Given a sequence of continuous laws $P^{(n)}, n \in \mathbb{N}$, for (Z, S), $Z \perp\!\!\!\perp S$, and a limit law $P^{(\infty)}$, let:

$$
\begin{aligned}
\hat{\tau}(\theta) &= 2\binom{n}{2}^{-1} \sum_{1 \le j < k \le n} \left\{ \frac{Z_j - Z_k}{S_j - S_k} < \theta \right\} - 1 \\
\tau(\theta) &= \mathbb{E}^{(n)}\hat{\tau}(\theta) = 2\mathbb{P}^{(n)}\left(\frac{z - z'}{s - s'} < \theta\right) - 1 \\
\Pi^{(n)}\hat{\tau}(\theta) &= 2\left(\frac{1}{n}\sum_{j=1}^{n} 2P^{(n)}\left(\frac{Z_j - Z}{S_j - S} < \theta \,\middle|\, Z_j, S_j\right) - 1\right) - \left(2\mathbb{P}^{(n)}\left(\frac{z - z'}{s - s'} < \theta\right) - 1\right)
\end{aligned}
$$

Theorem 1. *Assuming:*

1. $\theta^{(n)} = E^{(n)}Z \to E^{(\infty)}Z = \theta^{(\infty)} = 0$
2. $E^{(n)}Z^2 \to E^{(\infty)}Z^2$
3. $E^{(\infty)}f_Z^{(\infty)}(Z) = \int \left(f_Z^{(\infty)}\right)^2 < \infty$
4. $f_{Z-Z'}^{(n)} \to f_{Z-Z'}^{(\infty)}$ *uniformly*
5. *the distribution of* S *under* $P^{(n)}$ *does not depend on* n
6. $\mu_p = E^{(n)}S^p = ES^p < \infty$ *for* $p = 1, 2$

Then:

$$
\begin{aligned}
&\sqrt{n}\left(\hat{\tau}(\hat{\theta}) - (\mathbb{E}^{(n)}\hat{\tau})\left(\frac{\mu_1}{\mu_2}\theta^{(n)}\right)\right) \\
&\qquad = \sqrt{n}(\Pi^{(n)}\hat{\tau})(0) + \sqrt{n}\left(\frac{\overline{ZS}}{\mu_2} - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) \cdot 2\mathbb{E}|S_1 - S_2|\mathbb{E}^{(\infty)}f_Z^{(\infty)}(Z) + o_{\mathbb{P}^{(n)}}(1).
\end{aligned}
$$

Proof. Show that:

1. $\sqrt{n}\left(\hat{\tau}(\hat{\theta}) - (\Pi^{(n)}\hat{\tau})(\hat{\theta})\right) = o_{\mathbb{P}^{(n)}}(1)$
2. $\sqrt{n}((\Pi^{(n)}\hat{\tau})(\hat{\theta}) - (\mathbb{E}^{(n)}\hat{\tau})(\hat{\theta})) = \sqrt{n}(\Pi^{(n)}\hat{\tau})(0) + o_{\mathbb{P}^{(n)}}(1)$
3. $\sqrt{n}\left((\mathbb{E}^{(n)}\hat{\tau})(\hat{\theta}) - (\mathbb{E}^{(n)}\hat{\tau})\left(\frac{\mu_1}{\mu_2}\theta^{(n)}\right)\right) = \sqrt{n}\left(\frac{\overline{ZS}}{\mu_2} - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) \cdot 2\mathbb{E}|S_1 - S_2|\mathbb{E}^{(\infty)}f_Z^{(\infty)}(Z)$.

1. Let
$$g^{(n)}(\theta) = \sqrt{n}\left(\hat{\tau}(\theta) - \Pi^{(n)}\hat{\tau}(\theta)\right),$$
so the left-hand side of the equation in step 1 is $g^{(n)}(\hat{\theta})$. So enough to show that $\sup_\theta |g^{(n)}(\theta)| = O(n^{-1/2})$.

