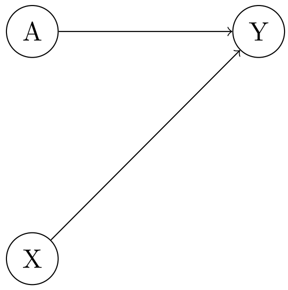
The data is modeled as:

$$
\begin{aligned}
(X_1, Y_1, A_1), \ldots, (X_n, Y_n, A_n) &\overset{iid}{\sim} \mathcal{O} \\
A &\perp X \\
P(A = 1 \mid X) = P(A = 1) = 1 - P(A = 0) &= p
\end{aligned} \tag{1}
$$

for some law $\mathcal{O}$.



The estimand is

$$\psi_0 = E(Y \mid A = 1) - E(Y \mid A = 0).$$

An estimator is obtained as the solution in ψ of

$$\sum_{i=1}^{n} U(Y_i, A_i; \psi) = 0,$$

where

$$U(Y, A; \psi) = (A - p)(Y - \psi A).$$

Consistency and asymptotic normality of this estimator follow from:

Lemma 0.1. $E(U(Y, A; \psi_0)) = 0$.

Proof.

$$
\begin{aligned}
E(U(Y, A; \psi_0)) &= E[(A - p)(Y - \psi_0 A)] \\
&= E[(A - p)(E(Y \mid A) - \psi_0 A)] \\
&= (E(Y \mid A = 1) - \psi_0 A)(1 - p)p + E(Y \mid A = 0)(-p)(1 - p) \\
&= p(1 - p)[E(Y \mid A = 1) - E(Y \mid A = 0) - \psi_0] = 0.
\end{aligned}
$$

□

We consider estimators obtained as solutions in ψ to equations of the form

$$\sum_i U(Y_i, A_i; \psi) + (A_i - p)h(X_i; \psi) = \sum_i (A_i - p)(Y_i - \psi A_i + h(X_i; \psi)) = 0 \tag{2}$$

for [arbitrary] functions h. It follows from Lemma 0.1 and (1) that

$$E(U(Y, A; \psi_0) + (A - p)h(X; \psi)) = 0,$$

so such estimators are also asymptotically normal. An additional benefit is that the asymptotic variance of the resulting estimator may be minimized by varying h, perhaps improving on the efficiency of the estimator obtained from $\sum_i U(Y, A; \psi) = 0$. In fact, the minimizing choice of h is determined by the estimating equation given by

$$W(X, Y, A; \psi) = U(Y, A; \psi) - E(U(Y, A; \psi) \mid A, X) + E(U(Y, A; \psi) \mid X). \tag{3}$$

A proof is given in Lemma 0.2, after rewriting the rhs of (3), as follows. The middle term on the rhs of (3) is,

$$
\begin{aligned}
E(U(Y,A;\psi) \mid A,X) &= E((A-p)(Y-\psi A) \mid A,X) \\
&= (E(Y \mid A,X) - \psi A)(A-p) \\
&= [E(Y \mid A=1,X)A + E(Y \mid A=0,X)(1-A)](A-p) - \psi A(A-p) \\
&= E(Y \mid A=1,X)A(1-p) - E(Y \mid A=0,X)(1-A)p - \psi A(1-p).
\end{aligned}
$$

The last term on the rhs of (3) is then,

$$
\begin{aligned}
E(U(Y,A;\psi) \mid X) &= pE(U(Y,A;\psi) \mid A=1,X) + (1-p)E(U(Y,A;\psi) \mid A=0,X) \\
&= p[E(Y \mid A=1,X)(1-p) - \psi(1-p)] - (1-p)[E(Y \mid A=0,X)p] \\
&= p(1-p)(E(Y \mid A=1,X) - E(Y \mid A=0,X) - \psi).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
W(X,Y,A;\psi) &= U(Y,A;\psi) - E(U(Y,A;\psi) \mid A,X) + E(U(Y,A;\psi) \mid X) \\
&= (A-p)(Y-\psi A) - (A-p)(1-p)E(Y \mid A=1,X) - (A-p)pE(Y \mid A=0,X) + \\
&\qquad (A-p)(1-p)\psi \\
&= (A-p)[Y - (1-p)E(Y \mid A=1,X) - pE(Y \mid A=0,X)] - p(1-p)\psi \\
&= (A-p)[Y - E(\tilde{Y} \mid X)] - p(1-p)\psi \\
&= U(Y_i, A_i; \psi) + (A-p)[\psi A - E(\tilde{Y} \mid X)] - p(1-p)\psi,
\end{aligned}
$$

