MEANS AND STANDARD DEVIATIONS OF A TRUNCATED NORMAL BIVARIATE DISTRIBUTION

H. WEILER[1]

C.S.I.R.O., Division of Mathematical Statistics, Sheep Biology Laboratory, Prospect, N.S.W.

1. Introduction and Summary. Consider the standardized normal bivariate distribution of probability density

$$\varphi(x,y;\rho)=\lambda(2\pi)^{-1}\exp\{-\tfrac{1}{2}\lambda^2(x^2-2\rho xy+y^2)\},$$

where ρ is the coefficient of correlation and $\lambda=(1-\rho^2)^{-\frac{1}{2}}$. If all the values outside the region $D\{a\leqslant x<\infty,\ b\leqslant y<\infty\}$ are discarded, the remaining truncated population has in D the probability density $\varphi(x,y;\rho)/P$, where

$$P=P(a,b;\rho)=\iint\limits_{D}\varphi(x,y;\rho)dxdy \tag{1}$$

is the " volume " under the normal bivariate surface over the base D.

Birnbaum and Meyer (1953) have derived formulae for the moments

$$E(x^sy^t)=\iint\limits_{D}x^sy^t\varphi(x,y;\rho)dxdy \tag{2}$$

in the general case of a multivariate normal distribution of given correlation matrix.

In this paper, a simplified proof is given for the first and second moments of the two-dimensional case. Charts and a table representing the relationships a,b with $P,E(x),E(y)$ and with the standard deviations of the truncated population are given for various values of ρ.

2. The First and Second Moments. Putting

$$S(x)=\int_b^\infty e^{-\frac{1}{2}\lambda^2(y^2-2\rho xy)}dy, \tag{3}$$

the mean of x can be written in the form

$$E(x)=\frac{\lambda}{2\pi P}\int_a^\infty xS(x)e^{-\frac{1}{2}\lambda^2x^2}dx. \tag{4}$$

Integration by parts then gives

$$\lambda^2PE(x)=\lambda^2P\rho E(y)+\lambda(2\pi)^{-1}S(a)e^{-\frac{1}{2}\lambda^2a^2},$$

and a change of variables $t=\lambda(y-\rho a)$ in the integral $S(a)$ reduces this to

$$\lambda^2PE(x)=\rho\lambda^2PE(y)+Z(a)Q(\lambda[b-\rho a]), \tag{5}$$

[1] Received for publication December 11, 1958; revised September 26, 1959.

where

$$(6) \qquad Z(t)=(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}t^2},\ Q(t)=\int_t^\infty Z(x)dx.$$

An equation similar to (5) is obtained by interchanging x with y and a with b. The two equations can then be solved for $E(x)$ and $E(y)$, giving

$$(7) \qquad \begin{aligned} P(a,b;\rho)E(x)&=Z(a)Q(A)+\rho Z(b)Q(B)\\ P(a,b;\rho)E(y)&=Z(b)Q(B)+\rho Z(a)Q(A), \end{aligned}$$

where

$$(8) \qquad A=\lambda(b-\rho a),\ B=\lambda(a-\rho b).$$

Again, by a similar integration by parts, we obtain

$$(9) \quad \lambda^2PE(x^2)=\frac{\lambda^3}{2\pi}\int_a^\infty x^2S(x)e^{-\frac{1}{2}\lambda^2x^2}dx=\lambda^2P\rho E(xy)+aZ(a)Q(A)+P.$$

Putting

$$T(x)=\int_b^\infty ye^{-\frac{1}{2}\lambda^2(y^2-2\rho xy)}dy,$$

we have

$$E(xy)=\frac{\lambda}{2\pi P}\int_a^\infty xT(x)e^{-\frac{1}{2}\lambda^2x^2}dx,$$

and a further integration by parts gives

$$\lambda^2PE(xy)=\lambda^2P\rho E(y^2)+\lambda(2\pi)^{-1}T(a)e^{-\frac{1}{2}\lambda^2a^2}.$$

The change of variables $t=\lambda(y-\rho a)$ in the integral $T(a)$ reduces this to

$$(10) \qquad \lambda^2PE(xy)=\lambda^{-2}\varphi(a,b;\rho)+\rho aZ(a)Q(A)+\lambda^2P\rho E(y^2).$$