$$
\begin{aligned}
g^{(n)}(\theta) &= \sqrt{n}\left(\hat{\tau}(\theta) - \Pi^{(n)}\hat{\tau}(\theta)\right) \\
&= \sqrt{n}\binom{n}{2}^{-1} \sum_{1 \le j < k \le n} \left(\left\{ \frac{Z_j - Z_k}{S_j - S_k} < \theta \right\} - P^{(n)}\left(\frac{Z_j - Z_k}{S_j - S_k} \middle| z_j, s_j \right) \right. \\
&\qquad \left. - P^{(n)}\left(\frac{Z_j - Z_k}{S_j - S_k} \middle| z_k, s_k \right) + P^{(n)} \otimes P^{(n)}\left(\frac{Z_j - Z_k}{S_j - S_k} \right) \right).
\end{aligned}
$$

For any fixed θ and n, the last line is a U-statistic with bivariate kernel

$$
((z, s), (z', s')) \mapsto \left\{ \frac{z - z'}{s - s'} \right\} - P^{(n)}\left(\frac{z - z'}{s - s'} \middle| z, s \right) - P^{(n)}\left(\frac{z - z'}{s - s'} \middle| z', s' \right) + P^{(n)} \otimes P^{(n)}\left(\frac{z - z'}{s - s'} \right). \tag{1}
$$

Let $\tilde{\mathcal{F}}^{(n)}$ denote the class of functions (1) as $\theta \in \mathbb{R}$ varies. The kernels in this class give rise to degenerate U-statistics with respect to $\mathbb{P}^{(n)}$, i.e., for any $f \in \tilde{\mathcal{F}}^{(n)}$ and (z, s),

$$
P^{(n)} f((z, s), (z', s') \mid (z, s)) = 0.
$$

Nolan and Pollard (1987, 1988) provides a bound for the supremum of $|g^{(n)}(\theta)|$ over a class of degenerate bivariate kernels such as $\tilde{\mathcal{F}}^{(n)}$. Given an IID sample $x_1, \ldots, x_{2n}$, let T_n denote the random measure that places mass 1 on all pairs of observations of the form $(x_{2j}, x_{2k}), (x_{2j-1}, x_{2k-1}), (x_{2j-1}, x_{2k})$, or (x_{2j}, x_{2k-1}), where $j \neq k, 1 \le j, k \le n$. Given a class of real function $\mathcal{F}$, measure μ, and $u > 0$, let $N(u, \mu, \mathcal{F})$ denote the $L^2(\mu)$ covering number of $\mathcal{F}$, and let $J(\mu, \mathcal{F}) = \int_0^1 \log N(u, \mu, \mathcal{F}) du$ denote the associated covering integral. Let F denote a bound for the functions in $\tilde{\mathcal{F}}^{(n)}$. By Theorem 6 of Nolan and Pollard (1987), there is a constant c such that

$$
\mathbb{P}^{(n)} \sup_{\tilde{\mathcal{F}}^{(n)}} |g^{(n)}| \le \frac{c}{\sqrt{n}} \mathbb{P}^{(n)} \left(\frac{1}{4} \sup_{\tilde{\mathcal{F}}^{(n)}} \sqrt{\frac{T_n f^2}{n^2}} + \sqrt{\frac{T_n F^2}{n^2}} J\left(T_n, \tilde{\mathcal{F}}^{(n)}\right) \right).
$$

With two applications of the Cauchy-Schwarz inequality,

$$
\mathbb{P}^{(n)} \sup_{\tilde{\mathcal{F}}^{(n)}} |g^{(n)}| \le \frac{c}{\sqrt{n}} \sqrt{\mathbb{P}^{(n)} \frac{T_n F^2}{n^2}} \left(1 + \sqrt{\mathbb{P}^{(n)} J(T_n, \tilde{\mathcal{F}}^{(n)})^2} \right).
$$

Taking the bound $F = 4$ for $f \in \tilde{\mathcal{F}}^{(n)}$, $\sqrt{\mathbb{P}^{(n)} \frac{T_n F^2}{n^2}} \le 64$. Since $\tilde{\mathcal{F}}^{(n)} \subset \mathcal{F} + 2P^{(n)}\mathcal{F} + P^{(n)} \otimes P^{(n)}\mathcal{F}$, the subadditive property of covering numbers, Lemma 16 of Nolan and Pollard (1987), implies

$$
N(u, T_n, \tilde{\mathcal{F}}^{(n)}) \le N(u/4, T_n, \mathcal{F}) \cdot N(u/16, T_n, P^{(n)}\mathcal{F}) \cdot N(u/64, T_n, P^{(n)}\mathcal{F}) \cdot N(u/64, T_n, P^{(n)}P^{(n)}\mathcal{F}).
$$

