The data is modeled as:

$$(X_1, Y_1, A_1), \ldots, (X_n, Y_n, A_n) \overset{iid}{\sim} \mathcal{O}$$
$$A \perp X \tag{1}$$
$$P(A = 1 \mid X) = P(A = 1) = 1 - P(A = 0) = p$$

for some law $\mathcal{O}$.

The estimand is

$$\psi_0 = E(Y \mid A = 1) - E(Y \mid A = 0).$$

One estimator is the semiparametric efficient estimator, augmented for covariates and optimized. It is the solution in ψ of

$$\sum_i (A_i - p)[Y - (1-p)E(Y \mid A = 1, X_i) - pE(Y \mid A = 0, X_i)] - p(1-p)\psi$$

Another estimator solves for β_1 the system of equations,

$$0 = \sum_{i=1}^{n} \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \hat{\beta}_0 - A_i\hat{\beta}_1)$$

The latter estimator is the OLS solution to the equation

$$E(\tilde{Y} \mid A) = \beta_0 + \beta_1 A.$$

where

$$\tilde{Y} = Y - E(Y \mid X).$$

Then $\beta_0 + \beta_1 = E(\tilde{Y} \mid A = 1) = E(Y \mid A = 1) - E(Y)$ and $\beta_0 = E(\tilde{Y} \mid A = 0) = E(Y \mid A = 0) - E(Y) = E(Y \mid A = 0, X) - pE(Y \mid A = 1, X) - (1-p)E(Y \mid A = 0, X) = p(E(Y \mid A = 0, X) - E(Y \mid A = 1, X))$, so

$$\beta_0 = -p\psi_0$$
$$\beta_1 = \psi_0.$$

The

An estimator is obtained as the solution in ψ of

$$\sum_{i=1}^{n} U(Y_i, A_i; \psi) = 0,$$

where

$$U(Y, A; \psi) = (A - p)(Y - \psi A).$$

We consider estimators obtained as solutions in ψ to equations of the form

$$\sum_i U(Y_i, A_i; \psi) + (A_i - p)h(X_i; \psi) = \sum_i (A_i - p)(Y_i - \psi A_i + h(X_i; \psi)) = 0 \tag{2}$$

for arbitrary measurable h.The minimizing choice of h is determined by the estimating equation given by

$$W(X, Y, A; \psi) = U(Y, A; \psi) - E(U(Y, A; \psi) \mid A, X) + E(U(Y, A; \psi) \mid X). \tag{3}$$

The middle term on the rhs of (3) is,

$$\begin{aligned} E(U(Y, A; \psi) \mid A, X) &= E((A - p)(Y - \psi A) \mid A, X) \\ &= (E(Y \mid A, X) - \psi A)(A - p) \\ &= [E(Y \mid A = 1, X)A + E(Y \mid A = 0, X)(1 - A)](A - p) - \psi A(A - p) \\ &= E(Y \mid A = 1, X)A(1 - p) - E(Y \mid A = 0, X)(1 - A)p - \psi A(1 - p). \end{aligned}$$

The last term on the rhs of (3) is then,

$$\begin{aligned} E(U(Y,A;\psi) \mid X) &= pE(U(Y,A;\psi) \mid A=1,X) + (1-p)E(U(Y,A;\psi) \mid A=0,X) \\ &= p[E(Y \mid A=1,X)(1-p) - \psi(1-p)] - (1-p)[E(Y \mid A=0,X)p] \\ &= p(1-p)(E(Y \mid A=1,X) - E(Y \mid A=0,X) - \psi). \end{aligned}$$