Interchanging x with y and a with b, we have

$$(11) \qquad \lambda^2PE(xy)=\lambda^{-2}\varphi(a,b;\rho)+\rho bZ(b)Q(B)+\lambda^2P\rho E(x^2),$$

and solving (9), (10), (11), we obtain

$$(12) \qquad \begin{aligned} PE(x^2)&=aZ(a)Q(A)+\rho^2bZ(b)Q(B)+\rho(1-\rho^2)\varphi(a,b;\rho)+P\\ PE(y^2)&=bZ(b)Q(B)+\rho^2aZ(a)Q(A)+\rho(1-\rho^2)\varphi(a,b;\rho)+P\\ PE(xy)&=\rho\{aZ(a)Q(A)+bZ(b)Q(B)+P\}+(1-\rho^2)\varphi(a,b;\rho). \end{aligned}$$

The values of $P(a,b;\rho)$ are tabulated in Pearson (1931) for positive values of a and b only, but the values for negative a and/or b can be deduced from the formulae

$$(13) \qquad \begin{aligned} P(a,-b;\rho)&=Q(a)-P(a,b;-\rho)\\ P(-a,b;\rho)&=Q(b)-P(a,b;-\rho)\\ P(-a,-b;\rho)&=1-Q(a)-Q(b)+P(a,b;\rho). \end{aligned}$$

These formulae, given in Pearson (1931), p. liii (iv), without proof, may be derived as follows:

By a change of variables $v=\lambda(y-\rho x)$ in $S(x)$, we obtain

$$(14) \qquad P(a,b;\rho)=\int_a^\infty Z(x)Q(\lambda[b-\rho x])dx.$$

A further substitution, $u=-x$, gives

$$\begin{aligned}P(a,b;\rho)&=\int_{-\infty}^{-a} Z(u)Q(\lambda[b+\rho u])du\\&=\int_{-\infty}^{-a} Z(u)\{1-Q(\lambda[-b-\rho u])\}du\\&=Q(a)-\int_{a}^{\infty} Z(x)Q(\lambda[-b+\rho x])dx\\&=Q(a)-P(a,-b;-\rho).\end{aligned}$$

Replacing ρ by $-\rho$ and interchanging a with b, we obtain the first two equations of (13); the third equation is obtained when b is replaced by $-b$.

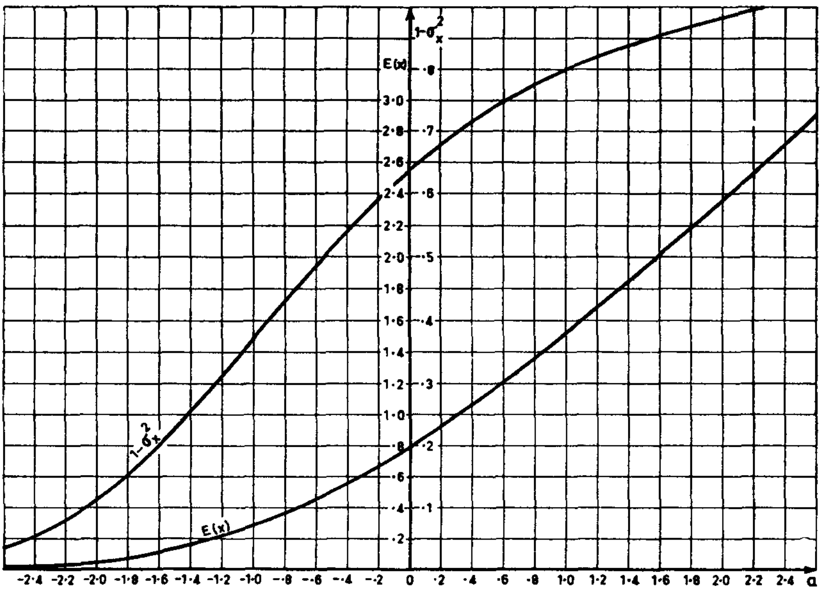

FIGURE 1

Mean and Variance of a Truncated Normal Variate

$E(x)$ and $1-\sigma_x^2$ are the deviations from the mean and variance of a standardized normal variate x when all values $x<a$ are discarded. When applied to a standardized normal bivariate (x,y) with correlation coefficient ρ, the corresponding values of $E(y)$ and $1-\sigma_y^2$ are obtained by multiplying $E(x)$ and $1-\sigma_x^2$ by ρ and ρ^2 respectively.

