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ASYMPTOTIC PROPERTIES OF TESTS OF HYPOTHESIS FOLLOWING A PRELIMINARY TEST

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Abstract

For the general multivariate linear models, based on least squares and rank statistics, test of hypothesis on a subset of parameters following a preliminary test on the complementary subset is considered. Asymptotic theory of the tests and the effects of preliminary test on the final test are studied and performance characteristics of the various tests are compared.

1. Introduction

Consider the general linear model (1.1)

$$\underset{p\times n}{\underset{\sim}{Y}} = \underset{p\times q}{\underset{\sim}{\beta}} \; \underset{q\times n}{\underset{\sim}{C}_n} + \underset{p\times n}{\underset{\sim}{\varepsilon}_n}$$

where the columns of $\underset{\sim}{\varepsilon}_n$ are independent and identically distributed random vectors (i.i.d.r.v.) with a continuous (unknown) distribution function (d.f.) $F(x)$, $x \in R^p$, the p-dimensional real field, $\underset{\sim}{\beta}$ is an unknown matrix of parameters and $\underset{\sim}{C}_n = (\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_n)$ is a known matrix of regression constants. Consider the partition of the regression matrix $\underset{\sim}{\beta}$ as

$$\underset{p\times q}{\underset{\sim}{\beta}} = (\underset{p\times q_1}{\underset{\sim}{\beta}_1}, \underset{p\times q_2}{\underset{\sim}{\beta}_2}), \; q_i \geq 0 \quad i = 1, 2, \quad q_1 + q_2 = q \tag{1.2}$$

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We are primarily interested in the test of hypothesis $H_0^{(0)}: \beta_1 = 0$. A class of likelihood ratio (LRT) and rank order tests (RPT) of β_1 (when β_2 may or may not be known) have been studied by Puri and Sen [7]. The test-statistics for testing $H_0^{(0)}$ are different in the two situations according as β_2 is specified or not. Let $\bar{L}_n$ and $L_n^{(0)}$ be respectively the test-statistics for testing $H_0^{(0)}: \beta_1 = 0$, based on Y, when we assume $\beta_2 = 0$ and when β_2 is unspecified. Usually, $\bar{L}_n$ performs better than $L_n^{(0)}$ when $\beta_2 = 0$. On the other hand, if the condition $\beta_2 = 0$ is not true, $\bar{L}_n$ may become inefficient and even inconsistent. Hence, one may advocate the use of $\bar{L}_n$ unless one has nigh confidence in the assumption $\beta_2 \neq 0$. When β_2 is unknown but by extraneous sources it is suspected that β_2 is close to 0, often a preliminary test on β_2 is made; if $H_0^*: \beta_2 = 0$ is tenable $\bar{L}_n$ is used while if H_0^* is not tenable then $L_n^{(0)}$ is used. We denote a preliminary test statistic by L_n^*. The object of the present investigation is to study the asymptotic properties of the test-statistic defined by

$$L_n = \begin{cases} \bar{L}_n & \text{if} \quad L_n^* < L_{n,\alpha^*}^* \\ L_n^{(0)} & \text{if} \quad L_n^* \geq L_{n,\alpha^*}^* \end{cases} \tag{1.3}$$

where α^* is the level of significance (size) of the preliminary tests. In particular, the effect of preliminary tests on the size and power of the ultimate tests is the main objective of the study. In some special problems, the effect of preliminary test on the size and power of some ultimate tests has been studied by Bechhofer [2] and Bozivich, Bancroft and Hartley [3], among others. Some non-parametric procedures are due to Tamura [14] and Saleh and Sen [9] among others. Sen [10] studied some asymptotic properties of maximum likelihood estimators and likelihood ratio tests after a preliminary test. Saleh and Sen [9] studied some least-squares and rank order estimators after a preliminary test on $\beta_2 = 0$. These results are incorporated here in the main study. In this context, the joint distribution theory of correlated quadratic forms studied by Jensen [4] and Khatri, Krishnaiah and Sen [6] plays an important role.

Along with preliminary notions, the proposed procedure is outlined in Section 2. Section 3 deals with the asymptotic distribution theory of various statistics involved in the proposed testing procedures. These results are then incorporated in Section 4 in the study of the <u>asymptotic size</u> and <u>asymptotic power function</u> of the test based on L_n.

2. Test of Hypothesis After A Preliminary Test

For the model (1.1) we first consider the least-squares estimator (LSE's) of $\underset{\sim}{\beta}$ given by

$$\underset{\sim}{\tilde{\beta}}_{(L)} = \underset{\sim}{Y}\,\underset{\sim}{C}_n'\,(\underset{\sim}{C}_n\underset{\sim}{C}_n')^{-1} = \underset{\sim}{Y}\,\underset{\sim}{C}_n'\underset{\sim}{D}_n^{-1} = (\underset{\sim}{\tilde{\beta}}_{1(L)},\underset{\sim}{\tilde{\beta}}_{2(L)}) \quad , \tag{2.1}$$

where $\underset{\sim}{D}_n = \underset{\sim}{C}_n\underset{\sim}{C}_n'$ and for normal F, this is also the M.L.E. of $\underset{\sim}{\beta}$. Also if we assume that the F possesses a finite and positive finite (p.d.) dispersion matrix, $\underset{\sim}{\Sigma}$, then

$$\underset{\sim}{S}_e^* = (n-q)^{-1}(\underset{\sim}{Y}-\underset{\sim}{\tilde{\beta}}_{(L)}\underset{\sim}{C}_n)(\underset{\sim}{Y}-\underset{\sim}{\tilde{\beta}}_{(L)}\underset{\sim}{C}_n)' \tag{2.2}$$

unbiasedly estimates $\underset{\sim}{\Sigma}$, where q is the rank of $\underset{\sim}{D}_n$. Let us consider the null hypotheses and their alternatives

$$\text{(i)}\quad H_0^{(0)}:\ \underset{\sim}{\beta}_1 = \underset{\sim}{0}\quad (\underset{\sim}{\beta}_2\ \text{unknown})\quad \text{Vs}\quad H_A^{(0)}:\ \underset{\sim}{\beta}_1 \neq \underset{\sim}{0} \tag{2.3a}$$

$$\text{(ii)}\quad \overline{H}_0:\ \underset{\sim}{\beta}_1 = \underset{\sim}{0}\quad (\text{and}\ \underset{\sim}{\beta}_2 = \underset{\sim}{0})\quad \text{Vs}\quad \overline{H}_A:\ \underset{\sim}{\beta}_1 \neq \underset{\sim}{0},\ \underset{\sim}{\beta}_2 = \underset{\sim}{0} \tag{2.3b}$$

$$\text{(iii)}\quad H_0^*:\ \underset{\sim}{\beta}_2 = \underset{\sim}{0}\quad \text{Vs}\quad H_A^{(*)}:\ \underset{\sim}{\beta}_2 \neq \underset{\sim}{0}\quad . \tag{2.3c}$$

Thus, under the null hypothesis $H_0^{(*)}$, the model (1.1) reduces to

$$\underset{\sim}{Y} = \underset{\sim}{\beta}_1\,\underset{\sim}{C}_{n1} + \underset{\sim}{\varepsilon}_n \quad \text{where} \quad \underset{\sim}{C}_n = (\underset{\sim}{C}_{n1},\underset{\sim}{C}_{n2}) \tag{2.4}$$

The LSE of $\underset{\sim}{\beta}_1$ under H_0^* is given by

$$\underset{\sim}{\hat{\beta}}_{1(L)} = \underset{\sim}{Y}\,\underset{\sim}{C}_{n1}'(\underset{\sim}{C}_{n1}\underset{\sim}{C}_{n1}')^{-1} \tag{2.5}$$

Further, the normal-theory likelihood ratio tests for H_0^* is given by

$$L_n^* = \{|\underset{\sim}{S}_e|/|\underset{\sim}{S}_e+\underset{\sim}{S}_{H^*}|\}^{n/2} \tag{2.6}$$

where $\underset{\sim}{S}_e = (n-q)\underset{\sim}{S}_e^*$ and $\underset{\sim}{S}_{H^*} = \underset{\sim}{\tilde{\beta}}_{2(L)}\,\underset{\sim}{D}_{n,22.1}\,\underset{\sim}{\tilde{\beta}}_{2(L)}'$, and

$$\underset{\sim}{D}_{n,22.1} = \underset{\sim}{D}_{n,22} - \underset{\sim}{D}_{n,21}\,\underset{\sim}{D}_{n,11}^{-1}\,\underset{\sim}{D}_{n,12} \tag{2.7}$$

with

$$D_n = \begin{pmatrix} D_{n,11} & D_{n,12} \\ D_{n,21} & D_{n,22} \end{pmatrix} = (C_n C_n') = \begin{pmatrix} C_{n1}C_{n1}' & C_{n1}C_{n2}' \\ C_{n2}C_{n1}' & C_{n2}C_{n2}' \end{pmatrix} \tag{2.8}$$

Similarly, the likelihood ratio test for $H_0^{(0)}$ is given by

$$L_n^{(0)} = \{|S_e|/|S_e + S_H^{(0)}|\}^{n/2} \tag{2.9}$$

where $S_H^{(0)} = \tilde{\beta}_{1(L)} D_{n,11.2} \tilde{\beta}_{1(L)}'$ and the likelihood ratio test for $\overline{H}_0$ is

$$\overline{L}_n = \{|S_e^{(1)}|/|S_e^{(1)} + S_{\overline{H}}|\}^{n/2} \tag{2.10}$$

where $S_e^{(1)} = (n-q)S_e^{*(1)}$, $S_e^{*(1)} = (n-q)^{-1}(Y-\hat{\beta}_{1(L)}C_{n1})(Y-\hat{\beta}_{1(L)}C_{n1})'$ and $S_{\overline{H}} = \hat{\beta}_{1(L)}(C_{n1}C_{n1}')\hat{\beta}_{1(L)}'$. By Theorem 8.6.2 of Anderson [1], it is known that for normal F, $-2 \log L_n^{(0)}$, $-2 \log \overline{L}_n$ and $-2 \log L_n^*$ converges in distribution to central chi-square distribution under $H^{(0)}$, $\overline{H}_0$ and H_0^* with pq_1, pq_1 and pq_2 degrees of freedom respectively. In what follows, we will see that the same holds when F is not normal but Σ is a finite positive definite matrix. Then, we finally formulate the test function $\nu_n^{(L)}$ corresponding to (1.3) as follows: Let $\overline{\alpha}(0<\overline{\alpha}<1)$ and $\alpha^0(0<\alpha^0<1)$ be positive numbers and α^* is defined as in (1.3). Then, we take

$$\nu_n^{(L)} = \begin{cases} 1 & \text{if } (-2 \log L_n^* < \ell_{n,\alpha^*}, \ -2 \log \overline{L}_n \geq \overline{\ell}_{n,\overline{\alpha}}) \\ & \text{or } (-2 \log L_n^* \geq \ell_{n,\alpha^*}^*, \ -2 \log L_n^{(0)} \geq \ell_{n,\alpha^0}^0 \\ 0, & \text{otherwise} \end{cases} \tag{2.11}$$

where $\overline{\ell}_{n,\overline{\alpha}}$, ℓ_{n,α^*}^* and ℓ_{n,α^0}^0 are respectively the upper $100\,\overline{\alpha}\%$, $100\,\alpha^*\%$, and $100\,\alpha^0\%$ points of the null distributions of $-2 \log \overline{L}_n$, $-2 \log L_n^*$ and $-2 \log L_n^0$. The size of the test of (1.3) and (2.11) depends on β_2, and hence, we may define it as

$$\alpha_n^{(L)} = E\{\nu_n^{(L)}|H_0^{(0)}\} \tag{2.12}$$

and its power (also dependent on both β_1, β_2) is given by

$$\pi_n^{(L)}(\beta) = E\{\nu_n^{(L)}|H_A^{(0)}\}, \quad (\beta_1, \beta_2) \in \Omega, \tag{2.13}$$

where $H_A^{(0)}$: $\beta_1 \neq 0$ (β_2 nuisance).

