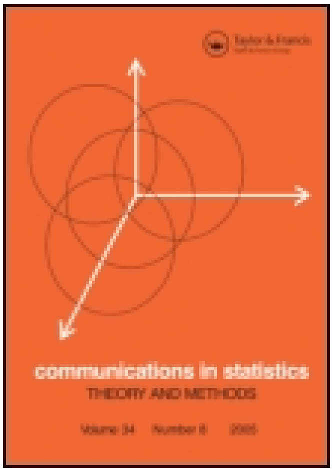

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Distribution functions and log-concavity

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DISTRIBUTION FUNCTIONS AND LOG-CONCAVITY

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ABSTRACT

This paper presents a collection of log-concavity results of one-dimensional cumulative distribution functions (cdf's) $F(x,\vartheta)$ and the related functions $\overline{F}(x,\vartheta) = 1 - F(x,\vartheta)$, $J_c(x,\vartheta) = F(x+c,\vartheta) - F(x,\vartheta)$, $c > 0$, in both $x \in \mathbf{R}$ or $x \in \mathbf{Z}$ and $\vartheta \in \Theta$, where $\mathbf{R}$ denotes the real line and $\mathbf{Z}$ the set of integers. We give a review of results available in the literature and try to fill some gaps in this field. It is well-known that log-concavity properties in x of a density f carry over to F, $\overline{F}$, and J_c in the continuous and discrete case. In addition, it

will be seen that the log-concavity of $g(y) = f(e^y)$ in y for a Lebesgue density f with $f(x) = 0$ for $x < 0$ implies the log-concavity of F. This criterion applies to many common densities. Moreover, a convex statistic T defined on $\mathbf{R}^n$ is shown to have a log-concave cdf whenever the underlying n-dimensional Lebesgue density h is log-concave. A slight generalization of the approach in Das Gupta & Sarkar (1984) is used to establish a connection between log-concavity in x of probability densities f or cdf's F and log-concavity of F, $\overline{F}$, and J_c in ϑ not only in the real, but also in the discrete case. Finally we apply the theory to the most common univariate distributions and discuss some further results obtained in the literature by different methods.

1. INTRODUCTION

The main purpose of this paper is the presentation of a collection of results concerning cumulative distribution functions (cdf's) $F(x)$ or $F_\vartheta(x) = F(x, \vartheta)$ and two related functions enjoying some log-concavity properties in $x \in \mathbf{X} \subseteq \mathbf{R}$ or in $\vartheta \in \Theta \subseteq \mathbf{R}$, where $\mathbf{R}$ denotes the real line. There exist several methods to derive results in this direction, e.g. Prekopa's theorem, convolution theorems, preservation theorems, and the concept of total positivity of order 2 or Pólya frequency functions (cf. Barlow et al. (1963), Karlin (1968), Marshall & Olkin (1979), and Das Gupta & Sarkar (1984)). Log-concavity results for probability density functions (pdf's) may be found e.g. in Keilson & Gerber (1971) and Sibuya (1988) for discrete distributions, in Bapat (1988) for multivariate discrete distributions, in Hansen (1988) for infinitely divisible distributions, in Huang & Ghosh (1982) for order statistics, or in Loh (1984) for a special class of scale mixtures. Most of the results for cdf's and related functions make use of the assumption that the underlying density is log-concave. Ibragimov (1956) introduced the concept of strong unimodality for absolutely continuous distributions and proved that the notions of strong unimodality for cdf's and log-concavity for the corresponding pdf's are equivalent, see also Dharmadhikari & Joag-dev (1988). pp. 17-23. The discrete analogue was proved by Keilson & Gerber (1971).

