Web-based Supplementary Materials for "Quantifying Publication Bias in Meta-Analysis" by

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Web Appendix: Proofs

Let $X_1, \ldots, X_n$ be iid random variables and denote the kth central moment $\beta_k = \mathrm{E}(X_1 - \beta)^k$, where $\beta = \mathrm{E}(X_1)$. Also, denote the sample kth central moment $m_k = n^{-1}\sum_{i=1}^n (X_i - \bar{X})^k$, where $\bar{X} = n^{-1}\sum_{i=1}^n X_i$. We have the following lemma regarding the asymptotic distribution of the sample kth central moment.

Lemma 1. *If $X_1, \ldots, X_n$ are iid with mean β and $\beta_{2k} < \infty$ for $k \geq 1$, then*

$$m_k - \beta_k = \frac{1}{n}\sum_{i=1}^n \left[(X_i - \beta)^k - \beta_k - k\beta_{k-1}(X_i - \beta)\right] + o_p(n^{-1/2}).$$

as $n \to \infty$.

Proof of Lemma 1. See page 72 in Serfling (1980). □

Now, let us back to the notation in the main text. Specifically, let $x_i = (s_i^2 + \tau^2)^{-1/2}$ and $z_i = y_i(s_i^2 + \tau^2)^{-1/2}$. The regression test is $z_i = \alpha + \mu x_i + \epsilon_i$, where ϵ_i's are iid following a distribution with mean zero; $\hat{\alpha}$ and $\hat{\mu}$ are the least squares estimates of α and μ respectively, and the residuals $\hat{\epsilon}_i = y_i - \hat{\mu}x_i - \hat{\alpha}$. Also, $\beta_k = \mathrm{E}(\epsilon_1 - \beta)^k$ is the kth central moment of ϵ_i's, where $\beta = \mathrm{E}(\epsilon_1) = 0$, and $m_k = n^{-1}\sum_{i=1}^n(\epsilon_i - \bar{\epsilon})^k$. The true skewness of ϵ_i's is $\gamma = \beta_3/\beta_2^{3/2}$. Let $\hat{m}_k = n^{-1}\sum_{i=1}^n(\hat{\epsilon}_i - \bar{\hat{\epsilon}})^k$ be the sample kth central moment by plugging in the residuals $\hat{\epsilon}_i$, where $\bar{\hat{\epsilon}} = n^{-1}\sum_{i=1}^n \hat{\epsilon}_i = 0$. The sample skewness of $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)^T$ is $\mathrm{Skew}(\boldsymbol{\epsilon}) = m_3/s^3$, where $s = \sqrt{nm_2/(n-1)}$, and $T_S = \mathrm{Skew}(\hat{\boldsymbol{\epsilon}})$ is obtained by plugging $\hat{\boldsymbol{\epsilon}} = (\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n)^T$ in $\mathrm{Skew}(\boldsymbol{\epsilon})$.

Proof of Proposition 1. First, we show that $\sqrt{n}(\mathrm{Skew}(\boldsymbol{\epsilon}) - \gamma) \xrightarrow{D} N(0, v)$ as $n \to \infty$, where

$$v = 9 + \frac{35}{4}\beta_2^{-3}\beta_3^2 - 6\beta_2^{-2}\beta_4 + \beta_2^{-3}\beta_6 + \frac{9}{4}\beta_2^{-5}\beta_3^2\beta_4 - 3\beta_2^{-4}\beta_3\beta_5.$$

Because $\mathrm{Skew}(\boldsymbol{\epsilon}) = [(n-1)/n]^{3/2}m_3/m_2^{3/2}$, $\mathrm{Skew}(\boldsymbol{\epsilon})$ have the same asymptotic distribution as $m_3/m_2^{3/2}$. By Lemma 1, we have

$$\begin{bmatrix} m_2 \\ m_3 \end{bmatrix} - \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} = \frac{1}{n}\sum_{i=1}^n \begin{bmatrix} \epsilon_i^2 - \beta_2 \\ \epsilon_i^3 - \beta_3 - 3\beta_2\epsilon_i \end{bmatrix} + o_p(n^{-1/2}).$$

Therefore,

$$\sqrt{n}\left(\begin{bmatrix} m_2 \\ m_3 \end{bmatrix} - \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix}\right) \xrightarrow{D} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \beta_4-\beta_2^2 & \beta_5-4\beta_2\beta_3 \\ \beta_5-4\beta_2\beta_3 & \beta_6-\beta_3^2-6\beta_2\beta_4+9\beta_2^3 \end{bmatrix}\right).$$

