

$$\hat{\theta} \cdot \sum_{ij} q_{ij}/n^2 = \sum_{ij} q_{ij}/n^2 + \frac{1}{n} \sum_{ij} q_{ij}/n^2.$$

$$v(x_i) \cdot \sum_j q_{ij}/n, \quad v(y_j) \cdot \sum_i q_{ij}/n = g_{\text{true}} \quad v(x_i) = q_{i1}/n, \quad v(y_j) = q_{j1}/n$$

$$S_x = \sum_i (v(x_i) - \hat{\theta})^2, \quad S_y = \sum_j (v(y_j) - \hat{\theta})^2$$

$$S_{xy} = \sum_{ik} (v(x_k) - \hat{\theta})(v(y_k) - \hat{\theta})$$

$$\hat{\theta}^2 = S_x/n + S_y/n + 2S_{xy}/n^2 - M^2 \left\{ (v(x_1) + v(y_1)) - 2\hat{\theta} \right\}^2$$

$$\theta_{-i} = \sum_{j \neq i, k \neq i} \frac{q_{ik}}{(n-1)^2}, \quad \hat{\theta}_j = M\hat{\theta} - (M-1)\theta_{-j}, \quad \hat{\sigma}^2 = \frac{1}{(M-1)} \sum_{i,j} (\hat{\theta}_i - \bar{\theta})^2,$$

$$\bar{\theta} = M\hat{\theta} - \frac{M-1}{M} \sum_j \theta_{-j}, \quad \text{diag}(M-1)^2 \theta_{-i} = \sum_k q_{ik} - \sum_j q_{ij} - \sum_j q_{-j} + q_{ii}.$$

$$= M^2 \hat{\theta} - q_{ii} - q_{-i} + q_{ii} \quad \sum_j \theta_{-j} = \frac{1}{(M-1)^2} \hat{\theta} - \frac{M^2 \hat{\theta}}{(M-1)^2} = \frac{\text{diag } \hat{\theta}}{(M-1)^2},$$

$$\bar{\theta} = \hat{\theta} \left( M - \frac{M}{M-1} + \frac{M}{M-1} \right) - \frac{\text{diag } \hat{\theta}}{M(M-1)}, \quad M-1 - \frac{M}{M-1} \frac{M}{M-1} = \frac{M-2}{M-1},$$

$$\bar{\theta} = \hat{\theta} \frac{1}{n-1} \left( \hat{\theta}' - \text{diag } \hat{\theta}/M \right), \quad \hat{\theta}_j - \bar{\theta} = M\hat{\theta} - (M-1)\theta_{-j} - \frac{1}{n-1} \hat{\theta} + \frac{\text{diag } \hat{\theta}}{M(M-1)}$$

$$= - \frac{M^2 \hat{\theta}}{M-1} + \frac{\text{diag } \hat{\theta}}{M(M-1)} + \frac{1}{n-1} (M^2 \hat{\theta} - q_{ii} - q_{-i} + q_{ii})$$

$$= - \frac{M^2 \hat{\theta}}{M-1} + \frac{\text{tr } \hat{\theta}}{M(M-1)} + \frac{q_{ii} + q_{-i} - q_{ii}}{M-1} = 2M^2 \hat{\theta} - \text{tr } \hat{\theta},$$

$$G_{\text{true}} = M \left( \frac{\text{tr } \hat{\theta}}{n(M-1)} - \frac{2M}{n-1} \hat{\theta} \right)^2 + 2 \left( \frac{\text{tr } \hat{\theta}}{n(M-1)} - \frac{2M}{n-1} \hat{\theta} \right) \frac{2M^2 \hat{\theta} - \text{tr } \hat{\theta}}{n-1} + \sum_j \frac{(q_{ij} + q_{-i} - q_{ii})^2}{(n-1)^2}$$

$$= \sum_j \frac{(q_{ij} + q_{-i} - q_{ii})^2}{(n-1)^2} - M \left( \frac{2M^2 \hat{\theta} - \text{tr } \hat{\theta}}{n(M-1)} \right)^2 = 2M(M-1) \hat{\sigma}_{jj}^2 + \sum_j \left( \frac{q_{ij}}{n} + \frac{q_{-i}}{n} - 2\hat{\theta} \right)^2$$

$$= M^{-2} \sum_j (q_{ij} + q_{-i} - q_{ii})^2 + 2M^2 \hat{\theta}^2 = \text{tr } \hat{\theta} - 4M^2 \hat{\theta}^2 + 2 \sum_j (q_{ij} + q_{-i} - q_{ii})^2$$

$$+ 4M^2 \hat{\theta}^2 = 2M(M-1) \hat{\theta}^2 + 4M^2 \hat{\theta}^2, \quad \sum_j (q_{ij} + q_{-i} - q_{ii}) (q_{ij} - 2M\hat{\theta}) =$$

$$= 2M\hat{\theta} (2M^2 \hat{\theta} - \text{tr } \hat{\theta}) + 2(q_{ij} + q_{-i} - q_{ii}) + \sum_j q_{ij} q_{ii} - \text{tr } \hat{\theta} = 4M^3 \hat{\theta}^2 + 2 \sum_j (q_{ij} + q_{-i} - q_{ii})$$

$$+ 2(q_{ij} + q_{-i} - q_{ii})^2 = \frac{1}{2} (q_{ij} + q_{-i} - q_{ii})^2 - \frac{\text{tr } \hat{\theta}}{2} - 4M^2 \hat{\theta}^2$$

$$M(M-1) \hat{\sigma}_{\text{obs}}^2 = \sum_j (\varphi_{j1} + \varphi_{j2} - 2\hat{\theta})^2 = M^{-2} \left( \sum_j \varphi_{j1}^2 + \sum_j \varphi_{j2}^2 + 2 \sum_j \varphi_{j1} \varphi_{j2} \right)$$

$$= 4\hat{\theta} \sum_j (\varphi_{j1} + \varphi_{j2}) + 4M\hat{\theta}^2, \quad M^{-2} \mathbb{H}^T (\varphi^T \varphi + \varphi \varphi^T + 2\varphi \varphi) \mathbb{H}$$

~~$$\text{rank } M = 4\hat{\theta} \frac{1}{M} \cdot 2M^2 \hat{\theta} + 4M\hat{\theta}^2 = M^{-2} \mathbb{H}^T (-) \mathbb{H} - 4M\hat{\theta}^2 =$$~~

$$M^{-2} \mathbb{H}^T (\varphi^T \varphi + \varphi \varphi^T + 2\varphi \varphi - 4/M \varphi \mathbb{H}^T \varphi) \mathbb{H} \quad \frac{1}{(M-1)^2} - \frac{1}{M^2} = \frac{2M-2}{(M(M-1))^2}$$

$$\text{Div? } \frac{2M-2}{(M(M-1))^2} \mathbb{H}^T (\varphi^T \varphi + \varphi \varphi^T + 2\varphi \varphi - \frac{4}{M} \varphi \mathbb{H}^T \varphi) \mathbb{H} - \frac{2}{(M-1)^2} \mathbb{H}^T (\varphi + \varphi^T - \mathbb{I}/2)$$

$$- \frac{2}{M} \varphi \mathbb{H} \mathbb{H}^T + \text{diag } \varphi \mathbb{H}^T / (2M) \cdot \text{diag } \varphi$$

$$\frac{M-1}{M^2} \mathbb{H}^T (\varphi^T \varphi + \varphi \varphi^T + 2\varphi \varphi - \frac{4}{M} \varphi \mathbb{H} \mathbb{H}^T \varphi) \mathbb{H} - \mathbb{H}^T (\varphi + \varphi^T - \frac{\mathbb{I}}{2} - \frac{2}{M} \varphi \mathbb{H} \mathbb{H}^T +$$

$$\text{diag } \varphi \mathbb{H}^T / (2M) \text{ diag } \varphi = \frac{1}{2} (M-1)^2 M (\sigma_{j_1}^2 - \sigma_{j_{20}}^2)$$

$$(q = q^T) \quad \mathbb{H}^T (\mathbb{I} - \mathbb{I} - \frac{1}{M} \mathbb{H} \mathbb{H}^T) \varphi \mathbb{H} = \sum s_j^2 - \frac{1}{M} (\sum s_j)^2 \quad \text{Var}(\mathbb{H}^T \varphi) = \overbrace{\text{Var}(\mathbb{H}^T \varphi^T)}$$

$$\approx \frac{M}{2} ((M-1)^2 + (M)^2) = \frac{M^3}{4}$$

$$\mathbb{H}^T (\varphi^T \varphi - \varphi \mathbb{H} \mathbb{H}^T \varphi / M) \mathbb{H} = \mathbb{H}^T (\varphi^T - \varphi \mathbb{H} \mathbb{H}^T / M) \varphi \mathbb{H}, \quad \mathbb{H}^T \varphi \cdot \text{Proj}_{\mathbb{H}^\perp} \varphi \mathbb{H}$$

$$= \mathbb{H}^T \varphi (\vec{r} - \bar{\vec{r}}) = (\vec{c}, \vec{r} - \bar{\vec{r}}) \perp (\vec{c}, \vec{r} - \bar{\vec{c}}), \quad (\vec{c}, \vec{r}) = (\sum \varphi_{ij}) / M,$$

$$(\vec{c}, \vec{r}) = \sum_j (\sum_i \varphi_{ij}) (\sum_i \varphi_{ji}) = \sum_j \sum_{i,k} \varphi_{ij} \varphi_{jk}, \quad \sum_{i,k} \sum_j \varphi_{ij} \varphi_{jk} = \sum_{i,k} \sum_{j \neq k} \varphi_{ij} \varphi_{jk} / M$$

$$- \sum_{i \neq j} (\varphi_{ij} \varphi_{jk} - \sum_k \varphi_{ij} \varphi_{ik} / M) = \sum_{i \neq j} \varphi_{ij} (\varphi_{jk} - \varphi_{ik})$$

$$\begin{aligned}
 M(M-1) \sigma_{\text{obs}}^2 &= M^{-2} \sum_j (\varphi_{j.} + \varphi_{.j} - 2M\hat{\theta})^2, \quad \varphi_{j.} = 1 - \varphi_{jj}, \quad \varphi_{.j} = \sum_i \varphi_{ij} \\
 &= \sum_j (1 - \varphi_{jj})^2 = M - \varphi_{..}, \quad M^{-2} \sum_j (M - 2M\hat{\theta})^2 = M - 4M\hat{\theta} + 4\hat{\theta}^2 = M(1 - 2\hat{\theta})^2, \\
 \sigma_{\text{obs}}^2 &= \frac{(M-2\hat{\theta})^2}{M-1} \\
 n(n-1) \sigma_{jk}^2 &= \frac{n}{(n-1)^2} - M \left( \frac{-2M\hat{\theta}}{M(n-1)} \right)^2 = \frac{n^2}{(M-1)^2} - \frac{4M^3\hat{\theta}^2}{(M-1)^2} = \frac{M^2}{(M-1)^2} (1 - 4\hat{\theta}^2), \\
 \sigma_{jk}^2 &= \frac{M^2}{(M-1)^2} (1 - 4\hat{\theta}^2)
 \end{aligned}$$

$$\sum_j \varphi_{.j} = \sum_i P(X_i < \tau_j) = \sum_i P(X_i > \tau_j)$$

$$M^{-2} \sum_j (\varphi_{j.} + \varphi_{.j} - \varphi_{jj})^2 - \text{tr} \varphi / M^2 - 4M\hat{\theta}^2 + \frac{2}{M^2} \sum_j \varphi_{jj} (\varphi_{j.} + \varphi_{.j}) = ?$$

$$(M-1)^{-2} \sum_j (\varphi_{j.} + \varphi_{.j} - \varphi_{jj})^2 - M^{-1} \left( \frac{\text{tr} \varphi - 2M^2\hat{\theta}}{M(M-1)} \right)^2$$

$$\cancel{\sum_j (\varphi_{j.} + \varphi_{.j} + \varphi_{jj})^2} \cancel{+ 2M\hat{\theta}\cancel{\sum_j (\varphi_{j.} + \varphi_{.j})}}, \quad \sum_j \varphi_{j.}^2 = \sum_j (\sum_i \varphi_{ji})^2 = \sum_j (\sum_i \varphi_{ji} + 2 \sum_{i < k} \varphi_{ji} \varphi_{jk})$$

$$\varphi_{jj} \varphi_{jk} = M^2\hat{\theta} + 2 \sum_{i < k} (\varphi_{ji} \varphi_{ik}), \quad \sum_j \varphi_{j.}^2 = \mathbb{1}^\top \varphi^\top \varphi \mathbb{1}, \quad \sum_j \varphi_{.j}^2 = \mathbb{1}^\top \varphi \varphi^\top \mathbb{1},$$

$$\sum_j \varphi_{j.} \varphi_{.j} = \sum_j (\sum_i \varphi_{ji}) (\sum_k \varphi_{kj}) = \sum_j \sum_{i,k} \varphi_{ji} \varphi_{kj} = \sum_{i,k} \sum_j \varphi_{ji} \varphi_{kj} = \mathbb{1}^\top \varphi \varphi^\top \mathbb{1},$$

$$\sum_j (\varphi_{j.} + \varphi_{.j} - \varphi_{jj})^2 = \mathbb{1}^\top (\varphi^\top \varphi + \varphi \varphi^\top + 2\varphi \varphi) \mathbb{1} + \text{tr} \varphi - 2 \mathbb{1}^\top \mathbb{1} (\varphi + \varphi^\top).$$

$$\text{diag } \varphi = \mathbb{1}^\top (\varphi^\top \varphi + \varphi \varphi^\top + 2\varphi \varphi) \mathbb{1} - 2 \mathbb{1}^\top (\varphi - \mathbb{I}/2 + \varphi^\top) \text{diag } \varphi,$$

$$\begin{aligned}
 \text{tr} \varphi - 2M^2\hat{\theta} &= \text{tr} \varphi - 2 \mathbb{1}^\top \varphi \mathbb{1}, \quad \text{then } (M-1)^{-2} \left\{ \sum_j (\varphi_{j.} + \varphi_{.j} - \varphi_{jj})^2 - \frac{(\text{tr} \varphi)^2}{M} \right\} = M - 4M^3\hat{\theta}^2 + 4M\hat{\theta} \text{tr} \varphi \}
 \end{aligned}$$

$$\begin{aligned}
 (M-1)^{-2} \left\{ \mathbb{1}^\top (\varphi^\top \varphi + \varphi \varphi^\top + 2\varphi \varphi - 4M^{-1}\varphi \mathbb{1} \mathbb{1}^\top \varphi / \mathbb{1}) - 2 \mathbb{1}^\top (\varphi - \mathbb{I}/2 + \varphi^\top - 2/M \varphi \mathbb{1} \mathbb{1}^\top \right. \\
 \left. + \text{diag } \varphi \cdot \mathbb{1}^\top / 2M) \text{diag } \varphi \right\} = M(M-1) \hat{\sigma}_{jk}^2
 \end{aligned}$$

$$\int_0^{\infty} \frac{y}{\theta} e^{-\frac{y}{\theta}} = \left[ \theta \left( \frac{y}{\theta} - y e^{-\frac{y}{\theta}} \right) \right]_0^{\infty} + \left[ e^{-\frac{y}{\theta}} \right]_0^{\infty} = -\theta e^{-\frac{y}{\theta}} \Big|_0^{\infty} = 0, \quad \int_0^{\infty} \frac{y^2}{\theta} e^{-\frac{y}{\theta}} =$$

$$= \frac{1}{2} \cdot y_0 e^{-\frac{y_0}{\theta}} \left[ \frac{y^2}{\theta} - \left[ 0 e^{-\frac{y}{\theta}} \right] \right]_0^{\infty} = -\theta^2 e^{-\frac{y_0}{\theta}} \Big|_0^{\infty} = \theta^2$$

$$\sum (Y_i - 2\bar{Y})(Y_i - \bar{Y}) = \sum Y_i^2 - n\bar{Y}^2, \quad \text{as } \frac{(n-1)\sigma^2}{\theta^2} \sim \chi^2(n-1), \quad \left(\frac{n}{\theta^2}\right)^2 \sim S^2 = 2(n-1)$$

$$\text{Var} S = \frac{2\sigma^4}{n-1}, \quad I \sim ((Y_i^2 - 2Y_i \bar{Y}_i + \bar{Y}_i^2)), \quad \text{I.e. } Y_i \sim N(0, 2\sigma^2), \quad \frac{Y_i - \bar{Y}_i}{\sqrt{2\sigma^2}} \sim Z,$$

$$\text{as } \frac{1}{2\sigma^2} \text{Var}(Y_i - \bar{Y}_i) = 1, \quad \text{as } \left( \frac{Y_i - \bar{Y}_i}{\sqrt{2\sigma^2}} \right)^2 \sim \text{Var } \chi^2(1) = 2, \quad \text{Var}(Y_i - \bar{Y}_i)^2 \\ = 8\sigma^4, \quad \text{approximate}$$

$$\left\{ \partial y^\theta = \frac{\theta}{\theta+1} y^{\theta+1} \right\} \Rightarrow \frac{\theta}{\theta+1}$$

$$2EY_{ij}^2 = 2\lambda(1+\lambda) - 2\lambda^2 = 2\lambda, \quad \text{Var}((Y_i - \bar{Y}_i)^2)$$

$$\frac{\theta+1}{\theta} y^{\theta+1} \Big|_0 = \frac{\theta+1}{\theta}, \quad \frac{\theta+1}{\theta+2} y^{\theta+2} \Big|_\infty = \frac{\theta+1}{\theta+2} = \bar{x}, \quad \theta(1-\bar{x}) = (\bar{x}-1), \quad \hat{\theta} = \frac{2\bar{x}-1}{1-\bar{x}} = \frac{\theta}{p} = \theta$$

$$\sigma^2 + \mu^2 = \bar{x}^2 = \frac{q}{p} = (n-p) \left\{ \frac{1}{(1-p)^2} \right\} \bar{x}^{2n}/\theta, \quad \bar{x}^n e^{\theta \bar{x}^2 / \theta}, \quad \text{by } -n \log \theta$$

$$- \sum x_i / \theta, \quad -n/\theta + \sum x_i / \theta, \quad \hat{\theta} = \bar{x}, \quad \left( \frac{\bar{x}}{2\theta+1} \right)^n \{ X_{(n)} \leq 2\theta+1 \}, \quad \hat{\theta} = \frac{1}{2}(X_{(n)} - 1),$$

$$\frac{1}{2\theta+1} \left\{ \bar{x}^2 + \frac{1}{3}(2\theta+1)^2 - \frac{1}{4}(2\theta+1)^2 \right\} = \frac{1}{12}(2\theta+1)^2, \quad \theta^{-2n} \prod y_i e^{-\theta y_i}, \quad -2n \log \theta + \sum y_i - 2 \bar{x} / \theta,$$

$$-2\bar{x}_0 + 2\bar{y}_{\theta}^2, \quad \hat{\theta} = \bar{Y}_2,$$

9.36 a

a)  $Y_1, \dots, Y_n$  iid  $\sim N(\mu, \sigma^2)$ , pdf  $f(y) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$

joint pdf  $\prod_{j=1}^n f(y_j) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{j=1}^n (y_j - \mu)^2\right)$

$$(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{j=1}^n (y_j^2 - 2\mu y_j + \mu^2)\right)$$

$$(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum y_j^2 + \frac{\mu}{\sigma^2} \sum y_j - \frac{n\mu^2}{2\sigma^2}\right)$$

$$f(y_1, \dots, y_n) = \prod f(y_j) = (2\pi\sigma^2)^{-n/2} \exp\left(\frac{\mu}{\sigma^2} \sum y_j - \frac{n\mu^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2} \sum y_j^2\right)$$

$$(2\pi\sigma^2)^{-n/2} \underbrace{\exp\left(\frac{n\mu}{\sigma^2} \bar{y} - \frac{n\mu^2}{2\sigma^2}\right)}_{\text{depends on } y_1, \dots, y_n \text{ only through } \bar{y}} \underbrace{\exp\left(-\frac{1}{2\sigma^2} \sum y_j^2\right)}_{\text{does not depend on target } \mu}$$

depends on  $y_1, \dots, y_n$  only through  $\bar{y}$  does not depend on target  $\mu$

b)  $f(y_1, \dots, y_n) = \prod f(y_j) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{j=1}^n (y_j - \mu)^2\right) \cdot 1$

depends on  $y_1, \dots, y_n$  only through  $\sum (y_j - \mu)^2$  does not depend on target  $\sigma^2$

c)  $f(y_1, \dots, y_n) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum y_j^2 + \frac{\mu}{\sigma^2} \sum y_j - \frac{n\mu^2}{2\sigma^2}\right) \cdot 1$

depends on  $y_1, \dots, y_n$  only through  $(\sum y_j^2, \sum y_j)$  does not depend on  $(\mu, \sigma^2)$

Q.39  $Y_1, \dots, Y_n \sim \text{Poi}(\lambda)$ , pmf is  $f_Y(j) = e^{-\lambda} \frac{\lambda^j}{j!}$  for  $j=0, 1, \dots$

also  $\sum_{j=1}^n Y_j \sim \text{Poi}(n\lambda)$ , pmf is then  $f_{\sum Y_j}(j) = e^{-n\lambda} \frac{(n\lambda)^j}{j!}$ .

Then  $P(Y_1=y_1, \dots, Y_n=y_n | \sum Y_j = \sum y_j)$

$$\frac{P(Y_1=y_1, \dots, Y_n=y_n \text{ and } \sum Y_j = \sum y_j)}{P(\sum Y_j = \sum y_j)}$$

$$= \frac{P(Y_1=y_1, \dots, Y_n=y_n)}{P(\sum Y_j = \sum y_j)} = \frac{\prod_{j=1}^n f_Y(y_j)}{f_{\sum Y_j}(\sum y_j)}$$

$$= \frac{\prod_{j=1}^n \left( e^{-\lambda} \frac{\lambda^{y_j}}{y_j!} \right)}{e^{-n\lambda} \frac{(n\lambda)^{\sum y_j}}{(\sum y_j)!}} = \frac{e^{-n\lambda} \frac{\lambda^{\sum y_j}}{\prod y_j!}}{e^{-n\lambda} \frac{(n\lambda)^{\sum y_j}}{(\sum y_j)!}}$$

$$= \frac{(\sum y_j)!}{\prod y_j!} \quad \text{which doesn't depend on } \lambda$$

Q.40  $Y_1, \dots, Y_n \sim \text{Rayleigh}(\theta)$ , pdf is  $f_Y(y) = \left(\frac{y}{\theta}\right) e^{-\frac{y^2}{\theta}}$  for  $y > 0$ ,

$$\therefore f(y_1, \dots, y_n) = \prod_{j=1}^n f_Y(y_j) = \prod_{j=1}^n \left(\frac{y_j}{\theta}\right) e^{-\frac{y_j^2}{\theta}} = \left(\frac{y}{\theta}\right)^n (\prod y_j) e^{-\frac{\sum y_j^2}{\theta}}$$

$$= \underbrace{\left(\frac{y}{\theta}\right)^n e^{-\frac{\sum y_j^2}{\theta}}}_{\text{depends on } y_1, \dots, y_n \text{ only through } \sum y_j^2} \cdot \underbrace{(\prod y_j)}_{\text{doesn't depend on } \theta}$$

9.41

$y_1, \dots, y_n \sim \text{Weibull}(\alpha)$ , pdf  $\Rightarrow f_Y(y) = \frac{1}{\alpha} \alpha y^{\alpha-1} e^{-y/\alpha}$  for  $y > 0$

$$f(y_1, \dots, y_n) = \prod_{j=1}^n f_Y(y_j) = \prod_{j=1}^n \left( \frac{1}{\alpha} \alpha y_j^{\alpha-1} e^{-y_j/\alpha} \right) = \left( \frac{n}{\alpha} \right)^n (\prod y_j)^{\alpha-1} e^{-\sum y_j/\alpha}$$

$$= \underbrace{\left( \frac{n}{\alpha} \right)^n e^{-\sum y_j/\alpha}}_{\substack{\text{dependence on} \\ \text{sample thru } \sum y_j}} \cdot \underbrace{(\prod y_j)^{\alpha-1}}_{\text{no } \alpha \text{ dependence}}$$

9.42 - HW

9.43

$$f(y_1, \dots, y_n) = \prod_{j=1}^n f_{Y_j}(y_j) = \prod_{j=1}^n \left( \alpha y_j^{\alpha-1} / \theta^\alpha \right) = \underbrace{\left( \frac{\alpha}{\theta^\alpha} \right)^n (\prod y_j)^{\alpha-1}}_{\substack{\text{depends on} \\ \text{sample thru } \prod y_j}} \cdot \underbrace{1}_{\substack{\text{no } \alpha \text{ dependence}}}$$

9.44

$$f(y_1, \dots, y_n) = \prod_{j=1}^n f_Y(y_j) = \prod_{j=1}^n \left( \alpha \beta^\alpha y_j^{-(\alpha+1)} \right) = \underbrace{(\alpha \beta^\alpha)^n (\prod y_j)^{-(\alpha+1)}}_{\text{ditto}} \cdot 1$$

$$f(y_1, \dots, y_n) = \prod_{j=1}^n f_Y(y_j) = \prod_{j=1}^n \left( a(\theta) b(y_j) e^{-c(\theta) d(y_j)} \right)$$

$$= \underbrace{a(\theta)^n e^{-c(\theta) \sum d(y_j)}}_{\substack{\text{depends on} \\ \text{sample thru } \sum d(y_j)}} \cdot \underbrace{\prod b(y_j)}_{\text{no } \theta \text{ dependence}}$$

7.46 pdf  $\mapsto f_y(y) = \frac{1}{\beta} e^{-y/\beta}$  which has form

$a(\theta) b(y) e^{-c(\theta)d(y)}$  by taking  $\theta = \beta$ ,  $c(\theta) = \frac{1}{\beta}$ ,  $a(\theta) = \frac{1}{\beta}$ ,  $b(y) = 1$ .

7.47 pdf  $\alpha y^{\alpha-1}/\bar{\theta}^\alpha = \bar{\theta}^{-\alpha} \alpha \exp((\alpha-1)\log y)$  has form

$a(\theta) b(y) e^{-c(\theta)d(y)}$  by taking  $\theta = \alpha$ ,  $d(y) = \log y$ ,  $c(\theta) =$   
 $\alpha-1$ ,  $a(\theta) = \bar{\theta}^{-\alpha} \cdot \alpha$ . So by 7.45  $\sum_j d(y_j) = \sum \log y_j$

is sufficient for  $\alpha$ . No contradiction since  $\sum \log y_j$   
 $= \log(\prod y_j)$  and "log" is one-to-one on  $\mathbb{R}^+$

7.48 pdf  $\alpha \beta^\alpha y^{-(\alpha+1)} = \alpha \beta^\alpha \exp(-(\alpha+1)\log y)$  has form

$a(\theta) b(y) e^{-c(\theta)d(y)}$  by taking  $\theta = \alpha$ ,  $d(y) = \log y$ ,

$c(\theta) = \alpha+1$ ,  $a(\theta) = \alpha \beta^\alpha$ ,  $b(y) = 1$ . Suf. suff. is

then  $\sum d(y_j) = \sum \log y_j = \log(\prod y_j)$ .

$$\begin{aligned}
& f'(i) \leq \frac{1+1}{1} = 1, \quad f(z) = \frac{1}{1+z-i}, \quad |1+z-i| \geq 10^2, \quad Re^{i\theta} + R^{-1}e^{-i\theta} = (R/\rho) \cos \theta \\
& + i(R - \frac{1}{\rho}) \sin \theta, \quad R^2 + \frac{1}{\rho^2} + 2(\cos^2 - \sin^2) = R^2 + \frac{1}{R^2} + 2 \cos 2\theta, \quad z = \frac{1}{R}, \quad R^2 + \frac{1}{R^2} - 2 \cos 2\theta, \\
& (R^2 + R^{-2}) R^2 + 2 \cos \frac{\theta}{2}, \quad z^{1/4} + \bar{z}^{-1/4} = \frac{z^{1/4} + 1}{z^{1/4}} = \bar{z}^1 (z^{5/4} + z^{3/4}), \quad z^\alpha + \bar{z}^{-\alpha} = \\
& z^{-\alpha} (z^{2\alpha} + 1), \quad \frac{1}{(z-a)(z-b)}, \quad \left. \frac{1}{(z-a)^2 (z-b)} + \frac{1}{(z-a)(z-b)^2} \right|_{z=i} : (i-a)^{-1} (i-b)^{-1} + (i-a)^{-1} (i-b)^{-2} \\
& = \frac{2i-a-b}{(i-a)^2 (i-b)^2} \text{ and } 12i\pi, \quad \frac{2i - (x_1 - iy + x_2 - iy)}{(i-x_1+iy)^2 (i-x_2+iy)^2} = \frac{-(x_1+x_2) + i(2+y+\bar{y})}{(-x_1+i(1+y))^2 (-x_2+i(1+y))^2}, \\
& ((x_1+x_2)^2 + 16)^{1/2} \geq (x_1^2 + 4)(x_2^2 + 4); \quad 2+y+\bar{y} \geq? (x_1^2 + (1+y)^2)(x_2^2 + (1+y)^2) \geq \\
& (2+y+\bar{y})^2 ...
\end{aligned}$$

More  $\{f(x) = 0 \Rightarrow |f(x)| = \sum_{j=0}^{m+1} |f(j)|\}, |f(x)| \leq \inf_{y \in D} |f(y)| \Rightarrow f(y) \neq 0$ ,

$$y \in D, \quad |f(x)| \geq \sup_{y \in \overline{D \setminus x}} |f(y)|, \quad \begin{cases} f = \text{const.} \\ f(x) \neq 0, \quad x \in D \end{cases} \Rightarrow \max_{\overline{D}} |f(x)|$$

$$\max_{\bar{D}} |f(x)| = c = \min_{\bar{D}} |f(x)|, \quad |f(x)| = c, \quad x \in \bar{D}, \quad u^2(x,y) + v^2(x,y) = c, \quad uu_x + vv_x$$

$$u u_y + v v_y = 0, \quad \frac{v_x}{u_x} = \frac{v_y}{u_y}, \quad \text{with } u_x - \frac{u_y}{u_x} = \frac{u_x^2}{u_y}, \quad u_x^2 = -u_y^2, \quad u_x = v_y = 0,$$

$$u_y = v_x = 0, \quad f(x) = \{m\}, \quad \rightarrow \text{#}44//$$

$$m, n > n_0 \Rightarrow \max_{z \in D} |t_n(z) - f_m(z)| \text{ with } \max_{z \in D} |z| = \frac{1}{2} \quad N(z) := \max_{\substack{z \in D \\ \operatorname{Re} z = x}} |f_h(z)|,$$

$$\text{Given } y_1 > \frac{\beta}{\eta\varepsilon} \Rightarrow |f_{\neq h_\varepsilon}| \leq \frac{\beta}{\varepsilon} \cdot \frac{1\varepsilon}{\beta} = 1, \quad R := \{(x, y) : a \leq x \leq b, \quad \frac{\beta}{\eta\varepsilon} \leq y \leq \frac{\beta}{\eta\varepsilon}\},$$

$$x_0 : \{x < x_0\} \cap R \Rightarrow |f(x)| < \eta, \quad |f(z)| < \eta, \quad x < x_0 \Rightarrow N(x) < \eta,$$

$$N(x) \rightarrow 0 \quad \text{as } x \rightarrow a^+, \quad \forall a < x < \xi < b \Rightarrow N(\xi) \leq N(x)^{\frac{b-\xi}{b-x}} N(b)^{\frac{\xi-a}{b-x}} \rightarrow 0, \quad N(x) \equiv 0,$$

$$h_\varepsilon(z) \neq 0, \quad z \in \Omega, \quad f = 0 \quad \text{iff} \quad M(r)^{\log^q r} \leq ? \quad M(n)^{\log^q n}, \quad M_4 = M_3$$

$$z_1 \Rightarrow M(r) \leq 1; \text{ then } \sup_{Re z=x} \log |f(z)| = \log \sup_{Re z=x} |f(z)|, \quad \left(\frac{r}{a}\right)^{\log M(r)} \leq \left(\frac{r}{a}\right)^{\log M(a)} \left(\frac{r}{a}\right)^{\log}$$

$$(f(x) = t), \log^{\log x} \leq ? \quad \log \log \log x, \quad \log \log \log x - \log \log \log x$$