The functions in each of the classes $\mathcal{F}, P^{(n)}\mathcal{F}$, and $P^{(n)} \otimes P^{(n)}\mathcal{F}$ are monotonic in θ, so each has a linear discriminating polynomial, $p(x) = x + 1$. By the Approximation Lemma, II.25 of Pollard (1984), there exist constants A, W, depending only on the discriminating polynomial, such that

$$N(u, \tilde{\mathcal{F}}^{(n)}) \leq A\left(\frac{u}{4}\right)^{-W} \cdot A\left(\frac{u}{16}\right)^{-W} \cdot 2A\left(\frac{u}{64}\right)^{-W} = 2A^3\left(\frac{u^3}{4^6}\right)^{-W}.$$

Therefore,

$$J(T_n, \tilde{\mathcal{F}}^{(n)}) = \int_0^1 \log N(u, T_n, \tilde{\mathcal{F}}^{(n)}) du \leq \int_0^1 \log\left(2A^3\left(\frac{u^3}{4^6}\right)^{-W}\right) du$$

is bounded uniformly in n.

2. Let $P_n^{(n)}$ denote the empirical measure on a sample of size n under $P^{(n)}$. For $\theta \in \mathbb{R}$, let $h_\theta^{(n)}$ denote the function $(z,s) \mapsto P^{(n)}\left(\frac{z-z'}{s-s'} < \theta \middle| z, s\right)$. Then

$$\sqrt{n}\left((\Pi^{(n)}\hat{\tau})(\theta) - (\mathbb{E}^{(n)}\hat{\tau})(\theta)\right) = \sqrt{n}\left(\frac{4}{n}\sum_{j=1}^{n} P^{(n)}\left(\frac{Z_j - Z}{S_j - S} < \theta \middle| Z_j, S_j\right) - 4P^{(n)} \otimes P^{(n)}\left(\frac{Z - Z'}{S - S'} < \theta\right)\right)$$

$$= \sqrt{n}\left(P_n^{(n)} - P^{(n)}\right)(h_\theta^{(n)}).$$

For fixed n, letting θ vary, $\sqrt{n}\left((\Pi^{(n)}\hat{\tau})(\theta) - (\mathbb{E}^{(n)}\hat{\tau})(\theta)\right)$ is therefore an empirical process indexed by the class of functions

$$P^{(n)}\mathcal{F} = \left\{h_\theta^{(n)} : \theta \in \mathbb{R}\right\}.$$

In terms of this process, the equation in step 2 is

$$\sqrt{n}\left(P_n^{(n)} - P^{(n)}\right)(f_\theta - f_0) = o_{\mathbb{P}^{(n)}}(1),$$

which follows on showing

(a) $\sqrt{n}\left(P_n^{(n)} - P^{(n)}\right)$ is stochastically equicontinuous along $P^{(n)}$, that is, for any $\eta > 0, \epsilon > 0$, there is $\delta > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}^{(n)}\left(\sup_{[\delta]^{(n)}} |\sqrt{n}\left(P_n^{(n)} - P^{(n)}\right)(h_\theta - h_{\theta'})| > \eta\right) < \epsilon,$$

where $[\delta]^{(n)} = \{(\theta, \theta') \in P^{(n)}\mathcal{F}^{(n)} : P^{(n)}(h_\theta - h_{\theta'})^2 \leq \delta^2\}$,

(b) For any $\epsilon > 0$, there is $\delta > 0$ such that $|\theta| < \delta$ implies $\limsup P^{(n)}(h_\theta - f_0)^2 < \epsilon$, and

(c) $\hat{\theta}$ is consistent along $P^{(n)}$, i.e., for any $\delta > 0, \epsilon > 0$, $\lim P^{(n)}(|\hat{\theta}| > \delta) < \epsilon$.

(a) Were $P^{(n)}$ and $\mathcal{F}^{(n)}$ fixed, the stochastic equicontinuity of $\sqrt{n}\left(P_n^{(n)} - P^{(n)}\right)$ would follow from standard empirical process theory. A small variation of the Equicontinuity Lemma, Theorem VII.15 of Pollard (1984), accommodates changing probability measures and function classes. The variation presented below ignores measurability qualifications that are not relevant for the present application.

