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ON THE PROPERTIES OF U-STATISTICS WHEN THE OBSERVATIONS ARE NOT INDEPENDENT

PART ONE

ESTIMATION OF NON-SERIAL PARAMETERS IN SOME STATIONARY STOCHASTIC PROCESS

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1. Introduction

The use of U-statistics for the estimation of regular functionals of distribution function, based on independent samples from it, is well-known (cf. Halmos, 1946, and Hoeffding, 1948). It is the purpose of the present investigation to extend the applications of these statistics, when the observations are not independent. In this article, we have considered the case of an m-dependent (with $m \geqslant 1$) stationary stochastic process and the various properties of the usual U-statistics as estimators of estimable non-serial parameters (explained later on) have been studied. In a subsequent communication will be considered the case of a random sample drawn without replacement from a finite universe and the role of U-statistics as estimators of estimable parameters.

Let $X_1,\ldots,X_n$ be a sequence of real, vector-valued random variables. We say the sequence of random variables is m-dependent, if $(X_1,\ldots,X_r)$ is stochastically independent of $(X_s, X_{s+1}, \ldots)$, whenever $s-r>m$. This sequence will be termed a stationary process, if the joint distribution of $X_i, \ldots, X_{i+r}$ is independent of i, for all r. Let us now designate the joint distribution of $X_i, X_{i+1}, \ldots$ by $F_\theta(X_i, X_{i+1}, \ldots)$, where θ indexes the distribution. Let $f(X_{\alpha_1}, \ldots, X_{\alpha_k})$ be a statistic symmetric in the arguments $X_{\alpha_1}, \ldots, X_{\alpha_k}$; $\alpha_1 < \cdots < \alpha_k$. Its expectation will naturally depend upon the nature of the subscripts $\alpha_1, \ldots, \alpha_k$. Let us then define

$$g(\theta) = E\{f(X_{\alpha_1}, \ldots, X_{\alpha_k}) \mid \alpha_{j+1} - \alpha_j > m \text{ for } j = 1, \ldots, k-1\}, \quad \ldots(1.1)$$

and $g(\theta \mid l, v_1, \ldots, v_l) = E\{f(X_{\alpha_1}, \ldots, X_{\alpha_k})\}$, ...(1.2)

where $\alpha_{j+1} - \alpha_j = v_j$, $0 < v_j \leqslant m$ for $j = j_1 \neq \cdots \neq j_l$, while for the remaning $k-1-l$ values of j, $\alpha_{j+1} - \alpha_j > m$, for $l = 1, \ldots, k-1$. Obviously, $g(\theta) = g(\theta/0)$. Then the set of parameters $\{g(\theta \mid l, v_1, \ldots v_l), 0 < v_1, \ldots, v_l \leqslant m, l = 1, \ldots k-1\}$ will be termed *serial parameters*, and $g(\theta)$ in (1.1), a *non-serial* parameter.

The object of the peresent investigation is to provide suitable estimators of $g(\theta)$, following the lines of Hoeffding (1948) who has given a systematic account of the various properties of a class of estimators, termed by him, the U-statistics, in the particular case of $m = 0$ i.e., the sample observations being all independent. The asymptotic normality of the proposed class of estimates has been proved and its consistency established even under less stringent regularity conditions; a suitable estimate of its variance has also been supplied. Finally, the findings are extended to the case of more than one sample, and a few important applications have also been considered.

For our purpose, here we réquire a set of theorems on some combinatorial problems, which have been presented in the appendix and have been used throughout the main part of this article.

2. U-Statistics and their asymptotic properties

The statistic $f(X_{\alpha_1}, \ldots, X_{\alpha_k})$ will be termed a *non-serial statistic*, whenever $\alpha_{j+1} - \alpha_j > m$ for $j = 1, \ldots, k-1$. We then define the corresponding symmetric estimator $U_0(X_1, \ldots, X_n)$ based on n observations by

$$U_0(X_1, \ldots, X_n) = \binom{n - km + m}{k}^{-1} \sum_{S_0} f(X_{\alpha_1}, \ldots, X_{\alpha_k}), \quad \ldots(2.1)$$

where the summation S_0 extends over all possible $\binom{n - km + m}{k}$ sets of $\alpha_1, \ldots, \alpha_k$, satisfying $\alpha_{j+1} + \alpha_j > m$ (cf. Theorem 4.2). Let us also define

$$U(X_1, \ldots, X_n) = \binom{n}{k}^{-1} \sum_{S} f(X_{\alpha_1}, \ldots, X_{\alpha_k}), \quad \ldots(2.2)$$

where the summation S extends over all $1 \leqslant \alpha_1 < \cdots < \alpha_k \leqslant n$. Thus $U(X_1, \ldots, X_n)$ is Hoeffding's (1948) U-statistic, in the particular case: $m = 0$. We further assume that for all $(\alpha_1, \ldots, \alpha_k)$ such that $0 < \alpha_{j+1} - \alpha_j \leqslant m + 1$, for $j = 1, \ldots, k-1$

$$E\{\ |f(X_{\alpha_1}, \ldots, X_{\alpha_k})|^2\} < \infty \qquad \ldots(2.3)$$

Now, for estimating $g(\theta)$, $U_0(X_1, \ldots, X_n)$ is unbiased but we can also use $U(X_1, \ldots, X_n)$ with its usual asymptotic propeties, studied by Hoeffding (1948), though the latter may fail to be unbiased. In fact, we have

Theorem 2.1. *If* (2.3) *holds, then*

(i) $E_\theta\{U(X_1, \ldots, X_n)\} = g(\theta) + O(n^{-1})$, *for any finite* n, *and*

(ii) $n^{\frac{1}{2}}\{U_0(X_1, \ldots, X_n) - U(X_1, \ldots, X_n)\} \xrightarrow{P} 0.$

(*where* $\xrightarrow{P}$ *means convergence in probability*).

PROOF. It follows from (1.1), (1.2), (2.2) and theorem 4.3 that

$$E_\theta\{U(X_1, \ldots, X_n) - g(\theta)\}$$

$$= \binom{n}{k}^{-1} \Big\{\sum_{l=1}^{k-1} \binom{k-1}{l} \sum_{v_1=1}^{m} \cdots \sum_{v_l=1}^{m} \binom{n - v_1 - \cdots - v_l - (k-l-1)m}{k-l}$$

$$[g(\theta \mid l, v_1, \ldots, v_l) - g(\theta)]\Big\} \qquad \ldots(2.4)$$

Now, by the Liapounoff's inequality of moments, we have

$$|g(\theta \mid l, v_1, \ldots, v_l) - g(\theta)| \leqslant 2[E\{\ |f(X_{\alpha_1}, \ldots, X_{\alpha_k})|^2\}]^{\frac{1}{2}} < \infty, \quad \ldots(2.5)$$

for all $(\alpha_1, \ldots, \alpha_k)$ for which $\alpha_{j+1} - \alpha_j \leqslant m+1$, for all $j = 1, \ldots, l-1$; as (2.3) holds. Thus, on denoting by $g_s(\theta)$ the supremum (over $\alpha_1, \ldots, \alpha_k$) of the quantities on the lefthand side of (2.5), we get from (2.4) and (2.5) and after some simplifications that

$$|E_\theta\{U(X_1, \ldots, X_n) - g(\theta)\}| \leqslant \binom{n}{k}^{-1}\left[\binom{n}{k} - \binom{n-km+m}{k}\right] g_s(\theta)$$

$$\leqslant \frac{mk(k-1)}{n}\ g_s(\theta) = O(n^{-1}). \qquad \ldots(2.6)$$

Hence (1) is proved.

Also $n^{\frac{1}{2}}\left\{U(X_1, \ldots, X_n) - \binom{n}{k}^{-1}\binom{n-km+m}{k} U_0(X_1, \ldots, X_n)\right\}$

$$= kn^{-\frac{1}{2}}\binom{n-1}{k-1}^{-1} \sum_{S-S_0} f(X_{\alpha_1}, \ldots, X_{\alpha_k}), \qquad \ldots(2.7)$$

where the summation $S - S_0$ extends over all possible $\binom{n}{k} - \binom{n-km+m}{k}$ terms, where $\alpha_{j+1} - \alpha_j \leqslant m$, for at least one $j = 1, \ldots, k-1$. It now follows from lemma 4.7 that $\binom{n-1}{k-1}^{-1}\left[\binom{n}{k} - \binom{n-km+m}{k}\right] \leqslant m(k-1)$

and hence using (2.3), we get following some simple algebraic manipulations that

$$E\left\{\left[kn^{-\frac{1}{2}}\binom{n-1}{k-1}^{-1}\sum_{S-S_0} f(X_{\alpha_1}, \ldots, X_{\alpha_k})\right]^2\right\} = O(n^{-1}), \qquad \ldots(2.8)$$

and hence by an application of Tshebysheff's lemma, we get that the righthand side of (2.7) converges to zero in probability i.e.,

$$n^{\frac{1}{2}}\left\{U(X_1, \ldots, X_n) - \binom{n}{k}^{-1}\binom{n-km+m}{k} U_0(X_1, \ldots, X_n)\right\} \xrightarrow{P} 0 \quad \ldots(2.9)$$

Also, using lemma 4.7, we get through a well known convergence theorem by Cramér (1946, pp. 253 – 254) that

$$\binom{n}{k}^{-1}\binom{n-km+m}{k} U_0(X_1, \ldots, X_n) \xrightarrow{P} U_0(X_1, \ldots, X_n) \qquad \ldots(2.10)$$

From (2.9) and (2.10), we get that

$$n^{\frac{1}{2}}\{U(X_1, \ldots, X_n) - U_0(X_1, \ldots, X_n)\} \xrightarrow{P} 0$$

Hence, the theorem.

