximal face is self-dual, every face is self-dual. Let x be an extremal of K and let $F = [\phi(x)]^D$. Then F is a simplicial cone of dimension $n-1$. We claim that $F \oplus \phi(x) = K$. Let $K_1 = F \oplus \phi(x)$. Then by the lemma K_1 is self dual. But $K_1 \subset K$ and K is self-dual; so $K_1 = K_1^* \supset K^* = K$, and thus $K_1 = K$, and the theorem is proved. ■

Theorem 2 provides a method for constructing n-dimensional self-dual cones from $(n-1)$-dimensional ones. The method permits the construction of polyhedral cones of any dimension greater than 2 which are not isometric to orthants.

THEOREM 2. *Let K_0 be a closed, pointed cone in E^n such that $K_0 = K_0^V$ and* $\dim K_0 = n-1$. *Let K_+ be a closed, pointed, full cone which has no points in one of the open half spaces determined by $H = \operatorname{span} K_0$. Suppose further that $K_0 \subset K_+ \subset (K_+)^*$. Then there is a self-dual cone K such that $K_+ \subset K$ and such that $K \setminus K_+$ has no points in the half space containing K_+.*

REMARK That the cone K satisfying the conditions of this theorem is unique follows from (i)⇒(iii) of Theorem 4.

Proof. We may without loss of generality assume that H is the subspace of E^n consisting of all vectors with nth coordinate zero. In addition, if $e = (0,\dots,0,1)$ then we may assume that for all $y \in K_+$, $(y,e) \geqslant 0$. Set $K = \{x : x \in (K_+)^* \text{ and } (e,x) \leqslant 0\} \cup K_+$. In addition, we let $H_+ = \{ y : (y,e) \geqslant 0\}$ and define H_- analagously. We claim that $K = K^*$. From this it is obvious that K is a closed, pointed, full cone. Let $u \in K$. We show $u \in K^*$. So let $v \in K$.

(i) If $u \in K_+$, $v \in K_+$, then $(u,v) \geqslant 0$, since $K_+ \subset (K_+)^*$.

(ii) If $u \in K_+$, $v \in K \cap H_-$ or if $u \in K \cap H_-$, $v \in K_+$, then $(u,v) \geqslant 0$, since $K \cap H_- = (K_+)^* \cap H_-$.

(iii) If $u,v \in K \cap H_-$, we let $\bar{u} = u - (e,u)e$, $\bar{v} = v - (e,v)e$. Then it is easy to check that $\bar{u}, \bar{v} \in K_0^* \cap H = K_0$. Thus $(\bar{u},\bar{v}) \geqslant 0$. But $(\bar{u},\bar{v}) = (u,v) - (u,e)(v,e) \geqslant 0$, so $(u,v) \geqslant (u,e)(v,e) \geqslant 0$. Thus $u \in K^*$ and $K \subset K^*$.

To show that $K^* \subset K$ take $z \in K^*$.

(i) If $z \in H_-$, then since $K^* \subset (K_+)^*$ we have $z \in (K_+)^* \cap H_- \subset K$.

(ii) If $z \in H$, then since $K^* \subset K_0^*$ we have $z \in K_0^* \cap H = K_0 \subset K$.

(iii) If $z \in H_+$, but $z \notin K$, then there is a $w \in K^*$ such that $(w,z) < 0$ and $(w,x) \geqslant 0$ for all $x \in K$. Applying the previous cases to w, we see that $w \in H_+$ as well. Let

$$\bar{z} = z - (e,z)e, \qquad \bar{w} = w - (e,w)e.$$

Clearly $(e,z) > 0$, $(e,w) > 0$. Also

$$\bar{z}, \bar{w} \in K^* \cap H = K_0 = K_0^V,$$

so $(\bar{z},\bar{w}) \geqslant 0$. Thus, $(z,w) = (\bar{z},\bar{w}) + (e,z)(e,w) > 0$, a contradiction. Therefore $z \in K$ and $K^* \subset K$. ∎

Every two dimensional self-dual cone is isometric with the two dimensional orthant. Some examples of three dimensional, polyhedral, self-dual cones are:

(1) The cone K with extremals $\phi(v_i)$, where

$$v_0=(1,1,1), \qquad v_1=(0,1,1), \qquad v_2=(-1,0,1),$$

$$v_3=(0,-1,1), \quad \text{and} \quad v_4=(1,-1,1).$$

It is readily determined that $(v_i,v_j)\geqslant 0$ and that $\phi(v_i)^D=\phi(v_p,v_q)$, where $p\equiv i+2 \bmod 5$, $q\equiv i+3 \bmod 5$. From this it follows that $\forall x\in K$, $\exists i \ni (x,v_i)<0$ and thus $K=K^*$.