The paper is organized as follows. For the sake of completeness we start section 2 with a summary of some well-known log-concavity (log-convexity) results in $x \in \mathbf{X}$ for $F(x)$, $\overline{F}(x) = 1 - F(x)$, and for $J_c(x) = F(x+c) - F(x)$. Moreover, a useful device indicated in Das Gupta & Sarkar (1984) for establishing the log-concavity of F in certain cases where the pdf f fails to be log-concave in x is presented in Lemma 2.5. Furthermore, assuming a log-concave Lebesgue density on $\mathbf{R}^n$ we obtain log-concavity in x for a wide range of cumulative distribution functions F of convex statistics T defined on $\mathbf{R}^n$. A counterexample reveals that a corresponding result cannot hold in the discrete case. In section 3 we consider log-concavity in $\vartheta \in \Theta$ for $F(x, \vartheta)$, $\overline{F}(x, \vartheta)$, and $J_c(x, \vartheta)$. Here the approach of Das Gupta & Sarkar (1984) for continuous positive random variables is extended to the discrete case. While log-concavity (log-convexity) in x is of interest in reliability theory the corresponding properties in ϑ may be useful in connection with the log-likelihood function (cf. e.g. Pratt (1981)) or to describe the shape of power functions of one- and two-sided tests. In section 4 we summarize log-concavity results for the most common univariate distributions, which can be derived with the methods provided in sections 2 and 3. Moreover, the connection between distributions like the Binomial, the Negative Binomial, the Beta, and the F distribution is used to obtain some further log-concavity results for various parameters of these distributions. A few results depending on the special distribution under consideration are left to the concluding remarks and will be reviewed briefly. Here we also discuss the relationship between the inequalities resulting from log-concavity and an inequality in Barlow & Proschan (1975) and Finner (1992).

2. LOG-CONCAVITY IN x

To set notation, let $(\mathbf{X}, \mathbf{A}, \mu)$ denote a measure space which in general is assumed (unless specified otherwise) to be equal to $(\mathbf{R}, \mathbf{B}, \lambda)$ or $(\mathbf{Z}, \mathcal{P}(\mathbf{Z}), \kappa)$, where λ denotes the Lebesgue measure on the Borel σ-field $\mathbf{B}$ and κ denotes the counting measure on the power set $\mathcal{P}(\mathbf{Z})$ of the set of integers $\mathbf{Z}$. Let $\Theta \subseteq \mathbf{R}$ denote a parameter space and let $f(x,\vartheta)$, $\vartheta \in \Theta$, be pdf's with respect to μ. Furthermore, let $F(x,\vartheta)$ denote the corresponding cdf and set $\overline{F}(x,\vartheta) = 1 - F(x,\vartheta)$ and $J_c(x,\vartheta) = F(x+c,\vartheta) - F(x,\vartheta)$, $c > 0$. If interest is focussed on the dependence on x (as in this section) we often use the notation $f(x)$, $F(x)$, $\overline{F}(x)$, and $J_c(x)$, respectively.

A function $g : A \to [0,\infty)$, $A \subseteq \mathbf{R}^n$, is said to be log-concave (log-convex) in $x \in A$ (short: g is lcc(x) (lcx(x))), if for all $x_1, x_2 \in A$ and all $\alpha \in [0,1]$ such that $\alpha x_1 + (1-\alpha)x_2 \in A$ we have

$$g(\alpha x_1 + (1-\alpha)x_2) \;\geq\; (\leq)\; g(x_1)^{\alpha} g(x_2)^{1-\alpha}.$$

Setting $\log 0 = -\infty$, this means that $\log g$ is concave (convex).

For a collection of some basic and fundamental results concerning log-concavity we refer to Eaton (1987), Chapter 4. One of the main results discussed there is

Proposition 2.1 (Prekopa (1973)). Let f be defined on $\mathbf{R}^n \times \mathbf{R}^m$ and suppose that f is log-concave. Then, the function h defined on $\mathbf{R}^n$ by $h(x) = \int f(x,y)d\lambda^m(y)$, assumed to be finite on $\mathbf{R}^n$, is lcc(x).

This result will be used at the end of this section to derive log-concavity results for the cdf of real-valued (convex) statistics.

A well-known method to derive log-concavity results which applies not only for the Lebesgue measure is based on the concept of total positivity of order 2 (cf. Karlin (1968)). A function $g : A \to [0,\infty)$, $A \subseteq \mathbf{R}^2$, is said to be totally positive of order 2 (short: $g(x,y)$ is $\mathrm{TP}_2(x,y)$) if for all $x_1 < x_2$, $y_1 < y_2$, $(x_i, y_j) \in A$, $i,j = 1,2$,

$$f(x_1,y_2)f(x_2,y_1) \leq f(x_1,y_1)f(x_2,y_2).$$

Furthermore, $f : \mathbf{X} \to [0,\infty)$ is called a Pólya frequency function of order 2 (PF_2) if $K(x,y) = f(x-y)$, $x,y \in \mathbf{X}$, is $\mathrm{TP}_2(x,y)$. We note that a location family generated by a pdf f has monotone likelihood ratio iff f is PF_2.

For $\mathbf{X} = \mathbf{Z}$ a PF_2 function f is also said to be a Pólya frequency sequence of order 2.

The following result which can be found e.g. in Marshall & Olkin (1979), Chapter 18, may be considered as a basic tool to derive log-concavity results.

