$\odot$

$\mathbb{E}(Y_a \mid L) - \mathbb{E}(Y_a) =: U(a,L)$, $\mathbb{E}(U(a,L)) = 0$, $\mathbb{E}(Y \mid A, L) - \mu(A) =: U(A,L)$. $U(a,L)$

$= L^a - \mathbb{E}(L^a)$, $U(A,L) = L^A - \mathbb{E}(L^a)|_{a=A}$. $\mathbb{E}(Y_a \mid a_1, \bar{L}_2) - \mathbb{E}(\mathbb{E}(Y_a \mid a_1, \bar{L}_2) \mid a_1, L_1)$

$\mathbb{E}(Y \mid A, L) = \mathbb{E}\left(\sum_a Y_a \{A=a\} \mid A, L\right) = \sum_a \{A=a\} \mathbb{E}(Y_a \mid L)$, $\mathbb{E}(Y \mid A, L) - \mu_\beta(A) = \sum_a \{A=a\}(\mathbb{E}(Y_a \mid L) - \mu_\beta(a))$

$=: \sum_a \{A=a\} U_a(L)$, $\mathbb{E}(U_a(L)) = 0$, $\mathbb{E}(Y \mid \bar{A}, \bar{L}) - \mu(\bar{A}) = \sum_{\bar{a}} \{\bar{A} = \bar{a}\}(\mathbb{E}(Y \mid \bar{a}, \bar{L}) - \mu(\bar{a}))$

$= \sum_{\bar{a}} \{\bar{A}=\bar{a}\} \sum_{j=1}^J \left(\mathbb{E}(Y_{\bar{a}} \mid \bar{a}_j, \bar{L}_j) - \mathbb{E}(Y_{\bar{a}} \mid \bar{a}_{j-1}, \bar{L}_{j-1})\right) = \sum_{\bar{a}} \{\bar{A}=\bar{a}\} \sum_{j=1}^J \left(\mathbb{E}(Y_{\bar{a}} \mid \bar{a}_j, \bar{L}_j) - \mathbb{E}(\mathbb{E}(Y_{\bar{a}} \mid \bar{a}, \bar{L}_j) \mid \bar{a}_{j-1}, \bar{L}_{j-1})\right)$

$=: \sum_{\bar{a}} \{\bar{A}=\bar{a}\} \sum_{j=1}^J \left(\varphi_{\bar{a}_j}(\bar{L}_j) - \mathbb{E}(\varphi_{\bar{a}_j}(\bar{L}_j) \mid \bar{a}_{j-1}, \bar{L}_{j-1})\right) = \sum_{\bar{a}} \{\bar{A}=\bar{a}\} \sum_{j=1}^J \left(\varphi_{\bar{a}_j}(\bar{L}_j) - \mathbb{E}(\varphi_{\bar{a}_j}(\bar{L}_j) \mid \bar{A}_{j-1},\right.$

$\left.\bar{L}_{j-1})\right)$, $\sum_{\bar{a}} \{\bar{A}=\bar{a}\} \sum_{j=1}^J \left(\varphi_{\bar{a},j}(\bar{A}_{j-1}, \bar{L}_j) - \mathbb{E}(\varphi_{\bar{a},j}(\bar{A}_{j-1}, \bar{L}_j) \mid \bar{A}_{j-1}, \bar{L}_{j-1})\right) =:$

$\sum_{\bar{a}} \sum_{j=1}^J (M_j^{(\bar{a})} - M_{j-1}^{(\bar{a})})$, $M_j^{(\bar{a})} := \sum_{k=1}^j \{\varphi_k^{(\bar{a})}(\bar{A}_{k-1}, \bar{L}_k) - \mathbb{E}(\varphi_k^{(\bar{a})}(\bar{A}_{k-1}, \bar{L}_k) \mid \bar{A}_{k-1}, \bar{L}_{k-1})\}$,

$\mathcal{F}_j := \bigvee_{k=1}^J \{(\bar{A}_{k+1}, \bar{L}_{k+1})\}$, $\mathbb{E}(M_j^{(a)} \mid \mathcal{F}_{j-1}) = M_{j-1}^{(a)}$, $\mathbb{E}(Y \mid \bar{A}, \bar{L}) - \mu(\bar{A})$

$= \sum_{\bar{a}} \{\bar{A}=\bar{a}\} M_J^{(\bar{a})}$

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$\mathbb{E}\left\{\sum_j^J \{\bar{A}=\bar{a}\}\left(\varphi_j(\bar{A}_{j-1}, \overline{ZU}_j) - \mathbb{E}(\varphi_j \mid \overline{UAL}_{j-1})\right) \prod_j \frac{(-1)^{1-z_j}}{f(z_j \mid \overline{AZ}_{j-1}, \bar{L}_j)\Delta(\bar{L}_j, \overline{AZ}_{j-1})}\right\}$

$= \mathbb{E}\left\{\{\bar{A}_{j-1} = \bar{a}_{j-1}\} P(A_j \mid \overline{ZLU}_j, \overline{AZ}_{j-1}) \sum_j (\ldots) \prod (\ldots)\right\} = \mathbb{E}\left\{\{\bar{A}_{j-1} = \bar{a}_{j-1}\} \sum_j (\varphi_j - \mathbb{E}\varphi_j) \prod_j^{J-1} \frac{(-1)^{1-z_j}}{f(z_j \mid \ldots)\Delta(\ldots)}\right\}$

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$\mathbb{E}\left(\frac{Y - \mu_\beta(A)}{\prod_j (-1)^{1-z_j} \Delta_j(A_j \mid \bar{z}_{j-1}, \bar{A}_{j-1}, \bar{L}_j) f_z(z_j \mid \bar{L}_j, \overline{AZ}_{j-1})}\right) = \mathbb{E}\left(\sum_a \{\bar{A}=a\}\left(\frac{Y_{\bar{a}} - \mu_\beta(\bar{a})}{\ldots}\right)\right)$, $z_j \perp Y_{\bar{a}} \mid \bar{U}_j, \overline{ZA}_{j-1}, \bar{L}$

$\mathbb{E}\left(\sum_{\bar{a}} \{\bar{A} = \bar{a}\} \prod \Delta_j^{-1} w_j^{-1} \mathbb{E}\left(\frac{Y_{\bar{a}} - \mu_\beta(\bar{a})}{(-1)^{1-z_j} f_z(z_j)} \mid \overline{ALU}_j\right)\right)$, $= \mathbb{E}\left(\sum_{\bar{a}} \{\bar{A}=\bar{a}\} \frac{\mathbb{E}(Y_{\bar{a}} - \mu_\beta(\bar{a}) \mid \overline{AZLU}_j)}{\ldots}\right)$

$\mathbb{E}\left(\{\bar{A}=\bar{a}\}\frac{Y_{\bar{a}}}{W}\right) = \mathbb{E}\left(\{\bar{A}_{J-1} = \bar{a}_{J-1}\}(\bar{W}_{J-1}\bar{\Delta}_J)^{-1}\right)$, $\mathbb{E}\frac{\{\bar{A}=\bar{a}\}Y_{\bar{a}}}{\bar{W}_J} = \mathbb{E}\frac{\{\bar{A}=\bar{a}\}\mathbb{E}(Y_{\bar{a}} \mid \overline{AZLU}_J)}{\bar{W}_J}$

$= \mathbb{E}\left\{\frac{\{\bar{A}_{J-1} = \bar{a}_{J-1}\} P(A_J = a_J \mid \bar{A}_{J-1}, \overline{ZLU}_J)\mathbb{E}(Y_{\bar{a}} \mid \bar{A}_{J-1}, \overline{ZLU}_J)}{\bar{W}_J}\right\} = \mathbb{E}\frac{\{\bar{A}_{J-1} = \bar{a}_{J-1}\}\mathbb{E}(Y_{\bar{a}} \mid \overline{AZ}_{J-1}, \overline{LU}_J) P(A_J = a_J \mid \bar{A}_{J-1}, \overline{ZLU}_J)}{\bar{W}_{J-1}(-1)^{1-z_J} f(z_J \mid \bar{A}_{J-1}, \bar{z}_{J-1}, \bar{L}_J)\Delta_J(\overline{ZLU}_J, \overline{AZ}_{J-1})}$

