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BAHADUR—KIEFER THEOREMS FOR UNIFORM SPACINGS PROCESSES

Weak laws for Bahadur — Kiefer type processes based on uniform spacings are established. The basic techniques are provided by the paper of Deheuvels and Mason [9] on Bahadur — Kiefer processes for independent identically distributed (i.i.d.) random variables. Our results constitute a futher contribution to the study of weighted empirical processes, which in a certain sense was begun by Chibisov.

1. Introduction and main results. Let $U_1, U_2, \ldots,$ be a sequence of independent uniform (0, 1) random variables. For $n \geqslant 0$, set $U_{0,n} = 0$, $U_{n+1,n} = 1$, and denote by $0 \leqslant U_{1,n} \leqslant \ldots \leqslant U_{n,n} \leqslant 1$ the order statistics of $U_1, \ldots, U_n$ for $n \geqslant 1$. The corresponding uniform spacings are defined for $n \geqslant 1$ by

$$D_{i,n} = U_{i,n-1} - U_{i-1,n-1}, \quad i = 1, \ldots, n. \tag{1.1}$$

There has been a continuous interest in the study of uniform spacings since the pioneering papers of Pyke [15, 16]. In Deheuvels [8], more than sixty references are given on this subject, with statistical applications such as testing uniformity or goodness of fit tests. In this paper, we shall be concerned with limiting properties of empirical processes related to spacings.

It is well known (see e. g. Pyke [15]) that for any $n \geqslant 1$, the spacings $D_{i,n}$, $i = 1, \ldots, n$, form an exchangeable set of random variables such that, for $1 \leqslant i \leqslant n$ and $t > 0$,

$$\mathbf{P}(nD_{i,n} \leqslant t) = \mathbf{P}(nD_{1,n} \leqslant t) = 1 - (1 - t/n)^{n-1} \to \\ \to 1 - e^{-t} \quad \text{as } n \to \infty. \tag{1.2}$$

Let $F(t) = 1 - e^{-t}$ for $t \geqslant 0$, and $F(t) = 0$ for $t < 0$. By (1.2), it is natural to define the empirical distribution function of the normalized spacings by

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty, t]}(nD_{i,n}), \quad \text{for} \quad -\infty < t < \infty, \tag{1.3}$$

and to consider the corresponding empirical process defined by

$$\delta_n(t) = n^{1/2}(F_n(t) - F(t)), \text{ for } -\infty < t < \infty. \tag{1.4}$$

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Consider $G(s) = \inf\{t \geqslant 0\colon F(t) \geqslant s\} = -\log(1 - s)$ and $G_n(s) = \inf\{t \geqslant 0\colon F_n(t) \geqslant s\}$ for $0 \leqslant s \leqslant 1$, where we use the notation $G(1) = -\log 0 = \infty$ and $F(\infty) = 1$. For $0 \leqslant s \leqslant 1$, $G_n(s)$ defines the empirical quantile function of the normalized spacings. The corresponding quantile process is defined by

$$\gamma_n(s) = n^{1/2}(G_n(s) - G(s))/G'(s) \text{ for } 0 \leqslant s < 1. \tag{1.5}$$

It is convenient to consider the following rescaled versions of δ_n and γ_n. Set for $0 \leqslant s \leqslant 1$

$$U_n(s) = F_n(G(s)) \text{ and } V_n(s) = F(G_n(s)) = \inf\{u \geqslant 0\colon U_n(u) \geqslant s\}. \tag{1.6}$$

We consider

$$a_n(s) = n^{1/2}(U_n(s) - s) = \delta_n(G(s)), \tag{1.7}$$

and

$$b_n(s) = n^{1/2}(V_n(s) - s), \text{ for } 0 \leqslant s \leqslant 1. \tag{1.8}$$

Throughout, we will use the notation, for any function f defined on $(0, 1)$,

$$\|f\| = \sup_{0<s<1} |f(s)|, \quad \text{and} \quad \|f\|_n = \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} |f(s)|.$$

Limiting properties of the processes δ_n, γ_n and a_n have been investigated by a number of authors, among which we cite Shorack [18], Rao and Sethuraman [17], Durbin [10], Beirlant [2] Aly, Beirlant and Horváth [1] and Einmahl and van Zuijlen [11]. It is noteworthy that $b_n(s)$ remains always close to $\gamma_n(s)$ as shown in the following proposition.

Proposition 1. *For any* $0 < \varepsilon < 1/2$,

$$\|b_n - \gamma_n\|_n = O_p(n^{\varepsilon - 1/2}) \quad \text{as } n \to \infty. \tag{1.9}$$

The proof of Proposition 1 is postponed until Section 3. In the sequel, we shall make use of the following notation (see e. g. M. Csörgö, S. Csörgö, Horváth and Mason (CsCsHM) [6]). Let Q denote the class of all positive and continuous functions on $(0, 1)$ which are nondecreasing in a neighborhood of zero, nonincreasing in a neighborhood of one and bounded away from zero on $(\delta, 1 - \delta)$ for any $0 < \delta < 1/2$ For any $q \in Q$ and $\varepsilon > 0$, let

$$I(q, \varepsilon) = \int_0^1 \frac{1}{t(1-t)} \exp\left(-\frac{\varepsilon q^2(t)}{t(1-t)}\right) dt. \tag{1.10}$$

Any $q \in Q$ such that $I(q, \varepsilon)$ is finite for all $\varepsilon > 0$ will be called a Chibisov — O'Reilly [COR] function. Likewise, any $q \in Q$ such that $I(q, \varepsilon)$ is finite for all $\varepsilon > 0$ sufficiently large will be called an Erdős — Feller — Kolmogorov — Petrovskii [EFKP] function.

For any Brownian bridge $\{B(s), 0 \leqslant s \leqslant 1\}$, define the process $\Gamma = T(B)$, by

$$\Gamma(s) = B(s) - G(S)e^{-G(s)} \int_0^1 B(u)\,dG(u) \text{ for } 0 \leqslant s \leqslant 1. \tag{1.11}$$

Beirlant [2] and Aly, Beirlant and Horváth [1] have proved that it is possible to define $U_1, U_2, \ldots$ on a probability space on which sit two sequences B'_n and B''_n, $n = 1, 2, \ldots$, of Brownian bridges such that, if $\Gamma'_n = T(B'_n)$ and $\Gamma''_n = T(B''_n)$, for $n = 1, 2, \ldots$, with probability one

$$\|a_n - \Gamma'_n\| = O(n^{-1/4}(\log n)^{3/4}) \text{ and } \|\gamma_n - \Gamma''_n\| = O(n^{-1/4}(\log n)^{3/4}) \text{ as } n \to \infty. \tag{1.12}$$

They also extended the results of Shorack [18], by proving that, for any $q \in Q$,

$$\|(a_n - \Gamma_n')/q\| \xrightarrow{P} 0 \text{ and } \|(\gamma_n - \Gamma_n'')/q\|_n \xrightarrow{P} 0 \text{ as } n \to \infty, \qquad (1.13)$$

if and only if q is COR.

The main purpose of this paper is to prove the following two theorems.

Theorem 1. *We have*

$$n^{1/4}(\log n)^{-1/2}\|a_n + b_n\|/\|b_n\|^{1/2} \xrightarrow{P} 1 \text{ as } n \to \infty. \qquad (1.14)$$

Let $\log^+ u = \max(\log u, 1)$, and let Γ be as in (1.11).

Theorem 2. *Let* $q \in Q$. *Then* $n^{1/4}\|(a_n + b_n)/\{2q \log^+(n^{1/2}/q)\}^{1/2}\|_n = O_p(1)$ *as* $n \to \infty$, *if and only if* q *is EFKP, in which case we have*

$$n^{1/4}\left\|(a_n + b_n)\Big/\left\{2q\log^+\left(\frac{n^{1/2}}{q}\right)\right\}^{1/2}\right\|_n \xrightarrow{d} \|\Gamma/q\|^{1/2} \text{ as } n \to \infty. \qquad (1.15)$$

In addition, if $q \in Q$ *is not EFKP, then*

$$n^{1/4}\left\|(a_n + b_n)\Big/\left\{2q\log^+\left(\frac{n^{1/2}}{q}\right)\right\}^{1/2}\right\|_n \xrightarrow{P} \infty \text{ as } n \to \infty. \qquad (1.16)$$

Moreover, if $q \in Q$, *is COR, then*

$$n^{1/4}(\log n)^{-1/2}\|(a_n + b_n)/q^{1/2}\|_n \xrightarrow{d} \|\Gamma/q\|^{1/2} \text{ as } n \to \infty. \qquad (1.17)$$

The following corollary will be shown to be a simple consequence of Theorem 1.

Corollary 1. *We have*

$$n^{1/4}(\log n)^{-1/2}\|a_n + b_n\| \xrightarrow{d} \|\Gamma\|^{1/2} \text{ as } n \to \infty. \qquad (1.18)$$

Proposition 1 and Theorem 1 will imply the next corollary which provides a Bahadur representation for the order statistics of uniform spacings.

Corollary 2. *We have*

$$n^{1/4}(\log n)^{-1/2}\|\gamma_n + a_n\|/\|a_n\|^{1/2} \xrightarrow{P} 1 \text{ as } n \to \infty.$$

These results give Bahadur—Kiefer-type representations for the empirica processes based on uniform spacings which are the analogues of the corresponding results obtained for the usual iniform empirical and quantile processes by Deheuvels and Mason [9]. The statement (1.14) of Theorem 1 is due to Kiefer [12] for the uniform empirical process (see also Shorack [19]).

The proofs of Theorems 1 and 2 will be given in Section 3. In Section 2, we will be concerned with the limiting finite — dimensional distributions of the Bahadur—Kiefer-type process $a_n(s) + b_n(s)$ as $n \to \infty$.

2. Weak convergence of finite dimensional distributions. Let $c_n(s) = a_n(s) + b_n(s)$ for $0 \leqslant s \leqslant 1$. In this section we prove the following theorem.