Proposition 2.2. (i) A measurable function $g : \mathbf{X} \to [0, \infty)$ is PF_2 if and only if g is $\mathrm{lcc}(x)$.
(ii) Let $g, h : \mathbf{X} \to [0, \infty)$ be measurable PF_2 functions. Then the convolution $k(x) = \int g(x-y)h(y)d\mu(y)$ is PF_2.

The subsequent theorem summarizes some well-known results concerning F, $\overline{F}$ and J_c, cf. e.g. Barlow et al. (1963), Karlin (1968), Barlow & Proschan (1975), and Marshall & Olkin (1979).

Theorem 2.3. (i) Let $f : \mathbf{X} \to [0, \infty)$ be a log-concave pdf w.r.t. μ. Then F, $\overline{F}$, and J_c are $\mathrm{lcc}(x)$.
(ii) Let $f : \mathbf{X} \to [0, \infty)$ be a pdf w.r.t. μ with $f(x) = 0$ for all $x \leq a$, $a \in \mathbf{X}$, which is $\mathrm{lcx}(x)$ on $(a, \infty) \cap \mathbf{X}$. Then $\overline{F}$ is $\mathrm{lcx}(x)$ on $(a, \infty) \cap \mathbf{X}$.

The most general approach to prove this theorem is the basic composition formula (cf. Karlin (1968), Chapter 18), which also implies Proposition 2.2(ii), but there exist several methods of proof for parts of the theorem, e.g. (i) with $\mu = \lambda$ is a consequence of Prekopa's theorem (Prekopa (1973)) whereas (ii) can easily be concluded from Artin's theorem (cf. Karlin (1968), Chapter 16). Furthermore, Theorem 2.3(i) follows from Proposition 2.2(ii) since indicator functions of convex sets are $\mathrm{lcc}(x)$. It should be noticed that in the discrete case log-concavity (log-convexity) of f, F, $\overline{F}$, and J_c is only defined on $\mathbf{X} = \mathbf{Z}$. In this case log-concavity (log-convexity) of e.g. f is equivalent to the requirements that the support of f is an interval in $\mathbf{Z}$ and $f(x)^2 \geq (\leq)\ f(x-1)f(x+1)$, $x \in \mathbf{Z}$. Log-concavity (Log-convexity) of the (reliability or survival) function $\overline{F}$ plays an important role in reliability theory; for more details see Barlow & Proschan (1975), Chapter 3.5, or Marshall & Olkin (1979), Chapter 18.

Finally, we note the following useful application of Proposition 2.2(ii) (and Theorem 2.3(i)).

Corollary 2.4. Let $X_1, \ldots, X_n$ be independent random variables with values in $\mathbf{X}$ and log-concave pdf's f_i, $i = 1, \ldots, n$. Then the pdf $f(x, n)$ of $T_n(X_1, \ldots, X_n) = \sum_{i=1}^n X_i$ and the corresponding functions $F(x, n)$, $\overline{F}(x, n)$, and $J_c(x, n)$ are lcc(x).

Although Theorem 2.3 is a very powerful tool to derive log-concavity results for F, $\overline{F}$, and J_c, there exist several distributions depending on a parameter $\vartheta \in \Theta$ such that f is not lcc(x) for a certain subset of ϑ-values. A method used by Das Gupta & Sarkar (1984) in case of the chi-square distribution turns out to be applicable for many other distributions on $\mathbf{X} = \mathbf{R}$ and is formalized in the following

Lemma 2.5. Let X be a random variable with pdf f defined on $\mathbf{X} = \mathbf{R}$ with $f(x) = 0$ for all $x < 0$. If the pdf of $Y = \log X$ is log-concave, then the cdf F of X is log-concave. Since a pdf of Y is given by $g(y) = e^y f(e^y)$, it is obvious that F is log-concave if $h(y) = f(e^y)$ is lcc(y).