Therefore,

$$\begin{aligned} W(X,Y,A;\psi) &= U(Y,A;\psi) - E(U(Y,A;\psi) \mid A,X) + E(U(Y,A;\psi) \mid X) \\ &= (A-p)(Y-\psi A) - (A-p)(1-p)E(Y \mid A=1,X) - (A-p)pE(Y \mid A=0,X) + \\ &\quad (A-p)(1-p)\psi \\ &= (A-p)[Y - (1-p)E(Y \mid A=1,X) - pE(Y \mid A=0,X)] - p(1-p)\psi \\ &= (A-p)[Y - E(\tilde{Y} \mid X)] - p(1-p)\psi \\ &= U(Y_i,A_i;\psi) + (A-p)[\psi A - E(\tilde{Y} \mid X)] - p(1-p)\psi, \end{aligned}$$

where $\tilde{Y}$ is determined by the transformation

$$Y = \tilde{Y}\frac{p^A(1-p)^{1-A}}{(1-p)^A p^{1-A}} = \tilde{Y}\left(\frac{p}{1-p}\right)^{2A-1}.$$

In case $p = P(A=1) = P(A=0) = 1/2$,

$$W(X,Y,A;\psi) = (A-1/2)(Y - E(Y \mid X)) - \psi/4. \tag{4}$$

0.1. **Regression estimator.** Define

$$\tilde{Y} = Y - E(Y \mid X)$$

and consider the regression

$$E(\tilde{Y} \mid A) = \beta_0 + \beta_1 A.$$

Then $\beta_0 + \beta_1 = E(\tilde{Y} \mid A=1) = E(Y \mid A=1) - E(Y)$ and $\beta_0 = E(\tilde{Y} \mid A=0) = E(Y \mid A=0) - E(Y) = E(Y \mid A=0,X) - pE(Y \mid A=1,X) - (1-p)E(Y \mid A=0,X) = p(E(Y \mid A=0,X) - E(Y \mid A=1,X))$, so

$$\begin{aligned} \beta_0 &= -p\psi_0 \\ \beta_1 &= \psi_0. \end{aligned}$$

The influence function of (β_0, β_1) is obtained as:

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \hat{\beta}_0 - A_i \hat{\beta}_1) \\
&= \sum_{i=1}^{n} \left\{ \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \beta_0 - A_i \beta_1) + \begin{pmatrix} -1 & -A_i \\ -A_i & -A_i \end{pmatrix} \begin{pmatrix} \hat{\beta}_0 - \beta_0 \\ \hat{\beta}_1 - \beta_1 \end{pmatrix} \right\} \\
n^{1/2} \begin{pmatrix} \hat{\beta}_0 - \beta_0 \\ \hat{\beta}_1 - \beta_1 \end{pmatrix} &= \left(\frac{1}{n} \sum_i \begin{pmatrix} 1 & A_i \\ A_i & A_i \end{pmatrix} \right)^{-1} n^{-1/2} \sum_i \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \beta_0 - A_i \beta_1) \\
&= \begin{pmatrix} 1 & p \\ p & p \end{pmatrix}^{-1} n^{-1/2} \sum_i \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \beta_0 - A_i \beta_1) + o_P(1) \\
n^{1/2}(\hat{\beta}_1 - \beta_1) &= n^{-1/2} \sum_i \begin{pmatrix} \frac{-1}{1-p} & \frac{1}{p(1-p)} \end{pmatrix} \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \beta_0 - A_i \beta_1) + o_P(1) \\
&= \frac{n^{-1/2}}{p(1-p)} \sum_i (A_i - p)(\tilde{Y}_i - \beta_0 - A_i \beta_1) + o_P(1) \\
&= \frac{n^{-1/2}}{p(1-p)} \sum_i (A_i - p)(\tilde{Y}_i - (A_i - p)\psi_0) + o_P(1) \\
&= \frac{n^{-1/2}}{p(1-p)} \sum_i \left\{ (A_i - p)(Y_i - E(Y \mid X_i)) - p^2 \left(\frac{1-p}{p} \right)^{2A_i} \psi_0 \right\} + o_P(1).
\end{aligned}
$$

In case $p = 1/2$,

$$
n^{1/2}(\hat{\beta}_1 - \beta_1) = 4n^{-1/2} \sum_i \{(A_i - 1/2)(Y_i - E(Y \mid X_i)) - \psi_0/4\} + o_P(1).
$$