Theorem 3. *Let* $k \geqslant 1$ *and* $0 < s_1 < \ldots < s_k < 1$ *be fixed. Then, as* $n \to \infty$,

$$(n^{1/4}c_n(s_1), \ldots, n^{1/4}c_n(s_k)) \xrightarrow{d} (Z_1|\Gamma(s_1)|^{1/2}, \ldots, Z_k/\Gamma(s_k)|^{1/2}), \qquad (2.1)$$

where Γ *is as in* (1.11), *and* $Z_1, \ldots, Z_k$ *are independent* $N(0, 1)$ *random variables, independent of* Γ.

R e m a r k 1. It will become obvious from the proof of Theorem 3 that (2.1) is valid for the usual Bahadur — Kiefer process R_n (see e. g. Deheuvels and Mason [9]) based on the uniform empirical and quantile processes with the formal changes of c_n by R_n and of Γ by a Brownian bridge. The corresponding result is also new in this case.

P r o o f. The proof of Theorem 3 is captured in the following sequence of lemmas.

Lemma 2.1. *Let $0 < s < 1$ be fixed. It is possible to define the sequence $\{c_n, n \geq 1\}$ on a probability space jointly with a sequence $\{W_n, n \geq 1\}$ of standard Wiener processes extended on $(-\infty, \infty)$ in such a way that, as $n \to \infty$,*

$$n^{1/4} | c_n (s) - \{W_n (s - n^{-1/2} m_n (s)) - W_n (s - n^{-1/2} (\Gamma_n (s) + m_n (s)))\} | = O_p (1), \tag{2.2}$$

where

$$m_n (s) = G (s) e^{-G(s)} \int_0^1 B_n (u) dG (u), \quad \Gamma_n (s) = B_n (s) - m_n (s),$$

and $B_n (s) = W_n (s) - sW_n (1)$.

The proof of Lemma 2.1 is postponed to the end of Section 3.

Our next lemma gives a general statement of independent interest. First we introduce some notation.

Let $(W, V_1, \ldots, V_m)$ denote a $(m + 1)$-variate Gaussian process, where $\{W (s), -\infty < s < \infty\}$ is a standard Wiener process extended on $(-\infty, \infty)$, and $V_1, \ldots, V_m$ are continuous Gaussian processes on $[0, 1]$. Set

$$R_j (s, t) = \mathrm{Cov} (W (s), V_j (t)), \ s \in \mathbf{R}, \ t \in [0, 1], \ j = 1, \ldots, m.$$

For any choice of $k \geq 1$ and $-\infty < s_1 < \ldots < s_k < \infty$ fixed, let for $n \geq 1$

$$W_n^{(i)} (x_i) = n^{1/4} \left(W \left(s_i + \frac{x_i}{\sqrt{n}}\right) - W (s_i)\right), \quad x_i \in \mathbf{R}, \quad i = 1, \ldots, k.$$

Lemma 2.2. *Whenever*

$$n^{1/4} \left(R_j \left(s + \frac{x}{\sqrt{n}}, t\right) - R_j (s, t)\right) \to 0 \text{ as } n \to \infty \tag{2.3}$$

for all $1 \leq j \leq m$, $-\infty < s, x < \infty$ and $0 \leq t \leq 1$, then, for any fixed $k \geq 1$ and $-\infty < s_1 < \ldots < s_k < \infty$,

$$(W_n^{(1)}, \ldots, W_n^{(k)}, V_1, \ldots, V_m) \xrightarrow{d} (W^{(1)}, \ldots, W^{(k)}, V_1, \ldots, V_m) \tag{2.4}$$

as $n \to \infty$, where $W^{(1)}, \ldots, W^{(k)}$ are independent standard Wiener processes defined on $(-\infty, \infty)$, independent of $V_1, \ldots, V_m$.

P r o o f. ($\xrightarrow{d}$ in (2.4) denotes the usual weak convergence of probability measures on the space $\mathscr{C}$ of all continuous functions from $\mathbf{R}^k \times [0, 1]^m$ to $\mathbf{R}^{k+m}$ of the form

$$f (x_1, \ldots, x_k, t_1, \ldots, t_m) = (f_1 (x_1), \ldots, f_k (x_k), f_{k+1} (t_1), \ldots, f_{k+m} (t_m)),$$

where $f_1, \ldots, f_{k+m}$ are continuous). Choose any $0 < T < \infty$. Calculating the covariance function of $W_n^{(i)}$, $i = 1, \ldots, k$, we see that for each $n \geq 1$ and $i = 1, \ldots, k$, $W_n^{(i)}$ is a standard Wiener process on $[-T, T]$, which shows that the sequence of probability measures indiced by $(W_n^{(1)}, \ldots, W_n^{(k)}, V_1, \ldots, V_m)$ on $\mathscr{C}$ is tight (cf. Problem 6, p. 41 in Billingsley [4]).

Moreover, for all large enough n,

$\mathrm{E}\,(W_n^{(i)}(x_i), W_n^{(j)}(x_j)) = 0$ for all $-T \leqslant x_i, x_j \leqslant T$, $1 \leqslant i < j < k$.

Finally, for each $x_i \in [-T, T]$, $t \in [0, 1]$, $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant m$

$$\mathrm{Cov}\,(W_n^{(i)}(x_i), V_j(t)) = n^{1/4}\left(R_j\left(s_i + \frac{x_i}{\sqrt{n}}, t\right) - R_j(s_i, t)\right) \to 0 \text{ as } n \to \infty.$$

Thus the sequence of processes $(W_n^{(1)}, \ldots, W_n^{(k)}, V_1, \ldots, V_m)$ restricted to $[-T, T]^k \times [0, 1]^m$ converges weakly to $(W^{(1)}, \ldots, W^{(k)}, V_1, \ldots, V_m)$ restricted to $[-T, T]^k \times [0, 1]^m$ as $n \to \infty$. Since T can be chosen arbitrarily large, this finishes the proof.

Lemma 2.3. *Let* $m = 2$. *Under the assumptions of Lemma 2.2, we have for any fixed* $k \geqslant 1$, $-\infty < s_1 < \ldots < s_k < \infty$ *and* $0 \leqslant t_1 < \ldots < t_k \leqslant 1$

$$\begin{aligned}(W_n^{(1)}(V_1(t_1)) - W_n^{(1)}(V_1(t_1) + V_2(t_1)), \ldots, W_n^{(k)}(V_1(t_k)) -\\ - W_n^{(k)}(V_1(t_k) + V_2(t_k)))\\ \xrightarrow{d} (W^{(1)}(V_2(t_1)), \ldots, W^{(k)}(V_2(t_k))).\end{aligned} \tag{2.5}$$

Proof. From Lemma 2.2 we can conclude by a random time-change argument (cf. Billingsely [4] p. 145) that the left side of (2.5) converges in distribution to

$$\begin{aligned}(W^{(1)}(V_1(t_1)) - W^{(1)}(V_1(t_1) + V_2(t_1)), \ldots\\ \ldots, W^{(k)}(V_1(t_k)) - W^{(k)}(V_1(t_k) + V_2(t_k))).\end{aligned} \tag{2.6}$$

Noticing that the random vector in (2.6) is equal in distribution to that on the right side of (2.5) completes the proof.

Let W be any standard Wiener process on $\mathbf{R}$ and write

$$B(s) = W(s) - sW(1), \quad m(s) = G(s)\, e^{-G(s)} \int_0^1 B(u)\, dG(u),$$

so that

$$\Gamma(s) = B(s) - m(s) \text{ for } 0 \leqslant s \leqslant 1.$$

Since $(W_n, \Gamma_n, m_n) \overset{d}{=} (W, \Gamma, m)$ for $n = 1, 2, \ldots$, we obtain from Lemma 2.1 that the left side of (2.1) is equal in distribution to

$$\begin{aligned}(n^{1/4}(W(s_1 - n^{-1/2}m(s_1)) - W(s_1 - n^{-1/2}(\Gamma(s_1) +\\ + m(s_1))) + o_p(1), \ldots,\\ n^{1/4}(W(s_k - n^{-1/2}m(s_k)) - W(s_k - n^{-1/2}(\Gamma(s_k) + m(s_k))) + o_p(1)).\end{aligned}$$

Now applying Lemma 2.3 with $V_1 = -m$ and $V_2 = -\Gamma$ [elementary computations show that condition (2.3) is fulfilled], we obtain that the left side of (2.1) converges in distribution to

$$(W^{(1)}(-\Gamma(s_1)), \ldots, W^{(k)}(-\Gamma(s_k))). \tag{2.7}$$

Finally, it is easily seen that the expression in (2.7) is equal in distribution to the righ side of (2.1). This finishes the proof of Theorem 3.

3. Proofs of Theorems 1 and 2. Let $D_{i,n}$, $i = 1, \ldots, n$ be as in (1.1), and let $0 \leqslant M_{1,n} \leqslant \ldots \leqslant M_{n,n} \leqslant 1$ denote the order statistics of $D_{1,n}, \ldots, D_{n,n}$. If F_n is as in (1.3), we have

$$F_n(t) = \begin{cases} 0 & \text{for } t < nM_{1,n}, \\ i/n & \text{for } nM_{i,n} \leqslant t < nM_{i+1,n}, \\ 1 & \text{for } nM_{n,n} \leqslant t, \end{cases} \tag{3.1}$$

so that

$$U_n(s)=\begin{cases}0 & \text{for } s<1-\exp(-nM_{1,n}),\\ i/n & \text{for } 1-\exp(-nM_{i,n})\leqslant s<1-\exp(-nM_{i+1,n}),\\ 1 & \text{for } 1-\exp(-nM_{n,n})\leqslant s.\end{cases} \tag{3.2}$$

It follows from (3.1) that $G_n(s)=\inf\{u\geqslant 0\colon F_n(u)\geqslant s\}$ and $V_n(s)=F(G_n(s))$ with $F(t)=1-e^{-t}$ for $t\geqslant 0$ satisfy

$$G_n(s)=\begin{cases}0 & \text{for } s=0,\\ nM_{i,n} & \text{for } (i-1)/n<s\leqslant i/n,\end{cases} \tag{3.3}$$

and

$$V_n(s)=\begin{cases}0 & \text{for } s=0,\\ 1-\exp(-nM_{i,n}) & \text{for } (i-1)/n<s\leqslant i/n.\end{cases} \tag{3.4}$$

Let $a_n(s)=n^{1/2}(U_n(s)-s)$, $b_n(s)=n^{1/2}(V_n(s)-s)$ and set $\|f\|=\sup\limits_{0\leqslant s\leqslant 1}|f(s)|$.