Proof. Let $y \in \mathbf{R}$ and $G(y) = F(e^y) = \int_{(-\infty, y]} g(z) d\lambda(z)$. Since $h(z, y) = g(z) I_{(-\infty, y]}(z)$ is lcc(z, y), Prekopa's theorem implies that G is lcc(y). Now let $x_i \in (0, \infty)$ and set $y_i = \log x_i$, $i = 1, 2$. Then for all $\alpha \in [0, 1]$

$$\begin{aligned} F(\alpha x_1 + (1-\alpha) x_2) &\geq F(x_1^\alpha x_2^{1-\alpha}) = G(\alpha y_1 + (1-\alpha) y_2) \\ &\geq G(y_1)^\alpha G(y_2)^{1-\alpha} = F(x_1)^\alpha F(x_2)^{1-\alpha} \end{aligned}$$

where the first inequality follows since $u(x) = x$ is lcc(x) for $x > 0$, hence F is lcc(x). $\square$

To give a first example consider the Beta density $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$, $\alpha, \beta > 0$. It can easily be seen that f is lcc(x) for $\alpha, \beta \geq 1$ but not if $\alpha < 1$ or $\beta < 1$. Since h defined by $h(y) = f(e^y)$ is lcc(y) for all

$\alpha > 0$, $\beta \geq 1$ we obtain with Lemma 2.5 that F is lcc(x) for all $\alpha > 0$, $\beta \geq 1$. For $0 < \beta < 1$ one can easily find examples where F is not lcc(x), e.g. for $\alpha = 1/2$, $\beta = 1/2$, which is the arcsin distribution. Further applications of Lemma 2.5 can be found in section 4.

The following result is an immediate consequence of Prekopa's theorem.

Theorem 2.6. Let P be a probability measure with Lebesgue density h on $\mathbf{R}^n$, and let T be a convex function defined on $\mathbf{R}^n$. Then the cdf of T, i.e., $F(z) = P(T \leq z)$ is lcc(z).

Proof. Since the epigraph of T, i.e., the set $A = \{(x, z) \in \mathbf{R}^{n+1} : T(x) \leq z\}$ is a convex set iff T is convex on $\mathbf{R}^n$ (cf. Marshall & Olkin (1979), p. 449), the function $g = I_A$ is an indicator function of a convex set, hence log-concave. In view of $F(z) = \int I_A(x, z)h(x)d\lambda^n(x)$, Prekopa's theorem yields the desired result. □

The last theorem is widely applicable. The pdf f may be any multivariate log-concave Lebesgue density, e.g. a multivariate normal density. Furthermore, many of the common statistics are convex, e.g. the statistic $\sum_{i=1}^n x_i$ or maxima of linear combinations of the x_i as are the maximum statistics $\max_{1\leq i\leq n} x_i$ and $\max_{1\leq i\leq n} |x_i|$, the range statistic $\max_{1\leq i,j\leq n} |x_i - x_j|$, the one-sided range statistic $\max_{1\leq i<j\leq n}(x_i - x_j)$, the statistic $\max_{1\leq i\leq n} |x_i - \overline{x}.|$ with $\overline{x}. = \sum_{i=1}^n x_i/n$, or, more generally, the statistic $T(x) = \sup_{c\in C} c'x$ where $C \subseteq \mathbf{R}^n$ is arbitrary. Other examples are the chi-square-type statistic $\sum_{i=1}^n (x_i - \overline{x}.)^2$, the mean deviation statistic $\sum_{i=1}^n |x_i - \overline{x}.|$ or, more generally, the so called Ψ_t-statistic, e.g. $\Psi_t(x) = (\sum_{i=1}^n |x_i - \overline{x}.|^t/n)^{1/t}$, $t \in [1, \infty]$, or quadratic forms like $x'Ax$ where A is a positive semidefinite $n \times n$ matrix. Whenever the underlying Lebesgue density is log-concave the corresponding cdf's of these statistics are log-concave.

Remark 2.7. A discrete analogue of Theorem 2.6 cannot exist even in the one-dimensional case. Let e.g. $h(x)$, $x \in \mathbf{Z}$, be the pdf of the discrete uniform distribution on $\{0,1,2\}$ and $T(x) = x^2$, $x \in \mathbf{Z}$. Then $F(z) = P(T \le z)$ is not lcc(z) on $\mathbf{Z}$.

3. LOG-CONCAVITY IN ϑ

In this section we discuss distributions depending on a real or discrete parameter ϑ. If we are concerned with a location family with densities $f(x,\vartheta) = h(x-\vartheta)$, $x, \vartheta \in \mathbf{X}$, then Theorem 2.3 yields that $F(x,\vartheta)$, $\overline{F}(x,\vartheta)$, and $J_c(x,\vartheta)$ are lcc(x) and lcc(ϑ) if h is lcc(x). In this case only the assumption lcc(x) on the original pdf $h(x)$ is required. If $f(x,\vartheta)$ has a more complicated structure, additional properties are necessary.