$= \mathbb{E}\frac{\{\bar{A}_{J-1} = \bar{a}_{J-1}\}\mathbb{E}(Y_{\bar{a}} \mid \overline{AZ}_{J-1}, \overline{LU}_J)}{\bar{W}_{J-1}\Delta(\bar{L}_J, \overline{AZ}_{J-1})}(-1)^{1-a_J}\Delta^{\vee}(\overline{LU}_J, \overline{AZ}_{J-1})$

(2)

$$\mathbb{E}\left(\frac{Y}{\overline{W}}\right) = \sum_{\bar a} \mathbb{E}\left(\{\bar A = \bar a\}\frac{Y_{\bar a}}{\overline{W}}\right), \quad \mathbb{E}\left(\{\bar A = \bar a\}\frac{Y_{\bar a}}{f_{z_j}\Delta_j(-1)^{1-z_j}}\overline{W}_{j-1}^{-1}\right) = \mathbb{E}\left(\{\bar A = \bar a\}\frac{\mathbb{E}(Y_{\bar a}\mid \bar A_{j-1}, \bar L, \bar U)}{f_{z_j}\Delta_j(-1)^{1-z_j}}\overline{W}_{j-1}^{-1}\right)$$

$$= \mathbb{E}\left(\{\bar A_{j-1} = \bar a_{j-1}\}\overline{W}_{j-1}^{-1}\frac{\mathbb{P}(A_j = a_j \mid \bar A_{j-1}, \overline{LUz_j})\,\mathbb{E}(Y_{\bar a}\mid \bar A_{j-1}, \overline{LU})}{f_{z_j}\Delta_j(-1)^{1-z_j}}\right) = \mathbb{E}\left(\{\bar A_{j-1} = \bar a_{j-1}\}\overline{W}_{j-1}^{-1}\,\mathbb{E}(Y_{\bar a}\mid \bar A_{j-1}, \overline{LU})\right)$$

$$= \mathbb{E}\left(\{\bar A_{j-1} = \bar a_{j-1}\}\overline{W}_{j-1}^{-1}\,\mathbb{E}(Y_{\bar a}\mid \overline{ALU}_{j-1})\right) = \mathbb{E}\left(\{\bar A_{j-2} = \bar a_{j-2}\}W_{j-2}^{-1}\frac{\{\bar A_{j-1} = a_{j-1}\}\mathbb{E}(Y_{\bar a}\mid \overline{ALU}_{j-1})}{(-1)^{1-z_{j-1}}f_{z_{j-1}}\Delta_{j-1}}\right)$$

$$= \mathbb{E}\left(\{\bar A_{j-2} = \bar a_{j-2}\}W_{j-2}^{-1}\frac{\mathbb{P}(A_{j-1} = a_{j-1}\mid \bar A_{j-2}, \overline{ZLU}_{j-1})\,\mathbb{E}(Y_{\bar a}\mid \bar A_{j-2}, \overline{LU}_{j-1})}{(-1)^{1-z_{j-1}}f_{z_{j-1}}\Delta_{j-1}}\right) = \mathbb{E}\left(\{\bar A_{j-2} = \bar a_{j-2}\}\overline{W}_{j-2}^{-1}\,\mathbb{E}(Y_{\bar a}\mid \overline{ALU}_{j-2})\right)$$

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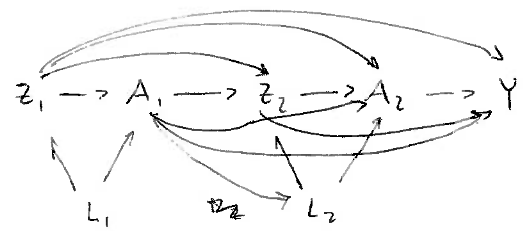


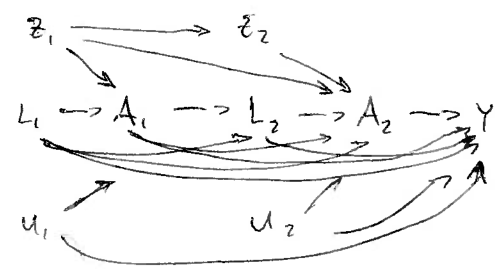


MS — use this version of DAG

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MS bias calculations

$$Y = A\beta + L + U + \varepsilon, \quad \hat\beta_{OLS} = (A^TA)^{-1}A^TA\beta + (A^TA)^{-1}(L+U+\varepsilon),$$

$$\mathbb{E}(\hat\beta_{OLS} - \beta) = \mathbb{E}((A^TA)^{-1}(L+U)), \quad \hat\beta_{SRA} = (h^TW_{SRA}A)^{-1}h^TW_{SRA}Y = \left(\sum_j h_j\frac{A_j}{f(A_j\mid \bar L_j \bar A_{j-1})}\right)^{-1}\sum_j h_j\frac{A_j\beta + L_j + U_j + \varepsilon_j}{f(A_j\mid \bar L_j \bar A_{j-1})}$$

$$= \beta + \left(\sum_j h_j\frac{A_j}{f(A_j\mid \bar L_j\bar A_{j-1})}\right)^{-1}\sum_j h_j\frac{L_j + U_j + \varepsilon_j}{f(A_j\mid \bar L_j\bar A_{j-1})} \qquad \mathbb{E}\left(h_j\frac{L_j}{f(A_j\mid \bar L_j\bar A_{j-1})}\right) = \mathbb{E}\left(L_j\,\mathbb{E}\left(\frac{h_j(A_j)}{f(A_j\mid \bar L_j, \bar A_{j-1})}\,\Big|\, \bar L_j, \bar A_{j-1}\right)\right) :$$ remove one from product

$$v(h_j(A_j))\,\mathbb{E}(L_j) = 0, \quad \mathbb{E}\left(h_j\frac{U_j}{f(A_j\mid \bar L_j\bar A_{j-1})}\right) = \mathbb{E}\left(h_j\frac{U_j - \mathbb{E}(U_j\mid \bar L_j \bar A_{j-1}) + \mathbb{E}(U_j\mid \bar L_j, \bar A_{j-1})}{f(A_j\mid \bar L_j, \bar A_{j-1})}\right) = \cancel{v}h(A)) \cdot 0$$

$$+\ \mathbb{E}\left(h_j\frac{U_j - \mathbb{E}(U_j\mid \bar L_j, \bar A_{j-1})}{f(A_j\mid \bar L_j, \bar A_{j-1})}\right) \qquad \mathbb{E}\left(h(A)\frac{L}{\Delta f_z(z)(-1)^{1-z}}\right) = \sum_a \mathbb{E}\left(\{A = a\}h(a)\frac{L}{\Delta f_z(-1)^{1-z}}\right)$$

$$= \sum_a \mathbb{E}\left(\mathbb{P}(A = a\mid L, z, U)\,h(a)\frac{L}{\Delta f_z(-1)^{1-z}}\right) \overset{(\text{IV indep})}{=} \sum_a \mathbb{E}\left(\frac{L}{\Delta(L)}\left(\mathbb{P}(A=a\mid L,U,z=1) - \mathbb{P}(A=a\mid L,U,z=0)\right)\right) \overset{(U\perp z\mid L)}{=} \sum_a \mathbb{E}(L) = 0,$$

$$\sum_a \mathbb{E}\left(\frac{U}{\Delta(L)}\left(\mathbb{P}(A=a\mid L,U,z=1) - \mathbb{P}(A=a\mid L,U,z=0)\right)\right) \overset{(\text{ICT})}{=} 0, \quad \mathbb{E}(Y_a\mid L,U) - \mathbb{E}(Y_0\mid L,U) \perp U \Rightarrow$$

$$\mathbb{E}(Y\mid A=1, L, U) - \mathbb{E}(Y\mid A=0, L, U) = \beta + U = \beta.$$ Assu.: $\mathbb{E}(Y_{a_2=1} - Y_{a_2=0}\mid A_1, \bar L_2, \bar z_2, \bar U_2)$

$$= f(A_1, \bar L_2, \bar z_2); \quad \mathbb{E}(Y_{a_2=1} - Y_{a_2=0}\mid A_1, \overline{LZU}_2) = \sum_{a_2}\mathbb{E}\left((-1)^{1-a_2}Y\mid \overline{LZU}_2, A_1, A_2 = a_2\right)$$

