LINEAR COMBINATIONS OF UNIFORM VARIATES, THE VOLUME OF A SIMPLEX, EULERIAN NUMBERS, AND e.

(I)

Let S_n denote the simplex given by

$$S_n = \left\{ (x_1, x_2, \ldots, x_n) \,\middle|\, \sum_{j=1}^{n} a_j x_j = k, \text{ where } x_j \geq 0,\ a_j > 0, \text{ and } k \geq 0 \right\}$$

and let $V(S_n)$ denote the n-dimensional volume of S_n. Show that

$$V(S_n) = \frac{k^n}{a_1 \cdot a_2 \cdots a_n \cdot n!}.$$

(II)

Suppose $U_1, U_2, \ldots, U_n$ are independent random variables and that each is distributed uniformly on the interval $(0, \delta)$. Show for fixed positive $a_1, \ldots, a_n$ and $k \geq 0$,

$$P(a_1U_1 + a_2U_2 + \ldots + a_nU_n \leq k) =$$

$$\frac{1}{\delta^n \cdot (a_1 \cdot a_2 \cdots a_n) \cdot n!} \times$$

$$\Bigg(k^n - \sum_{r=1}^{n} \sum_{\mathbb{C}_r} (-1)^{r-1} \Big(k - \delta(a_{j_1} + a_{j_2} + \ldots + a_{j_r}) \Big)^n$$

$$\mathbb{I}_{(0,k)} \Big(\delta(a_{j_1} + a_{j_2} + \ldots + a_{j_r}) \Big) \Bigg).$$

The inner sum is over all $(j_1, \ldots, j_r) \in \mathbb{C}_r$, where $\mathbb{C}_r$ is defined as the set of all samples of size r drawn without replacement from $\{1, 2, \ldots, n\}$, when the order of sampling is considered unimportant. We note that when $a_1 = \ldots = a_n = 1$, the general result simplifies to

$$P(U_1 + U_2 + \ldots + U_n \leq k) = \frac{1}{\delta^n \cdot n!} \sum_{r=0}^{\lfloor k/\delta \rfloor} (-1)^r \binom{n}{r} (k - r\delta)^n.$$

(III)

Suppose $V_1, V_2, \ldots, V_n$ are independent random variables and that V_j is distributed uniformly on the interval $(0, \delta_j)$. Show

$$P(V_1 + V_2 + \ldots + V_n \leq k) =$$

$$\frac{1}{(\delta_1 \cdot \delta_2 \cdots \delta_n) \cdot n!} \times$$

$$\Bigg(k^n - \sum_{r=1}^{n} \sum_{\mathbb{C}_r} (-1)^{r-1} \Big(k - (\delta_{j_1} + \delta_{j_2} + \ldots + \delta_{j_r}) \Big)^n$$

$$\times\ \mathbb{I}_{(0,k)}\Big(\ \delta_{j_1} + \ \delta_{j_2}\ + \ \ldots\ + \delta_{j_r}\ \Big)\Bigg).$$

(IV)

Suppose U_1, $U_2, \ldots, U_n$ are independent random variables and that each is distributed uniformly on the interval $(0,1)$. For integer $k \in \{1, \ldots, n\}$ show that

$$P(\ k-1\ \leq\ U_1 + \ldots + U_n\ \leq\ k\ )\ =\ \tfrac{1}{n!}\ A(n,k)$$

where $A(n,k)\ =\ \sum_{j=0}^{k}(-1)^j \binom{n+1}{j}(k-j)^n$ are the Eulerian Numbers defined in Problem 7.???.

(V)

In Problem 7.??? we defined the Eulerian number $A(n,k)$ to be the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k rises. In (IV) we noticed that Eulerian numbers pop up in the probability that the sum of uniform variates is between the hyperplanes $U_1 + \ldots + U_n = k-1$ and $U_1 + \ldots + U_n = k$ but the proof does not clue us in to why rises and sums of uniform variates are related. In this part of the problem we will establish a one - to - one relationship between rises and sums of discrete uniform variates.

Let $\mathbb{C}_m(k,n)$ be the set of all $(X_1, \ldots, X_n)$ such that

i) $X_i \in \{1, \ldots, m-1\}$

ii) $X_i \neq X_j$ for any $1 \leq i < j \leq n$

iii) $(0, X_1, \ldots, X_n)$ has exactly k rises.