Thus for the estimation of the non-serial parameter $g(\theta)$, the two estimators in (2.1) and (2.2) asymptotically converge to the same variable. Let us now introduce the following notations. Put

$$\phi_c(x_{\alpha_1}, \ldots, x_{\alpha_c}) = E_\theta\left\{f(x_{\alpha_1}, \ldots, x_{\alpha_c}, X_{\alpha_{c+1}}, \ldots, X_{\alpha_k})\right\} - g(\theta), \ldots(2.11)$$

for $c=0, \ldots, k$, where $\alpha_{j+1}-\alpha_j > m$, for all $j=1, \ldots, k-1$, and let

$$\zeta_{c\cdot 0} = E_\theta\left\{\phi_c^2(X_{\alpha_1}, \ldots, X_{\alpha_c})\right\}; \qquad \ldots(2.12)$$

$$\zeta_{c.(h_1, \ldots, h_c)} = E_\theta\left\{\phi_c(X_{\alpha_1}, \ldots, X_{\alpha_c}).\ \phi_c(X_{\beta_1}, \ldots, X_{\beta_c})\right\}$$

for $c=0, \ldots, k$; where $|\alpha_i - \beta_i| = h_i$; $0 < h_i \leqslant m$ for $i=1, \ldots, c$, but $|\alpha_i - \alpha_j| > m$, $|\beta_i - \beta_j| > m$ and $|\alpha_i - \beta_j| > m$ for all $i \neq j = 1, \ldots, m$. Finally, let

$$\zeta_1 = \zeta_{1\cdot 0} + 2\,\Sigma_{h=1}^{m}\ \zeta_{1\cdot h} \qquad \ldots(2.13)$$

Then, we have the following.

Theorem 2.2. *If* $E_\theta\{\,|\phi_1(X_i)|^3\} < \infty$, *and* (2.3) *holds, then either of the following* : (i) $n^{\frac{1}{2}}\{U(X_1, \ldots, X_n) - g(\theta)\}$ *or* $n^{\frac{1}{2}}\{U_0(X_1, \ldots\ldots, X_n) - g(\theta)\}$ *has asymptotically the same normal distribution with zero mean and variance* $k^2\zeta_1$.

PROOF. By virtue of theorem 2.1, it is sufficient to show that the standardised form of $U_0(X_1, \ldots, X_n)$ has asymptotically a normal distribution.

Let us first find an expression for the variance of $U_0(X_1, \ldots, X_n)$.

Now $$V_\theta\{U_0(X_1, \ldots, X_n)\} = \binom{n-km+m}{k}^{-2} \sum_{S_0^*} \mathrm{Cov}_\theta \left\{f(X\alpha_1, \ldots, X\alpha_k),\; f(X\beta_1, \ldots, X\beta_k)\right\}, \quad \ldots(2.14)$$

where the summation S_0^* extends over all possible $\binom{n-km+m}{k}^2$ terms of the type $\alpha_{j+1}-\alpha_j > m$ for all j; $1 \leqslant \alpha_1 < \cdots < \alpha_k \leqslant n$, and $\beta_{j+1}-\beta_j > m$ for all j; $1 \leqslant \beta_1 < \cdots \beta_k \leqslant n$.

Thus, for every pair of sets $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_k)$ for which $|\alpha_i - \beta_j| > m$ for all $i, j = 1, \ldots, k$, $\mathrm{Cov}_\theta\{f(X\alpha_1, \ldots, X\alpha_k), f(X\beta_1, \ldots, X\beta_k)\} = 0$, and by theorem 4.4 there are $\binom{2k}{k}\binom{n-2km+m}{k}$ such terms – whose contribution to (2.14) will thus be nil. Let us next consider the pairs of sets $(\alpha_1, \ldots, \alpha_k)$, $(\beta_1, \ldots, \beta_k)$, such that for exactly one (i, j), say (i_0, j_0), $|\alpha_{i_0} - \beta_{j_0}| = h$; $0 \leqslant h \leqslant m$, while for all other (i, j), $|\alpha_i - \beta_j| > m$. The number of such pairs of sets has been shown in theorems 4.4 and 4.5, to be equal to

$$\binom{2k-1}{k-1}\binom{n-2km+2m}{2k-1}, \text{ if } h=0; \quad \ldots(2.15)$$

and $$2\binom{2k-1}{k-1}\binom{k}{1}\binom{n-2km+2m-h}{2k-1}, \text{ if } 1 \leqslant h \leqslant m.$$

If $h=0$, the covariance of $f(X\alpha_1, \ldots, X\alpha_k)$ and $f(X\beta_1, \ldots, X\beta_k)$ will be equal to the variance of $\phi_1(X_i)$ i.e., equal to $\zeta_{1\cdot 0}$, while for any other $1 \leqslant h \leqslant m$, this covariance will similarly be $\zeta_{1\cdot h}$. Thus, the contribution of these pairs of sets to (2.14) will be equal to

$$\binom{2k-1}{k-1}\binom{k}{1}\binom{n-km+m}{k}^{-2}\left\{\binom{n-2km+2m}{2k-1}\zeta_{1\cdot 0} + 2\sum_{h=1}^{m}\binom{n-2km+2m-h}{2k-1}\zeta_{1\cdot h}\right\} \quad \ldots(2.16)$$

and using lemmas 4.7 and 4.8, we have the following

$$k^{-1}(n-km+m)\binom{2k-1}{k-1}\binom{n-2km+2m-h}{2k-1}\binom{n-km+m}{k}^{-2} = 1 + O(n^{-1}) \text{ for all } h = 0, \ldots, m;$$

(2.16) reduces to

$$\frac{k^2}{n-km+m}\left\{\zeta_{1\cdot 0}+2\sum_{h=1}^{m}\zeta_{1\cdot h}\right\}+O(n^{-2})=\frac{k^2\zeta_1}{n}+O(n^{-2}). \qquad \ldots(2.17)$$

Let us finally consider the pair of sets $(\alpha_1, \ldots, \alpha_k)$, $(\beta_1, \ldots, \beta_k)$ for which we have $|\alpha_i-\beta_j|\leqslant n$, for at least two different (i, j); the number of such pairs of sets has been shown in theorems 4.4 and 4.5, to be of the order n^{2k-2} at most, and hence, the contribution of these terms to (2.14), is of the order n^{-2}, at most. Hence, from (2.14) through (2.17), we get that

$$V_\theta\{U_0(X_1, \ldots, X_n)\}=\frac{k^2}{n}\zeta_1+O(n^{-2}), \qquad \ldots(2.18)$$

and the same expression also holds for $V_\theta\{U(X_1, \ldots, X_n)\}$.

Now to establish the asymptotic normality of $n^{\frac{1}{2}}\{U_0(X_1, \ldots, X_n)-g(\theta)\}$, we will apply Hoeffding's (1948) ingenious technique, as applied in the particular case of $m=0$. For this, we write

$$Y_n=kn^{-\frac{1}{2}}\,\Sigma_{i=1}^{n}\,\phi_1(X_i) \qquad \ldots(2.19)$$

Since $\{\phi_1(X_i)\}$ forms an m-dependent stationary stochastic process, whose third (absolute) moment has been assumed to be finite, it follows from a Central Limit Theorem for m-dependent random variables, by Hoeffding and Robbins (1948), that Y_n has asymptotically a normal distribution with zero mean and variance

$$k^2(\zeta_{1\cdot 0}+2\,\Sigma_{h=1}^{m}\,\zeta_{1\cdot h})=k^2\zeta_1$$

Hence, if we can show that $n^{\frac{1}{2}}\{U_0(X_1, \ldots, X_n)-g(\theta)\}\xrightarrow{P}Y_n$, then the asymptotic normality of the former would follow from that of the latter. It now follows from (2.11), (2.12) and (2.19) that

$$V_\theta\{Y_n\}=k^2\zeta_1+O(n^{-1}). \qquad \ldots(2.20)$$

Also, $\mathrm{Cov}_\theta\{Y_n, n^{\frac{1}{2}}[U_0(X_1, \ldots, X_n)-g(\theta)]\}=k\binom{n-km+m}{k}^{-1}\times$

$$\sum_{i=1}^{n}\sum_{S_0}E_\theta\left\{\phi_1(X_i)\,.\,\phi_k(X_{\alpha_1}, \ldots, X_{\alpha_k})\right\} \qquad \ldots(2.21)$$

where the summation S_0 extends over all $\alpha_{j+1}-\alpha_j>m$ for all j; $1\leqslant\alpha_1<\cdots<\alpha_k\leqslant n$. Now

$$E_\theta\{\phi_1(X_i)\phi_k(X_{\alpha_1}, \ldots, X_{\alpha_k})\}$$
$$=\zeta_{1\cdot h} \text{ if } i=\alpha_j\pm h,\ 0\leqslant h\leqslant m, \text{ for some } j=1, \ldots, k;$$
$$=0, \text{ if } |i-\alpha_j|>m, \text{ for all } j=1, \ldots, k.$$

Thus using theorems 4.6 and (4.11), (2.21) reduces after some simplifications to $k^2\zeta_1+O(n^{-1})$. Combining this with (2.18) and (2.20), we get by simple adjustments that

$$E_\theta\{n^{\frac{1}{2}}[U_0(X_1, \ldots, X_n)-g(\theta)]-Y_n\}^2=O(n^{-1}),$$

and hence, an application of Tshebysheff's lemma yields that

$$Y_n \xrightarrow{P} n^{\frac{1}{2}}\{U_0(X_1, \ldots, X_n)-g(\theta)\} \qquad \ldots(2.22)$$

Hence, from the asymptotic normality of Y_n, the same follows for the variable $n^{\frac{1}{2}}\{U_0(X_1, \ldots, X_n)-g(\theta)\}$.

