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# Unimodality of Differences

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*Summary*: The convolution of two unimodal densities is not in general unimodal. In [1953] *Chung* [see also his translation of *Gnedenko/Kolmogorov*] gave an example of i.i.d. random variables $X$, $Y$, both with an unimodal density $f$, where $X+Y$ has no unimodal density. *Wintner* [1938] had shown that the convolution of two symmetrical unimodal densities is again symmetrical unimodal. *Ibragimov* [1956] proved the strong unimodality for the convolution of strongly unimodal densities.

For the difference $X - Y$ of two i.i.d. random variables with arbitrary density $f$ it is known and easily proved that it has a density which is symmetrical and maximal at 0. It seems to be not yet known and is proved in this paper that this density of $X - Y$ is unimodal if $f$ is unimodal.

*Definition*: A density function $f$ is called *unimodal*, iff there is a real $m$ with

$$f(x_1) \leqslant f(x_2) \leqslant f(m) \qquad \text{for } x_1 \leqslant x_2 \leqslant m$$

and

$$f(m) \geqslant f(x_1) \geqslant f(x_2) \qquad \text{for } m \leqslant x_1 \leqslant x_2 .$$

We call every such $m$ a *modular value.*

$f$ is not supposed to be continuous and $m$ is in general not unique; there may be an interval of modular values and then we take

$$m^* = \sup \{m \mid m \text{ is modular value}\}$$

and call $m^*$ the *mode* of $f$. For each modular value $m$, $f(m)$ is the absolute maximum $a$ of $f$. If $f(m^*) < a$ (which might happen if $f$ is discontinuous at $m^*$) we define $f(m^*) = a$, thus altering, if necessary, the function in one point which doesn't affect any of the integrals in the following (s. Fig. 1).

Now let $X$, $Y$ be independent random variables, both with a density $f$. Then

$$g(z) = \int_{-\infty}^{\infty} f(x) f(x+z)\, dz$$

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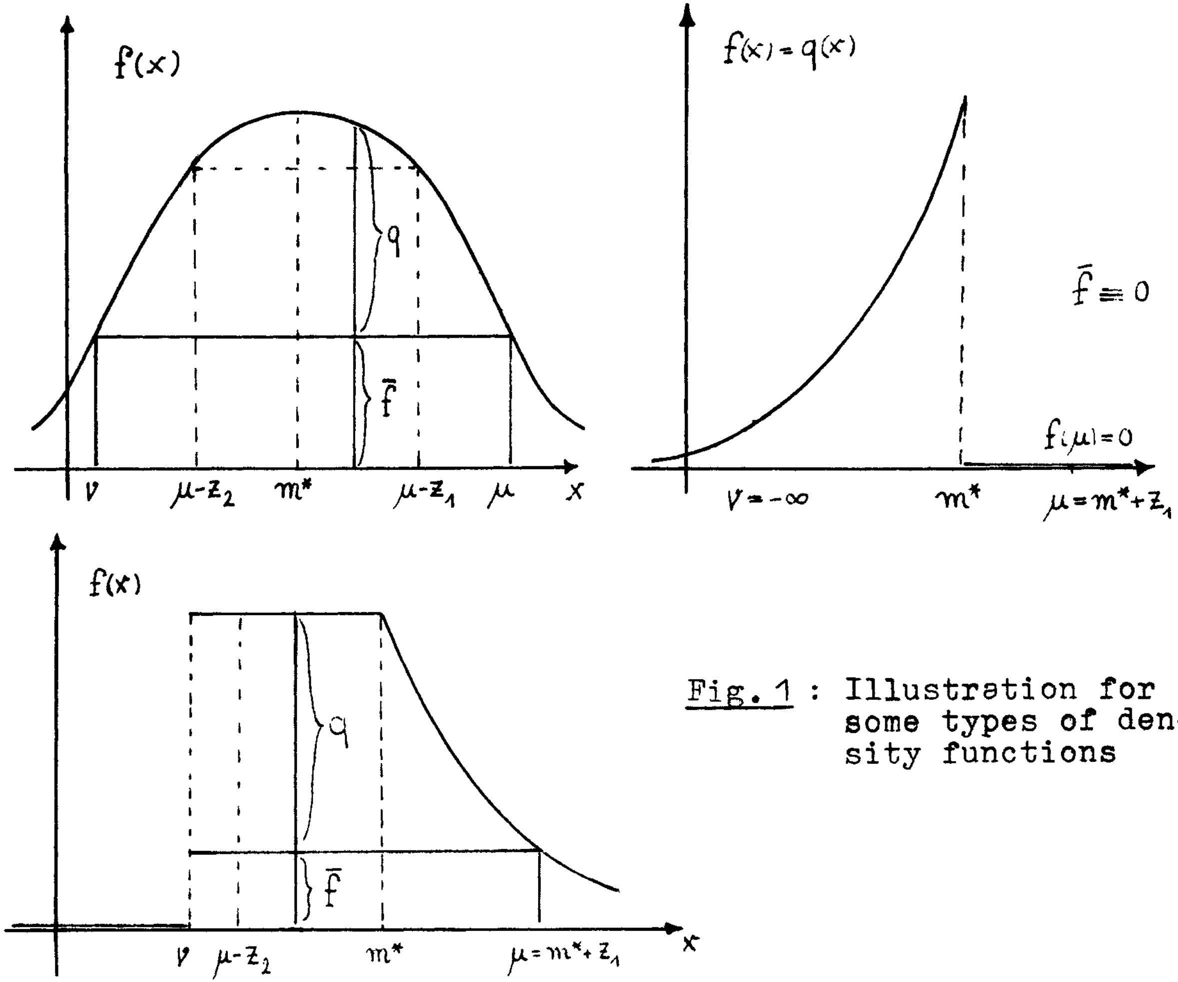


