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Exact Distribution for the Product of Two Correlated Gaussian Random Variables

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***Abstract*—This paper considers the distribution of product for two correlated real Gaussian random variables with non zero-means and arbitrary variances, which arises widely in radar and communication societies. We determine the exact probability density function (PDF) in terms of an infinite sum of modified Bessel functions of second kind, which includes some existent results, i.e., zero-means and/or independent variables, as special cases. Then, we study the approximation error and convergence rate when finite summations are exploited in practice. Finally, we evaluate the PDF behaviors of the derived expression as well as the Monte Carlo simulations.**

***Index Terms*—Distribution; two correlated real Gaussian random variables; non zero-means and arbitrary variances; exact probability density function.**

I. Introduction

The product of the random variables is of great interest in radar and communication societies [1]–[6]. For example, in a communication system, it can model the keyhole or pinhole channel where both the transmitter and the receiver are encompassed by multipath scattering [2], [3]. In time reversal detection [4], [5], the aggregate random channel is often described as the product of two complex Gaussian random variables. Moreover, it can be also employed to evaluate the detection performance for radar system [6].

The distribution for the product of the random variables has received considerable attention. In [7], the exact probability density function (PDF) is developed for the product of independent generalized Gamma variables with the same shape parameter. The distribution of the product of independent nonnegative random variables is investigated in [8] where one obeys a subexponential distribution. Considering the product of the classic distributions, e.g., Rician, Rayleigh, and chi-squared, the PDFs are developed for some special conditions in [9], [10]. In [11], [12], the numerical methods are exploited to evaluate the distribution of the product of two random variables.

Gaussian random variables commonly arise in radar and communication applications, and the distribution for the product of Gaussian or complex Gaussian random variables has also received huge attention. For example, in [13], the joint characteristic functions of the inner product for two independent complex Gaussian vectors is studied. In [14], the joint (amplitude, phase) distribution of the product of two independent non zero-mean complex Gaussian random variables is developed. In [15], the PDF of the product of two normal random variables with zero-means and unit variances is derived. The PDFs for the products involving real Gaussian random variables are summarized in [9], whereas they require some specified conditions, i.e., zero-mean or independent.

However, to the authors' best knowledge, the distribution for the product of correlated Gaussian random variables with non zero-means and arbitrary variances has never been considered in open literature. In this paper, we try to fill this gap and consider the problem of deriving an exact expression for the PDF of the product of two correlated real Gaussian random variables. It is determined in terms of an infinite sum of modified Bessel functions of second kind, which includes some existent results as special cases [9], [15]. At the analysis stage, we study the approximation error and convergence rate, showing that the derived PDF converges quickly and finite summations of few terms can lead to high accurate approximations. In addition, the PDF behaviors are consistent with Monte Carlo simulations.

The remainder of the paper is organized as follows. In Section II, we derive the aforementioned PDF. In Section III, we evaluate the approximation error, convergence rate and PDF behaviors. Finally, in Section IV, we provide some concluding remarks.

II. Exact PDF Derivation

Theorem 2.1: Let X_1 and X_2 be respectively two real Gaussian random variables with means μ_1, μ_2 and variances σ_1^2, σ_2^2, i.e., $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, and denote by ρ the correlation coefficient

$$\rho = \frac{E\{(X_1 - \mu_1)(X_2 - \mu_2)\}}{\sigma_1 \sigma_2}, \tag{1}$$

where $E\{\cdot\}$ denotes the statistical expectation. Then, the exact PDF $f_X(x)$ of the product $X = X_1 X_2$ is given by

$$f_X(x) = \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} - \frac{2\rho(x+\mu_1\mu_2)}{\sigma_1\sigma_2}\right)\right\} \times \sum_{n=0}^{\infty}\sum_{m=0}^{2n} \frac{x^{2n-m}|x|^{m-n}\sigma_1^{m-n-1}}{\pi(2n)!(1-\rho^2)^{2n+1/2}\sigma_2^{m-n+1}} \binom{2n}{m} \left(\frac{\mu_1}{\sigma_1^2} - \frac{\rho\mu_2}{\sigma_1\sigma_2}\right)^m \times \left(\frac{\mu_2}{\sigma_2^2} - \frac{\rho\mu_1}{\sigma_1\sigma_2}\right)^{2n-m} K_{m-n}\left(\frac{|x|}{(1-\rho^2)\sigma_1\sigma_2}\right). \tag{2}$$

where $|\cdot|$ is the absolute value operation and $K_v(\cdot)$ denotes the modified Bessel function of the second kind and order v.