An interesting special case occurs when truncation is affected on x alone, which amounts to putting $b=-\infty$. When b tends to $-\infty$, $Q(B)$, $Z(b)$, and $bZ(b)$ tend to zero, while $Q(A)$ tends to unity. Hence, equation (14) reduces to

$$P(a,-\infty;\rho)=Q(a),$$

equation (7) gives (for $b=-\infty$)

$$E(x)=Z(a)/Q(a),\ E(y)=\rho E(x), \tag{16}$$

and equation (12) gives (for $b=-\infty$)

$$1-\sigma_x^2=E(x)\{E(x)-a\},\ 1-\sigma_y^2=\rho^2(1-\sigma_x^2). \tag{17}$$

In Fig. 1, $E(x)$ and $1-\sigma_x^2$ are represented as functions of the truncation point $x=a$. These functions are independent of the variate y and thus can be applied to any standardized normal variate. The corresponding values of $E(y)$ and $1-\sigma_y^2$ can be obtained by multiplication with ρ and ρ^2 respectively.

TABLE 1

Standard Deviations of σ of a Truncated Standardized Normal Bivariate (x,y) for $\rho=0$, $\pm 0\cdot 3$, $\pm 0\cdot 5$

b	a					ρ
	—2	—1	0	+1	+2	
—2	0·94	0·79	0·60	0·45	0·34	+0·5
—1	0·92	0·80	0·61	0·45	0·34	
0	0·90	0·81	0·63	0·46	0·34	
+1	0·90	0·84	0·69	0·50	0·36	
+2	0·88	0·86	0·76	0·57	0·41	
—2	0·94	0·79	0·60	0·45	0·34	+0·3
—1	0·94	0·80	0·61	0·45	0·34	
0	0·94	0·82	0·63	0·46	0·35	
+1	0·94	0·85	0·67	0·49	0·36	
+2	0·95	0·88	0·71	0·53	0·39	
—2	0·94	0·79	0·60	0·45	0·34	0
—1	0·94	0·79	0·60	0·45	0·34	
0	0·94	0·79	0·60	0·45	0·34	
+1	0·94	0·79	0·60	0·45	0·34	
+2	0·94	0·79	0·60	0·45	0·34	
—2	0·94	0·79	0·60	0·44	0·33	—0·3
—1	0·92	0·77	0·58	0·42	0·32	
0	0·89	0·73	0·54	0·40	0·30	
+1	0·86	0·68	0·50	0·37	0·28	
+2	0·82	0·63	0·46	0·34	0·27	
—2	0·93	0·78	0·58	0·43	0·32	—0·5
—1	0·89	0·73	0·54	0·39	0·29	
0	0·83	0·66	0·48	0·34	0·25	
+1	0·77	0·58	0·41	0·30	0·24	
+2	0·69	0·50	0·35	0·27	0·16	

For other values of a and b, the means $E(x)$ and $E(y)$ are given graphically in the charts Figs. 2–5, whose construction and use are explained in the next section. Values of the standard deviation

$$\sigma_x=\sqrt{E(x^2)-\{E(x)\}^2}$$

are given in Table 1; they are calculated by means of the formulae (7) and (12). This table can be used also to obtain σ_y by simply interchanging a and b.

The charts and the table in this paper refer to the coefficients of correlation $\rho=\pm 0\cdot 3$, $\pm 0\cdot 5$ only, but further charts are planned for publication in a later issue.

The table gives the standard deviation σ_x of x when x and y are truncated at a and b respectively. Intermediate values may be obtained with sufficient accuracy by linear interpolation.