where $\tilde{Y}$ is determined by the transformation

$$
Y = \tilde{Y} \frac{p^A (1-p)^{1-A}}{(1-p)^A p^{1-A}} = \tilde{Y} \left(\frac{p}{1-p} \right)^{2A-1}.
$$

In case $p = P(A=1) = P(A=0) = 1/2$,

$$
W(X,Y,A;\psi) = (A-1/2)(Y - E(Y \mid X)) - \psi/4. \tag{4}
$$

Lemma 0.2. *The asymptotic variance of the estimator obtained as the solution in ψ to*

$$
\sum_i U(Y_i, A_i; \psi) + (A_i - p)h(X_i;\psi) = \sum_i (A_i - p)(Y_i - \psi A_i + h(X_i;\psi)) = 0 \tag{5}
$$

is minimized over arbitrary functions h of X at

$$
h_0(X;\psi) = (A-p)[\psi A - E(\tilde{Y} \mid X)] - p(1-p)\psi.
$$

Proof. We give the $p = P(A=1) = 1/2$ case, in which case

$$
h_0(X;\psi) = (A-1/2)[\psi A - E(Y \mid X)] - \psi/4.
$$

Under suitable regularity conditions, the asymptotic variance of the solution to the estimating equation (5) is given by the variance of its influence function. Since

$$
E\frac{\partial}{\partial\psi}[U(Y_i, A_i;\psi) + (A_i - p)h(X_i;\psi)] = E\frac{\partial}{\partial\psi}U(Y_i, A_i;\psi),
$$

the influence function of (5) is

$$
-\left(E\, \frac{\partial}{\partial\psi} U(Y,A;\psi)\bigg|_{\psi_0} \right)^{-1} (U(Y,A;\psi_0) + (A-1/2)h(X;\psi)).
$$

Thus we wish to show

$$\operatorname{Var}\left[\left(E\frac{\partial}{\partial\psi}U(Y,A;\psi_0)\right)^{-1}(U(Y,A;\psi_0)+(A-1/2)h(X;\psi))\right] \geq$$
$$\operatorname{Var}\left[\left(E\frac{\partial}{\partial\psi}U(Y,A;\psi_0)\right)^{-1}(U(Y,A;\psi_0)+(A-1/2)h_0(X;\psi))\right]$$

or

$$\begin{aligned} &E[(A-1/2)^2h^2(X)]+2E[U(Y,A;\psi_0)(A-1/2)h(X;\psi)] \geq \\ &E[(A-1/2)^2h_0^2(X)]+2E[U(Y,A;\psi_0)(A-1/2)h_0(X;\psi)]. \end{aligned} \tag{6}$$

Since A, X are uncorrelated, and noting that $(A-1/2)(-1)^{1-A} = 1/2$, the lhs is

$$\begin{aligned} &E[(A-1/2)^2h^2(X)]+2E[U(Y,A;\psi_0)(A-1/2)h(X;\psi)] \\ &= \operatorname{Var}(A)Eh^2(X)+2E[(A-1/2)E(U(Y,A;\psi_0)h(X;\psi)\mid A)] \\ &= Eh^2(X)/4+2E[(A-1/2)E((Y-\psi_0A)(-1)^{1-A}h(X;\psi)\mid A)] \\ &= Eh^2(X)/4+E[E((Y-\psi_0A)h(X;\psi)\mid A)] \\ &= Eh^2(X)/4+E((Y-\psi_0/2)h(X;\psi)). \end{aligned}$$