Lemma 3.1. *We have* $\|a_n\|=\|b_n\|$.

Proof. Obviously by (3.2) and (3.4), we have

$$\|a_n\|=\|b_n\|=\max_{1\leqslant i\leqslant n}\max\left(\left|1-\exp(-nM_{i,n})-\frac{i}{n}\right|,\right.$$
$$\left.\left|1-\exp(-nM_{i,n})-\frac{i-1}{n}\right|\right). \tag{3.5}$$

Let $c_n(s)=a_n(s)+b_n(s)$ for $0\leqslant s\leqslant 1$.

Lemma 3.2. *We have*

$$\sup_{0\leqslant s\leqslant 1}|c_n(s)-(a_n(s)-a_n(s+n^{-1/2}b_n(s)))|=n^{-1/2}\quad a.\ s. \tag{3.6}$$

Proof. For $(i-1)/n<s\leqslant i/n$, $V_n(s)=F(nM_{i,n})$ and $U_n(V_n(s))=i/n$. Hence in this case

$$b_n(s)=n^{1/2}(V_n(s)-s)=n^{1/2}(V_n(s)-U_n(V_n(s)))+n^{1/2}(i/n-s)\quad a.\ s.$$

It follows that with probability one

$$c_n(s)=a_n(s)+b_n(s)=$$
$$=a_n(s)-a_n(V_n(s))+n^{1/2}\left(\frac{i}{n}-s\right)\quad \text{for } \frac{i-1}{n}<s\leqslant\frac{i}{n}. \tag{3.7}$$

This, jointly with the observation that $|i/n-s|<1/n$ for $(i-1)/n<s\leqslant i/n$, proves (3.6).

In the sequel, we will make use of the following facts whose proofs are given in Theorems 3.3, 3.4, 4.2.1 and 4.2.3 in [6]. Let $B(s)$ for $0\leqslant s\leqslant 1$ denote a Brownian bridge.

Fact 1. *$q\in Q$ is an EFKP function if and only if there exist with probability one β_0 and β_1 such that*

$$\beta_0=\limsup_{s\downarrow 0}|B(s)|/q(s)<\infty \text{ and } \beta_1=\limsup_{s\downarrow 0}|B(1-s)|/q(1-s)<\infty. \tag{3.8}$$

Furthermore, $q\in Q$ is a COR function if and only if $\beta_0=\beta_1=0$ a.s.

Fact 2. *If $q\in Q$ is an EFKP function, then*

$$q(s)/s^{1/2}\to\infty \text{ and } q(1-s)/s^{1/2}\to\infty \text{ as } s\downarrow 0. \tag{3.9}$$

Fact 3. *Let $\alpha_n(s)$ for $0\leqslant s\leqslant 1$ be the usual uniform empirical process (based on a uniform $(0,1)$ sample of size n). Then $q\in Q$ is an EFKP function if and only if the random variable $\|\alpha_n/q\|$ converges in distribution as*

$n \to \infty$ to a nondegenerate random variable which is then equal in distribution to $\| B/q \|$.

The following lemma provides a weighted version of (3.6).

Lemma 3.3. *Let $q \in Q$ be an EFKP function. We have*

$$\lim_{n\to\infty} \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{1/4} \frac{|c_n(s)-(a_n(s)-a_n(s+n^{-1/2}b_n(s)))|}{\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = 0 \quad a.s. \tag{3.10}$$

P r o o f. Note that (3.9) implies that $\tau_2 := \inf_{0<s\leqslant 1/2} (q(s)/s^{1/2}) > 0$.

Now we have by (3.6) that

$$\sup_{1/(n+1)\leqslant s\leqslant 1/2} n^{1/4} \frac{|c_n(s)-(a_n(s)-a_n(s+n^{-1/2}b_n(s)))|}{\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} \leqslant$$
$$\leqslant n^{-1/4} \sup_{1/(n+1)\leqslant s\leqslant 1/2} (\tau_2 s^{1/2}\log^+(n^{1/2}/\tau_2 s^{1/2}))^{-1/2} \leqslant$$
$$\leqslant n^{-1/4}\left\{\frac{\tau_2}{\sqrt{n+1}}\log^+\left(\frac{n^{1/2}(n+1)^{1/2}}{\tau_2}\right)\right\}^{-1/2} \to 0 \text{ as } n\to\infty.$$

In both inequalities we used the fact that $u\log^+(n^{1/2}/u)$ is increasing in $u>0$. A similar proof works for $1/2\leqslant s\leqslant n/(n+1)$.

The general meaning of Lemmas 3.2 and 3.3 is that, in the proofs of our theorems, we may replace $c_n(s) = a_n(s) + b_n(s)$ by $c_n'(s) = a_n(s) - a_n(s+n^{-1/2}b_n(s))$. Our next step will be to replace $c_n'(s)$ by another process which will be of identical distribution and easier to analyze. We will make use of the following fact.

Fact 4. (see e.g. Pyke [15] p. 403) *Let $\omega_1, \omega_2, \ldots$ be a sequence of independent and identically distributed random variables having an exponential distribution with mean one, and let for $n \geqslant 1$, $S_n = \omega_1 + \ldots + \omega_n$. Then $\{D_{i,n}, 1\leqslant i\leqslant n\} \overset{d}{=} \{\omega_i/S_n, 1\leqslant i\leqslant n\}$, where $\overset{d}{=}$ denotes identity of distributions.*

Since our theorems concern only weak laws, there is no loss of generality in assuming from the start that $D_{i,n} = \omega_i/S_n$ for all $1\leqslant i\leqslant n$. From now on, we shall assume that this convention holds.

Observe that $F(\omega_1), F(\omega_2), \ldots$ are independent and uniformly distributed on (0, 1). Denote by $\alpha_n(s)$ the corresponding uniform empirical process, defined by

$$\alpha_n(s) = n^{-1/2}\sum_{i=1}^{n}\{1_{(-\infty,s]}(F(\omega_i)) - s\} := n^{1/2}(H_n(s)-s). \tag{3.11}$$

We will now relate $a_n(s)$ to $\alpha_n(s)$. For this, set $\bar{S}_n = n^{-1}S_n$, and recall that $G(s) = -\log(1-s)$.

Lemma 3.4. *We have*

$$\bar{S}_n = 1 - n^{-1/2}\int_0^1 \alpha_n(u)\,dG(u). \tag{3.12}$$

P r o o f. By integrating by parts,

$$n^{-1/2}\int_0^1 \alpha_n(u)\,dG(u) = -\int_0^1 G(u)\,dH_n(u) + \int_0^1 G(u)\,du =$$
$$= -n^{-1}\sum_{i=1}^{n} G(F(\omega_i)) - \int_0^1 \log(u)\,du = 1 - \bar{S}_n.$$

Fact 5. *We have*

$$a_n(s) = \alpha_n(F(\bar{S}_n G(s))) + n^{1/2}(F(\bar{S}_n G(s)) - s). \tag{3.13}$$

The proof of (3.13) is obvious from (3.11) and the chain of equalities:

$$a_n(s) = n^{-1/2} \sum_{i=1}^{n} \{1_{(-\infty, s]}(F(\omega_i/\bar{S}_n)) - s\} =$$

$$= n^{-1/2} \sum_{i=0}^{n} \{1_{\{\omega_i \leqslant \bar{S}_n G(s)\}} - s\} = n^{-1/2} \sum_{i=1}^{n} \{1_{(-\infty, F(\bar{S}_n G(s))]}(F(\omega_i)) - s\}.$$

For $n \geqslant 1$, denote by $\omega_{1,n} \leqslant \ldots \leqslant \omega_{n,n}$ the order statistics of $\omega_1, \ldots, \omega_n$. Obviously $nM_{i,n} = \omega_{i,n}/\bar{S}_n$ for $1 \leqslant i \leqslant n$. Denote by $\beta_n(s)$ the empirical quantile process of $F(\omega_1), \ldots, F(\omega_n)$, i.e. $\beta_n(s) = n^{1/2}(K_n(s) - s)$, where

$$K_n(s) = \begin{cases} 0 & \text{for } s = 0, \\ 1 - \exp(-\omega_{i,n}) & \text{for } (i-1)/n < s \leqslant i/n. \end{cases} \tag{3.14}$$

Fact 6. *We have*

$$b_n(s) = n^{1/2}(F(G(s + n^{-1/2}\beta_n(s))/\bar{S}_n) - s) \text{ for } 0 \leqslant s \leqslant 1. \tag{3.15}$$

The proof of (3.15) follows from (3.4) and (3.14), jointly with the chain of equalities:

$$b_n(s) = n^{1/2}(F(nM_{i,n}) - s) = n^{1/2}(F(G(K_n(s))/\bar{S}_n) - s) =$$
$$= n^{1/2}(F(G(s + n^{-1/2}\beta_n(s))/\bar{S}_n) - s) \quad \text{for } (i-1)/n < s \leqslant i/n.$$

Lemma 3.5. *For any* $0 \leqslant \gamma < 1$,

$$\sup_{0<s<1} \{s^{-2}(1-s)^{-\gamma} \,|\, F(\bar{S}_n G(s)) - s - (\bar{S}_n - 1) G(s) e^{-G(s)} \,|\} = O_p(n^{-1}) \text{ as } n \to \infty. \tag{3.16}$$

P r o o f. By a simple Taylor expansion, we see that

$$F(\bar{S}_n G(s)) = s + (\bar{S}_n - 1) G(s) e^{-G(s)} - (\bar{S}_n - 1)^2 (\log(1-s))^2 (1-s)^\theta/2,$$

where $\theta = \theta(n, s)$ is between 1 and $\bar{S}_n$. By the central limit theorem, $n^{1/2}(\bar{S}_n - 1) \xrightarrow{d} N(0, 1)$ as $n \to \infty$, so that $(\bar{S}_n - 1)^2 = O_p(1/n)$. In addition, for any $0 < \varepsilon < 1$, there exists a constant C_ε and an n_ε such that, uniformly over $0 < s < 1$, $n \geqslant n_\varepsilon$ implies that $\mathbf{P}(|\theta - 1| \geqslant C_\varepsilon/\sqrt{n}) \leqslant \varepsilon$. It follows evidently that

$$\sup_{0<s<1} (s^{-2}(1-s)^{-\gamma}(\log(1-s))^2(1-s)^\theta) < \infty$$

with probability larger than $1 - \varepsilon$. Since $0 < \varepsilon < 1$ is arbitrary, we have (3.16).