3. Construction and Use of the Charts. With the help of the formulae (7) and (13) and Pearson's tables (1930, 1931), $P(a,b;\rho)$, $E(x)$ and $E(y)$ can be calculated for any fixed value of ρ and given values of a and b. Auxiliary curves are then obtained by plotting $E(x)$ and $E(y)$ against $P=P(a,b;\rho)$ for a number of fixed values of b; for every fixed value of b, one curve $E(x)=f_1(P)$ and one curve

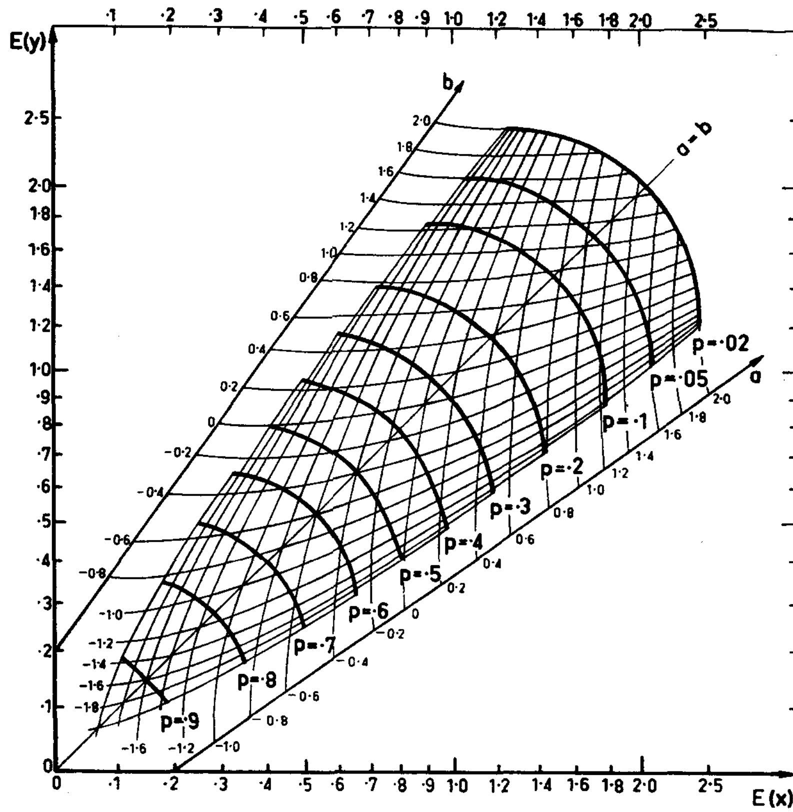

FIGURE 2

Means E(x),E(y) *and retained proportion* P *for a truncated standardized normal bivariate* (x,y) *when* $\rho=0\cdot5$

Each P-curve (heavily drawn) corresponds to a fixed proportion P of the population retained after truncation. To each point on such a curve corresponds a value $E(x)$, obtained by projection on the axis of $E(x)$, and a value of $E(y)$, obtained by projection on the axis of $E(y)$. The corresponding truncation points $x=a$, $y=b$ are given by the lightly drawn curves. Thus, for the truncation points $a=-0\cdot4$, $b=0\cdot7$, we find $P=0\cdot2$ and $E(x)=0\cdot9$, $E(y)=1\cdot3$, approximately.

$E(y)=f_2(P)$ is obtained. With these curves, the relationship between $E(x)$ and $E(y)$ is established for any fixed P. Every such relationship leads to a curve $P=$const., as shown in Figs. 2–5. Again, on each curve $P=$const. is a point for which $b=b_0$ (say). Joining all these points, we obtain one of the curves $b=$const. The curves $a=$const. are the symmetric images of the curves $b=$const. with respect to the line $E(x)=E(y)$.

The charts can be used in various ways:

(1) For a given truncation at $x=a$, $y=b$, the volume $P(a,b;\rho)$ can be read off (e.g., for $\rho=0\cdot5$, $a=-0\cdot4$, $b=+0\cdot2$, we find $P=0\cdot35$ from Fig. 2). This is useful because the charts can replace Pearson's tables for $P(a,b;\rho)$ when great accuracy is not required, and further, because $P(a,b;\rho)$ can also be obtained for negative values of a and/or b without recourse to the formulae (13).