We obtain an expression for the rhs by substituting $h(X;\psi) := h_0(X;\psi) = -(2E(Y \mid X)-\psi_0)$,

$$\begin{aligned} &E[(A-1/2)^2h_0^2(X)]+2E[U(Y,A;\psi_0)(A-1/2)h_0(X;\psi)] \\ &= Eh_0^2(X)/4+E((Y-\psi_0/2)h_0(X;\psi)) \\ &= E[h_0(X;\psi)(h_0(X;\psi)/4+Y-\psi_0/2)] \\ &= E[h_0(X;\psi)(-(2E(Y\mid X)-\psi_0)/4+E(Y\mid X)-\psi_0/2)] \\ &= E[h_0(X;\psi)(E(Y\mid X)/2-\psi_0/4)] \\ &= -Eh_0^2(X)/4 \qquad (7) \\ &= -E[E(Y\mid X)^2]+\psi_0EY-\psi_0^2/4. \end{aligned}$$

Thus (6), which we wish to show, becomes

$$Eh^2(X)/4+E((Y-\psi_0/2)h(X;\psi))+E[E(Y\mid X)^2]-\psi_0EY+\psi_0^2/4 \geq 0.$$

This inequality follows by an application of the Cauchy-Schwarz inequality,

$$\begin{aligned} &Eh^2(X)/4+E((Y-\psi_0/2)h(X;\psi))+E[E(Y\mid X)^2]-\psi_0EY+\psi_0^2/4 \\ &= (1/4)E[(h(X;\psi)-\psi_0)^2]+E(Yh(X;\psi))+E[E(Y\mid X)^2]-\psi_0EY \\ &= (1/4)E[(h(X;\psi)-\psi_0)^2]+E[E(Y\mid X)^2]+E[E(Y\mid X)(h(X;\psi)-\psi_0)] \\ &\geq (1/4)E[(h(X;\psi)-\psi_0)^2]+E[E(Y\mid X)^2]-E[(h(X;\psi)-\psi_0)^2]^{1/2}E[E(Y\mid X)^2]^{1/2} \\ &= \{(1/2)E[(h(X;\psi)-\psi_0)^2]^{1/2}-E[E(Y\mid X)^2]^{1/2}\}^2 \geq 0. \end{aligned}$$

□

Remark. *From (7), $Eh_0^2(X)/4$ is the reduction in the asymptotic variance gained by using (5) over (2).*

The expression for $W(X,Y,A;\psi)$ in (3) contains a term of the form $E(Y \mid A, X)$, generally requiring estimation, whereas the equivalent expression in (4) only requires $E(Y \mid X)$ to be estimated. We investigate whether the second form is more resistant to bias by an unscrupulous analyst.

Consider the following strategies for estimating ψ:

□

FIGURE 1. Comparison of the 3 methods of estimating ψ.

□

FIGURE 2. Coverage rates of 3 ψ estimates when the sample size is $n = 1000$.

(1) The analyst first forms 2^{p+1} estimates of $E(Y \mid A, X)$ by fitting submodels of the linear model

$$E(Y \mid A, X) = A + X_1 + \ldots + X_p + \text{second-order interactions.}$$

For each estimate $E(Y \mid A, X)$, the analyst then obtains an estimate for ψ by solving the estimating equation (3), using the model estimate of $E(Y \mid A_i, X_i), i = 1, \ldots, n$. From the resulting estimates for ψ, the analyst reports the largest.

(2) One analyst is given the control data

$$\{(Y_i, X_i) : A_i = 0\}$$

from which he forms estimates of $E(Y \mid A = 0, X)$ using submodels of the linear models

$$E(Y \mid X) = X_1 + \ldots + X_p + \text{second-order interactions.}$$

Another analyst estimates $E(Y \mid A = 1, X)$ analogously. The models are combined to estimate $E(Y \mid A, X)$ and obtain estimates of ψ as before, from which the largest is chosen.

(3) The analyst proceeds as in (1), but omitting A_i from his models. I.e., the analyst first forms estimates of $E(Y \mid X)$ by fitting submodels of the linear model

$$E(Y \mid X) = X_1 + \ldots + X_p + \text{second-order interactions.}$$

For each estimate $E(Y \mid X)$, the analyst then obtains an estimate for ψ by solving the estimating equation (4), using the model estimate of $E(Y \mid X_i), i = 1, \ldots, n$. From the resulting estimates for ψ, the analyst reports the largest.