Let $\mathbb{D}_m(k,n)$ be the set of all $(Y_1, \ldots, Y_n)$ such that

i) $Y_i \in \left\{\frac{1}{m}, \ldots, \frac{m-1}{m}\right\}$

ii) $k-1 < Y_1 + \ldots + Y_n < k$

iii) $Y_i + \ldots + Y_j$ is not an integer for any $1 \leq i < j \leq n$.

Notice that the Y_i are not necessarily distinct. Establish a one - to - one relationship between $\mathbb{C}_m(k,n)$ and $\mathbb{D}_m(k,n)$.

Now suppose that $W_1, \ldots, W_n$ are independent discrete uniform random variables on $\{1, \ldots, m-1\}$. Then it follows that

$$P\Big(\ (W_1, \ldots, W_n) \in \mathbb{C}_m(k,n)\Big)\ =\ P\Bigg(\ \Big(\tfrac{W_1}{m}, \ldots, \tfrac{W_n}{m}\Big) \in \mathbb{D}_m(k,n)\ \Bigg).$$

However,

$$\lim_{m \to \infty} P\Big(\ (W_1, \ldots, W_n) \in \mathbb{C}_m(k,n)\Big)$$

$$=\ \lim_{m \to \infty}\left(\ \frac{\binom{m-1}{n} A(n,k)}{(m-1)^n}\ \right)$$

$$= \lim_{m \to \infty} \left(\frac{A(n,k)}{n!} \prod_{i=1}^{n} \left(\frac{m-i}{m-1} \right) \right) = \frac{A(n,k)}{n!}.$$

Also it is not hard to see that

$$\lim_{m \to \infty} P\left(\left(\frac{W_1}{m}, \ldots, \frac{W_n}{m} \right) \in \mathbb{D}_m(k,n) \right) = P\Big(k-1 < U_1 + \ldots + U_n < k \Big)$$

where $U_1, \ldots, U_n$ are independent uniform random variables on $(0,1)$. This confirms (IV).

(VI)

Suppose $U_1, U_2, \ldots$ are independent random variables and that each is distributed uniformly on the interval $(0,1)$ and suppose (the random variable) K_w is defined by the inequalities

$$U_1 + U_2 + \ldots + U_{K_w - 1} \le w \quad \text{and} \quad U_1 + U_2 + \ldots + U_{K_w} > w.$$

Show that

$$E(K_w) = \sum_{r=0}^{\lfloor w \rfloor} (-1)^r \frac{1}{r!} (w-r)^r e^{(w-r)}$$

and note that $E(K_1) = \mathbf{e}$.

The formula for $E(K_w)$ was derived by K. G. Russell, “On the Number of Uniform Random Variables Which Must be Added to Exceed a Given Level", *Journal of Applied Probability*, **20**, 172 - 177, 1983. However, the derivation given here is a considerable simplification.

(VII)

Show that

$$E(K_w) = 2w + \tfrac{2}{3} + o(1).$$

The approximation $E(K_w) \simeq 2w + \frac{2}{3}$ is actually quite good even for small w. In fact a numerical exercise shows that

$$\left| E(K_w) - (2w + \tfrac{2}{3}) \right| < 0.00011 \quad \text{for } w \ge 3.$$

This approximation is important because the values of w where the exact value of $E(K_w)$ can be calculated with any precision is limited.

Proof of (I)

$$V_n = \int_0^{c_1}\int_0^{c_2}\cdots\int_0^{c_n} dX_n \cdots dX_2\, dX_1$$

where $c_1 = \frac{k}{a_1}$, $c_2 = \frac{k - a_1X_1}{a_2}$, ... , $c_n = \frac{k - a_1X_1 - a_2X_2 - \ldots - a_{n-1}X_{n-1}}{a_n}$.

Now let $Y_j = a_j X_j$ for $j = 1, 2, \ldots, n$. Then

$$V_n = \int_0^{h_1}\int_0^{h_2}\cdots\int_0^{h_n} \frac{1}{a_1 \cdot a_2 \cdots a_n}\; dY_n \cdots dY_2\, dY_1$$

where $h_1 = k$, $h_2 = k - Y_1, \ldots, h_n = k - Y_1 - Y_2 - \ldots - Y_{n-1}$.