Lemma 2. *Given function classes $\mathscr{F}^{(n)}$ and probability measures $\mathscr{P}^{(n)}, n \in \mathbb{N}$. Assume for any $\epsilon > 0, \eta > 0$, there is $\gamma > 0$ such that*

$$\limsup_n \mathscr{P}^{(n)} \left(J\left(\gamma, \mathscr{P}^{(n)}, \mathscr{F}^{(n)}\right) > \eta \right) < \epsilon.$$

Then there exists $\delta > 0$ such that

$$\limsup_n \mathscr{P}^{(n)} \left(\sup_{[\delta]^{(n)}} |\left(\mathscr{P}_n^{(n)} - \mathscr{P}^{(n)}\right)(f-g)| > \eta \right) < \epsilon,$$

where $[\delta]^{(n)} = \left\{ (f,g) \in \mathscr{F}^{(n)} : \mathscr{P}^{(n)}(f-g)^2 < \delta^2 \right\}$.

The proof follows by superficial changes to the proof of the form for fixed $\mathscr{F}^{(n)}$ and $\mathscr{P}^{(n)}$ cited above, and is omitted.

The stochastic equicontinuity 2a follows from the conclusion of the Lemma by setting $\mathscr{P}^{(n)} = P^{(n)}, \mathscr{F}^{(n)} = \mathcal{P}^{(n)}\mathcal{F}$. The assumptions of the Lemma hold by similar arguments as in 1. That is, the functions in $\mathcal{F}^{(n)}$ are monotonic in θ, so the graphs have discriminating polynomial $p(x) = x+1$ for all n, and then it follows from the Approximation Lemma, II.25 of Pollard (1984), that $J\left(\gamma, P^{(n)}, \mathcal{F}^{(n)}\right)$ is $O(\gamma)$ deterministically.

(b) For $\theta > 0$,

$$\begin{aligned} P^{(n)}(h_\theta - h_0)^2 &= 16P^{(n)} \left(P^{(n)} \left(0 < \frac{Z-Z'}{S-S'} < \theta \,\middle|\, z, s \right) \right)^2 \\ &\leq 16P^{(n)} \left(0 < \frac{Z-Z'}{S-S'} < \theta \right) \\ &= 8\mathbb{E} \left(2F_{Z-Z'}^{(n)}(\theta|S-S'|) - 1 \right). \end{aligned}$$

Since $f_{Z-Z'}^{(n)} \to f_{Z-Z'}^{(\infty)}$ uniformly, the same holds for the convergence of the CDFs $F_{Z-Z'}^{(n)} \to F_{Z-Z'}^{(\infty)}$. Therefore, for any $\epsilon > 0$ and all sufficiently large n, $F_{Z-Z'}^{(n)}(\theta|S-S'|) = F_{Z-Z'}^{(n)}(0) + \epsilon/16 = \frac{1}{2} + \epsilon/16$, and the last expression of the above display is $\leq \epsilon$.

(c) Asymptotic normality is established in 3.

3. As $\left(\mathbb{E}^{(n)}\hat{\tau}\right)(\theta) = 2\mathbb{P}^{(n)}\left(\frac{Z-Z'}{S-S'} < \theta\right) - 1 = \mathbb{E}\left(2F_{Z-Z'}^{(n)}(\theta|S-S'|) - 1\right)$, its derivative is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathbb{E}^{(n)}\hat{\tau}\right)(\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}\left(2F_{Z-Z'}^{(n)}(\theta|S-S'|) - 1\right) \\ &= \mathbb{E}\left(2|S-S'|f_{Z-Z'}^{(n)}(\theta|S-S'|)\right) \\ &= 2f_{Z-Z'}^{(\infty)}(0)\mathbb{E}|S-S'| + o(1). \end{aligned}$$