Hence, the theorem.

It may be noted that $\zeta_{1\cdot h}$, $0<h\leqslant m$; in general, depends upon the unknown cdf. F_θ, through the m-dependence pattern of the stochastic process, though in many non-parametric problems, $\zeta_{1\cdot 0}$ does not depend on F_θ. Thus, in most of the cases, ζ_1 depends on the unknown cdf F_θ. But, a knowledge of the value of ζ_1 appears to be essential in many problems of inference, viz., to test a numerical value of $g(\theta)$ or to attach a confidence interval to it, based on the given data. For this, we require to estimate ζ_1, and the same has been considered below.

Let $V_j=B_1(j, m \mid n, k)^{-1} \sum\limits_{S_{0j}} f(X_j, X_{\alpha_1}, \ldots, X_{\alpha_{k-1}})$,

$$\text{for } j=1, \ldots, n; \qquad \ldots(2.23)$$

where $B_1(j, m \mid n, k)$ has been defined in theorem 4.6, and where the summation S_{0j} extends over all possible $1\leqslant\alpha_1<\cdots<\alpha_{k-1}\leqslant n$; $\alpha_{j+1}-\alpha_j>m$ and $|j-\alpha_i|>m$ for all i. Let us also write

$s_{V(j)}=(n-j)^{-1}\{\Sigma_{i=1}^{n-j}(V_i-U_0)(V_{i+j}-U)\}$ for $j=0, \ldots, m$;

and $s_V^2=s_{V(0)}+2\Sigma_{j=1}^{m} s_{V(j)}$. $\qquad \ldots(2.24)$

Then, we have the following.

Theorem 2.3. *If* (2.3) *holds, then* (i) $V_j \xrightarrow{P} \phi_1(X_j)+g(\theta)$, *uniformly in* $j=1, \ldots, n$; (ii) $s_{V(j)} \xrightarrow{P} \zeta_{1\cdot j}$ *for all* $j=0, \ldots, m$ (iii) $s_V^2 \xrightarrow{P} \zeta_1$ *and further, if* $E\{|\phi_1(X_i)|^3\}<\infty$, *then* $n^{\frac{1}{2}}\{U(X_1, \ldots, X_n)-g(\theta)\}/ks_V$ *has asymptotically a normal distribution with zero mean and unit variance.*

PROOF. It follows from (2.1) and (2.23) that

$$U_0(X_1, \ldots, X_n)=\frac{1}{k}\binom{n-km+m}{k}^{-1}\sum_{j=1}^{n} B_1(j, m \mid n, k)\, V_j.$$

Proceeding then precisely on the same line as in the proof of lemma 4.9 and some simple but somewhat lengthy steps, we have

$$U_0(X_1, \ldots, X_n) = n^{-1} \Sigma_{j=1}^{n} V_j + O_p(n^{-1}) \qquad \ldots(2.25)$$

It also follows from (2.23), theorem 4.6, lemma 4.9 and some simple adjustments that

$$V_\theta\{V_j\} = \zeta_{1\cdot 0} + O(n^{-1})$$

Also by direct computation,

$$V_\theta\{\phi_1(X_j)\} = \mathrm{Cov}_\theta\ \{\phi_1(X_j),\ V_j\} = \zeta_{1\cdot 0}.$$

Hence, from the last two expressions, we get

$$E_\theta\{V_j - \phi_1(X_j) - g(\theta)\}^2 = O(n^{-1}), \text{ uniformly in } j = 1, \ldots, n$$

Thus, we get by an application of Tshebysheff's lemma that

$$V_j \xrightarrow{P} \phi_1(X_j) + g(\theta), \text{ uniformly in } j = 1, \ldots, n. \qquad \ldots(2.26)$$

Again, it follows from (2.24) and (2.25) that

$$sv_{(j)} \xrightarrow{P} \frac{1}{n-j} \Sigma_{i=1}^{n-j} (V_i - g(\theta))(V_{i+j} - g(\theta)) - \{U_0 - g(\theta)\}^2$$

$$\xrightarrow{P} \frac{1}{n-j} \Sigma_{i=1}^{n-j} (V_i - g(\theta))(V_{i+j} - g(\theta)),$$

$$\text{for all } j = 0, \ldots, m; \qquad \ldots(2.27)$$

since, by (2.18), $\{U_0 - g(\theta)\}^2 \xrightarrow{P} 0$.

We would then prove first the following two lemmas.

Lemma 2.4. *If* $\{a_i\}$ *and* $\{b_i\}$ *be two sequences of random variables such that* (i) $n^{-1} \Sigma_{i=1}^{n} (a_i - b_i)^2 \xrightarrow{P} 0$ *and* (ii) $\frac{1}{n-j} \Sigma_{i=1}^{n-j} a_i a_{i+j} \xrightarrow{P} A_j$, *for all* $j = 0, \ldots, m$; $A_0 < \infty$; *then* $\frac{1}{n-j} \Sigma_{i=1}^{n-j} b_i\, b_{i+j} \xrightarrow{P} A_j$, *for all* $j = 0, \ldots, m$,

PROOF.

$$\left|\frac{1}{n-j} \Sigma_{i=1}^{n-j} (b_i b_{i+j} - a_i a_{i+j}) - \frac{1}{n-j} \Sigma_{i=1}^{n-j} (a_i - b_i)(a_{i+j} - b_{i+j})\right|$$

$$= \left|\frac{1}{n-j} \Sigma_{i=1}^{n-j} a_i(b_{i+j} - a_{i+j}) + \frac{1}{n-j} \Sigma_{i=1}^{n-j} a_{i+j}(b_i - a_i)\right|$$

$$\leqslant \left\{\frac{1}{n-j} \Sigma_{i=1}^{n-j} a_i^2\right\}^{\frac{1}{2}} \cdot \left\{\frac{1}{n-j} \Sigma_{i=1}^{n-j} (b_{i+j} - a_{i+j})^2\right\}$$

$$+ \left\{\frac{1}{n-j} \Sigma_{i=1}^{n-j} a_{i+j}^2\right\}^{\frac{1}{2}} \cdot \left\{\frac{1}{n-j} \Sigma_{i=1}^{n-j} (a_i - b_i)^2\right\}^{\frac{1}{2}}. \qquad \ldots(2.28)$$

Now, by our assumptions,

$$\frac{1}{n-j}\,\Sigma_{i=1}^{n-j}\, a_i^2 \xrightarrow{P} \frac{1}{n-j}\,\Sigma_{i=1}^{n-j}\, a_{i-j}^2 \xrightarrow{P} A_0 < \infty, \text{ and}$$

$$\frac{1}{n-j}\,\Sigma_{i=1}^{n-j}\,(a_i - b_i)^2 \xrightarrow{P} \frac{1}{n-j}\,\Sigma_{i=1}^{n-j}\,(a_{i+j} - b_{i+j})^2 \xrightarrow{P} 0. \qquad ...(2.29)$$

Also, by an application of Cauchy-Schwarz's inequality and by (2.29) we get

$$\frac{1}{n-j}\,\Sigma_{i=1}^{n-j}\,(a_i - b_i)(a_{i+j} - b_{i+j}) \xrightarrow{P} 0. \qquad ...(2.30)$$

Hence, from (2.28) through (2.30) we readily get that

$$\frac{1}{n-j}\,\Sigma_{i=1}^{n-j}\, b_i\, b_{i+j} \xrightarrow{P} \frac{1}{n-j}\,\Sigma_{i=1}^{n-j}\, a_i a_{i+j} \xrightarrow{P} A_j, \text{ for } \quad j = 0, \ldots, m.$$

An application of Poincare's theorem on total probability then completes the proof of the lemma.

Lemma 2.5. *If* (2.3) *holds, then* $\frac{1}{n-j}\,\Sigma_{i=1}^{n-j}\, \phi_1(X_i)\phi_1(X_{i+j}) \underset{a.s}{\xrightarrow{P}} \zeta_{1\cdot j}$, *for all* $j = 0, \ldots, m$.