Now, we consider the rank order statistics for the hypothesis $H_0^{(0)}$, $\bar{H}_0$ and H_0^*. Let $u(t)$ be equal to 1 or 0 according as t is ≥ 0 or <0 and let $R_{ji} = \sum_{s=1}^{n} u(Y_{ji}-Y_{js})$ be the rank of Y_{ji} among $Y_{j1},\ldots,Y_{jn}$. Since F is continuous, ties among the observations may be neglected in probability. For each $j(= 1,\ldots,p)$ and $n(\geq 1)$, we consider a set of real-valued scores $a_n^{(j)}(1),\ldots,a_n^{(j)}(n)$ generated by the score-generating function $\phi(u)$, $0<u<1$, in either of the following way

$$a_n^{(j)}(i) = \phi_j(\frac{i}{n+1}) \quad \text{or} \quad E\phi_j(U_{ni}) \ , \quad i = 1,\ldots,n \ , \tag{2.14}$$

where $\phi_j(u)$ is assumed to be absolutely continuous, non-decreasing and square-integrable inside $(0,1)$ and $U_{n1}<\ldots<U_{nn}$ are the ordered random variables of a sample size n from a rectangular distribution over $(0,1)$. Let us consider the hypothesis H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$, the test procedure is based on the following type of rank statistics:

$$\underset{\sim}{S}_n = ((S_{n,jk})) = ((\underset{\sim}{S}_n^{(1)},\underset{\sim}{S}_n^{(2)}));\ S_{n,jk} = \sum_{i=1}^{n} c_{ki} a_n^{(j)}(R_{ni}) \tag{2.15}$$

for $j = 1,\ldots,p$; $k = 1,2,\ldots,q$; and $\underset{\sim}{S}_n^{(r)}$ is of the order $p\times q_r$, $r = 1,2$. Let $\underset{\sim}{B} = ((b_{jk}))$ be a $p\times q$ matrix, and write $\underset{\sim}{\beta}' = (\underset{\sim}{b}_1',\ldots,\underset{\sim}{b}_p')$ where $\underset{\sim}{b}_j = (b_{j1},\ldots,b_{jq})$ for $j = 1,\ldots,p$. Let $\underset{\sim}{c}_i = (c_{1i},\ldots,c_{qi})'$, $i\geq 1$ be vectors of constants. Let then (for each $i(= 1,\ldots,n)$),

$$\underset{\sim}{Y}_i(\underset{\sim}{B}) = \underset{\sim}{Y}_i-\underset{\sim}{B}\underset{\sim}{c}_i = (Y_{1i}(\underset{\sim}{b}_1),\ldots,Y_{pi}(\underset{\sim}{b}_p)) \tag{2.16}$$

$$R_{ji}(\underset{\sim}{B}) = R_{ji}(\underset{\sim}{b}_j) = \sum_{s=1}^{n} u(Y_{ji}(b_j)-Y_{js}(b_j)) \qquad j = 1,\ldots,p \tag{2.17}$$

so that $R_{ji}(\underset{\sim}{B})$ is the rank of $Y_{ji}(\underset{\sim}{b}_j)$ among $Y_{js}(\underset{\sim}{b}_j)$, $s = 1,\ldots,n$ for $j = 1,\ldots,p$. Now, replace the R_{ji} in (2.15) by $R_{ji}(\underset{\sim}{b}_j)$ for $i = 1,\ldots,n$ and denote the corresponding matrix of rank order statistics by

$$\underset{\sim}{S}_n(\underset{\sim}{B}) = ((S_{n,jk}(\underset{\sim}{b}_j)))_{j = 1,\ldots,p;\ k = 1,\ldots,q} \tag{2.18}$$

Note that $\underset{\sim}{S}_n(\underset{\sim}{B})$ in (2.18) is viewed as a function of pq elements in $\underset{\sim}{B}$ and generates a pq-dimensional stochastic process in R^{pq}.

As in (1.2) we introduce the partition:

$$\underset{\sim}{B} = (\underset{\sim}{B}_1,\underset{\sim}{B}_2) \ , \quad \text{so that} \quad \underset{\sim}{c}_i' = [\underset{\sim}{c}_{i(1)}',\underset{\sim}{c}_{i(2)}'] \quad i = 1,\ldots,n \tag{2.19}$$

Then, under the null hypothesis H_0^*: $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$, we have the estimate $\hat{\underset{\sim}{\beta}}_1$ for the model (2.4) via rank order matrix of statistics:

$$\underset{\sim}{S}_n^{(1)}(\underset{\sim}{B}_1,\underset{\sim}{0}) = ((S_{n,jk}(\underset{\sim}{b}_j^{(1)}))) \quad j = 1,\ldots,p; \quad k = 1,\ldots,q_1 \tag{2.20}$$

where

$$b_j^{(1)'} = (b_{j1},\ldots,b_{jq_1},0,\ldots,0) \;, \quad j = 1,\ldots,p \quad . \tag{2.21}$$

Now define

$$A_{n(1)} = \{\underset{\sim}{\beta}_1 : \; \Sigma_{j=1}^{p}\Sigma_{k=1}^{q_1}|S_{n,jk}(b_j^{(1)})| = \text{minimum}\} \quad , \tag{2.22}$$

Then, the estimate of β_1 defined as in Sen and Puri [13] and Jureckova [5] is given by

$$\hat{\underset{\sim}{\beta}}_{1(R)} = \text{centre of gravity of } A_{n(1)} \quad . \tag{2.23}$$

The estimate $\hat{\underset{\sim}{\beta}}_{1(R)}$ is translation-invariant, robust and consistent when (1.4) holds and is also asymptotically normal.

To estimate $\underset{\sim}{\beta}$, define

$$A_n = \{\beta : \; \Sigma_{j=1}^{p}\Sigma_{k=1}^{q} \; |S_{n,jk}(b)| = \text{minimum}\} \tag{2.24}$$

Then, the estimate of $\underset{\sim}{\beta}$ under model (1.1) is given by

$$\tilde{\underset{\sim}{\beta}}_{(R)} = \text{centre of gravity of } A_n \quad . \tag{2.25}$$

Under model (1.1) $\tilde{\underset{\sim}{\beta}}_{(R)}$ is translation-invariant robust, consistent and asymptotically normally distributed estimator of $\underset{\sim}{\beta}$. Thus, the estimator of $(\underset{\sim}{\beta}_1,\underset{\sim}{\beta}_2)$ is $(\tilde{\underset{\sim}{\beta}}_1,\tilde{\underset{\sim}{\beta}}_2)$, where the subscript (R) has been dropped for convenience.

To estimate $\underset{\sim}{\beta}_2$, define

$$A_{n(2)} = \{\underset{\sim}{\beta}_2 : \; \Sigma_{j=1}^{p}\Sigma_{k=q+1}^{q} \; |S_{njk}(\underset{\sim}{b}_j^{(2)})| = \text{minimum}\} \tag{2.26}$$

where

$$b_j^{(2)} = (0,\ldots,0,b_{jq_1+1},\ldots,b_{jq})' \tag{2.27}$$

and

$$\underset{\sim}{S}_n^{(2)}(\underset{\sim}{0},\underset{\sim}{B}_2) = ((S_{njk}(\underset{\sim}{b}_j^{(2)}))) \quad j = 1,\ldots,p; \quad k = q_1+1,\ldots,q \tag{2.28}$$

Then,

$$\hat{\underset{\sim}{\beta}}_2 = \text{centre of gravity of } A_{n(2)} \quad . \tag{2.29}$$

$\hat{\underset{\sim}{\beta}}_2$ is also translation-invariant robust, consistent and asymptotically normally distributed estimator of $\underset{\sim}{\beta}_2$ [under $H_0^{(0)}$].