(4) *Not yet done.* As in (2), two analysts each obtain 2^p estimates of $E(Y \mid A = 1, X)$ and $E(Y \mid A = 1, X)$ separately, but now the maximum estimate of ψ is obtained ranging over all 2^{p+1} pairs of the 2 analysts' sets of estimates.

The results of a simulation are in Figure 1. From the simulation, the reported ψ estimates are less biased when computed under method (3) and (2) than method (1), though all methods are seriously biased. Also plotted is the average, rather than maximum ψ estimate, over the sets of models considered under each method. These are close to the true value $\psi = 1$, particularly for larger n, as expected.

Since all methods are so biased, confidence intervals at this range of sample sizes (≤ 300) are useless. Increasing the sample size to 1000 allows some comparison of error rates. Figure 2 gives the coverage rates of the 3 methods for as the number of covariates given to the analysts ranges.

0.1. **Equivalence to regression estimator.** Define

$$\tilde{Y} = Y - E(Y \mid X)$$

and consider the regression

$$E(\tilde{Y} \mid A) = \beta_0 + \beta_1 A.$$

Then $\beta_0 + \beta_1 = E(\tilde{Y} \mid A = 1) = E(Y \mid A = 1) - E(Y)$ and $\beta_0 = E(\tilde{Y} \mid A = 0) = E(Y \mid A = 0) - E(Y) = E(Y \mid A = 0)/2 - E(Y \mid A = 1)/2$, so

$$\begin{aligned}\beta_0 &= -\psi/2\\ \beta_1 &= \psi.\end{aligned}$$

The influence function of (β_0, β_1) is obtained as:

$$\begin{aligned}
0 &= \sum_{i=1}^{n} \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \hat{\beta}_0 - A_i\hat{\beta}_1)\\
&= \sum_{i=1}^{n} \left\{ \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \beta_0 - A_i\beta_1) + \begin{pmatrix} -1 & -A_i \\ -A_i & -A_i \end{pmatrix} \begin{pmatrix} \hat{\beta}_0 - \beta_0 \\ \hat{\beta}_1 - \beta_1 \end{pmatrix} \right\}\\
n^{1/2} \begin{pmatrix} \hat{\beta}_0 - \beta_0 \\ \hat{\beta}_1 - \beta_1 \end{pmatrix} &= \left(\frac{1}{n} \sum_i \begin{pmatrix} 1 & A_i \\ A_i & A_i \end{pmatrix} \right)^{-1} n^{-1/2} \sum_i \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \beta_0 - A_i\beta_1)\\
&= \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}^{-1} n^{-1/2} \sum_i \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \beta_0 - A_i\beta_1) + o_P(1)\\
n^{1/2}(\hat{\beta}_1 - \beta_1) &= n^{-1/2} \sum_i \begin{pmatrix} -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ A_i \end{pmatrix} (\tilde{Y}_i - \beta_0 - A_i\beta_1) + o_P(1)\\
&= 4n^{-1/2} \sum_i (A_i - 1/2)(\tilde{Y}_i - \beta_0 - A_i\beta_1) + o_P(1)\\
&= 4n^{-1/2} \sum_i (A_i - 1/2)(\tilde{Y}_i - (A_i - 1/2)\psi) + o_P(1)\\
&= 4n^{-1/2} \sum_i \{(A_i - 1/2)(Y - E(Y \mid X)) - \psi/4\} + o_P(1).
\end{aligned}$$

By comparison with (4), we find that $\hat{\psi}$, the augmented estimator, is asymptotically equivalent to $\hat{\beta}_1$.

On the other hand, the OLS solution to the above regression gives

$$\hat{\beta}_1 = \hat{\mathrm{Cov}}(\tilde{Y}, A)/\hat{\mathrm{Var}}(A) = \frac{\overline{A\tilde{Y}}/\bar{\tilde{Y}} - \bar{\tilde{Y}}}{1 - \bar{A}}$$

whereas from (4),

$$\hat{\psi} = \bar{\tilde{Y}}/2 - \overline{A\tilde{Y}},$$

so the two estimators are not equal.