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$|H|^2 = \exp(\log|z|)$ , where  $e^{p_x+i\beta_x} = e^{p_x}e^{ip_y} + e^{-p_x}e^{-ip_y} = (e^{p_x} + e^{-p_x})\cos\beta_y$   
 $+ i(e^{p_x} - e^{-p_x})\sin\beta_y$ , i.e.  $e^{2p_x} + e^{-2p_x} + 2\cos\beta_y + 2i\sin\beta_y = 2\cosh 2p_x$   
 $\pm 2i\sin(\beta_y + \frac{\pi}{4}) < 0$ ,  $x > 0 \Rightarrow |e^{p_x} + e^{-p_x}|^2 > 0 > 0$ , then  $\beta_y + \frac{\pi}{4} = \frac{2k+1}{2}\pi$ ,  $y =$   
 $\beta_y \frac{1}{2}(2k+1)\pi$ ,  $k \in \mathbb{Z}$ ,  $Y = \{\beta_y \frac{1}{2}(2k+1)\pi : k \in \mathbb{Z}\}$ , and  $e^{p_x+i\beta_x} =$   
 $\exp(\log|z|) \cdot e^{p_x+i\beta_x} = 2\cosh 2p_x$ ,  $|e^{p_x+i\beta_x}|^2 = 2\cosh 2p_x +$   
 $2i\sin(2p_x + \frac{\pi}{4})$ ,  $|e^{x+iy}| = |\cos y + i\sin y|$ ,  $|\cos y + i\sin y + e^y e^{ix} + e^{-y} e^{-ix}|$   
 $= |e^x|$ ,  $x+i(y+x) = x+y+iy$ ,  $|e^{x+iy}| = e^{x+y}$ ,  $e^{x+iy} + e^{x-y} = e^{xy} e^{ix+y} + e^{xy} e^{ix-y}$   
 $= (e^{xy}) \cos(x+y) + i(\dots)$ ,  $\exists \varepsilon \in \Pi(C \setminus 0)$ ,  $\lim_{z \rightarrow 0} R^{\varepsilon i\theta} e^{z^{\varepsilon i\theta}} \in$   
 $H(C \setminus 0)$ , i.e.  $\exists \alpha > 0$ ,  $z^{1-\varepsilon} = R^{1-\varepsilon} e^{i(\frac{\pi}{2}-\varepsilon\theta)} \Rightarrow \operatorname{Re} z^{1-\varepsilon} =$   
 $R^{1-\varepsilon} \cos((\frac{\pi}{2}-\varepsilon\theta)\pi) > R^{1-\varepsilon} \cos((\frac{\pi}{2}-\varepsilon\theta)\frac{\pi}{2}) \Rightarrow R^{1-\varepsilon} \delta > 0$ ,  $|\exp(-\varepsilon R^{\varepsilon i\theta} z^{1-\varepsilon})|$   
 $\leq \exp(-\varepsilon \operatorname{Re} z^{1-\varepsilon}) \leq \exp(-\varepsilon R^{1-\varepsilon} \delta) < 1 \quad (z \in \Pi)$ ,  $h_\varepsilon(z) := e^{-\varepsilon z^{1-\varepsilon}}$ ,  
 $z \in \Pi$ ,  $\forall \varepsilon \in \Pi(\pi)$ ,  $|f h_2| \leq 1$ ,  $|f h_2| \leq A \exp(|z|^\alpha - \varepsilon \operatorname{Re} z^{1-\varepsilon})$   
 $\underset{|z|=R \rightarrow \infty}{\rightarrow} 0$ ,  $R_0 := |z| > R_0 \Rightarrow |f h_2| < 1$ ,  $\max_{z \in \Pi \cap \{|z| > R_0\}} |f h_2| = \max_{z \in \partial \Pi} |f h_2|$   
 $= 1$ ,  $\|f h_2\| \leq 1 \quad (z \in \Pi, \varepsilon > 0)$ ,  $\|f\| \leq 1$ .  $(\alpha=1)$  let  $f(z) := e^z$ ,  
 $|f(z)| \cdot e^{\operatorname{Re} z} \leq e^{|z|}$ ,  $|f(z)| = e^r \underset{r \rightarrow \infty}{\rightarrow} \infty$ .  ~~$\theta$~~   $(\frac{\pi}{2} + \theta) \alpha < \frac{\pi}{2}$ ,  $\alpha < \frac{\pi}{\frac{\pi}{2} + \theta}$

$|e^z| \leq 1$ ,  $|f(iy)e^{iy}| \leq 1$ ,  $|f(re^{i\alpha})e^{nr e^{i\alpha}}| = |e^{nr \cos \alpha} |f(re^{i\alpha})| = \exp(nr \cos \alpha$   
 $+ \log |f(re^{i\alpha})|) = \exp(r(n \cos \alpha + r^{-1} \log |f(re^{i\alpha})|))$ ,  $n \cos \alpha + r^{-1} \log |f(re^{i\alpha})| \rightarrow -\infty$   
 $A := \max_r \exp(r(n \cos \alpha + r^{-1} \log |f(re^{i\alpha})|))$ ,  $|f(re^{i\alpha})e^{nr e^{i\alpha}}| \leq A$ .  
~~(\*)~~  $\alpha < \frac{\pi}{2-\theta}$ ,  $\alpha < \beta < \frac{\pi}{2-\theta}$ ,  $\operatorname{Re} z^\beta = r^\beta \cos(\beta \arg z) \geq r^\beta \cos(\beta(\frac{\pi}{2}-\theta))$ ,  
 $\Rightarrow$   $|f(z)| \leq (e^{-i\theta} z)^\beta$ ,  $\operatorname{Re} (e^{-i\theta} z)^\beta > 0 \Leftrightarrow z \in \Pi$ ,  $h_\varepsilon(z) := \exp(-\varepsilon (e^{-i\theta} z)^\beta) < 1$ ,  
 $|f h_2| \leq A \exp(|z|^\alpha - \varepsilon r^\beta \cos(\beta(\frac{\pi}{2}-\theta))) \underset{r \rightarrow \infty}{\rightarrow} 0$ ,  $|e^z| \leq A e^{|z|^\alpha + \varepsilon r^\beta}$

$$(S_2 \cdot C) \quad T = 0, M = ? \quad (x_0 + \Omega) \sup_{z \in \Omega} |f(z)| = \sup_{\theta \in \Omega - x_0} |f(z + x_0)|, \quad \text{Def}$$

$$\Omega - x_0 \quad (0 \notin \Omega) \quad \text{def} \quad \inf_{z \in \Omega} |f(z)|, \quad r > 0, \quad \sup_{z \in \Omega} |f(z)| = \sup_{z \in \Omega} |f(z/r)|,$$

$$r \in \Omega \cap V \neq \emptyset, \quad M^n \vee \frac{B^n}{V}, \quad \left| \frac{f(z)}{z^n} \right| \leq \max_{1 \leq n} \left| \frac{f(z)}{z^n} \right| + \max_{n \geq 1} \left| \frac{f(z)}{z^n} \right| \leq \max(M^n, B^n) \rightarrow M^n, \quad \|f\|_{M^n} \leq \|f\|_{M^n}, \quad n \in \mathbb{Z},$$

$$|f(z)| \leq M \quad \text{def} \quad \dim E_n < \infty \Rightarrow \|f\|_{E_n} \leq \|f\|_{M^n} = n, \quad E_n = \emptyset; \quad \|f\|_{E_n} \leq n \quad \text{def}$$

$$\Rightarrow \|f\|_{E_n} \leq \|f\|_{M^n} = n, \quad E_n = \emptyset; \quad \text{def} \quad \gamma \in \mathbb{C} \setminus \mathbb{R}, \quad \gamma(1 - \gamma) \in E_n, \quad \gamma(1 - \gamma)$$

$$\Rightarrow |\gamma(1 - \gamma)| > 1 \quad \text{def} \quad e^{\gamma} \rightarrow \infty, \quad e^{\gamma} \rightarrow 0, \quad \operatorname{Re} \gamma \rightarrow \infty \rightarrow e^{-\infty}, \quad \operatorname{Re} \gamma \rightarrow -\infty$$

$$\Rightarrow e^{\gamma} \rightarrow 0, \quad \operatorname{Re} \gamma \neq \infty \Rightarrow \operatorname{Im} \gamma = 0, \quad e^{\gamma} \neq L. \quad e^{i\theta} - e^{-i\theta} = e^{2i\theta} - e^{-2i\theta}$$

$$\in (e^y - e^{-y}) \cos x + i(e^y + e^{-y}) \sin x, \quad |e^{i\theta} - e^{-i\theta}|^2 = e^{2y} + e^{-2y} - 2 \rightarrow \infty$$

$$\text{def} \quad (y \neq \infty) \wedge \infty, \quad e^{i\theta} - e^{-i\theta} \neq L \in \mathbb{C} \setminus \mathbb{R} \quad \text{def} \quad g(z) := f(\frac{1}{z}), \quad \text{Thm 10.21},$$

$$(\text{def}: \hat{g}(0) := c, \hat{g}(z) := g(z), z \neq 0, \hat{g} \in H(\mathbb{C})) \quad z_n \rightarrow \infty, \quad g(\frac{1}{z_n}) \rightarrow c, \\ f(z_n) \rightarrow c, \quad f = \text{const}; \quad (g(z) = \sum_{j=-m}^{\infty} c_j z^j) \quad z \rightarrow 0 \Rightarrow c = \lim_{z \rightarrow 0} g(z) - \sum_{j=-m}^1 c_j z_n^j \\ \rightarrow c, \quad f(w_n) - \sum_{j=1}^m c_{-j} w_n^j \underset{w_n \rightarrow \infty}{\rightarrow} c, \quad f(w_n) - \sum_{j=1}^m c_{-j} w_n^j = \text{const.}, \quad f(w_n) = \sum_{j=0}^m \tilde{c}_j w_n^j, \quad \operatorname{Im} f = \mathbb{C}; \quad (\operatorname{cl}(g(D(0; r))) = \mathbb{C}) \quad c \in \mathbb{C}, \quad z_n: z_n \rightarrow 0, \quad g(z_n) \rightarrow c,$$

$$f(\frac{1}{z_n}) \rightarrow c \quad \text{def}$$

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$$|A| = \infty, \quad \lim_{r \rightarrow 0^+} |D(r, r) \cap A| = 0, \quad \forall c \in \partial A \Rightarrow |A| < \infty, \quad f(z) = \sum_{j=1}^{|A|} R_j$$

$$\in H(\mathbb{C}) \quad \text{def} \quad \varphi: D(\frac{1}{2}, \frac{1}{2}) \rightarrow V, \quad \varphi \neq \varphi' \in C^1, \quad \Phi: 1 + \alpha \sqrt{1 - \cos \theta}, \quad \Phi: 1 + \alpha \sqrt{1 - \cos \theta}$$

$$\mapsto 1 + (\cos \theta + \frac{\alpha}{2}) v(\theta), \quad 0 < \alpha < 2 \pi \sqrt{\theta}, \quad \bar{v}(\theta) := -1 - \frac{1}{\cos \theta}, \quad v(\theta) := \bar{v}(\theta) / |\bar{v}(\theta)|, \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

$$f(z) := \frac{1}{z - \frac{1}{2}}, \quad \inf_{\Omega} |f| = \frac{2}{3}, \quad \sup_{\Omega} |P - f| < \frac{2}{3}, \quad |(z - \frac{1}{2})P - 1| < \frac{1}{2}, \quad \sup_{\Omega} |(z - \frac{1}{2})P - 1| \leq \frac{1}{2},$$

$$\sup_{D(\frac{1}{2}, \frac{1}{2})} |(z - \frac{1}{2})P - 1| \leq \sup_{\partial D(\frac{1}{2}, \frac{1}{2})} |(z - \frac{1}{2})P - 1| = 1 \quad \text{def}$$

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$$\text{II} \quad .95 = \mathbb{P}(L < \mu < U) = \mathbb{P}(X-L > X-\mu > X-U), \quad X-\mu \sim N(0,1),$$

~~z~~

$$X-L = z_{.025}, \quad L = X-z_{.025}, \quad X-U = -z_{.025}, \quad U = X+z_{.025}$$

$$\text{y} \quad .56 \pm z_{.005} \sqrt{\frac{.56 \cdot .44}{100^2}} = .56 \pm z_{.005} \frac{4}{100} \sqrt{.154}$$

$$\text{3)} \quad \text{a) } E(X-\mu) = 0 \quad \text{b) } 1 - 2\Phi(-1) = 2\Phi(1) - 1 \quad \text{c) } \mathbb{P}(|X_1 - \mu| > \epsilon) >$$

$$\mathbb{P}(|X_1 - \mu| > 1) = 2\Phi(1) - 1 \rightarrow 0$$

$$\text{4)} \quad \text{a) } t_\mu(x_1, \dots, x_n) = \prod_{j=1}^n t_\mu(x_j) = (2\pi)^{-n/2} \exp\left(-\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^2\right) = \\ (2\pi)^{-n/2} \exp\left(-\frac{1}{2} \sum x_j^2 + \mu \sum x_j + \mu^2/2\right) = (2\pi)^{-n/2} \underbrace{\exp(\mu n \bar{x} + \mu^2/2)}_{\text{depends on data}} \underbrace{\exp(-\frac{1}{2} \sum x_j^2)}_{\text{doesn't depend on } \mu}$$

$$\text{b) } \mathbb{E}(\bar{Y}^2) = \text{Var}(\bar{Y}) + (\mathbb{E}(\bar{Y}))^2 = \frac{1}{n} + \mu^2, \quad \mathbb{E}(\bar{Y}^2) - \frac{1}{n} = \mu^2$$

$$\text{c) } \text{LLN: } \bar{Y} \xrightarrow{P} \mu, \quad \text{so} \quad \bar{Y}^2 \xrightarrow{P} \mu^2 \quad \text{since } x \mapsto x^2 \text{ is continuous.}$$

$$\text{5)} \quad 1-\alpha = \mathbb{P}(L < \sigma^2 < U) = \mathbb{P}\left(\frac{(n-1)\sum x^2}{L} > \frac{(n-1)\sum x^2}{\sigma^2} > \frac{(n-1)\sum x^2}{U}\right),$$

$$\frac{(n-1)\sum x^2}{L} := \chi^2_{n-1, \frac{1}{2}(1-\alpha)}, \quad L = \frac{(n-1)\sum x^2}{\chi^2_{n-1, \frac{1}{2}(1-\alpha)}}, \quad U = \frac{(n-1)\sum x^2}{\chi^2_{n-1, \frac{\alpha}{2}}}.$$

$$\text{b) } \text{a) } (-1, \infty) \quad \text{b) } \mathbb{E} Y = \int_0^\infty y(\theta+1)y^\theta dy = \frac{\theta+1}{\theta+2} := \bar{Y}, \quad \hat{\theta}(1-\bar{Y}) = 2\bar{Y}-1,$$

$$\hat{\theta}_{\text{mom}} = \frac{2\bar{Y}-1}{1-\bar{Y}} \quad \text{c) } (\theta+1)^n \left(\prod y_j\right)^\theta, \quad n \log(\theta+1) + \theta \log \sum y_j, \quad \frac{n}{\theta+1} + \log \sum y_j$$

$$= 0, \quad \hat{\theta}_{\text{MLE}} = -\frac{n}{\log \sum y_j} - 1 \quad \text{d) } \hat{\theta}_{\text{MLE}}^2 \quad \text{(*)}$$

$$\text{7.84) } \theta^{-2n} \prod y_j e^{-\sum y_j/\theta}, \quad -2n \log \theta - \sum y_j/\theta, \quad -\frac{2n}{\theta} + \sum y_j/\theta^2 = 0, \quad \theta = \sum y_j/n$$

$$\text{Because } \mathbb{E} Y = \theta^{-2} \int y^2 e^{-y/\theta} = \theta^{-2} \left\{ -\theta e^{-y/\theta} y^2 \Big|_0^\infty + 2y \theta e^{-y/\theta} \Big|_0^\infty \right\} = 2\theta$$

WPF)  $\Sigma y_j$

$$\frac{(\pi y_j)^{\Sigma y_j}}{0^n} e^{-\Sigma y_j^2/\theta}, \quad -n \log \theta = \Sigma y_j^2 / \theta, \quad -\frac{n}{\theta} + \frac{\Sigma y_j^2}{\theta^2}, \quad \hat{\theta} = \frac{1}{n} \sum y_j^2,$$

$$\int y^2 e^{-y^2/\theta} = y \left( 1 - \frac{\theta}{2} e^{-y^2/\theta} \right) + \frac{\theta}{2} y e^{-y^2/\theta} = \left( \frac{\theta}{2} \right)^2$$

$$P^{(1-p)} e^{\Sigma y_j - n}, \quad -\cancel{\frac{\partial \ln L}{\partial \theta}} \approx \ln p + (\Sigma y_j - n) \ln(1-p), \quad \hat{p} \approx 1 - \frac{\Sigma y_j - n}{n} = 0$$
$$0 = n(1-p) - p(\Sigma y_j - n) = n - p(\Sigma y_j - n + 1), \quad p = \frac{n}{\Sigma y_j - n + 1}$$

$$\prod_{j=1}^n (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2}(x_j - \mu)^2\right) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum (x_j - \mu)^2\right),$$

$$\frac{L(\mu_0, \sigma_0)}{L(\mu_s)} = \exp\left(-\frac{1}{2\sigma^2} \left\{ (x_0 - \mu_0)^2 - (x_s - \mu_s)^2 \right\}\right) = \exp\left(\frac{1}{2\sigma^2} \left\{ -2x_s \mu_s + 2x_s \mu_0 + \mu_0^2 - \mu_s^2 \right\}\right) < k, \quad + 2x_s(\mu_0 - \mu_s) + \mu_0^2 - \mu_s^2 < 2\sigma^2 \log k,$$
$$\exp\left(\frac{1}{2\sigma^2} \left\{ -2n\bar{x}(\mu_0 - \mu_s) + n(\mu_s^2 - \mu_0^2)\right\}\right) < k, \quad \bar{x} < \frac{1}{2n\cdot s} \left\{ 2\sigma^2 \log k - n(\mu_s^2 - \mu_0^2) \right\}, \quad P(\bar{x} < k) = \alpha_s = P(Z < \frac{k - \mu_0}{\sqrt{\sigma^2/n}}), \quad k = \mu_0 + z_{1-\alpha} \sqrt{\sigma^2/n}$$

$$= \frac{1}{\theta_0} - \frac{1}{\theta_s} \frac{s}{\sqrt{n}}, \quad (2\theta)^{-n} (\pi y_j^2) e^{-\theta^2 \Sigma y_j^2} \frac{L(\theta_0)}{L(\theta_s)} = \left(\frac{\theta_s}{\theta_0}\right)^{3n} e^{-\Sigma y_j^2 (\frac{1}{\theta_0} - \frac{1}{\theta_s})} < k,$$

$$-1 \cdot \Sigma y_j (\frac{1}{\theta_0} - \frac{1}{\theta_s}) < k, \quad \Rightarrow \Sigma y_j > k, \quad y_j \sim \Gamma(3, \theta), \quad \Sigma y_j \sim \Gamma($$

$$P(Y_1 + Y_2 < c) = \int_0^c F_2(t) F_1(c-t) dt = \int_0^c \frac{t^2}{2} dt = -\frac{t^3}{3} \Big|_0^c = -\frac{c^3}{3}, \quad P(Y_1 + Y_2 < 1) = c + \int_c^\infty F_2(1-t) dt$$

$$= c + \frac{3}{2} (1-c) + \frac{1}{2} c^2 = c + \frac{1}{2} - \frac{1}{3} c^2, \quad f = 1-c, \quad F_{Y_1+Y_2} = \frac{1}{2} (2f + f^2)$$

Andreas

$$a^3 = n+3, \quad b^3 = n^2 + 3n + 3, \quad b^3 - a^3 = \frac{n(n+2)}{n^2 + 2n}, \quad b^3 + a^3 = n^2 + 4n + 6 = (n+2)^2 + 2$$

$$b^3 = (n+3)(n+1) - n = (n+2)(n+1) + 1, \quad b^3 = a^3(n+1) - n, \quad b^3 = n^2 + 2n + a^3,$$

$$n^2 + 2n = b^3 - a^3 = (b-a)(b^2 + ab + a^2) = n(n+2), \quad \cancel{\text{what do you do with } n+2}$$

$$(2|n) \Rightarrow a \equiv 2 \pmod{2}, \quad 2|b, \quad \cancel{\text{what do you do with } n+2} = \frac{n+2}{n} + \frac{2}{n(n+2)},$$

$$b^3 = a^6 - 6a^3 + 9 + 3a^3 - 6 = a^6 - 3a^3 + 3 = (a^3 - 3)(a^3 - 1) + a^3 = (a^3 - 2)^2$$

$$+ a^3 - 1, \quad b^3 - a^3 = (a^3 - 3)(a^3 - 1) = \cancel{\text{what do you do with } a^3 - 1} (a^3 - 3)(a - 1)(a^2 + a + 1)$$

$$2|n \Rightarrow 2|b, \quad 2|n \Rightarrow 2|b, \quad 2|(n^2 + 3n) = n(n+3), \quad b^3 - 3 = a^3(a^3 - 3),$$

$$b^3 = (n+2)(an) + 1, \quad (b-1)(b^2 + b + 1) = \cancel{(n+1)(n+2)}, \quad a^3 = m, \quad b^3 = n^2 - 6m + 9 + 3m - 9 + 3$$

$$= m^2 - 3m + 3, \quad a^3 = m+1, \quad b^3 = n^2 - 4m + 4 + 3m - 6 + 3 = m^2 - m + 1, \quad n^2 + 6n + 9,$$

$$(a^3 - b^3) = 3n + 6 = 3(n+2), \quad (n+3)^4 = n^4 + 4n^3 + 6n^2 \cdot 9 + 4 \cdot 27n + 81, \quad n^4 + 9n^2 + 9$$

$$+ (3n^3 + 3n^2 + 9n)2 = n^4 + 6n^3 + 15n^2 + 8n + 9, \quad a^3 - 6n^2 + 12n + 9, \quad (n+1)^3 =$$

$$n^3 + 3n^2 + 3n + 1, \quad (n+2)^3 = n^3 + 6n^2 + 12n + 8 \dots$$

$$A^p = I, \quad B^q = 0, \quad \text{or } A^{p-k} = A^{-k}, \quad A^k = A^{-(p-k)}, \quad \& (\text{what } p=2, \quad q=1) (I + B)^{-1}$$

$$= I + 2B, \quad \cancel{B + I} \quad A^{-1} = I, \quad (A+I)^{-1} = \sum (-A)^j$$

$$S^{5^{n+2}} + S^{5^{n+1}} - S^{5^{n+1}} - S^{5^n} = S^{5^{n+2}} - S^{5^n} : S^{5^n} (S^{5^{n+2}-1}), \quad (n=0) \quad \cancel{25 \cdot 625} + 6$$

$$= 15625 = 5 \cdot 28233 = 49 \cdot 319 \cdot 5^2 \cdot 11 \cdot 29, \quad S^{5^{n+2}-1} = 1 \quad (n=0) \quad S^5 + 6 = 3131$$

$$= 31 \cdot 101, \quad 245^2 - 1 = 24, \quad 124 = 31 \cdot 4 = S^2 - 1, \quad S^{5^{n+2}-3} - 1 = (S^2 - 1) \left( \sum_{j=0}^{S^2-2} S^j \right)$$

$$31 \mid (S^{5^{n+2}} - S^{5^n}) \quad // \quad S = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} S^{n-j-1}, \quad S^5 = \sum_{j=0}^{n-1} \binom{n}{j} (-1)^j S^{n-j} - (-1)^n,$$

$$S^5 - 1 = 6^n, \quad S = \frac{1}{2}(6^n + 1), \quad (n=5) \quad 36 \cdot 216 + 1 = 7777 = 77 \cdot 1111 = 7 \cdot 11 \cdot 101$$

$$\frac{116}{36} \quad S^5 - 1 = 4^n, \quad (n=5) \quad 4^n + 1 = 1025 = 25 \cdot 41, \quad S_5 = 5 \cdot 41, \quad \text{then } S_{2k+3}$$

$$4^{2k+1} + 1 \leq 5p \stackrel{(3)}{\leq} 2p$$

$$(a_0^2 + b_0^2 + c_0^2)(s_0^2 + t_0^2 + u_0^2) \geq (3as_0)^2 \stackrel{(*)}{\iff} \sum_{j=1}^n a_j^4 \stackrel{(*)}{\leq} \prod_{j=1}^n (a_j^2 + b_j^2 + c_j^2)$$

$$\geq (\sum \frac{s_{ij}}{\sqrt{n}})^2, \quad (\sum a_j s_{ij})^2 \leq n, \quad \sum a_j^4 \leq \sum a_j, \quad n = \sum a_j \leq \bar{a}(\sum a_j^2)^{\frac{1}{2}}$$

$$n \leq \sum a_j^2 \leq n^2 (\sum a_j^2)^{-1}, \quad \sum a_j^4 \leq n \stackrel{\text{using } (1)}{\leq} \sum a_j^2 - 2(\sum f_j)(\sum A_j x_j^2) \geq (\sum b_j x_j)^2,$$

$$(\sum f_j)(\sum f_j x^2) \geq (\sum \sqrt{f_j} \cdot \sqrt{f_j} x)^2 \stackrel{(*)}{\iff} L_k = S_{n-k}, \quad 1 \leq \sum k_j, \quad 1 \leq \sum k_j^2 \sum k_j^{-4}$$

$$1 \leq \sum k_j^3, \quad S_{n-k} \leq (\sum k_j^2)(\sum k_j^3)^2 = (\sum k_j^3)^2, \quad \sum k_j^3 \geq \frac{1}{3n-4}, \quad \frac{1}{6} \geq \frac{1}{3n-4},$$

$$1 \leq S_{n-k} - 4n + 8, \quad 16 + 20c, \quad n^2 \leq \sum k_j^2 \leq \sum k_j^{-2}, \quad n^2 \leq (\sum k_j^2 k_j^{-4})^2 \leq \sum k_j \sum \frac{1}{k_j} =$$

$$3n-4, \quad (n-1)(n-4) \leq 0, \quad 1 \leq n \leq 4, \quad \#(n-1) k_1 \cdot k_2 \quad (n-2) k_1 \cdot k_2 = k_2 \quad (n=3) k_1 \cdot k_2 =$$

$$k_3 = 3, \quad \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{2} \pi, \quad 2(k_1, k_2) = k_1 k_2, \quad (n=4) k_1 \cdot k_2 \cdot k_3 \cdot k_4 = \frac{1}{k_1} \cdot \frac{1}{k_2} \cdot \frac{1}{k_3} \cdot \frac{1}{k_4},$$

$$k_3 = 4 \stackrel{(*)}{\iff} \frac{\sin^3 a}{\sin b} + \frac{\cos^3 a}{\cos b} \geq ? \quad \frac{1}{\cos(a-b)} = \frac{1}{\cos a \cos b + \sin a \sin b}, \quad 1 \leq \left( \frac{\sin^3 a}{\sin b} + \frac{\cos^3 a}{\cos b} \right)$$

$$(\cos a \cos b + \sin a \sin b) \leq \cos a + \cos b \stackrel{(*)}{\iff} \cos a + \cos b \leq \frac{1}{\cos(a-b)}$$

$$\# P(x) = (x-r_1)(x-\bar{r}_1)(x-r_2)(x-\bar{r}_2)(x-r_3), \quad 35 = 2\pi r_1^2 - |r_1 r_2|^2 r_3, \quad -10$$

$$= \bar{r}_1 |r_2|^2 r_3 + r_1 |r_2|^2 \bar{r}_3 + |r_1|^2 \bar{r}_2 r_3 + |r_1|^2 r_2 \bar{r}_3 + |r_1 r_2|^2 \stackrel{(*)}{=} 2 \operatorname{Re} r_1 \cdot |r_2|^2 r_3$$

$$+ 2 \operatorname{Re} r_2 (|r_1|^2 r_3 + |r_1 r_2|^2), \quad \# \frac{2}{7} = \frac{2 \operatorname{Re} r_1}{|r_1|^2} + \frac{2 \operatorname{Re} r_2}{|r_2|^2} + \frac{1}{r_3}, \quad \frac{1}{7} = \frac{\operatorname{Re} r_1}{|r_1|^2} + \frac{\operatorname{Re} r_2}{|r_2|^2} + \frac{1}{r_3}$$

$$\frac{2}{7} = \sum \frac{1}{r_j} \text{ where ...}$$

$$(n^{n+1}(n!)^2)^{1/2} \leq \frac{1}{2}(n^{n+1} + n^2 - 2n + 1), \quad \leq \frac{1}{2}(n^{n+1}(n-1) + n-1) = \frac{1}{2}n^n(n-1) \stackrel{(*)}{\leq} n^n - 1 = (n-1) \sum_{j=0}^{n-1} n^j$$

$$\# \frac{2}{7} \geq (n-1)n \left( \prod_{j=0}^{n-1} n^j \right)^{1/n} = (n-1)n^{1+\frac{n-1}{2}} = (n-1)n^{\frac{n+1}{2}} \stackrel{(*)}{\leq} 1 + (r_1 \cdots r_n)^{\frac{1}{n}} \leq ? \cdot ((1+r_1) \cdots (1+r_n))^{\frac{1}{n}}$$

$$\log(1 + (\pi r_j)^n) \leq ? \cdot \frac{1}{n} \sum \log(1 + r_j) \leq (\pi \log(1 + r_j))^n, \quad \log(1 + (\pi r_j)^n) \leq \log(1 + \pi)$$

$$= \log\left(\frac{1}{n} \sum (1 + r_j)\right) \stackrel{(*)}{\leq} \frac{1}{n} \sum \log(1 + r_j) \stackrel{(*)}{\leq} \log((1 + r_1) \cdots (1 + r_n))^{\frac{1}{n}} = \frac{1}{n} \log((1 + r_1) \cdots (1 + r_n))^{\frac{1}{n}},$$

$$\frac{1}{n} \sum \log \log(1 + r_j) \geq ? \cdot \log \log(1 + (\pi r_j)^n) \quad 1 + \pi r_j \leq ? \cdot ((1+r_1)(1+r_2))^{\frac{1}{2}} \stackrel{(*)}{\leq} ((1+r_1)(1+r_2))^{\frac{1}{2}}$$