The derivative may be brought inside the expectation in the second equality since the derivative $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(2F_{Z-Z'}^{(n)}(\theta|S-S'|) - 1\right) = 2|S-S'|f_{Z-Z'}^{(n)}(\theta|S-S'|)$ is nonnegative. The interchange of limits in the last equality is justified by the assumed uniform convergence $f^{(n)}_{Z-Z'} \to f^{(\infty)}_{Z-Z'}$ and uniform continuity of the latter:

$$\begin{aligned} f^{(\infty)}_{Z-Z'}(x) - f^{(\infty)}_{Z-Z'}(y) &= \int \left(f^{(\infty)}_{Z}(x+\xi) - f^{(\infty)}_{Z}(y+\xi) \right) f^{(\infty)}_{Z}(\xi) d\xi \\ &\leq \left| \left(f^{(\infty)}_{Z}(x+\cdot) - f^{(\infty)}_{Z}(y+\cdot) \right) \right|_{L^2} \left| f^{(\infty)}_{Z} \right|_{L^2} \\ &\leq 2 \left| f^{(\infty)}_{Z} \right|^2_{L^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sqrt{n} \left(\left(\mathbb{E}^{(n)}\hat{\tau}\right)(\hat{\theta}) - \left(\mathbb{E}^{(n)}\hat{\tau}\right)\left(\frac{\mu_1}{\mu_2}\theta^{(n)}\right) \right) \\ &\quad = \sqrt{n}\left(\hat{\theta} - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\mathbb{E}^{(n)}\hat{\tau}\right)(\theta)\Big|_{\theta=\frac{\mu_1}{\mu_2}\theta^{(n)}} + \sqrt{n} \cdot o\left(\hat{\theta} - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) \\ &\quad = \sqrt{n}\left(\hat{\theta} - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) \cdot 2\mathbb{E}|S_1 - S_2| \mathbb{E}^{(\infty)} f^{(\infty)}_{Z}(Z) + \sqrt{n} \cdot o\left(\hat{\theta} - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) \\ &\quad = \sqrt{n}\left(\frac{1}{\mu_2 n}\sum_j Z_j S_j - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) \cdot 2\mathbb{E}|S_1 - S_2| \mathbb{E}^{(\infty)} f^{(\infty)}_{Z}(Z) + o_{\mathbb{P}^{(n)}}(1). \end{aligned}$$

The last line follows by rewriting $\sqrt{n}\left(\hat{\theta} - \frac{\mu_1}{\mu_2}\theta^{(n)}\right)$ as an IID sum and establishing asymptotic normality. By assumption [[ref consistency assumption]] $n^{-1/2}\sum_j Z_j S_j \left(\frac{n}{\sum_j S_j^2} - \frac{1}{\mu_2}\right) = n^{-1/2}\sum_j S_j S_j \cdot o_{\mathbb{P}}(1)$. The right-hand side tends to 0 along $\mathbb{P}^{(n)}$ as long as $n^{-1/2}\sum_j Z_j S_j = O_{\mathbb{P}^{(n)}}(1)$, which holds since $\mathbb{E}^{(n)}\left(n^{-1/2}\sum_j Z_j S_j\right)^2 = ES^2 E^{(n)} Z^2$ is uniformly bounded by [[ref assumption on means squares]]. Therefore $\sqrt{n}\left(\hat{\theta} - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) = \sqrt{n}\left(\frac{1}{\mu_2 n}\sum_j Z_j S_j - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) + o_{\mathbb{P}^{(n)}}(1)$, and the common distributional limit is given by a traingular array CLT,

$$\begin{aligned} \mathbb{E}^{(n)} \frac{1}{\mu_2 n}\sum_j Z_j S_j &= \theta^{(n)}\frac{\mu_1}{\mu_2} \\ \mathrm{Var}^{(n)} \frac{1}{\mu_2 n}\sum_j Z_j S_j &= \frac{\mathrm{Var}^{(n)} Z}{\mu_2} + \left(\theta^{(n)}\right)^2 \left(\frac{1}{\mu_2} - \left(\frac{\mu_1}{\mu_2}\right)^2\right) \\ \sqrt{n}\left(\frac{1}{\mu_2 n}\sum_j Z_j S_j - \frac{\mu_1}{\mu_2}\theta^{(n)}\right) &\overset{\mathbb{P}^{(n)}}{\rightsquigarrow} \mathcal{N}\left(0, \frac{\mathrm{Var}^{(\infty)} Z}{\mu_2}\right) \end{aligned}$$

□

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