Finally, let

$$\begin{aligned} T_1 &= Y_n \\ T_2 &= k - Y_1 - \ldots - Y_{n-1} \\ &\vdots \\ T_{n-1} &= k - Y_1 - Y_2 \\ T_n &= k - Y_1 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} Y_n &= T_1 \\ Y_{n-1} &= T_3 - T_2 \\ &\vdots \\ Y_2 &= T_n - T_{n-1} \\ Y_1 &= k - T_n. \end{aligned}$$

Then, with this multivariate change of variable, we have

$$V_n = \frac{1}{a_1 \cdot a_2 \cdots a_n} \cdot \left(\int_0^{k}\int_0^{T_n}\cdots\int_0^{T_2} dT_1 \cdots dT_{n-1}\, dT_n \right)$$

$$= \frac{k^n}{a_1 \cdot a_2 \cdots a_n} \cdot \left(\int_0^{k}\int_0^{T_n}\cdots\int_0^{T_2} \frac{1}{k^n}\; dT_1 \cdots dT_{n-1}\, dT_n \right).$$

We leave it to the reader to verify that the absolute value of the associated Jacobian of this transformation equals 1. The Jacobian is easily determined because the matrix of partial derivatives is in triangular form.

We will use a probabilistic argument to illustrate that this remaining integral simply equals $\frac{1}{n!}$.

Suppose that $T_1, T_2, \ldots, T_n$ are independent observations from a probability distribution with density function $f(\cdot)$. Let $T_{(1:n)} \le T_{(2:n)} \le \ldots \le T_{(n:n)}$ be the ordered values of $T_1, T_2, \ldots, T_n$. Because the T_j's are independent and identically distributed, $(T_{(1:n)}, T_{(2:n)}, \ldots, T_{(n:n)})$ is equally likely to be any one of the $n!$ possible orderings of $T_1, T_2, \ldots, T_n$. That is,

$$P\Big(\big(T_{(1:n)}, T_{(2:n)}, \ldots, T_{(n:n)}\big) = \big(T_1, T_2, \ldots, T_n\big) \Big) = \frac{1}{n!}.$$

However,

$$\mathrm{P}\Big(\big(\mathrm{T}_{(1:n)}, \mathrm{T}_{(2:n)}, \ldots, \mathrm{T}_{(n:n)} \big) = \big(\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_n \big) \Big)$$

$$= \mathrm{P}\big(\mathrm{T}_1 \le \mathrm{T}_2 \le \ldots \le \mathrm{T}_n \big)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\mathrm{T}_n} \cdots \int_{-\infty}^{\mathrm{T}_3} \int_{-\infty}^{\mathrm{T}_2} \left(\prod_{j=1}^{n} \mathrm{f}(\mathrm{T}_j) \right) d\mathrm{T}_1\, d\mathrm{T}_2 \cdots d\mathrm{T}_{n-1}\, d\mathrm{T}_n.$$

In the case of iid Uniform$(0, k)$ observations, this simplifies to

$$\mathrm{P}\Big(\big(\mathrm{T}_{(1:n)}, \mathrm{T}_{(2:n)}, \ldots, \mathrm{T}_{(n:n)} \big) = \big(\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_n \big) \Big)$$

$$= \int_{0}^{k} \int_{0}^{\mathrm{T}_n} \cdots \int_{0}^{\mathrm{T}_3} \int_{0}^{\mathrm{T}_2} \tfrac{1}{k^n}\, d\mathrm{T}_1\, d\mathrm{T}_2 \cdots d\mathrm{T}_{n-1}\, d\mathrm{T}_n.$$

Therefore,

$$\mathrm{V}_n = \frac{k^n}{\mathrm{a}_1 \cdot \mathrm{a}_2 \cdots \mathrm{a}_n} \cdot \left(\int_{0}^{k} \int_{0}^{\mathrm{T}_n} \cdots \int_{0}^{\mathrm{T}_2} \tfrac{1}{k^n}\, d\mathrm{T}_1 \cdots d\mathrm{T}_{n-1}\, d\mathrm{T}_n \right).$$

$$= \frac{k^n}{\mathrm{a}_1 \cdot \mathrm{a}_2 \cdots \mathrm{a}_n \cdot n!}. \qquad \square$$

Note : One standard "proof" of this result involves having the geometric insight that V_n must be proportional to the product of the sides. However this insight is substantiated only by knowing that it leads to the correct solution, making the argument somewhat circular.