## Woolridge

$$\hat{y}_2 = \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy^T), \hat{v}_2 = y_2 - z\hat{\Pi}_z(y_2), \hat{x} = (\overset{\text{"x"}}{z_1, y_2}) - \hat{\Pi}_{\hat{v}_2}(\overset{\text{"x"}}{z_1, y_2}), \hat{\Pi}_{\hat{v}_2}(\overset{\text{"x"}}{z_1, y_2})$$

$$g = \hat{v}_2 \mathbb{E}(\hat{v}_2 \hat{v}_2^T) \mathbb{E}(\hat{v}_2 (z_1 y_2)^T) \text{ 和 } \hat{\Pi}_{\hat{x}}(y) = \mathbb{E}(\hat{x} \hat{x}^T)^{-1} \mathbb{E}(\hat{x} y^T) \text{ 和 } \mathbb{E}(\hat{x} z^T)$$

$$= \mathbb{E}(xx^T) - \mathbb{E}(xx^T \Pi_{\hat{v}_2} \hat{v}_2^T) - \mathbb{E}(\hat{v}_2 \Pi_{\hat{v}_2} xx^T) + \mathbb{E}(\hat{v}_2 \Pi_{\hat{v}_2} xx^T \Pi_{\hat{v}_2} \hat{v}_2^T) =$$

$$\mathbb{E}(xx^T) - \mathbb{E}(xx^T \Pi_{\hat{v}_2} y_2^T) + \mathbb{E}(xx^T \Pi_{\hat{v}_2} y_2^T \Pi_z z^T) - \mathbb{E}(y_2 \Pi_z y_2 \Pi_{\hat{v}_2} xx^T) + \mathbb{E}(z \Pi_z y_2 \Pi_{\hat{v}_2} xx^T)$$

$$+ \mathbb{E}(y_2 \Pi_{\hat{v}_2} xx^T \Pi_{\hat{v}_2} y_2^T) - \mathbb{E}(y_2 \Pi_{\hat{v}_2} xx^T \Pi_{\hat{v}_2} y_2^T \Pi_z z^T) - \mathbb{E}(z \Pi_z y_2 \Pi_{\hat{v}_2} xx^T \Pi_{\hat{v}_2} y_2^T)$$

$$+ \mathbb{E}(z \Pi_z y_2 \Pi_{\hat{v}_2} xx^T \Pi_{\hat{v}_2} y_2^T \Pi_z z^T); \Pi_{\hat{x}} y = ? \quad \Pi_{\hat{x}} y = \mathbb{E}(\hat{x} \hat{x}^T)^{-1} \mathbb{E}(\hat{x} y),$$

$$A = \mathbb{E}(\Pi_z x - z \Pi_z x), \Pi_{\hat{x}} y = \mathbb{E}(\Pi_z x - z \Pi_z x) \mathbb{E}(\Pi_z \mathbb{E}(xx^T) \Pi_z) \cdot \Pi_z \mathbb{E}(xy),$$

$$\mathbb{P}_{\hat{x}} y = \mathbb{E}(y \hat{x}^T) \mathbb{E}(\hat{x} \hat{x}^T)^{-1} \hat{x}, \hat{x} = P_z x = \mathbb{E}(xz) \mathbb{E}(zz^T)^{-1} z, \hat{y}_1 = \mathbb{E}(y_1 z) \mathbb{E}(zz^T)^{-1} (z z^T)$$

$$\{\mathbb{E}(xz) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zz^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(z z^T) \mathbb{E}(zz^T)^{-1} z = \mathbb{E}(P_z y \cdot x^T) \cdot \mathbb{E}(P_z x \cdot x^T)^{-1}$$

$$\cdot P_z x, \quad \mathbb{P}_{\hat{x}} y = \mathbb{E}(y \hat{x}^T) \mathbb{E}(\hat{x} \hat{x}^T)^{-1} \hat{x}, \hat{x} = x - P_z \hat{v}_2 = \underset{\hat{v}_2}{\cancel{y_2 - \mathbb{E}(y_2)}} \quad \hat{v}_2 = \bar{P}_z y_2,$$

$$\bar{P}_{\hat{v}_2} x = x - \mathbb{E}(x \hat{v}_2^T) \mathbb{E}(\hat{v}_2 \hat{v}_2^T)^{-1} \hat{v}_2 = \underset{\hat{v}_2^T = \mathbb{E}(\hat{v}_2 \hat{v}_2^T) \mathbb{E}(\hat{v}_2 \hat{v}_2^T)^{-1} \hat{v}_2}{\cancel{x - \mathbb{E}(x \hat{v}_2^T) \mathbb{E}(\hat{v}_2 \hat{v}_2^T)^{-1} \hat{v}_2}} \quad \hat{v}_2 = y_2 - \bar{P}_z y_2 = y_2 - \mathbb{E}(y_2)$$

$$\mathbb{E}(zz^T)^{-1} z, \quad \mathbb{E}(y_2 \hat{v}_2^T P_x^T) = \mathbb{E}(y_2 (y_2 - z \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T)) \mathbb{E}(\hat{v}_2 \hat{v}_2^T)^{-1} \mathbb{E}(v_2 v_2^T)), \quad z = \begin{pmatrix} z_1 \\ y_2 \end{pmatrix}$$

$$P(x|\hat{v}_2) = \mathbb{E}(x \hat{v}_2^T) \mathbb{E}(\hat{v}_2 \hat{v}_2^T)^{-1} \hat{v}_2, \quad \mathbb{E}\left(\begin{pmatrix} z_1 \\ y_2 \end{pmatrix} | y_2\right) (y_2 - \mathbb{E}(y_2 z^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T) - \mathbb{E}(zy_2^T) z) = \begin{pmatrix} 0 \\ \mathbb{E}y_2^T - \mathbb{E}y_2 P_y \end{pmatrix}$$

$$\mathbb{E}(y_2 z^T) - \mathbb{E}(y_2 P(y_2|z)^T) = \mathbb{E}y_2^T - \mathbb{E}(y_2 z^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T) = \mathbb{E}y_2^T - \mathbb{E}(y_2 z^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T) \mathbb{E}(zz^T)$$

$$\mathbb{E}(y_2 z^2) = \mathbb{E}(y_2^2) - \mathbb{E}(P(y_2|z)^2) = \mathbb{E}\hat{v}_2^2, \quad P(y_2 z^2 - \hat{v}_2^2) \mathbb{E}(x \hat{v}_2^T) = \begin{pmatrix} 0 \\ \mathbb{E}\hat{v}_2^2 \end{pmatrix}, \quad ?(x|\hat{v}_2) = \begin{pmatrix} 0 \\ \hat{v}_2 \end{pmatrix}$$

$$\hat{x} = \mathbb{E}(y_2 z^T) z - P(x|\hat{v}_2) = \begin{pmatrix} z_1 \\ \hat{v}_2 \end{pmatrix}, \quad (\hat{v}_1, \hat{v}_2) = \mathbb{E}(y \hat{x}^T) \mathbb{E}(\hat{x} \hat{x}^T)^{-1} + \mathbb{E}(y \hat{x}^T) = (\mathbb{E}y_2^T, \mathbb{E}y_2 z^T)$$

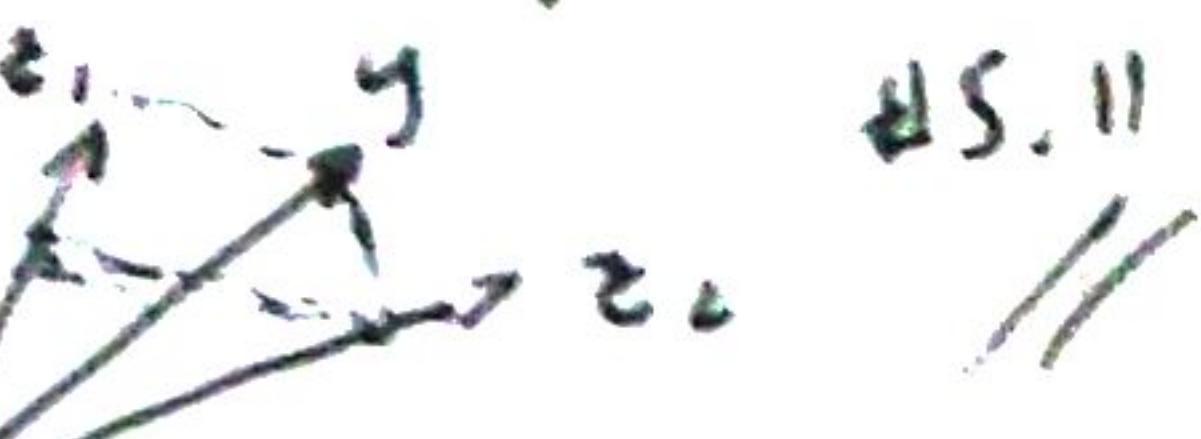
$$\mathbb{E}(yy_2 - \mathbb{E}y_2 \hat{v}_2^T) \begin{pmatrix} \mathbb{E}z_1 z_1^T & \mathbb{E}y_2 \\ \mathbb{E}y_2 z_1^T & \mathbb{E}(y_2 \hat{v}_2^T) \end{pmatrix}^{-1} = ? \quad \mathbb{E}(P(y_2|z) \cdot x^T) \cdot \mathbb{E}(P(x|z) \cdot x^T)^{-1},$$

$$\mathbb{E}(P(y_2|z) \cdot x^T) = \mathbb{E}(y_2 z^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T) = (\mathbb{E}y_2^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T) \mathbb{E}(zz^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T) = \mathbb{E}(y_2 z^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T)$$

$$\mathbb{E}(zy_2^T) = \mathbb{E}(y_2 z^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T) = (\mathbb{E}y_2^T, \mathbb{E}(y_2 P(y_2|z)^T)) = (\mathbb{E}y_2^T, \mathbb{E}(y_2 (y_2 - \hat{v}_2)))$$

$$\mathbb{E}(P(x|z) \cdot x^T) = \mathbb{E}(P(x|z) P(x|z)^T) = \mathbb{E}\left(\begin{pmatrix} z_1 \\ \hat{v}_2 \end{pmatrix} \otimes z\right)^T \# 5. // \quad \hat{x} = P(x|z), \quad \hat{x} = P(x|z)$$

$$= \mathbb{E}(xz) \mathbb{E}(zz^T)^{-1} z, \quad \hat{v}_1 = \mathbb{E}(y \hat{x}) \mathbb{E}(\hat{x} \hat{x}^T)^{-1} = \mathbb{E}(xz) \mathbb{E}(zz^T)^{-1} \mathbb{E}(yz) \mathbb{E}(zz^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(yz)$$

$\hat{\beta}_0 = E(yz)/E(xz)$ ,  $\hat{\beta}_1 = \Sigma yz/\Sigma xz = (Exz)^{-1} (\beta_0 Ex + \beta_1 \Sigma xz + \Sigma z^2) = \beta_0 + \hat{\beta}_{xz}$   
 $+ \beta_1 + 2xu/\Sigma xz$ ,  $\sqrt{N}(\hat{\beta}_1 - \beta_1) = \beta_0 \sqrt{N} \bar{z}/\sqrt{z^2} + \sqrt{N} \bar{xu}/\sqrt{z^2} \rightsquigarrow \frac{\beta_0}{\sigma_{xz}} N(0, 0) + \sqrt{N}(\hat{\beta}_1 - \beta_1)$   
 $= \frac{\sqrt{N} \bar{xu}}{\bar{z}^2} \rightsquigarrow \sigma_{xz}^{-1} N(0, \text{Var } xu) = N(0, \sigma_z^2 \sigma_u^2 / \sigma_{xz}^2) = N(0, r_{xz}^2 \frac{\sigma_u^2}{\sigma_x^2}) \text{ #S. 10}$   
 $\tilde{q}_2 = P(y_2 | z_2) = E(y_2 | z_2) E(z_2 | z_1)^{-1} \epsilon_2$ ,  $(\tilde{\delta}_1, \tilde{\alpha}_1) = E(y_1 | (z_1, \tilde{y}_2)) E((\frac{\epsilon_1}{\tilde{q}_2})^{(z_1^T, \tilde{y}_2)})^{-1}$   
 $E\left\{(\delta_1, \alpha_1) \left(\frac{z_1}{y_2}\right)^{(z_1^T, \tilde{y}_2)} + \lambda_1 (z_1^T, \tilde{y}_2)\right\} E\left(\frac{z_1^T \epsilon_1}{z_1^T \tilde{y}_2} \frac{\epsilon_1^T \tilde{y}_2}{\tilde{y}_2^2}\right)^{-1} = (\delta_1, \alpha_1) E((\frac{\epsilon_1}{\tilde{q}_2})^{(z_1^T, \tilde{y}_2)})$   
 $E\left(\left(\frac{z_1}{\tilde{q}_2}\right)^{\otimes 2}\right)^{-1}$ ,  $E(\tilde{q}_2^2) = E(y_2 \tilde{y}_2)$ ,  $E(y_2 z_1) \neq E(\tilde{y}_2 z_1)$   #S. 11

$$\text{Cov}(x_{-k} x_k^\top, E x_{-k} z^\top) = E x_{-k} x_k^\top - E x_{-k} x_k^\top \frac{E x_{-k} z^\top}{E z z^\top}, \quad E z x_k^\top = E x_{-k} x_k^\top - E x_{-k} x_k^\top \frac{E z x_k^\top}{E z z^\top}, \quad (\delta, \theta) = E(x_k | x_{-k}, z^\top) E^{-1} (x_{-k} | z^\top)$$

$\hat{\beta} = 0 \Rightarrow \text{K}_{\hat{\beta} \hat{\beta}} \geq 0$  且  $E((x_k x_{-k}^T, x_k z^T)) \in (\hat{\beta} x_k z^T + \sum_{j=1}^{n-1} x_j z)$  (,  $K := 1, \dots, n$ ) .

$$\left( \mathbb{E} X_{-k} X_{-k}^T, \mathbb{E} X_{-k} z^T \right) \perp E^{-1} \begin{pmatrix} x_{-k} \\ z \end{pmatrix}^{\otimes 2} \left[ \mathbb{E} X_{-k} X^T, \mathbb{E} z X^T \right] = k-1.$$

$$\left( \begin{matrix} E_{X_{-k} X_k} \\ E_{Z X_k} \end{matrix} \right) \in \text{sp} \left( \begin{matrix} E_{X_{-k} X_{-k}} \\ E_{Z X_{-k}} \end{matrix} \right) \Rightarrow \left( \begin{matrix} E_{X_{-k} X_k} \\ E_{Z X_k} \end{matrix} \right) \perp \left( \begin{matrix} E_{\cdot \cdot (X_k)}^{\otimes 2} \\ z \end{matrix} \right), \quad \Theta = 0 \quad \# S.12$$

$$\text{exzyz ratio } \beta_1 = C_w(y, z) / C_w(x, z) = (\text{IE}(yz) - \text{IE}y\text{IE}z) / (\text{IE}(xz) - \text{IE}x\text{IE}z)$$

$$(\sum y_{\{z=1\}} - \bar{y} \cdot \{\bar{z}/n\}) / (\sum x_{\{z=1\}} - \bar{x} \cdot \{\bar{z}/n\}), \quad Eyz = \beta_0 E z + \beta_1 Exz,$$

$$\text{Var}(z) = E((z - \mu_z)^2) = E(z^2) - \mu_z^2$$

$$\text{Ansatz: } E\left(\left(\frac{1}{z}\right)^{(1-x)}\right) \cdot E(y_z) = E\left(\frac{1}{z} \cdot \frac{x}{x-1} \cdot E(y_z)\right) = ((w(x,z)) \cdot E\left(\frac{xy}{z}; y_z\right)),$$

$$\hat{r}_j = \frac{\mathbb{E} y_j^2 - (\mathbb{E} y_j)^2}{\mathbb{E} x_j^2 - (\mathbb{E} x_j)^2}, \quad \frac{1}{n} (\sum_j \varepsilon_j^2) = \frac{1}{n} (2y_{j=1}^2 + \sum_{j \neq 1} y_j^2),$$

$$\sum_{j=1}^n y_j (\{z_j=1\} - \{z_j=0\}) = \sum_{j=1}^n y_j \{z_j=1\} - \bar{y} \sum_{j=1}^n \{z_j=1\}$$

$$+ \bar{y} \bar{N}_{\bar{z}} \bar{x} - \bar{z} \{ y(\{t=1\} + \{t=2\}) = (1-\bar{z}) \{ y(t=1) - \bar{z} n(1-\bar{z}) \bar{y}_1 + \bar{z} n(1-\bar{z}) \bar{y}_2 \quad \text{#S-13}$$

$$L(y|z) = L\left(\sum_{j=1}^K \beta_j x_j + u|z\right) = \sum_{j=1}^{K-1} \beta_j x_j + \beta_K L(x_K|z), \quad L(x_K|z) = \sum_{j=1}^L \delta_j x_j + \{\theta_j\},$$

$$\Theta \neq 0, \det(zx^*) = k \quad \text{BS14}$$

$$\frac{dP_n}{dQ_n} = \exp\left(\frac{1}{2}(x-\mu_n)^2 - \frac{1}{2}\lambda^2\right) = \exp(-x\mu_n + \frac{\lambda^2}{2}), \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} < \delta\right) = P_{Q_n}\left(x > \frac{1}{\mu_n}(\log \delta - \frac{\lambda^2}{2})\right) = P_{Q_n}\left(z > -\frac{\log \delta}{\mu_n} + \frac{\lambda^2}{2}\right), \quad \frac{\partial}{\partial c} = \varphi\left(-\frac{\log \delta}{\mu_n} + \frac{\lambda^2}{2}\right) \text{ c.p.}, \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} > 0\right)$$

$$2R \xrightarrow{n_1 \rightarrow \infty} P_{Q_n}\left(\exp(-x\mu_n + \frac{\lambda^2}{2}) > 0\right) = 1 \iff -x\mu_n + \frac{\lambda^2}{2} > -\infty \text{ up to } 1 - Q_\infty,$$

$$x\mu_n + \frac{\lambda^2}{2} \sim \frac{x\mu_n - \frac{\lambda^2}{2}}{\lambda^2} > 0 \text{ up to } \iff \mu_n \in [-k, k] \cdot E_{Q_n} \frac{dP_n}{dQ_n}$$

$$= \int \exp(-x\mu_n + \frac{\lambda^2}{2}) (2\pi)^{-1/2} \exp(-\frac{1}{2}(x - \mu_n)^2) dx = (2\pi)^{-1/2} \int \exp(-\frac{x^2}{2}) \geq 1 \quad \#1$$

$$\frac{dP_n}{dQ_n} = (2\pi \theta_n)^{-1/2} \exp(-\inf(x - \theta_n)^2 - x^2) = \frac{1}{\sqrt{2\pi \theta_n}} \exp(-\theta_n^2 - 2x\theta_n + 2n), \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} < \delta\right) \\ (\theta_n = O(n^{-1/2}))$$

$$= P_{Q_n}\left(\theta_n^2 - 2x\theta_n + \log(\frac{1}{\delta \sqrt{2\pi \theta_n}}) < \log(\frac{1}{\delta \sqrt{2\pi}})\right) = P\left(N(-\frac{\theta_n^2}{2\theta_n}, \frac{\theta_n^2}{2\theta_n^2}) < \log(\frac{1}{\delta \sqrt{2\pi}})\right)$$

$$= \Phi\left(\left\{\log(\frac{1}{\delta \sqrt{2\pi}}) + \frac{\theta_n^2}{2\theta_n}\right\}/\sqrt{\theta_n}\right) = \Phi\left(O(\log n)(\log(\frac{1}{\delta \sqrt{2\pi}}) + O(1))\right) \xrightarrow{n \rightarrow \infty} 0$$

$$= \Phi\left(O(n^{1/2} \log \log n / \sqrt{\theta_n})\right) \quad \frac{dP_n}{dQ_n} = (2\pi)^{-1/2} \exp\left(\frac{1}{2}(\theta_n^2 - 2x\theta_n)\right), \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} < \delta\right)$$

$$= P\left(N(-\frac{\theta_n^2}{2\theta_n}, \frac{\theta_n^2}{2\theta_n^2}) < \log(\frac{1}{\delta})\right) = \Phi\left(\left\{\log(\frac{1}{\delta}) + \frac{\theta_n^2}{2\theta_n}\right\}/\sqrt{\theta_n}\right)$$

$$= \mathbb{P}\left(O(\sqrt{n})\left\{\log(\frac{1}{\delta}) + \frac{\theta_n^2}{2\theta_n}\right\}\right) \xrightarrow{n \rightarrow \infty} \frac{\log(\frac{1}{\delta})}{\sqrt{n}} = \frac{\log(\frac{1}{\delta})}{\sqrt{n}}, \quad \left(\frac{n}{\sqrt{n}} \frac{\log(\frac{1}{\delta})}{\sqrt{n}}\right)/\left(\frac{1}{2}\sqrt{n}\right) \rightarrow 0,$$

$$N \xrightarrow{\delta \rightarrow 0} 0, \quad \frac{dP_n}{dQ_n} \xrightarrow{Q_n} V \Rightarrow P(V=0)=0, \quad \text{if } P_{P_n}\left(\frac{dQ_n}{dP_n} < \delta\right) =$$

$$P\left(N(-\frac{\theta_n^2}{2\theta_n}, \frac{\theta_n^2}{2\theta_n^2}) < \log \delta\right) \xrightarrow{\delta \rightarrow 0, n \rightarrow \infty} 0, \quad \frac{dQ_n}{dP_n} \xrightarrow{P_n} W \Rightarrow P(W=0)=0, \quad P_n \ll Q_n,$$

$$\theta_n = O(n^{-1/2}), \quad \limsup \theta_n n^{1/2} = \infty; \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} < \delta\right) = \Phi\left(\frac{\log \delta}{\sqrt{n}\theta_n} + \frac{\theta_n^2}{2}\right)$$

$$\xrightarrow{n_k} 1, \quad \leq \theta_n n^{1/2} \xrightarrow{k \rightarrow \infty} \frac{1}{2} \quad \frac{dP_n}{dQ_n} = \frac{1}{n+1} \{[0,1]\} \xrightarrow{n \rightarrow \infty} V \text{ a.s.}, \quad P_{Q_n}(V < \delta)$$

$$\lim_{k \rightarrow \infty} P_{Q_{n_k}}\left(\frac{dP_{n_k}}{dQ_{n_k}} < \delta\right) = \lim_{k \rightarrow \infty} \left\{ \frac{n_k}{n_k + 1} < \delta \right\} = 0 \iff \delta < \frac{1}{2}, \quad Q_n \ll P_n. \quad \frac{dQ_n}{dP_n} = \frac{n+1}{n} \{[0,1]\},$$

$$P_{P_n}\left(\frac{dQ_n}{dP_n} < \delta\right) = P_{P_n}\left(\{0 \leq X \leq 1\} \subset \frac{\delta n}{n+1}\right) = P_{P_n}(X \in [0,1]) = \frac{1}{n+1} \xrightarrow{n \rightarrow \infty} 0, \quad R_n \ll Q_n$$

$$P_n(A) \rightarrow 0 \Rightarrow Q_n(A) = P_n(A) + Q_n(A) - P_n(A) \leq P_n(A) + \|P_n - Q_n\|_{TV} \rightarrow 0 \quad \#4$$

$$P_n(0) = 1 - \varepsilon, \quad P_n(1) = \varepsilon, \quad Q_n(0) = \varepsilon, \quad Q_n(1) = 1 - \varepsilon, \quad \|P_n - Q_n\| = 1 - 2\varepsilon, \quad P_n(A) \rightarrow 0 \Rightarrow A \cap \Sigma_{Q_n} \neq \emptyset$$

## Andreeescu

$$a+b+c = 2, \geq 3(abc)^{1/3}, \quad (1-a)(1-b)(1-c) = 1-a-b-c + ab+bc+ac - abc =$$

$$-1 + abc \leq 0 \Leftrightarrow (1-a)(1-b)(1-c) \geq 0, \quad a+b \geq c, \quad 1-a \geq -b < 1-c \Leftrightarrow (c-a)(a+b) \geq 0$$

$$\Rightarrow a-1 \leq \left(\frac{1}{3}(a+b+c)\right)^3 = \frac{8}{27}, \quad a \leq \frac{35}{27} \quad \left(\frac{1}{3}(1-a-b-c)\right)^3 = \frac{1}{27}, \quad a \leq \frac{26}{27}.$$

$$abc \geq ab+bc+ac - \left(\frac{1}{3}(a+b+c)\right)^3 = abc + ab + ac - \frac{8}{27}, \quad ((a-1)(b-1)(c-1))^{1/3}$$

$$= (b-1)(a-1)(b+a-1), \quad 0 \leq ? (1-a)(1-b)(1-c), \quad i>1 \Rightarrow b+c = 2-a < 1,$$

$$a \geq b+c, \quad a, b, c < 1 \quad \frac{365^{25}}{\prod_{j=1}^{25} (365-j)} \leq ? \frac{1}{2}, \quad \left(\frac{1}{25} \sum (365-j)\right)^{25} \leq ?$$

$$365^{25}/2, \quad 365 - \frac{25 \cdot 26}{50} \leq ? 365/2^{25}, \quad 185 \frac{352}{352-365} 2^{25} \leq ? \frac{365}{352} \approx \frac{13}{11 \cdot 32} + 1$$

$$\geq 1 + \frac{1}{32} \approx 1.02, \quad \frac{13}{352} \approx \frac{13}{360} = \frac{1}{30}, \quad (1.03)^{25} \geq 2, \quad 1.03^4 = 1.0609, \quad 1.12, 1.25,$$

$$1.56, \quad 9: 1.2025, \quad \frac{156}{1293} \dots 0.125/9 \approx (4:3) \quad 1, \quad a_m \cdot 5 = 1 + (2-1) + \text{back}, \quad 2 = 5 + 1 + 2k, \quad 13 +$$

$$3 \cdot k = 22, \quad a_{m+1} \stackrel{?}{=} n-1, \quad a_n - a_{n-1} \stackrel{?}{=} 1, \quad a_n - a_{n-1} = ? \frac{1}{k(n-1)} + 1, \quad a_n = \sum_{m=1}^{n-1} \frac{1}{k(m-1)} + 1$$

$$= n + \frac{n(n-1)}{2} \quad \frac{8697}{f(0) + 2f(f(0)) = 5, \quad f(f(1)) = \frac{1}{2}(5-f(0)) = f(\frac{5-f(0)}{2})}, \quad 5-f(0) \neq 0,$$

$$2| (5-f(0)), \quad f(0) \in \{1, 3, 5\} \quad (f(0)=1) \quad f(1) = 2 - f(\frac{5-f(0)}{2}) = f(\frac{5-f(0)}{2}), \quad f(2) = f(f(1))$$

$$= \frac{1}{2}(3+5-f(1)), \quad f(2) = f^2(1) = 3, \quad f(3) = f^2(2) \quad \therefore f(n+1) = f(f(n)) = \frac{1}{2}(3+5-(n+1))$$

$$= n+2, \quad f(n) = n+1 \quad (f(0)=3) \quad f^2(1) = 3, \quad f(3) \in \mathbb{N}_+$$

$$(f(0)=5) \quad f(5)=0, \quad f(6)= \dots \quad \therefore f(n+1) = f(f(n)) = \frac{1}{2}(3+5-(n+1))$$

$$f(0) \geq -1, \quad f(n) \geq 0, \quad (n \geq 1) \quad f(n+1) = f(f(n)) = \frac{1}{2}(3+5-(n+1)) = 0, \quad f(n^2) =$$

$$2f(n) + 3f(n)^2 = (3f(n)) \quad f(2n) = f(2) + f(n) + 3f(2)f(n),$$

$$f(mn) = 1 + \frac{f(mn)}{f(m)} + \dots + \frac{f(mn)}{f(n)} \geq 0 \Rightarrow f(n) \geq 1, \quad f(6) \geq 3+6=9, \quad \text{absturde} \Rightarrow f(4)$$

$$= 2f(2) + 3f(2)^2, \quad f(4) \quad f(9) + f(2) + 3f(4)f(2) = 3f(2) + 3f(2)^2 + 6f(2)^2 + 9f(2)^3$$

$$= 3f_2 + 9f_2^2 + 9f_2^3, \quad \text{absturde} \Rightarrow f(n)f'(n) = 1 + f(n) \text{ abstrurde}$$

## advarnt ch6 cont

$P(A_n) \rightarrow 0, \quad Q(A_n) \neq 0 \Rightarrow \{A_{n_k}\}_k, \quad k \geq 0: \quad Q(A_{n_k}) > \delta, \quad \{A_{n_k}\}_k, \quad P(A_{n_k}) < 2^{-k}, \quad B_m := \bigcup_{j=m}^{\infty} A_{n_j}, \quad P(B_m) \leq 2^{-m+1} \xrightarrow{m \rightarrow \infty} 0, \quad B_m \downarrow B_\infty, \quad \text{Ges. } 0 = \lim P(B_m)$

$\therefore P(\lim B_m) = Q(\lim B_m) = \lim Q(B_m), \quad Q(B_m) > 0, \quad \leftarrow \frac{dQ}{dP_p} \left( \frac{dQ}{dP_p}, 0 \right)$

$= P_{P_p}(dQ > 0) = 1 - P_p(dQ = 0) = 1 \quad \text{ qed}$

Andreeescu

$$(mx+ny)^2 + 10(jx+ky)^2 = (m^2 + 10j^2)x^2 + (2mn + 20jk)xy + (n^2 + 10k^2)y^2, \quad mn = -10jk$$

$$m^2 + 10j^2 \leq 1, \quad n^2 + 10k^2 \leq 10, \quad n^2 + 10k^2 = 10(m^2 + 10j^2), \quad 2mn + 20jk = 0, \quad mn = -10jk$$

$$\begin{aligned} m^2 + 10j^2 &\leq 1, \quad n^2 + 10k^2 \leq 10, \\ m^2 + 10j^2 &= 10(n^2 + 10k^2), \quad 2mn + 20jk = 0, \quad mn = -10jk \end{aligned}$$

$$m^2j^2k^2 = n^2(u^2 - 10j^2), \quad 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, \dots, 1, 4, 9, 5, 6, \quad \{3, 2, 7, 5, 8\}$$

$$x^2 + 10y^2 = -\{1, 4, 9, 5, 6\}, \quad \text{and } n = \{5\}, \quad \text{also } 5|1, 5|y, 5|z \quad \frac{\#706}{(2x+2y)(2x-2y) = t^2 - t^2}$$

$$= (t+1)(t-1), \quad 4|(t^2 - 1^2), \quad x^2 - y^2 = \frac{1}{4}(t^2 - 1^2), \quad 2ty^2 = 2t(x^2 - y^2), \quad \text{and } \frac{x^2}{y^2} = -\frac{1}{4}\left(\frac{t-1}{t+1}\right)\left(\frac{t^2}{t^2}\right)$$

$$24x^2 = 5t^2 - z^2, \quad 24y^2 = 5z^2 - t^2, \quad 5|(t-x)(t+x), \quad \cancel{4|t^2 - 5|z^2}, \quad (t-x+5k, t+y+5j)$$

$$5y^2 = 10bx + 25k^2, \quad y^2 = 2ab^2kx + 5k^2, \quad \text{and } x^2 = 2jy + 5j^2, \quad 5|(2x)(2+y) = 5y$$

$$= y_1y_2, \quad t = \frac{1}{2}(y_1 + y_2), \quad s = \frac{1}{2}(y_2 - y_1), \quad (t-y)(t+y) = 5x^2 = x_1x_2, \quad t = \frac{1}{2}(x_1 + x_2)$$

$$y = \frac{1}{2}(x_2 - x_1), \quad 4y^2 = x_1^2 + x_2^2 - 10x^2 = \frac{4}{5}(z^2 - x^2), \quad -\frac{4b}{5}x^2 + x_1^2 + x_2^2 = z^2, \quad 5|x,$$

$$5|z, \quad 5|t, \quad \cancel{4|t^2 + 5y^2} \quad 25\left(\frac{x}{5}\right)^2 + 5y^2 = 25\left(\frac{z}{5}\right)^2, \quad 5x_{(1)}^2 + y^2 = 5z_{(1)}^2,$$

$$x^2 + 5y_{(1)}^2 = t_{(1)}, \quad 5|?y, \quad 25|(z^2 - x^2), \quad 5|y, \quad \left(\frac{x}{5}\right)^2 + 5\left(\frac{y}{5}\right)^2 = \left(\frac{z}{5}\right)^2,$$

$$5\left(\frac{x}{5}\right)^2 + \left(\frac{y}{5}\right)^2 = \left(\frac{z}{5}\right)^2 \quad \frac{\#707}{p|x \Rightarrow p|(x-y)(x+y), \quad \text{ply}, \quad x=y} \quad \frac{\#714}{x_j^2, x_{j+1}^2, x_{j+2}^2, \dots},$$

$$x_{j+1}^2 - x_j^2 = d, \quad x_{j+1} - x_j = \frac{d}{x_j + x_{j+1}} \rightarrow 0 \quad \frac{\#719}{n^3 - 4n^2 + 1 + n^2 - 1 + 4 = n^3 + n^2 + 3 + (1-4)n - (1+4)n^2 - 2n^3}, \quad (2n-1)|(n^3 + n^2 + 3)$$

$$(2n-1)|(2n^3 - n^2 - 2n^2 + 4) = n^2(2n-1) - n^3 - 2n^2 + 4, \quad (2n-1)|(n^3 + 2n^2 - 4), \quad (2n-1)|(n^2 - 7); \quad n^3 - 2n^2 - n^2 - n + n + 4 = -n(2n-1) + n^3 - n^2 - n + 4, \quad (2n-1)|$$

$$(n^3 - n^2 - n + 4), \quad (2n-1)|(2n^2 + n - 1) = (2n-1)(n+1), \quad n^2 = -\{1, 4, 9, 6, 5, 0\}, \quad n^2 - 7 = -\{4, 7, 2, 9, 6, 3\}$$

$$\cancel{2n-1 \mid 4n^2 - 1}; \quad (2n-1)|(n^3 - 17); \quad (2n-1)|(5n^2 - 4), \quad (2n-1)|(-27),$$

$$\cancel{2n-1 \in \{1, 3, 9\}} \quad \frac{\#724}{27 \mid 1! \cdot 2! \cdot \dots \cdot 100! = 1^{100} 2^{99} 3^{98} \dots 99^2 \cdot 100^1},$$

$$x^2(2 \cdot 4 \cdot 6 \dots 100) = x^2 50! 2^{50} = y^2 50! \quad \frac{\#715}{\text{and } a_j = m + j}, \quad \text{and } m | a_k =$$

$$(a_j, a_k) = m, \quad (a_j, a_{j+1}) = j \Rightarrow j|m, \quad (a_j, a_m) = j, \quad a_j = b_j \cdot j, \quad (j, k) = (b_j \cdot 1, b_k \cdot k) \Rightarrow (b_j, b_k) = 1$$

$$b_j \neq 1, \quad (a_{j+1}, a_{j+2}) = (j, b_{j+1}), \quad b_{j+1} | j, \quad j = b_{j+1} \cdot a_{j+1} \quad \frac{\#727}{a_{j+1} = 1}$$