Denote the asymptotic covariance matrix above as $\mathbf{\Sigma}$. Let $g(r,s)=s/r^{3/2}$, then $g'(r,s)=\left(-\frac{3}{2}sr^{-5/2}, r^{-3/2}\right)^T$. By the delta method,

$$\sqrt{n}(g(m_2,m_3)-g(\beta_2,\beta_3)) \xrightarrow{D} N(0,[g'(\beta_2,\beta_3)]^T\mathbf{\Sigma}[g'(\beta_2,\beta_3)]);$$

that is,

$$\sqrt{n}(\text{Skew}(\boldsymbol{\epsilon})-\gamma) \xrightarrow{D} N(0,v).$$

Second, we show that $\sqrt{n}(T_S-\text{Skew}(\boldsymbol{\epsilon})) \xrightarrow{D} 0$ as $n\to\infty$. We write $\text{Skew}(\boldsymbol{\epsilon})=[(n-1)/n]^{3/2}f(\boldsymbol{\delta})$, where $f(\boldsymbol{\delta})=m_3/m_2^{3/2}$ is a continuous and differentiable function of $\boldsymbol{\delta}=(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5)^T=(\bar{\epsilon}^2,\bar{\epsilon}^3,\bar{\epsilon}\cdot\overline{\epsilon^2},\overline{\epsilon^2},\overline{\epsilon^3})^T$; here, $\overline{\epsilon^k}=n^{-1}\sum_{i=1}^n \epsilon_i^k$. Specifically, $f(\boldsymbol{\delta})=(\delta_5-3\delta_3+2\delta_2)(\delta_4-\delta_1)^{-3/2}$; it is free of n. Also, $T_S=\text{Skew}(\hat{\boldsymbol{\epsilon}})=[(n-1)/n]^{3/2}f(\hat{\boldsymbol{\delta}})$, where $\hat{\boldsymbol{\delta}}=\left(\left(\bar{\hat{\epsilon}}\right)^2,\left(\bar{\hat{\epsilon}}\right)^3,\bar{\hat{\epsilon}}\cdot\overline{\hat{\epsilon}^2},\overline{\hat{\epsilon}^2},\overline{\hat{\epsilon}^3}\right)^T$, and $\overline{\hat{\epsilon}^k}=n^{-1}\sum_{i=1}^n\hat{\epsilon}_i^k$. Because the average of the residuals is $\bar{\hat{\epsilon}}=0$, we have $\hat{\boldsymbol{\delta}}=\left(0,0,0,\overline{\hat{\epsilon}^2},\overline{\hat{\epsilon}^3}\right)^T$. By multivariate Taylor expansion,

$$f(\hat{\boldsymbol{\delta}})=f(\boldsymbol{\delta})+[\boldsymbol{h}(\boldsymbol{\delta})]^T(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta})+O_p(\|\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}\|^2),$$

where $\boldsymbol{h}(\boldsymbol{\delta})=\bigtriangledown f(\boldsymbol{\delta})$ is the gradient of $f(\boldsymbol{\delta})$ and $\|\cdot\|$ is the Euclidean norm. Specifically,

$$\boldsymbol{h}(\boldsymbol{\delta})=\begin{bmatrix} h_1(\boldsymbol{\delta}) \\ h_2(\boldsymbol{\delta}) \\ h_3(\boldsymbol{\delta}) \\ h_4(\boldsymbol{\delta}) \\ h_5(\boldsymbol{\delta}) \end{bmatrix}=\begin{bmatrix} \frac{3}{2}(\delta_5-3\delta_3+2\delta_2)(\delta_4-\delta_1)^{-5/2} \\ 2(\delta_4-\delta_1)^{3/2} \\ -3(\delta_4-\delta_1)^{3/2} \\ -\frac{3}{2}(\delta_5-3\delta_3+2\delta_2)(\delta_4-\delta_1)^{-5/2} \\ (\delta_4-\delta_1)^{3/2} \end{bmatrix}.$$

Since $\delta_1,\delta_2,\delta_3 \xrightarrow{P} 0$, $\delta_4 \xrightarrow{P} \beta_2>0$, and $\delta_5 \xrightarrow{P} \beta_3$, we have $h_j(\boldsymbol{\delta})=O_p(1)$ for $j=1,\dots,5$. Now, we focus on

$$\hat{\boldsymbol{\delta}}-\boldsymbol{\delta}=\left(-\bar{\epsilon}^2,-\bar{\epsilon}^3,-\bar{\epsilon}\cdot\overline{\epsilon^2},\overline{\hat{\epsilon}^2}-\overline{\epsilon^2},\overline{\hat{\epsilon}^3}-\overline{\epsilon^3}\right)^T.$$