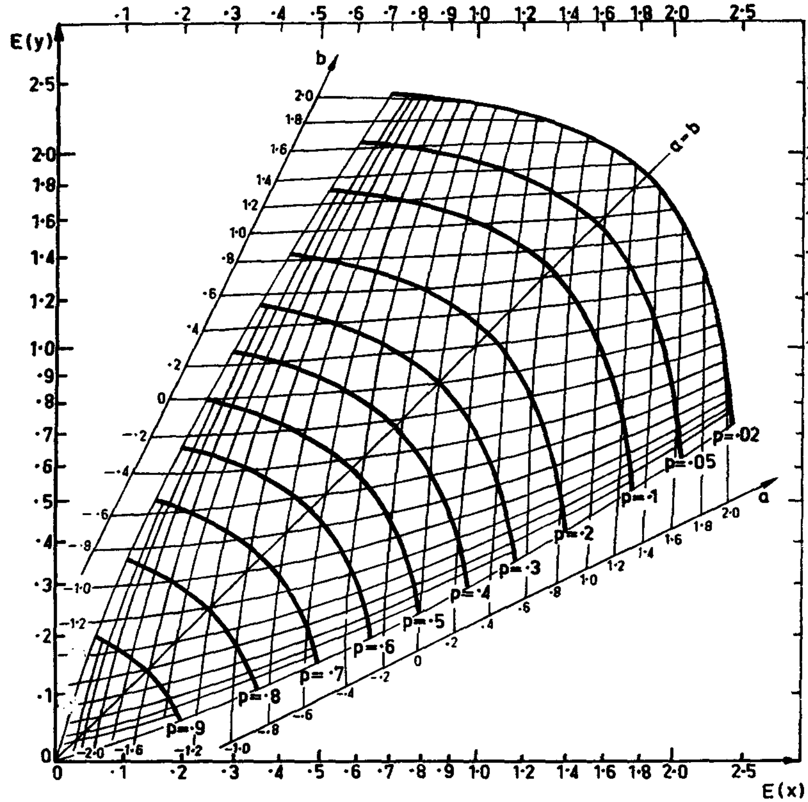

FIGURE 3

Means E(x),E(y) *and retained proportion* P *for a truncated standardized normal bivariate* (x,y) *when* $\rho=0\cdot3$

(The use of this chart is similar to that of Fig. 2.)

(2) For a given truncation at $x=a$, $y=b$, the means $E(x)$, $E(y)$ of the truncated population can be read off without recourse to the formulae (7) and (13) (e.g., for $\rho=-0\cdot5$, $a=-0\cdot55$, $b=-2\cdot2$, Fig. 5 gives $E(x)=0\cdot47$ and $E(y)=-0\cdot20$).

(3) For given means $E(x)$, $E(y)$, the required truncation $x=a$, $y=b$ and the proportion P of the population retained can be read off (e.g., for $\rho=-0\cdot3$, $E(x)=1\cdot1$, $E(y)=0\cdot20$, Fig. 4 gives $a=0\cdot50$, $b=-0\cdot75$).

(4) For a given proportion P retained from the original population, the most suitable combination $E(x)$, $E(y)$ can be selected by moving