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Proof: The PDF $f_X(x)$ of the product $X = X_1X_2$ can be computed as [16]

$$f_X(x) = \int_{-\infty}^{+\infty} \frac{1}{|t|} f_{X_1X_2}(t, x/t)dt, \tag{3}$$

where $f_{X_1X_2}(x_1, x_2)$ is the joint PDF of X_1 and X_2, i.e.,

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\times \left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right\}. \tag{4}$$

Substituting (4) into (3), it becomes

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ &\times \int_{-\infty}^{+\infty} \frac{1}{|t|} \exp\left\{-\frac{1}{2(1-\rho^2)}\times \left(\frac{(t-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(t-\mu_1)(x/t-\mu_2)}{\sigma_1\sigma_2} + \frac{(x/t-\mu_2)^2}{\sigma_2^2}\right)\right\} dt \\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\times \left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} - \frac{2\rho(x+\mu_1\mu_2)}{\sigma_1\sigma_2}\right)\right\}\times \\ &\int_{-\infty}^{+\infty} \frac{1}{|t|} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2}t^2 + \frac{x^2}{\sigma_2^2t^2} - \left(\frac{2\mu_1}{\sigma_1^2} - \frac{2\rho\mu_2}{\sigma_1\sigma_2}\right)t - \left(\frac{2x\mu_2}{\sigma_2^2} - \frac{2\rho\mu_1 x}{\sigma_1\sigma_2}\right)\frac{1}{t}\right]\right\} dt. \end{aligned} \tag{5}$$

Based on the expansion of the exponential, we have

$$\begin{aligned} &\exp\left\{\frac{\left(\frac{\mu_1}{\sigma_1^2} - \frac{\rho\mu_2}{\sigma_1\sigma_2}\right)}{1-\rho^2}t + \frac{\left(\frac{x\mu_2}{\sigma_2^2} - \frac{\rho\mu_1 x}{\sigma_1\sigma_2}\right)}{1-\rho^2}\frac{1}{t}\right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\left(\frac{\mu_1}{\sigma_1^2} - \frac{\rho\mu_2}{\sigma_1\sigma_2}\right)}{1-\rho^2}t + \frac{\left(\frac{x\mu_2}{\sigma_2^2} - \frac{\rho\mu_1 x}{\sigma_1\sigma_2}\right)}{1-\rho^2}\frac{1}{t}\right]^n \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{x^{n-m}}{n!(1-\rho^2)^n}\binom{n}{m}\left(\frac{\mu_1}{\sigma_1^2} - \frac{\rho\mu_2}{\sigma_1\sigma_2}\right)^m \times \left(\frac{\mu_2}{\sigma_2^2} - \frac{\rho\mu_1}{\sigma_1\sigma_2}\right)^{n-m} t^{2m-n}. \end{aligned} \tag{6}$$

Substituting (6) into (5), and exchanging the order of integral and sum operations, it yields

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\times \left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} - \frac{2\rho(x+\mu_1\mu_2)}{\sigma_1\sigma_2}\right)\right\} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{x^{n-m}}{n!(1-\rho^2)^n}\binom{n}{m} \\ &\times \left(\frac{\mu_1}{\sigma_1^2} - \frac{\rho\mu_2}{\sigma_1\sigma_2}\right)^m \left(\frac{\mu_2}{\sigma_2^2} - \frac{\rho\mu_1}{\sigma_1\sigma_2}\right)^{n-m} \times \\ &\int_{-\infty}^{+\infty} \frac{t^{2m-n}}{|t|} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2}t^2 + \frac{x^2}{\sigma_2^2t^2}\right]\right\} dt. \end{aligned} \tag{7}$$