1. Other estimands

As above, the full data is $Y_i^* = (Y_i^*(0), Y_i^*(1)), i = 1, \ldots, n$, the observed data is $(Y_i, A_i, X_i), i = 1, \ldots, n$, and we assume

$$\begin{aligned}&Y = AY^*(1) + (1 - A)Y^*(0),\\ &P(A = 1) = p \in (0, 1)\\ &A \perp X, A \perp Y^*.\end{aligned}$$

Besides the mean treatment difference $E(Y \mid A = 1) - E(Y \mid A = 0)$ discussed above, we consider other estimands:

(1) $\psi_0 = \log \frac{E(Y^*(1))}{E(Y^*(0))} = \log \frac{E(Y|A=1)}{E(Y|A=0)}$
(2) the slope in the model
$$\mathrm{logit}(AE(Y^*(1)) + (1 - A)E(Y^*(0))) = \mathrm{logit}(P(Y = 1 \mid A)) = \psi_0 + \psi_1 A,$$
for a binary-valued response Y

In each case, we obtain the efficient augmented influence function following the approach of [[Tsiatis ch. 13]]:

(1) obtain a full-data influence function $\phi^F(Y^*)$
(2) obtain an observed data influence function $\phi(Y, A, X)$ corresponding to ϕ^F under the mapping $\phi \mapsto E(\phi \mid Y^*)$
(3) compute the efficient augmentation term

$$\begin{aligned} h^*(Y, A, X) &= (A - p)(E(\phi \mid A = 1, X) - E(\phi \mid A = 0, X)) \\ &= E(\phi \mid A, X) - E(\phi \mid X) \end{aligned}$$

We then eliminate regressions on treatment level, i.e., the terms $E(Y \mid A = 1, X)$ and $E(Y \mid A = 0, X)$.

1.1. $\log \frac{E(Y|A=1)}{E(Y|A=0)}$. The problem is to estimate

$$\psi_0 = \log \frac{E(Y^*(1))}{E(Y^*(0))} = \log \frac{E(Y \mid A = 1)}{E(Y \mid A = 0)}.$$

A full-data estimator is given by the solution to

$$\sum_i (Y_i^*(1) - e^{\psi_0} Y_i^*(0)) = 0,$$

with influence function

$$\begin{aligned} \phi^F(Y, A, X; \psi) &= (e^{\psi} E(Y^*(0)))^{-1}(Y^*(1) - e^{\psi} Y^*(0)) \\ &= (E(Y^*(1)))^{-1}(Y^*(1) - e^{\psi} Y^*(0)). \end{aligned}$$

An influence functions ϕ of the observed data satisfies

$$\begin{aligned} E(Y_i^*(1))\phi(Y, A, X) &= \left(\frac{A}{p} - e^{\psi_0}\frac{1 - A}{1 - p}\right) Y + h(Y, A, X) \\ &= (A - p)\left(\frac{A}{(A - p)p} - e^{\psi_0}\frac{1 - A}{(A - p)(1 - p)}\right) Y + h(Y, A, X) \\ &= \frac{A - p}{p(1 - p)}(A + e^{\psi_0}(1 - A))Y + h(Y, A, X) \\ &= \frac{A - p}{p(1 - p)} e^{(1-A)\psi_0} Y + h(Y, A, X), \end{aligned}$$

where h satisfies $E(h(Y, A, X) \mid Y^*) = 0$. The minimizing h is given by subtracting out

$$\begin{aligned} h^*(Y, A, X) &= (A - p)[E(\frac{A - p}{p(1 - p)} e^{(1-A)\psi_0} Y \mid A = 1, X) - E(\frac{A - p}{p(1 - p)} e^{(1-A)\psi_0} Y \mid A = 0, X)] \\ &= (A - p)[\frac{1}{p} E(Y \mid A = 1, X) + \frac{e^{\psi_0}}{1 - p} E(Y \mid A = 0, X)]. \end{aligned}$$