In the following we consider a slight generalization of the approach in Das Gupta & Sarkar (1984) which applies for non-negative random variables with values in $[0,\infty)$ or $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, where $\mathbf{N}$ denotes the set of positive integers. Let $\Theta \in \{(0,\infty), [0,\infty), \mathbf{N}, \mathbf{N}_0\}$ and $g : \mathbf{X} \times \Theta \to [0,\infty)$ be measurable in the first component with $g(x,\vartheta) = 0$ for all $x < 0$.

Definition 3.1. $g(x,\vartheta)$ is said to have the reproductive property in $\vartheta \in \Theta$ (short: $g(x,\vartheta)$ has RP(ϑ)) if for every $\eta \in \Theta$ there exists a probability measure P_η on $(\mathbf{X}, \mathbf{A})$ with $P_\eta([0,\infty) \cap \mathbf{X}) = 1$ such that for all $\vartheta \in \Theta$ and $x \in \mathbf{X}$

$$\int_{[0,x]\cap\mathbf{X}} g(x-y,\vartheta)\,dP_\eta(y) = g(x,\vartheta+\eta).$$

Now let $f(x,\vartheta)$, $x \in \mathbf{X} \in \{\mathbf{R},\mathbf{Z}\}$, be probability density functions with the same properties as $g(x,\vartheta)$ defined above. In terms of random variables, the reproductive property of the corresponding pdf's $f(x,\vartheta)$ can be characterized as follows: $f(x,\vartheta)$ has RP(ϑ) iff there exists a stochastic process $(Y_\vartheta)_{\vartheta\in\Theta}$ with stationary, independent increments and $Y_\vartheta \ge 0$, $\vartheta \in \Theta$, such that Y_ϑ has the pdf $f(x,\vartheta)$, $\vartheta \in \Theta$. In the special case $\Theta = \mathbf{N}$ (or similarly for $\Theta = \mathbf{N}_0$ with obvious modifications) the random variable Y_ϑ can be expressed as $Y_\vartheta = Y_1 + \sum_{i=2}^{\vartheta} X_i$, $\vartheta \ge 2$, where the X_i, $i \ge 2$, are independent, identically distributed (iid) random variables being independent of Y_1.

The following two theorems generalize the results obtained for $\mathbf{X} = \mathbf{R}$ in Das Gupta & Sarkar (1984).

Theorem 3.2. (i) If $f(x,\vartheta)$ has RP(ϑ), then $F(x,\vartheta)$ has RP(ϑ).
(ii) If $f(x,\vartheta)$ is $\mathrm{TP}_2(x,\vartheta)$, so are $F(x,\vartheta)$, $\overline{F}(x,\vartheta)$, and $J_c(x,\vartheta)$.

The proof of (i) is straightforward and (ii) is a direct application of the basic composition formula (Karlin (1968), p. 17). Alternatively, (ii) can be proved by using an integral inequality given in Theorem 2 of Wijsman (1985).

It should also be mentioned that this integral inequality yields that $F(x,\vartheta)$ ($\overline{F}(x,\vartheta)$) is decreasing (increasing) in ϑ if $f(x,\vartheta)$ is $\mathrm{TP}_2(x,\vartheta)$.

Theorem 3.3. Suppose $F(x,\vartheta)$ is Borel-measurable in $\vartheta \in \Theta$ and has RP(ϑ).
(i) If $F(x,\vartheta)$ ($\overline{F}(x,\vartheta)$, $J_c(x,\vartheta)$) is lcc(x), then $F(x,\vartheta)$ ($\overline{F}(x,\vartheta)$, $J_c(x,\vartheta)$) is $\mathrm{TP}_2(x,\vartheta)$.
(ii) If $F(x,\vartheta)$ ($\overline{F}(x,\vartheta)$, $J_c(x,\vartheta)$) is $\mathrm{TP}_2(x,\vartheta)$, then $F(x,\vartheta)$ ($\overline{F}(x,\vartheta)$, $J_c(x,\vartheta)$) is lcc(ϑ).

Part (i) can be proved in the same way as in Theorem 3 of Das Gupta & Sarkar (1984), whereas the proof of (ii) is similar to the one of Theorem 1(ii) in the same paper. However, it should be noticed that the integrations occurring in the proofs have to be carried out over the set $[0,\infty)\cap\mathbf{X}$ to obtain the results. Secondly, it should be mentioned that the proof of Theorem 1(ii) in Das Gupta & Sarkar (1984) only yields e.g. $F(x,\vartheta_2+\eta)F(x,\vartheta_1) \le F(x,\vartheta_1+\eta)F(x,\vartheta_2)$ for all $\vartheta_1 < \vartheta_2$, $\eta > 0$. To conclude now that $F(x,\vartheta)$ is lcc(ϑ), the measurability of $F(x,\vartheta)$ in ϑ is necessary (cf. Roberts and Varberg (1973), pp. 221-224).