Lemma 3.6. *We have*

$$\lim_{\rho \uparrow \infty} \left\{ \liminf_{n \to \infty} \mathbf{P}\left(\rho^{-1} \leqslant \frac{V_n(s)}{s} \leqslant \rho \colon (n+1)^{-1} \leqslant s \leqslant 1\right\} = 1 \tag{3.17}$$

and

$$\lim_{\rho \uparrow \infty} \left\{ \liminf_{n \to \infty} \mathbf{P}\left(\rho^{-1} \leqslant \frac{1 - V_n(1-s)}{s} \leqslant \rho \colon (n+1)^{-1} \leqslant s \leqslant 1\right\} = 1. \tag{3.18}$$

P r o o f. Following the methods of Beirlant, van der Meulen, Ruymgaart and van Zuijlen [3] one obtains (see their Theorem 3.2)

$$\liminf_{n \to \infty} \mathbf{P}(V_n(s) \leqslant \rho s;\ 1/n \leqslant s \leqslant 1) =$$

$$= \liminf_{n \to \infty} \mathbf{P}(U_n(s) \geqslant \rho^{-1} s;\ F(nM_{1,n}) \leqslant s \leqslant 1) \geqslant 1 - \frac{e\rho \exp(-\rho)}{(1 - e\rho \exp(-\rho))}.$$

Furthermore using the negative orthant dependence property of uniform spacings

$$\mathbf{P}\,(V_n\,(s) \leqslant \rho s;\ 1/(n+1) \leqslant s \leqslant 1/n) \geqslant \mathbf{P}\,(F\,(nM_{1,n}) \leqslant \rho/(n+1)) \geqslant$$
$$\geqslant 1 - \mathbf{P}^n\,(nD_{1,n} < \rho/(n+1)) \to 1 - e^{-\rho} \text{ as } n \to \infty.$$

From this it follows readily that

$$\lim_{\rho\uparrow\infty}\{\liminf_{n\to\infty} \mathbf{P}\,(V_n\,(s) \leqslant \rho s;\ 1/(n+1) \leqslant s \leqslant 1)\} = 1.$$

In the same way using the results of Beirlant et al. [3] one obtains

$$\lim_{\rho\uparrow\infty}\{\liminf_{n\to\infty} \mathbf{P}\,(\rho^{-1}s \leqslant V_n\,(s);\ 0 \leqslant s \leqslant 1)\} = 1,$$

$$\lim_{\rho\uparrow\infty}\{\liminf_{n\to\infty} \mathbf{P}\,(1 - \rho\,(1-s) \leqslant V_n\,(s);\ 0 \leqslant s \leqslant n/(n+1))\} = 1,$$

and

$$\lim_{\rho\uparrow\infty}\{\liminf_{n\to\infty} \mathbf{P}\,(V_n\,(s) \leqslant 1 - \rho^{-1}\,(1-s);\ 0 \leqslant s \leqslant 1)\} = 1.$$

Lemma 3.7. *We have*

$$\lim_{n\to\infty} \mathbf{P}\,(\rho^{-1} \leqslant F\,(\bar{S}_n G\,(s))/s \leqslant \rho\colon 0 \leqslant s \leqslant 1) = 1, \text{ for any } \rho > 1, \quad (3.19)$$

$$\lim_{\rho\uparrow\infty}\{\liminf_{n\to\infty} \mathbf{P}\,(\rho^{-1} \leqslant F\,(\bar{S}_n G\,(V_n\,(s)))/s \leqslant \rho\colon 1/(n+1) \leqslant s \leqslant 1)\} = 1, \quad (3.20)$$

for any $\rho > 1$

$$\lim_{n\to\infty} \mathbf{P}\,(\rho^{-1} \leqslant (1 - F\,(\bar{S}_n G\,(1-s)))/s \leqslant \rho\colon 1/(n+1) \leqslant s \leqslant 1) = 1, \quad (3.21)$$

and

$$\lim_{\rho\uparrow\infty}\{\liminf_{n\to\infty} \mathbf{P}\,(\rho^{-1} \leqslant (1 - F\,(\bar{S}_n G\,(V_n\,(1-s)))/s \leqslant \rho : 1/(n+1) \leqslant s \leqslant 1)\} = 1. \quad (3.22)$$

P r o o f. Note that (3.20) follows from (3.19) and (3.17). For the proof of (3.19), we see by (3.16) and using the fact that $(\bar{S}_n - 1) = O_p\,(n^{-1/2})$, that for any fixed $0 < \gamma < 1$, we have, uniformly over $0 \leqslant s \leqslant 1$,

$$F\,(\bar{S}_n G\,(s))/s = 1 - (\bar{S}_n - 1)\,(1-s)\log\,(1-s)/s + sO_p\,(n^{-1}) = 1 + O_p\,(n^{-1/2})$$

which shows that, uniformly over $0 \leqslant s \leqslant 1$

$$F\,(\bar{S}_n\,G\,(s))/s - 1 = O_p\,(n^{-1/2}). \quad (3.23)$$

This suffices for (3.19).

The proof of (3.21) and (3.22) is identical, with the following evaluation replacing (3.23):

$$(1 - F\,(\bar{S}_n G\,(1-s)))/s - 1 = O_p\,(n^{-1/2}\,(\log n)) \quad (3.24)$$

uniformly over $1/(n+1) \leqslant s \leqslant 1$.

Consider now the decomposition for $0 < s < 1$

$$c_n'\,(s) = a_n\,(s) - a_n\,(s + n^{-1/2}b_n\,(s)) = C_{n,1}\,(s) + C_{n,2}\,(s), \quad (3.25)$$

where

$$C_{n,1}\,(s) = \alpha_n\,(F\,(\bar{S}_n G\,(s))) - \alpha_n\,(F\,(\bar{S}_n G\,(s + n^{-1/2}b_n\,(s)))) \quad (3.26)$$

and

$$C_{n,2}\,(s) = n^{1/2}\,\{F\,(\bar{S}_n G\,(s)) - F\,(\bar{S}_n G\,(s + n^{-1/2}b_n\,(s)))\} + b_n\,(s). \quad (3.27)$$

As a consequence of Lemma's 3.5 and 3.6 it follows that whenever $0 \leqslant \gamma < 1$ is fixed, we have

$$\sup_{0<s<1} (s^{-2}(1-s)^{-\gamma}) \left| C_{n,2}(s) + n^{1/2}(\bar{S}_n - 1)\{\Lambda(s + n^{-1/2} b_n(s)) - \Lambda(s)\}\right| = O_p(n^{-1/2}) \tag{3.28}$$

as $n \to \infty$, where

$$\Lambda(s) = G(s)\exp(-G(s)) = -(1-s)\log(1-s).$$

In order to evaluate these expressions, we will assume from now on and without loss of generality that the random variables $\omega_1, \omega_2, \ldots$ are defined on the probability space of Theorem 3 in Komlós, Major and Tusnády [KMT] [13] on which there sits a sequence $B_1, B_2, \ldots$ of Brownian bridges such that

$$\|\alpha_n - B_n\| = O(n^{-1/2}\log n) \text{ a.s. as } n \to \infty. \tag{3.29}$$

By enlarging this probability space if necessary, we can and do assume that each of these Brownian bridges is of the form

$$B_n(s) = W_n(s) - sW_n(1), \tag{3.30}$$

where $W_1, W_2, \ldots$ is a sequence of standard Wiener processes.

We will make use of the following property that this probability space possesses (see Deheuvels and Mason [9], Lemma 5, and Mason and Van Zwet [14]).