The efficient influence function is therefore

$$\begin{aligned} \phi^*(Y, A, X) &= (E(Y^*(1)))^{-1}\phi(Y, A, X) - h^*(Y, A, X) \\ &= (E(Y^*(1)))^{-1}(A - p)\left(\frac{e^{(1-A)\psi_0}}{p(1 - p)} Y - \frac{1}{p} E(Y \mid A = 1, X) - \frac{e^{\psi_0}}{1 - p} E(Y \mid A = 0, X)\right). \end{aligned}$$

Let $\hat{\psi}_n$ be a consistent estimator of ψ_0. Under the transformation

$$Y = \frac{p^{2A}(1 - p)^{2(1-A)}}{e^{(1-A)\hat{\psi}_n}} \tilde{Y}$$

the efficient influence function may be rewritten

$$\phi^*(Y,A,X) = (E(Y^*(1)))^{-1}(A-p)\left(\frac{e^{(1-A)\psi}}{p(1-p)}Y - E(\tilde{Y} \mid X)\right) + o_P(1).$$

In case $p = 1/2$, $\tilde{Y} = 4e^{(1-A)\hat{\psi}_n}Y$, and

(8)
$$\phi^*(Y,A,X) = 2(E(Y^*(1)))^{-1}(2A-1)[e^{(1-A)\psi}Y - E(e^{(1-A)\hat{\psi}_n}Y \mid X)] + o_P(1).$$

1.1.1. *Two-step regression.* Let

$$Z = Y - e^{(A-1)\psi_0}[E(e^{(1-A)\psi_0}Y \mid X) - E(Y \mid A=1)]$$

and consider the log-linear regression model

(9)
$$\log(E(Z \mid A)) = \beta_0 + \beta_1 A.$$

Then

$$E(e^{(1-A)\psi_0}Y) = (1/2)[e^{\psi_0}E(Y \mid A=0) + E(Y \mid A=1)] = E(Y \mid A=1)$$

implies

$$\begin{aligned} E(Z \mid A=1) &= E(Y \mid A=1) - E[E(e^{(1-A)\psi_0}Y \mid X) - E(Y \mid A=1) \mid A=1] \\ &= E(Y \mid A=1) - E(e^{(1-A)\psi_0}Y) + E(Y \mid A=1) \\ &= E(Y \mid A=1), \end{aligned}$$

and similarly

$$\begin{aligned} E(Z \mid A=1) &= E(Y \mid A=0) - E[E(e^{(1-A)\psi_0}Y \mid X) - E(Y \mid A=1) \mid A=0] \\ &= E(Y \mid A=0). \end{aligned}$$

Therefore, under model (9),

$$\begin{aligned} \beta_0 &= \log(E(Z \mid A=0)) = \log(E(Y \mid A=0)), \\ \beta_0 + \beta_1 &= \log(E(Z \mid A=1)) = \log(E(Y \mid A=0)), \\ \beta_1 &= \frac{\log(E(Y \mid A=1))}{\log(E(Y \mid A=0))} = \psi_0. \end{aligned}$$

An estimator $(\hat{\beta}_0, \hat{\beta}_1)$ under the log-linear regression model (9) is given by the estimating equations

$$0 = \sum_{i=1}^{n} \begin{pmatrix} 1 \\ A_i \end{pmatrix} (Z_i - e^{\hat{\beta}_0 + \hat{\beta}_1}).$$

The influence function of $\beta_1 = \psi_0$ is computed to be

$$\begin{aligned} \phi_{\beta_1}(Y,A,X;\beta) &= 2(2A-1)(e^{-\beta_0-\beta_1 A}Z - 1) \\ &= 2(2A-1)\left(\frac{Z}{E(Z \mid A)} - 1\right) \\ &= 2(2A-1)\left(\frac{e^{(1-A)\beta_1}Z}{E(Z \mid A=1)} - 1\right) \\ &= 2(2A-1)\frac{e^{(1-A)\beta_1}Y - E(e^{(1-A)\psi_0}Y \mid X) + E(Y \mid A=1) - E(Z \mid A=1)}{E(Z \mid A=1)} \\ &= 2(2A-1)\frac{e^{(1-A)\beta_1}Y - E(e^{(1-A)\psi_0}Y \mid X)}{E(Y \mid A=1)} \\ &= 2(E(Y^*(1)))^{-1}(2A-1)[e^{(1-A)\psi_0}Y - E(e^{(1-A)\psi_0}Y \mid X)]. \end{aligned}$$

This influence function is the same as the influence function (8), so the estimator $\hat{\beta}_1$ under the log-linear regression is asymptotically equivalent to the efficient estimator $\hat{\psi}$.