(3)

$\hat\beta = (h^T W A)^{-1} h^T W Y$, $W = (-1)^{1-z}$, $h = \mathbb{1}$, $\hat\beta = \left(\sum_j A_j (-1)^{1-z_j}\right)^{-1} \sum (-1)^{1-z_j} Y_j = \dfrac{\sum Y_j\{z_j=1\} - \sum Y_j\{z_j=0\}}{\sum A_j\{z_j=1\} - \sum A_j\{z_j=0\}}$ Wald estimator

$W^{-1} = (-1)^{1-z} \Delta(L)$

$\frac{1}{n}\sum_j h(A_j) \frac{L_j}{f(A_j \mid L_j)}$, $\mathbb{P}(A_j = 1 \mid L) = \sigma(\alpha_0 + \alpha_1 L) = \frac{e^{\alpha_0+\alpha_1 L}}{1+e^{\alpha_0+\alpha_1 L}}$, $\mathbb{P}(A=1\mid L) = \Phi(\alpha_0+\alpha_1 L)$, $f(A\mid L)$

$= \Phi(\alpha_0+\alpha_1 L)^A (1-\Phi(\alpha_0+\alpha_1 L))^{1-A}$, $\mathrm{Var}\left(\frac{L}{f(A\mid L)} \middle| L = \ell\right) = \ell^2 \sum_a \frac{f(a\mid\ell)}{f(a\mid\ell)^2} - \left(\ell \sum_a 1\right)^2 = \ell^2 \sum_a \frac{1}{f(a\mid\ell)} - \ell^2 |A|^2$

$\mathbb{E}\left(\frac{L}{f(A\mid L)} \middle| L=\ell\right) = \ell|A|$, $\mathrm{Var}\,\mathbb{E}(\cdot\cdot \mid L) = |A|^2 \mathrm{Var}\, L$, $\sum_a \mathbb{E}\left(\frac{L^2}{f(a\mid L)}\right)$, $\int \frac{x^2}{\Phi(\alpha_0+\alpha_1 x)} \varphi(x)\,dx$

$\approx \int \frac{x^2 \varphi(x)}{1-\varphi(\alpha_0+\alpha_1 x)}dx$, $g(x) - g(\mu) \approx g'(\mu)(x-\mu) + \frac{g''(\mu)}{2}(x-\mu)^2$, $\mathbb{E}\, g(x) \approx g(\mu) + \frac{g''(\mu)}{2}\mathrm{Var}\, X$,

$g: \ell \mapsto \frac{\ell^2}{\sigma(\alpha_0+\alpha_1\ell)}$, $g'(\ell) = \frac{2\ell\sigma(\alpha_0+\alpha_1\ell) - \ell^2\sigma'(\alpha_0+\alpha_1\ell)\alpha_1}{\sigma^2(\alpha_0+\alpha_1\ell)} = \frac{2\ell}{\sigma^2(\alpha_0+\alpha_1\ell)} - \alpha_1 \ell^2 (\ )$, $g'' = \frac{2\sigma^2(\alpha_0+\alpha_1\ell)}{\sigma^4(\alpha_0+\alpha_1\ell)}\Big|_{\ell=0}$

$= \frac{2}{\sigma^2(\alpha_0)}$, $\mathbb{E}\, g(L) \approx \frac{2}{\sigma^2(\alpha_0)} \mathrm{Var}\, L$, $g: \ell \mapsto \frac{\ell^2}{f^2(a\mid\ell)}$, $g'(\ell) = \frac{2\ell}{f^2(a\mid\ell)}$, $g''(\ell) = \frac{2}{f^2(a\mid\ell)}$,

$\mathbb{E}\frac{L^2}{f(a\mid L)^2} \approx \frac{1}{f^2(a\mid 0)} \mathrm{Var}\, L$ ✓ #29, $g'(\ell) = \frac{2\ell}{f^2(a\mid\ell)} - \frac{2\ell^2 f'(a\mid\ell)}{f^3(a\mid\ell)}$, $g''(\ell) = \frac{2}{f^2(a\mid\ell)} - \frac{4\ell f'(a\mid\ell)}{f^3(a\mid\ell)} - \frac{4\ell f' + 2\ell f''}{f^3}$

$+ \frac{6\ell^2(f')^2}{f^4}$, $g'''(\ell) = -\frac{4f'}{f^3} - 4\frac{f'(a\mid\ell)+\ell f''}{f^3} - 12\frac{\ell(f')^2}{f^4} - \frac{4f'}{f^3} - 12\frac{\ell(f')^2}{f^4} - 2\frac{f''}{f^3} - 6\frac{\ell f' f''}{f^4}$

$+ \left(\frac{6\ell^2(f')^2}{f^4}\right)'$, $g'''(0) = -\frac{4f'}{f^3} - \frac{4f'}{f^3} - \frac{4f'}{f^3} - \frac{2f''}{f^3} = -\frac{12f'}{f^3} - \frac{2f''}{f^3}$, $f = \frac{e^x}{1+e^x}$,

$f' = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} = f\cdot(1-f)$, $f'' = f' - 2ff' = f(1-f) - 2f^2(1-f) = f - 3f^2 + 2f^3$,

$f''' = f' - 6ff' + 6f^2 f' = f(1-f) - 6f^2(1-f) + 6f^3(1-f) = f - 7f^2 + 12f^3 - 6f^4$,

$\mathbb{E}\frac{L^2}{f(a\mid L)^2} \approx \frac{1}{f^2(a\mid 0)}\mathrm{Var}\, L - \left(\frac{2f'}{f^3} + \frac{f''}{3f^3}\right)\mathrm{Var}(L^3) = \frac{1}{f^2(a\mid 0)} - \frac{1}{f^3(a\mid 0)}\left(2f(1-f) + \frac{1}{3}f - f^2 + \frac{2}{3}f^3\right) = \frac{1}{f^2} - \frac{1}{f^3}\left(\frac{7}{3}f - 3f^2 + \frac{2}{3}f^3\right) = -\frac{4}{3}\frac{1}{f^2} + \frac{3}{f} - \frac{2}{3}$, $0 = \mathbb{E}L^3$, $\cancel{g^{(4)}(\ell) = -\frac{4f''}{f^3} + \frac{12(f')^2}{f^4}}$

$g^{(3)}(\ell) = -\frac{12f'}{f^3} - 4\frac{\ell f''}{f^3} - 24\frac{\ell(f')^2}{f^4} - 2\frac{f''}{f^3} - 6\frac{\ell f' f''}{f^4} + \frac{12\ell(f')^2 + 12\ell^2 f' f''}{f^4} - \frac{24\ell^2(f')^2}{f^5}$,

$g^{(4)}(\ell) = -\frac{12f''}{f^3} + \frac{36f'}{f^4} - \frac{4(f''+\ell f''')}{f^3} \cancel{- 48\frac{\ell(f')^2 f''}{f^4}} - 24\frac{(f')^2}{f^4} - 2\frac{f'''}{f^3} + 6\frac{f''}{f^4} - 6\frac{f'f''}{f^4} + \frac{12(f')^2}{f^4} + \ell\cdot f$