The integral term in the above equation can be recast as

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{t^{2m-n}}{|t|} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{1}{\sigma_1^2}t^2 + \frac{x^2}{\sigma_2^2t^2}\right)\right\} dt \\ &= \int_{-\infty}^{0} \frac{t^{2m-n}}{|t|} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{1}{\sigma_1^2}t^2 + \frac{x^2}{\sigma_2^2t^2}\right)\right\} dt \\ &+ \int_{0}^{+\infty} \frac{t^{2m-n}}{|t|} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{1}{\sigma_1^2}t^2 + \frac{x^2}{\sigma_2^2t^2}\right)\right\} dt \\ &= (1+(-1)^n)\times \\ &\int_{0}^{+\infty} t^{2m-n-1} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{1}{\sigma_1^2}t^2 + \frac{x^2}{\sigma_2^2t^2}\right)\right\} dt. \end{aligned} \tag{8}$$

Solving the above integral using formula [17, 3.478-4, p. 370], we obtain

$$\begin{aligned} &\int_{0}^{+\infty} t^{2m-n-1} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{1}{\sigma_1^2}t^2 + \frac{x^2}{\sigma_2^2t^2}\right)\right\} dt \\ &= \left(\frac{\sigma_1|x|}{\sigma_2}\right)^{(2m-n)/2} K_{(2m-n)/2}\left(\frac{|x|}{(1-\rho^2)\sigma_1\sigma_2}\right). \end{aligned} \tag{9}$$

Substituting (9) into (8), it becomes

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{t^{2m-n}}{|t|} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{1}{\sigma_1^2}t^2 + \frac{x^2}{\sigma_2^2t^2}\right)\right\} dt \\ &= (1+(-1)^n)\times \\ &\left(\frac{\sigma_1|x|}{\sigma_2}\right)^{(2m-n)/2} K_{(2m-n)/2}\left(\frac{|x|}{(1-\rho^2)\sigma_1\sigma_2}\right). \end{aligned} \tag{10}$$

Substituting (10) into (7), and elaborating on, the PDF $f_X(x)$ can be written

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\times \left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} - \frac{2\rho(x+\mu_1\mu_2)}{\sigma_1\sigma_2}\right)\right\} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(1+(-1)^n)\, x^{n-m}}{n!(1-\rho^2)^n} \\ &\times \binom{n}{m}\left(\frac{\mu_1}{\sigma_1^2} - \frac{\rho\mu_2}{\sigma_1\sigma_2}\right)^m \left(\frac{\mu_2}{\sigma_2^2} - \frac{\rho\mu_1}{\sigma_1\sigma_2}\right)^{n-m} \\ &\times \left(\frac{\sigma_1|x|}{\sigma_2}\right)^{(2m-n)/2} K_{(2m-n)/2}\left(\frac{|x|}{(1-\rho^2)\sigma_1\sigma_2}\right). \end{aligned} \tag{11}$$

Exploiting the equality

$$1+(-1)^n = \begin{cases} 2, & \text{for an even } n, \\ 0, & \text{for an odd } n, \end{cases} \tag{12}$$

and substituting (12) into (11), after some algebraic manipulations, $f_X(x)$ can be finally obtained. ■

Expression (2) indicates that the exact PDF requires the computation of an infinite number of terms, each of them involving the modified Bessel function of second kind. From a practical point of view, only a finite number of addends will be considered. To this end, resorting to several numerical simulations which are given in section III, a high precision approximation of the PDF (2) can be obtained using a finite sum with a small number of addends.