PROOF. Let us write $Z_i = \phi_1(X_i)\phi_1(X_{i+j})$, for $i = 1, \ldots, n-j$. Then Z_i forms an $(m+j)-$dependent stationary stochastic process, with a finite mean $\zeta_{1\cdot j}$. For any given $j(=0, \ldots, m)$, let $N_{i,j}$ be the greatest integer, which satisfies $i + (N_{i,j} - 1)(m+j+1) \leqslant n-j$, for $i = 1, \ldots, m+j$. It then follows readily that

$$\lim_{n=\infty} \quad N_{i,j}/(n-j) = 1/(j+m), \text{ as } N_{i,j} = (n-j)/(m+j) \text{ for all } i \leqslant i_0$$

while $N_{i,j} = (n-m-2j)/(m+j)$ for $i > i_0$, where $0 < i_0 < m+j$. Let us then write

$$\bar{Z}_{(i)} = \left\{Z_i + Z_{i+(m+j+1)} + \cdots + Z_{i+(N_{i,j}-1)(m+j+1)}\right\}/N_{i,j}, \qquad ...(2.31)$$

fof $i = 1, \ldots, m+j$. Since $\bar{Z}_{(i)}$ is the the mean of $N_{i,j}$ identically distributed and independent random variables, and as $N_{i,j} \to \infty$ with $n \to \infty$, it follows from Kintchine's law of Large Numbers that $\bar{Z}_{(i)} \underset{a.s.}{\xrightarrow{P}} \zeta_{1\cdot j}$, for any $i = 1, \ldots, m+j$. Since m and j are given positive quantities, it follows again by Poincare's theorem that $\bar{Z}_{(i)} \underset{a.s}{\xrightarrow{P}} \zeta_{1\cdot j}$ simultaneously for $i = 1, \ldots, m+j$. Hence,

2

$$\frac{1}{n-j}\,\Sigma_{i=1}^{n-j} Z_i = \Sigma_{i=1}^{m+j} \bar{Z}_{(i)} \frac{N_{i,j}}{n-j} \overset{P}{\rightarrow} \zeta_{1 \cdot j}$$

for any $j = 0, \ldots, m$ ...(2.32)

Since m is a given positive quantity, from (2.32) we again get by Poincare's theorem that

$$\frac{1}{n-j}\,\Sigma_{i=1}^{n-j} \phi_1(X_i)\phi_1(X_{i+j}) \underset{a.s.}{\overset{P}{\rightarrow}} \zeta_{1 \cdot j}, \text{ for all } j = 0, \ldots, m.$$

Hence, the lemma.

Let us now return to the proof of our theorem. Thus, if we let $a_i = \phi_1(X_i)$ and $b_i = V_i - g(\theta)$, we get from the last two lemmas and (2.27) that for $n \geqslant n_0(\epsilon, \eta)$, where ϵ and η are arbitrarity small positive quantities,

$$P\left\{\bigcup_{j=o}^{m}\left[\ |s_{V(j)} - \zeta_{1 \cdot j}| > \epsilon\right\} < \eta, \qquad \text{...(2.33)}$$

and consequently $s_V^2 \overset{P}{\rightarrow} \zeta_1$.

Finally, to prove the last part of the theorem, we note that

$$n^{\frac{1}{2}}\{U(X_1, \ldots, X_n) - g(\theta)\}/ks_V$$
$$= \frac{n^{\frac{1}{2}}\{U(X_1, \ldots, X_n) - g(\theta)\}/k\zeta_1^{\frac{1}{2}}}{(s_V^2/\zeta_1)^{\frac{1}{2}}} \qquad \text{...(2.34)}$$

where by theorem 2.2, the numerator of the righthand side of (2.34) has asymptotically a normal distribution, and by (2.23), the denominator converges to 1 in probability. Consequently, we get through a well-known convergence theorem by Cramér (1946, pp 253-254) that (2.34) has asymptotically a normal distribution.

Hence, the theorem.

It may be noted that the consistency of $U(X_1, \ldots, X_n)$ or $U_0(X_1, \ldots, X_n)$ as an estimator of $g(\theta)$ follows readily from the asymptotic normality of their standardised forms, sketched in theorem 2.2. This however requires the existence of the third (absolute) moment of $\phi_1(X_i)$. But, for the consistency, we need not require the existence of third order moments, as the same follows under much less restrictive regularity conditions.

Theorem 2.6. *For any unbiased estimator $f(X_{\alpha_1}, \ldots, X_{\alpha_k})$ of an estimable parameter $g(\theta)$, the corresponding $U(X_1, \ldots, X_n)$ or $U_0(X_1, \ldots, X_n)$ converges to $g(\theta)$ in probability.*

PROOF. The condition of estimability of $g(\theta)$, implies that $E_\theta\{\ |f(X_{\alpha_1}, \ldots, X_{\alpha_k})|\ \} < \infty$, for all $\alpha_1, \ldots, \alpha_k$ such that $0 < \alpha_{j+1} - \alpha_j$

$\leqslant m+1$; for $j=1, \ldots, k-1$. We will now apply the classical method of truncation (c.f. Feller (1950, p 232)) to prove the theorem.

We write $f(X_{\alpha_1}, \ldots, X_{\alpha_k}) = g(X_{\alpha_1}, \ldots, X_{\alpha_k}) + h(X_{\alpha_1}, \ldots, X_{\alpha_k})$, ...(2.35) where $h=0$ if $|f| \leqslant n\epsilon$, and $g=0$ if $|f| > n\epsilon$, ϵ being any arbitrarily small positive quantity. It then follows that by increasing n sufficiently, say for $n > n_0(\epsilon, \delta)$, δ being another arbitrarily small positive quantity ; we have for all $\alpha_{j+1} - \alpha_j > m$;

(i) $| E_\theta\{g(X_{\alpha_1}, \ldots, X_{\alpha_k}) - g(\theta)\} | \leqslant \delta$, and

(ii) $E_\theta\{ | h(X_{\alpha_1}, \ldots, X_{\alpha_k}) | \} < \delta$. Let us write now

$$U(X_1, \ldots, X_n) = U_g(X_1, \ldots, X_n) + U_h(X_1, \ldots, X_n), \qquad \ldots(2.37)$$

where U_g and U_h are the corresponding U-statistics of g and h respectively. Now, proceeding as in (2.18) and noting that

$$V_\theta\{g\} \leqslant n\epsilon\, E_\theta \, |g| \, ,$$

we get following some lengthy seps that

$$V_\theta\{U_g\} \leqslant \frac{n\epsilon A}{n} = \epsilon A, \text{ where } A < \infty .$$

Thus, by an application of Tshebysheff's lemma, we directly get that

$$P\{ \, | U_g - E\{g\} | > \epsilon' \} < \delta', \text{ for } n \geqslant n_0(\epsilon, \delta), \qquad \ldots(2.38)$$

where ϵ', δ' are also arbitrarily small. Again, from (2.36), we have,

$$E_\theta\{ \, | U_h | \, \} \leqslant E_\theta\{ \, | h | \, \} < \delta, \text{ for } n \geqslant n_0(\epsilon, \delta),$$

and hence, again by Tshebysheff's lemma, we get that $U_h \xrightarrow{P} 0$. Finally, from (2.36), we get $E_\theta\{g\} \to g(\theta)$. Hence, from (2.36), (2.37) and (2.38), we have for $n \geqslant n_0(\epsilon, \delta)$

$$P\{ \, | U(X_1, \ldots, X_n) - g(\theta) | > \epsilon'' \} < \delta'',$$

where ϵ'', δ'' are two arbitrarily small positive quantities.

Hence, the theorem.

Extension to the case of more than one sample.

Now all these findings can readily be extended to the case of more than one sample, each being a realisation from a stationary stochastic process. By way of summary, we state here briefly, the two-sample case, while the results may directly be extended to the case of c-samples ($c \geqslant 2$).

Let $X_1, \ldots, X_{n_1}$ be a realisation from an m_1-dependent stationary stochastic process with a cdf $F_{1\theta}$, and let $Y_1, \ldots, Y_{n_2}$ be another realisation from an m_2-dependent stationary stochastic process. We further assume these two processes to be mutually independent. Now, corresponding to an estimator $f(X_{\alpha_1}, \ldots, X_{\alpha_{k1}}, Y_{\beta_1}, \ldots, Y_{\beta_{k2}})$ of an estimable non-serial parameter $g(\theta)$, we define the U_0-statistic and the U-statistic based on our samples, by

$$U_0(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})$$
$$= \binom{n_1 - k_1 m_1 + m_1}{k_1}^{-1} \binom{n_2 - k_2 m_2 + m_2}{k_2}^{-1} \sum_{S_0} f(X_{\alpha_1}, \ldots, X_{\alpha_{k1}}, Y_{\beta_1}, \ldots, Y_{\beta_{k2}}), \quad \ldots(2.39)$$

where the summation S_0 extends over all possible $1 \leqslant \alpha_1 < \cdots < \alpha_{k1} \leqslant n_1$ with $\alpha_{j+1} - \alpha_j > m_1$ for all j, and $1 \leqslant \beta_1 < \cdots < \beta_{k2} \leqslant n_2$ with $\beta_{j+1} - \beta_j > m_2$ for all j; and

$$U(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}) = \binom{n_1}{k_1}^{-1} \binom{n_2}{k_2}^{-1} \times \sum_{S} f(X_{\alpha_1}, \ldots, X_{\alpha_{k1}}, Y_{\beta_1}, \ldots, Y_{\beta_{k2}}), \quad \ldots(2.40)$$

where the summation S extends over all possible $1 \leqslant \alpha_1 < \cdots < \alpha_{k1} \leqslant n_1$, $1 \leqslant \beta_1 < \cdots < \beta_{k2} \leqslant n_2$. Then as in theorem 2.1, we arrive at the following.