Fact 7. *On the probability space of* (3.29) *for all* $0 < \nu_1 < 1/2$

$$\sup_{0<s<1} n^{\nu_1} |\alpha_n(s) - B_n(s)| / (s(1-s))^{1/2-\nu_1} = O_p(1), \tag{3.31}$$

and for all $0 < \nu_2 < 1/4$

$$\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{\nu_2} |\beta_n(s) + B_n(s)| / (s(1-s))^{1/2-\nu_2} = O_p(1) \textit{ as } n \to \infty. \tag{3.32}$$

Let in the sequel

$$\Gamma_n(s) = B_n(s) - G(s)e^{-G(s)} \int_0^1 B_n(u)\,dG(u). \tag{3.33}$$

Lemma 3.8. *For all* $0 < \nu_2 < 1/4$,

$$\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{\nu_2} |b_n(s) + \Gamma_n(s)| / (s(1-s))^{1/2-\nu_2} = O_p(1) \textit{ as } n \to \infty. \tag{3.34}$$

P r o o f. By (3.15),

$$b_n(s) = n^{1/2}\{F[G(K_n(s)) + (1/S_n - 1)G(K_n(s))] - s\},$$

and hence, by a simple Taylor expansion, we obtain

$$b_n(s) = \beta_n(s) + n^{1/2}(1/\bar{S}_n - 1)G(K_n(s))(1 - K_n(s)) + n^{1/2}(1/S_n - 1)^2 G(K_n(s))^2 (1 - K_n(s))^{\rho}/2,$$

where $\rho = \rho(n, s)$ is between 1 and $1/\bar{S}_n$. By (3.14), this expression becomes

$$b_n(s) = \beta_n(s) + \tau_n(s) + R_{1,n}(s) + R_{2,n}(s),$$

where, for $(i-1)/n < s \leqslant i/n$,

$$\tau_n(s) = -n^{1/2}(\bar{S}_n - 1)\omega_{i,n}\exp(-\omega_{i,n}) = -n^{1/2}(\bar{S}_n - 1)G(K_n(s))e^{-G(K_n(s))},$$

$$R_{1,n}(s) = n^{1/2}(\bar{S}_n - 1)^2(\bar{S}_n)^{-1}\omega_{i,n}\exp(-\omega_{i,n}), \tag{3.35}$$

and

$$R_{2,n}(s) = -\frac{1}{2}n^{1/2}(\bar{S}_n - 1)^2(\bar{S}_n)^{-2}\omega_{i,n}^2\exp(-\rho\omega_{i,n}).$$

Notice that, for $i = 1$ and 2,

$$\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} \frac{n^{\nu_2}|R_{i,n}(s)|}{(s(1-s))^{1/2-\nu_2}} \leqslant 2(n+1)^{1/2} \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} |R_{i,n}(s)| = O_p(1)$$
as $n \to \infty$.

By (3.32), it follows that, for (3.34), al we need is to show that

$$T_n = \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{\nu_2}\left|\tau_n(s) - G(s)e^{-G(s)} \int_0^1 B_n(u)dG(u)\right|/(s(1-s))^{1/2-\nu_2} = O_p(1). \tag{3.36}$$

Observe by (3.12) that

$$T_n \leqslant \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{\nu_2}\left|\int_0^1 \alpha_n(u)\,dG(u)\right|\left|G(K_n(s))e^{-G(K_n(s))} - \right.$$
$$\left. - G(s)e^{-G(s)}\right|/(s(1-s))^{1/2-\nu_2} + \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{\nu_2}\left|\int_0^1 (\alpha_n(u) - \right.$$
$$\left. - B_n(u))\,dG(u)\right|\left|G(s)e^{-G(s)}\right|/(s(1-s))^{1/2-\nu_2} := T_n' + T_n''. \tag{3.37}$$

By (3.12) and the central limit theorem $\int_0^1 \alpha_n(u)dG(u) = O_p(1)$, so that a Taylor expansion yields

$$T_n' = O_p(1)\,n^{\nu_2-1/2} \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} |\beta_n(s)(1+\log(1-\delta))|(s(1-s))^{1/2-\nu_2},$$

where $\delta = \delta(n, s)$ is between s and $K_n(s)$. Since

$$-\log(1 - K_n(n/(n+1))) = \omega_{n,n} = O_p(\log n) \text{ as } n \to \infty \tag{3.38}$$

it follows that

$$T_n' = O_p(n^{-1/2}\log n) \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{\nu_2}|\beta_n(s)|/(s(1-s))^{1/2-\nu_2},$$

which by (3.32) is equal to

$$O_p(n^{-1/2}\log n)\{\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{\nu_2}|B_n^{(s)}|/(s(1-s))^{1/2-\nu_2} + O_p(1)\},$$

which, in turn, is by Fact 1, $O_p(n^{\nu_2-\frac{1}{2}}\log n) = o_p(1)$ as $n \to \infty$. In order to complete the proof of Lemma 3.8, we need the following lemma.

Lemma 3.9. *We have, on the probability space of* (3.29),

$$\left|\int_0^1 (\alpha_n(u) - B_n(u))\,dG(u)\right| = O_p(n^{-1/2}\log^2 n) \text{ as } n \to \infty. \tag{3.39}$$

P r o o f. We use the inequality

$$\left|\int_0^1 (\alpha_n(u) - B_n(u))\,dG(u)\right| \leqslant \sup_{0\leqslant s\leqslant 1} |\alpha_n(s) - B_n(s)| \int_0^{1-1/n} dG(u) +$$
$$+ \left|\int_{1-1/n}^1 \alpha_n(u)\,dG(u)\right| + \left|\int_{1-1/n}^1 B_n(u)\,dG(u)\right| := A_1 + A_2 + A_3. \tag{3.40}$$

Since $G(1 - 1/n) = \log n$, (3.29) yields

$$A_1 = O_p(n^{-1/2}\log^2 n) \text{ as } n \to \infty. \tag{3.41}$$

Next, since $\mathbf{E}A_2^2 = \mathbf{E}A_3^2 = 2(1 - 1/(2n))/n$, we have

$$A_2 = O_p(n^{-1/2}) \text{ and } A_3 = O_p(n^{-1/2}) \text{ as } n \to \infty. \tag{3.42}$$

The proof of (3.39) follows from (3.41) and (3.42).

Consider now T''_n: by (3.39), we have

$$T''_n = O_p(n^{-1/2}\log^2 n) \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{\nu_2}(G(s)e^{-G(s)})/(s(1-s))^{1/2-\nu_2} =$$
$$= O_p(n^{\nu_2-1/2}\log^2 n) \text{ as } n\to\infty.$$

By all this, we have proved that $T_n = O_p(n^{\nu_2-\frac{1}{2}}\log^2 n)$ as $n\to\infty$, which, in view of (3.36) proves (3.34) and completes the proof of Lemma 3.8.

Lemma 3.10. *$q \in Q$ is an EFKP function if and only if there exists with probability one γ_0 and γ_1 such that, with Γ being as in* (1.11),

$$\gamma_0 = \limsup_{s\downarrow 0} |\Gamma(s)|/q(s) < \infty \text{ and } \gamma_1 = \limsup_{s\downarrow 0} |\Gamma(1-s)|/q(1-s) < \infty. \tag{3.43}$$

Furthermore, $q\in Q$ is a COR function if and only if $\gamma_0=\gamma_1=0$ in (3.43).

P r o o f. The result is an obvious consequence of Facts 1 and 2, jointly with the remark that, whenever $0\leqslant\gamma<1$,

$$\sup_{0<s<1} |G(s)e^{-G(s)}|/(s(1-s))^{\gamma} < \infty. \tag{3.44}$$

Another consequence of (3.44) when combined with Theorem 1.4.1 in M. Csörgö and Révész [7] is that

$$\limsup_{s\downarrow 0} |\Gamma_n(s)|/\{2s\log_2(1/s)\}^{1/2} = \limsup_{s\downarrow 0} |\Gamma_n(1-s)|/\{2s\log_2(1/s)\}^{1/2} = 1 \tag{3.45}$$

almost surely, where $\log_j$ denotes the j-th iterared logarithm. Observe that for $0<s<1/n$, $b_n(s) = n^{1/2}(1-e^{-nM_{1,n}}-s)$ and $b_n(1-s) = n^{1/2}(s-$ $-\exp(-nM_{n,n}))$. Since $\mathbf{P}(M_{1,n}>c) = (1-nc)^{n-1}$ and $\mathbf{P}(nM_{n,n}-$ $-\log n\leqslant t)\to\exp(-e^{-t})$ as $n\to\infty$ for all $t>0$, this, jointly with (3.45) proves that, for any $0<\varepsilon<1/2$,

$$\sup_{0\leqslant s\leqslant 1/(n+1)} n^{1/2-\varepsilon}|b_n(s)+\Gamma_n(s)| = o_p(1), \tag{3.46}$$

and

$$\sup_{n/(n+1)\leqslant s\leqslant 1} n^{1/2-\varepsilon}|b_n(s)+\Gamma_n(s)| = o_p(1) \text{ as } n\to\infty. \tag{3.47}$$

A joint use of (3.34), (3.46) and (3.47) proves the following lemma.

Lemma 3.11. *For all $0\leqslant\nu_2<1/4$,*

$$n^{\nu_2}\|b_n+\Gamma_n\| = o_p(1) \text{ as } n\to\infty. \tag{3.48}$$

Lemma 3.12. *Let $C_{n,2}(s)$ be as in* (3.27). *For any EFKP function $q\in Q$;*

$$\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{1/4}|C_{n,2}(s)|/\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2} = o_p(1) \text{ as } n\to\infty. \tag{3.49}$$

P r o o f. By Fact 2 and routine arguments paralleling the proof of Lemma 3.3,

$$\sup_{0<s<1} (s(1-s))^{1/4}/\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}\to 0 \text{ as } n\to\infty. \tag{3.50}$$

This, together with (3.28) taken with $\gamma=1/4$, shows that (3.49) reduces to

$$\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{1/4}|\Lambda(s+n^{-1/2}b_n(s))-\Lambda(s)|/\{s(1-s)\}^{1/4} = o_p(1). \tag{3.51}$$

Since $\Lambda(s) = -(1-s)\log(1-s)$ has derivative $\Lambda'(s) = 1+\log(1-$ $-s)$, an application of Taylor's formula gives for some $\theta\,(=\theta_{n,s})$ between

s and $V_n(s)$

$$\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{1/4}|\Lambda(s+n^{-1/2}b_n(s))-\Lambda(s)|/\{s(1-s)\}^{1/4}\leqslant$$
$$\leqslant n^{-1/4}\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)}|\Lambda'(\theta)|\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)}\frac{|b_n(s)|}{(s(1-s))^{1/4}}=$$
$$=O_p\left(n^{-3/8}\log n\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)}\frac{n^{1/8}|b_n(s)|}{(s(1-s))^{3/8}}\right), \qquad (3.52)$$

where for the last equality Lemma 3.6 is applied. In view of (3.34) with $\nu_2=1/8$ we can bound the right side of (3.52) by

$$O_p\left(n^{-1/4}(\log n)\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)}\frac{|\Gamma_n(s)|}{(s(1-s))^{3/8}}\right)+O_p(n^{-1/4}\log n), \qquad (3.53)$$

which in turn by Lemma 3.10 is bounded by $O_p(n^{-1/4}\log n)$.