To define the test statistics for the null hypothesis $H_0^{(0)}$, $\overline{H}_0$ and H_0^*, we consider the alligned rank statistics:

$$\hat{S}_{n(1)}^{(1)} = ((\hat{S}_{njk}^{(1)})) \quad j = 1,\ldots,p; \quad k = q_1+1,\ldots,q \tag{2.30}$$

where $$\hat{S}^{(1)}_{njk} = \sum_{i=1}^{n} c_{ki} a_n^{(j)}(\hat{R}^{(1)}_{ni}) \tag{2.31}$$

$$\hat{R}^{(1)}_{ji} = R_{ji}(0,\hat{\beta}_2), \quad j = 1,\dots,p; \quad i = 1,2,\dots,n \quad . \tag{2.32}$$

Similarly, define

$$S_n^{(2)} = ((S^{(2)}_{njk})) \quad j = 1,\dots,p; \quad k = 1,\dots,q_1 \quad , \tag{2.33}$$

where $$\hat{S}^{(2)}_{njk} = \sum_{i=1}^{n} C_{ki} a_n^{(j)}(\hat{R}^{(2)}_{ni}) \quad , \tag{2.34}$$

$$\hat{R}^{(2)}_{ji} = \hat{R}_{ji}(\hat{\beta}_1, \underset{\sim}{0}); \quad j = 1,\dots,p; \quad i = 1,\dots,n \tag{2.35}$$

Now, we introduce the proposed statistics by just defining

$$\underset{\sim}{M}_n = ((m_{jj',n})) \quad j,j' = 1,\dots,p = \begin{pmatrix} \underset{\sim}{M}_{n11} & \underset{\sim}{M}_{n12} \\ \underset{\sim}{M}_{n21} & \underset{\sim}{M}_{n22} \end{pmatrix}; \tag{2.36}$$

$$m_{jj',n} = (n-1)^{-1}\{\Sigma a_n^{(j)}(R_{ji}) a^{(j')}_{n(R_{j'i})} - n\bar{a}_n^{(j)} \cdot \bar{a}_n^{(j')}\} \tag{2.37}$$

where $\bar{a}_n^{(j)} = n^{-1}\Sigma_{i=1}^n a^{(j)}_{n(i)}$. Also replacing R_{ji} by $\hat{R}^{(1)}_{ji}$ and by $\hat{R}^{(2)}_{ji}$ we obtain the corresponding $\underset{\sim}{M}_n$-matrices $\hat{\underset{\sim}{M}}^{(1)}_n$ and $\hat{\underset{\sim}{M}}^{(2)}_n$ respectively. Let then $\underset{\sim}{D}_{n,22.1}$ be defined by (2.7) and

$$\underset{\sim}{D}_{n11.2} = \underset{\sim}{D}_{n11} - \underset{\sim}{D}_{n12}\underset{\sim}{D}_{n22}\underset{\sim}{D}_{n21} \tag{2.38}$$

$$\hat{\underset{\sim}{G}}^{(1)}_n = \hat{\underset{\sim}{M}}^{(1)}_n \otimes \underset{\sim}{D}_{n11.2} \quad \text{and} \quad \hat{\underset{\sim}{G}}^{(2)} = \hat{\underset{\sim}{M}}^{(2)}_n \otimes \underset{\sim}{D}_{n22.1} \tag{2.39}$$

$$\bar{\underset{\sim}{G}}_n = \underset{\sim}{M}_{n11} \otimes \underset{\sim}{D}_{n11} \tag{2.40}$$

$$\hat{\underset{\sim}{H}}^{(1)}_n = ((\hat{S}^{(1)}_{nkj}\hat{S}^{(1)}_{nk'j'})) \quad j,j' = 1,\dots,p; \quad k,k' = 1,\dots,q_1 \tag{2.41}$$

$$\hat{\underset{\sim}{H}}^{(2)}_n = ((\hat{S}^{(2)}_{njk}\hat{S}^{(2)}_{nj'k'})) \quad j,j' = 1,\dots,p; \quad k,k' = q_1+1,\dots,q \tag{2.42}$$

$$\bar{\underset{\sim}{H}}_n = ((S_{nkj}S_{nk'j'})) \quad j,j = 1,\dots,p; \quad k,k' = 1,\dots,q_1 \quad , \tag{2.43}$$

where $\hat{H}^{(1)}_n$ and $\hat{H}^{(2)}_n$ are of order $pq_1 \times pq_1$ and $pq_2 \times pq_2$ respectively. Then the proposed test-statistics are

$$L_n^{(0)} = \mathrm{Tr}(\underset{\sim}{H}_n^{(1)}(\hat{\underset{\sim}{G}}_n^{(1)})^{-1}) \ , \quad L_n^* = \mathrm{Tr}(\underset{\sim}{H}_n^{(2)}(\hat{\underset{\sim}{G}}_n^{(2)})^{-1}) \tag{2.44}$$

$$\overline{L}_n = \mathrm{Tr}(\overline{\underset{\sim}{H}}_n(\overline{\underset{\sim}{G}}_n)^{-1}) \quad . \tag{2.45}$$

Finally, we define the test function as follows:

$$\nu_n^{(R)} = \begin{cases} 1 \ \text{ if } \ (L_n^* < L_{n,\alpha^*}^* \ , \ \ \overline{L}_n \geq \overline{L}_{n,\overline{\alpha}}) \ \text{ or} \\ \qquad (L_n^* \geq L_{n,\alpha^*} \ , \ \ L_n^{(0)} \geq L_{n,\alpha^0}^{(0)}) \\ 0, \text{ otherwise.} \end{cases} \tag{2.46}$$

Our objective is to study the asymptotic size and power of $\nu_n^{(R)}$, which may be written as in (2.12) and (2.13) respectively.

3. Asymptotic Distribution Theory of $-2 \log L_n^0$, $-2 \log L_n^*$ and $-2 \log \overline{L}_n$.

To study the asymptotic properties of $E(\nu_{n_0}^{(R)})$ we need to study first, the asymptotic joint distributions of $(-2 \log L_n^0, -2 \log L_n^*)$ and $(-2 \log \overline{L}_n, -2 \log L_n^*)$ when the null hypotheses $H_0^{(0)}$, $\overline{H}_0$ and H_n^* may or may not hold. First we consider the likelihood ratio type tests based on least squares estimators of $\underset{\sim}{\beta}$.

We make the following assumptions

$$\text{(i)} \ \max_{1 \leq i \leq n} (\underset{\sim}{c}_i' \underset{\sim}{D}_n^{-1} \underset{\sim}{c}_i) \to 0 \quad \text{as} \quad n \to \infty \tag{3.0}$$

(ii) $\underset{\sim}{D}_n^{-1}$ exists and $(\lim n^{-1}\underset{\sim}{D}_n)^{-1} = \underset{\sim}{\Delta}$ is finite and p.d. Similarly $n^{-1}\underset{\sim}{D}_n$ converges to $\underset{\sim}{D}$ which is p.d. and finite.

(iii) $\underset{\sim}{\Sigma}$ is positive definite and $0<|\underset{\sim}{\Sigma}|<\infty$. Then, under (i) - (iii) and under $H_0^{(0)}$ or H_0^*, $\underset{\sim}{S}_e \to \underset{\sim}{\Sigma}$ in probability as $n \to \infty$ and under $\overline{H}_0$, $\underset{\sim}{S}_e^{(1)} \to \underset{\sim}{\Sigma}$ in probability as $n \to \infty$. Also, as $n \to \infty$, (1) $-2 \log L_n^{(0)} + \mathrm{Tr}(\underset{\sim}{S}_{H^{(0)}} \underset{\sim}{S}_e^{-1}) \to 0$ in probability, (2) $-2 \log \overline{L}_n + \mathrm{Tr}(\underset{\sim}{S}_{\overline{H}_0} \underset{\sim}{S}_e^{(1)^{-1}}) \to 0$ in probability, and (3) $-2 \log L_n^* + \mathrm{Tr}(\underset{\sim}{S}_{H_0^*} \underset{\sim}{S}_e^{-1}) \to 0$ in probability as $n \to \infty$. Note that by (1.1) and (2.1)

$$\sqrt{n}(\tilde{\underset{\sim}{\beta}}_{(L)} - \underset{\sim}{\beta}) \sim \eta_{pq}(\underset{\sim}{0}, \underset{\sim}{\Sigma} \otimes \underset{\sim}{\Delta}) \tag{3.1}$$

where $\otimes$ stands for Kronecker product. Similarly, under H_0^*,

$$\sqrt{n}(\hat{\underset{\sim}{\beta}}_{1(L)} - \underset{\sim}{\beta}_1) \sim \eta_{pq_1}(\underset{\sim}{0}, \underset{\sim}{\Sigma} \otimes \underset{\sim}{D}_{11}^{-1}) \tag{3.2}$$

Thus, we have

$$-2\log L_n^{(0)} \xrightarrow{\mathcal{D}} \chi^2_{pq_1} \quad (\text{under } H_0^{(0)}) , \quad -2\log \bar{L}_n \xrightarrow{\mathcal{D}} \chi^2_{pq_1} \quad (\text{under } \bar{H}_0) \tag{3.3}$$

and $$-2\log L_n^* \xrightarrow{\mathcal{D}} \chi^2_{pq_2} \quad (\text{under } H_0^*)$$

Further, $-2\log \bar{L}_n$ and $-2\log L_n^*$ are asymptotically independent under $\bar{H}_0$. Then, we have

$$\ell^0_{n,0} \to \chi^2_{pq_1,\alpha 0} , \quad \bar{\ell}_{n\bar{\alpha}} \to \chi^2_{pq_1,\bar{\alpha}} , \quad \text{and} \quad \ell^*_{n\alpha^*} \to \chi^2_{pq_2,\alpha^*} , \tag{3.4}$$

where $\chi^2_{t,\beta}$ is the upper $100\beta\%$ point of the chi square distribution with t DF. The random variables $-2\log L_n^{(0)}$ and $-2\log L_n^*$ are not independent but they are jointly correlated quadratic forms with marginal central chi square distribution with pq_1 and pq_2 degrees of freedoms respectively. We consider the case where $\bar{H}_0$ holds, then using results of Jensen [4] and Khatri, Krishnaiah and Sen [6] we consider the expression to be used in the joint distribution of correlated chi square variables,

$$\phi(\underset{\sim}{u}:\underset{\sim}{b}) = \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{m!\,\Gamma\frac{1}{2}} \sum_{\underset{\sim}{0} \le \underset{\sim}{\alpha} \le \underset{\sim}{m}} a_{\underset{\sim}{\alpha}} \frac{a_1!\,a_2!\,\Gamma b_1\,\Gamma b_2}{\Gamma(b_1+\alpha_1)\,\Gamma(b_2+\alpha_2)} \psi(\underset{\sim}{u}:\underset{\sim}{b}) L_{\underset{\sim}{\alpha}}(\underset{\sim}{u}:\underset{\sim}{b}) \tag{3.5}$$

where

$$\left.\begin{array}{ll} \underset{\sim}{u} = (u_1,u_2) \ge 0 & \underset{\sim}{\alpha} = (\alpha_1,\alpha_2) \\ \underset{\sim}{b} = (b_1,b_2) \ge 0 & \underset{\sim}{m} = (m_1,m_2) \end{array}\right\} \tag{3.6}$$

$$\psi(u:b) = \prod_{i=1}^{2} \{e^{-u_i} u_i^{b_i - 1} / \Gamma b_i\} , \quad \underset{\sim}{0} \le \underset{\sim}{u} \le \underset{\sim}{\infty}, \ \underset{\sim}{b} > \underset{\sim}{0} \tag{3.7}$$

The Laguerre ploynomials $L_{\underset{\sim}{\alpha}}$ are defined by

$$\alpha_1!\,\alpha_2!\,\psi(\underset{\sim}{u};\underset{\sim}{b}) L_{\underset{\sim}{\alpha}}(\underset{\sim}{u}:\underset{\sim}{b}) = \left(-\frac{d}{d\underset{\sim}{u}}\right)^{\underset{\sim}{\alpha}} [u_1^{\alpha_1} u_2^{\alpha_2} \ (\underset{\sim}{u};\underset{\sim}{b})] \tag{3.8}$$

$\underset{\sim}{\alpha} \ge \underset{\sim}{0}$ and $a_{\underset{\sim}{\alpha}}$ are suitable constants.