(2) If a cone K_n is formed over the regular $(2n+1)$-sided polygon, so that each extremal is perpendicular to the face determined by the opposite side of the polygon, then K_n is self-dual. In particular, let K_n be determined by $\phi(v_j)$, where $r=(2n+1)^{-1}$ and for $j=0,1,\dots,2n$,

$$v_j=\left(\cos 2\pi jr, \sin 2\pi jr, \sqrt{-\cos 2\pi nr}\,\right).$$

Again it is readily determined that $(v_i,v_j)\geqslant 0$ and that $\phi(v_i)^D=\phi(v_p,v_q)$, where $p=j+n \bmod 2n+1, q=j+n+1 \bmod 2n+1$; from this it follows that $K_n=K_n^*$.

The examples in (2) above show that there are polyhedral, self-dual cones in E_3 with any odd number of extremals. Conversely, we have the following:

THEOREM 3. *Every self-dual, polyhedral cone in E^3 has an odd number of extremals.*

Proof. Let K be a polyhedral, self-dual cone in E^3. It is readily seen that there are two extremals of K, $\phi(v_0)$ and $\phi(v_1)$, such that $(v_0,v_1)=0$. Suppose v_0 and v_1 are selected so that $(v_0,v_0)=(v_1,v_1)=1$. Let H be the plane determined by v_0, v_1, and 0. Then $\phi(v_0)^D$ and $\phi(v_1)^D$ are faces of K which lie on the same side of H. To see this, let $F_1=\phi(v_0)^D=\phi(v_1,w_1)$ and $F_2=\phi(v_1)^D=\phi(v_0,w_0)$. Let $e=w_1-(w_1,v_1)v_1$. Then $(e,v_0)=(e,v_1)=0$ and e is perpendicular to H. But $(e,w_1)\geqslant 0$ and $(e,w_0)=(w_1,w_0)\geqslant 0$, so that w_0 and w_1 lie on the same side of H.

Let u_i, $i=1,\dots,k$, be the extremals of K satisfying $(u_i,e)<0$. (In the event that there are no u_i, the cone is an orthant.) Each u_i determines a face $\phi(u_i)^D$, and these faces lie on the e-side of H. To see this, let $\bar{w}=w-(w,v_0)$

$v_0 - (w,v_1)v_1$, where $w \in K$ and w is perpendicular to a particular $u_i = u$. Then $(\bar{w},v_0) = (\bar{w},v_1) = 0$, and $\bar{w}$ is perpendicular to H. But $(u,\bar{w}) = (u,w) - (w,v_0)(u,v_0) - (w,v_1)(u,v_1)$, and since $(u,w) = 0$, $(u,\bar{w}) \leqslant 0$. Since $(e,w) = (e,\bar{w})$, it follows that w and u are on opposite sides of H.

Since $\phi(v_0)^D$, $\phi(v_1)^D$, $\phi(u_i)^D$, $i = 1,\dots,k$, are the faces of K on the e-side of H, it follows that there are exactly $k+1$ extremals at the successive intersections of these $k+2$ faces. Thus K has k extremals u_i, 2 extremals v_0 and v_1, and $k+1$ extremals on the e-side of H; i.e., K has $2k+3$ extremals. Theorem 3 is thus proved. ∎

We pose the following questions:

(1) What number of extremals are possible for a self-dual, polyhedral cone in E^n, $n > 3$?

(2) Does every n-dimensional, self dual cone K contain an $(n-1)$-dimensional cone $K_0 = K_0^V$?

The answer to the second question is clearly yes for $n = 3$. The following theorem characterizes the relationship which holds between such cones K and K_0.

Theorem 4. *Let H be a hyperplane determined by a vector e, where $(e,e) = 1$. Let K be a self-dual cone, $H_+ = \{x : (x,e) \geqslant 0\}$, $H_- = \{x : (x,e) \leqslant 0\}$, $K_+ = K \cap H_+$, $K_- = K \cap H_-$, $K_0 = K_+ \cap K_-$. Define $K_H = \{z \in K : z - (z,e)e \in K\}$. Then K_H is a cone, and the following are equivalent*:

(i) $K_0 = K_0^V$,
(ii) $K_H = K$,
(iii) $(K_+)^* \cap H_- = K_-$ and $(K_-)^* \cap H_+ = K_+$.

Proof. *K_H is a cone.* Let $u,v \in K_H$; $a,b \geqslant 0$. Then $u - (u,e)e \in K$ and $v - (v,e)e \in K$. Consequently, $au - a(u,e)e + bv - b(v,e)e \in K$. That is, $(au + bv) - (au + bv,e)e \in K$, and K_H is a cone.

(i) *implies* (ii). Let $z \in K$, $\bar{z} = z - (z,e)e$; then $\bar{z} \in H$. Let $y \in K_0$. Then $(y,e) = 0$ and hence $(y,\bar{z}) = (y,z) \geqslant 0$. Since $y \in K_0$ is arbitrary, $\bar{z} \in K_0^*$; hence,

$$\bar{z} \in K_0^* \cap H = K_0^V = K_0 \subseteq K.$$

Since z is arbitrary, $K \subset K_H$. However, $K_H \subset K$ by the definition of K_H. Thus $K_H = K$.

