Limit Theorems for Exchangeable Random Variables via Martingales

Neville Weber,
University of Sydney.

May 15, 2006

- A sequence of random variables $\{X_1, X_2, .., X_N\}$ is said to be exchangeable if

$$\{X_i\} \stackrel{\mathcal{D}}{=} \{X_{\pi(i)}\}$$

for every permutation π of $\{1, 2, .., N\}$. An infinite sequence is said to be exchangeable if every finite subset is exchangeable.

- For an infinite exchangeable sequence if we condition on an appropriate σ-field then the sequence behaves like a sequence of independent and identically distributed random variables (de Finetti (1930)).
Blum et al. (1958), Bühlmann (1958) used this result to obtain the central limit theorem for sums of exchangeable random variables.

- For an array of random variables where each row consists of finitely exchangeable variables the above technique does not apply. Martingale methods provide a unified approach to both situations. The martingale (or reversed martingale) idea was suggested by Loynes (1969) in the context of U-statistics and developed by Eagleson (1979, 1982), Eagleson and Weber (1978), and Weber (1980) for exchangeable variables.

Martingales

- Let $\mathcal{F} = \{\mathcal{F}_n\}$ be an increasing sequence of σ-fields, i.e. $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and let $S_1, S_2, ..$ be a sequence of random variables such that S_i is $\mathcal{F}_i$-measurable. $\mathcal{F}_n$ can be thought of as the history of the process up to time n. $\{S_n\}$ is a martingale with respect to $\mathcal{F}$ if $E|S_n| < \infty$, and $E(S_n|\mathcal{F}_m) = S_m$ for all $m < n$.

- Martingales include sums of zero mean, i.i.d. random variables, $S_n = \Sigma_{i=1}^n X_i$, and many of the properties of sums carry across to martingales under suitable conditions.

- Flexibility in the choice of $\mathcal{F}_n$ makes martingale methods very useful. We will show that we can exploit the symmetry of various exchangeable situations to construct σ-fields that allow us to form martingales and to evaluate key conditional expectations.

Let $\{Y_{nj}\}$ be a triangular array of martingale differences with respect to $\{\mathcal{F}_{nj}\}$.

Theorem 1. [Scott (1973), Theorem 2] *Suppose*

(i) $\sum_{j=1}^{n} EY_{nj}^2 \longrightarrow 1$,

(ii) *max*$_{j \leq n}|Y_{nj}| \xrightarrow{p} 0$, *and*

(iii) $\sum_{j=1}^{n} Y_{nj}^2 \xrightarrow{p} 1$.

Then

$$S_n = \sum_{j=1}^{n} Y_{nj} \xrightarrow{\mathcal{D}} N(0,1).$$

A Poisson convergence result follows from the results for infinitely divisible laws developed by Brown and Eagleson (1971). See also Freedman (1974). Let $\{A_{nj}\}$ be an array of events with A_{nj} being $\mathcal{F}_{nj}$-measurable.

Theorem 2. *If, for some constant* λ*,*

(i) $\sum_{j=1}^{n} P(A_{nj}|\mathcal{F}_{n,j-1}) \xrightarrow{p} \lambda$, *and*

(ii) $\max_{j \le n} P(A_{nj}|\mathcal{F}_{n,j-1}) \xrightarrow{p} 0$,

then

$$T_n = \sum_{j=1}^{n} I(A_{nj}) \xrightarrow{\mathcal{D}} \textit{Poisson}(\lambda).$$

Exchangeable Random Variables

Consider an array of row-wise finitely or infinitely exchangeable random variables $\{X_{nj} : j = 1, .., m_n;\ n = 1, 2, ..\}$. Construct an array of σ-fields such that X_{nj} is $\mathcal{F}_{nj}$-measurable and $\mathcal{F}_{n,j-1} \subseteq \mathcal{F}_{nj},\ j = 1, 2, ..$