<u>Fig. 1</u> : Illustration for some types of density functions

is a density for $X - Y$. By the substitution $u = x + z$ in the integral we see the symmetry of $g$ relative to 0:

$$g(z) = g(-z) \tag{1}$$

and by the Cauchy-Schwarz-inequality:

$$g(z) \leqslant g(0) \quad \text{for all } z \tag{2}$$

since

$$\int_{-\infty}^{\infty} f(x) f(x+z)\,dx \leqslant \sqrt{\int_{-\infty}^{\infty} (f(x))^2\,dx}\,\sqrt{\int_{-\infty}^{\infty} (f(x+z))^2\,dx} =$$

$$= \int_{-\infty}^{\infty} (f(x))^2\,dx = g(0).$$

(1) and (2) hold for any density $f$.

*Lemma*: If $f$ is unimodal, then $g(-z_2) \leqslant g(-z_1)$ for $0 \leqslant z_1 \leqslant z_2$. The unimodality of $g$ follows then immediately from (1) and (2).

*Proof*: $g(-z_2) \leqslant g(-z_1)$ is equivalent to

$$\int_{-\infty}^{\infty} f(x)\,[f(x - z_1) - f(x - z_2)]\,dx \geqslant 0. \tag{3}$$

This inequality is trivial if an equality sign holds in $0 \leqslant z_1 \leqslant z_2$. For $0 < z_1 < z_2$ let $\mu = \inf\{x \mid f(x - z_1) < f(x - z_2)\}$; from the definition of unimodality it follows that

$$\mu \geqslant m^* + z_1, \quad \text{since } f(x - z_1) \geqslant f(x - z_2) \quad \text{for all } x \leqslant m^* + z_1,$$

and

$$\mu \leqslant m^* + z_2, \quad \text{since } f(x - z_1) < f(x - z_2) \quad \text{for } x = m^* + z_2,$$

$$\text{because } f(m^*) = a > f(x) \quad \text{for } x > m^*.$$

By the definition of $\mu$ it is clear that $f(x - z_1) \geqslant f(x - z_2)$ for $x < \mu$. A little proof is needed for the fact that we have

$$f(x - z_1) \leqslant f(x - z_2) \quad \text{for all } x > \mu. \tag{4}$$

If $x > \mu$, there is a $x'$, $\mu < x' < x$, with $f(x' - z_1) < f(x' - z_2)$ by the definition of