4

$g^{(4)}(0) = -16\frac{f''}{f^3} + 36\frac{f'}{f^4} - 12\frac{(f')^2}{f^4} - 2\frac{f'''}{f^3} + 6\frac{f''}{f^4} - 6\frac{f'f''}{f^4} = -16\left(\frac{1}{f^2} - \frac{3}{f} + 2\right)$

$+36\left(\frac{1}{f^3} - \frac{1}{f^2}\right) - 12\left(\frac{1}{f^2} + 1 - \frac{2}{f}\right) - 2\left(\frac{1}{f^2} - \frac{7}{f} + 12 - 6f\right) + 6\left(\frac{1}{f^3} - \frac{3}{f^2} + \frac{2}{f}\right) - 6\left(\frac{1}{f^2} - \frac{3}{f} + 2\right.$

$\left. - \frac{1}{f} + 3 - 2f\right) = \frac{42}{f^3} + \frac{1}{f^2}(-90) + \frac{1}{f}\cdot 122 - 98 + f\cdot 24$, ... $g(\ell) := \frac{\ell^2}{f(a|\ell)}$, $g(\ell) \approx g(0) + \ell g'(0)$

$+\frac{1}{2}\ell^2 g''(0) + \frac{1}{6}\ell^3 g'''(0) + \frac{1}{24}\ell^4 g^{(4)}(0)$, $\mathbb{E}\, g(\ell) \approx \frac{\mathbb{E}(\ell^2)}{2} g''(0) + \frac{1}{24}\mathbb{E}(\ell^4)\, g^{(4)}(0)$, $g' = \frac{2\ell}{f} - \frac{\ell^2 f'}{f^2}$

$g'' = \frac{2}{f} - \frac{2\ell f'}{f^2} - \frac{2\ell f' + \ell^2 f''}{f^2} + \frac{2\ell^2 (f')^2}{f^3} = \frac{2}{f} - \frac{4\ell f'}{f^2} - \frac{\ell^2 f''}{f^2} + \frac{2\ell^2 (f')^2}{f^3}$,

$g''' = -\frac{2f'}{f^2} - \frac{4f' + 4\ell f''}{f^2} + \frac{8\ell (f')^2}{f^3} - \frac{2\ell f'' + \ell^2 f'''}{f^2} + \frac{2\ell^2 f''}{f^3} f' + \frac{4\ell (f')^2 + 4\ell^2 f' f''}{f^3} -$

$\frac{6\ell^2 (f')^3}{f^4} = \frac{-6f' - 6\ell f'' - \ell^2 f'''}{f^2} + \frac{12\ell (f')^2 + 6\ell^2 f'' f'}{f^3} + \frac{6\ell^2 f'}{f^2}$, $g^{(4)}(0) = \frac{-6f'' - 6f''}{f^2}$

$+ \frac{12(f')^2}{f^3} + \frac{12(f')^2}{f^3} + \alpha\ell = \frac{-12 f''}{f^2} + \frac{24(f')^2}{f^3}$, $\mathrm{Var}\left(\frac{L}{f(A|L)}\right) = |A|^2\, \mathrm{Var}\, L +$

$\sum_a \mathbb{E}\frac{L^2}{f(a|L)} - |A|^2\, \mathrm{Var}\, L = \sum_a \mathbb{E}\left(\frac{L^2}{f(a|L)}\right) \approx \sum_a \left\{ \mathrm{Var}\, L / f(a|0) + \frac{\mathbb{E}(L^4)}{24}\left(\frac{-12 f''(a|0)}{f^2(a|0)} + \frac{24(f'(a|0))^2}{f^3}\right)\right\}$

✓ 29c

$= \sum_a \left\{ \frac{\mathrm{Var}\, L}{f(a|0)} + \mathbb{E}(L^4)\left(\frac{(f'(a|0))^2}{f^3} - \frac{f''(a|0)}{2f^2(a|0)}\right)\right\}$, $f(\ell) := \sigma(\alpha_0 + \alpha_1 \ell)$, $f'(\ell) = f(1-f)\alpha_1$,

$f'' = \alpha_1(\alpha_1 f(1-f) - 2f\alpha_1 f(1-f)) = \alpha_1^2(f - 3f^2 + 2f^3)$, $\mathrm{Var}\left(\frac{L}{f(a|L)}\right) \approx \frac{1}{f(0)} + \frac{3\alpha_1^2 f(0)^2 (1-f(0))^2}{f(0)^3}$

$- \frac{3\alpha_1^2 (f(0) - 3f^2(0) + 2f^3(0))}{2f^2(0)}$, $g(\ell) \approx g(0) + \frac{\ell^2}{2}\left(\frac{2}{f}\right) + \frac{\ell^3}{6}\left(\frac{-6f'}{f^2}\right) = g(0) + \frac{\ell^2}{f} - \ell^3 \frac{1-f}{f}\alpha_1$,

$\alpha_1^2 \ell^4 \left(\frac{(1-f)^2}{f} - \frac{1}{2f} + \frac{3}{2} - f\right)$, $g'''(0) = -\frac{6f'}{f^2} = -6\alpha_1 \frac{1-f}{f}$, $f_0(\ell) = 1 - \sigma(\alpha_0 + \alpha_1 \ell)$, $f_0' = -\alpha f(1-f)$

$f_0'' = -\alpha(-\alpha f(1-f) + 2\alpha f^2(1-f)) = \alpha^2(f - 3f^2 + 2f^3)$.

$\mathbb{E}\frac{L}{f(A|L)} = \mathbb{E}\left(\mathbb{E}\frac{L}{f(A|L)} \,|\, L=\ell\right) = \mathbb{E}(L|A|) = 0$, $\mathbb{E}\frac{LA}{f(A|L)} = \mathbb{E}(\mathbb{E}(\cdots|L)) = \mathbb{E}\left(L\sum_a \frac{a}{f(a|L)}\right) = \sum_a a\,\mathbb{E}\left(\frac{L}{f(a|L)}\right) = 0$.

$\mathrm{Cov}\left(\frac{L}{f(A|L)}, \frac{A}{f(A|L)}\right) = 0$ 29d, $\mathbb{E}\frac{L/f(A|L)}{\sum_j A_j/f(A_j|L_j)} = \sum_a \frac{\ell/f(a_1|\ell_1)}{\sum_j a_j/f(a_j|\ell_j)} \prod_j f(a_j|\ell_j)$, $\mathbb{E}\frac{A}{f(A|L)}$

$\mathbb{E}\left(\frac{A}{f(A|L)}|L\right) = \sum_a a$, $\mathrm{Var}\left(\frac{A}{f(A|L)}|L=\ell\right) = \mathbb{E}\left(\left|\frac{A}{f(A|\ell)}\right|^2 | L=\ell\right) - \mathbb{E}\left(\frac{A}{f(A|L)}|L\right)^2 = \sum_a \frac{a^2}{f(a|L)} - \left(\sum_a a\right)^2$

$\mathrm{Var}\left(\frac{A}{f(A|L)}\right) = \sum_a a^2\, \mathbb{E}\left(\frac{1}{f(a|L)}\right) - \left(\sum_a a\right)^2$, $\frac{1}{f(L)} \approx \frac{1}{f(0)} + L\left(-\frac{f'(0)}{f^2(0)}\right) + \frac{L^2}{2}$, $\frac{1}{f}$, $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$

$\left(\frac{1}{f}\right)'' = -\frac{f''}{f^2} + \frac{2(f')^2}{f^3}$, $\left(\frac{1}{f}\right)''' = -\frac{f'''}{f^2} + \cancel{\frac{2f''f'}{f^3}} + \frac{6f'f''}{f^3} - \frac{6(f')^3}{f^4}$, $\left(\frac{1}{f}\right)^{(4)} = -\frac{f^{(4)}}{f^2} + \frac{2f'''f'}{f^3} + \frac{6((f'')^2 + f'f''')}{f^3}$