We finish this section with a widely applicable result concerning the distribution of sums of iid random variables.

Corollary 3.4. Let X_i, $i \in \mathbf{N}$, be non-negative iid random variables with values in $\mathbf{X}$ and log-concave pdf. Then the pdf $f(x,n)$ of $T_n(X_1,\ldots,X_n) = \sum_{i=1}^{n} X_i$ is lcc(x) and has RP(n), so that the corresponding functions $F(x,n)$, $\overline{F}(x,n)$, and $J_c(x,n)$ are lcc(n).

4. EXAMPLES

In this section we consider the most common univariate distributions enjoying some log-concavity properties. To identify the distributions under consideration we introduce some abbreviations in terms of the defining parameters. The following list is a part of a table of distributions given in Kokoska and Nevison (1989): Binomial $B(n,p)$, Negative Binomial $NB(s,p)$ with $s > 0$, $s \in \mathbf{R}$, Hypergeometric $Hyp(N,M,n)$, Poisson $Po(\mu)$, Discrete Uniform $DU(n)$, Beta(α,β), Uniform $U(a,b)$, Chi(α), Gamma(α,β), Chi-Square χ^2_n, Exponential Exp(λ), Extreme Value Extr(α,β), Half-Normal HN(θ), LaPlace (Double Exponential) DExp(α,β), Logistic Log(α,β), Log-Normal LN(μ,σ^2), Normal N(μ,σ^2), Pareto(a,θ), Rayleigh(σ), Weibull Wei(α,β). In addition we consider the Inverse Normal InvN(α,β) with density $f(x,\alpha,\beta) = (\beta/(2\pi x^3))^{1/2} \exp(-\beta(x-\mu)^2/(2\mu^2 x))$, $x > 0$, $\alpha,\beta > 0$, and the Power distribution Pow(a,θ) with density $f(x,a,\theta) = \theta x^{\theta-1}/a^{\theta}$, $0 < x < a$, $\theta > 0$.

First we note that the cdf's of these distributions except Beta(α,β) for $0 < \beta < 1$ are lcc(x). Except NB(s,p) for $0 < s < 1$ where a different argument is used (see below) this may easily be seen by showing that f is lcc(x), or, if not, by showing that $g(y) = f(e^y)$ is lcc(y). In the following cases the pdf is not lcc(x) but $g(y) = f(e^y)$ is lcc(y): Beta(α,β) for (α,β) with $\alpha < 1$ and $\beta \geq 1$, Chi(α) for $0 < \alpha < 1$, Gamma(α,β) for $0 < \alpha < 1$, χ^2_1, LN(μ,σ^2) for all μ,σ, Pareto(a,θ) for all a,θ, Wei(α,β) for $0 < \alpha < 1$, and InvN(α,β) for all α,β. In all other cases except Beta(α,β) for (α,β) with $0 < \alpha,\beta < 1$ we obtain that f is lcc(x), hence $\overline{F}$ and J_c are also lcc(x) in these cases. Furthermore, for all distributions with a location parameter ϑ (say) we have that f, F, $\overline{F}$ and J_c are lcc(ϑ) if the underlying density f is lcc(x), e.g. N(μ,σ^2) in μ, Extr(α,β) in α, etc.

The t distribution with ν degrees of freedom (including the Cauchy distribution for $\nu = 1$) may serve as an example where neither f nor F, $\overline{F}$, and J_c, respectively, are lcc(x) (cf. Corrigendum (1982) of Pratt (1981)).

In the following cases $\overline{F}$ is lcx(x) since its pdf f is lcx(x) on an unbounded interval: Gamma(α,β) for $0 < \alpha \leq 1$, Wei(α,β) for $0 < \alpha \leq 1$ (cf. Barlow & Proschan (1975), Chapter 3.5), NB(s,p) for $0 < s \leq 1$, and Pareto(a,θ) for all a,θ.