Proof of (II)

The sample space for $(\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_n)$ is $\Omega = \Big\{ (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n) \subseteq \Re^n \,\Big|\, 0 \le \mathrm{x}_j \le \delta, \quad j = 1, 2, \ldots, n \Big\}$. It follows that $\mathrm{V}(\Omega)$, the n-dimensional volume of Ω, equals δ^n. By definition of the uniform distribution, all points in Ω are equally likely, and hence for any event $\mathcal{C} \subseteq \Omega$,

$$\mathrm{P}(\mathcal{C}) = \frac{\mathrm{V}(\mathcal{C})}{\mathrm{V}(\Omega)} = \frac{\mathrm{V}(\mathcal{C})}{\delta^n}.$$

As we have illustrated previously, the General Probability Theorem can be applied with any countably additive set function and is not limited to the special case of a probability measure. In particular, in this problem we will illustrate its application when our additive set function is n - dimensional volume.

Before proceeding we note that the General Probability Theorem is applied on a fixed sample space. And when our additive set function is a probability measure, changing the sample space changes the probability of the set we are measuring. In contrast, the volume of a set $\mathcal{C} \subseteq \Omega$ is not changed by changing the sample space from Ω to Ω^*, provided $\mathcal{C} \subseteq \Omega^*$. For some problems, including this one, it is possible to simplify the calculation of $\mathrm{V}(\mathcal{C})$ by a judicious choice of a new sample space Ω^*.

For fixed $k \geq 0$, and $\mathrm{a}_j > 0$ $(j = 1, 2, \ldots, n)$, and sample space Ω, define

$$\mathcal{C} = \left\{ (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n) \subseteq \Omega \,\middle|\, \sum_{j=1}^{n} \mathrm{a}_j \mathrm{x}_j \leq k \right\}.$$

Then,

$$\mathrm{P}(\, \mathrm{a}_1 \mathrm{U}_1 + \mathrm{a}_2 \mathrm{U}_2 + \ldots + \mathrm{a}_n \mathrm{U}_n \leq k) = \tfrac{\mathrm{V}(\mathcal{C})}{\delta^n}.$$

Now define a new sample space Ω^*,

$$\Omega^* : \left\{ (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n) \subseteq \Re^n \,\middle|\, \sum_{j=1}^{n} \mathrm{a}_j \mathrm{x}_j \leq k \text{ and } 0 \leq \mathrm{x}_j \leq \infty \text{ for } j = 1, 2, \ldots, n \right\}.$$

Clearly, $\mathcal{C} \subseteq \Omega^*$. Also, we recognize that Ω^* is the simplex described in (I). Now, for $j = 1, 2, \ldots, n$, define,

$$\mathrm{A}_j : \left\{ (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n) \subseteq \Omega^* \,\middle|\, \delta \leq \mathrm{x}_j \leq \infty \right\}.$$

Then, with sample space Ω^* in mind, $\mathcal{C} = \overline{\mathrm{A}}_1 \cap \overline{\mathrm{A}}_2 \cap \cdots \cap \overline{\mathrm{A}}_n$. By the General Probability Theorem, we have

$$\begin{aligned} \mathrm{V}(\mathcal{C}) &= \mathrm{V}(\, \overline{\mathrm{A}}_1 \cap \overline{\mathrm{A}}_2 \cap \cdots \cap \overline{\mathrm{A}}_n) \\ &= \mathrm{V}(\Omega^*) - \sum_{r=1}^{n} \sum_{\mathbb{C}_r} (-1)^{r-1}\, \mathrm{V}(\, \mathrm{A}_{i_1} \cap \mathrm{A}_{i_2} \cap \cdots \cap \mathrm{A}_{i_r}). \end{aligned}$$

where we define $\mathbb{C}_r$ to be the set of all samples of size r drawn without replacement from $\{1, 2, \ldots, n\}$, when the order of sampling is considered unimportant. From our formula for the volume of a simplex,

$$\mathrm{V}(\Omega^*) = \frac{k^n}{\mathrm{a}_1 \cdot \mathrm{a}_2 \cdots \mathrm{a}_n \cdot n!}.$$