(5)

$$-\cancel{\frac{18(f')^2 f''}{f^4}} - \frac{\overset{36}{\cancel{18}}(f')^2 f''}{f^4} + \frac{24(f')^4}{f^5} = -\frac{f^{(4)}}{f^2} + \frac{\overset{6(f'')^2 + 8f'''f'}{\cancel{8f''' + 6((f'')^2 + f'f''')}}}{f^3} - \frac{\overset{36(f')^2 f''}{\cancel{16 f' f'' + (f')^2 f''}}}{f^4} + \frac{24(f')^4}{f^5},$$

$f = \sigma(\alpha_0 + \alpha_1 x)$, $f' = f(1-f)\alpha_1$, $f'' = \alpha^2\left(f(1-f) - 2f^2(1-f)\right) = \alpha^2\left(f - 3f^2 + 2f^3\right)$,

$f''' = \alpha^3\left(f - f^2 - 6f^2(1-f) + 6f^3(1-f)\right) = \alpha^3\left(f - 7f^2 + 12f^3 - 6f^4\right)$, $f^{(4)} = \alpha^4\big(f - f^2 - 14f^2(1-f) + 36f^3(1-f) - 24f^4(1-f)\big) = \alpha^4\left(f - 15f^2 + 50f^3 - 60f^4 + 24f^5\right)$ (28a) $f_0 = 1 - f_1$,

$f_0' = -f_1'$, $\left(\text{ar var of } \frac{A}{f(A|L)}\right)$

---

$\mathbb{E}_0(n) = \mathbb{E}\left(\frac{L}{f(A|L)}\right) = \mathbb{E}(\mathbb{E}(\cdot \mid L)) = \mathbb{E}(L \cdot |A|) = 0$, $g(x,y) = \frac{x}{y} \approx \frac{\mu_x}{\mu_y} + g_x(x-\mu_x) + g_y(y-\mu_y) + \frac{1}{2}\cancel{g_{xx}}\overset{0}{}(x-\mu_x)^2 + g_{xy}(x-\mu_x)(y-\mu_y) + g_{yy}(y-\mu_y)^2$,

$\mathbb{E}\left(\left(\frac{x}{y}\right)^2\right) \approx \left(\frac{\mu_x}{\mu_y}\right)^2 + g_x^2 \operatorname{Var} X + g_y^2 \operatorname{Var} Y + g_{xy}^2\, \cancel{\operatorname{Cov}(X,Y)}\, \mathbb{E}(x-\mu_x)^2(y-\mu_y)^2 + \ldots$,

$\operatorname{Var} \frac{\sum L/f(A|L)}{\sum A/f(A|L)} \approx \frac{\operatorname{Var}\left(\sum \frac{L}{f(A|L)}\right)}{\left(\mathbb{E}\sum \frac{A}{f(A|L)}\right)^2} = \frac{\operatorname{Var}\left(\sum \frac{L}{f(A|L)}\right)}{(n\sum a)^2} = \frac{\operatorname{Var}\frac{L}{f(A|L)}}{n(\sum a)^2}$ (29f) $= \frac{1}{n(\sum a)^2}\sum_a\left(\frac{1}{f(a|0)} + 3\left(\frac{f'(a|0)^2}{f(a|0)^3} - \frac{f''(a|0)}{2f(a|0)^2}\right)\right)$

---

$\operatorname{Var}\frac{h(A)L}{f(A|L)} = \operatorname{Var}\left(\sum_a h(a)L\right) + \mathbb{E}\left(\sum_a \frac{h(a)^2L^2}{f(a|L)}\right) - \left(\sum_a h(a)L\right)^2 = \sum_a h(a)^2\, \mathbb{E}\left(\frac{L^2}{f(a|L)}\right)$

$\operatorname{Var}\frac{\sum h(A)L/f(A|L)}{\sum h(A)A/f(A|L)} \approx \frac{n\operatorname{Var}\left(h(A)L/f(A|L)\right)}{\left(\mathbb{E}\sum h(A)A/f(A|L)\right)^2} = \frac{\sum_a h(a)^2\,\mathbb{E}\left(L^2/f(a|L)\right)}{n\left(\sum h(a)a\right)^2}$, $\frac{h_0^2 e_0 + h_1^2 e_1}{h_1^2} = \frac{h_0^2}{h_1^2}e_0 + e_1$, $\underset{h_0/h_1}{\operatorname{argmin}} = 0$; ~~h(A) = A~~ $h(a) = a$

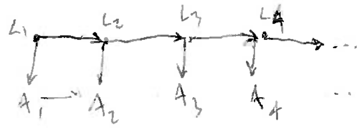
---

should be $L_t - \mathbb{E}(L_t \mid \bar{A}_{t-1}, \bar{L}_{t-1})$

$$\sum_j h(\vec{A}_j) \frac{B^T \vec{L}_j}{\prod_{t=1}^T f(A_j(t) \mid \vec{L}_j(t), \vec{A}_{j-1}(t))}, \quad \mathbb{E}\frac{B^T \vec{L}}{\prod_{t=1}^T f(A_t \mid \vec{L}_t, \vec{A}_{t-1})} = \sum_t \mathbb{E}\frac{B_t L_t}{\prod_{t'=1}^T f(A_{t'} \mid \vec{L}_{t'}, \vec{A}_{t'-1})}$$

$$\mathbb{E}\left(\mathbb{E}\frac{1}{\prod_{t=u}^T f(A_t \mid \vec{L}_t, \vec{A}_{t-1})} \,\Big|\, \vec{L}_T, \vec{A}_{T-1}\right) = \left(\frac{1}{\prod_{t=u}^{T-1} f(A_t \mid \vec{L}_t, \vec{A}_{t-1})} \sum_{a_T} \frac{f(a_T \mid \bar{L}_T, \bar{A}_{T-1})}{f(a_T \mid \bar{L}_t, \bar{A}_{t-1})}\right) = |A_T|\, \mathbb{E}\left(\frac{1}{\prod_{t=u}^{T-1} f(A_t \mid \bar{L}_t, \bar{A}_{t-1})}\right)$$

$$= \ldots = \prod_{t=u}^T |A_t|, \quad \mathbb{E}\frac{L_t}{\prod_{t'=1}^t f(A_{t'} \mid \bar{L}_{t'}, \bar{A}_{t'-1})} = |A_t|\, \mathbb{E}\frac{L_t}{\prod_{t'=1}^{t-1} f(A_{t'} \mid \bar{L}_{t'}, \bar{A}_{t'-1})}$$



$f(\bar{A} \mid \bar{L}) = \prod_t f(A_t \mid L_t)$, $\mathbb{E}\left(\frac{B_t L_t}{\prod_{t'=1}^T f(A_{t'} \mid \bar{L}_{t'}, \bar{A}_{t'-1})} \,\Big|\, \bar{L}_T = \bar{l}\right) = \mathbb{E}\frac{B_t l_t}{\prod_{t'} f(A_{t'} \mid \bar{l}_{t'})} \,|\,$

assumption (DAG) $A_t \not\to L_{t+1}$

$f(\bar{A}_T \mid \bar{L}_T) = f(A_T \mid \bar{A}_{T-1}, \bar{L}_T)\, f(\bar{A}_{T-1} \mid \bar{L}_T) = f(A_T \mid \bar{A}_{T-1}, \bar{L}_T)\, f(\bar{A}_{T-1} \mid \bar{L}_{T-1})$,