IV

$$\frac{1}{2} \frac{\partial T}{\partial \gamma} = \left( \frac{1}{2} \frac{\partial T}{\partial \gamma} - \frac{\partial D}{\partial \gamma} \right) \frac{1}{2} (T^2/4 - D)^{-1/2} = 0, \quad T = p^2 \left( \frac{\gamma^2}{\delta_0^2} + \frac{(1-\gamma)^2}{\delta_1^2} \right) + (1-p)^2 \left( \frac{\gamma^2}{\delta_1^2} + \frac{(1-\gamma)^2}{\delta_0^2} \right),$$

$$\frac{\partial T}{\partial \gamma} = 2p^2 \left( \frac{\gamma}{\delta_0} - \frac{1-\gamma}{\delta_1} \right) + 2(1-p)^2 \left( \frac{\gamma}{\delta_1} - \frac{1-\gamma}{\delta_0} \right), \quad D = \frac{1}{2} (p^4 + (1-p)^4) \cdot (2p-1)^2$$

$$T = p^2 \left( \frac{\gamma}{\delta_0} + \frac{1-\gamma}{\delta_1} \right) + (1-p)^2 \left( \frac{\gamma}{\delta_1} + \frac{1-\gamma}{\delta_0} \right), \quad \frac{\partial T}{\partial \gamma} = p^2 \left( \frac{1}{\delta_0} - \frac{1}{\delta_1} \right) + (1-p)^2 \left( \frac{1}{\delta_1} - \frac{1}{\delta_0} \right)$$

$$\geq \left( \frac{1}{\delta_0} - \frac{1}{\delta_1} \right) (2p-1), \quad T^2 = p^4 \left( \frac{\gamma^2}{\delta_0^2} + \frac{(1-\gamma)^2}{\delta_1^2} + 2 \frac{\gamma(1-\gamma)}{\delta_0 \delta_1} \right) + (1-p)^4 \left( \frac{\gamma^2}{\delta_1^2} + \frac{(1-\gamma)^2}{\delta_0^2} + 2 \frac{\gamma(1-\gamma)}{\delta_0 \delta_1} \right)$$

$$+ 2p^2(1-p)^2 \left( \frac{\gamma^2}{\delta_0 \delta_1} + \frac{(1-\gamma)^2}{\delta_0 \delta_1} + \gamma_0 \gamma_1 (\frac{1}{\delta_0^2} + \frac{1}{\delta_1^2}) \right) \quad (\delta_0 = \delta_1) \quad T = p^2 \delta^{-2} (\gamma_0^2 + \gamma_1^2) (2p^2 - 2p + 1),$$

$$D = (2p-1)^2 \delta^{-4} (\gamma_0 \gamma_1)^2, \quad (\omega_0 \gamma_0 + \omega_1 \gamma_1)/\delta_0 + (\omega_0 \gamma_1 + \omega_1 \gamma_0)/\delta_1$$

$$= ((\omega_0 - \omega_1) \gamma_0 + \omega_1)/\delta_0 + ((\omega_1 - \omega_0) \gamma_0 + \omega_0)/\delta_1,$$

$$P(A_{t+1}^a | L_t^l, V_t^u) = \sum_b P(A_{t+1}^a | L_t^l, V_t^u, B_{t+1}^b) = P(B_t^a | P(A_{t+1}^a | L_t^l) + P(A_{t+1}^u | L_t^l)$$

$$V_t^u = u) = \frac{1}{2} P_B P_{LA}^{a=l} (1-P_{LA})^{a \neq l} + (1-P_B) P_{AU}^{a=u} (1-P_{AU})^{a \neq u}, \quad P(L_{t+1}^l, V_{t+1}^u |$$

$$A_t^a) = P_B P(L_{t+1}^l, V_{t+1}^u | A_t^l = a, B_t^l = b) + (1-P_B) P(L_{t+1}^l, V_{t+1}^u | A_t^u = a),$$

$$B_t^a) = P_B \frac{1}{2} P_B P_{AL}^{a=l} (1-P_{AL})^{a \neq l} + \frac{1}{2} (1-P_B) P_{AU}^{a=u} (1-P_{AU})^{a \neq u}, \quad P(L_t^l | A_t^a)$$

$$= P_B P(L_t^l | A_t^l = a) + (1-P_B) P(L_t^l) = P_B P_{AL}^{a=l} (1-P_{AL})^{a \neq l} + \frac{1}{2} (1-P_B),$$

$$P(A_t^a | L_t^l) = P_B P_{LA}^{a=l} (1-P_{LA})^{a \neq l} + (1-P_B)/2$$

$$P(L_t^l | L_{t-1}^l = l) = \sum_a P(L_t^l | A_{t-1}^a = a) P(A_{t-1}^a | L_{t-1}^l = l) \quad S_j^{-2} \rightarrow S_j$$

$$T = p \left( \frac{1}{\delta_0^2} + \frac{1}{\delta_1^2} \right), \quad \Delta = \frac{p^2}{\delta_0^2 \delta_1^2} - (1-p)^2 \frac{1}{\delta_0^2 \delta_1^2} = (2p-1)/\delta_0^2 \delta_1^2, \quad T^2/4 - \Delta =$$

$$\frac{1}{4} (p^2 (\delta_0^2 + \delta_1^2) + 2 \delta_0 \delta_1) - (2p-1) \delta_0 \delta_1 = \frac{p^2}{4} (\delta_0^2 + \delta_1^2) + \delta_0 \delta_1 (p^2/2 - 2p + 1)$$

$$= \frac{p^2}{4} (\delta_0 - \delta_1)^2 + \delta_0 \delta_1 (p^2 - 2p + 1) = \frac{p^2}{4} (\delta_0 - \delta_1)^2 + \delta_0 \delta_1 (p-1)^2 \quad (\delta_0 = \delta_1)$$

$$p \left( \frac{1}{\delta_0^2} + \frac{1}{\delta_1^2} \right) \pm \frac{p}{\delta_0^2} \pm (1-p)/\delta_0^2 = \frac{1}{\delta_0^2}, \quad \frac{(2p-1)}{4} \frac{1}{\delta_0^2}$$

$$\text{tr } \hat{\delta}_0 \hat{\gamma}_1 = s, \quad T = p^2 ((\gamma_0 - \gamma_1) / (\delta_0 + \gamma_1 s)) + (1-p)^2 ((\gamma_1 - \gamma_0) / (\delta_0 + \gamma_0 s))$$
$$= (\gamma_0 - \gamma_1) \delta_0^{-1} (2p-1) + s(p^2 \gamma_1 + (1-p)^2 \gamma_0),$$

Andreas

$$n^a - 1 = (n^{(a,b)} - 1)(1 + n^{a-b} + n^{2(a-b)} + \dots + n^{(a,b)-1}), \quad a' := \frac{a}{(a,b)}, \quad n^{(a,b)-1} \mid (n^a - 1, n^b - 1). \quad \text{Z d} \mid (n^a - 1, n^b - 1),$$

$$d \mid (n^a - n^b) = (n^{(a,b)a'} - n^{(a,b)b'}) = (n^{a'} - n^{b'}) (n^{a'((a,b)-1)} + n^{a'((a,b)-2)} n^{b'} + \dots + n^{b'((a,b)-1)})$$

$$= (n^{a'} - n^{b'}) (n^{a-a'} + n^{a-2a'+b'} + n^{a-3a'+2b'} + \dots + n^{b-b'}), \quad (a' + b' + \dots + a - 1 = n^a - 1) \sum_{j=0}^{a'-1} n^{aj}$$

$$n^b - 1 = (n^a - 1) \sum_{j=a}^{a'-1} n^{aj}, \quad (b > a) \quad \text{Z } n^b - 1 - (n^a - 1) = (n^a - 1) \sum_{j=a}^{a'-1} n^j, \quad \text{Z } \cancel{n^a - 1} \sum_{j=a}^{a'-1} n^j$$

$$(a', b') = 1, \quad a'x + b'y = 1, \quad \begin{matrix} 1+n+n^2 \\ 1+n+n^2+n^3+n^4 \end{matrix} \text{ (irrational)}, \quad 1+n \rightarrow 1 \neq (\sum_{j=1}^{a'-1} - n^2 \sum_{j=1}^{b'-1}) (-n) + \sum_{j=1}^{a'-1} = 1 - n \sum_{j=1}^{a'-1} +$$

$$(n^a - 1) \sum_{j=0}^{a'-1} n^j, \quad S_{a'-1} := \sum_{j=0}^{a'-1} n^j, \quad \alpha S_{a'-1} + \beta S_{b'-1} = ? 1, \quad \alpha, \beta \in \mathbb{Z}, \quad (a'x + b'y = 1)$$

$$(\sum_{j=0}^{a'-1} n^{ja'}) S_{a'-1} = (\sum_{j=0}^{a'-1} n^{ja'}) \sum_{j=0}^{a'-1} n^j = \sum_{j=0}^{a'-1} n^j, \quad (\sum_{j=0}^{a'-1} n^{jb'}) S_{b'-1} = \sum_{j=0}^{a'-1} n^j = \sum_{j=0}^{a'-1} n^j, \quad \cancel{\sum_{j=0}^{a'-1} n^{ja'}} \quad \cancel{\sum_{j=0}^{a'-1} n^{jb'}}$$

$$(\sum_{j=0}^{a'-1} n^{ja'}) S_{a'-1} - n (\sum_{j=0}^{a'-1} n^{jb'}) S_{b'-1} = 1 \quad // \quad \text{now } g = (a, b), \quad a = ga', \quad b = gb', \quad (a', b') = 1,$$

$$(2^a - 1, 2^b - 1) = (2^a + 1) \sum_{j=0}^{a'-1} 2^{aj}, \quad 2^b - 1 = (2^b + 1) \sum_{j=0}^{b'-1} 2^{bj}, \quad (\# 727) \Rightarrow \alpha, \beta \in \mathbb{Z},$$

$$\alpha \sum_{j=0}^{a'-1} 2^{aj} + \beta \sum_{j=0}^{b'-1} 2^{bj} = 1, \quad 2^a - 1 = \alpha (2^a + 1) + \beta (2^b + 1), \quad (2^a + 1, 2^b + 1) \mid (2^a - 1)$$

$$(2^a - 1, \quad b' = 2^k b'' \quad \text{, } 2^b - 1) \quad 2^a - 1 = (2^a + 1) \sum_{j=0}^{a'-1} (-2^a)^j, \quad 2^b - 1 = (2^b + 1) \sum_{j=0}^{b'-1} (-2^b)^j$$

$$\alpha, \beta \in \mathbb{N} : \alpha (2^a + 1) \sum_{j=0}^{a'-1} (-2^a)^j = (2^a + 1) \left\{ \beta \sum_{j=0}^{b'-1} (-2^b)^j - 1 \right\}, \quad 2^{a+b+k} \alpha 2^{k+1} 2^k.$$

$$= \sum_{j=0}^{a'-1} 2^{aj} + 1 = \sum_{j=1}^{a'-1} 2^{aj} + 2, \quad (\alpha : (2^a + 1)) ? \beta : (2^a + 1) \sum_{j=0}^{b'-1} (-2^b)^j = \beta \sum_{j=0}^{a'-1} (-2^a)^j - 1;$$

$$HS = \sum_{j=0}^{b'-1} (-1)^j 2^{2^k b'' (j+1)} + \sum_{j=0}^{b'-1} (-1)^j (2^{2^k b''})^j = - \sum_{j=1}^{b'-1} (-1)^j 2^{2^k b'' j} + \sum_{j=0}^{b'-1} (-1)^j (2^{2^k b''})^j$$

$$= 1 - (-2^{2^k b''})^{b''} = ? \beta \sum_{j=0}^{a'-1} (-2^a)^j - 1, \quad (-2^{2^k b''})^{b''} = ? \beta \sum_{j=0}^{a'-1} (-2^a)^j, \quad \cancel{\text{not possible}}$$

$$? \alpha', \beta : \{ (2^a + 1) - 1 \} \alpha' = \beta \sum_{j=0}^{a'-1} (-2^a)^j - 1 = \beta \frac{2^a - 1}{2^a + 1} - 1, \quad \alpha' (2^a + 1) (2^a - 1) = ?$$

$$\beta (2^a + 1) - 1, \quad \alpha' (2^{a+g} + 2^g (1 + 2^{b-g}) + 1) = ? \beta 2^a + \beta - 1; \quad 2^g \{ \alpha' (1 + 2^{b-g} + 2^g)$$

$$- \beta 2^{a-g} \} = ? \beta - 1 - \alpha'$$

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$$L(\mu) = \left(2\pi\frac{\sigma^2}{n}\right)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right), \quad \frac{L(\mu_0)}{L(\mu_n)} = \exp\left(\frac{1}{2\sigma^2} \sum (x_i - \mu_n)^2 - (x_i - \mu_0)^2\right)$$

$$\therefore \exp\left(\frac{1}{2\sigma^2} \sum (2x_i(\mu_0 - \mu_n) + \mu_0^2 - \mu_n^2)\right) < k, \quad 2(\mu_0 - \mu_n)n\bar{x} + n(\mu_0^2 - \mu_n^2) < k \\ \bar{x} < k, \quad \alpha = P_{\mu=\mu_0}(\bar{x} < k) = P\left(Z < \frac{k-10}{5/\sqrt{n}}\right), \quad k = 10 - \frac{5}{\sqrt{n}} \approx 0.025, \quad P_{\mu=\mu_0}(\bar{x} < \\ 10 - \frac{5}{\sqrt{n}}) = P\left(Z < \frac{5 - 5\sqrt{n}\tau_\alpha}{5/\sqrt{n}}\right) = P(Z < \sqrt{n} - \tau_\alpha), \quad \frac{+0.018}{.01} > \chi^2_{0.05}$$

$$\left(\frac{\sigma_n}{\sigma_0}\right)^n \exp\left(-\sum (x_i - \mu)^2 \left(\frac{1}{2\sigma_n^2} - \frac{1}{2\sigma_0^2}\right)\right) < k, \quad \sum (x_i - \mu)^2 \left(\frac{1}{2\sigma_n^2} - \frac{1}{2\sigma_0^2}\right) < k$$

$$\sum (x_i - \mu)^2 > k, \quad \chi^2(n) > k, \quad \chi^2(n) = \chi^2_\alpha, \quad \theta^{2N_1} (2\theta(1-\theta))^{N_2} (1-\theta)^{2N_3}$$

$$\text{treatment} \quad L(\theta) = \frac{\theta^n}{\theta^{2N_1} (1-\theta)^{N_2} (1-\theta)^{2N_3}}, \quad L(\theta) = \left(\frac{\theta}{\theta_n}\right)^{2N_1+N_2} \left(\frac{1-\theta}{1-\theta_n}\right)^{N_2+2N_3} < k$$

$$\text{first continue to } \theta_n \text{ only} \quad \chi^2(2N_1+N_2) \log\left(\frac{\theta_n}{\theta_n}\right) + (N_2+2N_3) L(\theta) < k$$

$$\text{strike with DAGs not known a's} \quad L(\lambda) = e^{-n\lambda} \frac{\lambda^n}{n!}, \quad \frac{dL(\lambda)}{d\lambda} =$$

$$\text{interpolate side} \quad \therefore e^{n(\lambda_n - \lambda_0)} \left(\frac{\lambda_0}{\lambda_n}\right)^{n\bar{x}} < k, \quad \bar{x} \log(\lambda) < k, \quad \bar{x} > k,$$

$$\alpha = P(\sum x_i > k), \quad \binom{24}{26} p^{26} (1-p)^{44} \binom{14}{53} p^{53} (1-p)^{44} \dots = (1-p)^2 (1-p)^2 \dots \frac{1}{(2\pi)^{3/2} (\sigma_1^2 \sigma_2^2 \dots)}$$

$$\sigma_0^2)^{-1/2} \exp\left(-\frac{1}{2\sigma_0^2} \sum (x_i - \mu_1)^2 - \frac{1}{2\sigma_1^2} \sum (x_i - \mu_2)^2 - \frac{1}{2\sigma_2^2} \sum (x_i - \mu_3)^2\right), \quad L(\theta_0) = (2\pi)^{-3/2} \sigma_0^{-3}$$

$$\exp\left(-\frac{1}{2\sigma_0^2} \left(\hat{\Sigma}_1 + \hat{\Sigma}_2 + \hat{\Sigma}_3\right)\right), \quad \text{but } \log = -\frac{3}{2} \Rightarrow \log \sigma^2 - \frac{1}{2\sigma_0^2} \left(\hat{\Sigma}_1 + \hat{\Sigma}_2 + \hat{\Sigma}_3\right),$$

$$\frac{\partial \log}{\partial \sigma^2} = -\frac{3}{2\sigma_0^2} + \frac{1}{2\sigma_0^4} (\Sigma - \dots) = 0, \quad \sigma^{-2} = 3/(\Sigma - \dots), \quad \hat{\sigma}^2 = \frac{1}{3} (\Sigma + \dots), \quad \frac{\partial \log}{\partial \mu_1} = \frac{1}{2\sigma_0^2} \sum (x_i - \mu_1)$$

$$\mu_1 = \bar{x}, \quad L(\theta) =$$

$$\frac{1}{\|u\|^2} (u, u) = u^T u / \|u\|^2, \quad \lambda(A) = 1, \quad A^2 = uu^T u u^T = A \quad \text{if } uu^T = \begin{pmatrix} 4/9 & 1/9 & 2/9 \\ 1/9 & 4/9 & 2/9 \\ 2/9 & 2/9 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 5/9 & -1/9 & -2/9 \\ -1/9 & 5/9 & -2/9 \\ -2/9 & -2/9 & 8/9 \end{pmatrix}$$

$$P_b = \begin{pmatrix} 5/9 & -1/9 & -2/9 \\ -1/9 & 5/9 & -2/9 \\ -2/9 & -2/9 & 8/9 \end{pmatrix}, \quad \lambda(P) = \{0, 1, 1\}, \quad P^2 = I - 2uu^T + uu^T uu^T = P$$

$$\text{If } H = \begin{pmatrix} 4/9 & 1/9 & 5/9 \\ 1/9 & 4/9 & 2/9 \\ 5/9 & 2/9 & 1 \end{pmatrix}, \quad \lambda(H) = \{1, 1, -1\}, \quad H^2 = I - 4uu^T + 4uu^T uu^T = I \quad \text{if } R(H) = c \cdot (1, 1),$$

$$N(A) = (1, 1, 2)^\perp, \quad R(A^T) = \{(2, 1, 1)\}, \quad N(A^T) = (1, 1)^\perp = c \cdot (1, -1)$$

$$A^T A = w v^T v w^T = (v^T v) w w^T, \quad A^T A w = \|v\|^2 \|w\|^2 w \quad \text{if } I = B B^{-1} = I - (c+1) v w^T$$

$$+ c v w^T v w^T = I + (-c-1 + c w^T v) v w^T, \quad c(w^T v - 1) = 1, \quad c = (w^T v - 1)^{-1}$$

$$B B^{-1} = I - c v w^T A^T - v w^T A^{-1} + c v w^T A^{-1} v w^T A^{-1} = I + (-c-1) v w^T A^{-1} +$$

$$c w^T A^{-1} v v w^T A^{-1} = I + v w^T A^{-1} (-c-1 + c w^T A^{-1} v), \quad c = (w^T A^{-1} v - 1).$$

~~$$v = w = (1, 0), \quad c = \bar{a}_{11} - 1, \quad c \left( \frac{\bar{a}_{11}}{\bar{a}_{21}} \right) (\bar{a}_{11}, \bar{a}_{12}) \approx 6.85$$~~

$$\lambda(A)=0, \quad \{1, -1\}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \{1, 1\}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d-b \\ -c-a \end{pmatrix} \frac{1}{ad-bc}$$

$$= \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \begin{pmatrix} d-b \\ -c-a \end{pmatrix} \frac{1}{ad-bc} = \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} \cdot \frac{1}{\Delta}, \quad a^2=ac, \quad a=c, \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -a & a \\ -a & a \end{pmatrix} \cdot \frac{1}{d-b}, \quad a=c=-1, \quad d=1, \quad b=0, \quad \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \cdot -1 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = A$$

$$A^T = (S^T)^{-1} \wedge S^T, \quad (S^T)^{-1} \stackrel{6.22}{=} A^T \{1, 1, 1\} = 0, \quad N(A^T) = \{(1, 1, 1)\}, \quad Q(A^T) = \{(1, 0, -1),$$

$$(1, -1, 0) \} \stackrel{6.25}{=} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} A, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = A \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & -15/4 \\ 4 & 0 \end{pmatrix} = A \begin{pmatrix} 1 & -8/4 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 4 & 5 \\ 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} r_1 g_{11} & r_2 g_{11} + r_3 g_{21} \\ r_1 g_{21} & r_2 g_{21} + r_3 g_{31} \end{pmatrix} = A, \quad g_1 \propto \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

$$g_2 \propto \begin{pmatrix} 3 \\ 4 \end{pmatrix} \perp \propto \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} / 5, \quad g_2 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} / \sqrt{5} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad r_1 = 5, \quad r_2 \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} + r_3 \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$: \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \quad r_2 = -\frac{1}{3} r_3, \quad \frac{1}{5} - \frac{3}{5} \cdot 1 \cdot r_3 = 5, \quad r_3 = \frac{-25}{19}, \quad r_2 = \frac{100}{57}, \quad A = \begin{pmatrix} 3/5 & 1/5 \\ 1/5 & -2/5 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -25 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}^2 = \begin{pmatrix} b_1^2 + b_2^2 & b_1 b_2 + b_2 b_3 \\ b_1 b_2 + b_2 b_3 & b_2^2 + b_3^2 \end{pmatrix}, \quad b_1^2 = b_3^2, \quad b_1 = b_3, \quad b_1^2 + b_2^2 = 25,$$

$$2b_1 b_2 = 20, \quad b_1^2 + \frac{100}{b_1^2} = 25, \quad b_1^4 - 25b_1^2 + 100 = 0, \quad b_1^2 = \frac{1}{2}(25 \pm \sqrt{15}) = 20, 5, \quad B = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$\textcircled{2} Q = A B^{-1} = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} \sqrt{5} & -2\sqrt{5} \\ 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -\sqrt{5} & \sqrt{5} \\ 2\sqrt{5} & \sqrt{5} \end{pmatrix} \stackrel{4+\sqrt{5}\sqrt{2}=3\sqrt{2}}{\parallel} (3-\lambda)(5-\lambda) = \cancel{25+4\sqrt{5}\lambda+\lambda^2}, \quad \text{or } (1, -2),$$

$$(0, 1), \quad A = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(25-\lambda)^2 - 400 = \lambda^2 - 50\lambda + 225, \quad \frac{1}{2}(50 \cancel{+ 40}) = 45, 5, \quad \cancel{400} + 25 + 20g = 95, \quad g = 1, \quad g_1 = (1, 1)$$

$$g_2 = (1, -1), \quad A^T A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \stackrel{A = \cancel{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \begin{pmatrix} 3\sqrt{10} & -\sqrt{10} \\ \sqrt{10} & \sqrt{10} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{=} \cancel{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

$$= \cancel{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 9 & 10 \\ 10 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \stackrel{\cancel{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3\sqrt{10} & -\sqrt{10} \\ \sqrt{10} & \sqrt{10} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}}{=} \cancel{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \begin{pmatrix} 3\sqrt{10} & -\sqrt{10} \\ \sqrt{10} & \sqrt{10} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{2\sqrt{10}}{\sqrt{10}} \frac{\sqrt{10}}{\sqrt{10}}$$

$$AA^T = \begin{pmatrix} 9 & 12 \\ 12 & 9 \end{pmatrix}, \quad (9-\lambda)(41-\lambda) - 144 = \lambda^2 - 50\lambda + 225 = (\lambda-45)(\lambda-5), \quad 9+12g = 45, \quad g_1 = (1, 3),$$

$$g_2 = (1, -3), \quad A = ? \begin{pmatrix} \sqrt{10} & \sqrt{5} \\ 3\sqrt{10} & -2\sqrt{5} \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \sqrt{10} & \sqrt{5} \\ \sqrt{10} & -\sqrt{5} \end{pmatrix} = \begin{pmatrix} \sqrt{10} & \sqrt{5} \\ 3\sqrt{10} & -3\sqrt{5} \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} \sqrt{10} & \sqrt{5} \\ \sqrt{10} & -\sqrt{5} \end{pmatrix} = 1$$

$$A = ? \begin{pmatrix} \sqrt{10} & \sqrt{5} \\ 3\sqrt{10} & -2\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ \sqrt{5} & -\sqrt{5} \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 6\sqrt{5} & 0 \\ 6\sqrt{5} & 10\sqrt{5} \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} = \cancel{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 45 & 70 \\ 70 & 70 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} = \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix} \cancel{\frac{6\sqrt{5}}{\sqrt{10}}}$$

$$\hat{\theta} = \sum_i \varphi_{ij} / M^2 = \sum_j \varphi_{ij} / \binom{M}{2} = \sum_j \varphi_{ij} / M^2.$$

$$V(x_i) = \sum_j \varphi_{ij} / M, \quad V(y_j) = \sum_i \varphi_{ij} / M = \cancel{\text{fix}} \quad V(x_i) = \varphi_{ii} / M, \quad V(y_j) = \varphi_{jj} / M$$

$$S_x = \sum_i (V(x_i) - \hat{\theta})^2, \quad S_y = \sum_j (V(y_j) - \hat{\theta})^2$$

$$S_{xy} = \sum_k (V(x_k) - \hat{\theta})(V(y_k) - \hat{\theta})$$

$$\hat{\sigma}^2 = S_x / M^2 + S_y / M^2 + 2S_{xy} / M^2 = M^2 \sum_j (V(x_i) + V(y_i) - 2\hat{\theta})^2 / (M(M-1))^2$$

$$\theta_{-i} = \sum_{j \neq i, k \neq i} \varphi_{jk} / \binom{M-1}{2}, \quad \hat{\theta}_j = M\hat{\theta} - (M-1)\theta_{-j}, \quad \hat{\sigma}^2 = M^2 / \sum_j (\hat{\theta}_j - \bar{\theta})^2 / (M(M-1))^2$$

$$\bar{\theta} = M\hat{\theta} - \frac{M-1}{M} \sum_j \theta_{-j}, \quad (M-1)^2 \theta_{-i} = \sum_{jk} \varphi_{jk} - \sum_j \varphi_{ij} - \sum_i \varphi_{ij} + \varphi_{ii}$$

$$= M^2 \hat{\theta} - \varphi_{ii} - \varphi_{ji} + \varphi_{ii}, \quad \sum_j \theta_{-j} = \frac{M^3}{(M-1)^2} \hat{\theta} - \frac{2M^2 \hat{\theta}}{(M-1)^2} + \frac{\text{diag } \varphi}{(M-1)^2},$$

$$\bar{\theta} = \hat{\theta} \left( M - \frac{M^2}{M-1} + \frac{2M}{M-1} \right) - \frac{\text{diag } \varphi}{M(M-1)}, \quad 1 - \frac{M^2}{M-1} \frac{M-2}{n-1} = \frac{1}{M-1},$$

$$\bar{\theta} = \frac{1}{M-1} \left( \hat{\theta} - \frac{\text{diag } \varphi}{M} \right), \quad \hat{\theta}_j - \bar{\theta} = M\hat{\theta} - (M-1)\theta_{-j} - \frac{M}{M-1} \hat{\theta} + \frac{\text{diag } \varphi}{M(M-1)}$$

$$= \frac{M\hat{\theta}(M-2)}{M-1} - \frac{\text{diag } \varphi}{M(M-1)} - \frac{M^2}{M-1} (M^2 \hat{\theta} - \varphi_{jj} - \varphi_{ij} + \varphi_{jj})$$

$$= - \frac{2M}{M-1} \hat{\theta} + \frac{\text{tr } \varphi}{M(M-1)} + \frac{\varphi_{jj} + \varphi_{ij} - \varphi_{jj}}{M-1}, \quad \sum_j (\varphi_{jj} + \varphi_{ij} - \varphi_{jj}) = 2M^2 \hat{\theta} - \text{tr } \varphi,$$

$$\cancel{\text{tr } \varphi} = M \left( \frac{\text{tr } \varphi}{n(n-1)} - \frac{2M}{n-1} \hat{\theta} \right)^2 + 2 \left( \frac{\text{tr } \varphi}{n(n-1)} - \frac{2M}{n-1} \hat{\theta} \right) \frac{2M^2 \hat{\theta} - \text{tr } \varphi}{M-1} + \frac{\sum_j (\varphi_{jj} + \varphi_{ij} - \varphi_{jj})}{(M-1)^2}$$

$$= \frac{\sum_j (\varphi_{jj} + \varphi_{ij} - \varphi_{jj})^2}{(M-1)^2} - M \left( \frac{\text{tr } \varphi - 2M^2 \hat{\theta}}{M(M-1)} \right)^2 = \frac{2M(M-1) \hat{\sigma}_{jk}^2}{M-1}, \quad \sum_j \left( \frac{\varphi_{jj}}{M} + \frac{\varphi_{ij}}{M} - 2\hat{\theta} \right)^2$$

$$= M^{-2} \sum_j (\varphi_{jj} + \varphi_{ij} - \varphi_{jj} - 2M\hat{\theta})^2, \quad \sum_j (\varphi_{jj} - 2M\hat{\theta})^2 = \cancel{\text{tr } \varphi} - 4M\hat{\theta} \text{tr } \varphi$$

$$+ 4M^2 \hat{\theta}^2 = 4M(M^2 \hat{\theta}^2 + 4M^2 \hat{\theta}^2) \cancel{\text{tr } \varphi} \quad \sum_j (\varphi_{jj} + \varphi_{ij} - \varphi_{jj}) (\varphi_{jj} - 2M\hat{\theta}) =$$

$$= 2M\hat{\theta} (2M^2 \hat{\theta} - \text{tr } \varphi) + \sum \varphi_{jj} \varphi_{ij} + \sum \varphi_{ij} \varphi_{jj} - \text{tr } \varphi - \text{tr } \varphi - 4M^3 \hat{\theta}^2 + 2 \sum \varphi_{jj} (\varphi_{jj} + \varphi_{ij})$$

$$+ \sum_j (\varphi_{jj} + \varphi_{ij} - \varphi_{jj})^2, \quad \frac{\sum_j (\varphi_{jj} + \varphi_{ij} - \varphi_{jj})^2}{2} - \frac{\text{tr } \varphi}{2} - 4M^2 \hat{\theta}^2$$

$$M(M-1) \hat{\sigma}_{\text{obs}}^2 = \sum_j \left( \frac{q_{j.} + q_{.j}}{M} - 2\hat{\theta} \right)^2 = M^{-2} \left( \sum_j q_{j.}^2 + \sum_j q_{.j}^2 + 2 \sum q_{j.} q_{.j} \right)$$

$$- 4\hat{\theta} \frac{1}{M} \sum_j (q_{j.} + q_{.j}) + 4M\hat{\theta}^2 = M^{-2} \mathbf{1}^\top (\mathbf{q}^\top \mathbf{q} + \mathbf{q} \mathbf{q}^\top + 2\mathbf{q} \mathbf{q}) \mathbf{1}$$