Theorem 2.7. *If the estimator* $f(X_{\alpha_1}, \ldots, X_{\alpha_{k1}}, Y_{\beta_1}, \ldots, Y_{\beta_{k2}})$ *has a finite second order moment for all* $0 < \alpha_{j+1} - \alpha_j \leqslant m_1 + 1$, $0 < \beta_{j+1} - \beta_j \leqslant m_2 + 1$, *then*

(i) $E_\theta\{U(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})\} = g(\theta) + O(n^{-1})$,

for any $n_1 = O(n) = n_2$,

and (ii) $n^{\frac{1}{2}}\{U(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}) - U_0(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})\} \xrightarrow{P} 0.$

Let us then introduce the following notations.

$$\phi_{c_1 c_2}(x_{\alpha_1}, \ldots, x_{\alpha_{c1}}, y_{\beta_1}, \ldots, y_{\beta_{c2}})$$
$$= E_\theta\{f(x_{\alpha_1}, \ldots, x_{\alpha_{c1}}, X_{\alpha_{c1+1}}, \ldots, X_{\alpha_{k1}}, y_{\beta_1}, \ldots, y_{\beta_{c2}}, Y_{\beta_{c2+1}}, \ldots, Y_{\beta_{k2}})\} - g(\theta) \text{ for } 0 \leqslant c_1 \leqslant k_1,\ 0 \leqslant c_2 \leqslant k_2;$$

where $\alpha_{j+1} - \alpha_j > m_1$ and $\beta_{j+1} - \beta_j > m_2$ for all j. Let then

$$\zeta_{c_1 c_2} = E_\theta\{\phi^2_{c_1 c_2}(X_{\alpha_1}, \ldots, X_{\alpha_{c1}}, Y_{\beta_1}, \ldots, Y_{\beta_{c2}})\}, \text{ for } 0 \leqslant c_1 \leqslant k_1,\ 0 \leqslant c_2 \leqslant k_2;$$

and $\zeta_{c_1 c_2}(h_{11}, \ldots, h_{1c_1}; h_{21}, \ldots, h_{2c_2})$

$$= E_\theta\{\phi_{c_1 c_2}(X\alpha_1, \ldots, X\alpha_{c_1}, Y\beta_1, \ldots, Y\beta_{c_2}).$$

$$\phi_{c_1 c_2}(X\alpha'_1, \ldots, X\alpha'_{c_1}, Y\beta'_1, \ldots, Y\beta'_{c_2})\},$$

where $|\alpha'_i - \alpha_i| = h_{1i}$; $0 \leqslant h_{1i} \leqslant m_1$, for all $i = 1, \ldots, c_1$,

and $|\beta'_j - \beta_j| = h_{2j}$; $0 \leqslant h_{2j} \leqslant m_2$, for all $j = 1, \ldots, c_2$,

but $|\alpha_i - \alpha'_j| > m_1$ for all $i \neq j = 1, \ldots, c_1$,

and $|\beta_i - \beta'_j| > m_2$, for all $i \neq j = 1, \ldots, c_2$. Finally, let

$$\zeta_{10} = \zeta_{10(0\,;\,0)} + 2\sum_{h=1}^{m_1} \zeta_{10(h\,;\,0)} \text{ and } \zeta_{01} = \zeta_{01(0\,;\,0)} + 2\sum_{h=1}^{m_2} \zeta_{01(0\,;\,h)} \qquad \ldots(2.41)$$

Then we have the following.

Theorem 2.8. *If* $E_\theta\{|\phi_{11}(X_i, Y_j)|^3\} < \infty$, *for all* (i, j), *then* $n_1^{\frac{1}{2}}\{U - g(\theta)\}$ *or* $n_1^{\frac{1}{2}}\{U_0 - g(\theta)\}$ *has asymptotically a normal distribution with zero mean and variance* $k_1^2\,\zeta_{10} + k_2^2\,\rho\zeta_{01}$, *where* n_1, n_2 *both tend to* ∞, *with* $n_1/n_2 = \rho$; $0 < \rho < \infty$.

Again to estimate ζ_{10} and ζ_{01} we consider the following. Let

$$V_{j0} = [B_1(j, m_1 \mid n_1, k_1)]^{-1} \binom{n_2 - k_2 m_2 + m_2}{k_2}^{-1} \sum_{S_{j0}} f(X_j, X\alpha_1, \ldots, X\alpha_{k_1-1}, X\beta_1, \ldots, Y\beta_{k_2}),$$

where the summation S_{j0} extends over all possible $\alpha_{i+1} - \alpha_i > m_1$, $|j - \alpha_i| > m_1$ for all $i = 1, \ldots k_1 - 1$, and $\beta_{j+1} - \beta_j > m_2$, for all $j = 1, \ldots, k_2 - 1$. Similarly, let

$$V_{0j} = \binom{n_1 - k_1 m_1 + m_1}{k_1}^{-1} [B_1(j, m_2 \mid n_2, k_2)]^{-1} \sum_{S_{0j}} f(X\alpha_1, \ldots, X\alpha_{k_1}, Y_j, Y\beta_1, \ldots, Y\beta_{k_2-1}),$$

where the summation S_{0j} extends over all $\alpha_{i+1} - \alpha_i > m_1$ for $i = 1, \ldots, k_1 - 1$, $|j - \beta_i| > m_2$, $\beta_{i+1} - \beta_i > m_2$, for all $i = 1, \ldots, k_2 - 1$, and where $B_1(j, m \mid n, k)$ has been defined in theorem 4.6. Further let

$$s_{V(j)0} = \frac{1}{n_1 - j} \sum_{i=1}^{n_1 - j} (V_{i0} - U_0)(V_{i+j\,0} - U_0), \qquad \text{for } j = 0, \ldots, m_1 ;$$

and $$s_{V0(j)} = \frac{1}{n_2 - j} \sum_{i=1}^{n_2 - j} (V_{0i} - U_0)(V_{0\,i+j} - U_0), \qquad \text{for } j = 0, \ldots, m_2 ; \qquad \ldots(2.42)$$

and $$s^2_{V0} = s_{V(0)0} + 2\sum_{h=1}^{m_1} s_{V(h)0},$$

$$s^2_{0V} = s_{V0(0)} + 2\sum_{h=1}^{m_2} s_{V0(h)}. \qquad \ldots(2.43)$$

Then, we have the following.

Theorem 2.9. *If* $f(X_{\alpha_1}, \ldots, X_{\alpha_{k_1}}, Y_{\beta_1}, \ldots, Y_{\beta_{k_2}})$ *has a finite second order moment, then* (i) $V_{j0} \xrightarrow{P} \phi_{10}(X_j) + g(\theta)$ *uniformly in* $j = 1, \ldots, n_1$; (ii) $V_{0j} \xrightarrow{P} \phi_{01}(Y_j) + g(\theta)$ *uniformly in* $j = 1, \ldots, n_2$; (iii) $s^2_{V0} \xrightarrow{P} \zeta_{10}$ *and* $s^2_{0V} \xrightarrow{P} \zeta_{01}$, *and if further* $E_\theta\{\,|\phi_{11}(X_i, Y_j)|^{\,3}\} < \infty$, *then* (iv) $n_1^{\frac{1}{2}}\{U - g(\theta)\} \,/\, \{k_1^2\, s^2_{V0} + \rho k_2^2\, s^2_{0V}\}^{\frac{1}{2}}$ *has asymptotically a normal distribution with zero mean and unit variance, provided* $n_1, n_2 \to \infty$, *with* $n_1/n_2 = \rho$; $0 < \rho < \infty$. ((iv) *also applies to* U_0).

Finally, as regards the consistency of U-statistics, we have the following.

Theorem 2.10. *For any unbiased estimator of an estimable non-serial parameter* $g(\theta)$, *the corresponding* $U(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})$ *or* $U_0(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})$ *converges to* $g(\theta)$ *in probability.*

With these properties of the U-statistics, we would next discuss some important applications of these statistics, in some problems of statistical inference.

3. Some Useful Illustrations

I. Estimation of non-serial moments of a stationary stochastic process.

Let $X_1, \ldots, X_n$ be a realisation from an m-dependent stationary stochastic process, with a continuous cdf $F_\theta(x)$, whose mean is μ_θ, variance σ^2_θ, and, in general, whose k-th central moment is $\mu_{k\theta}$, for any $k \geqslant 0$ — these parameters incidentally do not depend on X_i, as the stochastic process is assumed to be stationary. Then, obviously, these parameters are estimable, and they are non-serial ones. In the particular case of $m=0$, symmetric, unbiased and optimum estimates of these parameters are available with Halmos (1946), and Hoeffding (1948). However, in the case of m-dependent process (with $m \geqslant 1$), these estimators (excepting the sample mean) become biased ones, and nothing in detail, was known about their various properties. Accordingly, these will be studied here. For simplicity and brevity, we will consider only the case of σ_θ^2 and the general case of $\mu_{k\theta}$ would then follow precisely on the same line.