Then, by Theorem 2 of Jensen [4] we conlude that under $\bar{H}_0$

$$\lim_{n\to\infty} P\{-2\log L_n^{(0)} \ge \ell^{(0)}_{n,\alpha(0)}, \ -2\log L_n^* \ge \ell^*_{n,\alpha^*}\}$$

$$= \int_{\chi^2_{pq_1,\alpha^0}}^{\infty} \int_{\chi^2_{pq_1,\alpha^*}}^{\infty} \phi(u_1,u_2;\ \tfrac{1}{2}pq_1,\tfrac{1}{2}pq_2)\,du_1 du_2 \tag{3.9}$$

where $\phi(u_1,u_2;\ \frac{r}{2},\frac{s}{2})$ is defined by (3.4). Therefore, the asymptotic size of the test is given by

$$\lim_{n\to\infty} E\{\nu_n^{(L)} | H_0^{(0)}\} = \bar{\alpha}(1-\alpha^*) + \int_{\chi^2_{pq_1,\alpha^0}}^{\infty} \int_{\chi^2_{pq_2,\alpha^*}}^{\infty} \phi(u;\ \tfrac{1}{2}pq_1,\tfrac{1}{2}pq_2)\,du_1 du_2 \tag{3.10}$$

In the above development we confined ourselves to the case where appropriate null hypothesis holds. We consider now the joint distribution of $(-2\log L_n^{(0)}, -2\log L_n^*)$ and $(-2\log \bar{L}_n, -2\log L_n^*)$ under local alternatives defined by

$$K_n:\ \underset{\sim}{\beta}_1 = n^{-\frac{1}{2}}\underset{\sim}{\gamma}_1,\ \ \underset{\sim}{\beta}_2 = n^{-\frac{1}{2}}\underset{\sim}{\gamma}_2 \tag{3.11}$$

where $\underset{\sim}{\gamma}_1$ and $\underset{\sim}{\gamma}_2$ are pq_1 and pq_2 matrices of constants. Under this alternative, the statistics $-2\log \bar{L}_n$, $-2\log L_n^*$ remain independent but $-2\log L_n^{(0)}$ and $-2\log L_n^*$ remain correlated as non-central chi square variables. For this we consider the asymptotically <u>equivalent</u> statistics

$$\mathrm{Tr}(\Sigma^{-1}\tilde{\underset{\sim}{\beta}}_{1(L)}\underset{\sim}{D}_{11.2}\tilde{\underset{\sim}{\beta}}{}'_{1(L)}) \text{ and } \mathrm{Tr}(\Sigma^{-1}\tilde{\underset{\sim}{\beta}}_{2(L)}D_{22.1}\tilde{\beta}'_{2(L)}) \tag{3.12}$$

which may be expressed as

$$\tilde{\underset{\sim}{\beta}}_{1(L)}\underset{\sim}{D}_{11.2}\tilde{\underset{\sim}{\beta}}{}'_{1(L)} = \underset{\sim}{Z}^{(1)}\underset{\sim}{Z}^{(1)'} \text{ and } \tilde{\underset{\sim}{\beta}}_{2(L)}\underset{\sim}{D}_{22.1}\tilde{\underset{\sim}{\beta}}{}'_{2(L)} = \underset{\sim}{Z}^{(2)}\underset{\sim}{Z}^{(2)'} \tag{3.13}$$

where $\underset{\sim}{Z}^{(1)}$ and $\underset{\sim}{Z}^{(2)}$ may be obtained by non-singular transformations of $(\tilde{\underset{\sim}{\beta}}_1,\tilde{\underset{\sim}{\beta}}_2)$ using $\underset{\sim}{Z}^{(i)} = \tilde{\underset{\sim}{\beta}}_i\underset{\sim}{E}_i$, $i=1,2$, so that

$$\underset{\sim}{E}'_1\underset{\sim}{D}_{22.1}\underset{\sim}{E}_1 = \underset{\sim}{I}_{q_1},\ \ \underset{\sim}{E}'_2\underset{\sim}{D}_{11.2}\underset{\sim}{E}_2 = \underset{\sim}{I}_{q_2}\ . \tag{3.14}$$

Thus, define the vector $\underset{\sim}{\xi} = (\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2)'$ as the mean vector of the transformed variables. Then

$$\underset{\sim}{\xi}_i = \underset{\sim}{\beta}_i\underset{\sim}{E}_i;\ \ (i = 1,2) \tag{3.15}$$

and the asymptotic joint distribution of $(\tilde{\underset{\sim}{\beta}}_1\underset{\sim}{E}_1,\tilde{\underset{\sim}{\beta}}_2\underset{\sim}{E}_2)'$ is

$$\eta_{pq}\left\{\begin{pmatrix}\xi_1\\ \\ \xi_2\end{pmatrix},\quad \Sigma \otimes \begin{pmatrix} I_{q_1} & E_1'D_{12}E_2 \\ \\ E_2'D_{21}E_1 & I_{q_2}\end{pmatrix}\right\} \tag{3.16}$$

Define

$$\Sigma^* = \Sigma \otimes \begin{pmatrix} I_{q_1} & E_1'D_{12}E_2 \\ \\ E_2'D_{21}E_1 & I_{q_2}\end{pmatrix} \tag{3.17}$$

Choose

$$\Omega = \Sigma \otimes \begin{pmatrix} \delta_1 I_{q_1} & 0 \\ \\ 0 & \delta_2 I_{q_2}\end{pmatrix} \tag{3.18}$$

where $(\delta_1,\delta_2) = \delta > 0$ and

$$R = I_{pq} - \Omega^{-1}\Sigma^* \tag{3.19}$$

$$= I_{pq} - I_p \otimes \begin{pmatrix} \delta_1 I_{q_1} & 0 \\ \\ 0 & \delta_2 I_{q_2}\end{pmatrix}^{-1} \begin{pmatrix} I_{q_1} & E_1'D_{12}E_2 \\ \\ E_2'D_{21}E_1 & I_{q_2}\end{pmatrix}$$

$$= I_p \otimes \left[I_q - \begin{pmatrix} \delta_1 I_{q_1} & 0 \\ \\ 0 & \delta_2 I_{q_2}\end{pmatrix}^{-1} \begin{pmatrix} I_{q_1} & E_1'D_{12}E_2' \\ \\ E_2'D_{21}E_1 & I_{q_2}\end{pmatrix}\right] \tag{3.20}$$

and $\beta = \Omega^{-1}\xi^0\xi^0$, where ξ^0 is pq-rolled vector corresponding to ξ in (3.15). The choice of δ is arbitrary and the convergence rate of the expansion to follow depends on the choice of δ and define

$$b_0 = |I_{pq}-R|^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}Tr[I_{pq}-R)^{-1}B]\} \tag{3.21}$$

and for every $k>0$

$$\phi_i(w,k) = \{(2\delta_i)^k\,\Gamma k\}^{-1} w^{k-1} e^{-w/2\delta_i}\,, \quad i=1,2 \tag{3.22}$$

Finally, let

$$\phi^*(w_1,w_2) = b_0 \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} b_{j_1} b_{j_2} \phi_1(w_1; \tfrac{1}{2}pq_1+j_1)\phi_2(w_2,\tfrac{1}{2}pq_2+j_2) \quad (3.23)$$

where the b_{j_k}'s depend on $\underset{\sim}{\Omega}$ and $\underset{\sim}{R}$ and some formulae to compute b_{j_1} and b_{j_2} are given in Khatri, Krishnaiah and Sen [6]. Then we define by $H_t(x,\delta)$ the noncentral chi square distribution function with t DF and non-centrality parameter δ, and obtain

$$E(\nu_n^{(L)}|H_A^{(0)}) = H_{pq_2}(\chi^2_{pq_2,\alpha^*};\ \theta^*_{(L)})\{1-H_{pq_1}(\chi^2_{pq_1,\overline{\alpha}};\ \overline{\theta}_{(L)})\}$$
$$+ \int_{\chi^2_{pq_1,\alpha^0}}^{\infty} \int_{\chi^2_{pq_2,\alpha^*}}^{\infty} \psi^*(w_1,w_2)dw_1dw_2 \quad (3.24)$$

where

$$\theta^*_{(L)} = \mathrm{Tr}[\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\gamma}_2\underset{\sim}{\Delta}_{22}\underset{\sim}{\gamma}_2')] \quad (3.25)$$

$$\overline{\theta}_{(L)} = \mathrm{Tr}[\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\gamma}_1-\underset{\sim}{\gamma}_2\underset{\sim}{\Delta}_{22}^{-1}\underset{\sim}{\Delta}_{21})\underset{\sim}{\Delta}_{11}(\underset{\sim}{\gamma}_1-\underset{\sim}{\gamma}_2\underset{\sim}{\Delta}_{22}^{-1}\underset{\sim}{\Delta}_{21})'] \quad (3.26)$$

$$\theta^{(0)}_{(L)} = \mathrm{Tr}[\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\gamma}_1\underset{\sim}{\Delta}_{11.2}\underset{\sim}{\gamma}_1')]\ ;\ \underset{\sim}{\Delta}_{11.2} = \underset{\sim}{\Delta}_{11}-\underset{\sim}{\Delta}_{12}\underset{\sim}{\Delta}_{22}^{-1}\underset{\sim}{\Delta}_{21} \quad . \quad (3.27)$$

4. Asymptotic Distribution Theory of $(\overline{L}_n$, L_n^0 and $L_n^*)$

In this section, we study the asymptotic joint distribution of $(\overline{L}_n,L_n^*)$ and (L_n^0,L_n^*) which will in turn determine the asymptotic size and power of the rank test $\nu_n^{(R)}$ defined in (2.46). In this respect, if $\underset{\sim}{\beta}_2 \neq \underset{\sim}{0}$, by consistency of the preliminary test based on L_n^*, $\nu_n^{(R)}$ will be equivalent (in probability) to the test based on $L_n^{(0)}$. Hence, the asymptotic distribution of $L_n^{(R)}$, the test statistic corresponding to $\nu_n^{(R)}$ will be chi square with pq_1 degrees of freedom. On the other hand if $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$ or near $\underset{\sim}{0}$, the asymptotic distribution of $L_n^{(R)}$ will depend on all the three statistics, namely, $\overline{L}_n$, L_n^0 and L_n^* as we have seen in the case of least squares procedures. For this, we consider the following assumptions:

(i) $\phi(u)$ is non-decreasing and square integrable inside $(0,1)$ and

(ii) $\int_0^1 |\phi(u)|\{u(1-u)\}^{-\frac{1}{2}}du < \infty$. (4.1)

(iii) $\lim n^{-1}\underset{\sim}{D}_n = \underset{\sim}{D}$ exists and is p.d. (4.2)

(iv) For every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ such that for $n > n_0$

$$n^{-1}D_{nkk} > \varepsilon\{\max_{1\le i\le n}|c_{ki}^2|\} \quad \text{for} \quad k = 1,\ldots,q \tag{4.3}$$

(v) We assume that for each j, the marginal df $F_{[j]}$ of F has an absolutely continuous pdf $f_{[j]}$ with finite Fisher information

$$I_j = I(f_{[j]}) = \int_{-\infty}^{\infty} [f'_{[j]}(x)/f_{[j]}(x)]^2 f_{[j]}(x)dx\ , \quad j = 1,\ldots,p \tag{4.4}$$

Denote by $\bar{\phi}_j = \int_0^1 \phi_j(u)du$ and let

$$\lambda_{jj'}(F) = \int_{-\infty}^{\infty} \phi_j(F_{[j]}(x))\phi_{j'}(F_{[j']}(y))dF_{[j,j']}(x,y) - \bar{\phi}_j\bar{\phi}_j' \tag{4.5}$$

where $F_{[jj']}(x,y)$ is the marginal bivariate df of the (j,j')th components, for $j,j' = 1,\ldots,p$.