~~$$M - 4\hat{\theta} \frac{1}{M} \cdot 2M^2 \hat{\theta} + 4M\hat{\theta}^2 = M^{-2} \mathbf{1}^\top (-) \mathbf{1} - 4M\hat{\theta}^2 =$$~~

$$M^{-2} \mathbf{1}^\top (\mathbf{q}^\top \mathbf{q} + \mathbf{q} \mathbf{q}^\top + 2\mathbf{q} \mathbf{q} - \frac{4}{M} \mathbf{q} \mathbf{1} \mathbf{1}^\top \mathbf{q}) \mathbf{1} \quad \frac{1}{(M-1)^2} - \frac{1}{M^2} = \frac{2M-2}{(M(M-1))^2}$$

$$\text{Div? } \frac{2M-2}{(M(M-1))^2} \mathbf{1}^\top (\mathbf{q}^\top \mathbf{q} + \mathbf{q} \mathbf{q}^\top + 2\mathbf{q} \mathbf{q} - \frac{4}{M} \mathbf{q} \mathbf{1} \mathbf{1}^\top \mathbf{q}) \mathbf{1} - \frac{2}{(M-1)^2} \mathbf{1}^\top (\mathbf{q} + \mathbf{q}^\top - \mathbf{I}/2$$

$$- \frac{2}{M} \mathbf{q} \mathbf{1} \mathbf{1}^\top + \text{diag}(\mathbf{q}^\top \mathbf{q} / 2n) \cdot \text{diag}(\mathbf{q})$$

$$\frac{M-1}{M^2} \mathbf{1}^\top (\mathbf{q}^\top \mathbf{q} + \mathbf{q} \mathbf{q}^\top + 2\mathbf{q} \mathbf{q} - \frac{4}{M} \mathbf{q} \mathbf{1} \mathbf{1}^\top \mathbf{q}) \mathbf{1} - \mathbf{1}^\top (\mathbf{q} + \mathbf{q}^\top - \frac{\mathbf{I}}{2} - \frac{2}{M} \mathbf{q} \mathbf{1} \mathbf{1}^\top +$$

$$\text{diag}(\mathbf{q}^\top \mathbf{q} / 2n) \text{diag}(\mathbf{q}) = \frac{1}{2}(M-1)^2 M (\sigma_{j_k}^2 - \sigma_{\text{obs}}^2)$$

$$(q=q^\top) \mathbf{1}^\top (\mathbf{I} - \frac{1}{M} \mathbf{1} \mathbf{1}^\top) \mathbf{q} \mathbf{1} = \sum s_j^2 - \frac{1}{M} (\sum s_j)^2 = \overbrace{\text{var}(\mathbf{q}^\top \mathbf{q})}^{(M-1)} = \overbrace{\text{var}(\mathbf{q}^\top \mathbf{q}^\top)}^{(M-1)}$$

$$\leq \frac{M}{2} ((M-n_2)^2 + (n_2)^2) = \frac{M}{2}$$

$$\mathbf{1}^\top (\mathbf{q}^\top \mathbf{q} - q \mathbf{1}^\top \mathbf{q} / M) \mathbf{1} = \mathbf{1}^\top (\mathbf{q}^\top - q \mathbf{1} \mathbf{1}^\top / M) \mathbf{q} \mathbf{1}, \quad \mathbf{1}^\top \mathbf{q} \cdot \text{Proj}_{\mathbf{1}^\perp} \mathbf{q} \mathbf{1}$$

$$= \mathbf{1}^\top \mathbf{q} (\bar{r} - \bar{\bar{r}}) = (\bar{c}, \bar{r} - \bar{\bar{r}}) = (\bar{c}, \bar{r} - \bar{\bar{c}}), \quad (\bar{c}, \bar{r}) = (\sum q_{ij}) / M,$$

$$(\bar{c}, \bar{r}) = \sum_j (\sum_i q_{ij}) / (\sum_i q_{ji}) = \sum_j \sum_{i,k} q_{ij} q_{jk}, \quad \sum_{i,k} \sum_j q_{ij} q_{jk} - \sum_{i,k} \sum_{j,l} q_{ij} q_{jk} / M$$

$$- \sum_{i \neq j} (q_{ij} q_{jk} - q_{ij} q_{jk} / M) = \sum_{i \neq j} q_{ij} (q_{jk} - q_{jk} / M)$$

$$\begin{aligned}
M(M-1) \sigma_{\text{obs}}^2 &= M^{-2} \sum_j (\varphi_{j.} + \varphi_{.j} - 2M\hat{\theta})^2, \quad \varphi_{j.} = 1 - \varphi_{j.j}, \quad \varphi_{.j} = \sum_i \varphi_{ij} \\
&= \sum_j (1 - \varphi_{j.j})^2 = M - \varphi_{.j.}, \quad M^{-2} \sum_j (M - 2M\hat{\theta})^2 = M - 4M\hat{\theta} + 4M\hat{\theta}^2 = M(1 - 4\hat{\theta}^2), \\
&\sigma_{\text{obs}}^2 = \frac{(M-2\hat{\theta})^2}{M-1} \\
n(n-1) \sigma_{jk}^2 &= \frac{n^2}{(n-1)^2} - M \left( \frac{-2M\hat{\theta}}{M(n-1)} \right)^2 = \frac{n^2}{(M-1)^2} - \frac{4M^3\hat{\theta}^2}{(M-1)^2} = \frac{M^2}{(M-1)^2} (1 - 4\hat{\theta}^2), \\
\sigma_{jk}^2 &= \frac{M^2}{(M-1)^2} (1 - 4\hat{\theta}^2)
\end{aligned}$$

$$\sum_j \varphi_{j.} = \sum_i P(X_i < \tau_j) = \cancel{\sum_i} n - \sum_i P(X_i > \tau_j)$$

$$M^{-2} \sum_j (\varphi_{j.} + \varphi_{.j} - \varphi_{j.j})^2 - \text{tr} \varphi / M^2 - 4M\hat{\theta}^2 + \frac{2}{M^2} \sum_j \varphi_{j.j} (\varphi_{j.} + \varphi_{.j}) = ?$$

$$(M-1)^{-2} \sum_j (\varphi_{j.} + \varphi_{.j} - \varphi_{j.j})^2 - M^{-2} \left( \frac{\text{tr} \varphi - 2M^2\hat{\theta}}{M(n-1)} \right)^2$$

$$\cancel{\sum_j} (\varphi_{j.} + \varphi_{.j} + \varphi_{jj})^2, \quad \sum_j \varphi_{j.}^2 = \sum_j (\sum_i \varphi_{ji})^2 = \sum_j (\sum_i \varphi_{ji} + 2 \sum_{i < k} \varphi_{ji} \varphi_{jk})$$

$$\varphi_{j.} \varphi_{jk} = M^2 \hat{\theta} + 2 \sum_{i < k} (\varphi_{ji} \varphi_{ik}), \quad \sum_j \varphi_{j.}^2 = \mathbb{1}^\top \varphi \mathbb{1}, \quad \sum_j \varphi_{.j}^2 = \mathbb{1}^\top \varphi \varphi^\top \mathbb{1},$$

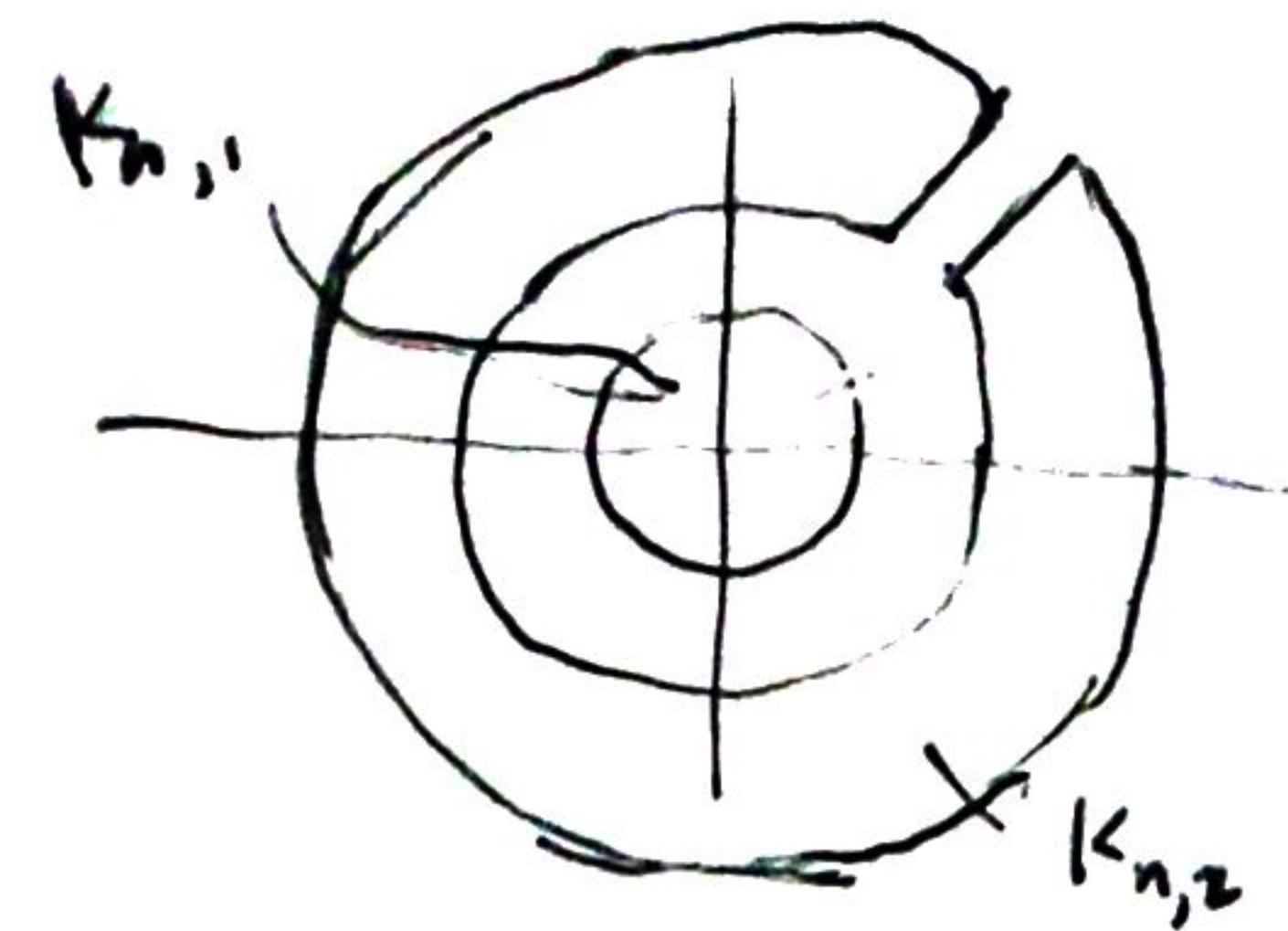
$$\sum_j \varphi_{j.} \varphi_{.j} = \sum_j (\sum_i \varphi_{ji}) (\sum_k \varphi_{kj}) = \sum_j \sum_{i,k} \varphi_{ji} \varphi_{kj} = \sum_{i,k} \sum_j \varphi_{ji} \varphi_{kj} = \mathbb{1}^\top \varphi \varphi^\top \mathbb{1},$$

$$\sum_j (\varphi_{j.} + \varphi_{.j} - \varphi_{j.j})^2 = \mathbb{1}^\top (\varphi^\top \varphi + \varphi \varphi^\top + 2 \varphi \varphi) \mathbb{1} + \text{tr} \varphi - 2 \mathbb{1}^\top (\varphi + \varphi^\top) \mathbb{1}$$

$$\text{diag } \varphi = \mathbb{1}^\top (\varphi^\top \varphi + \varphi \varphi^\top + 2 \varphi \varphi) \mathbb{1} - 2 \mathbb{1}^\top (\varphi - \mathbb{I}/2 + \varphi^\top) \text{diag } \varphi,$$

$$\begin{aligned}
\text{tr} \varphi - 2M^2\hat{\theta} &= \text{tr} \varphi - 2 \mathbb{1}^\top \varphi \mathbb{1}, \quad \text{then } (M-1)^{-2} \left\{ \sum_j (\varphi_{j.} + \varphi_{.j} - \varphi_{j.j})^2 - \frac{(\text{tr} \varphi)^2}{M} \right\} = \\
&= 4M^3\hat{\theta}^2 + 4M\hat{\theta} \text{tr} \varphi \}, \quad \hat{\theta}^2 = \mathbb{1}^\top \varphi \mathbb{1} \mathbb{1}^\top \varphi \mathbb{1} / M^2, \quad \hat{\theta} \text{tr} \varphi = \mathbb{1}^\top \varphi \mathbb{1} \mathbb{1}^\top \text{diag } \varphi, \quad \Rightarrow \\
&(M-1)^2 \left\{ \mathbb{1}^\top (\varphi^\top \varphi + \varphi \varphi^\top + 2 \varphi \varphi - 4M^{-1}\varphi \mathbb{1} \mathbb{1}^\top \varphi) \mathbb{1} - 2 \mathbb{1}^\top (\varphi - \mathbb{I}/2 + \varphi^\top - 2/M \varphi \mathbb{1} \mathbb{1}^\top \right. \\
&\quad \left. + \text{diag } \varphi \cdot \mathbb{1}^\top / 2M) \text{diag } \varphi \right\} = M(M-1) \hat{\sigma}_{jk}^2
\end{aligned}$$

Rudin ch 13, cont

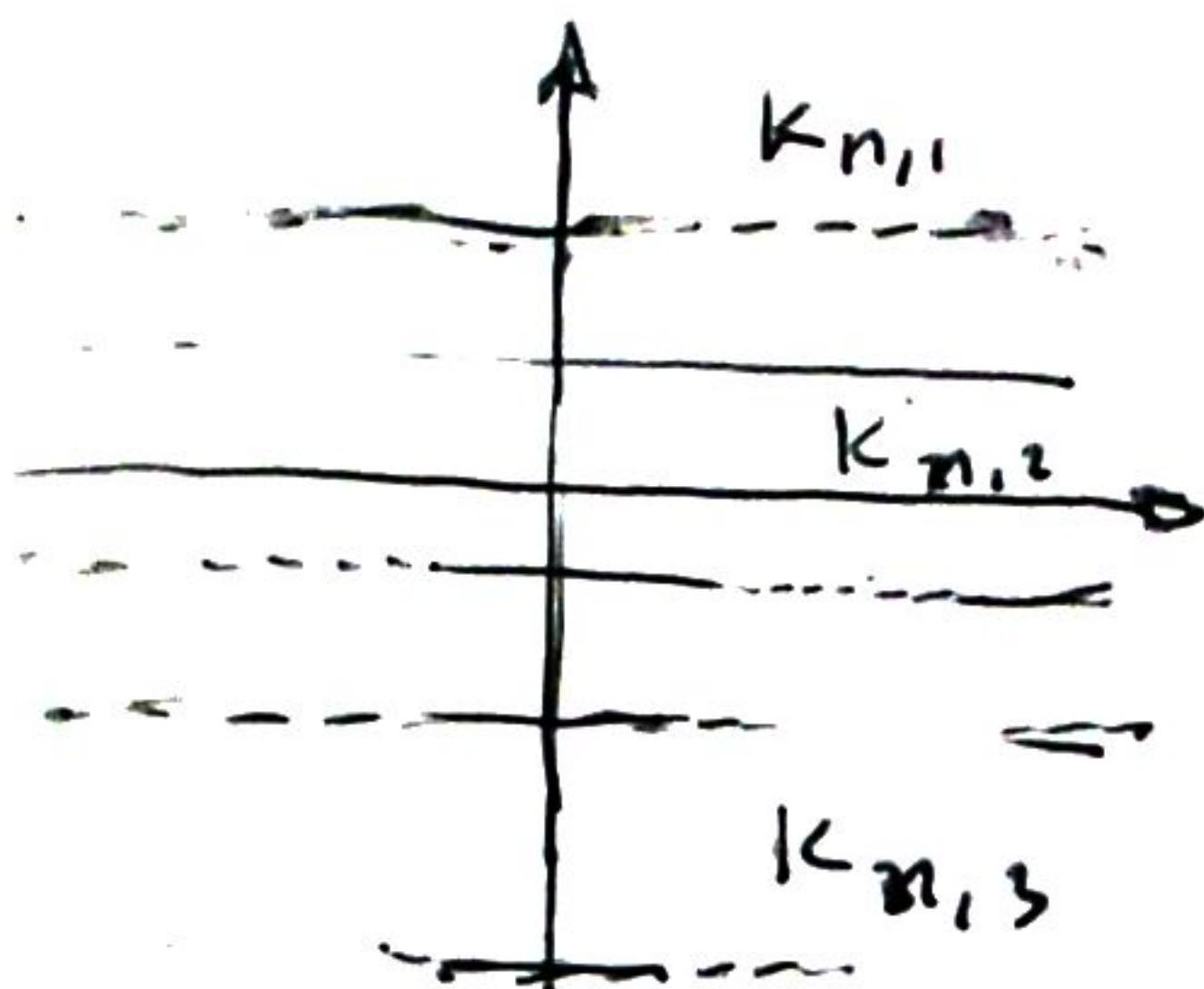


$$K_n := \bar{D}(0, \frac{1}{n}) \cup \left\{ re^{i\theta} : \frac{1}{n} + \frac{1}{n^2} \leq r \leq n, \theta \in \sum_{j=1}^{n-1} \frac{1}{n^2} \cdot \frac{1}{2\pi} \right\}$$

$$\Rightarrow \theta \notin \left[ \frac{\pi}{2}, \frac{1}{2\pi} \left( \sum_{j=1}^{n-1} \frac{1}{j^2} \right), \sum_{j=1}^{n-1} \frac{1}{j^2} \right], P_n(z) := z \in K_n \Rightarrow$$

$$|P_n(z) - f(z)| < \frac{1}{n}, f_n(z) = \begin{cases} 0, & z \in K_{n,1} \\ 1, & z \in K_{n,2} \end{cases},$$

$$P_n(z) \xrightarrow[n \rightarrow \infty]{} \begin{cases} 1, & z=0 \\ 0, & z \neq 0 \end{cases}, \tilde{P}_n(z) := P_n(z)/P_n(0)$$



$$K_n := \{ |mz| \leq \frac{1}{n} \} \cup \{ \frac{1}{n} + \frac{1}{n^2} \leq |mz| \leq n \} \cup \{ -\frac{1}{n} - \frac{1}{n^2} \geq |mz| \geq -n \} =: K_{n,1} \cup K_{n,2} \cup K_{n,3}, f_n(z) = \begin{cases} 1, & z \in K_{n,1} \\ 0, & z \in K_{n,2} \\ -1, & z \in K_{n,3} \end{cases}$$

$$P_n(z) := |f - P_n|_{K_n} < \frac{1}{n} \quad \text{#4} \quad a \in \partial \Omega \Rightarrow a \notin K_n \#7$$

$$A_n := A \cap \{ n < |z| \leq n+1 \}, |A_n| < \infty, Q_n := \sum_{\alpha \in A_n} P_\alpha(z),$$

$$Q_n \in H(\bar{D}(0, n)), \sum_{j=0}^{\infty} a_j z_j : |\sum a_j z_j - Q_n|_{\bar{D}(0, n)} \rightarrow 0, m := |\sum_{j=1}^m a_j z_j - Q_n(z)|_{\bar{D}(0, n)} < 2^{-n}, \tilde{Q}_n(z) := \sum_{j=1}^m a_j z_j, f(z) := Q_1 + \sum_{j=2}^{\infty} (Q_j(z) - \tilde{Q}_j(z)),$$

$$(z \in \bar{D}(0, n)), \left| \sum_{j=n}^{\infty} (Q_j(z) - \tilde{Q}_j(z)) \right|_{\bar{D}(0, n)} \leq 2^{-n+1}, f \in H(\bar{D}),$$

$$\sum_{j=n}^{\infty} (Q_j(z) - \tilde{Q}_j(z)) \in H(\bar{D}(0, n)), f(z) - \sum_{j=1}^{n-1} Q_j(z) \in H(\bar{D}(0, n)) \quad \#8$$

$$n_k \xrightarrow[k \rightarrow \infty]{} \infty, z^{2n_k} \rightarrow 0, b_j, b_j > b_0 \Rightarrow z^{2n_k} < s^{-1}, s^k z^{2n_k} < 1, \sum_{k=0}^{\infty} s^k z^{2n_k}$$

$$< \sum (s z^{2n_k})^k < \infty. |z| = 1 - \frac{1}{n_m}, |h(z)| \leq \sum_{k=0}^{\infty} s^k \left( \frac{n_m-1}{n_m} \right)^{n_k}, (z = \frac{n_m-1}{n_m})$$

$$\sum_{k=0}^{\infty} s^k \left( \frac{n_m-1}{n_m} \right)^{n_k} = s^m \sum_{k=0}^{\infty} s^{k-m} \left( \frac{n_m-1}{n_m} \right)^{2k+n_k-1}, \lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} s^{k-m} \left( \frac{n_m-1}{n_m} \right)^{2k+n_k-1} > 0?$$

$$(1 - \frac{1}{n_m})^{nm} \rightarrow e^{-1}, (|z| = 1 - \frac{1}{n_m}) \sum_{k=0}^{m-1} s^{k-m} \left( \frac{n_m-1}{n_m} \right)^{n_k-n_m} < \sum_{k=0}^{m-1} s^{k-m} \left( \frac{n_m}{n_{m-1}} \right)^{nm}$$

$$= \left( \frac{1}{e} + o(1) \right) \sum_{k=0}^{m-1} s^{k-m} \frac{1-s^{2m}}{1-s^2} \approx \frac{1}{3} \left( \frac{1}{e} + o(1) \right), \sum_{k=m+1}^{\infty} s^{k-m} \left( \frac{n_m-1}{n_m} \right)^{n_k-n_m}$$

$$< \sum \left( s \cdot \frac{n_m-1}{n_m} \right)^{n_k-n_m} \left( \frac{n_m-1}{n_m} \right)^{n_k-n_m} \approx s^m \sum_{k=0}^{\infty} s^k (1 - \frac{1}{n_m})^{n_k+m-n_m} = \sum_{k=0}^{\infty} s^k \left( \frac{1}{e} + o(1) \right)^{\frac{n_k+m-n_m}{n_m}}$$

$$= o(1), \left| \sum_{k=0}^{\infty} s^k z^{2n_k} \right| \geq s^m |z|^m - \sum_{j=m}^{\infty} s^j |z|^j \geq s^m M \text{ for some } C //$$

16

$$\int_0^\infty \frac{y}{\theta} e^{-\frac{y}{\theta}} = \left[ \theta y - y^2 e^{-\frac{y}{\theta}} \right]_0^\infty + \int_0^\infty y^2 e^{-\frac{y}{\theta}} = -\frac{\theta}{2} e^{-\frac{y}{\theta}} \Big|_0^\infty = \theta, \quad \int_0^\infty \frac{y^2}{\theta} e^{-\frac{y}{\theta}}$$

$$= \frac{1}{2} \cdot y \theta e^{-\frac{y}{\theta}} \Big|_0^\infty + \int_0^\infty \theta e^{-\frac{y}{\theta}} = -\theta^2 e^{-\frac{y}{\theta}} \Big|_0^\infty = \theta^2$$

$$\sum (Y_j^2 - 2\bar{Y}Y_j + \bar{Y}^2) = \sum Y_j^2 - n\bar{Y}^2, \quad \frac{(n-1)\sigma^2}{\theta^2} \sim \chi^2(n-1), \quad \left(\frac{n}{\theta}\right)^2 \text{Var } S^2 = 2(n-1)$$

$$\text{Var } S^2 = \frac{2\sigma^4}{n-1}, \quad I \sim (Y_1^2 - 2Y_1 Y_2 + Y_2^2), \quad Y_1 - Y_2 \sim N(0, 2\sigma^2), \quad \frac{Y_1 - Y_2}{\sqrt{2}\sigma} \sim Z,$$

$$I \sim \frac{1}{2\sigma^2} \text{Var}(Y_1 - Y_2), \quad \text{Var}\left(\frac{Y_1 - Y_2}{\sqrt{2}\sigma}\right)^2 \sim \text{Var } \chi^2(1) = 2, \quad \text{Var}(Y_1 - Y_2)^2$$

$$= 8\sigma^4, \quad \text{Var } S^2 = 8\sigma^4$$

$$\left\{ \theta y^\theta = \frac{\theta}{\theta+1} y^{\theta+1} \right\}_0^\infty = \frac{\theta}{\theta+1}$$

$$2EY_{ij}^2 = 2\lambda(1+\lambda) - 2\lambda^2 = 2\lambda, \quad \text{Var}((Y_i - Y_j)^2)$$

$$\frac{\theta+1}{\theta} y^{\theta+1} \Big|_0^\infty = \frac{\theta+1}{\theta}, \quad \frac{\theta+1}{\theta+2} y^{\theta+2} \Big|_\infty = \frac{\theta+1}{\theta+2} = \bar{x}, \quad \theta(1-\bar{x}) = \bar{x} - 1, \quad \hat{\theta} = \frac{2\bar{x}-1}{1-\bar{x}} = \frac{2\bar{x}-1}{p} = \theta$$

$$\theta^2 + \mu^2 = \bar{x}^2 - \frac{2}{p} = -(n-p) \left\{ \frac{1}{(1-p)^2} \right\} \bar{x}^2, \quad \bar{x}^n / \theta, \quad \theta^{-n} e^{\theta^{-1} \sum x_i}, \quad \log -n \log \theta$$

$$- \sum x_i / \theta, \quad -n/\theta + \sum x_i / \theta^2, \quad \hat{\theta} = \bar{x}, \quad , \quad \left( \frac{1}{2\theta+1} \right)^n \{ X_{(n)} \leq 2\theta+1 \}, \quad \hat{\theta} = \frac{1}{2}(X_{(n)} - 1),$$

$$\frac{1}{2\theta+1} \left\{ \sum_{j=1}^{2n} y_j^2 \right\} + \frac{1}{3} (2\theta+1)^2 - \frac{1}{4} (2\theta+1)^2 = \frac{1}{12} (2\theta+1)^2, \quad \theta^{-2n} \prod y_j e^{-\sum y_j / \theta}, \quad -2n \log \theta + \sum y_j - \sum y_j / \theta,$$

$$-2n\hat{\theta} + \sum y_j \hat{\theta}^2, \quad \hat{\theta} = \bar{Y}/2,$$

9.36 a

a)  $y_1, \dots, y_n$  iid  $\sim N(\mu, \sigma^2)$ , pdf  $f(y) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$

Joint pdf  $\prod_{j=1}^n f(y_j) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{j=1}^n (y_j - \mu)^2\right)$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{j=1}^n (y_j^2 - 2\mu y_j + \mu^2)\right)$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum y_j^2 + \frac{\mu}{\sigma^2} \sum y_j - \frac{n\mu^2}{2\sigma^2}\right)$$

$$f(y_1, \dots, y_n) = \overline{\prod} f(y_j) = (2\pi\sigma^2)^{-n/2} \exp\left(\frac{\mu}{\sigma^2} \sum y_j - \frac{n\mu^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2} \sum y_j^2\right)$$

$$= \underbrace{(2\pi\sigma^2)^{-n/2} \exp\left(\frac{n\mu}{\sigma^2} \bar{y} - \frac{n\mu^2}{2\sigma^2}\right)}_{\text{depends on } y_1, \dots, y_n \text{ only through } \bar{y}} \underbrace{\exp\left(-\frac{1}{2\sigma^2} \sum y_j^2\right)}_{\text{does not depend on target } \mu}$$

b)  $f(y_1, \dots, y_n) = \overline{\prod} f(y_j) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{j=1}^n (y_j - \mu)^2\right) \cdot 1$

$$\underbrace{\text{depends on } y_1, \dots, y_n}_{\text{only through } \sum (y_j - \mu)^2} \cdot 1 \quad \begin{matrix} \text{does not} \\ \text{depend on} \\ \text{target } \sigma^2 \end{matrix}$$

c)  $f(y_1, \dots, y_n) = \underbrace{(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum y_j^2 + \frac{\mu}{\sigma^2} \sum y_j - \frac{n\mu^2}{2\sigma^2}\right)}_{\text{depends on } y_1, \dots, y_n \text{ only through } (\sum y_j^2, \sum y_j)} \cdot 1$

$$\cdot 1 \quad \begin{matrix} \text{does not} \\ \text{depend on} \\ (\mu, \sigma^2) \end{matrix}$$

Q.39  $Y_1, \dots, Y_n \sim \text{Poi}(\lambda)$ , pmf is  $f_Y(j) = e^{-\lambda} \frac{\lambda^j}{j!}$  for  $j=0, 1, \dots$

also  $\sum_{j=1}^n Y_j \sim \text{Poi}(n\lambda)$ , pmf is then  $f_{\sum Y_j}(j) = e^{-n\lambda} \frac{(n\lambda)^j}{j!}$ .

Then  $P(Y_1=y_1, \dots, Y_n=y_n | \sum Y_j = \sum y_j)$

$$\frac{P(Y_1=y_1, \dots, Y_n=y_n \text{ and } \sum Y_j = \sum y_j)}{P(\sum Y_j = \sum y_j)}$$

$$= \frac{P(Y_1=y_1, \dots, Y_n=y_n)}{P(\sum Y_j = \sum y_j)} = \frac{\prod_{j=1}^n f_Y(y_j)}{f_{\sum Y_j}(\sum y_j)}$$

$$= \frac{\prod_{j=1}^n \left( e^{-\lambda} \frac{\lambda^{y_j}}{y_j!} \right)}{e^{-n\lambda} \frac{(n\lambda)^{\sum y_j}}{(\sum y_j)!}} = \frac{e^{-n\lambda} \lambda^{\sum y_j}}{\prod y_j!} / \frac{e^{-n\lambda} \frac{(n\lambda)^{\sum y_j}}{(\sum y_j)!}}{e^{-n\lambda} \frac{(n\lambda)^{\sum y_j}}{(\sum y_j)!}}$$

$$= \frac{(\sum y_j)!}{\prod y_j!} \quad \text{which doesn't depend on } \lambda$$

Q.40  $Y_1, \dots, Y_n \sim \text{Rayleigh}(\theta)$ , pdf is  $f_Y(y) = \left(\frac{y}{\theta}\right) e^{-y^2/\theta}$  for  $y > 0$ ,

$$\text{so } f(y_1, \dots, y_n) = \prod_{j=1}^n f_Y(y_j) = \prod_{j=1}^n \left(\frac{y_j}{\theta}\right) e^{-y_j^2/\theta} = \left(\frac{y}{\theta}\right)^n (\prod y_j) e^{-\sum y_j^2/\theta}$$

$$= \underbrace{\left(\frac{y}{\theta}\right)^n e^{-\sum y_j^2/\theta}}_{\text{depends on } y_1, \dots, y_n \text{ only through } \sum y_j^2} \cdot \underbrace{(\prod y_j)}_{\text{doesn't depend on } \theta}$$

9.41

$$Y_1, \dots, Y_n \sim \text{Weibull}(\alpha), \quad \text{pdf} \Rightarrow f_Y(y) = \frac{1}{\alpha} \alpha y^{\alpha-1} e^{-y^\alpha/\alpha} \quad \text{for } y > 0$$

$$\begin{aligned} f(y_1, \dots, y_n) &= \prod_{j=1}^n f_Y(y_j) = \prod_{j=1}^n \left( \frac{1}{\alpha} \alpha y_j^{\alpha-1} e^{-y_j^\alpha/\alpha} \right) = \left( \frac{n}{\alpha} \right)^n (\pi y_j)^{\alpha-1} e^{-\sum y_j^\alpha/\alpha} \\ &= \underbrace{\left( \frac{n}{\alpha} \right)^n e^{-\sum y_j^\alpha/\alpha}}_{\text{dependence on sample thru } \sum y_j^\alpha} \cdot \underbrace{(\pi y_j)^{\alpha-1}}_{\text{no } \alpha \text{ dependence}} \end{aligned}$$

9.42 - HW

9.43

$$f(y_1, \dots, y_n) = \prod_{j=1}^n f_Y(y_j) = \prod_{j=1}^n \left( \alpha y_j^{\alpha-1} / \theta^\alpha \right) = \underbrace{\left( \frac{\alpha}{\theta^\alpha} \right)^n}_{\text{depends on sample thru } \pi y_j} \underbrace{(\pi y_j)^{\alpha-1}}_{\text{no } \alpha \text{ dependence}} \cdot 1$$

9.44

$$f(y_1, \dots, y_n) = \prod_{j=1}^n f_Y(y_j) = \prod_{j=1}^n \left( \alpha \beta^\alpha y_j^{-(\alpha+1)} \right) = \underbrace{(\alpha \beta^\alpha)^n}_{\text{ditto}} \underbrace{(\pi y_j)^{-(\alpha+1)}}_{1} \cdot 1$$

$$f(y_1, \dots, y_n) = \prod_{j=1}^n f_Y(y_j) = \prod_{j=1}^n \left( a(\theta) b(y_j) e^{-c(\theta) d(y_j)} \right)$$

$$= \underbrace{a(\theta)^n e^{-c(\theta) \sum d(y_j)}}_{\text{depends on sample thru } \sum d(y_j)} \cdot \underbrace{\prod_{j=1}^n b(y_j)}_{\text{no } \theta \text{ dependence}}$$

9.46 pdf  $\Rightarrow f_y(y) = \frac{1}{\beta} e^{-y/\beta}$  which has form

$a(\theta) b(y) e^{-c(\theta)d(y)}$  by taking  $\theta = \beta$ ,  $c(\theta) = \frac{y}{\beta}$ ,  $a(\theta) = \frac{1}{\beta}$ ,  $b(y) = 1$ .