Thus, it follows from our results in section 2 that the: unbiased, symmetric estimator of σ_θ^2 is

$$U_0(X_1, \ldots, X_n) = \tfrac{1}{2}\binom{n-m}{2}^{-1} \sum_{S_0} (X_i - X_j)^2, \qquad \ldots(3.1)$$

where the summation S_0 extends over all $1 \leqslant i < j-m\,;\ j \leqslant n$. By theorem 2.1, this estimator again asymptotically converges to the usual U-statistic

$$U(X_1, \ldots, X_n) = \tfrac{1}{2}\binom{n}{2}^{-1} \sum_{1 \leqslant i < j \leqslant n} (X_i - X_j)^2$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \frac{1}{n}\, \Sigma_{i=1}^{n}\, X_i\right)^2$$

$$= s^2,$$

where s^2 is the sample variance. Thus s^2 is a consistent estimator of σ_θ^2 and by theorem 2.2, $n^{\frac{1}{2}}\,(s^2 - \sigma^2)$ has asymptotically a normal distribution with zero mean, and variance $4\zeta_1$, where $\zeta_1 = \zeta_{10} + 2\Sigma_{h=1}^{m}\, \zeta_{1\cdot h}$, and $\zeta_{1\cdot h} = \mathrm{Cov}_\theta\ \{\tfrac{1}{2}(X_i - X_j)^2,\ \tfrac{1}{2}(X_{i+h} - X_k)^2\}$, for $h=0, \ldots, m$; where $|i-j| > m$, $|k-i-h| > m$ and $|j-k| > m$. Also, by theorem 2.3, we have a consistent estimate of ζ_1 as s_V^2, where $s_V^2 = s_{V(0)} + 2\Sigma_{h=1}^{m}\, s_{V(h)}$;

$$s_{V(h)} = \frac{1}{n-h}\left\{\sum\nolimits_{i=1}^{n-h} (V_i - U_0)(V_{i+h} - U_0)\right\} \quad \text{for } h=0, \ldots, m\,;$$

and $$V_j = \frac{1}{2(n-m-j)} \sum\nolimits_{i=j+m+1}^{n} (X_i - X_j)^2, \quad \text{for } 1 \leqslant j \leqslant m+1,$$

$$= \frac{1}{2(n-2m)} \left\{ \sum_{i=1}^{j-m-1} (X_i - X_j)^2 + \sum_{i=j+m+1}^{n} (X_i - X_j)^2 \right\}$$

for $m+2 \leqslant j \leqslant n-m-1$;

$$=\frac{1}{2(j-m-1)}\sum_{i=1}^{j-m-1}(X_i-X_j)^2 \quad \text{for } n-m\leqslant j\leqslant n.$$

Finally, for large samples, $n^{\frac{1}{2}}\{s^2-\sigma_\theta^2\}/2s_V$ has a normal distribution with zero mean and unit variance, and this property may therefore be used to set confidence limits to σ_θ^2, or to test a numerical value of it.

2. *A non-parametric test against trend.*

Let $X_1, \ldots\ldots, X_n$ be a realisation from an m-dependent stochastic process (not necessarily stationary). In the particular case of $m=0$, the following test has been used by Mann (1945) to test for the randomness of the series against trend alternatives with any monotonically decreasing trend, and the test has further been studied by Hoeffding (1947).

$$U(X_1, \ldots\ldots, X_n)=\binom{n}{2}^{-1}\sum_{1\leqslant i<j\leqslant n}\phi(X_i,\ X_j),$$

$$\text{where}\quad \phi(a,\ b)=1 \text{ if } a<b$$
$$=0 \text{ if } a\geqslant b. \qquad \ldots(3.3)$$

Now, in actual practice, often we want to test against trend alternatives when the actual series is an m-dependent process. For example, let Y_i be a sequence of independent random variables, where $Y_i=M_i+\epsilon_i$, M_i being the systematic component and ϵ_i the residual error component, which we suppose to have mean zero and variance σ^2, and to have the common cdf. F_θ for all i. If now, we operate a moving average trend with a period m, the resultant series is known to be an m-dependent process (as follows from Slutzsky-Yule effect). This m-dependent process will be a stationary one, if and only if, the moving average trend coincides everywhere with the actual one. Thus, any test for the goodness of fit of the moving average trend, as to be made on these trend-deflated values, is based on an m-dependent process, and as such, Mann's (1945) test is not applicable here, as the variance of U in (3.3) will then be different from the one obtained by Mann. Accordingly, we have studied this problem thoroughly, and the relevant results are to be published elsewhere, (for the intended brevity of our discussion here, and the somewhat lengthy steps in this deduction, these are not reproduced).

3. *An analogue of Wilcoxon-Mann-Whitney's test.*

Let $X_1, \ldots\ldots, X_{n_1}$ be a realisation from an m_1-dependent stationary stochastic process, with a continuous cdf. $F_{1\theta}$, and let $Y_1, \ldots\ldots, Y_{n_2}$

be an independent realisation from a (second) m_2-dependent stationary stochastic process, with a cdf. $F_{2\theta}$. Suppose we want to estimate $P\{X<Y\}$ or to test the hypothesis that X is stochastically larger (or smaller) than Y. In the particular case : $m_1=m_2=0$, the test by Wilcoxon (1945) and later developed by Mann and Whitney (1947) is usually employed for this purpose.

It now follows from our results in section 2 that

$$U(X_1,\ldots\ldots, X_{n_1}, Y_1,\ldots\ldots, Y_{n_2})=\frac{1}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\phi(X_i, Y_j),$$

where $\phi(a, b)=1$ if $a<b$, and $=0$, otherwise ; ...(3.4)

is an unbiased estimate of $P\{X<Y\}$, though the variance of U, depends upon the unknown cdf's $F_{1\theta}$ and $F_{2\theta}$. But, it follows from theorem 2.9 and some simple but lengthy computations that

$$t=\frac{[U(X_1,\ldots\ldots, X_{n_1}, Y_1,\ldots\ldots, Y_{n_2})-P\{X<Y\}]}{\left\{\frac{1}{n_1}\, s_{VO}^2+\frac{1}{n_2}\, s_{OV}^2\right\}^{\frac{1}{2}}} \qquad \ldots(3.5)$$

has asymptotically a normal distribution with zero mean and unit variance, where

$$s_{VO}^2=\frac{1}{n_1 n_2^2}\left\{\sum_{i=1}^{n_1}(R_i-i)^2+2\sum_{j=1}^{m_1}\frac{n_1}{n_1-j}\sum_{i=1}^{n_1-j}(R_i-i)(R_{i+j}-i-j)-\frac{n_1(2m_1+1)}{4}\right\},$$

and $$s_{OV}^2=\frac{1}{n_1^2 n_2}\left\{\sum_{i=1}^{n_2}(S_i-i)^2+2\sum_{j=1}^{m_2}\frac{n_2}{n_2-j}\sum_{i=1}^{n_2-j}(S_i-i)(S_{i+j}-i-j)-\frac{n_2(2m_2+1)}{4}\right\},$$

where R_i (and S_i) is the rank of the i-th largest first (and second) sample observation in the pooled sample of size n_1+n_2.

Thus, the test for $P\{X<Y\}=\frac{1}{2}$ or the confidence interval for it, may be obtained for large samples.

In the same way, the test for the identity of locations and/or scale parameters of $F_{1\theta}$ and $F_{2\theta}$, based on appropriate U-statistics, can also be made, when the sample sizes are large. For the extreme similarity in the approach, these are therefore not considered in detail.

4. Appendix on some combinatorial theorems.

Here we would prove certain theorems on some problems in combinatorial mathematics, which we used throughout the section 2.

Theorem 4.1. *For $k\geqslant 0$, and any set of non-negative integers $a_1,\ldots\ldots, a_{k+1}$*

$$\sum_{S_{kp}} \binom{i_1}{a_1} \cdots\cdots \binom{i_k - i_{k-1}}{a_k} \binom{p - i_k}{a_{k+1}} = \binom{p+k}{a_1 + \cdots\cdots + a_{k+1}},$$

where the summation S_{kp} extends over all possible $1 \leqslant i_1 < \cdots < i_k \leqslant p$.

PROOF. It is well-known (cf. Feller (1950. p. 62)) that for any $x \leqslant a,\ y \leqslant b$,

$$\sum_{i=x}^{p-y} \binom{i}{a}\binom{p-i}{b} = \binom{p+1}{a+b+1},$$

whence the theorem can readily be proved by induction.

Theorem 4.2. *Let $(i_1 < \cdots\cdots < i_k)$ be a combination of k integers from the n natural numbers $(1,\ldots\ldots, n)$. Let $C\ (l,\ m \mid n,\ k)$ be the number of combinations of $(i_1 < \cdots\cdots < i_k)$, such that $i_{j+1} - i_j \leqslant m$, for exactly l different values of j, while for the remaining $(k-1-l)$ values of j, $i_{j+1} - i_j > m$. Then*

$$C(l,\ m \mid n,\ k) = \binom{k-1}{l} \left\{ \sum_{r=0}^{l} (-1)^r \binom{l}{r} \binom{n-(k-l-1)m + rm}{k} \right\},$$

for $l = 0,\ldots\ldots, k-1$.

PROOF. Let us first consider the case of $l=0$, and prove the theorem by induction. We have by direct enumeration, $C(0,\ m \mid n,\ 1) = n$. For $k=2$, we note that when $i_1 = j$, i_2 may be any one of the $(n-j-m)$ values, $j+m+1,\ldots\ldots, n$; $1 \leqslant j \leqslant n-m-1$, Thus,

$$C(0,\ m \mid n,\ 2) = \sum_{j=1}^{n-m-1} (n-j-m) = \binom{n-m}{2}, \text{ (by theorem 4.1).}$$

Thus, the result holds true for $k=1$ and 2. Let it be assumed then that it holds for all $k=1,\ldots\ldots, r$. Let us then consider the case of $C(0,\ m/n,\ r+1)$. Let now i_r be equal to j, then i_{r+1} may assume any one of the $(n-j-m)$ values, $j+m+1,\ldots\ldots, n$, while we can select $i_1 < \cdots\cdots < i_{r-1}$ satisfying $i_{j+1} - i_j > m$ for $j = 1,\ldots\ldots, r-1$, in $\binom{j-(m-1)-(r-2)m}{r+1}$ ways ; and finally j may range from $(r-1)(m+1)+1$ to $n-m-1$.