(vi) $\underset{\sim}{\Lambda}(F) = ((\lambda_{jj'}(F)))$ is p.d. and finite. (4.6)

Then, under H_0^*: $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$, we have

$$F_i(\underset{\sim}{x}) = F(\underset{\sim}{x}-\underset{\sim}{\beta}_1\underset{\sim}{c}_{i(1)})\ , \quad i = 1,2,\ldots,n \tag{4.7}$$

and the estimate of $\underset{\sim}{\beta}_1$ is available using (2.22). Further, under H_0^*, $\underset{\sim}{S}_{n(1)}(\underset{\sim}{\beta}_1,\underset{\sim}{0})$ defined by (2.20) has the same distribution as $\underset{\sim}{S}_n^{(1)}(\underset{\sim}{0},\underset{\sim}{0})$ under $\bar{H}_0$: $\underset{\sim}{\beta} = \underset{\sim}{0}$; for the later case we can use the results of Puri and Sen [7] and obtain the following: (a) Under $H_0^{(2)*}$, $\underset{\sim}{S}_n^{(1)}(\underset{\sim}{\beta}_1,0)$ has mean $\underset{\sim}{0}$ and dispersion matrix $\underset{\sim}{\Lambda}(F) \otimes \underset{\sim}{D}_{n11}$ and as $n\to\infty$

$$n^{-\frac{1}{2}}\underset{\sim}{S}_n^{(1)}(\underset{\sim}{\beta}_1, 0) \xrightarrow{\mathcal{D}} \eta_{pq_1}(\underset{\sim}{0},\ \underset{\sim}{\Lambda}(F) \otimes \underset{\sim}{D}_{11}) \tag{4.8}$$

Similarly, under $H_0^{(1)*}$: $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$, we have

$$F_i(\underset{\sim}{x}) = F(\underset{\sim}{x}-\underset{\sim}{\beta}_2\underset{\sim}{c}_{i(2)})\ , \quad i = 1,\ldots,n \tag{4.9}$$

and the estimate of $\underset{\sim}{\beta}_2$ is available using (2.26). Further, under $H_0^{(1)*}$, $\underset{\sim}{S}_n^{(2)}(\underset{\sim}{0},\underset{\sim}{\beta}_2)$ defined by (2.28) has the same distribution as $\underset{\sim}{S}_n^{(2)}(\underset{\sim}{0},\underset{\sim}{0})$ under $\bar{H}_0$: $\underset{\sim}{\beta} = \underset{\sim}{0}$ and under $H_0^{(1)*}$, $\underset{\sim}{S}_n^{(2)}(\underset{\sim}{0},\underset{\sim}{\beta}_2)$ has mean $\underset{\sim}{0}$ and dispersion matrix $\underset{\sim}{\Lambda}(F) \otimes \underset{\sim}{D}_{n,22}$ and

$$n^{-\frac{1}{2}}\underset{\sim}{S}_n^{(2)}(\underset{\sim}{0},\underset{\sim}{\beta}_2) \to \eta_{pq_1}(\underset{\sim}{0},\ \underset{\sim}{\Lambda}(F) \otimes \underset{\sim}{D}_{22})\quad . \tag{4.10}$$

By arguments parallel to those of Jurečková [5]

$$\sup_{\underset{\sim}{\beta}_1 \in A_{n(1)}} \|\underset{\sim}{\beta}_1 - \hat{\underset{\sim}{\beta}}_{1(R)}| \xrightarrow{P} 0 \quad \text{as} \quad n\to\infty \tag{4.11}$$

and under H_0^*,

$$n^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_{1(R)} - \underset{\sim}{\beta}_1) \xrightarrow{\mathcal{D}} \eta_{pq_1}(0, \ \underset{\sim}{T}(F) \otimes D_{11}^{-1}) \tag{4.12}$$

where

$$\underset{\sim}{T}(F) = ((\lambda_{jj'}(F)/A_j A_{j'})) \tag{4.13}$$

and

$$A_j = \int_{-\infty}^{\infty} \frac{d}{dx} \phi_j(F_{[j]}(x)) dF_{[j]}(x) \ , \quad j = 1,\ldots,p \quad . \tag{4.14}$$

Similarly, consider the estimator of $\underset{\sim}{\beta}$ using (2.25), it may be shown that following Jurečková [5] that

$$\sup_{\underset{\sim}{\beta} \in A_n} |\underset{\sim}{\beta} - \underset{\sim}{\beta}_{(R)}| \xrightarrow{P} 0 \quad \text{as} \quad n\to\infty \tag{4.15}$$

$$n^{\frac{1}{2}}(\tilde{\underset{\sim}{\beta}}_{(R)} - \underset{\sim}{\beta}) \overset{\mathcal{D}}{\sim} \eta_{pq}(\underset{\sim}{0}, \ \underset{\sim}{T}(F) \otimes \underset{\sim}{D}^{-1})) \quad , \tag{4.16}$$

Further, define

$$\underset{\sim}{A} = \text{diag } (A_1,\ldots,A_p) \tag{4.17}$$

Since preliminary-test tests of $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$ are of interest when $\underset{\sim}{\beta}_2$ is suspected to be close to $\underset{\sim}{0}$, we confine ourselves to local alternatives $\{K_n\}$ defined by (3.11).

Further, under H_0^*: $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$ and the assumptions (3.26) ((i) - (vi))

$$\text{(a)} \ n^{-\frac{1}{2}}\{\hat{\underset{\sim}{S}}_n^{(2)} - \underset{\sim}{S}_n^{(1)}(\underset{\sim}{\beta}_1, 0) + \underset{\sim}{A}(\hat{\underset{\sim}{\beta}}_{1(R)} - \underset{\sim}{\beta}_1)\underset{\sim}{D}_{n12}\} \xrightarrow{P} \underset{\sim}{0} \tag{4.18}$$

$$\text{(b)} \ n^{-\frac{1}{2}}\{\underset{\sim}{S}_{n(1)}(\underset{\sim}{\beta}_1, \underset{\sim}{0}) - \underset{\sim}{A}(\underset{\sim}{\beta}_{1(R)} - \underset{\sim}{\beta}_1)\underset{\sim}{D}_{n11}\} \xrightarrow{P} \underset{\sim}{0} \tag{4.19}$$

$$\text{(c)} \ n^{-\frac{1}{2}}\{\hat{\underset{\sim}{S}}_n^{(2)} - \underset{\sim}{S}_n^{(2)}(\underset{\sim}{\beta}_1, \underset{\sim}{0}) + \underset{\sim}{S}_n^{(1)}(\underset{\sim}{\beta}_1, \underset{\sim}{0})\underset{\sim}{D}_{n11}^{-1}\underset{\sim}{D}_{n12}\} \xrightarrow{P} \underset{\sim}{0} \tag{4.20}$$

by lemma 3.2 - 3.4 of Sen and Puri [14]. Similarly under $H_0^{(0)}\underset{\sim}{\beta}_1 = \underset{\sim}{0}$, as $n\to\infty$, we have,

$$\text{(d)} \ n^{-\frac{1}{2}}\{\underset{\sim}{S}_n^{(1)} - \underset{\sim}{S}_n^{(2)}(\underset{\sim}{0}, \underset{\sim}{\beta}_2) + \underset{\sim}{A}(\hat{\underset{\sim}{\beta}}_{2(R)} - \underset{\sim}{\beta}_2)\underset{\sim}{D}_{n21}\} \xrightarrow{P} \underset{\sim}{0} \tag{4.21}$$

$$\text{(e)} \ n^{-\frac{1}{2}}\{\hat{\underset{\sim}{S}}_n^{(2)}(\underset{\sim}{0}, \underset{\sim}{\beta}_2) - \underset{\sim}{A}(\hat{\underset{\sim}{\beta}}_{2(R)} - \underset{\sim}{\beta}_2)\underset{\sim}{D}_{n22}\} \xrightarrow{P} \underset{\sim}{0} \tag{4.22}$$

$$\text{(f)}\quad n^{-\frac{1}{2}}\{\hat{\underset{\sim}{S}}_n^{(1)} - \underset{\sim}{S}_{n(1)}(\underset{\sim}{0},\underset{\sim}{\beta}_2) + \underset{\sim}{S}_n^{(2)}(\underset{\sim}{0},\underset{\sim}{\beta}_2)\underset{\sim}{D}_{n22}^{-1}\underset{\sim}{D}_{n21}\} \xrightarrow{p} \underset{\sim}{0} \quad . \tag{4.23}$$

Note that under H_0^*: $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$, $n^{-\frac{1}{2}}(\underset{\sim}{S}_n^{(1)}(\underset{\sim}{\beta}_1,\underset{\sim}{0}),\ \underset{\sim}{S}_n^{(2)}(\underset{\sim}{\beta}_1,\underset{\sim}{0}))$ have the same distribution as of $n^{-\frac{1}{2}}\underset{\sim}{S}_n$ under H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ and since the later is asymptotically multinormal with $\underset{\sim}{0}$ mean and dispersion matrix $\underset{\sim}{\Lambda}(F) \otimes \underset{\sim}{D}$, it follows that under H_0^*: $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$ we have

$$L\{n^{-\frac{1}{2}}\underset{\sim}{S}_n^{(2)}(\underset{\sim}{\beta}_1,\underset{\sim}{0}) - \underset{\sim}{S}_n^{(1)}(\underset{\sim}{\beta}_1,\underset{\sim}{0})\underset{\sim}{D}_{n11}^{-1}\underset{\sim}{D}_{n12}\} \to \eta_{pq_2}(\underset{\sim}{0},\underset{\sim}{\Lambda}\ (F) \otimes \underset{\sim}{D}_{22.1}) \tag{4.24}$$