Up to now, by Lemma 3.3, (3.25) and Lemma 3.12, we have proved that, for any EFKP function $q\in Q$, we have

$$n^{1/4}\|(c_n-C_{n,1})/\{2q\log^+(n^{1/2}/q)\}^{1/2}\|_n=o_p(1)\ as\ n\to\infty, \qquad (3.54)$$

where c_n is as in (3.7) and $C_{n,1}$ is as in (3.26). We now concentrate on

$$C_{n,1}(s)=-\alpha_n(F(\bar{S}_nG(s+n^{-1/2}b_n(s))))+\alpha_n(F(\bar{S}_nG(s))) \qquad (3.55)$$

Lemma 3.13. *Let B_n be as in* (3.29). *Then, for any EFKP function $q\in$ $\in Q$,*

$$n^{1/4}\|(C_{n,1}-D_n)/\{2q\log^+(n^{1/2}/q)\}^{1/2}\|_n=o_p(1)\ as\ n\to\infty, \qquad (3.56)$$

where

$$D_n(s)=-B_n(F(\bar{S}_nG(s+n^{-1/2}b_n(s))))+B_n(F(\bar{S}_nG(s))). \qquad (3.57)$$

P r o o f. Choose in (3.31) any $1/4\leqslant\nu_1<1/2$, and let $\rho>1$ be an arbitrary constant. We have, uniformly in $1/(n+1)\leqslant s\leqslant 1/2$ and $1/\rho\leqslant$ $\leqslant\lambda\leqslant\rho$,

$$n^{1/4}\frac{|\alpha_n(\lambda s)-B_n(\lambda s)|}{\{q(s)\log^+(n^{1/2}/q(s))\}^{1/2}}=\left(\frac{(ns)^{1/4-\nu_1}}{\{q(s)s^{-1/2}\log^+(n^{1/2}/q(s))\}^{1/2}}\right)O_p(1).$$

By (3.9), it is straightforward that this expression is uniformly $o_p(1)$. This, jointly with Lemmas 3.6, 3.7, and a similar argument applied to $1/2\leqslant s\leqslant n/(n+1)$, completes the proof of (3.56).

Let W_n be as in (3.30), D_n as in (3.57), and consider the decomposition

$$D_n(s)=D_{n,1}(s)+D_{n,2}(s),\quad \text{where}$$
$$D_{n,1}(s)=-W_n(F(\bar{S}_nG(s+n^{-1/2}b_n(s))))+W_n(F(\bar{S}_nG(s))), \qquad (3.58)$$

and

$$D_{n,2}(s)=(-F(\bar{S}_nG(s+n^{-1/2}b_n(s))+F(\bar{S}_nG(s)))W_n(1). \qquad (3.59)$$

Lemma 3.14. *For any EFKP function $q\in Q$,*

$$n^{1/4}\|D_{n,2}/\{2q\log^+(n^{1/2}/q)\}^{1/2}\|_n=o_p(1)\ as\ n\to\infty. \qquad (3.60)$$

P r o o f. Recall that $V_n(s)=s+n^{-1/2}b_n(s)$. By (3.16), we have, uniformly over $1/(n+1)\leqslant s\leqslant n/(n+1)$,

$$D_{n,2}(s)=\{n^{-1/2}b_n(s)+(\bar{S}_n-1)(\Lambda(V_n(s))-\Lambda(s))+(s(1-s))^{\gamma}O_p(n^{-1})+$$
$$+(V_n(s)(1-V_n(s))^{\gamma}O_p(1/n)\}W_n(1),$$

where $0<\gamma<1$ is an arbitrary constant.

Note that $|\Lambda'(s)|\leqslant 1-\log(1-s)=O_p(\log n)$ uniformly over all $1/(\rho(n+1))\leqslant s\leqslant 1-1/(\rho(n+1))$ for any fixed $\rho>1$.

Fix any $0 < \varepsilon < 1$, and choose $\rho > 1$ and $n_0 \geqslant 1$ such that, for $n \geqslant n_0$, we have with probability larger than $1 - \varepsilon$ the event

$$\rho^{-1} \leqslant V_n(s)/s \leqslant \rho \text{ and } \rho^{-1} \leqslant (1 - V_n(s))/(1 - s) \leqslant \rho, \tag{3.61}$$

for all $1/(n+1) \leqslant s \leqslant n/(n+1)$. Such a choice is possible by Lemma 3.6.

On the event of (3.61), we have uniformly over $1/(n+1) \leqslant s \leqslant n(n+1)$, as $n \to \infty$

$$D_{n,2}(s) = n^{-1/2} b_n(s) + n^{-1} O_p(\log n) b_n(s) + (s(1-s))^{\gamma} O_p(n^{-1}) : = \\ : = D'_{n,2}(s) + D''_{n,2}(s) + D'''_{n,2}(s). \tag{3.62}$$

First we see by a similar argument as in the proof of Lemma 3.3, using (3.9) and $\| b_n \| = O_p(1)$, that uniformly over $1/(n+1) \leqslant s \leqslant 1/2$,

$$n^{1/4} D'_{n,2}(s)/\{2q(s) \log^+(n^{1/2}/q(s))\}^{1/2} \leqslant n^{-1/4} \| b_n \| / \{2q(s) \log^+(n^{1/2}/q(s))\}^{1/2} = o_p(1).$$

Likewise, one shows that on the event (3.61), we have uniformly over $1/2 \leqslant s \leqslant n/(n+1)$

$$D'_{n,2}(s)/\{2q(s) \log^+(n^{1/2}/q(s))\}^{1/2} = o_p(1) \text{ as } n \to \infty. \tag{3.63}$$

Since this event holds with probability arbitrarily close to one it follows that (3.63) holds without this restriction. Of course in the same way

$$n^{1/4} D''_{n,2}(s)/\{2q(s) \log^+(n^{1/2}/q(s))\}^{1/2} = o_p(1) \; as \; n \to \infty \tag{3.64}$$

uniformly over $1/(n+1) \leqslant s \leqslant n/(n+1)$.

Next we have uniformly over $1/(n+1) \leqslant s \leqslant 1/2$ with $0 < \gamma < 1$ arbitrary but fixed

$$n^{1/4} D'''_{n,2}(s)/\{2q(s) \log^+(n^{1/2}/q(s))\}^{1/2} = \frac{(ns)^{-1/4} s^{\gamma} n^{-1/2} O_p(1)}{\{2q(s) \log^+(n^{1/2}/q(s))\}^{1/2}}, \tag{3.65}$$

which is uniformly $o_p(1)$ as $n \to \infty$. A similar argument holds for $1/2 \leqslant s \leqslant n/(n+1)$. Statements (3.62)—(3.65) together complete the proof of (3.60).

By (3.54), (3.56) and (3.60), we have just shown that

$$n^{1/4} \| (c_n - D_{n,1})/\{2q \log^+(n^{1/2}/q)\}^{1/2} \|_n = o_p(1) \; as \; n \to \infty \tag{3.66}$$

where $D_{n,1}$ is as in (3.58).

Finally, to handle $D_{n,1}$ we introduce the process

$$\widetilde{W}_n(s) = W_n\left(s - n^{-1/2} \Lambda(s) \int_0^1 B_n(u)\, dG(u)\right). \tag{3.67}$$

In order to show that in (3.58) we can replace $W_n(F(\bar{S}_n G(\cdot)))$ by $\widetilde{W}_n(\cdot)$ we need the following result (see Deheuvels and Mason [9]).

Fact 8. *For any EFKP function $q \in Q$, any $0 < \nu < 1/2$ and $k > 0$, we have*

$$n^{1/4} \sup_{1/(n+1) \leqslant s \leqslant n/(n+1)} \; \sup_{|x-y| \leqslant K\psi_n(s)} \frac{|W_n(s+x) - W_n(s+y)|}{\{q(s) \log^+(n^{1/2}/q(s))\}^{1/2}} = o_p(1) \tag{3.68}$$

as $n \to \infty$, where $\psi_n(s) = n^{-\nu - 1/2}(s(1-s))^{1/2 - \nu}$.

Lemma 3.15. *For any EFKP function $q \in Q$,*

$$n^{1/4} \sup_{1/(n+1) \leqslant s \leqslant n/(n+1)} |(W_n(F(\bar{S}_n G(s))) - \widetilde{W}_n(s))/\{2q(s) \log^+(n^{1/2}/q(s))\}^{1/2} = o_p(1) \tag{3.69}$$

and

$$n^{1/4} \sup_{1/(n+1)\leq s\leq n/(n+1)} \Big(\Big| W_n(F(\bar{S}_n G(V_n(s)))) - W_n\Big(s - n^{-1/2}\Gamma_n(s) - - n^{-1/2}\Lambda(s)\int_0^1 B_n(u)\,dG(u)\Big)\Big|\Big) \Big/ \{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2} = o_p(1) \quad (3.70)$$

as $n \to \infty$.