Due to $\bar{\epsilon}=O_p(n^{-1/2})$, we have $\hat{\delta}_1-\delta_1=-\bar{\epsilon}^2=O_p(n^{-1})$, $\hat{\delta}_2-\delta_2=-\bar{\epsilon}^3=O_p(n^{-3/2})$, and $\hat{\delta}_3-\delta_3=-\bar{\epsilon}\cdot\overline{\epsilon^2}=O_p(n^{-1/2})$. Note that

$$\hat{\epsilon}_i=(\alpha-\hat{\alpha})+(\mu-\hat{\mu})x_i+\epsilon_i,$$

and $\hat{\alpha}-\alpha=O_p(n^{-1/2})$, $\hat{\mu}-\mu=O_p(n^{-1/2})$. Also, by the assumption that x_i's have finite third moment and the weak law of large numbers, $\frac{1}{n}\sum_{i=1}^n x_i^k=O_p(1)$ for $k=1,2,3$. Consequently, we have

$$\begin{aligned}
\hat{\delta}_4-\delta_4 &= \overline{\hat{\epsilon}^2}-\overline{\epsilon^2} \\
&= \frac{1}{n}\sum_{i=1}^n[(\alpha-\hat{\alpha})+(\mu-\hat{\mu})x_i+\epsilon_i]^2-\frac{1}{n}\sum_{i=1}^n\epsilon_i^2 \\
&= (\alpha-\hat{\alpha})^2+(\mu-\hat{\mu})^2\frac{\sum_{i=1}^n x_i^2}{n}+2(\alpha-\hat{\alpha})(\mu-\hat{\mu})\frac{\sum_{i=1}^n x_i}{n} \\
&\quad +2(\alpha-\hat{\alpha})\bar{\epsilon}+2(\mu-\hat{\mu})\frac{\sum_{i=1}^n x_i\epsilon_i}{n} \\
&= O_p(n^{-1}),
\end{aligned}$$

and

$$
\begin{aligned}
\hat{\delta}_5 - \delta_5 &= \overline{\hat{\epsilon}^3} - \overline{\epsilon^3} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[(\alpha-\hat{\alpha}) + (\mu-\hat{\mu})x_i + \epsilon_i\right]^3 - \frac{1}{n}\sum_{i=1}^{n}\epsilon_i^3 \\
&= (\alpha-\hat{\alpha})^3 + (\mu-\hat{\mu})^3\frac{\sum_{i=1}^n x_i^3}{n} + 3(\alpha-\hat{\alpha})^2(\mu-\hat{\mu})\frac{\sum_{i=1}^n x_i}{n} \\
&\quad + 3(\alpha-\hat{\alpha})(\mu-\hat{\mu})^2\frac{\sum_{i=1}^n x_i^2}{n} + 3(\alpha-\hat{\alpha})^2\bar{\epsilon} + 3(\mu-\hat{\mu})^2\frac{\sum_{i=1}^n x_i^2\epsilon_i}{n} \\
&\quad + 6(\alpha-\hat{\alpha})(\mu-\hat{\mu})\frac{\sum_{i=1}^n x_i\epsilon_i}{n} + 3(\alpha-\hat{\alpha})\overline{\epsilon^2} + 3(\mu-\hat{\mu})\frac{\sum_{i=1}^n x_i\epsilon_i^2}{n} \\
&= 3(\alpha-\hat{\alpha})\overline{\epsilon^2} + 3(\mu-\hat{\mu})\frac{\sum_{i=1}^n x_i\epsilon_i^2}{n} + O_p(n^{-1}) \\
&= O_p(n^{-1/2}).
\end{aligned}
$$

Therefore, $O_p(\|\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\|^2) = O_p(n^{-1})$, implying

$$
\begin{aligned}
f(\hat{\boldsymbol{\delta}}) - f(\boldsymbol{\delta}) &= [\boldsymbol{h}(\boldsymbol{\delta})]^T(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}) + O_p(n^{-1}) \\
&= \sum_{j=1}^{5} h_j(\boldsymbol{\delta})(\hat{\delta}_j - \delta_j) + O_p(n^{-1}) \\
&= h_3(\boldsymbol{\delta})(\hat{\delta}_3 - \delta_3) + h_5(\boldsymbol{\delta})(\hat{\delta}_5 - \delta_5) + O_p(n^{-1}) \\
&= 3(\delta_4-\delta_1)^{3/2}\cdot\bar{\epsilon}\cdot\overline{\epsilon^2} + (\delta_4-\delta_1)^{3/2}\cdot\left[3(\alpha-\hat{\alpha})\overline{\epsilon^2} + 3(\mu-\hat{\mu})\frac{\sum_{i=1}^n x_i\epsilon_i^2}{n}\right] + O_p(n^{-1}) \\
&= 3(\delta_4-\delta_1)^{3/2}\left\{\left[(\alpha-\hat{\alpha}) + \bar{\epsilon}\right]\overline{\epsilon^2} + (\mu-\hat{\mu})\frac{\sum_{i=1}^n x_i\epsilon_i^2}{n}\right\} + O_p(n^{-1})
\end{aligned}
$$