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Corollary 2.2: Let X_1 and X_2 be two independent Gaussian random variables, the PDF $f_X(x)$ is

$$f_X(x) = \exp\left(-\frac{\mu_1^2}{2\sigma_1^2} - \frac{\mu_2^2}{2\sigma_2^2}\right) \times \sum_{n=0}^{\infty}\sum_{m=0}^{2n}\binom{2n}{m}\frac{\mu_1^m \mu_2^{2n-m} x^{2n-m}|x|^{m-n}}{\pi(2n)!\sigma_1^{n+m+1}\sigma_2^{3n-m+1}} K_{m-n}\left(\frac{|x|}{\sigma_1\sigma_2}\right). \tag{13}$$

Proof: Substituting $\rho = 0$ into (2), (13) can be obtained. ■

Remark: Theorem 2.1 provides a general analytic PDF expression for the product of two correlated real Gaussian random variables, which includes all the special solutions in [9], [15]. In the following, we provide some special instances of the obtained result showing its agreement with [9], [15].

1) *Independent Gaussian* $(\times)$ *Gaussian (both zero-mean)*: For $\mu_1 = 0$ and $\mu_2 = 0$, except for the first term with $m = n = 0$ in the summation at the right hand of (13) are all zeros, the expression (13) can be recast as (see also [9, Ch. 6, eq. 6.2] and [18])

$$f_X(x) = \frac{1}{\pi\sigma_1\sigma_2} K_0\left(\frac{|x|}{\sigma_1\sigma_2}\right). \tag{14}$$

2) *Gaussian* $(\times)$ *Gaussian (both zero-mean)*: Since only the first term ($m = n = 0$) in the summation at the right hand of (2) is non-zero for $\mu_1 = 0$ and $\mu_2 = 0$, the PDF can be rewritten as (see also [9, Ch. 6, eq. 6.15][1] and [15])

$$f_X(x) = \frac{1}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(\frac{\rho x}{\sigma_1\sigma_2(1-\rho^2)}\right) \times K_0\left(\frac{|x|}{\sigma_1\sigma_2(1-\rho^2)}\right). \tag{15}$$

3) *Independent Gaussian* $(\times)$ *Gaussian (One has zero-mean, and both have identical variance)*: Without loss of generality, suppose $\mu_1 \neq 0$, $\mu_2 = 0$, and $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Substituting them into (13), only first term ($m = 2n$) is reserved, given by

$$f_X(x) = \exp\left(-\frac{\mu_1^2}{2\sigma^2}\right) \sum_{n=0}^{\infty} \frac{\mu_1^{2n}|x|^n}{\pi(2n)!\sigma^{4n+2}} K_n\left(\frac{|x|}{\sigma^2}\right). \tag{16}$$

In the following, we highlight that (16) is the same as (6.28) in reference [9]. Exploiting the fact

$$\Gamma(2n) = \frac{2^{2n-1}\Gamma(n)\Gamma(n+1/2)}{\sqrt{\pi}}, \tag{17}$$

the factorial $(2n)!$ can be expressed as

$$(2n)! = 2n\Gamma(2n) = \frac{4^n n!\Gamma(n+1/2)}{\sqrt{\pi}}, \tag{18}$$

[1]Notice that the term $\sqrt{1-\rho^2}$ was missed at the denominator in reference [9].

Substituting (18) into (16), $f_X(x)$ becomes

$$f_X(x) = \exp\left(-\frac{\mu_1^2}{2\sigma^2}\right) \times \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi}n!\Gamma(n+1/2)\sigma^2}\left(\frac{\mu_1^2|x|}{4\sigma^4}\right)^n K_n\left(\frac{|x|}{\sigma^2}\right). \tag{19}$$

III. Numerical Results

In this section, we discuss the convergence and the behavior of the derived PDF expression. Finally we also provide Monte Carlo simulations to assess the effects of a truncation on the infinite series.

A. Approximation error and convergence rate

According to (2), the evaluation of $f_X(x)$ requires the computation of an infinite number of addends. In practice, the series (2) has to be approximated as the sum of the first N terms. This implies an approximation error, which can be controlled through a careful selection of N. Define the approximation error function with respect to the truncation length N as

$$\epsilon(N) \text{ (in dB)} = 20\log_{10}|P_N - 1|, \tag{20}$$

where P_N is area under the truncated PDF $\bar{f}_X(x;N)$

$$P_N = \left|\int_{-\infty}^{+\infty} \bar{f}_X(x;N)dx - 1\right| \approx \left|\int_{-x_0}^{+x_0} \bar{f}_X(x;N)dx - 1\right|, \tag{21}$$

whereas x_0 considers the truncation of the integration interval. In practice, the integral term in (21) is often computed by numerical approach, which would lead to an additional error. To ensure a high integral precision, the values of x_0 and the quantization steps in x-axis should be large enough.