$\mu$. Since $m^* < x' - z_1 < x - z_1$ we have $f(x - z_1) \leqslant f(x' - z_1)$ and if $x - z_2 \leqslant m^*$ it follows that $f(x' - z_2) \leqslant f(x - z_2)$ and thus

$$f(x - z_1) \leqslant f(x' - z_1) < f(x' - z_2) \leqslant f(x - z_2).$$

If $x - z_2 > m^*$, then $f(x - z_1) \leqslant f(x - z_2)$ follows by the definition of $m^*$. In general there is a $\nu$, $-\infty < \nu \leqslant m^*$, $\nu = \inf \{x \mid x \leqslant m^*$ and $f(x) > f(\mu)$; if $f(\mu) = 0$ and $f(x) > 0$ for all $x \leqslant m^*$, then we put $\nu = -\infty$ (see Fig. 1 which gives a qualitative illustration for some types of density functions).

Hence if $x < \nu$, the inequality $f(x) \leqslant f(\mu)$ is true. We regard now the function

$$\bar{f}(x) = \min (f(x), f(\mu))$$

and put

$$f(x) = \bar{f}(x) + q(x);$$

$q(x) = 0$ outside of the interval $[\nu, \mu]$ and clearly $f(x_1) \leqslant f(x_2)$ implies $\bar{f}(x_1) \leqslant \bar{f}(x_2)$. For the integral in (3) we write now

$$\int_{-\infty}^{\infty} (\bar{f}(x) + q(x)) [\bar{f}(x - z_1) + q(x - z_1) - \bar{f}(x - z_2) - q(x - z_2)] \, dx =$$

$$= I_1 + I_2 + I_3$$

with $I_1 = \int_{-\infty}^{\infty} \bar{f}(x) [\bar{f}(x - z_1) - \bar{f}(x - z_2)] \, dx$, $I_2 = \int_{-\infty}^{\infty} q(x) [f(x - z_1) - - f(x - z_2)] \, dx$ and $I_3 = \int_{-\infty}^{\infty} \bar{f}(x) [q(x - z_1) - q(x - z_2)] \, dx$. $I_2$ and $I_3$ reduce, since $q(x) = 0$ outside of $[\nu, \mu]$, to

$$I_2 = \int_{\nu}^{\mu} q(x) [f(x - z_1) - f(x - z_2)] \, dx,$$

$$I_3 = \int_{\nu + z_1}^{\mu + z_2} \bar{f}(x) [q(x - z_1) - q(x - z_2)] \, dx.$$

We show that all three integrals $I_1$, $I_2$ and $I_3$ are nonnegative:
First we split $I_1$ up into

$$I_1 = \int_{-\infty}^{\mu} \bar{f}(x) [\bar{f}(x - z_1) - \bar{f}(x - z_2)] \, dx + \int_{\mu}^{\infty} \bar{f}(x) [\bar{f}(x - z_1) - \bar{f}(x - z_2)] \, dx.$$

In the former of the last two integrals we have always $\bar{f}(x) \geqslant \bar{f}(x - z_1)$, because in $[\nu, \mu]$ $\bar{f}(x) = f(\mu)$ and to the left of $[\nu, \mu]$ $\bar{f}(x) = f(x)$ is $\leqslant f(\mu)$ and increasing; the difference in the square brackets is nonnegative.

In the latter integral the difference in the square brackets is always $\leqslant 0$ since by (4) this inequality holds for $f$ instead of $\bar{f}$; but now $\bar{f}(x-z_1) \geqslant \bar{f}(x)$, since $x - z_1 \geqslant m^*$ for $x \geqslant \mu$ and because the same inequality holds then for $f$ instead of $\bar{f}$. Hence

$$I_1 \geqslant \int_{-\infty}^{\infty} \bar{f}(x-z_1)\,[\bar{f}(x-z_1)-\bar{f}(x-z_2)]\,dx =$$

$$= \int_{-\infty}^{\infty} (\bar{f}(x-z_1))^2\,dx - \int_{-\infty}^{\infty} \bar{f}(x-z_1)\bar{f}(x-z_2)\,dx$$

and this is $\geqslant 0$ which follows like (1) by means of the Cauchy-Schwarz-inequality.