9.47 pdf  $\alpha y^{\alpha-1}/\theta^\alpha = \bar{\theta}^{-\alpha} \alpha \exp((\alpha-1)\log y)$  has form

$a(\theta) b(y) e^{-c(\theta)d(y)}$  by taking  $\theta = \alpha$ ,  $d(y) = \log y$ ,  $c(\theta) =$   
 $\alpha-1$ ,  $a(\theta) = \bar{\theta}^{-\alpha} \cdot \alpha$ . So by 9.45  $\sum_j d(y_j) = \sum \log y_j$

is sufficient for  $\theta$ . No contradiction since  $\sum \log y_j$   
 $= \log(\prod y_j)$  and "log" is one-to-one on  $\mathbb{R}^+$

9.48 pdf  $\alpha \beta^\alpha y^{-(\alpha+1)} = \alpha \beta^\alpha \exp(-(\alpha+1)\log y)$  has form

$a(\theta) b(y) e^{-c(\theta)d(y)}$  by taking  $\theta = \alpha$ ,  $d(y) = \log y$ ,

$c(\theta) = \alpha+1$ ,  $a(\theta) = \alpha \beta^\alpha$ ,  $b(y) = 1$ . Suf. stat. is

then  $\sum d(y_j) = \sum \log y_j = \log(\prod y_j)$ .

$$\begin{aligned}
& f'(i) \leq \frac{1+i}{1} = 1, \quad f(z) = \frac{1}{1+z-i}, \quad |1+3-i| \geq 10e, \quad Re^{i\theta} + R^{-1}e^{-i\theta} = (R^2 \frac{1}{R}) \cos \theta \\
& + i(R - \frac{1}{R}) \sin \theta, \quad R^2 + \frac{1}{R^2} + 2(\cos^2 - \sin^2) = R^2 + \frac{1}{R^2} + 2 \cos 2\theta, \quad z = \frac{1}{R}, \quad R^2 + \frac{1}{R^2} - 2 \cos 2\theta, \\
& \left( \frac{1}{R^2} + \frac{1}{R^2} \right) R^2 + 2 \cos \frac{\theta}{2}; \quad z^{1/4} + z^{-1/4} = \frac{z^{1/2} + 1}{z^{1/4}} = z^1 (z^{5/4} + z^{3/4}), \quad z^\alpha + \bar{z}^\alpha = \\
& z^{-\alpha} (z^{2\alpha} + 1), \quad \frac{1}{(z-a)(z-b)}, \quad \left. \frac{1}{(z-a)^2 (z-b)} + \frac{1}{(z-a)(z-b)^2} \right|_{z=i} = (i-a)^2 (i-b)^{-1} + (i-a)^{-1} (i-b)^{-2} \\
& = \frac{2i-a-b}{(i-a)^2 (i-b)^2} \text{ and } 12i\pi, \quad \frac{2i - (x_1 - iy + x_2 - iy)}{(i-x_1 + iy)^2 (i-x_2 + iy)^2} = \frac{-(x_1 + x_2) + i(2+y + y)}{(-x_1 + i(1+y))^2 (-x_2 + i(1+y))^2} \\
& ((x_1 + x_2)^2 + 16)^{1/2} \geq (x_1^2 + 4)(x_2^2 + 4); \quad 2+y + y \geq ? \quad (x_1^2 + (1+y)^2)(x_2^2 + (1+y)^2) \geq \\
& (2+y+y)^2 ...
\end{aligned}$$

then  $|f(x)| = \min_{y \in D} |f(y)|$ , if  $|f(x)| \neq 0$ ,  $|f(x)| = \inf_{y \in D} |f(y)| \Rightarrow f(y) \neq 0$ ,

$$y \in D, \quad |f(x)| \geq \sup_{y \in D_2} |f(y)|, \quad f = \text{const.} \quad \Rightarrow \quad h(x) \neq 0, \quad x \in D \quad \Rightarrow \quad \max_{D_2} |f(x)|$$

$$\max_{\partial D} |H(x)| = c = \min_{\bar{D}} |f(x)|, \quad |f(z)| = c, \quad k \in \bar{D}, \quad u^2(x,y) + v^2(x,y) = c, \quad uu_x + vv_x$$

$$= u u_y + v v_y = 0, \quad \frac{v_x}{u_x} = \frac{v_y}{u_y}, \quad \text{so} \quad \frac{u_y}{u_x} = \frac{v_x}{v_y}, \quad u_x^2 = -u_y^2, \quad u_x = v_y = 0,$$

$$u_y = v_x = 0, \quad f(x) = \{ \text{const} \}, \quad \rightarrow \text{---} \quad \# 49 //$$

$$m, n > n_0 \Rightarrow \max_{z \in D} |f_n(z) - f_m(z)| < \epsilon, \quad N(x) := \max_{\substack{z \in D \\ \operatorname{Re} z = x}} |f_n(z)|,$$

$$\text{ges. f. } |y| > \frac{B}{\eta\varepsilon} \Rightarrow |f_\eta h_\varepsilon| \leq \frac{B}{\varepsilon} \cdot \frac{1}{B} = 1; \quad R := \{(x,y) : a \leq x \leq b, -\frac{B}{\eta\varepsilon} \leq y \leq \frac{B}{\eta\varepsilon}\},$$

$$x_0: \{x < x_0\} \cap R \Rightarrow |f(z)| < \eta, \quad |f(z)h_z| < \eta, \quad x < x_0 \Rightarrow N(x) < \eta,$$

$$N(x) \underset{x \rightarrow a^+}{\rightarrow} 0, \quad \forall a < x < \xi < b \Rightarrow N(\xi) \leq N(x)^{\frac{b-\xi}{b-x}} N(b)^{\frac{\xi-x}{b-x}} \underset{x \rightarrow a^+}{\rightarrow} 0, \quad N(\kappa) = 0,$$

$$h_\varepsilon(z) \neq 0, \quad z \in \Omega, \quad f \equiv 0 \quad \Leftrightarrow \quad M(r)^{\log^2 r} \leq ? \quad M(u)^{\log^2 u} \quad M(s)^{\log^2 s}, \quad M(u) = M(s)$$

$$z_1 \Rightarrow M(r) \leq 1, \quad (\text{why}) \quad \sup_{\operatorname{Re} z=x} |\log|f(z)|| = \log \sup_{\operatorname{Re} z=x} |f(z)|, \quad \left(\frac{z}{r}\right)^{\log M(r)} \leq ? \quad \left(\frac{z}{r}\right)^{\log M(r)} \left(\frac{z}{r}\right)^{\log}$$

$$(f(x) = z), \log^{\log z} \leq ? \quad \log^{\log z} \{ \log^z \}; \quad \log^b \log r - \log_a \log r \leq ? \quad \log_a \log_b - \log_a \log_b$$

$\log \log x + \log \log x - \log \log x$

$$|e^z|^{\alpha} = \exp(\alpha \log|z|), \quad \text{where } e^{bx} + e^{-bx} = e^{bx} e^{i\beta y} + e^{-bx} e^{-i\beta y} = (e^{bx} + e^{-bx}) \cos \beta y$$

$$+ i(e^{bx} - e^{-bx}) \sin \beta y, \quad \text{--- i.e. } e^{bx} + e^{-bx} = 2 \cosh 2\beta x$$

$$\pm 2 \cos(\beta y + \frac{\pi}{4}) \geq 0, \quad x > 0 \Rightarrow |e^{bx} + e^{-bx}|^{\alpha} > \delta > 0, \quad \text{for } \beta y + \frac{\pi}{4} = \frac{(2k+1)\pi}{2}, \quad y =$$

$$\beta^{-1} \frac{\pi}{2} (2k+1), \quad k \in \mathbb{Z}, \quad Y := \left\{ \beta^{-1} \frac{\pi}{2} (2k+1) : k \in \mathbb{Z} \right\}, \quad \text{so } e^{bx} + e^{-bx} =$$

$$\text{and } \ln z \in Y \Rightarrow e^{bx} + e^{-bx} = 2 \cosh 2\beta x, \quad |e^{bx} + e^{-bx}|^{\alpha} = 2 \cosh 2\beta y +$$

$$2 \cos(2\beta y + \frac{\pi}{4}), \quad |e^z + e^{iz}| = |e^x e^{iy} + e^y e^{ix}|, \quad |e^x e^{iy} + e^y e^{-ix} + e^y e^{ix} + e^x e^{-ix}|$$

$$\geq |e^x|, \quad z = x + iy = x + y + i(y-x), \quad |e^{z-iz}| = e^{x+y}, \quad e^{z-iz} + e^{iz-z} = e^{x+y} e^{i(y-x)} + e^{-x-y} e^{i(x-y)}$$

$$= (e^{x+y} + e^{-x-y}) \cos(x-y) + i(-1, \dots, 1)^T \in H(C \setminus 0), \quad \text{then } z \mapsto R^{\frac{2e^{iz}}{2}} e^{iz} \in$$

$$H(C \setminus 0), \quad (\text{arg } z = n\pi) \text{ for } n: 1 - \frac{1}{n} > \alpha, \quad z^{1-n} = R^{1-n} e^{i((1-\frac{1}{n})\theta)}, \quad \operatorname{Re} z^{1-n} =$$

$$R^{1-n} \cos((1-\frac{1}{n})\theta) > R^{1-n} \cos((1-\frac{1}{n})\frac{\pi}{2}) = R^{1-n} \delta > 0, \quad |\exp(-\varepsilon R^{\frac{2e^{iz}}{2}} z^{1-n})|$$

$$= \exp(-\varepsilon \operatorname{Re} z^{1-n}) < \exp(-\varepsilon R^{1-n} \delta) < 1 \quad (z \in \overline{\Omega}), \quad h_{\varepsilon}(z) := e^{-\varepsilon z^{1-n}}$$

$$z \in \overline{\Omega}, \quad \text{if } h_{\varepsilon} \in H(\Omega), \quad |f h_{\varepsilon}| < 1, \quad |f h_{\varepsilon}| < A \exp(|z|^{\alpha} - \varepsilon \operatorname{Re} z^{1-n})$$

$$\underset{|z|=R \rightarrow \infty}{\longrightarrow} 0, \quad R_0: |z| > R_0 \Rightarrow |f h_{\varepsilon}| < 1, \quad \max_{z \in \overline{\Omega} \cap \{|z| > R_0\}} |f h_{\varepsilon}| = \max_{z \in \partial(\Omega)} |f h_{\varepsilon}|$$

$$= 1, \quad |f h_{\varepsilon}| \leq 1 \quad (z \in \overline{\Omega}, \varepsilon > 0), \quad |f| \leq 1. \quad (\alpha=1) \quad \text{let } f(z) := e^z,$$

$$|f(z)| = e^{\operatorname{Re} z} \leq e^{|z|}, \quad |f(z)| = e^x \underset{x \rightarrow \infty}{\rightarrow} \infty. \quad \text{crossed out} \quad (\frac{\pi}{2} + \theta) \alpha < \frac{\pi}{2}, \quad \alpha < \frac{\pi}{2} - \theta$$

$$|e^{iz}| \leq 1, \quad |f(iz)| e^{n|z|} \leq 1, \quad |f(re^{i\alpha}) e^{nr e^{i\alpha}}| = |e^{nr \cos \alpha} f(re^{i\alpha})| = \exp(nr \cos \alpha + \log|f(re^{i\alpha})|) = \exp(r(n \cos \alpha + r^{-1} \log|f(re^{i\alpha})|))$$

$$A := \max_r \exp(r(n \cos \alpha + r^{-1} \log|f(re^{i\alpha})|)), \quad |f(re^{i\alpha}) e^{nr e^{i\alpha}}| \leq A.$$

$$\text{crossed out} \quad \alpha < \frac{\pi}{2} - \theta, \quad \alpha < \beta < \frac{\pi}{2} - \theta, \quad \operatorname{Re} z^{\beta} = r^{\beta} \cos(\beta \arg z) \geq r^{\beta} \cos(\beta(\frac{\pi}{2} - \theta)),$$

$$\text{crossed out} \quad z \mapsto (e^{-i\theta} z)^{\beta}, \quad \operatorname{Re} (e^{-i\theta} z)^{\beta} > 0 \Leftrightarrow z \in \Pi, \quad h_{\varepsilon}(z) := \exp(-\varepsilon (e^{-i\theta} z)^{\beta}) < 1,$$

$$|f h_{\varepsilon}| < A \exp(|z|^{\alpha} - \varepsilon r^{\beta} \cos(\beta(\frac{\pi}{2} - \theta))) \underset{|z|=r \rightarrow \infty}{\rightarrow} 0. \quad |e^{iz}| < A e^{1+|z| - \theta}$$

$$(\Omega = \mathbb{C}) \quad \Gamma = \emptyset, \quad M = ? \quad (x_0 \notin \Omega) \quad \sup_{z \in \Omega} |f(z)| = \sup_{z \in \Omega - x_0} |f(z+x_0)|, \quad 0 \notin$$

$$\Omega - x_0 \quad (0 \notin \Omega) \quad r := \inf_{z \in \Omega} |z|, \quad r \neq 0, \quad \sup_{z \in \Omega} |f(z)| = \sup_{z \in r\Omega} |f(z/r)|,$$

$r \Omega \cap V \neq \emptyset$ .



$$\left| \frac{f(z)}{z_0} \right| \leq \max_{V \ni z} \left| \frac{f(z)}{z} \right| \vee \max_{r \in V} \left| \frac{f(z)}{z} \right| \leq$$

$$M^n \vee \frac{B^n}{V}, \quad \max(M^n, \frac{B^n}{V}) \rightarrow M^n, \quad |f(z_0)|^n \leq |z_0|^n M^n, \quad n \in \mathbb{Z},$$

$$|f(z_0)| \leq M \quad \#11 \quad \text{diam } E_n < \infty \Rightarrow \|f\|_{E_n} \leq \|f\|_{\partial E_n} = n, \quad E_n = \emptyset; \quad \|f\|_{E_n} \leq n+1$$

$$\Rightarrow \|f\|_{E_n} \leq \|f\|_{\partial E_n} = n, \quad E_n \neq \emptyset; \quad \underbrace{\gamma}_{\gamma: Y(1-\frac{1}{n}) \rightarrow E_n}, \quad \gamma(1-\frac{1}{n}) \in E_n, \quad |\gamma(1-\frac{1}{n}) - \gamma(1-\frac{1}{n+1})| > 1 \quad \#12$$

$$\Rightarrow e^y \rightarrow \infty, \quad e^{-y} \rightarrow 0, \quad \operatorname{Re} \gamma \rightarrow \infty \Rightarrow e \rightarrow \infty, \quad \operatorname{Re} \gamma \rightarrow -\infty$$

$$\Rightarrow e^y \rightarrow 0, \quad \operatorname{Re} \gamma \neq \infty \Rightarrow \operatorname{Im} \gamma \rightarrow 0, \quad e^y \neq L. \quad e^{ix} - e^{-ix} = e^{-y+ix} - e^{-y-ix}$$

$$= (e^y - e^{-y}) \cos x + i(e^y + e^{-y}) \sin x, \quad |e^{ix} - e^{-ix}|^2 = e^{2y} + e^{-2y} - 2 \rightarrow \infty$$

$$\text{arg } g(z) \rightarrow (\operatorname{Im} z \neq 0) \rightarrow \infty, \quad e^{ix} - e^{-ix} \neq L \subseteq \mathbb{C} \cup \infty \quad \#13 \quad g(z) := f(\frac{1}{z}), \quad \text{Thm 10.21},$$

$$(c \in \mathbb{C}: \hat{g}(0) := c, \hat{g}(z) := g(z), z \neq 0, \hat{g} \in H(\mathbb{C})) \quad z_n \rightarrow \infty, \quad g(\frac{1}{z_n}) \rightarrow c,$$

$$f(z_n) \rightarrow c, \quad f = \text{const}; \quad (g(z) = \sum_{j=-m}^{\infty} c_j z^j) \quad z_n \rightarrow 0 \Rightarrow c \in \mathbb{C}: g(z_n) - \sum_{j=-m}^1 c_j z_n^j$$

$$\rightarrow c, \quad f(w_n) - \sum_{j=1}^m c_{-j} w_n^j \rightarrow c, \quad f(w_n) - \sum_{j=1}^m c_{-j} w_n^j = \text{const.}, \quad f(w_n) =$$

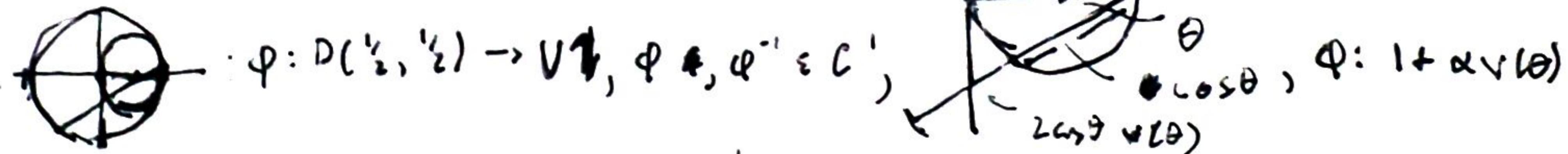
$$\sum_{j=0}^m \tilde{c}_j w_n^j, \quad \operatorname{Im} f = \mathbb{C}; \quad (\operatorname{cl}(g(D(0; r))) = \mathbb{C}) \quad c \in \mathbb{C}, \quad z_n: z_n \rightarrow 0, \quad g(z_n) \rightarrow c,$$

$$f(\frac{1}{z_n}) \rightarrow c \quad \#14$$

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$$|A| = \infty, \quad \lim_{r \rightarrow 0^+} |D(x, r) \cap A| > 0, \quad x \in \partial A \Rightarrow |A| < \infty, \quad f(z) - \sum_{j=1}^{|A|} R_j$$

$$\in H(\mathbb{C}). \quad \#15$$



$$\mapsto 1 + (\cos \theta + \frac{\alpha}{2}) v(\theta), \quad 0 < \alpha < 2 \omega \theta, \quad \bar{v}(\theta) := -1 - \frac{1}{\cos \theta} \bar{v}(\theta), \quad v(\theta) := \bar{v}(\theta) / |\bar{v}(\theta)|, \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

$$f(z) := \frac{1}{z - \frac{1}{2}}, \quad \inf_{\Omega} |f| = \frac{2}{3}, \quad \sup_{\Omega} |P - f| < \frac{2}{3}, \quad |(z - \frac{1}{2})P - 1| < \frac{1}{2}, \quad \sup_{\Omega} |(z - \frac{1}{2})P - 1| \leq \frac{1}{2},$$

$$\sup_{D(\frac{1}{2}, \frac{1}{2})} |(z - \frac{1}{2})P - 1| \leq \sup_{\partial D(\frac{1}{2}, \frac{1}{2})} |(z - \frac{1}{2})P - 1| = \frac{1}{2}, \quad \|(z - \frac{1}{2})P - 1\|_{z=\frac{1}{2}} = 1 \quad \#16$$

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$$1) .95 = P(L \leq \mu \leq U) = P(X-L \geq X-\mu \geq X-U), \quad X-\mu \sim N(0,1),$$



$$X-L = z_{.025}, \quad L = X-z_{.025}, \quad X-U = -z_{.025}, \quad U = X+z_{.025}$$

$$2) .56 \pm z_{.005} \sqrt{\frac{.56 \cdot .44}{100^2}} = .56 \pm .005 \sqrt{\frac{.56 \cdot .44}{100^2}} \sqrt{154}$$

$$3) a) E[X-\mu] = 0 \quad b) 1 - 2\Phi(-1) = 2\Phi(1) - 1 \quad c) P(|X_1 - \mu| > \epsilon) \geq$$

$$P(|X_1 - \mu| > 1) = 2\Phi(1) - 1 \rightarrow 0$$

$$4) a) f_\mu(x_1, \dots, x_n) = \prod_{j=1}^n f_\mu(x_j) = (2\pi)^{-n/2} \exp\left(-\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^2\right) = \\ (2\pi)^{-n/2} \exp\left(-\frac{1}{2} \sum x_j^2 + \mu \sum x_j + \mu^2/2\right) = (2\pi)^{-n/2} \underbrace{\exp(\mu n \bar{x} + \mu^2/2)}_{\text{depends on data}} \underbrace{\exp(-\frac{1}{2} \sum x_j^2)}_{\text{doesn't depend on } \mu}$$

$$b) E(\bar{Y}^2) = \text{Var}\bar{Y} + (E(\bar{Y}))^2 = \frac{1}{n} + \mu^2, \quad E(\bar{Y}^2) - \frac{1}{n} = \mu^2$$

$$c) LLN: \bar{Y} \xrightarrow{P} \mu, \quad \text{so} \quad \bar{Y}^2 \xrightarrow{P} \mu^2 \quad \text{since } x \mapsto x^2 \text{ is continuous.}$$

$$5) 1-\alpha = P(L \leq \sigma^2 \leq U) = P\left(\frac{(n-1)S^2}{L} \geq \frac{(n-1)S^2}{\sigma^2} \geq \frac{(n-1)S^2}{U}\right),$$

$$\frac{(n-1)S^2}{L} := \chi^2_{n-1, \frac{1}{2}(1-\alpha)}, \quad L = \frac{(n-1)S^2}{\chi^2_{n-1, \frac{1}{2}(1-\alpha)}}, \quad U = \frac{(n-1)S^2}{\chi^2_{n-1, \frac{\alpha}{2}}}.$$

$$b) a) (-1, \infty) \quad b) EY = \int_0^1 y(\theta+1)y^\theta dy = \frac{\theta+1}{\theta+2} := \bar{Y}, \quad \hat{\theta}(1-\bar{Y}) = 2\bar{Y}-1,$$

$$\hat{\theta}_{\text{MLE}} = \frac{2\bar{Y}-1}{1-\bar{Y}} \quad c) (\theta+1)^n (\prod y_j)^\theta, \quad n \log(\theta+1) + \theta \log \sum y_j, \quad \frac{n}{\theta+1} + \log \sum y_j$$

$$= 0, \quad \hat{\theta}_{\text{MLE}} = -\frac{n}{\log \sum y_j} - 1 \quad d) \hat{\theta}_{\text{MLE}}^2 \quad e)$$

$$6.84) \quad \theta^{-2n} \prod y_j e^{-\sum y_j/\theta}, \quad -2n \log \theta - \sum y_j/\theta, \quad -\frac{2n}{\theta} + \sum y_j/\theta^2 = 0, \quad \theta = \sum y_j/n$$

$$\text{MLE } EY = \theta^{-2} \int y^2 e^{-y/\theta} = \theta^{-2} \left\{ -\theta e^{-\theta} y^2 \Big|_0^\infty + 2y \theta e^{-\theta} \Big|_0^\infty \right\} = 2\theta$$

W-P approximation

$$\frac{\prod_{j=1}^n \pi_{y_j}}{\theta^n} e^{-\sum y_j/\theta}, \quad -n \log \theta = \sum y_j/\theta, \quad -\frac{n}{\theta} + \frac{\sum y_j^2}{\theta^2}, \quad \hat{\theta} = \frac{1}{n} \sum y_j^2,$$

$$\int y^2 e^{-y/\theta} = y \left( 1 - \frac{\theta}{2} e^{-y/\theta} \right) + \int \frac{\theta}{2} y e^{-y/\theta} = \left( \frac{\theta}{2} \right)^2$$

$$P^{(1-p)} \stackrel{\text{approx}}{\sim} n \log p + (\sum y_j - n) \log (1-p), \quad P \stackrel{\text{approx}}{\sim} \frac{\sum y_j - n}{1-p} = 0$$

$$0 = n(1-p) - p(\sum y_j - n) = n - p(\sum y_j - n + 1), \quad p = \frac{n}{\sum y_j - n + 1}$$

$$\prod_{j=1}^n (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2}(x_j - \mu)^2\right) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum (x_j - \mu)^2\right),$$

$$\frac{L(\mu_{10})}{L(\mu_0)} = \exp\left(-\frac{1}{2\sigma^2} \left\{ (x_0 - \mu_0)^2 - (x_j - \mu_{10})^2 \right\}\right) = \exp\left(\frac{1}{2\sigma^2} \left\{ -2x_0\mu_0 + 2x_j\mu_0 + \mu_0^2 - \mu_{10}^2 \right\}\right) < k, \quad -2x_0(\mu_0 - \mu_{10}) + \mu_0^2 - \mu_{10}^2 < 2\sigma^2 \log k,$$

$$\exp\left(\frac{1}{2\sigma^2} \left\{ -2n\bar{x}(\mu_0 - \mu_n) + n(\mu_n^2 - \mu_0^2) \right\}\right) < k, \quad \bar{x} < \frac{1}{2n\cdot 5} \left\{ 2\sigma^2 \log k - n(\mu_n^2 - \mu_0^2) \right\},$$

$$P(\bar{x} < k) = \alpha, \quad P(Z < \frac{k - \mu_0}{\sqrt{\sigma^2/n}}), \quad k = \mu_0 + z_{1-\alpha} \sqrt{\sigma^2/n}$$

$$= \frac{10}{\mu_0 - z_{\alpha} \sqrt{\sigma^2/n}}, \quad (2\pi\theta)^{-n} (\prod y_j) e^{-\theta^{-1} \sum y_j}, \quad \frac{L(\theta_0)}{L(\theta_n)} = \left(\frac{\theta_n}{\theta_0}\right)^{3n} e^{-\sum y_j (\theta_0 - \theta_n)} < k,$$

$$-(\sum y_j)(\theta_0 - \theta_n) < k, \quad \Rightarrow \sum y_j > k, \quad y_j \sim \Gamma(3, \theta), \quad \sum y_j \sim \Gamma($$

$$P(Y_1 + Y_2 < c) = \int_0^c F_2(\frac{c-y_1}{\theta}) = \int_0^c \frac{c-y_1}{\theta} \frac{e^{-y_1/\theta}}{\theta} = (-\frac{c}{\theta})^2, \quad P(Y_1 + Y_2 < 1\pi) = c + \int_c^\infty F_2(1\pi - y)$$

$$= c + \frac{1}{2} (1\pi - c) + \frac{1}{2} c(1\pi - 1 - c^2 - \frac{1}{2} + \frac{1}{2} c^2) = c + \frac{1}{2} - \frac{1}{2} c^2, \quad f = 1 - c, \quad \int_{\frac{c}{\theta}}^{\frac{1\pi}{\theta}} \frac{1}{2} (1\pi - y)^2$$

Andreasru

$$a^3 = n+3, \quad b^3 = n^2 + 3n + 3, \quad b^3 - a^3 = \frac{n(n+2)}{n^2 + 2n}, \quad b^3 + a^3 = n^2 + 4n + 6 = (n+2)^2 + 2$$

$$b^3 = (n+3)(n+1) - n = (n+2)(n+1) + 1, \quad b^3 = a^3(n+1) - n, \quad b^3 = n^2 + 2n + a^3,$$

$$n^2 + 2n = b^3 - a^3 = (b-a)(b^2 + ab + a^2) = n(n+2), \quad \text{cancel}$$

$$(2|n) \rightarrow a, \quad 2|b \quad (2|n) \rightarrow a, \quad 2|b, \quad \text{cancel} \rightarrow \frac{n+2}{n} + \frac{2}{n(n+2)},$$

$$b^3 = a^6 - 6a^3 + 9 + 3a^3 - 6 = a^6 - 3a^3 + 3 = (a^3 - 3)(a^3 - 1) + a^3 = (a^3 - 2)^2$$

$$+ a^3 - 1, \quad b^3 - a^3 = (a^3 - 3)(a^3 - 1) = (a^3 - 3)(a - 1)(a^2 + a + 1)$$

$$2|n \Rightarrow 2|b, \quad 2|n \Rightarrow 2|b, \quad 2|(n^2 + 3n) = n(n+3), \quad b^3 - 3 = a^3(a^3 - 3),$$

$$b^3 = (n+2)(n+1), \quad (b-1)(b^2 + b + 1) = (n+1)(n+2), \quad a^3 = m, \quad b^3 = m^2 - 6m + 9 + 3m - 9 + 3$$

$$= m^2 - 3m + 3, \quad a^3 = m+1, \quad b^3 = m^2 - 4m + 4 + 3m - 6 + 3 = m^2 - m + 1, \quad n^2 + 6n + 9,$$

$$(a^3 - b^3) = 3n + 6 = 3(n+2), \quad (n+3)^3 = n^4 + 4n^3 + 6n^2 \cdot 9 + 4 \cdot 27n + 81, \quad n^4 + 9n^2 + 9$$

$$+ (3n^3 + 3n^2 + 9n)2 = n^4 + 6n^3 + 15n^2 + 8n + 9, \quad n^3 - 6n^2 + 12n + 9, \quad (n+1)^3 =$$

$$n^3 + 3n^2 + 3n + 1, \quad (n+2)^3 = n^3 + 6n^2 + 12n + 8 \dots$$

$$A^p = I, \quad B^q = 0, \quad \text{or} \quad A^{p-k} = A^{-k}, \quad A^k = A^{-(p-k)}, \quad \text{if } (p-q=2, \quad p=1) \quad (I+B)^{-2}$$

$$= I + 2B, \quad \cancel{B^3} B + I, \quad A^{-p} = I, \quad (A+I)^{-1} = \sum (-A)^j$$

$$S^{5^{n+2}} + S^{5^{n+1}} - S^{5^{n+1}} - S^{5^n} = S^{5^{n+2}} - S^{5^n} - S^{5^n} (S^{5^n \cdot 24} - 1), \quad (n=0) \quad 25 \cdot 625 + 6$$

$$= 15625 = 7 \cdot 24233 = 49 \cdot 319 = 7 \cdot 24233 = 7^2 \cdot 11 \cdot 29, \quad S^{5^{n+2} \cdot 3} - 1 \quad (n=0) \quad S^5 + 6 = 3131$$

$$= 31 \cdot 101, \quad 24233 = 24, \quad 124 = 31 \cdot 4 = S^3 - 1, \quad S^{5^{n+2} \cdot 3} - 1 = (S^3 - 1) \left( S^{5^{n+2} \cdot 2} \right) \sum_{j=0}^{5^{n+2}-1} S^j,$$

$$31 \mid (S^{5^{n+2}} - S^{5^n}) \quad // \quad S = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} S^{n-j-1}, \quad S^5 = \sum_{j=0}^{n-1} \binom{n}{j} (-1)^j S^{n-j} - (-1)^n,$$

$$S^5 - 1 = 6^n, \quad S = \frac{1}{6}(6^n + 1), \quad (n=5) \quad 36 \cdot 216 + 1 = 7777 = 7 \cdot 11 \cdot 101$$

$$\frac{216}{36} \quad S^5 - 1 = 6^n, \quad (n=5) \quad 6^n + 1 = 1025 = 5 \cdot 401, \quad S_5 = 5 \cdot 41, \quad \text{then } S_{26+3}$$

$$4^{2k+1} + 1 \leq 5p \stackrel{(3)}{\leq} 2p$$

$$(a_b^2 + b^2 + c^2_n)(s_c^2 + c_n^2 + a^2) \geq (3as_c)^2 \stackrel{(*)}{\iff} \sum_{j=1}^n a_j^4 \geq ? \quad \text{and} \quad \sum_{j=1}^n a_j^2 = (a_j) / (\sum \frac{a_j^2}{n})$$

$$\geq (\sum \frac{a_j^2}{\sqrt{n}})^2, \quad (\sum a_j^2)^2 \geq n^2, \quad \sum a_j^4 \geq ? \quad \sum a_j, \quad n = \sum a_j \leq \sqrt[n]{(\sum a_j^2)^2}$$

$$n \leq \sum a_j^2 \leq n^2 (\sum a_j^2)^{-1}, \quad \sum a_j^2 \geq n \stackrel{\text{average}}{\iff} \sum a_j^2 - n(\sum a_j) (\sum a_j x_j^2) \geq (\sum a_j x_j)^2,$$

$$(\sum f_j)(\sum f_j x^2) \geq (\sum \sqrt{f_j} \cdot \sqrt{f_j} x)^2 \stackrel{(*)}{\iff} \sum k_j = s_{n-4}, \quad 1 \leq \sum \frac{1}{k_j} k_j, \quad 1 \leq \sum k_j^2 \sum k_j^{-4}$$

$$1 \leq \sum k_j^{-3}, \quad s_{n-4} \leq (\sum k_j^2)(\sum k_j^{-2}) = (\sum k_j^3)^2, \quad \sum k_j^3 \geq \frac{1}{3n-4}, \quad \frac{1}{6} \geq \frac{1}{5n-4},$$

$$0 \leq s_{n-4} - 4n + 8, \quad 16 + 20c, \quad n^2 \leq \sum k_j^2 \geq \sum k_j^{-2}, \quad n^2 \leq (\sum k_j^2 k_j^{-2})^2 \leq \sum k_j \sum \frac{1}{k_j} =$$

$$s_{n-4}, \quad (n-1)(n-4) \leq 0, \quad 1 \leq n \leq 4, \quad \text{if } (n=1) k_1 = k_2 \quad (n=2) k_1 = k_2 = k_3 \quad (n=3) k_1 = k_2 =$$

$$k_3 = 3, \quad \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{2} \pi, \quad 2(\cos \theta_{k_1} + \cos \theta_{k_2}) = \frac{1}{k_1} + \frac{1}{k_2}, \quad (n=4) k_1 : k_2 : k_3 : k_4 = \frac{1}{k_1} : \frac{1}{k_2} : \frac{1}{k_3} : \frac{1}{k_4},$$

$$k_3 = 4 \stackrel{(*)}{\iff} \frac{\sin^3 \alpha}{\sin \beta} + \frac{\cos^3 \alpha}{\cos \beta} \geq ? \quad \frac{1}{\cos(\alpha-\beta)} = \frac{1}{\cos \alpha \cos \beta + \sin \alpha \sin \beta}, \quad 1 \leq \left( \frac{\sin^3 \alpha}{\sin \beta} + \frac{\cos^3 \alpha}{\cos \beta} \right)$$

$$(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \geq \frac{\sin^3 \alpha + \cos^3 \alpha}{\sin \beta \cos \beta} \stackrel{(*)}{\iff} \frac{\sin^2 \alpha + \cos^2 \alpha}{\sin \beta \cos \beta} \geq 1, \quad \text{if } \alpha, \beta \in \mathbb{R},$$

$$\text{If } P(x) = (x-r_1)(x-\bar{r}_1)(x-r_2)(x-\bar{r}_2)(x-r_3), \quad 35 = -2\Re z - (r_1 r_2)^2 r_3, \quad -10$$

$$= \bar{r}_1 (r_2)^2 r_3 + r_1 (r_2)^2 \bar{r}_3 + (r_1)^2 \bar{r}_2 r_3 + (r_1)^2 r_2 \bar{r}_3 + (r_1 r_2)^2 \stackrel{(*)}{=} 2\Re r_1 \cdot (r_2)^2 r_3$$

$$+ 2\Re r_2 (r_1)^2 r_3 + (r_1 r_2)^2, \quad \Re \frac{z}{r} = \frac{z^2 \Re r_1}{(r_1)^2} + \frac{z^2 \Re r_2}{(r_2)^2} + \frac{1}{r_3}, \quad \frac{1}{r} = \frac{\Re r_1}{(r_1)^2} + \frac{\Re r_2}{(r_2)^2} + \frac{1}{r_3}$$

$$\frac{2}{r} = \sum \frac{1}{r_j} \quad \text{and so ...}$$

$$(n^{n+1}(n!)^2)^{1/2} \leq \frac{1}{2}(n^{n+1} + n^2 - 2n + 1), \quad \leq \frac{1}{2}(n^{n+1}(n-1) + n-1) \stackrel{(*)}{=} \frac{1}{2} n^2, \quad n^n - 1 = (n-1) \sum_{j=0}^{n-1} n^j$$

$$\text{and so } \geq (n-1) n \left( \prod_{j=0}^{n-1} n^j \right)^{1/n} = (n-1) n^{1 + \frac{n-1}{2}} = (n-1) n^{\frac{n+1}{2}} \stackrel{(*)}{\iff} 1 + (r_1 \cdots r_n)^{\frac{1}{n}} \leq ? \quad ((1+r_1) \cdots (1+r_n))^{\frac{1}{n}}$$

$$\log(1 + (\pi r_j)^n) \leq ? \quad \frac{1}{n} \sum \log(1 + r_j) \geq (\pi \log(1 + r_j))^{1/n}, \quad \log(1 + (\pi r_j)^n) \leq \log(1 + \pi)$$

$$= \log\left(\frac{1}{n} \sum (1 + r_j)\right) \stackrel{(*)}{\geq} \frac{1}{n} \sum \log(1 + r_j) \quad \log(1 + r_1) \log(1 + r_2) \stackrel{(*)}{\geq} (\log(1 + \sqrt{r_1 r_2}))^2,$$

$$\frac{1}{n} \sum \log \log(1 + r_j) \geq ? \quad \log \log(1 + (\pi r_j)^n) \quad 1 + \sqrt{r_1 r_2} \leq ? \quad ((1+r_1)(1+r_2))^{\frac{1}{2}} \stackrel{(*)}{\geq} (\pi - 1)(\pi - 2)$$