Hence, $C(0, m \mid n, r+1) = \sum_{j=(r-1)(m+1)+1}^{n-m-1} (n-j-m) \binom{j-(r-1)m-1}{r-1}$

$$= \sum_{\beta=r-1}^{n-mr-2} \binom{\beta}{r-1} \binom{n-mr-1-\beta}{1}$$

$$= \binom{n-mr}{r+1}, \text{ (by theorem 4.1)}$$

Hence, it follows by induction that for any $k = 1, 2, \ldots\ldots$

$$C(0, m \mid n, k) = \binom{n-km+m}{k}. \qquad \ldots(4.1)$$

Let us next consider the case of $l=1$. Let then i_j be equal to u and $i_{j+1} = u+v$, $1 \leqslant v \leqq m$; while for any other j, $i_{j+1} - i_j > m$. Then $i_1 < \ldots < i_{j-1}$ have to be selected from the $(u-m-1)$ integers $1, \ldots\ldots, u-m-1$, while $i_{j+2} < \ldots\ldots < i_k$ have to be selected from the $(n-u-v-m)$ integers $u+v+m+1, \ldots\ldots, n$. Now the first selection (i.e., of $i_1 < \cdots\cdots < i_{j-1}$) may be made in $\binom{u-(j-1)m-1}{j-1}$ ways (by 4.1), and the second selection (i.e., of $i_{j+2} < \cdots\cdots < i_k$) in $\binom{n-u-v(k-j-1)m}{k-j-1}$ ways, while v may range from 1 to m, and j from 1 to $k-1$. Hence,

$$C(1, m \mid n, k) = \sum_{j=1}^{k-1} \sum_{v=1}^{m} \sum_{u=(j-1)(m+1)+1}^{n-v-(k-j-1)(m+1)} \binom{u-(j-1)m-1}{j-1} \binom{n-u-v-(k-j-1)m}{k-j-1}$$

$$= \sum_{j=1}^{k-1} \sum_{v=1}^{m} \sum_{\alpha=j-1}^{n-v-2-(k-2)m} \binom{\alpha}{j-1} \binom{n-v-1-m(k-2)-\alpha}{k-j-1}$$

$$= \sum_{j=1}^{k-1} \sum_{v=1}^{m} \binom{n-v-(k-2)m}{k-1}, \text{ (by theorem 4.1)} \qquad \ldots(4.2)$$

Also, $\binom{n}{r} = \binom{n-1}{r-1} + \cdots\cdots + \binom{n-p}{r-1} + \binom{n-p}{r}$ for any $p > 1$,

and hence, from (4.2), we get

$$C(1, m \mid n, k) = (k-1) \left\{ \binom{n-(k-2)m}{k} - \binom{n-(k-1)m}{k} \right\} \qquad \ldots(4.3)$$

Similarly, for $C(2, m \mid n, k)$, we have either of the following two cases :

(i) $i_j + v_1 = i_{j+1}$, $i_{j+1} + v_2 = i_{j+2}$, $1 \leqslant v_1$, $v_2 \leqslant m$, while for all other r's, $\alpha_{r+1} - \alpha_r > m$;

(ii) $i_j + v_1 = i_{j+1}$, $i_{j'} + v_2 = i_{j'+1}$, $j' > j+1$, $1 \leqslant v_1$, $v_2 \leqslant m$, while for all other r's $\alpha_{r+1} - \alpha_r > m$;

Thus, proceeding precisely on the same line, as before, it follows that

$$C(2,\ m \mid n,\ k)=\sum_{j=1}^{k-2}\ \sum_{v_1=1}^{m}\ \sum_{v_2=1}^{m}\ \sum_{u=(j-1)(m+1)+1}^{(n-v_1-v_2)-(k-j-2)(m+1)} \binom{u-(j-1)m-1}{j-1}$$

$$\binom{n-u-v_1-v_2-(k-j-2)m}{k-j-2}+\sum_{j'>j=1}^{k-2}\ \sum_{v_1=1}^{m}\ \sum_{v_2=1}^{m}\ \sum_{u_2>u_1}$$

$$\binom{u_1-(j-1)m-1}{j-1}\binom{u_2-u_1-v_1-1-(j-j'-1)m}{j'-j-1}\binom{n-v_2-u_2-(k-j'-1)m}{k-j'-1}$$

$$=\sum_{j=1}^{k-2}\ \sum_{v_1=1}^{m}\ \sum_{v_2=1}^{m}\binom{n-v_1-v_2-(k-3)m}{k-2}+$$

$$\sum_{j'>j=1}^{k-2}\ \sum_{v_1=1}^{m}\ \sum_{v_2=1}^{m}\binom{n-v_1-v_2-(k-3)m}{k-2}\quad \text{(by theorem 4.1)}$$

$$=\binom{k-1}{2}\left\{\sum_{v_1=1}^{m}\left[\binom{n-v_1-(k-3)m}{k-1}-\binom{n-v_1-(k-2)m}{k-1}\right]\right\}$$

$$=\binom{k-1}{2}\left\{\binom{n-(k-3)m}{k}-2\binom{n-(k-2)m}{k}+\binom{n-(k-1)m}{k}\right\}\qquad \ldots(4.4)$$

Precisely on the same line, it follows that for any $l \geqslant 0$,

$$C(l,\ m \mid n,\ k)=\binom{k-1}{l}\left\{\sum_{r=0}^{l}(-1)^r\binom{l}{r}\binom{n-(k-1-l)m+rm}{k}\right\}\qquad \ldots(4.5)$$

Hence, the theorem.

Also, it may be noted here that

$$\sum_{l=0}^{k-1} C(l,\ m \mid n,\ a)=\sum_{l=0}^{k-1}\binom{n-lm}{k}\left\{\sum_{r=0}^{l}(-1)^r\binom{k+r-l-1}{r}\binom{k-1}{k+r-l-1}\right\}$$

$$=\sum_{l=0}^{k-1}\binom{n-lm}{k}\binom{k-1}{l}\left\{\sum_{r=0}^{l}(-1)^r\binom{l}{r}\right\}=\binom{n}{k},\qquad \ldots(4.6)$$

as it is to be expected.

Theorem 4.3. *Let* $(i_1<\cdots\cdots<i_k)$ *be a combination of* k *integers from the* n *natural numbers* $(1,\ldots\ldots,n)$. *Let* $d(v_1,\ldots\ldots,v_l,\ m \mid n,\ k)$ *be the number of combinations of* $(i_1<\cdots<i_k)$, *such that for exactly* l *of the values of* j, $\alpha_{j+1}-\alpha_j=v_j$, $1 \leqslant v_j \leqslant m$, *while for the remaining* $(k-1-l)$ *values of* j, $\alpha_{j+1}-\alpha_j>m$, $1 \leqslant l \leqslant k-1$, *then*

$$d(v_1,\ldots\ldots, v_l, m \mid n, k)=\binom{k-1}{l}\frac{l\,!}{\alpha_1\,!\ldots\ldots\alpha_{l'}\,!}\binom{n-v_1-\cdots-v_l-(k-1-l)m}{k-l}$$

where among $v_1,\ldots\ldots, v_l$, α_i *are equal to* v_i *for* $i=1,\ldots\ldots, l'$; $l'\leqslant l$, $\sum_{i=1}^{l'}\alpha_i=l$.

The proof follows precisely on the same line as in theorem 4.2, and hence is omitted.

Theorem 4.4. *Let* $(i_{11}<\cdots\cdots<i_{1k})$ *and* $(i_{21}<\cdots\cdots<i_{2k})$ *be two combinations of* k *integers, each from the set of natural numbers* $(1,\ldots, n)$; *such that* $i_{j(l+1)}-i_{jl}>m$, *for all* $j=1$, 2 *and* $l=1,\ldots\ldots, k-1$. *Then the number of such pairs of sets* $(i_{11}<\cdots\cdots<i_{1k})$ *and* $(i_{21}<\cdots\cdots<i_{2k})$, *for which exactly* l *of the two sets of* k *integers each are common, while the remaining* $k-l$ *integers in each set are apart from the ones in the other set by a gap of more than* m, *is*

$$A(l, m \mid n, k)=\binom{2k-l}{k-l}\binom{k}{l}\binom{n-(2k-l-1)m}{2k-l}$$

for $l=0,\ldots\ldots, k$.

PROOF. As there are l integers common to the two sets of k integers each, the number of distinct integers in the combined set is $2k-l$, which we denote by $u_1<\cdots\cdots<u_{2k-l}$ and which thus satisfy $u_{j+1}-u_j>m$, for all $j=1,\ldots\ldots, 2k-l-1$. Now, these $(2k-l)$ integers may be selected from $(1,\ldots\ldots, n)$ in $C(0, m \mid n, 2k-l)$ ways. Now, in the first set we can take any k of these $2k-l$ integers in $\binom{2k-l}{k}=\binom{2k-l}{k-l}$ ways, while the remaining $k-l$ integers may be combined with l integers selected from the first set in $\binom{k}{l}$ ways, to form the second set. Hence

$$A(l, m \mid n, k)=\binom{2k-l}{k-l}\binom{k}{l}\binom{n-(2k-l-1)m}{2k-l} \qquad \ldots(4.7)$$

for $l=0,\ldots\ldots, k$.