Note that $\sum_{i=1}^n \hat{\epsilon}_i = 0$, so $(\alpha-\hat{\alpha}) + \bar{\epsilon} = (\hat{\mu}-\mu)\frac{\sum_{i=1}^n x_i}{n}$. Consequently,

$$
\begin{aligned}
f(\hat{\boldsymbol{\delta}}) - f(\boldsymbol{\delta}) &= 3(\delta_4-\delta_1)^{3/2}\left\{(\hat{\mu}-\mu)\frac{\sum_{i=1}^n x_i}{n}\overline{\epsilon^2} - (\hat{\mu}-\mu)\frac{\sum_{i=1}^n x_i\epsilon_i^2}{n}\right\} + O_p(n^{-1}) \\
&= 3(\delta_4-\delta_1)^{3/2}(\hat{\mu}-\mu)\left\{\frac{\sum_{i=1}^n x_i}{n}\overline{\epsilon^2} - \frac{\sum_{i=1}^n x_i\epsilon_i^2}{n}\right\} + O_p(n^{-1}) \\
&= O_p(n^{-1/2})\left\{[\mathrm{E}(x_1) + O_p(n^{-1/2})][\beta_2 + O_p(n^{-1/2})] - [\mathrm{E}(x_1\epsilon_1^2) + O_p(n^{-1/2})]\right\} + O_p(n^{-1}) \\
&= O_p(n^{-1/2})\left\{[\mathrm{E}(x_1)\beta_2 + O_p(n^{-1/2})] - [\mathrm{E}(x_1)\beta_2 + O_p(n^{-1/2})]\right\} + O_p(n^{-1}) \\
&= O_p(n^{-1}).
\end{aligned}
$$

This leads to $\sqrt{n}(f(\hat{\boldsymbol{\delta}}) - f(\boldsymbol{\delta})) \xrightarrow{D} 0$; hence, $\sqrt{n}(T_S - \mathrm{Skew}(\boldsymbol{\epsilon})) \xrightarrow{D} 0$, and $\sqrt{n}(T_S - \gamma) \xrightarrow{D} N(0, v)$.

Finally, we show that $\hat{v} \xrightarrow{P} v$. By continuous mapping theorem, it is sufficient to show that $\hat{m}_k \xrightarrow{P} \beta_k$ for $k = 2, \ldots, 6$. Recall that $\beta_k = \mathrm{E}(\epsilon_1^k)$ and $\hat{m}_k = n^{-1}\sum_{i=1}^n \hat{\epsilon}_i^k$. Since $\hat{\epsilon}_i = (\alpha-\hat{\alpha}) + (\mu-\hat{\mu})x_i + \epsilon_i = \epsilon_i + O_p(n^{-1/2})$, we have $\hat{m}_k = n^{-1}\sum_{i=1}^n(\epsilon_i + O_p(n^{-1/2}))^k = n^{-1}\sum_{i=1}^n \epsilon_i^k + o_p(1) = \beta_k + o_p(1)$; that is, $\hat{m}_k \xrightarrow{P} \beta_k$. By Slutsky's theorem, $\sqrt{n}(T_S - \gamma)/\sqrt{\hat{v}} \xrightarrow{D} N(0, 1)$; this completes the proof. $\square$

Proof of Corollary 1. Under H_0'', we have $\epsilon_i \sim N(0, \sigma^2)$, so $\beta_{2k} = (2k-1)!!\sigma^{2k}$ and $\beta_{2k-1} = 0$ for $k \geq 1$. Here, $c!! = c \cdot (c-2) \cdot (c-4) \cdots$ is the double factorial. Specifically, $\beta_2 = \sigma^2$, $\beta_4 = 3\sigma^4$, and $\beta_6 = 15\sigma^6$. In the proof of Proposition 1, we showed that $\sqrt{n}(T_S - \gamma) \xrightarrow{D} N(0, v)$. Under H_0'', v is simplified as $v = 9 - 6(\sigma^2)^{-2} \cdot 3\sigma^4 + (\sigma^2)^{-3} \cdot 15\sigma^6 = 6$. This completes the proof. □

References

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics.* John Wiley & Sons, New York, NY.