Similarly, under H_0^0: $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$,

$$L\{n^{-\frac{1}{2}}\underset{\sim}{S}_n^{(1)}(0,\beta_2) - \underset{\sim}{S}_n^{(2)}(\underset{\sim}{0},\underset{\sim}{\beta}_2)\underset{\sim}{D}_{22}^{-1}\underset{\sim}{D}_{21}\} \to \eta_{pq_1}(\underset{\sim}{0},\underset{\sim}{\Lambda}\ (\underset{\sim}{F}) \otimes \underset{\sim}{D}_{11.2}) \tag{4.25}$$

Hence, we obtain under H_0^* as $n\to\infty$

$$L\{n^{-\frac{1}{2}}\hat{\underset{\sim}{S}}_n^{(2)}\} \to \eta_{pq_2}(\underset{\sim}{0},\underset{\sim}{\Lambda}\ (F) \otimes \underset{\sim}{D}_{22.1}) \tag{4.26}$$

and under $H_0^{(0)}$: $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$

$$L\{n^{-\frac{1}{2}}\hat{\underset{\sim}{S}}_n^{(1)}\} \to \eta_{pq_1}(\underset{\sim}{0},\underset{\sim}{\Lambda}\ (F) \otimes \underset{\sim}{D}_{11.2}) \tag{4.27}$$

Also, from Sen and Puri [14] we obtain that

(i) under $\overline{H}_0^*$: $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$ and the assumptions of section 3,

$$n\hat{\underset{\sim}{G}}_n^{(1)^{-1}} \xrightarrow{p} \underset{\sim}{\Lambda}^{-1}(F) \otimes \underset{\sim}{D}_{11.2}^{-1} \tag{4.28}$$

(ii) Under $H_{(0)}^*$: $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$ and same assumptions

$$n\hat{\underset{\sim}{G}}_n^{(2)^{-1}} \xrightarrow{p} \underset{\sim}{\Lambda}^{-1}(F) \otimes \underset{\sim}{D}_{22.1}^{-1} \tag{4.29}$$

(iii) Under $\overline{H}_0$: $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$ and same assumptions,

$$n\,\overline{G}_n^{-1} \xrightarrow{p} \underset{\sim}{\Lambda}^{-1}(F) \otimes \underset{\sim}{D}_{11}^{-1} \tag{4.30}$$

Thus, it follows from (4.26) and (4.27) and the asymptotic theory of quadratic forms associated with asymptotically multinormal vectors that (a) under H_0^*: $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$, $L_n^* \xrightarrow{D} \chi^2_{pq_2}$; (b) under $\overline{H}_0$: $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$, $\overline{L}_n \xrightarrow{D} \chi^2_{pq_1}$ and (c) under $H_0^{(0)}$: $\underset{\sim}{\beta}_1 = \underset{\sim}{0}$, $L_n^0 \xrightarrow{D} \chi^2_{pq_1}$, where χ^2_m denotes central chi square with m DF. Also, the joint distribution of $(\overline{L}_n, L_n^*)$ are asymptotically independent central chi square with pq_1 and pq_2 DF respectively, whereas the joint distribution of (L_n^0, L_n^*) is correlated chi square with pq_1 and pq_2 DF respectively. Therefore, using the results of section 3, we have

$$\lim E_{H_0^{(0)}}(\nu_n^{(R)}) = \lim P\{\overline{L}_n \geq \overline{L}_{n,\overline{\alpha}};\ L_n^* < L_{n,\alpha^*}\} + \lim P\{L_n^0 \geq L_{n,\alpha^0}^0, L_n^* \geq L_{n,\alpha^*}\}$$

$$= \overline{\alpha}(1-\alpha^*) + \int_{\chi^2_{pq_1,\alpha^0}}^{\infty} \int_{\chi^2_{pq_2,\alpha^*}}^{\infty} \phi\{(u_1,u_2),\ \tfrac{1}{2}(pq_1,pq_2)\} du_1 du_2 \qquad (4.31)$$

In order to find the joint distribution of $(\overline{L}_n, L_n^*)$ and (L_n^0, L_n^*) under $\{K_n\}$ we note that

$$\text{(i)}\ \operatorname{Tr}[\hat{\underset{\sim}{H}}_n^{(1)}(\hat{\underset{\sim}{G}}^{(1)})^{-1}] - \operatorname{Tr}[(\hat{\underset{\sim}{H}}_n^{(1)}(\underset{\sim}{\Lambda}(F) \otimes \underset{\sim}{D}_{11.2})^{-1}] \xrightarrow{P} 0, \qquad (4.32)$$

$$\text{(ii)}\ \operatorname{Tr}[\hat{\underset{\sim}{H}}_n^{(2)}(\hat{\underset{\sim}{G}}^{(2)})^{-1}] - \operatorname{Tr}[\hat{H}_n^{(2)}(\underset{\sim}{\Lambda}(F) \otimes D_{22.1})^{-1}] \xrightarrow{P} 0, \qquad (4.33)$$

$$\text{(iii)}\ \operatorname{Tr}[\overline{\underset{\sim}{H}}_n(\overline{G}_n)^{-1}] - \operatorname{Tr}[\overline{H}_n(\underset{\sim}{\Lambda}(F) \otimes \underset{\sim}{D})^{-1}] \xrightarrow{P} 0 \qquad (4.34)$$

as $n\to\infty$. Hence we consider the equivalent statistics given in (4.32), (4.33), and (4.34) instead of L_n^0, L_n^*, and $\overline{L}_n$ and make non-singular linear transformations $E_i (i = 1,2)$ such that

$$\underset{\sim}{E}_1'\underset{\sim}{D}_{11.2}\underset{\sim}{E}_1 = \underset{\sim}{I}_{q_1} \quad \text{and} \quad \underset{\sim}{E}_2'\underset{\sim}{D}_{22.1}\underset{\sim}{E}_2 = \underset{\sim}{I}_{q_2} \qquad (4.35)$$

so that

$$\operatorname{Tr}[\hat{\underset{\sim}{H}}_n^{(1)}(\underset{\sim}{\Lambda}(F) \otimes \underset{\sim}{D}_{11.2})^{-1}] = Z_n^{(1)}\underset{\sim}{\Lambda}^{-1}(F)\underset{\sim}{Z}_n^{(1)'} \qquad (4.36)$$

$$\operatorname{Tr}[\hat{\underset{\sim}{H}}_n^{(2)}(\underset{\sim}{\Lambda}(F) \otimes \underset{\sim}{D}_{22.1})^{-1}] = Z_n^{(2)}\underset{\sim}{\Lambda}^{-1}(F)\underset{\sim}{Z}_n^{(2)'} \qquad (4.37)$$

where $\underset{\sim}{Z}_n^{(1)}$ and $\underset{\sim}{Z}_n^{(2)}$ may be obtained by non-singular transformations

$$\underset{\sim}{Z}_n^{(1)} = \underset{\sim}{S}_n^{(1)}\underset{\sim}{E}_1\ ,\quad \underset{\sim}{Z}_n^{(2)} = \underset{\sim}{S}_n^{(2)}\underset{\sim}{E}_2 \qquad (4.38)$$

with centering values $\underset{\sim}{\xi} = (\underset{\sim}{\xi}^{(1)}, \underset{\sim}{\xi}^{(2)})$ of $(\underset{\sim}{Z}_n^{(1)}, \underset{\sim}{Z}_n^{(2)})$. Thus, as $n\to\infty$, the joint distrubution of $(\underset{\sim}{Z}_n^{(1)}, \underset{\sim}{Z}_n^{(2)})'$ is

$$\eta_{pq}\{(\underset{\sim}{\xi}^{(1)}, \underset{\sim}{\xi}^{(2)})',\ \underset{\sim}{\Lambda}(F) \otimes \begin{pmatrix} \underset{\sim}{I}_{q_1} & \underset{\sim}{E}_1'\underset{\sim}{D}_{12}\underset{\sim}{E}_2 \\ \underset{\sim}{E}_2'\underset{\sim}{D}_{21}\underset{\sim}{E}_1 & \underset{\sim}{I}_{q_2} \end{pmatrix} \qquad (4.39)$$

Define

$$\underset{\sim}{\Sigma}^* = \underset{\sim}{\Lambda}(F) \otimes \begin{pmatrix} \underset{\sim}{I}_{q_1} & \underset{\sim}{E}_1'\underset{\sim}{D}_{12}\underset{\sim}{E}_2 \\ \underset{\sim}{E}_2'\underset{\sim}{D}_{21}\underset{\sim}{E}_1 & \underset{\sim}{I}_{q_2} \end{pmatrix} \tag{4.40}$$

and choose

$$\underset{\sim}{\Omega} = \underset{\sim}{\Lambda}(F) \otimes \text{Diag}(\delta_1\underset{\sim}{I}_{q_1}, \delta_2\underset{\sim}{I}_{q_2}) \tag{4.41}$$

where $\underset{\sim}{\delta} = (\delta_1,\delta_2) \geq \underset{\sim}{0}$ and

$$\underset{\sim}{R} = \underset{\sim}{I}_{pq} - \underset{\sim}{\Omega}^{-1}\underset{\sim}{\Sigma}^* \quad , \quad \underset{\sim}{B} = \underset{\sim}{\Omega}^{-1}\underset{\sim}{\xi}^0\underset{\sim}{\xi}^{0'} \tag{4.42}$$

and $\underset{\sim}{\xi}^0$ is a rolled-out vector corresponding to $\underset{\sim}{\xi}$. Define

$$b_0 = |\underset{\sim}{I}_{pq} - \underset{\sim}{R}|^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\text{tr}(\underset{\sim}{I}-\underset{\sim}{R})^{-1}\underset{\sim}{B}\} \tag{4.43}$$

Finally, following the definition of functions as in (3.4), (3.20) and (3.21) and Saleh and Sen [9], we obtain

$$E(\nu_n^{(R)}|K_n) = H_{pq_2}(\chi^2_{pq_2,\alpha^*};\ \theta)\ 1-H_{pq_1}(\chi^2_{pq_1,\overline{\alpha}};\ \overline{\theta})\}$$
$$+ \int_{\chi^2_{pq_1,\alpha^0}}^{\infty} \int_{\chi^2_{pq_2,\alpha^*}}^{\infty} \phi^*(w_1,w_2)\,dw_1dw_2 \tag{4.44}$$

where the non-centrality parameters of the chi square variables are given by

$$\theta^*_{(R)} = \text{Tr}[\underset{\sim}{T}^{-1}(F)(\underset{\sim}{\gamma}_2\underset{\sim}{\gamma}_{22}\underset{\sim}{\gamma}_2')] \tag{4.45}$$

$$\overline{\theta}_{(R)} = \text{Tr}[\underset{\sim}{T}^{-1}(F)(\underset{\sim}{\gamma}_1-\underset{\sim}{\gamma}_2\underset{\sim}{\Delta}_{22}^{-1}\Delta_{21})\underset{\sim}{\Delta}_{11}(\underset{\sim}{\gamma}_1-\underset{\sim}{\gamma}_2\underset{\sim}{\Delta}_{22}^{-1}\Delta_{21})] \tag{4.46}$$

$$\theta^0_{(R)} = \text{Tr}[\underset{\sim}{T}^{-1}(F)(\underset{\sim}{\gamma}_1\underset{\sim}{\Delta}_{11.2}\underset{\sim}{\gamma}_1')] \quad , \tag{4.47}$$

and $\underset{\sim}{T}(F) = \underset{\sim}{A}^{-1}\underset{\sim}{\Lambda}(F)\underset{\sim}{A}^{-1}$.