Form a martingale difference array and apply Theorem 1 to obtain results for

$$S_n = \sum_{j=1}^{n} [X_{nj} - E(X_{nj}|\mathcal{F}_{n,j-1})].$$

For exchangeable sequences we can choose $\mathcal{F}_{nj}$ so that we can evaluate $E(X_{nj}|\mathcal{F}_{n,j-1})$. Thus we can obtain conditions under which $\sum_{j=1}^{n} X_{nj}$ converges in distribution to normal. Specifically, let

$$\mathcal{F}_{n,j-1} = \sigma\{X_{n1}, .., X_{n,j-1}, \sum_{i=j}^{m_n} X_{ni}\} = \sigma\{X_{n1}, .., X_{n,j-1}, \sum_{i=1}^{m_n} X_{ni}\}$$

$$
\begin{aligned}
\sum_{i=j}^{m_n} X_{ni} &= E(\sum_{i=j}^{m_n} X_{ni}|\mathcal{F}_{n,j-1}) \\
&= \sum_{i=j}^{m_n} E(X_{ni}|\mathcal{F}_{n,j-1}) \\
&= \sum_{i=j}^{m_n} E(X_{nj}|\mathcal{F}_{n,j-1}), \text{ by exchangeability} \\
&= (m_n - j + 1)E(X_{nj}|\mathcal{F}_{n,j-1}).
\end{aligned}
$$

Thus

$$
E(X_{nj}|\mathcal{F}_{n,j-1}) = (m_n - j + 1)^{-1} \sum_{i=j}^{m_n} X_{ni}.
$$

Theorem 3. *Given* $\{X_{ni}\}$ *as above with*

(i)	$nEX_{n1}^2 \to 1,$	*(iv)*	$\sum_{j=1}^n X_{nj}^2 \xrightarrow{p} 1,$
(ii)	$n^2 EX_{n1}X_{n2} \to 0,$	*(v)*	$n/m_n \to 0.$
(iii)	$\max_{j \le n} \lvert X_{nj} \rvert \xrightarrow{p} 0,$		

Then

$$S_n = \sum_{j=1}^n X_{nj} \xrightarrow{\mathcal{D}} N(0,1).$$

Proof. Let $Y_{nj} = X_{nj} - E(X_{nj}|\mathcal{F}_{n,j-1})$.

$$\sum_{j=1}^n EY_{nj}^2 = \sum_{j=1}^n EX_{nj}^2 - E(\sum_{j=1}^n E^2(X_{nj}|\mathcal{F}_{n,j-1})).$$

$$
\begin{aligned}
& E(\sum_{j=1}^{n} E^2(X_{nj}|\mathcal{F}_{n,j-1})) \\
= \quad & \sum_{j=1}^{n} EX_{n1}^2/(m_n - j + 1) + \sum_{j=1}^{n} \frac{m_n - j}{m_n - j + 1} EX_{n1}X_{n2} \\
\leq \quad & EX_{n1}^2 \log(n+1) + n|EX_{n1}X_{n2}| \to 0, \text{ by (i) and (ii).}
\end{aligned}
$$

$$
\max_{j \leq n} |Y_{nj}| \leq \max_{j \leq n} |X_{nj}| + \max_{j \leq n} |E(X_{nj}|\mathcal{F}_{n,j-1})|
$$

and

$$
\max_{j \leq n} |E(X_{nj}|\mathcal{F}_{n,j-1})| \leq [\sum_{j=1}^{n} E^2(X_{nj}|\mathcal{F}_{n,j-1})]^{1/2} \xrightarrow{p} 0,
$$

so (i), (ii) and (iii) imply $\max_{j \leq n} |Y_{nj}| \xrightarrow{p} 0.$
Similarly we have $\sum_{j=1}^{n} Y_{nj}^2 \xrightarrow{p} 1$. Thus

$$
\sum_{j=1}^{n} Y_{nj} \xrightarrow{\mathcal{D}} N(0,1).
$$

To obtain the final result we need to show that

$$\sum_{j=1}^{n} E(X_{nj}|\mathcal{F}_{n,j-1}) \xrightarrow{p} 0.$$

By exchangeability $E(X_{nj}|\mathcal{F}_{n,j-1}) = E(X_{nm_n}|\mathcal{F}_{n,j-1})$ and so for fixed n, $\{E(X_{nm_n}|\mathcal{F}_{n,j-1}), \mathcal{F}_{n,j-1}\}$ is a martingale. Given $\epsilon > 0$