P r o o f. From (3.16) and (3.68) it follows that for any $0 < \nu < 1/2$ and $0 < \gamma < 1$

$$\sup_{1/(n+1)\leq s\leq n/(n+1)} n^{\nu+1/2}(s(1-s)^{-1/2+\nu}\,|\,(\bar{S}_n - 1)\Lambda(s) + + n^{-1/2}\Lambda(s)\int_0^1 B_n(u)\,G(u) + O_p(n^{-1})\,s^2(1-s)^{\gamma}\,| = O_p(1) \text{ as } n\to\infty, \quad (3.71)$$

which yields (3.69). For the proof of (3.70), we use first (3.12) and (3.37) to show that

$$\Big|(\bar{S}_n - 1)\Lambda(s) + n^{-1/2}\Lambda(s)\int_0^1 B_n(u)\,dG(u)\Big| =$$

$$= n^{-1/2}\Lambda(s)\Big|\int_0^1 (\alpha_n(u) - B_n(u)\,dG(u)\Big| = \Lambda(s)\,O_p(n^{-1}\log^2 n) \text{ as } n\to\infty,$$

so that

$$\sup_{1/(n+1)\leq s\leq n/(n+1)} n^{\nu+1/2}(s(1-s))^{-1/2+\nu}\Big|(\bar{S}_n - 1)\Lambda(s) + n^{-1/2}\Lambda(s)\int_0^1 B_n(u)\,dG(u)\Big| =$$
$$= O_p(n^{\nu-1/2}\log^2 n)\sup_{1/(n+1)\leq s\leq n/(n+1)} (\Lambda(s)/(s(1-s))^{1/2-\nu}) = o_p(1) \text{ as } n\to\infty.$$

In the same way

$$\sup_{1/(n+1)\leq s\leq n/(n+1)} n^{\nu+1/2}(s(1-s))^{-1/2+\nu}\,O_p(n^{-1})\,s^2(1-s)^{\gamma} = o_p(1) \text{ as } n\to\infty$$

by choosing $\gamma \geq 1/2 - \nu$.

Now, restricting ourselves to the event (3.61) (which holds with probability arbitrarily close to one) one shows by the same method that

$$n^{1/4}\sup_{1/(n+1)\leq s\leq n/(n+1)} \frac{|W_n(F(\bar{S}_n G(V_n(s)))) - \tilde{W}_n(V_n(s))|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = o_p(1). \quad (3.72)$$

Next, we prove, that as $n\to\infty$

$$n^{1/4}\sup_{1/(n+1)\leq s\leq n/(n+1)} \frac{\Big|\tilde{W}_n(V_n(s)) - W_n\Big(s - n^{1/2}\Gamma_n(s) - n^{-1/2}\Lambda(s)\int_0^1 B_n(u)\,dG(u)\Big)\Big|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} = = o_p(1). \quad (3.73)$$

In view of (3.68), all we need for the proof of (3.73) is to show that for some $0 < \nu < 1/2$,

$$\sup_{1/(n+1)\leq s\leq n/(n+1)} n^{\nu}(s(1-s))^{-1/2+\nu}\,|\,b_n(s) + \Gamma_n(s)\,| = O_p(1) \quad (3.74)$$

and

$$\sup_{1/(n+1)\leq s\leq n/(n+1)} n^{\nu}(s(1-s))^{-1/2+\nu}\,|\,\Lambda(V_n(s)) - \Lambda(s)\,| = O_p(1) \quad (3.75)$$

as $n\to\infty$.

Statement (3.74) follows immediately from Lemma 3.8 with $\nu_2 = \nu \in \in (0, 1/4)$.

To prove (3.75), notice that on the event of (3.61), uniformly over $1/(n+1) \leqslant s \leqslant n/(n+1)$, we have, for some $\theta_{s,n}$ between s and $V_n(s)$,

$$|\Lambda(V_n(s)) - \Lambda(s)| = n^{-1/2} |b_n(s)(1 + \log(1-\theta_{s,n}))| \leqslant$$
$$\leqslant n^{-1/2} O(\log n) |b_n(s)| \text{ as } n \to \infty.$$

Hence on this event

$$\sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} n^{\nu}(s(1-s))^{-1/2+\nu} |\Lambda(V_n(s)) - \Lambda(s)| \leqslant$$
$$\leqslant O(n^{-1/2}\log n) \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} \frac{n^{\nu}|b_n(s)+\Gamma_n(s)|}{(s(1-s))^{1/2-\nu}} +$$
$$+ O(n^{-1/2+\nu}\log n) \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} \frac{|\Gamma_n(s)|}{(s(1-s))^{1/2-\nu}},$$

so that (3.75) follows from Lemma's 3.6, 3.8, and (3.10). Relation 3.70 is now a consequence of (3.72) and (3.73).

In view of (3.66), (3.69) and (3.70), we see that complete the proof of (1.15), it only remains to show that if q is EFKP

$$n^{1/4} \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} \{2q(s)\log^{+}(n^{1/2}/q(s))\}^{1/2} \Big| W_n\Big(s - n^{1/2}\Lambda(s)\int_0^1 B_n(u)\,dG(u)\Big) -$$
$$- W_n\Big(s - n^{-1/4}\Gamma_n(s) - n^{-1/2}\Lambda(s)\int_0^1 B_n(u)\,dG(u)\Big)\Big| \xrightarrow{d} \|\Gamma/q\|^{1/2} \tag{3.76}$$

as $n \to \infty$. The following generalization of Proposition 4 in Deheuvels and Mason [9], plays the central role in the proof of this last statement.

Lemma 3.16. *Let W be a standard Wiener process extended to $(-\infty, \infty)$. Then with probability one for all $f \in C[0,1]$, $K \in \mathbf{R}$, $q \in Q$ and $0 \leqslant \eta < 1/2$*

$$\lim_{T\to\infty} T^{1/4} \sup_{\eta<s\leqslant 1-\eta} \frac{\left|W\left(s + \dfrac{f(s)+K\Lambda(s)}{T^{1/2}}\right) - W\left(s + \dfrac{K\Lambda(s)}{T^{1/2}}\right)\right|}{\{2q(s)\log^{+}(T^{1/2}/q(s))\}^{1/2}} =$$
$$= \sup_{\eta<s\leqslant 1-\eta} |f(s)/q(s)|^{1/2}. \tag{3.77}$$

P r o o f. As in Deheuvels and Mason [9] the proof goes along the following steps:

(i) For any $0 \leqslant a < b \leqslant 1$, $K \in \mathbf{R}$ and $f \in C[a,b]$ we have with probability one

$$\lim_{t\to\infty} T^{1/4}(\log T)^{-1/2} \sup_{a\leqslant s\leqslant b} |W(s + T^{-1/2}f(s) + T^{-1/2}K\Lambda(s)) -$$
$$- W(s + T^{-1/2}K\Lambda(s))| = \sup_{a\leqslant s\leqslant b} |f(s)|^{1/2}. \tag{3.78}$$

(ii) For any $f \in C[0,1]$, $q \in Q$, $K \in \mathbf{R}$, and $0 < \eta < 1/2$ we have with probability one

$$\lim_{T\to\infty} T^{1/4} \sup_{\eta<s\leqslant 1-\eta} \frac{|W(s + T^{-1/2}f(s) + T^{-1/2}K\Lambda(s)) - W(s + T^{-1/2}K\Lambda(s))|}{\{2q(s)\log(T^{1/2}/q(s))\}^{1/2}} =$$
$$= \sup_{\eta<s\leqslant 1-\eta} |f(s)/q(t)|^{1/2}. \tag{3.79}$$

(iii) Whenever $f \in C[0,1]$, $q \in Q$ are such that

$$c_1 = \limsup_{s\downarrow 0} |f(s)/q(s)| < \infty \text{ and } c_2 =$$
$$= \limsup_{s\downarrow 0} |f(1-s)/q(1-s)| < \infty,$$

then with probability one for all $\varepsilon > 0$ and $K \in \mathbf{R}$ there exists a $0 < \eta < 1/2$ such that

$$\limsup_{T\to\infty} T^{1/4} \sup_{0<s\leqslant\eta} \frac{|W(s+T^{-1/2}f(s)+T^{-1/2}K\Lambda(s)) - W(s+T^{-1/2}K\Lambda(s))|}{\{2q(s)\log^+(T^{1/2}/q(s))\}^{1/2}} \leqslant \leqslant (1+\varepsilon)(c_1+\varepsilon)^{1/2} \tag{3.80}$$

and

$$\limsup_{T\to\infty} T^{1/4} \sup_{1-\eta\leqslant s\leqslant 1} \frac{|W(s+T^{-1/2}f(s)+T^{-1/2}K\Lambda(s)) - W(s+T^{-1/2}K\Lambda(s))|}{\{2q(s)\log^+(T^{1/2}/q(s))\}^{1/2}} \leqslant \leqslant (1+\varepsilon)(c_2+\varepsilon)^{1/2}. \tag{3.81}$$

For (i), (ii) and (iii) we make use of the fact that with probability one for all $0 \leqslant a < b \leqslant 1$

$$\lim_{h\downarrow 0} \sup_{\substack{a\leqslant u\leqslant b,\\ |u-v|\leqslant h}} \frac{|W(u)-W(v)|}{(2h\log(1/h))^{1/2}} = 1.$$

(See e.g. M. Csörgő and Révész [7], Theorem 1.1.1.) The proof of each of the preceeding steps only demands minor modifications ef the proofs in Deheuvels and Mason [9].

P r o o f o f T h e o r e m 2. From Lemma 3.3, relation (3.25), Lemmas 3.12 and 3.13, relations (3.58) and (3.59), Lemmas 3.14 and 3.15 we conclude that if q is EFKP, then

$$\begin{aligned} n^{1/4} \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} \{2q(s)\log^+(n^{1/2}/q(s))\}^{-1/2} \Big| (a_n(s)+b_n(s) - \\ - W_n\Big(s - n^{-1/2}\Lambda(s)\int_0^1 B_n(u)\,dG(u)\Big) + \\ + W_n\Big(s - n^{-1/2}\Big(\Gamma_n(s) + \Lambda(s)\int_0^1 B_n(u)\,dg(u)\Big)\Big)\Big| = o_p(1) \end{aligned} \tag{3.82}$$

as $n\to\infty$. Recall from Lemma 3.10 that $q \in Q$ is an EFKP function if and only if with probability one there exist γ_0 and γ_1 such that, for Γ as in (1.11),

$$\gamma_0 = \limsup_{s\downarrow 0} |\Gamma(s)|/q(s) < \infty \text{ and } \gamma_1 = = \limsup_{s\downarrow 0} |\Gamma(1-s)|/q(1-s) < \infty.$$

Hence using the fact that a. s. convergence implies convergence in distribution we obtain from Lemma 3.16 taken with f equal to $-\Gamma_n$ and relation (3.82) that if $q \in Q$ is EFKP, then

$$\left| n^{1/4} \sup_{1/(n+1)\leqslant s\leqslant n/(n+1)} \frac{|a_n(s)+b_n(s)|}{\{2q(s)\log^+(n^{1/2}/q(s))\}^{1/2}} - \|\Gamma_n/q\|_n^{1/2} \right| = o_p(1) \; as\, n\to\infty, \tag{3.83}$$

which proves (1.15).