Now let $\mathcal{I}_r = \{ i_1, i_2, \ldots, i_r \}$. Then

$\mathrm{A}_{i_1} \cap \mathrm{A}_{i_2} \cap \cdots \cap \mathrm{A}_{i_r}$

$$= \left\{ (x_1, x_2, \ldots, x_n) \subseteq \Omega^* \,\middle|\, x_j > \delta \text{ for } j \in \mathcal{I}_r \right\}$$

$$= \left\{ (y_1, y_2, \ldots, y_n) \,\middle|\, \sum_{j=1}^{n} a_j y_j \leq k - \delta(a_{i_1} + a_{i_2} + \ldots + a_{i_r}), \right.$$

$$\text{where } y_j \geq 0, \; a_j > 0, \text{ and } k$$

$$\left. \geq 0 \right\}$$

where

$$y_j = \begin{cases} x_j - \delta & j \in \mathcal{I}_r \\ \\ x_j & j \notin \mathcal{I}_r. \end{cases}$$

We recognize this set as a simplex, provided $k - \delta(a_{i_1} + a_{i_2} + \ldots + a_{i_r}) \geq 0$, and hence from the formula for the volume of a simplex,

$$V(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = \frac{\left(k - \delta(a_{i_1} + a_{i_2} + \ldots + a_{i_r})\right)^n}{a_1 \cdot a_2 \cdots a_n \cdot n!}.$$

We note that $V(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = 0$ if $k - \delta(a_{i_1} + a_{i_2} + \ldots + a_{i_r}) < 0$. Therefore,

$$P(a_1U_1 + a_2U_2 + \ldots + a_nU_n \leq k) =$$

$$\frac{1}{\delta^n \cdot (a_1 \cdot a_2 \cdots a_n) \cdot n!} \times$$

$$\left(k^n - \sum_{r=1}^{n} \sum_{\mathbb{C}_r} (-1)^{r-1} \left(k - \delta(a_{j_1} + a_{j_2} + \ldots + a_{j_r})\right)^n \right.$$

$$\left. \mathbb{I}_{(0,k)}\left(\delta(a_{j_1} + a_{j_2} + \ldots + a_{j_r})\right)\right). \square$$

Proof of (III)

Define $U_j = \frac{V_j}{\delta_j}$. We leave it to the reader to verify that $U_j \sim \text{Uniform}(0,1)$. It follows that

$$P(V_1 + V_2 + \ldots + V_n \leq k)$$

$$= P(\delta_1U_1 + \delta_2U_2 + \ldots + \delta_nU_n \leq k)$$

$$= \frac{1}{1^n \cdot (\delta_1 \cdot \delta_2 \cdots \delta_n) \cdot n!} \times$$

$$\left(k^n - \sum_{r=1}^{n} \sum_{\mathbb{C}_r} (-1)^{r-1} \Big(k - 1(\delta_{j_1} + \delta_{j_2} + \ldots + \delta_{j_r}) \Big)^n \right.$$

$$\left. \mathbb{I}_{(0,k)}\Big(1(\delta_{j_1} + \delta_{j_2} + \ldots + \delta_{j_r}) \Big) \right)$$

$$= \frac{1}{(\delta_1 \cdot \delta_2 \cdots \delta_n) \cdot n!} \times$$

$$\left(k^n - \sum_{r=1}^{n} \sum_{\mathbb{C}_r} (-1)^{r-1} \Big(k - (\delta_{j_1} + \delta_{j_2} + \ldots + \delta_{j_r}) \Big)^n \right.$$

$$\left. \mathbb{I}_{(0,k)}\Big(\delta_{j_1} + \delta_{j_2} + \ldots + \delta_{j_r} \Big) \right). \qquad \square$$

Proof of (IV)