$$\begin{aligned}
&P(|\textstyle\sum_{j=1}^{n} E(X_{nj}|\mathcal{F}_{n,j-1})| > \epsilon) \\
&\quad \leq \quad P(\max_{k\leq n} |\textstyle\sum_{j=1}^{k} E(X_{nj}|\mathcal{F}_{n,j-1})| > \epsilon) \\
&\quad \leq \quad P(n \max_{k\leq n} |E(X_{nm_n}|\mathcal{F}_{nk})| > \epsilon) \\
&\quad \leq \quad (n/\epsilon)^2 E[E(X_{nm_n}|\mathcal{F}_{nn})]^2, \quad \text{by Kolmogorov 's inequality} \\
&\quad \leq \quad (n/\epsilon)^2 [E(X_{n1}^2)/(m_n - n + 1) + |EX_{n1}X_{n2}|] \\
&\quad \to \quad 0 \quad \text{by (i) and (ii).}
\end{aligned}$$

□

Let $\mathcal{G}_{nj}$ be the σ-field of measurable sets involving $\{X_{n1}, .., X_{nm_n}\}$ that are unaffected by permuting of the indices $1, 2, .., j$, i.e.

$$\mathcal{G}_{nj} = \sigma\{(X_{n1}^{(1)}, .., X_{nj}^{(j)}), X_{n,j+1}, .., X_{nm_n}\}$$

where $(X_{n1}^{(1)}, .., X_{nj}^{(j)})$ denotes the order statistics of the first j terms in the nth row. Here

$$\mathcal{G}_{nj} \supseteq \mathcal{G}_{n,j+1},$$

and, by symmetry,

$$E(X_{n1}|\mathcal{G}_{nj}) = j^{-1}\sum_{i=1}^{j} X_{ni} = j^{-1}S_{nj}.$$

Thus $\{j^{-1}S_{nj}, \mathcal{G}_{nj}\}$ is a reversed martingale for each n.

Let

$$
\begin{aligned}
Z_{nj} &= E(X_{n1}|\mathcal{G}_{nj}) - E(X_{n1}|\mathcal{G}_{n,j+1}) \\
&= j^{-1}\{E(X_{n1}|\mathcal{G}_{n,j+1}) - X_{n,j+1}\}
\end{aligned}
$$

so

$$
\begin{aligned}
\textstyle\sum_{j=n}^{m_n-1} Z_{nj} &= E(X_{n1}|\mathcal{G}_{nn}) - E(X_{n1}|\mathcal{G}_{nm_n}) \\
&= n^{-1}\textstyle\sum_{j=1}^{n} X_{nj} - E(X_{n1}|\mathcal{G}_{nm_n}).
\end{aligned}
$$

If we have an exchangeable array with non-random row sums, $\sum_{j=1}^{m_n} X_{nj} = 0$, say, and

$$
n^{-1}\sum_{j=1}^{n} X_{nj} = \sum_{j=n}^{m_n-1} Z_{nj}.
$$

Thus we can represent $\sum_{j=1}^{n} X_{nj}$ as a sum of reversed martingale differences. The martingale CLT can be applied to the array $\{Z_{n,m_n-j+1}, \mathcal{G}_{n,m_n-j+1}\}_{j=1}^{m_n-n+1}$ to get the following variation of Chernoff and Teicher (1958).

Theorem 4. *Let $\{X_{ni}\}$ be an array of row wise exchangeable random variables with $\sum_{i=1}^{m_n} X_{ni} = 0$ for each n. If*

(i) $n/m_n \to \alpha$ *as* $n \to \infty, \quad \alpha \in [0, 1)$

(ii) $\max_{i \le m_n} |X_{ni}| \xrightarrow{p} 0,$

(iii) $\sum_{i=1}^{n} X_{ni}^2 \xrightarrow{p} 1, \quad$ *as* $n \to \infty,$

then

$$S_n \xrightarrow{\mathcal{D}} N(0, 1 - \alpha).$$

Similar methods can be applied to arrays of row wise exchangeable events to obtain Poisson limit laws. Let

$$\mathcal{H}_{nj} = \sigma\{A_{n1}, .., A_{nj}, \sum_{i=j+1}^{m_n} I(A_{ni})\}$$

then $P(A_{nj}|\mathcal{H}_{n,j-1}) = (m_n - j + 1)^{-1} \sum_{i=j}^{m_n} I(A_{ni})$. Applying Theorem 2 we obtain