In Fig. 1, we plot $\epsilon(N)$ (in dB) versus N for $\mu_1 = 0, \sigma_1 = 1, \mu_2 = 1, \sigma_2 = 1.2$ (Fig. 1(a)) and $\mu_1 = 0.5, \sigma_1 = 1, \mu_2 = 1, \sigma_2 = 1.2$ (Fig. 1(b)), respectively and different values of ρ. We fix the integration interval $x \in [-15, 15]$ and divide it equally into $L = 5000000$ sections to ensure enough integral precision in (21). The curves illustrate that the value of $\epsilon(N)$ decreases increasing N and convergence rate is very fast. Specifically, the values of N are about 10 for $\rho = -0.5, 0, 0.5$, and 0.9, respectively. In addition, the higher ρ, the larger the approximation error in correspondence of the same L. For instance, the approximation errors in Fig. 1(b) achieve the values of about -87 dB -94 dB, -64 dB, -56 dB for $\rho = -0.5, 0, 0.5, 0.9$, respectively. It implies that the derived PDF has very low approximation error with a finite sum characterized by a small number of addends assuming the fixed L and integration upper extreme $x_0 = 15$.

B. PDF behaviors with different parameters

In Figs. 2, 3 and 4, we study the PDF behaviors of the analytic expression as well as Monte Carlo simulations for different ρ, means and variances. We set the truncation length N to 30, which is enough to ensure a high precision.

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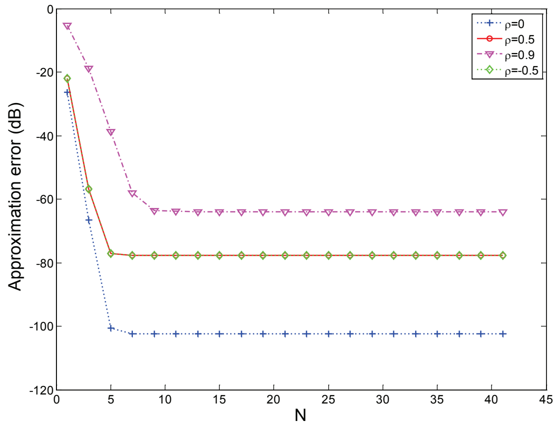
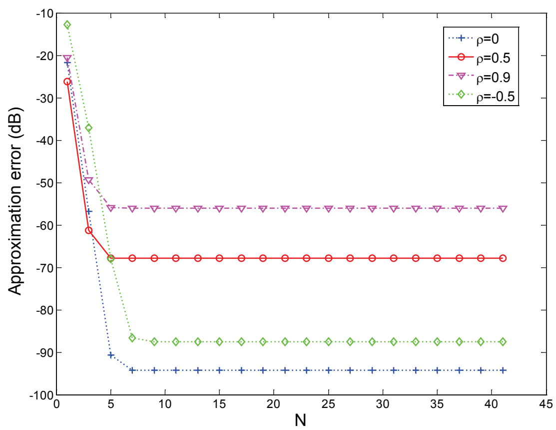


(a)



(b)

Fig. 1: Approximation error versus N for (a) $\mu_1 = 0, \sigma_1 = 1, \mu_2 = 1, \sigma_2 = 1.2$; (b) $\mu_1 = 0.5, \sigma_1 = 1, \mu_2 = 1, \sigma_2 = 1.2$.

Specifically, we consider that (a) both X_1 and X_2 are zero-mean random variables (in Fig. 2), (b) X_1 is zero-mean but X_2 is not zero-mean random variables (in Fig. 3), and (c) both X_1 and X_2 are not zero-mean random variables. The curves highlight that the analytic expressions perfectly overlap with the Monte Carlo simulations.