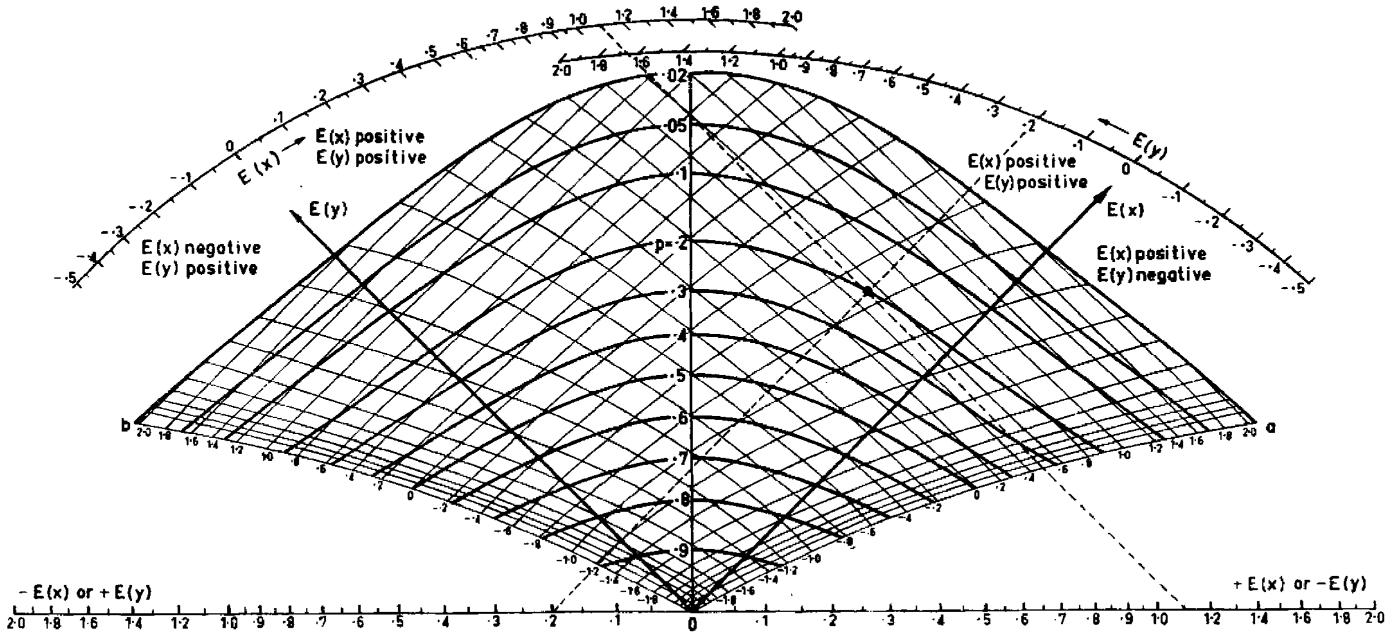

FIGURE 4

Means E(x),E(y) *and retained proportion* P *for a truncated standardized normal bivariate* (x,y) *when* $\rho=-0\cdot3$

This chart is similar to the two previous ones, except that for convenience the axes have been rotated. To obtain the values of $E(x)$ and $E(y)$ corresponding to any point, a ruler should be held parallel to the axes of $E(x)$ and $E(y)$. This is facilitated by joining corresponding points on the straight scale at the base and the circular scale at the top of the chart. For example, the point marked on the curve $P=0\cdot2$ has the coordinates $E(x)=1\cdot1$, $E(y)=0\cdot2$, and the corresponding truncation points are approximately $a=0\cdot50$ and $b=-0\cdot75$. Note also that $E(x)$ and $E(y)$ are either both positive or one is positive and the other negative, according to the sector in which the point is situated.