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$$\hat{y}_1 = \mathbf{z}^T \mathbf{F}(\mathbf{z}, \mathbf{z}^T)^{-1} \mathbf{F}(\mathbf{z}, \mathbf{y}^1), \quad \hat{v}_2 = y_2 - \mathbf{z}^T \Pi_{\mathbf{z}}(y_2), \quad \mathbf{z}^T \hat{x} = (\mathbf{z}_1, y_2) - \mathbf{z}_1^T \Pi_{\mathbf{z}_1}(\hat{v}_2(\mathbf{z}_1, y_2)), \quad \mathbf{z}_1^T \Pi_{\mathbf{z}_1}(\mathbf{z}_1, y_2)$$

$$g = \hat{v}_2^* E(\hat{v}_1 \hat{v}_2^*)^{-1} E(\hat{v}_2 (z_1 g_2)^{-1}) \text{ and } R \Pi_{\hat{x}}(y) = E(\hat{x} \hat{x}^*)^{-1} E(\hat{x} y^*) \in E(\hat{x} \hat{x}^*)$$

$$E(x x^T) - E(x x^T \Pi_{\hat{V}_2} \hat{V}_2^T) - E(\hat{V}_2 \Pi_{\hat{V}_2}^* x x^T) + E(\hat{V}_2 \Pi_{\hat{V}_2}^* x x^T \Pi_{\hat{V}_2} \hat{V}_2^T) =$$

$$\mathbb{E}(xx^T) = \mathbb{E}(xx^T \Pi_{Y_2} y_2^T) + \mathbb{E}(xx^T \Pi_{Y_2} y_2^T \Pi_z z^T) - \mathbb{E}(y_2 \Pi_{Y_2} xx^T) + \mathbb{E}(z \Pi_z y_2 \Pi_{Y_2} x)$$

$$+ E(y_2 \pi_{\hat{z}_2}^{\top} x x^T \pi_{\hat{z}_2}^{\top} y_2^T) - E(y_2 \pi_{\hat{z}_2}^{\top} x x^T \pi_{\hat{z}_2}^{\top} z^T) - E(2 \pi_z y_2 \pi_{\hat{z}_2}^{\top} x x^T \pi_{\hat{z}_2}^{\top} u^T)$$

$$E(z\pi_2 y \in \pi_1^{\perp} x y^T \pi_1^{\perp}) = \pi_2^T \pi_1^{\perp} z^T$$

$$\pi_1 = \text{ker } (\pi_2 x \mapsto \tilde{\pi}_2 x) \quad \pi_1(x,y) = \text{ker } (\pi_2 x \mapsto (\tilde{\pi}_2 x)^T E(x^*) \tilde{\pi}_2),$$

$$\hat{P}_{\hat{X}} y = E(y \hat{x}^T) E(\hat{x} \hat{x}^T)^{-1} \hat{x}, \quad \hat{x} = P_{\hat{Z}} x = E(x z^T) E(z z^T)^{-1} z, \quad \hat{\beta}_1 = \sqrt{2} (y \hat{x}^T) E(\hat{x} \hat{x}^T)^{-1} z$$

$$P_{\tilde{X}}y = \mathbb{E}(y\tilde{X}^T) \mathbb{E}(\tilde{X}\tilde{X}^T)^{-1}\tilde{X}, \quad \tilde{X} = I - P_{\tilde{X}}\tilde{v}_2 = \begin{matrix} \text{Top rows} \\ \hline \text{Bottom row} \end{matrix} \quad \tilde{v}_2 = \tilde{P}_{\tilde{X}}y_2,$$

$$\hat{P}_{\hat{v}_2} \delta = x - E(x \hat{v}_2^\top) E(\hat{v}_2 \hat{v}_2^\top)^{-1} \hat{v}_2$$

~~$\hat{v}_2 = E(x \hat{v}_2^\top)$~~

$$P_{x \hat{v}_2} = E(x \hat{v}_2^\top) C(v_2 v_2^\top)^{-1} \hat{v}_2$$

$$\mathbb{E}(\hat{\beta}_2 \hat{\beta}_2^T) = \mathbb{E}(y_i y_i^T \mathbb{E}(z z^T)^{-1} \mathbb{E}(z y_i^T)) \mathbb{E}(\hat{\beta}_2 \hat{\beta}_2^T)^{-1} \mathbb{E}(\hat{\beta}_2 x^T), \quad x = \begin{pmatrix} z \\ y_i \end{pmatrix}$$

$$P(x|\hat{v}_2) = E(x|\hat{v}_2)E(\hat{v}_2|\hat{v}_2)^{-1}\hat{v}_2; \quad E\left(\begin{pmatrix} x \\ y_2 \end{pmatrix} | y_2 - (x + y_2 \tilde{\alpha} \gamma_2^T - E(x + y_2 \tilde{\alpha} \gamma_2^T))\right) = \begin{pmatrix} 0 \\ E y_2^* - E y_2 P \end{pmatrix}$$

$$E(y_2^i) - E(y_2 P(y_2 | z)^*) = E[y_2^i - E(y_2 z^i) E(z z^i)^{-1} E(z y_2)] = E[y_2^i - E(y_2 z^i) E(z z^i)^{-1} E(z z^i) E(y_2^i)]$$

$$\text{E}(y_2^2) = \text{E}(y_2^2) - \text{E}(\text{E}(y_2|z)^2) = \text{E}(\hat{v}_2^2), \quad \text{D}(x|\hat{v}_2) = \text{E}(\hat{x}\hat{v}_2) = (\text{E}\hat{v}_2^2), \quad ?(x|\hat{v}_2) = \begin{pmatrix} 0 \\ \hat{v}_2 \end{pmatrix}$$

$$\hat{x} = \cancel{A_1 y_1 + A_2 y_2} \quad x - P(x|\hat{v}_2) = \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 - \hat{v}_1 \end{pmatrix}, \quad (\hat{o}_1, \hat{q}_1) = E(y\hat{x}^\top) E(\hat{x}\hat{x}^\top)^{-1} + \cancel{E(y\hat{x}^2)} = (E y_1 \hat{v}_1^\top,$$

$$(\mathbb{E} y_2 - \mathbb{E}_z y_2^*) \begin{pmatrix} \mathbb{E} z_1 z_1^\top & \mathbb{E} z_1 y_2 \\ \mathbb{E} y_2 z_1^\top & \mathbb{E} (y_2 - \mathbb{E}_z y_2)^2 \end{pmatrix}^{-1} = ? \quad \mathbb{E} (\mathbf{P}(y|z) \cdot x^\top) \cdot \mathbb{E} (\mathbf{P}(x|z) \cdot x^\top)^{-1}$$

$$\mathbb{E}(P(y_i|z_i \cdot x^i)) = \mathbb{E}(y_i z_i^\top) \mathbb{E}(z_i z_i^\top)^{-1} \mathbb{E}(z_i x^i) = \mathbb{E}(y_i z_i^\top) \mathbb{E}(z_i z_i^\top)^{-1} \mathbb{E}(z_i \cdot (x_i, y_i^\top)) = \mathbb{E}(y_i z_i^\top) [I : \mathbb{E}(z_i z_i^\top)]$$

$$\mathbb{E}(zy_2^T) = \mathbb{E}(y_2 z^T) = \mathbb{E}(y_2^T) \mathbb{E}(zz^T)^{-1} \mathbb{E}(zy_2^T) = (\mathbb{E}y_2^T, \mathbb{E}(y_2 P(y_2 | z)^T)) = (\mathbb{E}y_2^T, \mathbb{E}(y_2 (y_2 - \bar{y}_2))^T).$$

$$E(P(x|z) \cdot x^T) = E(P(x|z) P(x|z)^T) = I_2 \left( \begin{pmatrix} z_1 & z_2 \\ y_1 - z_1 & y_2 - z_2 \end{pmatrix} \otimes I_2 \right)$$

$$\hat{\beta}_1 = E(xz)E(z^2)^{-1}z, \quad \hat{\beta}_2 = E(yx)E(x^2)^{-1} = E(xz)E(z^2)^{-1}E(yz)E(z^2)E^{-2}(z^2)$$

$$\hat{\beta}_1 = \text{IE}(yz)/\text{IE}(xz), \quad \hat{\beta}_{2x} = \sum yz/\sum xz = (\sum xz)^{-1} (\beta_0 \bar{z} + \beta_1 \sum xz + \sum zy) \approx \beta_0 \bar{z} / \bar{xz}$$

$$+ \beta_1 + \bar{xz}/\sum xz, \quad \sqrt{N}(\hat{\beta}_1 - \beta_1) = \beta_0 \sqrt{N} \bar{z} / \sqrt{\bar{z}^2} + \sqrt{N} \bar{xz} / \sqrt{\bar{z}^2} \sim \frac{\beta_0}{\sqrt{\bar{z}^2}} N(0, \sigma_z^2) + \sqrt{N} (\hat{\beta}_1 - \beta_1)$$

$$= \frac{\sqrt{N} \bar{xz}}{\bar{z}^2} \sim \sigma_{\beta_{2x}}^{-1} N(0, \text{Var} z) = N(0, \sigma_z^2 \sigma_y^2 / \sigma_{2x}^2) = N(0, \rho_{2x}^2 \frac{\sigma_y^2}{\sigma_x^2}) \quad \text{BS.10}$$

$$\tilde{y}_2 = P(y_2 | z_2) = \text{IE}(y_2 z_2^T) \text{IE}(z_2 z_2^T)^{-1} z_2, \quad (\tilde{\delta}_1, \tilde{\alpha}_1) = \text{IE}(y_1(z_1, \tilde{y}_2)) \text{IE}((\frac{z_1}{\tilde{y}_2})^{(z_1^T, \tilde{y}_2)})^{-1}$$

$$= \left\{ (\delta_1, \alpha_1) \begin{pmatrix} z_1 \\ y_2 \end{pmatrix}^{(z_1^T, \tilde{y}_2)} + u_1(z_1^T, \tilde{y}_2) \right\} \text{IE} \begin{pmatrix} z_1 z_1^T & z_1 \tilde{y}_2 \\ z_1^T \tilde{y}_2 & \tilde{y}_2 \tilde{y}_2^T \end{pmatrix}^{-1} = (\delta_1, \alpha_1) \text{IE}((\frac{z_1}{y_2})^{(z_1^T, \tilde{y}_2)})$$

$$\text{IE}((\frac{z_1}{\tilde{y}_2})^{(\otimes 2)})^{-1}, \quad \text{IE}(\tilde{y}_2^2) = \text{IE}(y_2 \tilde{y}_2), \quad \text{IE}(y_2 z_1) \neq \text{IE}(\tilde{y}_2 z_1) \quad \text{BS.11}$$

$$\begin{pmatrix} \text{IE} x_{-k} x_{-k}^T & \text{IE} x_{-k} z^T \\ \text{IE} z x_{-k}^T & \text{IE} z z^T \end{pmatrix} \begin{pmatrix} \text{IE} x_{-k} x_{-k}^T & \text{IE} x_{-k} x_k \\ \text{IE} z x_k^T & \text{IE} z z^T \end{pmatrix}^{-1} = \begin{pmatrix} \text{IE} x_{-k} x_{-k}^T & \text{IE} x_{-k} x_k \\ \text{IE} z x_k^T & \text{IE} z z^T \end{pmatrix} \quad (\delta, \theta) = \text{IE}(x_k(x_{-k}^T, z^T)) \text{IE}((\frac{x_{-k}}{z})^{(\otimes 2)})$$

$$\Theta = 0 \Rightarrow \text{IE}(x_k x_{-k}^T, x_k z^T) \perp \left( \begin{pmatrix} \text{IE} x_{-k} z^T \\ \text{IE} z z^T \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} x_{-k} \\ z \end{pmatrix} \right)^{(\otimes 2)} [ , k:1, k+M ] .$$

$$(\text{IE} x_{-k} x_{-k}^T, \text{IE} x_{-k} z^T) \perp \text{IE}((\frac{x_{-k}}{z})^{(\otimes 2)} [ , k:1, k+M ]), \quad \text{rank} \begin{pmatrix} \text{IE} x_{-k} x_{-k}^T \\ \text{IE} z z^T \end{pmatrix} = k-1 .$$

$$\left( \begin{pmatrix} \text{IE} x_{-k} x_k \\ \text{IE} z x_k \end{pmatrix} \in \text{sp} \begin{pmatrix} \text{IE} x_{-k} x_{-k}^T \\ \text{IE} z z^T \end{pmatrix} \right) \Rightarrow \left( \begin{pmatrix} \text{IE} x_{-k} x_k \\ \text{IE} z x_k \end{pmatrix} \perp \text{IE}((\frac{x_{-k}}{z})^{(\otimes 2)} [ , k:1, k+M ]), \quad \Theta = 0 \right) \quad \text{BS.12}$$

$$\text{Cor}(y, z) = \beta_1 = \text{Cov}(y, z) / \text{Cov}(x, z) = (\text{IE}(yz) - \text{IE}y \text{IE}z) / (\text{IE}(xz) - \text{IE}x \text{IE}z) \doteq$$

$$(\sum y \{z=1\} - \sum y \cdot \sum z/n) / (\sum x \{z=1\} - \sum x \cdot \sum z/n), \quad \text{IE}yz = \beta_0 \text{IE}z + \beta_1 \text{IE}x, \quad \text{IE}xz$$

$$\text{IE}(yz|z) \text{IE}((\frac{1}{z})^{(1-z)}) \text{IE}((\frac{1}{z} \frac{z}{z-1}))^{-1} = \left( \begin{array}{cc} \text{IE} z & \\ \text{IE} x / \text{IE} z & \text{IE} z / \text{IE} z^2 \end{array} \right) \left( \begin{array}{cc} 1 & \text{IE} z \\ \text{IE} z & \text{IE} z^2 \end{array} \right)^{-1} = (\text{IE} z (\text{Var} z))^{-1} \left( \begin{array}{cc} 1 & \text{IE} z \\ \text{IE} x / \text{IE} z & -\text{IE} z \end{array} \right) \left( \begin{array}{cc} \text{IE} z^2 & -\text{IE} z \\ -\text{IE} z & 1 \end{array} \right)$$

$$\text{Cor} \text{IE}((\frac{1}{z})^{(1-x)})^{-1} \text{IE}(\frac{y}{z}) = \text{IE} \left( \frac{1}{z} \frac{x}{x-1} \right) \text{IE}(\frac{y}{z}) = ((\text{Cov}(x, z))^{-1} \text{IE} \left( \frac{xz}{z-1} \right)) \text{IE}(\frac{y}{z}),$$

$$\hat{\beta}_1 = \frac{\text{IE}yz - \text{IE}y \text{IE}z}{\text{IE}xz - \text{IE}x \text{IE}z}, \quad \frac{1}{n} (\sum y \sum z) = \frac{1}{n} (\sum y \{z=1\} + \sum_{j \neq k} y_j z_k), \quad \sum y_j (\{z_j=1\} - \bar{z}),$$

$$\sum y_j (\{z_j=1\} - \{z_j=0\}) = 2 \sum y_j \{z_j=1\} - \sum y_j, \quad \sum y \{z=1\} - \bar{y} \sum \{z=1\}, \quad \sum y \{z=1\}$$

$$-\bar{z} \sum y (\{z=1\} + \{z=0\}) = (1-\bar{z}) \sum y \{z=1\} - \bar{z} \sum y \{z=0\} = n\bar{z}(1-\bar{z})\bar{y}_1 - \bar{z}n(1-\bar{z})\bar{y}_0 \quad \text{BS.13}$$

$$L(y|z) = L \left( \sum_{j=1}^k \beta_j x_j + u | z \right) = \sum_{j=1}^{k-1} \beta_j x_j + \beta_k L(x_k | z), \quad L(x_k | z) = \sum_{j=1}^k \delta_j x_j + \sum_{j \neq k} \theta_j z_j,$$

$$\Theta \neq 0, \dim \text{IE}(zx^T) = k \quad \text{BS.14}$$

~~variance~~ Ch 6  
 $\frac{dP_n}{dQ_n} = \exp\left(\frac{1}{2}(x-\mu_n)^2 - \frac{1}{2}\theta_n^2\right) = \exp\left(-x\mu_n + \frac{\mu_n^2}{2}\right), \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} < \delta\right) = P_{Q_n}\left(x - \frac{1}{\mu_n}(\log \delta - \frac{\mu_n^2}{2})\right) = P_{Q_n}\left(x - \frac{\log \delta}{\mu_n} - \frac{\mu_n}{2}\right), \quad \frac{d}{dc} = \phi\left(-\frac{\log \delta}{\mu_n} - \frac{\mu_n}{2}\right) / \mu_n, \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} > \delta\right)$ 
 $\exists R \rightarrow \lim_{n \rightarrow \infty} P_{Q_n}\left(\exp(-x\mu_n + \frac{\mu_n^2}{2}) > 0\right) = 1 \iff -x\mu_n + \frac{\mu_n^2}{2} > -\infty \text{ w.p } 1 - Q_\infty,$ 
 $x\mu_n + \frac{\mu_n^2}{2} \sim Q_n - \mu_n - \frac{\mu_n^2}{2} > 0 \text{ w.p } 1 \iff \mu_n \in [-k, k]. \quad E_{Q_n}\left[\frac{dP_n}{dQ_n}\right]$ 
 $= \int \exp(-x\mu_n + \frac{\mu_n^2}{2}) (2\pi)^{-1/2} \exp\left(-\frac{1}{2}(x - \mu_n)^2\right) dx = (2\pi)^{-1/2} \int \exp(-x^2) \geq 1 \quad \#1$ 
 $\frac{dP_n}{dQ_n} = (2\pi \theta_n)^{-1} \exp\left(\frac{1}{2\theta_n^2}((x - \theta_n)^2 - x^2)\right) = \exp\left(\frac{1}{2\theta_n^2}(\theta_n^2 - 2x\theta_n + 2x^2)\right), \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} < \delta\right)$ 
 $(\theta_n = O(n^{-1/2}))$ 
 $= P_{Q_n}\left((\theta_n^2 - 2x\theta_n)/\theta_n < \log(\sqrt{\frac{2\pi}{\delta}})\right) = P\left(N(-\frac{\theta_n^2}{2\theta_n}, \frac{\theta_n^2}{n^2}) < \log(\sqrt{\frac{2\pi}{\delta}})\right)$ 
 $\Phi\left(\{\log(\sqrt{\frac{2\pi}{\delta}}) + \frac{\theta_n^2}{2\theta_n}\}/\sqrt{\frac{2\pi}{\delta}}\right) = \Phi\left(O(\sqrt{n})\left(\log(\sqrt{\frac{2\pi}{\delta}}) + O(1)\right)\right) \xrightarrow{n \rightarrow \infty} 0$ 
 $= \Phi(O(n^{1/2} \log \sqrt{\frac{2\pi}{\delta}})) \quad \frac{dP_n}{dQ_n} = \exp\left(\frac{1}{2}(\theta_n^2 - 2x\theta_n)\right), \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} < \delta\right)$ 
 $= P\left(N(-\frac{\theta_n^2}{2}, n\theta_n^2) < \log(\sqrt{\delta})\right) = \Phi\left(\{\log(\sqrt{\delta}) + \theta_n^2/2\}/\sqrt{n\theta_n}\right)$ 
 $\mathbb{E}[O(n^{1/2})\{\log(\sqrt{\delta}) + \theta_n^2/2\}] \xrightarrow{n \rightarrow \infty} \frac{\log(\sqrt{\delta})}{\sqrt{n}} + \left(\frac{n}{\sqrt{n}\delta} \frac{\sqrt{n}\delta}{n^2}\right)/\left(\frac{1}{2}\sqrt{n}\right) \rightarrow 0,$ 
 $\sqrt{n\theta_n} \xrightarrow{\delta \rightarrow 0} 0, \quad \frac{dP_n}{dQ_n} \xrightarrow{Q_n} V \Rightarrow P(V=0)=0, \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} < \delta\right) =$ 
 $P\left(N(-\frac{\theta_n^2}{2}, n\theta_n^2) < \log \delta\right) \xrightarrow{\delta \rightarrow 0, n \rightarrow \infty} 0, \quad \frac{dQ_n}{dP_n} \xrightarrow{P_n \ll Q_n} W \Rightarrow P(W=0)=0, \quad P_n \ll Q_n,$ 
 $\theta_n \neq O(n^{-1/2}), \quad \limsup \theta_n n^{1/2} = \infty, \quad P_{Q_n}\left(\frac{dP_n}{dQ_n} < \delta\right) = \Phi\left(\frac{\log \delta}{\sqrt{n}\theta_n} + \frac{\theta_n^2}{2}\right)$ 
 $\xrightarrow{n_k \rightarrow 1} \leq \theta_n n^{1/2} \xrightarrow{k \rightarrow \infty} \infty \quad \frac{dP_n}{dQ_n} = \frac{1}{n+1} \{[0,1]\} \xrightarrow{n \rightarrow \infty} V \text{ a.s.}, \quad P_{Q_n}(V < \delta) =$ 
 $\lim_{k \rightarrow \infty} P_{Q_{n_k}}\left(\frac{dP_{n_k}}{dQ_{n_k}} < \delta\right) = \lim_{k \rightarrow \infty} \left\{ \frac{n_k}{n_k+1} < \delta \right\} = 0 \iff \delta < \frac{1}{2}, \quad Q_n \ll P_n. \quad \frac{dQ_n}{dP_n} = \frac{1}{n+1} \{[0,1]\},$ 
 $P_{P_n}\left(\frac{dQ_n}{dP_n} < \delta\right) = P_{P_n}\left(\{0 \leq X \leq 1\} \subset \frac{\delta n}{n+1}\right) \stackrel{(S \subset)}{=} P_{P_n}(X \in [0,1]) = \frac{1}{n+1} \rightarrow 0, \quad R_n \ll Q_n \quad \#2$ 
 $P_n(A) \rightarrow 0 \Rightarrow Q_n(A) = P_n(A) + Q_n(A) - P_n(A) \leq P_n(A) + \|P_n - Q_n\|_{TV} \rightarrow 0 \quad \#3$ 
 $P_n(0) = 1 - \varepsilon, \quad P_n(1) = \varepsilon, \quad Q_n(0) = \varepsilon, \quad Q_n(1) = 1 - \varepsilon, \quad \|P_n - Q_n\| = 1 - 2\varepsilon, \quad P_n(A) \rightarrow 0 \Rightarrow A \cap Q_n \neq \emptyset$

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$$a+b+c = 2, \geq 3(abc)^{1/3}, \quad (1-a)(1-b)(1-c) = 1 - a - b - c + ab + bc + ca - abc = \\ -1 + abc \leq 0 \Leftrightarrow (1-a)(1-b)(1-c) \geq 0 \Leftrightarrow a+b \geq c, \quad 1-a \geq -b < 1-c \Leftrightarrow (a-b)(a-b+c) \geq 0$$

$$= a-1 \leq \left(\frac{1}{3}(a+b+c)\right)^3 = \frac{8}{27}, \quad a \leq \frac{35}{27} \quad \left(\frac{1}{3}(1-a-b-c)\right)^3 = \frac{1}{27}, \quad a \leq \frac{26}{27}.$$

$$\text{aber } a \geq b \geq c \geq 0 \Rightarrow abc + ac - \left(\frac{1}{3}(a+b+c)\right)^3 = abc + ac - \frac{8}{27}, \quad (1-a)(1-b)(1-c) \geq 0$$

$$= (b-1)(c-1)(b+c-1), \quad \geq 0 \Leftrightarrow (1-a)(1-b)(1-c), \quad \Rightarrow 1 \Rightarrow a+b+c = 2-a < 1,$$

$$a > b+c, \quad \Leftrightarrow a, b, c < 1 \quad \frac{365^{124}}{\prod_{j=1}^{124} (365-j)} \leq \frac{1}{2}, \quad \left(\frac{1}{25} \sum (365-j)\right)^{124} \leq ?$$

$$365^{125}/2, \quad 365 - \frac{25 \cdot 26}{50} \leq ? \quad 365/2^{125}, \quad 185 \quad 352 \quad 2^{125} \leq ? \quad \frac{365}{352} \approx \frac{13}{11 \cdot 32} + 1$$

$$\geq 1 + \frac{1}{32} \approx 1.02, \quad \frac{13}{352} \approx \frac{1}{30}, \quad (1.03)^{125} \geq 1, \quad 1.03^k = 1.0609, \quad 1.12, 1.25,$$

$$1.56, \quad 9: 1.2025, \quad \frac{156}{12025} \approx 0.0125/12 = 1, \quad a+n = 1 + (2-1) + \text{back}, \quad 12 = 5 + 1 + 2k, \quad 13 \neq$$

$$3 \cdot k = 22, \quad a_{n+1} = \frac{n-1}{k}, \quad a_n - a_{n-1} = 1, \quad a_n - a_{n-1} = k(n-1) + 1, \quad a_n = \sum_{m=1}^{n-1} (k(m-1) + 1)$$

$$= n + \frac{n(n-1)}{2} \quad f(0) + 2f(f(0)) = 5, \quad f(f(0)) = \frac{1}{2}(5-f(0)) = f\left(\frac{5-f(0)}{2}\right), \quad 5-f(0) \neq 0,$$

$$2|5-f(0), \quad f(0) \in \{1, 3, 5\} \quad (f(0)=1) \quad f(1) = 2 = f\left(\frac{1}{2}(5-f(0))\right) = f\left(\frac{1}{2}(5-1)\right) = f(2), \quad f(2) = f(f(1))$$

$$= \frac{1}{2}(3n+5-f(n)), \quad f(2) = f^2(1) = 3, \quad f(3) = f^2(2) = \dots \quad \therefore f(n+1) = f(f(n)) = \frac{1}{2}(3n+5-(n+1))$$

$$= n+2, \quad f(n) = n+1 \quad (f(0)=3) \quad \therefore f(n+1) = f(f(n)) = \frac{1}{2}(3n+5-(n+1)) = \frac{1}{2}(3n+3) = n+2, \quad (3) \in \mathbb{N}_+$$

$$(f(0)=5) \quad f(5)=0, \quad f(6)=\dots \quad \therefore f(n+1) = f(f(n)) = \frac{1}{2}(3n+5-(n+1)) = \frac{1}{2}(3n+4) = n+2, \quad 0=f(n) + 3f(0)f(n)$$

$$f(0) \geq -1, \quad f(n) \geq 0, \quad (n \geq 1) \quad f(n) \geq 0, \quad f(n+1) = f(n) + 3f(0)f(n) = 0, \quad f(n^2) =$$

$$2f(n) + 3f(n)^2 = (3f(n) + 3f(n)^2)/2, \quad f(2n) = f(2) + f(n) + 3f(2)f(n),$$

$$\frac{f(mn)}{f(m)} = 1 + \frac{f(mn)}{f(m)} + \dots \geq 0 \Rightarrow f(n) \geq 1, \quad f(6) \geq 3+6=9, \quad \text{absturde} \Rightarrow f(4)$$

$$= 2f(2) + 3f(2)^2, \quad f(3) = f(4) + f(2) + 3f(4)f(2) = 3f(2) + 3f(2)^2 + 0f(2)^2 + 9f(2)^3$$

$$= 3f_2 + 9f_2^2 + 9f_2^3, \quad \text{absturde} \Rightarrow f(n)f'(n) = 1 + f(n) \text{ abstr}$$

causal

$$P(1-P) = \frac{1}{4} - (P - \frac{1}{2})^2, \quad 1 + 4 \frac{P_2(1-P_2)}{P_1(1-P_1)}, \quad 1 + \frac{1-4(P_2-\frac{1}{2})^2}{\frac{1}{4} - (P_1-\frac{1}{2})^2} (P_1-\frac{1}{2})^2, \quad 1 + \frac{1}{4} \cancel{(P_1-\frac{1}{2})^2}$$

$$1 + \frac{1-4(P_2-\frac{1}{2})^2}{\frac{1}{4} - 1}, \quad 1 + \frac{P_2(1-P_2)}{P_1(1-P_1)} - 4P_2(1-P_2) = 4(P_2-\frac{1}{2})^2 + \frac{P_2(1-P_2)}{P_1(1-P_1)}$$

$$T = P(\delta_0 + \delta_1), \quad \Delta = P^2 \delta_0 \delta_1 - (1-P)^2 \delta_0 \delta_1 = \delta_0 \delta_1 (2P-1), \quad \frac{T^2}{4} - \Delta = \frac{P^2}{4} (\delta_0^2 + \delta_1^2 + 2\delta_0 \delta_1)$$

$$- \delta_0 \delta_1 (2P-1), \quad \frac{T^2}{4} - \Delta = (1-P)^2 \delta_0 \delta_1 + \frac{P^2}{4} (\delta_0^2 + \delta_1^2 + 2\delta_0 \delta_1) - P^2 \delta_0 \delta_1 = (1-P)^2 \delta_0 \delta_1,$$

$$+ \frac{P^2}{4} (\delta_0 - \delta_1)^2 = (\delta_0 - \delta_1)^2 \left( \frac{P^2}{4} + (1-P)^2 \frac{\delta_0 \delta_1}{\delta_0^2 - 2\delta_0 \delta_1 + \delta_1^2} \right) = (\delta_0 - \delta_1)^2 \left( \frac{P^2}{4} + (1-P)^2 \frac{1}{\frac{\delta_0}{\delta_1} + \frac{\delta_1}{\delta_0} + 2} \right).$$

$$\left( \frac{\delta_0}{\delta_1} + \frac{\delta_1}{\delta_0} - 2 \right)^{-1}, \quad \frac{T^2}{4} - \Delta = \frac{P^2}{4} (\delta_0 + \delta_1)^2 \left( 1 - \frac{4}{P^2} (2P-1) \frac{1}{\frac{\delta_0}{\delta_1} + \frac{\delta_1}{\delta_0} + 2} \right), \quad \text{if } \frac{\delta_0}{\delta_1} + \frac{\delta_1}{\delta_0} > 2, \quad \frac{T^2}{4} - \Delta \leq \frac{P^2}{4} (\delta_0 + \delta_1)^2$$

$$<? \quad \frac{x+\delta}{x} - 1 = \frac{\delta}{x}, \quad <? \quad \frac{x^2 + x\delta + \delta x}{x(x+\delta)} = 1 + \frac{\delta}{x+\delta}, \quad \text{if } \frac{\delta_0}{\delta_1} + \frac{\delta_1}{\delta_0} < 2, \quad \frac{T^2}{4} - \Delta \leq \frac{P^2}{4} (\delta_0 + \delta_1)^2$$

$$(1 - \frac{4}{P^2} (2P-1)) \leq \frac{1}{4} \cdot \frac{P^2}{4} (\delta_0 + \delta_1)^2 \quad \frac{T^2}{4} - \Delta \leq \frac{P^2}{4} (\delta_0 + \delta_1)^2 (1 - \frac{2}{P} + \frac{1}{P^2})$$

$$\cancel{\frac{T^2}{4}} \frac{P^2}{4} (\delta_0^2 + \delta_1^2)^2 + (1-P)^2 \delta_0 \delta_1, \quad \frac{T^2}{4} - \Delta \leq \frac{P^2}{4} (\delta_0 - \delta_1)^2 \delta_0 \delta_1 \cdot 2, \quad \sqrt{\frac{T^2}{4} - \Delta} \leq \frac{1}{2} |\delta_0 - \delta_1|$$

$$+ (1-P) \sqrt{\delta_0 \delta_1}, \quad \frac{1}{2} (\delta_0 + \delta_1) + (1-P) \sqrt{\delta_0 \delta_1} = \frac{1}{2} (\sqrt{\delta_0} - \sqrt{\delta_1})^2 + \sqrt{\delta_0 \delta_1}, \quad |\delta_0 - \delta_1| = |\sqrt{\delta_0} + \sqrt{\delta_1}| |\sqrt{\delta_0} - \sqrt{\delta_1}| \geq 1$$

$$\delta_0, \delta_1 \geq 1 \Rightarrow |\sqrt{\delta_0} - \sqrt{\delta_1}| = |\delta_0 - \delta_1|, \quad \lambda \leq \frac{1}{2} (\delta_0 + \delta_1) + \lambda \leq \frac{1}{2} (\sqrt{\delta_0} + \sqrt{\delta_1})^2 + \sqrt{\delta_0 \delta_1} + \frac{1}{2} |\delta_0 - \delta_1|$$

$$= \sqrt{\delta_0 \delta_1} + \frac{1}{2} (|\sqrt{\delta_0} - \sqrt{\delta_1}| |\sqrt{\delta_0} + \sqrt{\delta_1}| + (\sqrt{\delta_0} - \sqrt{\delta_1})^2) \leq \sqrt{\delta_0 \delta_1} + \frac{1}{2} |\sqrt{\delta_0} - \sqrt{\delta_1}| |\delta_0 - \delta_1| \cdot \frac{1}{2}$$