IV. Conclusion

In this paper, we have studied the distribution of product for two correlated real Gaussian random variables and have developed an exact PDF expression in terms of an infinite sum modified Bessel functions of second kind, which includes the existent results in open literature as special cases. We have studied the approximation error when only a finite number of terms is retained. The curves have illustrated that the series converges quickly. Finite summations of few terms can lead to high accurate approximations. Finally, we have evaluated the behavior of the derived analytic expression demonstrating also a perfect match with Monte Carlo simulations in correspondence of all the analyzed parameters.

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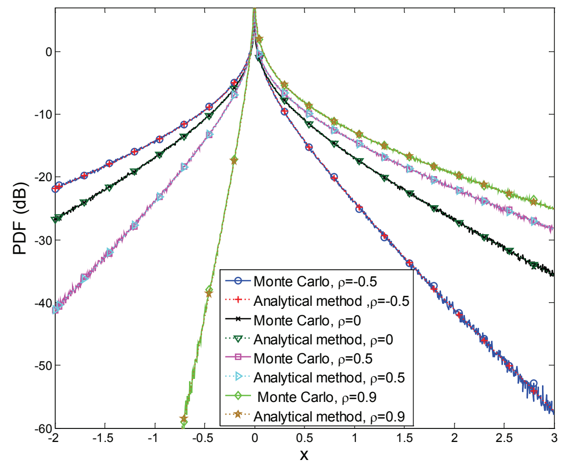

Fig. 2: PDF behavior versus x of the analytic expression and Monte Carlo method for $\mu_1 = 0, \sigma_1 = 1, \mu_2 = 0, \sigma_2 = 1.2$, and $N = 30$.

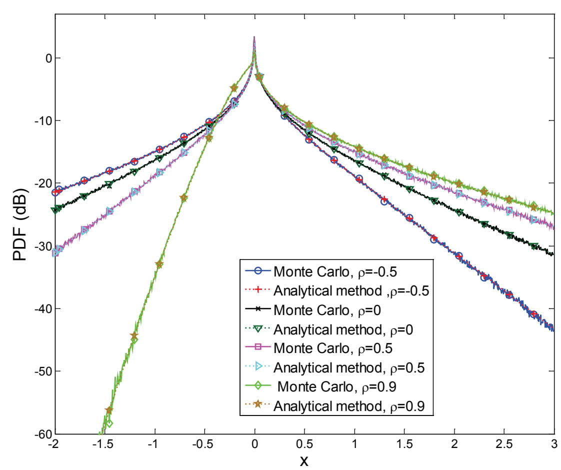

Fig. 3: PDF behavior versus x of the analytic expression and Monte Carlo method for $\mu_1 = 0, \sigma_1 = 1, \mu_2 = 1, \sigma_2 = 1.2$ and $N = 30$.

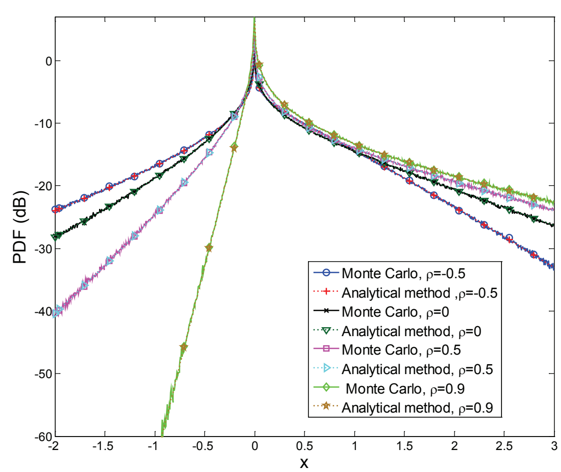

Fig. 4: PDF behavior versus x for the analytic expression and Monte Carlo method for $\mu_1 = 0.5, \sigma_1 = 1, \mu_2 = 1, \sigma_2 = 1.2$ and $N = 30$.

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