Theorem 5. [Ridler-Rowe (1967)] *If*

(i) $nP(A_{n1}) \to \lambda$,
(ii) $n^2 P(A_{n1} \cap A_{n2}) \to \lambda^2$
(iii) $n/m_n \to 0$

then

$$\sum_{j=1}^{n} I(A_{nj}) \xrightarrow{\mathcal{D}} \textit{Poisson}(\lambda).$$

To establish the result note

$$
\begin{aligned}
& E|P(A_{nj}|\mathcal{H}_{n,j-1}) - P(A_{nj})|^2 \\
= \quad & E(I(A_{nj}) \sum_{i=j}^{m_n} I(A_{ni})/(m_n - j + 1) - P(A_{nj})^2) \\
= \quad & P(A_{n1})/(m_n - j + 1) + \frac{m_n - j}{m_n - j + 1} P(A_{n1} \cap A_{n2}) - P(A_{n1})^2
\end{aligned}
$$

so

$$
\begin{aligned}
& \sum_{j=1}^{n} E|P(A_{nj}|\mathcal{H}_{n,j-1}) - P(A_{nj})| \\
\leq \quad & \frac{n}{(m_n - n)^{1/2}} P(A_{n1})^{1/2} + n|P(A_{n1} \cap A_{n2}) - P(A_{n1})^2|^{1/2} \\
\longrightarrow \quad & 0, \quad \text{using (i) - (iii).}
\end{aligned}
$$

Contractable Sequences

Kallenberg (2000) renewed interest in a more general structure than exchangeability. A sequence of random variables $(X_1, \ldots, X_n)$ taking values in some measurable space $(G, \mathcal{G})$ is **contractable** (sometimes referred to as **spreadable**) if for any finite $m < n$ and increasing subsequence of indices $k_1 < \ldots < k_m$,

$$(X_{k_1}, \ldots, X_{k_m}) =_{\mathcal{D}} (X_1, \ldots, X_m).$$

If the above is true for any finite subset of distinct, but not necessarily increasing indices, then the sequence is **exchangeable**.

- Contractability implies the variables are identically distributed.
- Exchangeability and contractability are equivalent in the infinite case (Ryll-Nardzewski (1957)) but not in the finite case.
- What extra features or conditions are needed for contractable sequences to be exchangeable?

Example (Kallenberg (2000))

(X_1, X_2, X_3) takes values in $\{0, 1, 2\}$. Joint distribution:

Value	Probability	Value	Probability
(0,0,1)	2/14	(1,2,2)	1/14
(0,1,2)	2/14	(2,0,0)	2/14
(0,2,0)	2/14	(2,0,2)	1/14
(1,0,2)	1/14	(2,1,0)	1/14
(1,2,0)	1/14	(2,2,1)	1/14

$$P(X_i = 0) = 6/14, \quad P(X_i = 1) = 3/14, \quad P(X_i = 2) = 5/14.$$

Note $(X_1, X_2) \stackrel{\mathcal{D}}{=} (X_1, X_3) \stackrel{\mathcal{D}}{=} (X_2, X_3)$, but $(X_1, X_2) \stackrel{\mathcal{D}}{\neq} (X_2, X_1)$

(eg $P(X_1 = 2, X_2 = 0) = 3/14$ whereas $P(X_1 = 0, X_2 = 2) = 2/14$).

$\mathcal{F}$ - Contractability

Given a discrete filtration $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, ...)$, the finite or infinite sequence $(X_1, X_2, \ldots,)$ is $\mathcal{F}$-contractable (exchangeable) if it is adapted to $\mathcal{F}$ and when conditioned on $\mathcal{F}_k$,

$$\theta_k \circ X = (X_{k+1}, \ldots)$$

is contractable (exchangeable) for every k.

- Every contractable sequence is $\mathcal{F}$-contractable wrt the minimal filtration.
- If X is $\mathcal{F}$-contractable then

$$E(X_n|\mathcal{F}_k) = E(X_{k+1}|\mathcal{F}_k), \quad \text{for all } n > k.$$

Specifically if $X_1, .., X_n$ are $\mathcal{F}$-contractable where

$$\mathcal{F}_{j-1} = \sigma\{X_1, .., X_{j-1}, \sum_{i=j}^{n} X_i\} = \sigma\{X_1, .., X_{j-1}, \sum_{i=1}^{n} X_i\}$$

then

$$E(X_j|\mathcal{F}_{j-1}) = (n-j+1)^{-1} \sum_{i=j}^{n} X_i.$$

This was a key observation in applying martingale weak limit results to obtain results for exchangeable sequences.