From (III) we have

$$\begin{aligned}
&\mathrm{P}(k-1 \le \mathrm{U}_1 + \ldots + \mathrm{U}_n \le k) \\
&\quad = \mathrm{P}(\mathrm{U}_1 + \ldots + \mathrm{U}_n \le k) - \mathrm{P}(\mathrm{U}_1 + \ldots + \mathrm{U}_n \le k-1) \\
&\quad = \frac{1}{n!} \sum_{j=0}^{k} (-1)^j \binom{n}{j} (k-j)^n - \frac{1}{n!} \sum_{j=0}^{k-1} (-1)^j \binom{n}{j} (k-1-j)^n \\
&\quad = \frac{1}{n!} \sum_{j=0}^{k} (-1)^j \binom{n}{j} (k-j)^n + \frac{1}{n!} \sum_{j=1}^{k} (-1)^j \binom{n}{j-1} (k-j)^n \\
&\quad = \frac{1}{n!} (-1)^0 \binom{n}{0} (k-0)^n + \frac{1}{n!} \sum_{j=1}^{k} (-1)^j \left(\binom{n}{j} + \binom{n}{j-1} \right) (k-j)^n \\
&\quad = \frac{1}{n!} (-1)^0 \binom{n+1}{0} (k-0)^n + \frac{1}{n!} \sum_{j=1}^{k} (-1)^j \binom{n+1}{j} (k-j)^n \\
&\quad = \frac{1}{n!} \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n \\
&\quad = \frac{1}{n!} \mathrm{A}(n,k). \qquad \square
\end{aligned}$$

Proof of (V)

Let $c = (x_1, \ldots, x_n) \in \mathbb{C}_m(k,n)$ and let $x_0 = 0$. Define the mapping $f(c) = (y_1, \ldots, y_n)$ where

$$y_i = \begin{cases} \frac{x_{i-1} - x_i}{m} & \text{if } x_i < x_{i-1} \\ \\ 1 + \frac{x_{i-1} - x_i}{m} & \text{if } x_i > x_{i-1}. \end{cases}$$

i) Clearly $f(c)$ is uniquely defined. Show that $f(c) \in \mathbb{D}_m(k, n)$.

1) Is $y_i \in \left\{ \frac{1}{m}, \dots, \frac{m-1}{m} \right\}$? Obvious.

2) Is $k - 1 < y_1 + \dots + y_n < k$? Yes!

We note that

$$y_1 + \dots + y_n = \sum_{i=1}^{n} \mathrm{I}_{(x_i > x_{i-1})} + \sum_{i=1}^{n} \frac{x_{i-1} - x_i}{m}$$

$$= \#\text{ rises in } (0, x_1, \dots, x_n) + \left(-\frac{x_n}{m}\right)$$

$$= k - \frac{x_n}{m}$$

and it is clear that $(k-1) < k - \frac{x_n}{m} < k$.

3) Is $y_i + \dots + y_j$ an integer for any $1 \le i < j \le n$? No!

$$y_i + \dots + y_j = \sum_{u=i}^{j} \mathrm{I}_{(x_u > x_{u-1})} + \sum_{u=i}^{j} \frac{x_{u-1} - x_u}{m}$$

$$= \Big(\#\text{ rises in } (x_{i-1}, \dots, x_j)\Big) + \frac{1}{m}(x_j - x_i).$$

We see that

$$0 < \left| \frac{1}{m}(x_j - x_i) \right| < 1$$

and hence cannot be an integer. However the number of rises must be an integer. It follows that their sum cannot be an integer. Therefore we can conclude that $f(c) \in \mathbb{D}_m(k, n)$.

ii) If $c \in \mathbb{C}, c^* \in \mathbb{C}$, and $c \neq c^*$, then show $f(c) \neq f(c^*)$.

Let $c = (x_1, \dots, x_n)$ and $c^* = (x_1^*, \dots, x_n^*)$ be distinct elements of $\mathbb{C}_m(k, n)$. Define $f(c) = (y_1, \dots, y_n)$ and let $f(c^*) = (y_1^*, \dots, y_n^*)$.

Suppose that $x_j \neq x_j^*$ and that $x_i = x_i^*$ for all $i < j$. It is easy to see that in this case $y_j \neq y_j^*$ and hence $f(c) \neq f(c^*)$.

iii) If $d \in \mathbb{D}$, then show there exists a $c \in \mathbb{C}$ such that $d = f(c)$.

Let $d = (y_1, \ldots, y_n) \in \mathbb{D}$ and define

$$x_i = m\Big(1 + \lfloor y_1 + \ldots + y_i \rfloor - (y_1 + \ldots + y_i)\Big) \qquad i = 1, \ldots, n.$$

We must show that $c = (x_1, \ldots, x_n) \in \mathbb{C}_m(k, n)$ and that $(y_1, \ldots, y_n) = f(c)$.