5. Asymptotic Performance of the Three Tests in LSPTT and ROPTT

In this section, we study the comparative performance of the least squares preliminary test tests (LSPTT) and the rank order preliminary test tests (ROPTT) described in Section 3 and 4 respectively. First, we consider the three tests related to LSPTT defined by

$$\nu^0_{n,\alpha^0} = \begin{cases} 1 & \text{if } -2\log L_n^{(0)} \geq \ell^0_{\alpha^0} \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

$$\bar{\nu}_{n,\bar{\alpha}} = \begin{cases} 1 & \text{if } -2\log \bar{L}_n \geq \bar{\ell}_{n,\bar{\alpha}} \\ 0, & \text{otherwise} \end{cases} \tag{5.2}$$

and

$$\nu^*_{n,\alpha^*} = \begin{cases} 1 & \text{if } -2\log L_n^* \geq \ell^*_{n,\alpha^*} \\ 0, & \text{otherwise} \end{cases} \tag{5.3}$$

By virtue of the regularity conditions in Section 3 (see (i) - (iii)), as $n\to\infty$, the critical values ℓ^0_{n,α^0}, $\bar{\ell}_{n,\bar{\alpha}}$ and ℓ^*_{n,α^*} converges to $\chi^2_{pq_1,\alpha^0}$, $\chi^2_{pq_1,\bar{\alpha}}$ and $\chi^2_{pq_2,\alpha^*}$ respectively as given in (3.4). Further, note that for any fixed $\alpha^*(0<\alpha^*<1)$ and under H_A^*

$$E\left\{\nu^*_{n,\alpha^*}\middle| H_A^*\right\} \to 1 \tag{5.4}$$

so that by (2.12) and (5.3) under H_A^*, $\nu_n^{(L)}$ is asymptotically equivalent to ν^0_{n,α^0} for every fixed α^*. The picture is different when $\underset{\sim}{\beta}_2$ is near $\underset{\sim}{0}$. This is the case, where we want to study the behaviour of ν^0_{n,α^0}, $\bar{\nu}_{n,\bar{\alpha}}$ and $\nu^{(L)}_{n,\alpha}$. Towards this end, we consider the sequences of alternatives $\{K_n\}$ defined by (3.11) so that $H_0^{(0)}$, $\bar{H}_0$ and H_0^* are characterized by $\underset{\sim}{\gamma}_1 = \underset{\sim}{0}$, $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$, and $\underset{\sim}{\gamma}_1 = \underset{\sim}{0}$, $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$ respectively.

Note that by (5.1) we have

$$E\{\nu^{(0)}_{n,\alpha^0} \mid H_0^{(0)}\} = \alpha^0 \quad \text{whatever be } \underset{\sim}{\gamma}_2 \tag{5.5}$$

Also note that for $\underset{\sim}{\gamma}_1 \neq 0$, the asymptotic distribution of $-2\log \bar{L}_n$ is non-central chi square with pq_1 degrees of freedom with non-centrality parameter

$$\bar{\theta}_{(L)} = \mathrm{Tr}[\Sigma_{11}^{-1}(\underset{\sim}{\gamma}_1-\underset{\sim}{\gamma}_2\underset{\sim}{\Delta}_{22}^{-1}\underset{\sim}{\Delta}_{21})\underset{\sim}{\Delta}_{11}(\underset{\sim}{\gamma}_1-\underset{\sim}{\gamma}_2\underset{\sim}{\Delta}_{22}^{-1}\underset{\sim}{\Delta}_{21})'] \ (\geq 0) \tag{5.6}$$

Recall that $\underset{\sim}{\Delta}_{11}^{-1} = \underset{\sim}{D}_{11} - \underset{\sim}{D}_{12}\underset{\sim}{D}_{22}^{-1}\underset{\sim}{D}_{21}$ and $-\underset{\sim}{\Delta}_{22}^{-1}\underset{\sim}{\Delta}_{21} = \underset{\sim}{D}_{21}\underset{\sim}{D}_{11}^{-1}$. Thus for $\underset{\sim}{\gamma}_1 = \underset{\sim}{0}$, $\bar{\theta}_{(L)} \geq 0$, where equality sign holds for $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$.

Hence, by (5.6) and (5.2)

$$E\{\bar{\nu}_{n,\bar{\alpha}} \mid \underset{\sim}{\gamma}_1 = \underset{\sim}{0}\} \to P\{\chi^2_{pq_1}(\bar{\theta}_{(L)}) \geq \chi^2_{pq_1,\bar{\alpha}}\} \quad (\geq \bar{\alpha}) \tag{5.7}$$

where the equality holds when $\bar{\theta}_{(L)} = 0$ (i.e. $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$). This explains the lack of roubstness of the $\bar{\nu}_n$-test based on $-2 \log \bar{L}_n$. Under $\bar{H}_0$ i.e. ($\underset{\sim}{\gamma}_1 = \underset{\sim}{0}$, $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$), $\bar{\theta}_{(L)} = 0$ and the size of the test is $\bar{\alpha}$. But, under H_0: $\underset{\sim}{\gamma}_1 = \underset{\sim}{0}$ when $\underset{\sim}{\gamma}_2$ is not specified, this may be $\geq \bar{\alpha}$. Thus, unless $\bar{\theta}(L)$ defined by (5.6) is very close to 0, the use of $-2 \log \bar{L}_n$ may result in a significance level greater than the specified level $\bar{\alpha}$. But, $E\{\bar{\nu}_{n,\bar{\alpha}} | \underset{\sim}{\gamma}_1 = \underset{\sim}{0}\}$ tends to 1 for the set of $\underset{\sim}{\gamma}_2$ leading to large values of $\bar{\theta}_{(L)}$. This is disquieting so that the test $\bar{\nu}_n$ may even be inconsistent against such alternatives.

Now, consider the size of the test $\nu_n^{(L)}$ given by (3.10) which is bounded above by

$$\bar{\alpha}(1-\alpha^*) + \alpha^* \wedge \alpha^0 \quad . \tag{5.8}$$

This may be equated to a desired level α. Actually both $\bar{\alpha}$ and α^0 are chosen very close to (but less than) α or α^*, then the upper bound provides a close approximation of the series (3.5) by few terms.

We now consider the size when $\underset{\sim}{\gamma}_1 = \underset{\sim}{0}$ but $\underset{\sim}{\gamma}_2$ may not be $\underset{\sim}{0}$. Then, the limiting size of the best $\nu_n^{(L)}$ may be expressed as

$$E\{\nu_n^{(L)} | \underset{\sim}{\gamma}_1 = \underset{\sim}{0}\} \doteq \{1-H_{pq_1}(\chi^2_{pq_1,\bar{\alpha}}; \bar{\theta}_{(L)})\}H_{pq_2}(\chi^2_{pq_2,\alpha^*}; \theta^*_{(L)})$$

$$+ b_0 \Sigma_{j_1 \geq 0} \Sigma_{j_2 \geq 0} b_{j_1} b_{j_2} [1-\phi_1(\chi^2_{pq_1,\bar{\alpha}}; \tfrac{1}{2}pq_1+j_1)]$$

$$[1-\phi_2(\chi^2_{pq_2,\alpha^*}; \tfrac{1}{2}pq_2+j_2)] \tag{5.9}$$

where $\bar{\theta}_{(L)}$ is defined by (3.26) is ≥ 0 (with equaltiy sign when $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$). Further, b_0, b_{j_1}, b_{j_2} as well as ϕ_1 and ϕ_2 are defined by (3.21) and (3.22). In this context we note that

$$(1-\alpha^* \geq) H_{pq_2}(\chi^2_{pq_2,\alpha^*}; \theta^*_{(L)}) \text{ is } \searrow \text{ in } \theta^*_{(L)} \ (\geq 0) \tag{5.10}$$

and

$$(\bar{\alpha} \leq) \{1-H_{pq_1}(\chi^2_{pq_1,\bar{\alpha}}; \bar{\theta}_{(L)}) \text{ is } \nearrow \text{ in } \bar{\theta}_{(L)} \ (\geq 0) \tag{5.11}$$

and the second term is bounded by

$$\alpha^0 \wedge \{1-H_{pq_2}(\chi^2_{pq_2,\alpha^*}; \theta^*_{(L)})\} \tag{5.12}$$

Thus, unlike (5.7), though (5.9) is affected by $\underset{\sim}{\gamma}_2 \neq \underset{\sim}{0}$ it may not converge to 1 as $\bar{\theta}_{(L)}$ or $\theta^*_{(L)}$ blows up. In other words, it is more robust against $\underset{\sim}{\gamma}_2 \neq \underset{\sim}{0}$ than the test with $-2 \log \bar{L}_n$. Hence, from the consideration of validity robustness, LSPTT, $\nu_n^{(L)}$ may be preferred to $\bar{\nu}_{n,\alpha}$.

If, in particular $\underset{\sim}{D}_{12} = \underset{\sim}{0}$, then (5.9) reduces to

$$\alpha^0 + H_{pq_2}(\chi^2_{pq_2,\alpha^*};\ \theta^*_{(L)})\{1-\alpha^0-H_{pq_1}(\chi^2_{pq_1,\bar{\alpha}};\ \bar{\theta}_{(L)})\} \tag{5.13}$$

and robustness picture becomes more clear. In this case, we have

$$\alpha = \alpha^0 + (1-\alpha)(\bar{\alpha}-\alpha^0) \tag{5.14}$$

so that letting $\alpha^0 = \bar{\alpha} = \alpha$, one may choose α^* arbitrarily.