To prove that (1.16) holds, i. e. that

$$n^{1/4}\|(a_n+b_n)/\{2q\log^+(n^{1/2}/q)\}^{1/2}\|_n \xrightarrow{P} \infty \; as\, n\to\infty$$

if q is not EFKP, consider $\hat q \in Q$ given by $\hat q = q$ on $[\eta, 1-\eta]$, $\hat q(s) = q(\eta)$, $0 \leqslant s < \eta$, and $\hat q(s) = q(1-\eta)$, $1-\eta \leqslant s \leqslant 1$. Then $\hat q$ is EFKP and for $\eta \in (0, 1/2)$ small enough, $\hat q \geqslant q$. Now from (3.83) we have

$$\left| n^{1/4} \left\| \frac{a_n+b_n}{\{2\hat q\log(n^{1/2}/\tilde q)} \right\|_n - \left\| \frac{\Gamma_n}{\hat q} \right\|_n^{1/2} \right| = o_p(1) \; as\, n\to\infty.$$

The conclusion (1.16) now follows from Lemma 3.10 and the arbitrary choice of $\eta \in (0, 1/2)$ small enough (see e. g. the proof of Theorem 2A in Deheuvels and Mason [9]).

To prove (1.17), i. e. that whenever $q \in Q$ is COR,

$$n^{1/4} (\log n)^{-1/2} \| (a_n + b_n)/q^{1/2} \|_n \xrightarrow{d} \| \Gamma/q \|^{1/2} \, as \, n \to \infty,$$

recall by (3.43) that in this case with probability one

$$\limsup_{s \downarrow 0} | \Gamma_n (s) |/q (s) = \limsup_{s \downarrow 0} | \Gamma_n (1 - s) |/q (1 - s) = 0. \quad (3.84)$$

so that there exists a. s. a $0 < \eta_0 < 1/2$ such that

$$\sup_{\eta_0 < s \leqslant 1 - \eta_0} | \Gamma_n (s)/q (s) |^{1/2} = \sup_{0 < s < 1} | \Gamma_n (s)/q (s) |^{1/2}.$$

Moreover as in the proof of corollary 2A of Deheuvels and Mason [9],

$$\frac{1}{2} \leqslant \liminf_{n \to \infty} \inf_{1/(n+1) \leqslant s \leqslant n/(n+1)} \frac{\log^+ (n^{1/2}/q (s))}{\log n} \leqslant$$

$$\leqslant \limsup_{n \to \infty} \sup_{1/(n+1) \leqslant s \leqslant n/(n+1)} \frac{\log^+ (n^{1/2}/q (s))}{\log n} \leqslant 1.$$

One can now follow the steps used in the proof of (3.83) to show that if $q \in Q$ is COR then (i)

$$n^{1/4} (\log n)^{-1/2} \sup_{1/(n+1) \leqslant s \leqslant n/(n+1)} \{q (s)\}^{-1/2} | (a_n (s) + b_n (s)) -$$

$$- W_n \left(s - n^{-1/2} \Lambda (s) \int_0^1 B_n (u) \, dG (u)\right) + W_n \left(s - n^{-1/2} \Gamma_n (s) -\right.$$

$$\left. - n^{-1/2} \Lambda (s) \int_0^1 B_n (u) \, dG (u)\right) \Big| = o_p (1) \, as \, n \to \infty.$$

(ii) For any $0 < \eta < 1/2$ one has with probability one

$$n^{1/4} (\log n)^{-1/2} \sup_{\eta < s \leqslant 1 - \eta} \{q (s)\}^{-1/2} \Big| - W_n \left(s + n^{-1/2} \Lambda (s) \int_0^1 B_n (u) \, dG (u)\right) +$$

$$+ W_n \left(s - n^{-1/2} \left(\Gamma_n (s) + \Lambda (s) \int_0^1 B_n (u) \, dG (u)\right)\right) \Big| = \sup_{\eta < s \leqslant 1 - \eta} | \Gamma_n (s)/q (s) |^{1/2} + o (1).$$

(iii) For any $\varepsilon > 0$ there exists with probability one a $0 < \eta_1 < 1/2$ such that

$$n^{1/4} (\log n)^{-1/2} \sup_{s \in (0, \eta_1) \cup [1 - \eta_1, 1)} \{q (s)\}^{-1/2} \Big| - W_n \left(s - n^{-1/2} \Lambda (s) \int_0^1 B_n (u) \, dG (u)\right) +$$

$$+ W_n \left(s - n^{1/2} \left(\Gamma_n (s) + \Lambda (s) \int_0^1 B_n (u) \, dG (u)\right)\right) \Big| \leqslant \varepsilon^{1/2} (1 + \varepsilon).$$

The conclusion (1.17) now follows in an obvious manner from (i—ii—iii) above. The proof of Theorem 2 is now completed.

P r o o f o f T h e o r e m 1. By Lemma 3.11, $\| b_n \| / \| \Gamma_n \| \xrightarrow{P} 1$ as $n \to \infty$. Thus by (3.83) taken with $q \equiv 1$, we obtain

$$n^{1/4} (\log n)^{-1/2} \| a_n + b_n \|_n / \| b_n \|_n^{1/2} \xrightarrow{P} 1 \, as \, n \to \infty. \quad (3.85)$$

Also, using the same techniques as in the proof of Lemma 3.6, we get

$$e^{-nM_{n,n}} = O_p (n^{-1}) \, and \, 1 - e^{-nM_{1,n}} = O_p (n^{-1}),$$

from which by letting $\| f \|_{a,b} = \sup_{a<s\leq b} | f(s) |$, it follows that

$$n^{1/4} (\log n)^{-1/2} \| b_n \|_{0,1/(n+1)} \leqslant n^{3/4} (\log n)^{-1/2} \left(\frac{1}{n+1} + (1 - e^{-nM_{1,n}} \right) = \\ = O_p (n^{-1/4} (\log n)^{-1/2}) \text{ as } n \to \infty. \tag{3.86}$$

In the same way

$$n^{1/4} (\log n)^{-1/2} \| b_n \|_{n/(n+1),1} = O_p (n^{-1/4} (\log n)^{-1/2}) \text{ as } n \to \infty. \tag{3.87}$$

Moreover by Lemma 3.6 and using the fact that U_n is the inverse of V_n we have

$$\lim_{\rho\uparrow\infty} \liminf_{n\to\infty} \mathbf{P} (U_n (s) \leqslant \rho s, s \in [0, 1]) = \\ = \lim_{\rho\uparrow\infty} \liminf_{n\to\infty} \mathbf{P} (V_n (s) \geqslant \rho^{-1} s, s \in [0, 1]) = 1,$$

from which it follows that

$$n^{1/4} (\log n)^{-1/2} \| a_n \|_{0,1/(n+1)} \leqslant n^{3/4} (\log n)^{-1/2} \left(U_n \left(\frac{1}{n+1} \right) + \frac{1}{n+1} \right) = \\ = O_p (n^{-1/2} (\log n)^{-1/2}), \text{ as } n \to \infty. \tag{3.88}$$

In the same way we get

$$\lim_{\rho\uparrow\infty} \liminf_{n\to\infty} \mathbf{P} (1 - U_n (s)) \leqslant \rho (1 - s), \quad s \in [0, 1]) = 1$$

and thus

$$n^{1/4} (\log n)^{-1/2} \| a_n \|_{n/(n+1),1} = O_p (n^{-1/4} (\log n)^{-1/2}) \text{ as } n \to \infty. \tag{3.89}$$

The proof of Theorem 1 is an obvious consequence of (3.85)—(3.89).

P r o o f o f C o r o l l a r y 1. Note that by Lemma 3.11 $\| b_n \|^{1/2} \xrightarrow{d} \| \Gamma \|^{1/2}$ as $n \to \infty$. The conclusion now follows from Slutsky's theorem (see e. g. Chow and Teicher [5, p. 249]).

P r o o f o f P r o p o s i t i o n 1. A two-term Taylor expansion gives

$$| b_n (s) - \gamma_n (s) | \leqslant n^{-1/2} \gamma_n^2 (s) (1 - s)^{-2} \exp (-\theta_n (s))/2,$$

where $\theta_n (s)$ lies between $G_n (s)$ and $G (s)$. On the event of (3.61) we have uniformly over $0 \leqslant s \leqslant n/(n + 1)$,

$$e^{-G_n(s)} = 1 - V_n (s) \leqslant \rho (1 - s),$$

which implies that on this event

$$(1 - s)^{-1} \exp (-\theta_n (s)) \leqslant \rho. \tag{3.90}$$

By Theorem 4.3 (b) in Aly et al. [1], we obtain that for any $\varepsilon > 0$

$$\| \gamma_n^2 (s) (1 - s)^{-1} \|_n = O_p (n^{\varepsilon}) \text{ as } n \to \infty.$$

This in combination with (3.90) completes the proof of Proposition 1.

P r o o f o f L e m m a 2.1. Statement (2.2) is nothing else but a pointwise version of (3.82) with $q \equiv 1$ and omitting the $(\log n)^{-1/2}$ term. Given the proof of (3.82) it is simple to construct the elementary steps needed for the proof of (2.2). Therefore we omit the details.

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