$1)$ Is $x_i \in \{1, \ldots, m-1\}$? Yes!

$$0 < 1 - \Big((y_1 + \ldots + y_i) - \lfloor y_1 + \ldots + y_i \rfloor\Big) < 1$$

and hence

$$0 < m\Big(1 + \lfloor y_1 + \ldots + y_i \rfloor - (y_1 + \ldots + y_i)\Big) < m.$$

Furthermore,

$$(y_1 + \ldots + y_i) = \tfrac{\alpha_1}{m} \quad \text{and} \quad \lfloor y_1 + \ldots + y_i \rfloor = \tfrac{\alpha_2}{m}$$

for some integers α_1 and α_2. Therefore

$$m\Big(1 + \lfloor y_1 + \ldots + y_i \rfloor - (y_1 + \ldots + y_i)\Big)$$

must be an integer. That is, $x_i \in \{1, \ldots, m-1\}$.

$2)$ Does $x_i = x_j$ for any $1 \le i < j \le n$? No!

$$\begin{aligned}
\frac{x_j - x_i}{m} &= \Big(1 + \lfloor y_1 + \ldots + y_j \rfloor - (y_1 + \ldots + y_j)\Big) \\
&\quad - \Big(1 + \lfloor y_1 + \ldots + y_i \rfloor - (y_1 + \ldots + y_i)\Big) \\
&= \Bigg(\lfloor y_1 + \ldots + y_j \rfloor - \lfloor y_1 + \ldots + y_i \rfloor\Bigg) - (y_{i+1} + \ldots + y_j). \\
&= \text{integer} - \text{noninteger} \neq 0.
\end{aligned}$$

It follows that $x_i \neq x_j$.

3) Does $(0, x_1, \ldots, x_n)$ have exactly k rises? Yes!

For integer $i \geq 2$,

$$\begin{aligned}
x_i - x_{i-1} &= m\Big(1 + \lfloor y_1 + \ldots + y_i \rfloor - (y_1 + \ldots + y_i)\Big) \\
&\quad - m\Big(1 + \lfloor y_1 + \ldots + y_{i-1} \rfloor - (y_1 + \ldots + y_{i-1})\Big)
\end{aligned}$$

$$= m\Big(\lfloor y_1 + \dots + y_i \rfloor - \lfloor y_1 + \dots + y_{i-1} \rfloor - y_i \Big).$$

Therefore,

$$x_i - x_{i-1} > 0 \Leftrightarrow \lfloor y_1 + \dots + y_{i-1} + y_i \rfloor - \lfloor y_1 + \dots + y_{i-1} \rfloor > y_i > 0$$

$$\Leftrightarrow \lfloor y_1 + \dots + y_{i-1} + y_i \rfloor - \lfloor y_1 + \dots + y_{i-1} \rfloor = 1.$$

That is, there is exactly one integer between $y_1 + \dots + y_{i-1}$ and $y_1 + \dots + y_i$. Put another way, there is a rise in $(x_1, \dots, x_n)$ everytime $\Big(y_1, (y_1 + y_2), (y_1 + y_2 + y_3), \dots, (y_1 + \dots + y_n) \Big)$ jumps to the next higher integer.

But $k - 1 < y_1 + \dots + y_n < k$. Hence, there must be exactly $k - 1$ rises in $(x_1, \dots, x_n)$ and exactly k rises in $(0, x_1, \dots, x_n)$. Therefore $c = (x_1, \dots, x_n) \in \mathbb{C}_m(k, n)$.

Finally, we must show that $f(c) = f(x_1, \dots, x_n) = (y_1, \dots, y_n)$ if we define

$$x_i = m\Big(1 + \lfloor y_1 + \dots + y_i \rfloor - (y_1 + \dots + y_i) \Big) \qquad i = 1, \dots, n.$$

To see this, let $f(x_1, \dots, x_n) = (t_1, \dots, t_n)$ where

$$t_i = \begin{cases} \frac{x_{i-1} - x_i}{m} & \text{if } x_i < x_{i-1} \\ \\ 1 + \frac{x_{i-1} - x_i}{m} & \text{if } x_i > x_{i-1}. \end{cases}$$