Hence, the theorem.

Theorem 4.5. *Let* $(i_{11}<\cdots\cdots<i_{1k})$ *and* $(i_{21}<\ldots\ldots<i_{2k})$ *be two combinations of* k *integers, each from the set of* n *natural integers* $(1,\ldots, n)$; *such that* $i_{j(l+1)}-i_{jl}>m$ *for all* $j=1$, 2 ; $l=1,\ldots\ldots, k-1$. *Then the number of such pairs of sets, for which* $\mid i_{1\alpha j}-i_{2\beta j}\mid =v_j$; $1\leqslant v_j\leqslant m$

where α_j, β_j *are some of* $(1, \ldots\ldots, k)$,) *for exactly* l *different values of* j, *while for the remaining* $(k-l)$ *values of* j, $|i_{1j} - i_{2j'}| > m$ *and* $|i_{1j'} - i_{2j}| > m$ *for all* $j' = 1, \ldots\ldots, k$; *is*

$$A(v_1, \ldots\ldots, v_l, m \mid n, k) = 2^l \frac{l\,!}{\alpha_1\,! \ldots\ldots \alpha_{l'}\,!} \binom{2k-l}{k-l}\binom{k}{l}$$
$$\binom{n - v_1 - \ldots\ldots - v_l - (2k-l-1)m}{2k-l},$$

where among the l *values* $v_1, \ldots\ldots, v_l$, α_i *are equal to* v_i *for* $i = 1, \ldots, l'$, $l' \leqslant l$, $\sum_{i=1}^{l'} \alpha_i = l$, *for all* $1 \leqslant v_1, \ldots\ldots, v_l \leqslant m$, $1 \leqslant l \leqslant k$.

The proof is a simple extension of the proof of the preceding theorem and hence is not considered here.

Theorem 4.6. *The number of combinations* $(i_1 < \ldots\ldots < i_k)$, *each containing a particular integer* j, *as well as satisfying* $i_{p+1} - i_p > m$ *for all* $p = 1, \ldots\ldots, k-1$, *is*

$$B_1(j, m \mid n, k) = \sum_{r=0}^{k-1} \binom{j - rm - 1}{r}\binom{n - j - (k-1-r)m}{k-1-r},$$

for $j = 1, \ldots\ldots, n$. *The number of pairs of sets i.e.,* $(i_{11} < \ldots\ldots < i_{1k})$ *and* $(i_{21} < \ldots\ldots < i_{2k})$ *both containing a particular integer* j, *as well as satisfying* $i_{p(l+1)} - i_{pl} > m$, *for* $p = 1, 2$, $l = 1, \ldots\ldots, k$ *and* $|i_{1l} - i_{2l'}| > m$ *for all* $l, l' = 1, \ldots\ldots, k$, (*excepting for* j *which is common*), *is*

$$B_2(j, m \mid n, k) = \binom{2k-2}{k-1} \sum_{r=0}^{2k-2} \binom{j - rm - 1}{r}\binom{n - j - (2k-2-r)m}{2k-2-r}$$

for $j = 1, \ldots\ldots, n$.

PROOF. The particular integer j may appear in any of the k ordered positions (i.e. 1st, ..., k-th one), and in each case, it will do so, in a certain number of ways. Let now $j = i_{r+1}$. Then $i_1 < \ldots\ldots < i_r$ have to be selected from the $(j - m - 1)$ integers $1, \ldots\ldots, j - m - 1$, and this can be done in $C(0, m \mid j - m - 1, r) = \binom{j - rm - 1}{r}$ ways as $\alpha_{p+1} - \alpha_p > m$ for all p. Similarly, $i_{r+2} < \ldots\ldots < i_k$ have to be selected from $j + m + 1, \ldots\ldots, n$, satisfying $\alpha_{p+1} - \alpha_p > m$ for all p, and this can be done in $\binom{n - j - (k - r - 1)m}{k - r - 1}$ ways. Hence

$$B_1(j, m \mid n, k) = \sum_{r=0}^{k-1} \binom{j - 1 - rm}{r}\binom{n - j - (k - r - 1)m}{k - r - 1)} \qquad \ldots(4.8)$$

for all $j = 1, \ldots\ldots, n$.

It also follows by simple arguments that

$$B_2(j,\ m \mid n,\ k) = \binom{2k-2}{k-1} B_1(j,\ m \mid n,\ 2k-1),$$

and hence, from (4.8), we get that

$$B_2(j,\ m \mid n,\ k) = \binom{2k-2}{k-1} \sum_{r=0}^{2k-2} \binom{j-1-rm}{r} \binom{n-j-(k-r-1)m}{k-r-1},$$

for $j = 1, \ldots\ldots, n$.

Hence, the theorem.

Lemma 4.7. $\left[\binom{n}{k} - \binom{n-km+m}{k} \mid \binom{n}{k}\right]^{-1} < \frac{mk(k-1)}{n} = O(n^{-1})$

Proof. Using the simple identity that for any $p > 1$

$$\binom{n}{r} = \binom{n-1}{r-1} + \cdots\cdots + \binom{n-p}{r-1} + \binom{n-p}{r}$$

we get directly that

$$\binom{n}{k}^{-1}\left[\binom{n}{k} - \binom{n-km+m}{k}\right] - \binom{n}{k}^{-1} \sum_{s=0}^{m(k-1)-1} \binom{n-(k-1)m+s}{k-1}$$

$$\leqslant \binom{n}{k}^{-1} m(k-1) \binom{n-1}{n-k} = \frac{mk(k-1)}{n} = O(n^{-1}).$$

Hence, the lemma.

Lemma 4.8. For any given $p',\ p > 0$, $\binom{2p}{p}\binom{N-p'}{2p}\binom{N}{p}^{-2}$

$$= 1 + O(N^{-1}).$$

Proof. $$\binom{2p}{p}\binom{N-p'}{2p}\binom{N}{p}^{-2} = \frac{(N-p')\ldots(N-p'-2p+1)}{\{N(N-1)\ldots(N-p+1)\}^2} = 1 + O(N^{-2})$$

Hence, the lemma.

Lemma 4.9. *With the notations in theorem 4.6, we have for all* $j = 1, \ldots\ldots, n$

$$\frac{B_2(j,\ m \mid n.\ k)}{\{B_1(j,\ m \mid n,\ k)\}^2} = 1 + O(n^{-1}).$$

Proof. We have from theorem 4.6 that

$$B_1(j,\ m \mid n,\ k) = \binom{n-(k-1)m-1}{k-1} \sum_{r=0}^{k-1} \binom{k-1}{r} \binom{n-(k-1)(m+1)-1}{j-r(m+1)-1} \Big/ \binom{n-(k-1)m-1}{j-rm-1} \qquad \ldots(4.9)$$

It follows similarly that

$$B_2(j,\ m \mid n,\ k) = \binom{2k-2}{k-1}\binom{n-2(k-1)m-1}{2k-2} \sum_{r=0}^{2k-2} \binom{2k-2}{r} \binom{n-2(k-1)(m+1)-1}{j-r(m+1)-1} \Big/ \binom{n-2(k-1)m-1}{j-rm-1} \qquad \ldots(4.10)$$

Thus, using lemma 4.8, we are only to show that, for any a, $p>0$

$$A_j(p,\ n)=\sum_{r=0}^{p}\binom{p}{r}\binom{n-p(a+1)-1}{j-r(a+1)-1}\binom{n-pa-1}{j-ra-1}^{-1}=1+O(n^{-1}), \quad \ldots(4.11)$$

for all $j=1,\ldots,\ n$, as the rest would then follow from (4.9) and (4.10).

$$\text{Now, } A_j\ (p,\ n)=\frac{\binom{n-1}{p}}{\binom{n-pa-1}{p}}\sum_{r=0}^{p}\binom{j-ra-1}{r}\binom{n-j-(p-r)a}{p-r}\Big/\binom{n-1}{p}$$

$$\leqslant\frac{\binom{n-1}{p}}{\binom{n-pa-1}{p}}\sum_{r=0}^{p}\frac{\binom{j-1}{r}\binom{n-j}{p-r}}{\binom{n-1}{p}}=\frac{\binom{n-1}{p}}{\binom{n-pa-1}{p}}$$

for all $j=1,\ldots\ldots,\ n$.

Similarly, $A_j(p,\ n)\geqslant\binom{n-2pa-1}{p}\binom{n-pa-1}{p}^{-1}$, for all $j=1,\ldots n$.

$$\text{Thus, } \frac{\binom{n-2pa-1}{p}}{\binom{n-pa-1}{p}}\leqslant A_j(p,\ n)\leqslant\frac{\binom{n-1}{p}}{\binom{n-pa-1}{p}} \text{ for all } j=1,\ldots,\ n. \quad \ldots(4.12)$$

Also, using lemma 4.8, we get directly that

$$\left|\binom{n-2pa-1}{p}\binom{n-pa-1}{p}^{-1}-1\right|=O(n^{-1})=\left|\binom{n-1}{p}\binom{n-pa-1}{p}^{-1}-1\right|, \quad \ldots(4.13)$$

i.e., $A_j(p,\ n)=1+O(n^{-1})$, for all $j=1,\ldots\ldots,\ n$.
Hence, the lemma.

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