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$$\mathbb{E}\left[\frac{L_t - \mathbb{E}(L_t \mid \overline{A}_{t-1})}{\prod_{t'}^{t-1} f(A_{t'} \mid \overline{A}_{t'-1}, \overline{L}_{t'})} \,\middle|\, \overline{L}_{t^*}, A_{t^*-1}\right] = 0, \qquad \mathbb{E}\, \frac{\cancel{B_t L_t B_{t+1} L_{t+1}}\; B_t (L_t - \mathbb{E}(L_t \mid \overline{AL}_{t-1}) \cdot B_{t+1} (L_{t+1} - \mathbb{E}(L_{t+1} \mid \overline{AL}_t))}{\prod_{t'=1}^{T} f(A_{t'} \mid \overline{L}_{t'}, \overline{A}_{t'-1})^2}$$

$$= \mathbb{E}\left(\mathbb{E}\left(\cdots \mid L_T, A_{T-1}\right)\right) = \frac{B_t B_{t+1}(\ )(\ )}{\left(\prod^{T-1} f(\ )\right)^2} \sum_{a_T} \frac{1}{f(a_T \mid \overline{L}_T, \overline{A}_{T-1})}, \qquad \mathbb{E}\, \frac{L_t - \mathbb{E}(L_t \mid A_{t-1})}{\left(\prod_{t'} f(A_{t'} \mid L_{t'})\right)^2}, \quad f(A_{T-1} \mid L_{T-1})^2 \frac{1}{f(a \mid L_T)}$$

$$\begin{array}{cc} L_1 & L_2 \\ \downarrow & \nearrow\ \downarrow \\ A_1 & A_2 \end{array}$$

$$\mathbb{E}(\cdot \mid \overline{L}_T, \overline{A}_{T-1}) \quad \mathbb{E}\, \frac{L_t - \mathbb{E}(L_t \mid A_{t-1})}{\left(\prod_{t'}^{T-1} f(A_{t'} \mid L_{t'})\right)^2} \sum_{a_T} \frac{1}{f(a_T \mid L_T)} \quad \xrightarrow{\mathbb{E}(\cdot \mid L_{T-1}, A_{T-1})} \quad \mathbb{E}\, \frac{L_t - \mathbb{E}(L_t \mid A_{t-1})}{\left(\prod^{T-1} f(A_{t'} \mid L_{t'})^2\right)}\, g(A_{T-1})$$

$$\xrightarrow{\mathbb{E}(\cdot \mid L_{T-1}, \overline{A}_{T-2})} \mathbb{E}\, \frac{\ }{\prod^{T-2}(\ )^2} \cdot \sum_{a_{T-1}} \frac{g(a_{T-1})}{f(a_{T-1} \mid L_{T-1})} \longrightarrow \mathbb{E}\, \frac{L_t - \mathbb{E}(L_t \mid A_{t-1})}{\prod^{T-2} f(A_{t'} \mid L_{t'})^2}\, g(A_{T-2})$$

$$\cdots \longrightarrow \mathbb{E}\, \frac{L_t - \mathbb{E}(L_t \mid A_{t-1})}{\prod^{t} f(A_{t'} \mid L_{t'})^2}\, g(A_t) \xrightarrow{\mid \overline{L}_t, \overline{A}_{t-1}} \mathbb{E}\, \frac{L_t - \mathbb{E}(L_t \mid A_{t-1})}{\prod^{t-1} f(A_{t'} \mid L_{t'})} \sum_{a_t} \frac{g(a_t)}{f(a_t \mid L_t)}$$

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MS: discuss implications

$$\cancel{\mathbb{P}(A=a, Z=z) = \mathbb{P}(A} \quad \mathbb{P}(A=1 \mid Z=1) - \mathbb{P}(A=1 \mid Z=0) = \Delta, \quad \mathbb{P}(A=0 \mid Z=1) - \mathbb{P}(A=0 \mid Z=0) = -\Delta,$$

$$\mathcal{L}(\alpha; A, Z, L) = \mathcal{L}(\alpha; A \mid Z, L)\, \mathcal{L}(Z \mid L)\, \mathcal{L}(L), \quad \cancel{\mathcal{L}(\alpha; A, Z, L) = \mathcal{L}(\alpha, A \mid Z, L) = \mathcal{L}(\alpha, A \mid Z, L)\ \mathcal{L}(\alpha, A, Z \mid L)}$$

$$\cancel{= \mathbb{P}(A=0} \quad \mathcal{L}(\alpha; A \mid Z, L) = \mathbb{P}_v(A=0 \mid Z=0, L)^{\{A=0, Z=0\}} \left(\mathbb{P}_v(A=0 \mid Z=0, L) - \Delta(\alpha)\right)^{\{A=0, Z=1\}}\, \mathbb{P}_v(A=1 \mid Z=0, L)^{\{A=1, Z=0\}}$$

$$\left(\mathbb{P}_v(A=1 \mid Z=0, L) + \Delta(\alpha)\right)^{\{A=1, Z=1\}}, \quad \frac{d}{d\alpha} l(\alpha; A, Z, L) = -\{A=0, Z=1\} \frac{\Delta'(\alpha)}{\mathbb{P}(A=0 \mid Z=0, L) - \Delta(\alpha)} +$$

$$\{A=1, Z=1\} \frac{\Delta'(\alpha)}{\mathbb{P}(A=1 \mid Z=0, L) + \Delta(\alpha)}, \ \cdots \qquad \frac{d\,(Y - \mu_\beta(A) / \overline{w}_{\alpha,\gamma})}{d\,\beta, \alpha, \gamma} = \left(-\overline{w}_{\alpha,\gamma}^{-1} \frac{d}{d\beta} \mu_\beta(A),\ -\frac{Y - \mu_\beta(A)}{\prod_t \left(f_{z_t,\gamma} (-1)^{1-z_t} \Delta_{t,\alpha}^2\right)} \cdot \frac{d}{d\alpha} \prod \Delta_{t,\alpha},\right.$$

$$\left. -\frac{Y - \mu_\beta(A)}{\prod_t \left(f_{z_t;\gamma}^2 \Delta_{t,\alpha} (-1)^{1-z_t}\right)} \cdot \frac{d}{d\gamma} \prod_t f_{z_t;\gamma}\right) = \left(-\overline{w}_{\alpha,\gamma}^{-1} \frac{d}{d\beta} \mu_\beta(A),\ -\frac{Y - \mu_\beta(A)}{\prod_t f_{z_t;\gamma} (-1)^{1-z_t}} \cdot \sum_t \frac{\frac{d}{d\alpha} \Delta_{t,\alpha}}{\Delta_t^2}\right.$$

$$\left. \cdot \prod_{t' \neq t} \frac{1}{\Delta_{t';\alpha}},\ -\frac{Y - \mu_\beta(A)}{\prod_t \Delta_{t,\alpha} (-1)^{1-z_t}} \cdot \sum_t \frac{\frac{d}{d\gamma} f_{z_t;\gamma}}{f_{z_t;\gamma}^2 \prod_{t' \neq t} f_{z_{t'};\gamma}}\right) = \left(-\overline{w}_{\alpha,\gamma}^{-1} \frac{d}{d\beta} \mu_\beta(A),\right.$$

$$\left. -\frac{Y - \mu_\beta(A)}{\overline{w}} \sum_t \frac{\frac{d}{d\alpha} \Delta_{t,\alpha}}{\Delta_t},\ -\frac{Y - \mu_\beta(A)}{\overline{w}} \sum_t \frac{\frac{d}{d\gamma} f_{z_t;\gamma}}{f_{z_t;\gamma}}\right), \quad J = \frac{d}{d\alpha, \beta, \gamma, v} \begin{pmatrix} h'(A) \frac{Y - \mu_\beta(A)}{\overline{w}_{\alpha,\gamma}} \\ l_\alpha \\ l_\gamma \\ l_v \end{pmatrix}$$

$$0 = \begin{cases} h'(A) \dfrac{Y - \mu_\beta(A)}{\overline{w}_{\alpha,\gamma}} \\ l'_{\alpha,1} \\ \vdots \\ l'_{\alpha,p_\alpha} \\ l'_{\gamma,1} \\ \vdots \\ l'_{\gamma,p_\gamma} \\ l'_{v,1} \\ \vdots \\ l'_{v,p_v} \end{cases}$$

$$\mathbb{P}_v(A=1 \mid Z=0, L) = \Phi(v^T(1, L, v))\,(1 - \Delta_\alpha), \ \ldots, \quad \cancel{\mathcal{L}(\alpha, v; A \mid Z, L)}$$

$$\pi_{\alpha,v} = \Phi(v^T X)(1 - \Phi(\alpha^T X)) + z\, \Phi(\alpha^T X), \qquad \pi'_{\alpha,v} = \frac{d\pi}{d\alpha, v} = \left((z - \Phi(v^T X))\, \varphi(\alpha^T X)\, X,\right.$$