1.2. $\operatorname{logit}(P(Y=1 \mid A)) = \psi_0 + \psi_1 A$**.** The target is the slope in the logistic model

$$\operatorname{logit}(P(Y=1 \mid A)) = \psi_0 + \psi_1 A.$$

This model is also a type of restricted moment model. Let σ denote the sigmoid function $x \mapsto e^x/(1+e^x)$ and $\sigma' = \sigma(1-\sigma)$ its derivative. The coefficients may be estimated as the root of

$$\sum_{i=1}^{n} \begin{pmatrix} 1 \\ A_i \end{pmatrix} [Y_i - \sigma(\psi_0 + \psi_1 A_i)].$$

The influence function is computed in the usual way as

$$U(Y, A, X; \psi) = \frac{A-p}{p(1-p)\sigma'(\psi_0)} \begin{pmatrix} A-1 \\ [\sigma'(\psi_0)/\sigma'(\psi_0+\psi_1)]^A \end{pmatrix} (Y - \sigma(\psi_0 + \psi_1 A)).$$

To minimize the asymptotic variance we subtract out $h^*(X)$ given by

$$\begin{aligned} (A-p)^{-1} h^*(X) &= E(U \mid A=1, X) - E(U \mid A=0, X) \\ &= \begin{pmatrix} -\frac{E(Y|A=0,X)-\sigma(\psi_0)}{(1-p)\sigma'(\psi_0)} \\ \frac{E(Y|A=1,X)-\sigma(\psi_0+\psi_1)}{p\sigma'(\psi_0+\psi_1)} + \frac{E(Y|A=0,X)-\sigma(\psi_0)}{(1-p)\sigma'(\psi_0)} \end{pmatrix}. \end{aligned}$$

The optimized influence function for the slope, $\phi^*_{\psi_1}$, is then given by

$$\begin{aligned} \frac{p(1-p)}{A-p} \phi^*_{\psi_1}(Y, A, X; \psi) &= \frac{Y}{\sigma'(\psi_0+\psi_1)^A \sigma'(\psi_0)^{1-A}} \\ &\quad - \left(\frac{(1-p)E(Y \mid A=1, X)}{\sigma'(\psi_0+\psi_1)} + \frac{pE(Y \mid A=0, X)}{\sigma'(\psi_0)} \right) \\ &\quad - (A - (1-p)) \left(\frac{1}{1-\sigma(\psi_0+\psi_1)} - \frac{1}{1-\sigma(\psi_0)} \right). \end{aligned}$$

Under the transformation

$$Y = \left(\frac{p}{1-p} \right)^{2A-1} \sigma'(\psi_0+\psi_1)^A \sigma'(\psi_0)^{1-A} \tilde{Y},$$

the optimized influence function may be rewritten as

$$\begin{aligned} \frac{p(1-p)}{A-p} \phi^*_{\psi_1}(Y, A, X; \psi) &= \left(\frac{p}{1-p} \right)^{2A-1} \tilde{Y} - E(\tilde{Y} \mid X) \\ &\quad - (A - (1-p)) \left(\frac{1}{1-\sigma(\psi_0+\psi_1)} - \frac{1}{1-\sigma(\psi_0)} \right). \end{aligned}$$

In case $p = 1/2$,

$$\phi^*_{\psi_1}(Y, A, X; \psi) = \frac{2(2A-1)}{\sigma'(\psi_0+\psi_1)^A \sigma'(\psi_0)^{1-A}} [Y - E(Y \mid X)] - \left(\frac{1}{1-\sigma(\psi_0+\psi_1)} - \frac{1}{1-\sigma(\psi_0)} \right).$$