Lemma 1. [Theorem 1.13, Kallenberg (2005)] *If $X = (X_1, X_2, .., X_n)$ is $\mathcal{F}$-contractable and $\Sigma_{i=1}^{n} X_i$ is $\mathcal{F}_0$-measurable then X is exchangeable given $\mathcal{F}_0$.*

Partially Exchangeable Arrays.

U-statistics

Given a sequence of iid random variables $\xi_1, \xi_2, ..$ and a function $h : R^m \to R$ which is symmetric in its arguments, the U-statistic with kernel h (of degree m) is

$$U_n = \binom{n}{m}^{-1} {\sum_n}' h(\xi_{i_1}, .., \xi_{i_m}),$$

where ${\sum}'_n$ denotes summation over $1 \le i_1 < i_2 < .. < i_m \le n$.

Examples:

1.	Sample moments	$h(x) = x^k$
2.	Sample variance	$h(x, y) = \frac{1}{2}(x - y)^2$
3.	Wilcoxon statistic	$h(x, y) = I(x + y > 0)$
4.	Rayleigh statistic	$h(x, y) = 2\cos(x - y), \quad x \in [0, 2\pi)$

Let

$$\mathcal{I}_n = \sigma\{(\xi_{(1)}, .., \xi_{(n)}),\ \xi_{n+1}, \xi_{n+2}, ..\}.$$

Then $\mathcal{I}_n \supseteq \mathcal{I}_{n+1}$ and

$$U_n = E(h(\xi_1, .., \xi_m)\ |\mathcal{I}_n).$$

Thus $\{U_n\}$ is a reversed martingale wrt $\{\mathcal{I}_n\}$.

If $E|h(\xi_1, .., \xi_m)| < \infty$ strong consistency follows from the reversed mg convergence theorem.

Loynes (1969) conjectured that the CLT for U-statistics, established by Hoeffding (1948), should follow via martingale techniques.

Jointly Exchangeable Arrays

- U-statistics are special cases of partial sums of jointly exchangeable random variables. The array $\{X_{i_1,i_2,..,i_m}\}$ is jointly (or weakly) exchangeable if

$$\{X_{i_1,i_2,..,i_m}\} \stackrel{\mathcal{D}}{=} \{X_{\pi(i_1),\pi(i_2),..,\pi(i_m)}\} \tag{1}$$

  where π is a permutation of the positive integers that leaves all but a finite number of terms fixed. Trivially

$$X_{i_1,i_2,..,i_m} = h(\xi_{i_1}, \xi_{i_2}, .., \xi_{i_m})$$

  satisfies (3).

- The arrays $\{h(\xi_{i_1}, .., \xi_{i_m})\}$ are infact dissociated arrays in that terms with disjoint indexing sets are independent. Limit theorems for dissociated arrays were established by Silverman (1976).

For the infinite, symmetric, jointly exchangeable array case Eagleson and Weber (1978) used

$$\mathcal{I}_n = \sigma\{T_n, T_{n+1}, ..\} \quad \text{where} \quad T_n = E(X_{1,2,..,m}|\mathcal{I}_n) = \binom{n}{m}^{-1} \sum_n{}' X_{i_1,i_2,..,i_m}.$$

$\{T_n, \mathcal{I}_n\}$ is a reversed martingale and martingale methods were used to establish the central limit theorem for T_n.