$X := (1, L)$

$$\left.(1 - \Phi(\alpha^T X))\, \varphi(v^T X)\, X\right), \quad \pi''_{\alpha,v}$$

$$\pi''_{\alpha,v} = \begin{pmatrix} (z - \Phi(v^T X))\, \varphi'(\alpha^T X)\, X X^T & -\varphi(v^T X)\, \varphi(\alpha^T X)\, X X^T \\ -\varphi(\alpha^T X)\, \varphi(v^T X)\, X X^T & (1 - \Phi(\alpha^T X))\, \varphi'(v^T X)\, X X^T \end{pmatrix}$$

$(-\alpha^T x)$, $(-v^T x)$

(MS: DAG more restrictive than assumptions, just one model of assumptions)

⑦

$\{\bar A_{T-1}=\bar a_{T-1}\}$

$$\mathbb{E}\frac{Y_a}{\bar W_T} = \mathbb{E}\frac{\mathbb{E}(Y_a|\bar A_{T-1},\bar Z\bar L U)\,\mathbb{P}(A_T=a_T|\bar A_{T-1},\bar Z\bar L U)}{\bar W_{T-1}\,(-1)^{1-z_T}\Delta_T\, f_Z(z_T)} = \mathbb{E}\frac{\mathbb{E}(Y_a|\bar A_{T-1},\bar Z_{T-1},\bar L U_T)}{\bar W_{T-1}}\frac{\tilde\Delta_T(U)(-1)^{1-a_T}}{\tilde\Delta_T}$$

$$= \mathbb{E}\frac{\sum_{a_T}(-1)^{1-a_T}\mathbb{E}(Y_{\bar a_{T-1},a_T}|\bar A\bar Z_{T-1},\bar L_T,\cancel{U_T})\,\Delta_T(U)}{\bar W_{T-1}\,\Delta_T} = \sum_{a_T}\mathbb{E}\frac{(-1)^{1-a_T}\mathbb{E}(Y_{\bar a_{T-1},a_T}|\bar A\bar Z_{T-1},\bar L_T)\{\bar A_{T-1}=\bar a_{T-1}\}}{\bar W_{T-1}}$$

$$= \cancel{\sum_{a_T}\mathbb{E}\,(-1)^{1-a_T}} = \sum_{a_T}(-1)^{1-a_T}\mathbb{E}\left(\frac{Y_{\bar a_{T-1},a_T}\{\bar A_{T-1}=\bar a_{T-1}\}}{\bar W_{T-1}}\right) = \sum_{a_T,a_{T-1}}(-1)^{1-a_T}(-1)^{1-a_{T-1}}\mathbb{E}\left(\frac{Y_{\bar a_T}\{\bar A_{T-2}=\bar a_{T-2}\}}{\bar W_{T-2}}\right)$$

---

$s=(s_\beta,s_\alpha,s_\gamma,s_\nu)$, $s_\beta=\begin{pmatrix}1\\h(\bar A)\end{pmatrix}\frac{Y-\mu_\beta(\bar A)}{\bar W_{\alpha,\gamma}}$, $I:=\mathbb{E}\frac{\partial s}{\partial\beta,\alpha,\gamma,\nu}$, $\frac{\partial s_\beta}{\partial\beta,\alpha,\gamma,\nu}=\Big(-\bar W^{-1}\frac{\partial}{\partial\beta}\mu_\beta(\bar A),$

$-\frac{Y-\mu_\beta(\bar A)}{\bar W}\sum_t\frac{\frac{\partial}{\partial\alpha}\Delta_\alpha(t)}{\Delta_\alpha(t)},\ -\frac{Y-\mu_\beta(\bar A)}{\bar W}\sum_t\frac{\frac{\partial}{\partial\gamma}f_{Z_t;\gamma}(z_t)}{f_{Z_t;\gamma}(z_t)}\Big)$, $\cancel{\mathbb{P}_{\alpha,\nu}(A=1|Z,L)=\Phi(\nu^TX)(1-\Phi(\alpha^TX))}$

$\pi_{\alpha,\nu}:=\mathbb{P}_{\alpha,\nu}(A=1|Z,L)=\Phi(\nu^TX)(1-\Phi(\alpha^TX))+Z\Phi(\alpha^TX)$, $X=(1,L)$,

$f_{A|Z,L}(A)=\pi_{\alpha,\nu}^A(1-\pi_{\alpha,\nu})^{1-A}$, $s_{\alpha,\nu}=\left(\frac{A}{\pi_{\alpha,\nu}}-\frac{1-A}{1-\pi_{\alpha,\nu}}\right)\pi'_{\alpha,\nu}$, $\frac{\partial}{\partial\alpha,\nu}s_{\alpha,\nu}=\left(-\frac{A}{\pi_{\alpha,\nu}^2}-\frac{1-A}{(1-\pi_{\alpha,\nu})^2}\right)\pi'^{\otimes 2}_{\alpha,\nu}$

$+\left(\frac{A}{\pi_{\alpha,\nu}}-\frac{1-A}{1-\pi_{\alpha,\nu}}\right)\frac{\partial^2}{\partial\alpha^2\nu^2}\pi_{\alpha,\nu}$, $\frac{\partial}{\partial\alpha,\nu}\pi_{\alpha,\nu}=\big((Z-\Phi(\nu^TX))\phi(\alpha^TX)X,$

$(1-\Phi(\alpha^TX))\phi(\nu^TX)X\big)$, $\frac{\partial^2\pi_{\alpha,\nu}}{\partial(\alpha,\nu)^2}=\begin{pmatrix}(Z-\Phi(\nu^TX))\phi(\alpha^TX)(-\alpha^TX)XX^T, & -\phi(\nu^TX)\phi(\alpha^TX)XX^T\\ -\phi(\alpha^TX)\phi(\nu^TX)XX^T, & (1-\Phi(\alpha^TX))\phi(\nu^TX)(-\nu^TX)XX^T\end{pmatrix}$

| | β | α | γ | ν |
|---|---|---|---|---|
| β | ✓ | ✓ | ✓ | 0 |
| α | 0 | ✓ | 0 | ✓ |
| γ | ~~0~~ | ~~0~~ | ~~0~~ | ~~0~~ |
| ν | 0 | ✓ | 0 | ✓ |

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$\mathbb{E}(g(Y_a))=\int_{\mathbb R}g(y)f_{Y_a}(y)\,dy$, $\mathbb{E}\left(\frac{g(Y\{A=a\})}{f(A|L)}\right)=\int\frac{g(y)}{f(a|\ell)}f_{Y\{A=a\},A,L}(y,a,\ell)\,dy\,da\,d\ell$

$f(Y\{A=a\},A,L)=f(Y\{A=a\}|A=a,L)f(A=a|L)f(L)$, $\cancel{f(Y\{A=a\},L)=\{A=a\}f}$

$f(Y\{A=a\},A,L)=f(Y\{A=a\},A=a,L)=f(Y_a|A=a,L)f(A=a|L)f(L)$, $\mathbb{E}(g(Y_a))=\int_{\mathbb R}\int_{\mathbb R}g(y)f_{Y_a|L}(y|\ell)f_L(\ell)$