We previously noticed that

$$x_i - x_{i-1} > 0 \Rightarrow \lfloor y_1 + \dots + y_{i-1} + y_i \rfloor - \lfloor y_1 + \dots + y_{i-1} \rfloor = 1.$$

$$x_i - x_{i-1} < 0 \Rightarrow \lfloor y_1 + \dots + y_{i-1} + y_i \rfloor - \lfloor y_1 + \dots + y_{i-1} \rfloor = 0.$$

and that

$$\frac{x_{i-1} - x_i}{m} = y_i - \Big(\lfloor y_1 + \dots + y_i \rfloor - \lfloor y_1 + \dots + y_{i-1} \rfloor \Big).$$

Therefore,

$$t_i = \begin{cases} y_i - 0 & \text{if } x_i < x_{i-1} \\ \\ 1 + (y_i - 1) & \text{if } x_i > x_{i-1}. \end{cases}$$

That is, $t_i = y_i$. $\square$

Proof of (VI)

$$\mathrm{P}(\, \mathrm{K}_w > k\,) \;=\; \mathrm{P}(\, \mathrm{U}_1 + \mathrm{U}_2 + \ldots + \mathrm{U}_k \le w\,) \;=\; \frac{1}{k!} \sum_{r=0}^{\lfloor w \rfloor} (-1)^r \binom{k}{r} (w-r)^k \qquad k \ge 1.$$

We note that

$$\frac{1}{0!} \sum_{r=0}^{\lfloor w \rfloor} (-1)^r \binom{0}{r} (w-r)^0 \equiv 1 \quad \text{for all } w$$

and hence the above formula is valid for the case $k = 0$ as well. Therefore

$$\mathrm{E}(\, \mathrm{K}_w\,) \;=\; \sum_{k=0}^{\infty} \mathrm{P}(\, \mathrm{K}_w > k\,)$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{k!} \sum_{r=0}^{\lfloor w \rfloor} (-1)^r \binom{k}{r} (w-r)^k \right)$$

$$= \sum_{r=0}^{\lfloor w \rfloor} (-1)^r \left(\sum_{k=0}^{\infty} \frac{1}{k!} \binom{k}{r} (w-r)^k \right)$$

$$= \sum_{r=0}^{\lfloor w \rfloor} (-1)^r \left(\sum_{k=r}^{\infty} \frac{1}{k!} \binom{k}{r} (w-r)^k \right) \qquad \left[\ \binom{k}{r} = 0 \text{ for } k < r.\ \right]$$

$$= \sum_{r=0}^{\lfloor w \rfloor} (-1)^r \frac{1}{r!} (w-r)^r \left(\sum_{k=r}^{\infty} \frac{1}{(k-r)!} (w-r)^{k-r} \right)$$

$$= \sum_{r=0}^{\lfloor w \rfloor} (-1)^r \frac{1}{r!} (w-r)^r\, \mathrm{e}^{(w-r)}. \qquad \square$$

Proof of (VII)

We can think of $\mathrm{S}_n = \mathrm{U}_1 + \ldots + \mathrm{U}_n$ as the time of the n^{th} arrival in a process. Define

$$\mathrm{N}(\, t\,) \;=\; \sup\{\, n : \mathrm{S}_n \le t\}.$$

Our interarrival times $\mathrm{U}_1, \mathrm{U}_2, \ldots$ are independent and identically distributed, so the counting process $\{\, \mathrm{N}(t),\ t \ge 0\}$ is in fact a renewal process. From the theory of renewal processes, we have that for any renewal process with a continuous interarrival time distribution with mean μ and finite variance σ^2

$$\mathrm{E}\Big(\mathrm{N}(t)\Big) = \tfrac{t}{\mu} + \tfrac{\sigma^2-\mu^2}{2\mu^2} + o(1).$$

[see Sheldon Ross, *Stochastic Processes*, Corollary 3.4.7].

We note that

$$\mathrm{E}(\mathrm{K}_w) = \mathrm{E}\Big(\mathrm{N}(w) + 1\Big) = \tfrac{t}{\mu} + \tfrac{\sigma^2+\mu^2}{2\mu^2} + o(1).$$

It is easy to demostrate that $\mu = \mathrm{E}(\mathrm{U}_1) = \frac{1}{2}$ and $\sigma^2 = \mathrm{var}(\mathrm{U}_1) = \frac{1}{12}$. Simplifying we have

$$\mathrm{E}(\mathrm{K}_w) = 2w + \tfrac{2}{3} + o(1). \qquad \square$$

Reference : The Mathematical Gazette, John Haigh.