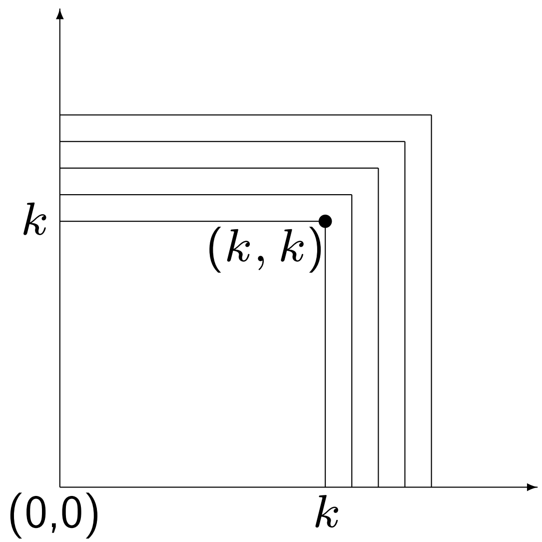

Figure 1: σ-field structure for $m = 2$

- Aldous (1981) extended de Finetti's result to arrays and proved that every infinite, jointly exchangeable array is a mixture of dissociated arrays. Thus the central limit result for partial sums can be deduced from this and Silverman (1976).

- Aldous' representation does not hold for finitely jointly exchangeable arrays. Martingale methods are particularly powerful in these cases.

- Results obtained for sequences of partial sums of finitely, jointly exchangeable random variables have direct application to U-statistics based on samples drawn without replacement from finite populations.

- To establish limit theorems we need to select filtrations that enable key conditional expectations to be evaluated.

Eagleson (1979) constructed an appropriate filtration to create forward martingales to establish Poisson limit laws.

Theorem 6. [Eagleson (1979), Theorem 3] *Let* $\{A_{ij}^{(n)} : 1 \leq i, j \leq m_n\}$ *be a sequence of symmetric, jointly exchangeable events. If, for some constant* λ*,*

(i) $\binom{n}{2} P(A_{12}^{(n)}) \to \lambda,$

(ii) $n^3 P(A_{12}^{(n)} \cap A_{13}^{(n)}) \to 0,$

(iii) $\binom{n}{2}^2 P(A_{12}^{(n)} \cap A_{34}^{(n)}) \to \lambda^2,$

(iv) $n/m_n \to 0,$

then

$$\sum_{1 \leq i < j \leq n} I(A_{ij}^{(n)}) \xrightarrow{\mathcal{D}} \textit{Poisson}(\lambda).$$

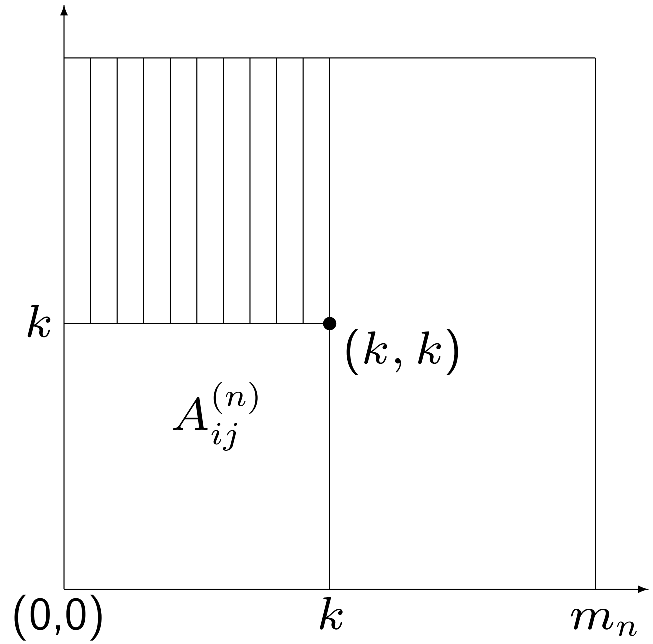

Figure 2: σ-field structure

The σ-field structure used in this case was

$$\mathcal{F}_{nk} = \sigma\{A_{ij}^{(n)}, 1 \le i, j \le k; U_{1k}^{(n)}, .., U_{kk}^{(n)}\}$$

where $U_{\alpha k}^{(n)} = \sum_{\beta=k+1}^{m_n} I(A_{\alpha\beta}^{(n)}),\ \alpha \le k.\quad \mathcal{F}_{n,k-1} \subseteq \mathcal{F}_{nk}$. For $i < j$

$$P(A_{ij}^{(n)} | \mathcal{F}_{n,j-1}) = U_{i,j-1}^{(n)}/(m_n - j + 1).$$

Symmetric statistics of U-type

By considering sequences of arrays we can obtain results for generalizations of U-statistics where the kernels depend on the sample size. Such statistics arise naturally in analysing spatial data. Given a sequence of iid random variables $\xi_1, \xi_2, ..$ define

$$S_n = \binom{n}{m}^{-1} \sum_n{}' \, h_n(\xi_{i_1}, .., \xi_{i_m}).$$

Asymptotic normality can be established via reversed martingale methods using the σ-fields

$$\mathcal{F}_{nj} = \sigma\{(\xi_{(1)}, .., \xi_{(j)}), \xi_{j+1}, \xi_{j+2}, ..\}$$

(Weber (1983)). Poisson limits were obtained by Silverman and Brown (1978).