(ii) *implies* (iii). Let $x\in(K_+)^*\cap H_-$; then $\forall y\in K_+$, $(y,x)\geqslant 0$. Suppose $y\in K_-$ and $\bar{y}\in H$, $\bar{y}=y-(e,y)e$. By hypothesis, $\bar{y}\in K$. Since $\bar{y}\in H$, $\bar{y}\in K_+$, and thus $(\bar{y},x)\geqslant 0$. But $(e,y)\leqslant 0$, $(e,x)\leqslant 0$, so that

$$(y,x)=(\bar{y},\bar{x})+(e,y)(e,x)\geqslant 0$$

and $x\in K^*=K$. Since $x\in H_-$, $x\in K_-$, and because x was an arbitrary element in $(K_+)^*\cap H_-$, it follows that $(K_+)^*\cap H_-\subset K_-$. However, $K_-\subset K^*\cap H_-\subset(K_+)^*\cap H_-$. The other half of condition (iii) follows from the parallel proof which exchanges the roles of H_+ and H_-.

(iii) *implies* (i). First note that

$$K_0^V=K_0^*\cap H=(K_+\cap K_-)^*\cap H\supset\left[(K_+)^*+(K_-)^*\right]\cap H\supset K\cap H=K_0.$$

But also

$$K_0^V\subset(K_+)^*\cap H_-=K_-,$$

and

$$K_0^V\subset(K_-)^*\cap H_+=K_+,$$

so that $K_0^V\subset K_-\cap K_+=K_0$. Thus $K_0^V=K_0$. ∎

Remark If K is self-dual, it is always possible to find an H such that $(K_+)^*\cap H_-\supset K_-$ and $(K_-)^*\cap H_+\supset K_+$. For let

$$A=\left\{e\in E^n\setminus\{0\}:(K_{e^+})[\cap H_{e^-}\supsetneq K_{e^-}\right\}$$

and

$$B=\left\{e\in E^n\setminus\{0\}:(K_{e^-})[\cap H_{e^+}\supsetneq K_{e^+}\right\}.$$

Either there exists an $e_0\in A^c\cap B^c$, or $A\cup B=E^n\setminus\{0\}$. However, $A^c=\{e\in E^n\setminus\{0\}:(K_{e+})^*\cap H_{e-}=K_{e-}\}$ and $B^c=\{e\in E^n\setminus\{0\}:(K_{e-})^*\cap H_{e+}=K_{e+}\}$. For such an e_0, $(K_+)^*\cap H_-=K_-$ and $(K_-)^*\cap H_+=K_+$. Because A^c and B^c are closed, we have in the latter case $E^n\setminus\{0\}$ is the union of two open sets, and since $E^n\setminus\{0\}$ is connected, there is a point $e^1\in A\cap B$ and e_1 provides an H for which $(K_+)^*\cap H_-\supsetneq K_-$ and $(K_-)^*\cap H_+\supsetneq K_+$.

We conclude with the following theorem, which shows that a cone which is contained in its dual is always contained in a self-dual cone.

THEOREM 5. *If K is a cone and $K \subset K^* \subset E^n$, then there exists a cone K_D such that $K \subset K_D = K_D^* \subset K^*$.*

Proof. Let $K_1 = K$ and let $\{x_i\}$ be an enumeration of a countable dense subset of E^n. Define K_i, $i = 2,3,\ldots$ inductively as follows:

$$\begin{array}{llll} \text{If} & x_i \in K_i \ \text{ or } \ x_i \not\in K_i^*, & \text{let} & K_{i+1} = K_i \\ \text{If} & x_i \in K_i^* \setminus K_i, & \text{let} & K_{i+1} = K_i \oplus t x_i, \quad t \geqslant 0. \end{array}$$

Then if $u, v \in K_i$ and $a, b \geqslant 0$ and $K_{i+1} = K_i \oplus t x_i$, it follows that $(u + a x_i, v + b x_i) \geqslant 0$. Consequently, $K_{i+1} \subset K_{i+1}^*$. Hence $\forall i$, $K_i \subset K_{i+1} \subset K_{i+1}^* \subset K_i^*$. Let $K_A = \cup K_i$. K_A is clearly a cone, and

$$y \in K_A^* \quad \Leftrightarrow$$

$$\forall x \in K_A, \quad (x, y) \geqslant 0 \quad \Leftrightarrow \quad \forall n, \quad y \in K_n^* \quad \Leftrightarrow \quad y \in \cap K_i^*.$$

Thus $K_A \subset K_A^* = \cap K_i^*$. Let K_D be the closure of K_A. Since $\forall x, y \in K_D$, $(x, y) \geqslant 0$, it follows that $K_D \subset K_D^*$. However, if $K_D \neq K_D^*$, then by observing that $K_D^* \setminus K_D$ has non-empty interior and thus contains some point x_i, a contradiction to the method of construction of K_D is obtained.

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