Now, we study the asymptotic power of the three tests ν_n^0, $\bar{\nu}_n$, and ν_n^*. As in (5.4) for fixed alternatives, there is not much interest in studying these, as the limits degenerate at α^0 or 1, ($\bar{\alpha}$ or 1) and (α^* or 1) respectively. Hence, we confine ourselves to local alternatives $\{K_n\}$ as in (3.11) for which the limits are different from 1.

First consider $\nu^{(0)}_{n,\alpha^0}$. Then,

$$\lim E\{\nu^{(0)}_{n,\alpha^0}|K_n\} = 1 - H_{pq_1}(\chi^2_{pq_1,\alpha^0};\ \theta^0_{(L)}) \tag{5.15}$$

Similarly, for $\bar{\nu}_{n,\bar{\alpha}}$, we have

$$\lim E\ \{\bar{\nu}_{n,\bar{\alpha}}|K_n\} = 1 - H_{pq_1}(\chi^2_{pq_1,\bar{\alpha}};\ \bar{\theta}_{(L)}) \tag{5.16}$$

For comparison of (5.15) and (5.16) we consider the difference $\bar{\theta}_{(L)}-\theta^0_{(L)}$ which is

$$\begin{aligned}\bar{\theta}_{(L)} - \theta^0_{(L)} &= \mathrm{Tr}[\Sigma^{-1}(\underset{\sim}{\gamma}_1-\underset{\sim}{\gamma}_2\underset{\sim}{\Delta}_{22}^{-1}\underset{\sim}{\Delta}_{21})\underset{\sim}{\Delta}_{11.2}(\underset{\sim}{\gamma}_1-\underset{\sim}{\gamma}_2\underset{\sim}{\Delta}_{22}^{-1}\underset{\sim}{\Delta}_{21})']\\ &\quad - \mathrm{Tr}[\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\gamma}_1\underset{\sim}{\Delta}_{11.2}\underset{\sim}{\gamma}_1')]\end{aligned} \tag{5.17}$$

From (5.17), we immediately claim that,

$$\underset{\sim}{\gamma}_2 = \underset{\sim}{0} \Longrightarrow \underset{\sim}{\bar{\theta}}_{(L)} \geq \theta^0_{(L)} \tag{5.18}$$

with equality holding for $\underset{\sim}{D}_{12} = \underset{\sim}{0}$. Hence, if H_0^*: $\underset{\sim}{\gamma}_2 = \underset{\sim}{0}$ holds, then $\bar{\nu}_{n,\alpha}$ has an asymptotic power (against $\underset{\sim}{\beta}_1 = n^{-\frac{1}{2}}\underset{\sim}{\gamma}_1$) greater than or equal to that of $\nu^{(0)}_{n,\alpha}$ for fixed size α. The picture may be different when H_0^* may not hold. For example, if $\gamma_2 \neq 0$ but $\gamma_1 = \gamma_2\Delta_{22}^{-1}\Delta_{21}$, then $\bar{\theta}_{(L)} = 0$ and $\theta^0_{(L)} > 0$ so that $\nu^{(0)}_{n,\alpha}$ performs better than $\bar{\nu}_{n,\alpha}$ for fixed size α. In general,

$$\bar{\theta}_{(L)} \geq \theta^0_{(L)} \quad \text{when} \quad ch_1(\Delta_{11.2}(\gamma_1-\gamma_2\Delta_{22}^{-1}\Delta_{21})(\gamma_1-\gamma_2\Delta_{22}^{-1}\Delta_{21})')$$

$$\geq ch_1(\Delta_{11}\gamma_1\gamma_1') \tag{5.19}$$

where ch_1 stands for the largest characteristic root. Clearly, in a neighborhood of $\gamma_2\Delta_{22}^{-1}\Delta_{21}$, this may not hold. This explains the lack of efficiency-robustness of $\bar{\nu}_n$ when H_0^* may not hold.

For the preliminary test test $\nu_n^{(L)}$, we obtain that

$$\lim_{n\to\infty} E\{\nu_n^{(L)}|K_n\} = P\{\chi^2_{pq_2}(\theta^*_{(L)}) < \chi^2_{pq_2,\alpha^*}\}P\{\chi^2_{pq_1}(\bar{\theta}_{(L)}) \geq \chi^2_{pq_1,\bar{\alpha}}\}$$

$$+ P\{\chi^2_{pq_2}(\theta^*_{(L)}) \geq \chi^2_{pq_2,\alpha^*};\chi^2_{pq_1}(\theta^0_{(L)}) \geq \chi^2_{pq_1,\alpha^0}\} \tag{5.20}$$

where $\chi^2_{pq_2}(\theta^*_{(L)})$ and $\chi^2_{pq_1}(\theta^0_{(L)})$ are jointly correlated chi square variables with non-centrality parameters $\theta^*_{(L)}$ and $\theta^0_{(L)}$ respectively. For the case $D_{12} = 0$, (5.20) reduces to

$$P\{\chi^2_{pq_1}(\theta^0_{(L)})\geq\chi^2_{pq_1,\alpha^0}\} + P\{\chi^2_{pq_2}(\theta^*_{(L)})<\chi^2_{pq_2,\alpha^*}\}[P\{\chi^2_{pq_1}(\bar{\theta}_{(L)}\leq\chi^2_{pq_1,\bar{\alpha}}\}$$

$$- P\{\chi^2_{pq_1}(\theta^0_{(L)})\geq\chi^2_{pq_1,\alpha^0}\}] \tag{5.21}$$

so that by arguments similar to (5.17) to (5.19) we conclude that (5.21) lies between (5.15) and (5.16). In particular, if $\alpha^0 = \alpha = \alpha$, then (5.15) reduces to

$$P\{\chi^2_{pq_1}(\theta^0_{(L)}) \geq \chi^2_{pq_1,\alpha^0}\}P\{\chi^2_{pq_2}(\theta^*_{(L)}) \geq \chi^2_{pq_2,\alpha^*}\} +$$

$$[1 - P\{\chi^2_{pq_2}(\theta^*_{(L)}) \geq \chi^2_{pq_2,\alpha^*}\}]P\{\chi^2_{pq_1}(\bar{\theta}_{(L)}) \geq \chi^2_{pq_1,\bar{\alpha}}\} \tag{5.22}$$

which is the weighted average of (5.15) and (5.16). In general $D_{12} \neq 0$, the second term on the right of (5.20) can be evaluated using (3.24) and it may be concluded that asymptotic power of $\nu^{(L)}_{n,\alpha}$ lies between $\nu^{(0)}_{n,\alpha}$ and $\bar{\nu}_{n,\alpha}$ and further, $\nu_n^{(L)}$ is more (less) efficiency-robust than $\bar{\nu}_n(\nu_n^{(0)}$ when H_0^* may not hold.

From the above analysis of size and power of the $\nu_n^{(L)}$- test, it is clear that the computation of size and power needs elaborate expansion as in (3.5) and (5.9). The situation is simpler when $\underset{\sim}{D}_{12} = \underset{\sim}{0}$. Also both $\bar{\nu}_{n,\alpha}$ and $\nu_n^{(L)}$ have size and power affected by the validity of H_0^*. A very similar situation holds for the rank based procedure based on $\nu_n^{(R)}$.

Apart from the difference in the dispersion matrices (Σ and $\underset{\sim}{T}(F)$) and the non-centrality parameters ($\bar{\theta}_{(L)}$ vs. $\bar{\theta}_{(R)}$, $\theta^*_{(L)}$ vs. $\theta^*_{(R)}$ and $\theta^0_{(L)}$ vs. $\theta^0_{(R)}$), the other things remain the same. So the validity-robustness and efficiency-robustness results apply to the rank case as well. However, if we want to study the relative efficiency results for the rank vs. the least squares solutions, we face a more complicated situation, where the right hand side of (5.20), particularly the second term, may not be readily compared (they depend on $\underset{\sim}{\Sigma}$ or $\underset{\sim}{\Lambda}(F)$ in a very involved way). Numerical studies may of course be made for various special cases.

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References

[1] Anderson, T.W.: Introduction to Multivariate Statistical Analysis, New York, John Wiley (1958).

[2] Bechhofer, R.E.: The effect of preliminary test of significance on the size and power of certain tests of univariate linear hypothesis. Ph.D. Thesis, (unpublished), Columbia University (1951).

[3] Bozivich, H., Bancroft, T.A. and Hartley, H.: Power of analysis of variance test procedures for certain incompletely specified models, I. Ann. Math. Statist. 27, 1017-1043 (1956).

[4] Jensen, D.R.: The joint distribution of traces of Wishart matrices and some applications. Ann. Math. Statist. 41, 133-145 (1970).

[5] Jurečková, J.: Nonparametric estimates of regression coefficients. Ann. Math. Statist. 42, 1328-1338 (1971).

[6] Khatri, C.G., Krishnaiah, P.R. and Sen, P.K.: A note on the joint distribution of correlated quadratic forms. Jour. Statist. Plann. Inf. 1, 299-307 (1977).

[7] Puri, M.L. and Sen, P.K.: A class of rank order tests for general linear hypothesis. Ann. Math. Statist. 40, 1325-1343 (1969).

[8] Saleh, A.K.Md. E. and Sen, P.K.: Nonparametric estimation of location parameter after a preliminary test on regression. Ann. Statist. 6. 154-

168 (1978).

[9] Saleh, A.K.Md. E. and Sen, P.K.: Nonparametric tests of location after a preliminary test on regression. Communication in Statistics 10, 639-652 (1982).

[10] Saleh, A.K.Md. E. and Sen, P.K.: Least-squares and rank order preliminary test estimation in general multivariate linear models. Carleton Mathematical Series No. 186, (1981).

[11] Saleh, A.K.Md. E. and Sen, P.K.: Nonparametric tests of location after a preliminary test on regression in the multivariate case. Communications in Statistics 12

[12] Sen, P.K.: On a class of rank order tests for parallelism of several regression lines. Ann. Math. Statist. 40, 1668-1683 (1969).

[13] Sen, P.K.: Asymptotic properties of likelihood ratio tests based on conditional specification (to appear in "Statistics and Decisions") (First Issue), 1982.

[14] Sen, P.K. and Puri, M.L.: Asymptotic theory of likelihood ratio and rank order tests in some multivariate linear models. Ann. Math. Statist. 41, 87-100 (1970).

[15] Tamura, R.: Nonparametric inferences with a preliminary test. Bull. Math. Statist. (Japan) 14, 39-61 (1965).

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