Examples

- **Interpoint distance statistic** - test for clusters

$$h_n(\xi_1, \xi_2) = I(d(\xi_1, \xi_2) \leq \beta n^{-\alpha})$$

for some constants α and β, $\xi_i \in R^k$. A normal limit exists for if $\alpha \in [0, 2/k)$ and a Poisson limit if $\alpha = 2/k$.

- **Test for collinearity**.
Kendall and Kendall (1980) suggest using an indicator function h_n and counting the number of triples where the largest interior angle is greater than $\pi - \beta n^{-\alpha}$. If $0 \leq \alpha < 2$ then a normal limit exists.

Separately Exchangeable Arrays

$$\{X_{ij}^{(n)} : 1 \leq i \leq c_n, 1 \leq j \leq r_n\} \stackrel{\mathcal{D}}{=} \{X_{\pi_n(i),\sigma_n(j)}^{(n)}\}$$

where σ_n is any permutation of $(1, 2, .., c_n)$ and π_n is any permutation of $(1, 2, .., r_n)$.

In these cases the arrays are rectangular and so typically when constructing the σ-fields we have to consider row sums, column sums plus the total of the upper rectangular region. Exchangeability within rows and within columns is exploited to evaluate the conditional expectations.

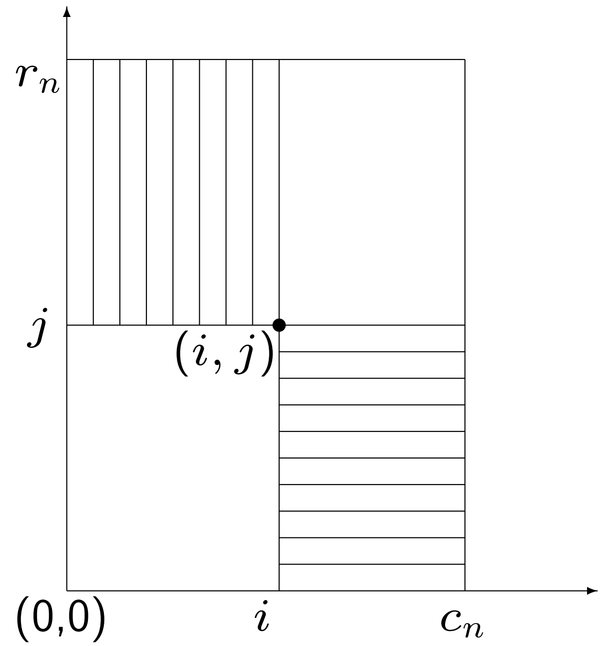

Figure 3: σ-field structure for separately exchangeable arrays.

This approach can also be used to construct relevant 1- and 2- martingales to obtain 2-d limits. For example,

Theorem 7. [Brown et al. (1986), Theorem 4.1] *Let* $\{A_{ij}^{(n)} : 1 \leq i \leq r_n,$ $1 \leq j \leq m_n\}$ *be a sequence of separately exchangeable events. If, for some constant* λ*,*

(i) $n^2 P(A_{11}^{(n)}) \to \lambda,$
(ii) $n^3 P(A_{11}^{(n)} \cap A_{12}^{(n)}) \to 0,$
(iii) $n^3 P(A_{11}^{(n)} \cap A_{21}^{(n)}) \to 0,$
(iv) $n^4 P(A_{11}^{(n)} \cap A_{22}^{(n)}) \to \lambda^2,$
(v) $n/m_n \to 0,$ *and* $n/r_n \to 0$

then

$$\sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} I(A_{ij}^{(n)})$$

converges in distribution to a Poisson process with intensity λ *times Lebesgue measure.*

References

[1] Aldous, D.J. (1981) Representations for partially exchangeable arrays of random variables. *J. Multivariate Anal.*, **11**, 581-598.