Families of distributions with a reproductive property are $B(n, p)$ (in n), $NB(s, p)$ (in s, cf. Feller (1968), p. 269), $Po(\mu)$ (in μ), $Gamma(\alpha, \beta)$ (in α), and χ_n^2 (in n) which is equivalent to $Gamma(n/2, 2)$, so we obtain that F, $\overline{F}$, and J_ε are $lcc(n)$ for $B(n, p)$, $lcc(s)$ for $NB(s, p)$, $lcc(\mu)$ for $Po(\mu)$, $lcc(\alpha)$ for $Gamma(\alpha, \beta)$, and $lcc(n)$ for χ_n^2. It should be noticed that the pdf of $Gamma(\alpha, \beta)$, is $TP_2(x, \alpha)$ for all $x, \alpha, \beta > 0$, and the pdf of $NB(s, p)$ is $TP_2(x, s)$ for all $s > 0$, $x \in \mathbf{N}_0$, $p \in (0, 1]$, so that Theorem 3.2(ii) and 3.3(ii) may be applied in this case.

Results for the quantiles of the chi-square distribution and a modified F distribution may be found in Sarkar (1983). Furthermore, in Das Gupta & Sarkar (1984) it is shown (by using a restricted reproductive property) that the cdf $F_{\text{Beta}}(x, \alpha, \beta)$ of $\text{Beta}(\alpha, \beta)$ is $lcc(\beta)$. It should be mentioned that Das Gupta & Sarkar (1984) probably used a definition of the Beta distribution (which is not given explicitly there) where the parameters are interchanged. However, using the relationship $F_{\text{Beta}}(x, \alpha, \beta) = \overline{F}_{\text{Beta}}(1 - x, \beta, \alpha)$ we also obtain that $\overline{F}$ is $lcc(\alpha)$. Moreover, the relationships between the Beta, Binomial, Negative Binomial, and F distribution can be exploited to derive several further results. Let $F_{\text{B}}(x, n, p)$ denote the cdf of $B(n, p)$, $F_{\text{NB}}(x, s, p)$ the cdf of $NB(s, p)$, and $G(x, n_1, n_2) = F_{\text{F}}(n_2 x/n_1, n_1, n_2)$, $n_i \in \mathbf{N}$, where $F_{\text{F}}(\cdot, n_1, n_2)$ denotes the cdf of the F distribution with n_1 and n_2 degrees of freedom, i.e., G is the cdf of the ratio of two independently distributed chi-square variables. Then it is well-known (cf. Patel et al. (1976), p. 198, p. 204 and p. 217) that

$$\begin{aligned}
F_{\text{Beta}}(x, \alpha, \beta) &= F_{\text{NB}}(\beta - 1, \alpha, x), \; x \in (0, 1], \; \alpha > 0, \; \beta \in \mathbf{N}, \\
F_{\text{B}}(x, n, p) &= \overline{F}_{\text{Beta}}(p, x + 1, n - x), \; x, n \in \mathbf{N}_0, \; 0 \le x < n, \; p \in [0, 1], \\
G(x, n_1, n_2) &= F_{\text{Beta}}(x/(1 + x), n_1/2, n_2/2), \; x \in [0, \infty), \; n_1, n_2 \in \mathbf{N}.
\end{aligned}$$

From these equations and the log-concavity results above it is obvious that $F_{\text{Beta}}(x, \alpha, \beta)$ is $lcc(\alpha)$ if $\beta \in \mathbf{N}$, $\overline{F}_{\text{Beta}}(x, \alpha, \beta)$ is $lcc(\beta)$ if $\alpha \in \mathbf{N}$, thus $G(x, n_1, n_2)$ is $lcc(n_1)$ if n_2 is even, and $\overline{G}(x, n_1, n_2)$ is $lcc(n_2)$ if n_1 is even. Furthermore, $G(x, n_1, n_2)$ is $lcc(n_2)$ for all $n_1 \in \mathbf{N}$, $\overline{G}(x, n_1, n_2)$ is $lcc(n_1)$ for all $n_2 \in \mathbf{N}$. For the Binomial and the Negative Binomial distribution we obtain that F_{B}, $\overline{F}_{\text{B}}$, F_{NB}, $\overline{F}_{\text{NB}}$ are $lcc(p)$.

Finally, since the cdf of $\text{Beta}(\alpha, \beta)$ is $lcc(\beta)$, the cdf of $NB(s, p)$ is $lcc(x)$ for all $s > 0$ as mentioned above.

In view of the above results it is obvious that the Beta distribution plays a central role in deriving results also for other distributions. Unfortunately, no complete solution concerning log-concavity properties of $F_{\text{Beta}}(x, \alpha, \beta)$ in α is available yet. However, it can be shown that $F_{\text{Beta}}(x, \alpha, \beta)$ is log-convex in α for $0 < \beta < 1$. In the light of the result for $\beta \in \mathbf{N}$ and some numerical investigations there is strong evidence for the conjecture that $F_{\text{Beta}}(x, \alpha, \beta)$ is log-concave in α for all $\beta > 1$.