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$$\begin{aligned} P(A_n) &\rightarrow 0, \quad Q(A_n) \neq 0 \Rightarrow \{A_{n_k}\}_k, \quad \delta > 0: \quad Q(A_{n_k}) > \delta, \quad \{A_{n_k}\}_k, \quad P(A_{n_k}) \\ &< 2^{-k}, \quad B_m := \bigcup_{j=m}^{\infty} A_{n_k}, \quad P(B_m) \leq 2^{-m+1} \xrightarrow[m \rightarrow \infty]{<} 0, \quad B_m \downarrow B_{\infty}, \quad \text{d.h. } 0 = \lim P(B_m) \\ &= P(\lim B_m) = Q(\lim B_m) = \lim Q(B_m), \quad Q(B_m) > 0, \quad \leftarrow \frac{dQ}{dP_p} \left( \frac{dQ}{dP_p} > 0 \right) \\ &= P_p(dQ > 0) = 1 \quad \text{d.h.} \end{aligned}$$

Andreeescu

$$(mx+ny)^2 + 10(jx+ly)^2 = (m^2 + 10j^2)x^2 + (2mn + 20jk)xy + (n^2 + 10k^2)y^2, \quad mn = -10jk$$

$$m^2 + 10j^2 \leq 1, \quad n^2 + 10k^2 \leq 10, \quad n^2 + 10k^2 = 10(m^2 + 10j^2), \quad 2mn + 20jk = 0, \quad mn = -10jk$$

$$\begin{aligned} m^2 + 10j^2 &\leq 1, \quad n^2 + 10k^2 \leq 10, \\ n^2 + 10k^2 &= 10(m^2 + 10j^2), \quad 2mn + 20jk = 0, \quad mn = -10jk \end{aligned}$$

$$m^2 + 10j^2 \leq 1, \quad n^2 + 10k^2 \leq 10, \quad n^2 + 10k^2 = 10(m^2 + 10j^2), \quad 2mn + 20jk = 0, \quad mn = -10jk$$

$$m^2 + 10j^2 = u^2, \quad n^2 + 10k^2 = v^2, \quad n^2 + 10k^2 = 10u^2, \quad \text{and } n^2u^2 - 10j^2v^2 = 100jk$$

$$m^2 + 10j^2 = u^2(u^2 - 10j^2), \quad 4, 9, 16, 25, 36, 49, 64, 41, 100, 121, 144, 169, \dots, 1, 4, 9, 5, 6, \quad 3z^2 = -\{3, 2, 7, 5, 8\}$$

$$x^2 + 10y^2 = -\{1, 4, 9, 5, 6\}, \quad \text{and } n = \{5\}, \quad \text{so } 5|x, 5|y, 5|z \quad \text{and } (2x+2y)(2x-2y) = z^2 - t^2$$

$$= (z+t)(z-t), \quad 4|(z^2 - t^2), \quad z^2 - t^2 = \frac{1}{4}(z^2 - t^2), \quad 24y^2 = 10u^2 - 5z^2 - t^2, \quad \text{and } \left(\frac{x^2}{y^2}\right) = -\frac{1}{4}\left(\frac{z-t}{z+t}\right)\left(\frac{z}{t}\right)$$

$$24x^2 = 5t^2 - z^2, \quad 24y^2 = 5z^2 - t^2, \quad 5|(z-x)(z+x), \quad \cancel{5|z+t+5k}, \quad (z = x+5k, t = y+5j)$$

$$5y^2 = 10kx + 25k^2, \quad y^2 = 20k^2 + 5k^2, \quad \text{and } x^2 = 2jy + 5j^2, \quad 5|(2x)(z+x) = 5y^2$$

$$= y_1y_2, \quad z = \frac{1}{2}(y_1 + y_2), \quad x = \frac{1}{2}(y_2 - y_1), \quad (t-y)(t+y) = 5x^2 = x_1x_2, \quad t = \frac{1}{2}(x_1 + x_2)$$

$$y = \frac{1}{2}(x_2 - x_1), \quad 4y^2 = x_1^2 + x_2^2 - 10x^2 = \frac{4}{5}(z^2 - x^2), \quad -\frac{46}{5}x^2 + x_1^2 + x_2^2 = z^2, \quad 5|x,$$

$$5|z, 5|t, \quad (5|k + 5y^2) \quad 25\left(\frac{x}{5}\right)^2 + 5y^2 \neq 25\left(\frac{z}{5}\right)^2, \quad 5x_{(1)}^2 + y^2 = 5z_{(1)}^2,$$

$$x^2 + 5y_{(1)}^2 = t_{(1)}, \quad 5|?y, \quad 25|(z^2 - x^2), \quad 5|y, \quad \left(\frac{x}{5}\right)^2 + 5\left(\frac{y}{5}\right)^2 = \left(\frac{z}{5}\right)^2,$$

$$5\left(\frac{x}{5}\right)^2 + \left(\frac{y}{5}\right)^2 = \left(\frac{z}{5}\right)^2 \quad \text{and } p|x \Rightarrow p|(y-x)(x+y), \quad p|y, \quad x=y \quad \text{and } x_j^2, x_{j+1}^2, x_{j+2}^2, \dots$$

$$x_{j+1}^2 - x_j^2 = d, \quad x_{j+1} - x_j = \frac{d}{x_j + x_{j+1}} \rightarrow 0 \quad \text{and } n^3 - 4n^2 + 1 + n^2 - 1 + 4 = n^3 + n^2 + 3 + (1-4n) -$$

$$(1+4n-2n), \quad (2n-1)|(n^3 + n^2 + 3), \quad 2n^3 - n^3 - n^2 - 2n^2 + 4 = n^2(2n-1) - n^3 = 2n^2 + 4, \quad (2n-1)|$$

$$(n^3 + 2n^2 - 4), \quad (2n-1)|(n^2 - 7), \quad n^3 - 2n^2 - n^2 - n + n + 4 = -n(2n-1) + n^3 - n^2 - n + 4, \quad (2n-1)|$$

$$(n^3 - n^2 - n + 4), \quad (2n-1)|(2n^2 + n - 1) = (2n-1)(n+1), \quad n^2 = -\{1, 4, 9, 6, 5, 0\}, \quad n^2 - 7 = -\{4, 7, 2, 9, 6, 3\}$$

$$\cancel{2n-1} \quad (2n-1)|(4n^2 - 1), \quad (2n-1)|(n^3 - 17), \quad (2n-1)|(5n^2 - 4), \quad (2n-1)|(-27),$$

$$2n-1 \in \{1, 3, 9\} \quad \text{and } 1! \cdot 2! \cdots 100! = 1^{100} 2^{99} 3^{98} \cdots 99^2 \cdot 100^1 =$$

$$x^2 (2 \cdot 4 \cdot 6 \cdots 100) = x^2 50! 2^{50} = y^2 50! \quad \text{and } a_j = m + j, \quad \text{and } m | a_k \Rightarrow$$

$$(a_j, a_k) = m, \quad (a_j, a_{2j}) = j \Rightarrow j | m, \quad (a_j, a_m) = j, \quad a_j = b_j \cdot j, \quad (j, k) = (b_j \cdot j, b_k \cdot k) \Rightarrow (b_j, b_k) = 1$$

$$b_j \neq 1, \quad (a_j, a_{b_j}) = (j, b_j), \quad b_j, b_{j_0} = (j_0, b_{j_0}), \quad b_{j_0} | j_0, \quad j_0 | b_j, \quad a_j = j^2$$

IV

$$\frac{1}{2} \frac{\partial T}{\partial \gamma} = \left( \frac{1}{2} \frac{\partial T}{\partial \gamma} - \frac{\partial D}{\partial \gamma} \right) \frac{1}{2} (T^2 / A - D)^{-1} = 0, \quad T = p^2 \left( \frac{\gamma^2}{\delta_0^2} + \frac{(1-\gamma)^2}{\delta_1^2} \right) + (1-p)^2 \left( \frac{\gamma^2}{\delta_1^2} + \frac{(1-\gamma)^2}{\delta_0^2} \right),$$

$$\frac{\partial T}{\partial \gamma} = 2p^2 \left( \frac{\gamma}{\delta_0} - \frac{1-\gamma}{\delta_1} \right) + 2(1-p)^2 \left( \frac{\gamma}{\delta_1} - \frac{1-\gamma}{\delta_0} \right), \quad \text{and } i(p^2 + (1-p)^2) \cdot (2p-1)^2$$

$$T = p^2 \left( \frac{\gamma}{\delta_0} + \frac{1-\gamma}{\delta_1} \right) + (1-p)^2 \left( \frac{\gamma}{\delta_1} + \frac{1-\gamma}{\delta_0} \right), \quad \frac{\partial T}{\partial \gamma} = p^2 \left( \frac{1}{\delta_0} - \frac{1}{\delta_1} \right) + (1-p)^2 \left( \frac{1}{\delta_1} - \frac{1}{\delta_0} \right)$$

$$= \left( \frac{1}{\delta_0} - \frac{1}{\delta_1} \right) (2p-1), \quad T^2 = p^4 \left( \frac{\gamma^2}{\delta_0^2} + \frac{(1-\gamma)^2}{\delta_1^2} + 2 \frac{\gamma(1-\gamma)}{\delta_0 \delta_1} \right) + (1-p)^4 \left( \frac{\gamma^2}{\delta_1^2} + \frac{(1-\gamma)^2}{\delta_0^2} + 2 \frac{\gamma(1-\gamma)}{\delta_1 \delta_0} \right)$$

$$+ 2p^2(1-p)^2 \left( \frac{\gamma^2}{\delta_0 \delta_1} + \frac{(1-\gamma)^2}{\delta_0 \delta_1} + \gamma_0 \gamma_1 (\frac{1}{\delta_0^2} + \frac{1}{\delta_1^2}) \right) \quad (\delta_0 = \delta_1) \quad T = p^2 \delta^{-2} (\gamma_0^2 + \gamma_1^2) (2p^2 - 2p + 1)$$

$$D = (2p-1)^2 \delta^{-4} (\gamma_0 \gamma_1)^2, \quad (\omega_0 \gamma_0 + \omega_1 \gamma_1) / \delta_0 + (\omega_0 \gamma_1 + \omega_1 \gamma_0) / \delta_1$$

$$= ((\omega_0 - \omega_1) \gamma_0 + \omega_1) / \delta_0 + ((\omega_1 - \omega_0) \gamma_0 + \omega_0) / \delta_1$$

$$P(A_{t+1}^a | L_t^l, V_t^u) = \sum_b P(A_{t+1}^a | L_t^l, V_t^u, B_{t+1}^b) = P(B_{t+1}^a | L_t^l) + P(A_{t+1}^a | L_t^l, V_t^u)$$

$$V_t^u = u = P_B P_{LA}^{a=l} (1-P_{LA})^{a \neq l} + (1-P_B) P_{VA}^{a=u} (1-P_{VA})^{a \neq u}, \quad P(L_{t+1}^l, V_{t+1}^u)$$

$$(A_t^a = a) = P_B P(L_{t+1}^l, V_{t+1}^u | A_t^l = a, B_t^l = b) + (1-P_B) P(L_{t+1}^l, V_{t+1}^u | A_t^u = a),$$

$$(B_t^a = 0) = P_B \frac{1}{2} P_B P_{AL}^{a=l} (1-P_{AL})^{a \neq l} + \frac{1}{2} (1-P_B) P_{AV}^{a=u} (1-P_{AV})^{a \neq u}, \quad P(L_t^l | A_t^a = a)$$

$$= P_B P(L_t^l | A_t^l = a) + (1-P_B) P(L_t^l | A_t^u = a) = P_B P_{AL}^{a=l} (1-P_{AL})^{a \neq l} + \frac{1}{2} (1-P_B),$$

$$P(A_t^a | L_t^l) = P_B P_{LA}^{a=l} (1-P_{LA})^{a \neq l} + (1-P_B)/2$$

$$P(L_t^l | L_{t-1}^l = l) = \sum_a P(L_t^l | A_{t-1}^a = a) P(A_{t-1}^a | L_{t-1}^l = l)$$

$$T = p \left( \frac{1}{\delta_0^2} + \frac{1}{\delta_1^2} \right), \quad \Delta = P \frac{\delta_0^2 \delta_1^2}{\delta_0^2 + \delta_1^2} - (1-p)^2 \frac{\delta_0^2 \delta_1^2}{\delta_0^2 + \delta_1^2} = (2p-1) / \delta_0^2 \delta_1^2, \quad T^2 / 4 - \Delta = \frac{\delta_0^2}{4} (p^2 (\delta_0^2 + \delta_1^2) + 2 \delta_0 \delta_1) - (2p-1) \delta_0 \delta_1 = \frac{p^2}{4} (\delta_0^2 + \delta_1^2) + \delta_0 \delta_1 (p^2 / 2 - 2p + 1)$$

$$= \frac{p^2}{4} (\delta_0 - \delta_1)^2 + \frac{p^2}{4} \delta_0 \delta_1 (p^2 - 2p + 1) = \frac{p^2}{4} (\delta_0 - \delta_1)^2 + \delta_0 \delta_1 (p-1)^2 \quad (\delta_0 = \delta_1)$$

$$\frac{1}{2} p \left( \frac{1}{\delta_0^2} + \frac{1}{\delta_1^2} \right) \pm \frac{P}{\delta_0^2} \pm (1-p) / \delta_0^2 = \frac{1}{\delta_0^2}, \quad \frac{(2p-1)}{4} \frac{1}{\delta_0^2}$$

$$t \in [t_0, t_{\delta_1}] = S, \quad T = p^2 ((\gamma_0 - \gamma_1) / (\delta_0 + \gamma_1 s)) + (1-p)^2 ((\gamma_1 - \gamma_0) / (\delta_0 + \gamma_0 s))$$
$$= (\gamma_0 - \gamma_1) \delta_0^{-1} (2p-1) + s(p^2 \gamma_1 + (1-p)^2 \gamma_0),$$

Andreas

$$\begin{aligned}
 n^a - 1 &= (n^{(a,b)} - 1)(1 + n^{a-b} + n^{2(a-b)} + \dots + n^{(a,b)-1}), \quad a' := \frac{a}{\text{lcm}(a,b)}, \quad n^{(a,b)-1} \mid (n^a - 1, n^b - 1). \quad \text{If } d \mid (n^a - 1, n^b - 1), \\
 d \mid (n^a - n^b) &\Rightarrow (n^{(a,b)a'} - n^{(a,b)b'}) = (n^{a'} - n^{b'}) (n^{a'((a,b)-1)} + n^{a'((a,b)-2)} n^{b'} + \dots + n^{b'((a,b)-1)}) \\
 &= (n^{a'} - n^{b'}) (n^{a-a'} + n^{a-2a'+b'} + n^{a-3a'+2b'} + \dots + n^{b-b'}), \quad (\text{because } a'-1 = b'-1 = \text{lcm}(a,b)) \sum_{j=0}^{a'-1} n^{jb'} \\
 n^b - 1 &= (n^a - 1) \sum_{j=0}^{a'-1} n^{jb'}, \quad (b > a) \quad \notin n^b - 1 - (n^a - 1) = (n^a - 1) \sum_{j=a'}^{a'-1} n^j, \quad \cancel{\text{cancel}} \sum_{j=a'}^{a'-1} n^j \\
 (a', b') = 1, \quad a'x + b'y = 1, \quad &\stackrel{1+n+n^2}{1+n+n^2+n^3+n^4} \stackrel{1+n+n^2}{1+n}, \quad 1 = (\sum_{j=0}^{a'-1} n^j) (-n) + \sum_{j=0}^{a'-1} n^j = 1 - n \sum_{j=0}^{a'-1} n^j + \\
 (n^a - 1) \sum_{j=0}^{a'-1} n^j, \quad S_{a'-1} := \sum_{j=0}^{a'-1} n^j, \quad \alpha S_{a'-1} + \beta S_{b'-1} = ? \quad 1, \quad \alpha, \beta \in \mathbb{Z}, \quad (a'x - b'y) = 1, \\
 \left(\sum_{j=0}^{a'-1} n^{ja'}\right) S_{a'-1} &= \left(\sum_{j=0}^{a'-1} n^{ja'}\right) \sum_{j=0}^{a'-1} n^j = \sum_{j=0}^{a'-1} n^j, \quad \left(\sum_{j=0}^{a'-1} n^{jb'}\right) S_{b'-1} = \sum_{j=0}^{a'-1} n^{jb'} = \sum_{j=0}^{a'-1} n^j, \quad \cancel{\text{cancel}} \\
 \left(\sum_{j=0}^{a'-1} n^{ja'}\right) S_{a'-1} - n \left(\sum_{j=0}^{a'-1} n^{jb'}\right) S_{b'-1} &= 1 \quad // \text{new } g = (a, b), \quad a = ga', \quad b = gb', \quad (a', b') = 1, \\
 (2^a + 1, 2^b + 1) &= (2^g + 1) \sum_{j=0}^{b'-1} 2^{gj}, \quad 2^b + 1 = (2^g + 1) \sum_{j=0}^{a'-1} 2^{gj}, \quad (\# 727) \Rightarrow \alpha, \beta \in \mathbb{Z}, \\
 \alpha \sum_{j=0}^{a'-1} 2^j + \beta \sum_{j=0}^{b'-1} 2^j &= 1, \quad \Rightarrow 2^b + 1 = \alpha (2^a + 1) + \beta (2^b + 1), \quad (2^a + 1, 2^b + 1) \mid (2^b + 1) \\
 (2^a + 1, \quad b = 2^k b'' \quad \&, \quad 2^b + 1) &= (2^g + 1) \sum_{j=0}^{a'-1} (-2^g)^j, \quad 2^b + 1 = (2^{2g} + 1) \sum_{j=0}^{b'-1} (-2^{2g})^j \\
 \alpha, \beta \in \mathbb{N} : \alpha (2^{2g} + 1) \sum_{j=0}^{a'-1} (-2^{2g})^j &= (2^g + 1) \left\{ \beta \sum_{j=0}^{b'-1} (-2^g)^j - 1 \right\}, \quad 2^{2g+1} \cancel{q^{2g+1}} 2^k. \\
 = \sum_{j=0}^{a'-1} 2^j + 1 &= \sum_{j=1}^{b'-1} 2^j + 2, \quad (\alpha = (2^g + 1)) ? \beta = (2^{2g} + 1) \sum_{j=0}^{b'-1} (-2^{2g})^j = \beta \sum_{j=0}^{a'-1} (-2^g)^j - 1, \\
 \sum_{j=0}^{b'-1} (-1)^j 2^{2g(j+1)} + \sum_{j=0}^{a'-1} (-1)^j (2^{2g})^j &= - \sum_{j=1}^{b'-1} (-1)^j 2^{2gj} + \sum_{j=0}^{a'-1} (-1)^j (2^{2g})^j \\
 &= 1 - (-2^{2g})^{b''} = ? \quad \beta \sum_{j=0}^{a'-1} (-2^g)^j - 1, \quad (-2^{2g})^{b''} = ? \quad \beta \sum_{j=0}^{a'-1} (-2^g)^j, \quad \cancel{\text{cancel}}
 \end{aligned}$$

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$$L(\mu) = (2\pi)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right), \quad \frac{L(\mu_0)}{L(\mu_n)} = \exp\left(\frac{1}{2\sigma^2} \sum [(x_i - \mu_n)^2 - (x_i - \mu_0)^2]\right)$$

$$\exp\left(\frac{1}{2\sigma^2} \sum (2x_i(\mu_0 - \mu_n) + \mu_0^2 - \mu_n^2)\right) < k, \quad 2(\mu_0 - \mu_n)n\bar{x} + n(\mu_0^2 - \mu_n^2) < k$$

$$\bar{x} < k, \quad \alpha = P_{\mu=10}(\bar{x} < k) = P\left(z < \frac{k-10}{\sigma/\sqrt{n}}\right), \quad k = 10 + \frac{\epsilon}{\sqrt{n}} \approx 0.025, \quad P_{\mu=5}(\bar{x} < 10 - \frac{\epsilon}{\sqrt{n}} t_\alpha) = P\left(z < \frac{5 - 10 - \epsilon t_\alpha}{\sigma/\sqrt{n}}\right) = P(z < \sqrt{n} - t_\alpha), \quad \frac{t(0.018)}{0.01} \approx \chi^2_{0.05}$$

$$\left(\frac{\tau_n}{\sigma_0}\right)^n \exp\left(-\sum (x_i - \mu)^2 \left(\frac{1}{2\sigma_n^2} - \frac{1}{2\sigma_0^2}\right)\right) < k, \quad \sum (x_i - \mu)^2 \left(\frac{1}{2\sigma_n^2} - \frac{1}{2\sigma_0^2}\right) < k$$

$$\sum (x_i - \mu)^2 > k, \quad \chi^2(n) > k, \quad \chi^2_{\alpha}, \quad \theta^{2N_1} (2\theta(1-\theta))^{N_2} (1-\theta)^{2N_3}$$

~~$$L(\theta) = \frac{\theta^{2N_1}}{(2\theta(1-\theta))^{N_2} (1-\theta)^{2N_3}}$$~~

$$L(\theta) = 2^{N_2} \theta^{2N_1+N_2} (1-\theta)^{N_2+2N_3}, \quad L(\theta) = \left(\frac{\theta}{\theta_n}\right)^{2N_1+N_2} \left(\frac{1-\theta}{1-\theta_n}\right)^{N_2+2N_3} < k$$

treatment

$$\text{first continue to } \theta_n \text{ s.t. } (2N_1+N_2) \log\left(\frac{\theta}{\theta_n}\right) + (N_2+2N_3) \log(1-\theta) < k$$

$$\text{strike with DAs not linear n's} \quad L(\lambda) = e^{-n\lambda} \frac{\lambda^n}{n!}, \quad \frac{dL(\lambda)}{d\lambda} = e^{n(\lambda-\lambda_0)} \left(\frac{\lambda_0}{\lambda}\right)^n < k, \quad \bar{\lambda} \log(\lambda) < k, \quad \bar{\lambda} > k,$$

$$\alpha = P(\sum x_i > k), \quad \left(\frac{p^4}{2^6}\right) p^{26} (1-p)^{48} \left(\frac{14}{53}\right) p^{53} (1-p)^{94} \dots = (1-p^2)^2 (1-p)^4 \dots (2\pi)^{3/2} (\sigma_1^2 \sigma_2^2 \dots)$$

$$e^{-\frac{1}{2\sigma_1^2} \sum (x_i - \mu_1)^2 - \frac{1}{2\sigma_2^2} \sum (x_i - \mu_2)^2 - \frac{1}{2\sigma_3^2} \sum (x_i - \mu_3)^2}, \quad L(\theta_0) = (2\pi)^{-3/2} \sigma_1^{-3}$$

$$\exp\left(-\frac{1}{2\sigma^2} (\hat{\Sigma}_1 + \hat{\Sigma}_2 + \hat{\Sigma}_3)\right), \quad \text{where } \hat{\Sigma}_j = -\frac{3}{2} \Rightarrow \hat{\Sigma}_j \sigma^2 - \frac{1}{2\sigma^2} (\hat{\Sigma}_1 + \hat{\Sigma}_2 + \hat{\Sigma}_3),$$

$$\frac{\partial \log}{\partial \sigma^2} = -\frac{3}{2\sigma^2} + \frac{1}{2\sigma^4} (\Sigma - \dots) = 0, \quad \sigma^{-2} = 3/(\Sigma - \dots), \quad \hat{\sigma}^2 = \frac{1}{3} (\Sigma + \dots), \quad \frac{\partial \log}{\partial \mu_j} = \frac{1}{2\sigma^2} \sum (x_i - \mu_j) \cdot 0,$$

$$u_1 = \bar{x}, \quad L(\theta) =$$

$$\frac{1}{\|u\|^2} (u, b) = u_1^2, \quad \lambda(A) = 1, \quad A^2 = uu^Tuu^T = A \quad \text{if } uu^T = \begin{pmatrix} 9 & 3 & 3 \\ 3 & 9 & 3 \\ 3 & 3 & 9 \end{pmatrix}, \quad P = \begin{pmatrix} 5/9 & -1/9 & -1/9 \\ -1/9 & 5/9 & -1/9 \\ -1/9 & -1/9 & 8/9 \end{pmatrix}$$

$$P_b = \begin{pmatrix} 5/9 & -1/9 & -1/9 \\ -1/9 & 5/9 & -1/9 \\ -1/9 & -1/9 & 8/9 \end{pmatrix}, \quad \delta(P) = \{0, 1, 2\}, \quad P^2 = I - 2uu^T + uu^Tuu^T = P$$

$$H = \begin{pmatrix} 4 & 4 & 5/9 \\ 4 & 4 & 5/9 \\ 5/9 & 5/9 & 7/9 \end{pmatrix}, \quad \delta(H) = \{1, 1, -1\}, \quad H^2 = I - 4uu^T + 4uu^Tuu^T = I \quad \text{if } R(H) = C \cdot (1, 1),$$

$$N(A) = (1, 1, 2)^\perp, \quad R(A^T) = \{(2, 1, 1)\}, \quad N(A^T) = (1, 1)^\perp = C \cdot (1, -1)$$

$$A^2 A = w v^T v w^T = (v^T v) w w^T, \quad A^2 A w = \|v\|^2 \|w\|^2 w \quad \text{if } I = BB^{-1} = I - (c+1)vw^T$$

$$+ cvw^Tvw^T = I + (-c-1 + cw^Tv)vw^T, \quad c(w^Tv - 1) = 1, \quad c = (w^Tv - 1)^{-1}$$

$$BB^{-1} = I - cvw^TA^{-1} - vw^TA^{-1} + cvw^TA^{-1}vw^TA^{-1} = I + (-c-1)vw^TA^{-1} +$$

$$cw^TA^{-1}vw^TA^{-1} = I + vw^TA^{-1}(-c-1 + cw^TA^{-1}v), \quad c = (w^TA^{-1}v - 1).$$

~~$$v = w = (1, 0), \quad c = \bar{a}_{11} - 1, \quad c \left( \begin{pmatrix} \bar{a}_{11} \\ \bar{a}_{21} \end{pmatrix} \right) \left( \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \end{pmatrix} \right) \approx 6.85$$~~

$$\lambda(A) = 0, \quad \{1, -1\}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \{1, 1\}, \quad \{1, 1\}, \quad \begin{pmatrix} 1 & 1 \\ 1 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ -c & a \end{pmatrix} \frac{1}{ad-bc}$$

$$= \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \begin{pmatrix} d & b \\ -c & a \end{pmatrix} \frac{1}{ad-bc} = \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} \cdot \frac{1}{\Delta}, \quad a^2 = ac, \quad a = c, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$1-b = -1, \quad M = \begin{pmatrix} 1 & b-1 \\ 1 & 0 \end{pmatrix} \frac{1}{\Delta}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \cdot -1 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -a & a \\ -a & a \end{pmatrix} \cdot \frac{1}{d-b}, \quad a=c=-1, \quad d=1, \quad b=0, \quad \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \cdot -1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = A \quad \text{??}$$

$$A^T = (S^T)^{-1} \wedge S^T, \quad (S^T)^{-1} \quad \text{??} \quad A^T \{1, 1, 1\} = 0, \quad N(A^T) = \{(1, 1, 1)\}, \quad Q(A^T) = \{(1, 0, -1),$$

$$(1, -1, 0) \} \quad \text{??} \quad \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} A, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = A \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & -15/4 \\ 4 & 0 \end{pmatrix} = A \begin{pmatrix} 1 & -8/4 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} r_1 g_{11} & r_2 g_{11} + r_3 g_{21} \\ r_1 g_{21} & r_2 g_{21} + r_3 g_{31} \end{pmatrix} = A, \quad g_1 \propto \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

$$g_2 \propto \begin{pmatrix} 3 \\ 4 \end{pmatrix} \perp \propto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}/5, \quad g_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}/\sqrt{5} = \begin{pmatrix} \sqrt{5} \\ -3\sqrt{5} \end{pmatrix}, \quad r_1 = 5, \quad r_2 \begin{pmatrix} \sqrt{5} \\ 3\sqrt{5} \end{pmatrix} + r_3 \begin{pmatrix} \sqrt{5} \\ -3\sqrt{5} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \quad r_2 = -\frac{1}{3}r_3, \quad \frac{1}{5}r_3 - \frac{3}{5} \cdot 9r_3 = 5, \quad r_3 = \frac{-25}{19}, \quad r_2 = \frac{100}{57}, \quad A = \begin{pmatrix} \frac{3}{5} & \frac{\sqrt{5}}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 5 & \frac{100}{57} \\ 0 & -25 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}^2 = \begin{pmatrix} b_1^2 + b_2^2 & b_1 b_2 + b_2 b_3 \\ b_1 b_2 + b_2 b_3 & b_2^2 + b_3^2 \end{pmatrix}, \quad b_1^2 = b_3^2, \quad b_1 = b_3, \quad b_1^2 + b_2^2 = 25,$$

$$2b_1 b_2 = 20, \quad b_1^2 + \frac{100}{b_1^2} = 25, \quad b_1^4 - 25b_1^2 + 100 = 0, \quad b_1^2 = \frac{1}{2}(25 \pm \sqrt{15}) = 20, 5, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Q} = A B^{-1} = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} \sqrt{5} & -2\sqrt{5} \\ 0 & \sqrt{5} \end{pmatrix} \frac{1}{-15} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} & \frac{5}{\sqrt{5}} \end{pmatrix} \quad \text{??} \quad (3-\lambda)(5-\lambda) = \cancel{25} \cancel{+ 15} + 20g = 95, \quad g = 1, \quad g_1 = (1, 1)$$

$$(0, 1), \quad A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 5^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(25-\lambda)^2 - 400 = \lambda^2 - 50\lambda + 225, \quad \frac{1}{2}(50 \pm \cancel{40}) = 45, 5, \quad \cancel{25} + 25 + 20g = 95, \quad g = 1, \quad g_1 = (1, 1)$$

$$g_2 = (1, -1), \quad A^T A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 45 & 5 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cancel{+} \cancel{A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3\sqrt{10} & -2\sqrt{5} \\ 2\sqrt{5} & \sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = (1, -1)}$$

$$= \cancel{\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2\sqrt{10} & -3\sqrt{10} \\ 3\sqrt{10} & 2\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}} + \cancel{\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 45 & 5 \\ 5 & 1 \end{pmatrix}} = \cancel{\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3\sqrt{10} & -2\sqrt{5} \\ 2\sqrt{5} & \sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}} = \frac{2\sqrt{10}}{\sqrt{10}} \frac{\sqrt{10}}{\sqrt{10}}$$

$$AA^T = \begin{pmatrix} 9 & 12 \\ 12 & 9 \end{pmatrix}, \quad (9-\lambda)(41-\lambda) - 144 = \lambda^2 - 50\lambda + 225 = (\lambda-45)(\lambda-5), \quad 9+12g = 45, \quad g_1 = (1, 3)$$

$$g_2 = (1, -1, 3), \quad A = ? \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 45 & 5 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 45 & 5 \\ 5 & 1 \end{pmatrix} \frac{1}{\sqrt{10}} = 1$$

$$A = ? \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ \sqrt{5} & -\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ \sqrt{5} & -\sqrt{5} \end{pmatrix} = \frac{1}{2\sqrt{5}} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ \sqrt{5} & -\sqrt{5} \end{pmatrix} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ \sqrt{5} & -\sqrt{5} \end{pmatrix} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2\sqrt{5}} = \frac{1}{20} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$