[2] Blum, J.R., Chernoff, H., Rosenblatt, M. and Teicher, H. (1958). Central limit theorems for interchangeable processes, *Canad. J. Math.*, **10**, 222 - 229.

[3] Brown, B.M. and Eagleson, G.K. (1971) Martingale convergence to infinitely divisible laws with finite variances. *Trans. Amer. Math. Soc.*, **162**, 449 - 453.

[4] Brown, T.C., Ivanoff, B.G. and Weber, N.C. (1986) Poisson convergence in two dimensions with application to row and column exchangeable arrays. *Stoch. Proc. Appl.*, **23**, 307 - 318.

[5] Bühlmann, H. (1958) Le problème "limit central" pour les variables alèatoires échongeables. *C.R. Acad. Sci. Paris*, **246**, 534 - 536.

[6] Chernoff, H. and Teicher, H. (1958) A central limit theorem for sums of interchangeable random variables. *Ann. Math. Statist.*, **29**, 118 - 130.

[7] De Finetti, B. (1930) Funzione caratteristica di un fenomeno aleatorio. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.*, **4**, 86 - 133.

[8] Eagleson, G.K. (1979) A Poisson limit theorem for weakly exchangeable events. *J. Appl. Probab.*, **16**, 794 - 802.

[9] Eagleson, G.K. (1982) Weak limit theorems for exchangeable random variables. In *Exchangeability in Probability and Statistics* Ed. G. Koch and F.Spizzichino, North-Holland, Amsterdam-New York, 251-268.

[10] Eagleson, G.K. and Weber, N.C. (1978). Limit theorems for weakly exchangeable arrays. *Math. Proc. Camb. Philos. Soc.*, **84**, 123 - 130.

[11] Freedman, D. (1974) The Poisson approximation for dependent events. *Ann. Probab.*, **2**, 256 - 269.

[12] Hall, P. and Heyde, C.C (1980) *Martingale Limit Theory and Its Application.* Academic Press, London

[13] Hoeffding, W. (1948) A class of statistics with asymptotically normal distribution. *Ann. Math. Statist.*, **19**, 293 - 325.

[14] Kallenberg, O. (1988) Spreading and predictable sampling in exchangeable sequences and processes. *Ann. Probab.*, **16**, 508 - 534.

[15] Kallenberg, O. (2000) Spreading-invariant sequences and processes on bounded index sets. *Probab. Theory Rel. Fields*, **118**, 211 - 250.

[16] Kallenberg, O. (2005) *Probabilistic Symmetries and Invariance Principles.* Springer, New York.

[17] Kendall, D.G. and Kendall, W.S. (1980) Alignments in two dimensional random sets of points. *Adv. Appl. Probab.*, **12**, 380 - 424.

[18] Loynes, R.M. (1969) The central limit theorem for backwards martingales. *Z. Wahrsch. Verw. Gebiete*, **13**, 1 - 8.

[19] Ridler-Rowe, C.J. (1967) On two problems on exchangeable events. *Studia Sci. Math. Hungar.*, **2**, 415 - 418.

[20] Ryll-Nardzewski, C. (1957) On stationary sequences of random variables and the de Finetti's equivalence. *Colloq. Math.*, **4**, 149 - 156.

[21] Scott, D.J. (1973) Central limit theorems for martingales and for processes with stationary increments using a Skorokhod representation approach. *Adv. Appl. Probab.*, **5**, 119 - 137.

[22] Silverman, B.W. (1976) Limit theorems for dissociated random variables. *Adv. Appl. Probab.*, **8**, 806-819.

[23] Silverman, B.W. and Brown, T.C. (1978) Short distances, flat triangles, and Poisson limits. *J. Appl. Probab.*, **15**, 815 - 825.

[24] Weber, N.C. (1980). A martingale approach to central limit theorems for exchangeable random variables. *J. Appl. Probab.*, **17**, 662 - 673.

[25] Weber, N.C. (1983). Central limit theorems for a class of symmetric statistics. *Math. Proc. Camb. Philos. Soc.*, **94**, 307 - 313.