5. CONCLUDING REMARKS

First we review briefly some other log-concavity results obtained in the literature. The following result has been derived for the distribution of order statistics by Bapat and Beg (1989) using permanents and later by Sathe and Bendre (1991) using a result concerning the occurrence of at least r independent events out of n. Let $X_1, \ldots, X_n$ be n independent real-valued random variables and let $F_{r,n}$ denote the cdf of the rth smallest order statistic. Then $F_{r,n}(x)$ and $\overline{F}_{r,n}(x) = 1 - F_{r,n}(x)$ are lcc(r), $r = 1, \ldots, n$. Notice that the X_i need not be identically distributed. However, this result can easily be obtained with Corollary 2.4. Let F_i denote the cdf of X_i and Y_i be defined as 0 or 1 according as $X_i > x$ or $X_i \leq x$, i.e., the Y_i are independently $\mathrm{B}(1, F_i(x))$ distributed. Denote the pdf and cdf of $\sum_{i=1}^{n} Y_i$ by $g_x(r)$ and $G_x(r)$, respectively. Since $F_{r,n}(x) = \overline{G}_x(r-1)$ and the pdf's of the Y_i are log-concave, we immediately obtain that $g_x(r)$, hence $G_x(r), \overline{G}_x(r)$, and therefore $F_{r,n}(x), \overline{F}_{r,n}(x)$ are lcc(r). Note that $g_x(r)$ is the probability of the occurrence of the event that exactly r of the X_i are $\leq x$. Therefore, the probability of the occurrence of exactly r independent events out of n is lcc(r). The corresponding problem for not necessarily independent events has been studied in a recent paper by Balasubramanian and Balakrishnan (1993).

Another result for the cdf of the range statistic of iid normally distributed random variables obtained by Hayter (1986) has been generalized by Royen (1990) for iid continuous random variables having log-concave (log-convex) continuously differentiable densities with support (a, b). If G_n denotes the cdf of the range statistic of n of these random variables, then Royen (1990) proved that G_n is lcc(n) (lcx(n)) if f is lcc(x) (lcx(x) and $b-a=\infty$). The proof of the log-concavity result is obviously based on the assumption that f has a unique mode, thus distributions like U(a,b) where the mode is not uniquely determined are excluded from consideration. However, some additional arguments show that G_n is lcc(n) whenever f is lcc(x). It can easily be seen by considering the B(1, p) distribution (cf. Finner (1990)) that an analogous result cannot hold for discrete distributions.

The proofs of the results for order statistics and range statistics require very specific methods, so it would be interesting to know whether there exist other conditions than RP (or the mixture property used by Das Gupta & Sarkar (1984)) to derive these and other log-concavity results for parameters of F, $\overline{F}$, or J_c from a more general point of view.

In a recent paper Finner (1992) obtained an inequality for cdf's of real-valued statistics $T_n(X_1,\ldots,X_n)$ depending on iid random variables which is closely related to the inequalities resulting from log-concavity in $n \in \mathbf{N}$, i.e., $F_n^2 \geq F_{n-1}F_{n+1}$. If T_n satisfies the simple monotonicity condition $T_{n+1}(x_1,\ldots,x_{n+1}) \geq T_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1})$ for all $j=1,\ldots,n+1$, $n \in \mathbf{N}$, then $F_n(x)^{1/n} \geq F_{n+1}(x)^{1/(n+1)}$ for all $n \in \mathbf{N}$, $x \in \mathbf{R}$, where F_n denotes the cdf of T_n. No further restrictions concerning the distribution of the X_i are necessary. However, if F_n is lcc(n) for $n \in \mathbf{N}_0$ with $F_0 \equiv 1$, it follows directly that $F_n^{1/n}$ is decreasing in $n \in \mathbf{N}_0$, i.e., log-concavity yields sharper inequalities than the inequality $F_n^{1/n} \geq F_{n+1}^{1/(n+1)}$. On the other hand, the condition $T_{n+1}(x_1,\ldots,x_{n+1}) \geq T_n(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1})$ is satisfied for many statistics, e.g. for the range statistic, the one-sided range statistic, the sum statistic $\sum_{i=1}^n x_i$ with $x_i \geq 0$, etc. It should be noted that the above inequality for the cdf of the sum statistic (with $x_i \geq 0$) is well-known in reliability theory (cf. Barlow & Proschan (1975), Chapter 4.3).

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