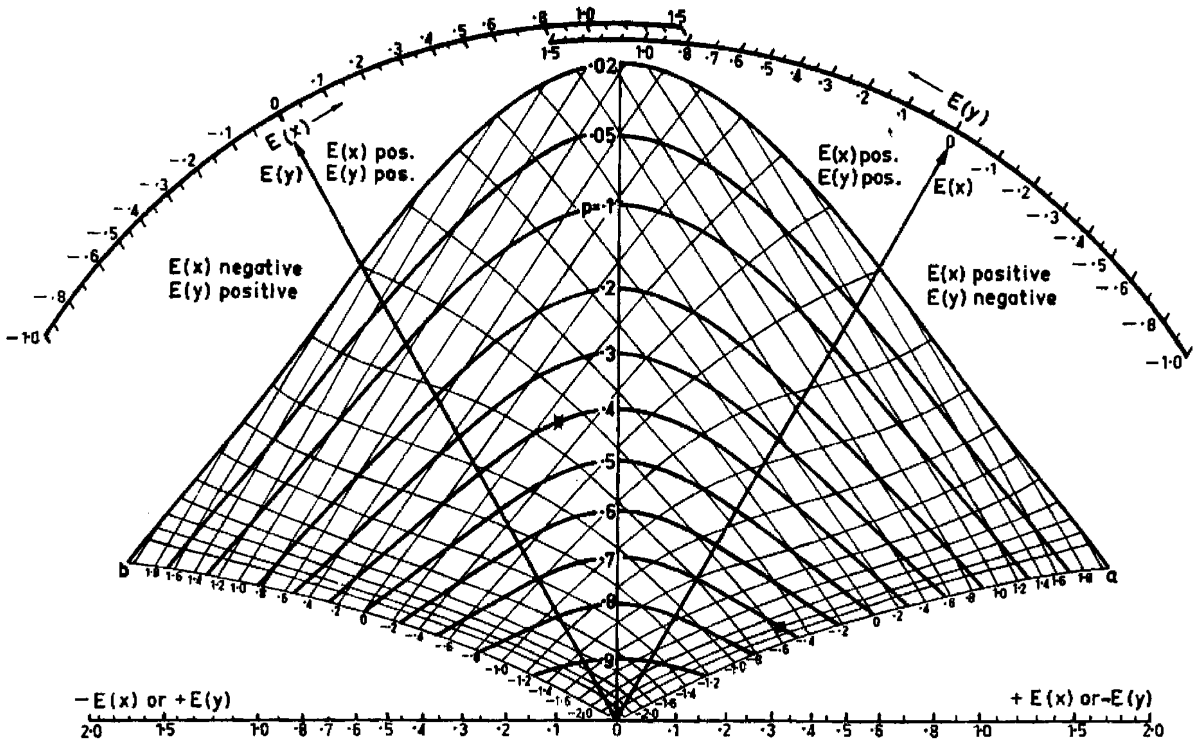

Figure 5

Means E(x),E(y) *and retained proportion* P *for a truncated standardized normal bivariate* (x,y) *when* $\rho=-0\cdot5$

This chart is similar to the previous one, except that the axes of $E(x)$ and $E(y)$ are at an angle of 60° instead of 90°. Its use is similar to that of Fig. 4. For example, the points marked on the curves $P=0\cdot4$ and $P=0\cdot7$ have the co-ordinates $E(x)=0\cdot2$, $E(y)=0\cdot5$ and $E(x)=0\cdot47$, $E(y)=-0\cdot20$, respectively; the corresponding truncation points are $a=-0\cdot7$, $b=-0\cdot3$ and $a=-0\cdot55$, $b=-2\cdot2$.

along the curve P=const. and reading off the corresponding values for $E(x)$, $E(y)$. Once a suitable combination has been chosen, the appropriate truncation points $x=a$, $y=b$ can be read off.

Birnbaum and Meyer (1953) give an application of the above theory, where x and y are two abilities to be tested in a personnel selection. It is assumed that the bivariate distribution (x,y) of the applicants is normal with known coefficient of correlation ρ. The scores x and y are given in standardized units. The charts, Figs. 2–5, provide an easy method of determining the " cutting scores " $x=a$, $y=b$ to obtain preassigned means $E(x)$, $E(y)$ and, moreover, give the proportion P of successful applicants. Their standard deviations can be obtained from Table 1.

Another application has been treated by Young and Weiler (in the press). It deals with the selection of animals or plants for breeding or production purposes. The selection is based on two correlated traits, while the proportion P of animals to be retained is preassigned. The most suitable combination of means $E(x)$, $E(y)$ is then determined as described above, and the culling levels $x=a$, $y=b$ are found.

4. Acknowledgments. The problem treated originated in an investigation of the effects of culling with respect to two characters in breeding experiments, which was suggested to the author by S. S. Y. Young, Division of Animal Health and Production, C.S.I.R.O. Figures 2–5 are reproduced from a paper by Young and Weiler (*loc. cit.*) entitled " Selection for two correlated traits by independent culling levels ", with the kind permission of the Editor of the *Journal of Genetics*.

Mrs. Jean Williams, Division of Mathematical Statistics, C.S.I.R.O., did the extensive computational work required for the charts, and Miss Patricia Doran, National Standards Laboratory, C.S.I.R.O., drew the charts.

References

Birnbaum, Z. W., and Meyer, P. L. (1953). " On the effect of truncation in some or all co-ordinates of a multinormal population." *Jour. Indian Soc. Agric. Statistics*, Vol. 5, pp. 17–28.

Pearson, K. (1930). *Tables for Statisticians and Biometricians*, Part I. Cambridge.

Pearson, K. (1931). *Tables for Statisticians and Biometricians*, Part II. Cambridge.

Young, S. S. Y., and Weiler, H. (Press). " Selection for two correlated traits by independent culling levels." *Journal of Genetics*.