$I_2 \geqslant 0$ is trivial because $f(x-z_1)-f(x-z_2) \geqslant 0$ and $q(x) \geqslant 0$ in $[\nu, \mu]$. In order to prove $I_3 \geqslant 0$ we remember $\bar{f}(x) = f(\mu) = \text{const.}$ in $[\nu, \mu]$ and decompose $I_3$ into

$$I_3 = \int_{\nu+z_1}^{\mu} f(\mu)\,[q(x-z_1)-q(x-z_2)]\,dx +$$

$$+ \int_{\mu}^{\mu+z_2} \bar{f}(x)\,[q(x-z_1)-q(x-z_2)]\,dx.$$

In the last integral $q(x-z_1)-q(x-z_2)$ is always $\leqslant 0$, since by (4) $f(x-z_1) \leqslant f(x-z_2)$ for $x > \mu$ i.e. $\bar{f}(x-z_1)+q(x-z_1) \leqslant \bar{f}(x-z_2)+$ $+q(x-z_2)$ and $\bar{f}(x-z_1) = f(\mu) \geqslant \bar{f}(x-z_2)$ wherever $q(x-z_1) > 0$. Further we know that $\bar{f}(x) \leqslant f(\mu)$, hence

$$I_3 \geqslant f(\mu)\,\{\int_{\nu+z_1}^{\mu} [q(x-z_1)-q(x-z_2)]\,dx +$$

$$+ \int_{\mu}^{\mu+z_2} [q(x-z_1)-q(x-z_2)]\,dx\}$$

$$= f(\mu)\,[\int_{\nu+z_1}^{\mu+z_2} q(x-z_1)\,dx - \int_{\nu+z_1}^{\mu+z_2} q(x-z_2)\,dx]$$

and this is 0 because both integrals in the great square brackets are equal to

$\int_{\nu}^{\mu} q(x)\,dx.$

This completes the proof of the Lemma. Our Lemma and the results (1) and (2) imply the following.

*Theorem*: If $X$, $Y$ are independent random variables each with the unimodal density $f$, then $X - Y$ has a unimodal density which is symmetrical to the modular value 0.

This theorem implies a corollary which might be useful in another context.

*Corollary*: If $f$ is an unimodal density, then for any real number $r$ the inequality

$$\int_{-\infty}^{r} f(x) f(x - z_1)\, dx \geqslant \int_{-\infty}^{r} f(x) f(x - z_2)\, dx \text{ holds for } 0 \leqslant z_1 \leqslant z_2.$$

*Proof*: The corollary holds trivially if $z_1 = z_2$ and it follows from the Cauchy-Schwarz-inequality if $0 = z_1$. For $0 < z_1 < z_2$ let $\mu$ be defined as before; then the inequality holds if $r < \mu$ because in this case $f(x - z_1) \geqslant f(x - z_2)$ for all $x \in (-\infty, r]$.

If $r \geqslant \mu$ we regard

$$\int_{-\infty}^{\infty} f(x) [f(x - z_1) - f(x - z_2)]\, dx =$$

$$= \int_{-\infty}^{r} f(x) [f(x - z_1) - f(x - z_2)]\, dx + \int_{r}^{\infty} f(x) [f(x - z_1) - f(x - z_2)]\, dx$$

which is $\geqslant 0$ by the theorem. Since $f(x - z_1) - f(x - z_2) \leqslant 0$ for $x > \mu$, the last integral is $\leqslant 0$. So the sum of the last two integrals couldn't be nonnegative if the former were negative. Hence this integral is $\geqslant 0$ for any real $r$ and this proves the corollary.

## Acknowledgement

I am indebted to Prof. K.L. Chung (Stanford, Cal.) who told me by a private communication a way to simplify the proof of the above theorem essentially. He uses also the truncation idea but assumes

$$g'(z) = \int_{-\infty}^{\infty} f(x) f'(x + z)\, dx.$$

This means a stronger assumption about $f$ than differentiability a.e. The latter is given by the monotonicity on both sides of $m^*$.

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