$d\mu(y,\ell)$, $f(Y\{A=a\},A,L)=f(Y\{A=a\},A=a,L)\overset{(\text{nuc})}{=}f(Y_a,L)f(A=a|L)$, $\cancel{f(Y,\bar A,\bar L)}$

$f(Y\{\bar A=\bar a\},\bar A,\bar L)=f(Y\{\bar A=\bar a\},\bar A=\bar a,\bar L)=f(Y_{\bar a}|\bar A=\bar a,\bar L)f(\bar A=\bar a|\bar L)f(\bar L)=f(Y_{\bar a}|\bar A_{j-1}=\bar a_{j-1},\bar L)f(A_j=a_j|\bar A_{j-1}=\bar a_{j-1},$

$\bar L)f(\bar a_{j-1}|\bar L)f(\bar L)\to f(Y_{\bar a}|\bar A_{j-1}=\bar a_{j-1},\bar L)f(\bar A_{j-1}=\bar a_{j-1}|\bar L)f(\bar L)$, $f(\bar A,\bar L)=f(A_j|\bar A_{j-1},\bar L_j)f(\bar L_j|\bar L_{j-1},$

$\bar A_{j-1})f(A_{j-1}|\bar A_{j-2},\bar L_{j-1})f(L_{j-1}|\bar A_{j-2},\bar L_{j-2})=\cdots=\prod_{j=1}^{J}f(A_j|\bar A_{j-1},\bar L_j)\prod_{j=1}^{J}f(L_j|\bar L_{j-1},\bar A_{j-1})$,

$\to f(Y_{\bar a},L_J|\bar A_{J-1},\bar L_{J-1})\prod_{j=1}^{J-1}f(L_j|\bar A\bar L_{j-1})$, $\mathbb{E}\left(\frac{g(Y_{\bar a})}{\bar W}\right)=\mathbb{E}\left(\frac{g(Y_{\bar a})}{\prod f(A_j|\bar L_j,\bar A_{j-1})}\right)$

$=\int\frac{g(y)}{\prod f(a_j|\bar\ell_j,\bar a_{j-1})}f(y,\bar a,\bar\ell)=\int g(y)f(y|\bar a,\bar\ell)\prod_j f(\ell_j|\bar a_{j-1},\bar\ell_{j-1})$

(8)

$\cancel{f(Y_{\bar a}, \bar A, \bar L) \prod_j f(A}$ $f(Y \mid \bar A, \bar L) \prod_j f(L_j \mid \overline{LA}_{j-1})$, $f(\{\bar A = \bar a\} Y_{\bar a} \mid \bar A, \bar L) \prod_j f(L_j \mid \overline{LA}_{j-1}) f(\bar A)$

$= f(Y_{\bar a} \mid \bar A = \bar a, \bar L) \prod_j f(L_j \mid \overline{La}_{j-1}) f(\bar a)$, $\int (\dots) = f(Y_{\bar a}) f(\bar a)$, $\mathbb{E}(Y_a) = \int y_a f(y_a \mid \ell) f(\ell \mid a)$

$f(a)$ ?, $\mathbb{E}\left(\frac{Y}{W}\right) = \int \frac{y}{f(a \mid \ell)} f(y \mid a, \ell) f(a \mid \ell) f(\ell) = \int y f(y \mid a, \ell) f(\ell)$

$\stackrel{?}{=} \sum_a \int y_a f(y_a \mid \ell) f(\ell) \mu(y_a)$, $\int_{\{A=a\}} y f(y \mid a, \ell) f(\ell) \mu(y) \mu(\ell) \mu(a) = \int \{A = a\} y f_{Y_a, L}(y, \ell)$

$f(\ell) \mu(\ell) \mu(y) \mu(a) = \mu(A = a) \mathbb{E}(Y_a)$, $\mathbb{E}\left(\frac{Y}{W}\right) = \int \frac{y}{\dots} f(y_a \mid a, \ell) f(\ell) \mu(y) \mu(\ell) \mu(a)$,

$\frac{f(Y, \bar A, \bar L)}{W} = f(Y \mid \bar A, \bar L) \prod_t f(L_t \mid \bar L_{t-1}, \bar A_{t-1})$, $\int_{\{\bar A = \bar a\}} g(y) f(y \mid \bar a, \bar \ell) \prod_t^T f(\ell_t \mid \bar\ell_{t-1}, \bar a_{t-1})$

$\mu(y) \mu(\bar\ell) \mu(\bar a) = \int_{\{\bar A = \bar a\}} g(y) f(y \mid \bar a_{t-1}, \bar\ell) \prod^T f(\ell_t \mid \bar\ell_{t-1}, \bar a_{t-1}) \mu(y) \mu(\bar\ell) \mu(\bar a) = \int_{\{\bar A = \bar a\}} g(y) f(y \mid \bar a_{t-1}, \bar\ell_{t-1})$

$\prod^{T-1} f(\ell_t \mid \bar\ell_{t-1}, \bar a_{t-1}) \mu(y) \mu(\bar\ell_{t-1}) \mu(\bar a) = \dots = \int_{\{\bar A = \bar a\}} g(y) f_{Y_{\bar a}}(y) \mu(y) \mu(\bar a) = \mu(\bar A = \bar a) \mathbb{E}(g(Y_{\bar a}))$,

$\mathbb{E}\left(\frac{g(Y)}{f(A \mid L)}\right) = \int \frac{g(y)}{f(a \mid \ell)} f(y \mid a, \ell) f(a \mid \ell) f(\ell) \mu(y) \mu(\ell) \mu(a) = \int \sum_{a'} g(y) \{a = a'\} f(y_{a'} \mid \ell) f(\ell) \mu(y) \mu(a) \mu(\ell) = \sum_{a'} \mu(a') \mathbb{E} g(Y_{a'})$

$\mathbb{E}\left(\frac{g(Y) h(A)}{f(A \mid L)}\right) = \sum_a h(a) \mathbb{E} g(Y_a)$, $\int_{\{a = a'\}} g(y) \sum_{a'} \{a = a'\} f(y_{a'} \mid \ell) f(\ell) \mu(y, a, \ell) = \int g(y) \{a = a''\} f(y_{a''} \mid \ell) f(\ell) \mu(y, a, \ell)$

$= \mu(a = a'') \mathbb{E} g(Y_{a''})$, $\mathbb{E}\left(\frac{g(Y\{A = a\}) h(A)}{f(A \mid L)}\right)$, $\tilde f = f_{Y_a, L}$, $f = f_{Y_a, L} \, f_{A \mid L}$,

$\tilde{\mathbb{E}}\, g(Y_a) \{A = a\} = \int_{\{A = a\}} g(y) f_{Y_a, L}(y, \ell) \mu_{Y_a} \times \mu_L \times \mu_A = \mu_A(A = a) \cdot \int g(y) f_{Y_a, L}(y, \ell) \mu_{Y_a} \times \mu_L = \mu_A(A = a) \mathbb{E} g(Y_a)$,

$\mathbb{E}\, g(Y_a) \{A = a\} = \int_{\{A = a\}} g(y) f_{Y_a, L}(y, \ell) f_{A \mid L}(a, \ell) \mu_{Y_a} \times \mu_L \times \mu_A = \int_{\{A = a\}} g(y) \underbrace{f_{Y \mid A = a, L}(y, \ell)}_{\text{``}f_{Y \mid A = a, L}\text{''}} f_L(\ell) f_{A \mid L}(a, \ell) \mu_{Y_a} \times \mu_L \times \mu_A$

$= \int_{\{A = a\}} g(y) f_{Y, A = a, L}(y, \ell) \underset{\mu_Y}{\mu_{Y_a}} \times \mu_L \times \mu_A = \mathbb{E}\, g(Y) \{A = a\}$, $\cancel{f = f_{Y_a \mid \bar A, \bar L}(Y_a \mid \bar A, \bar L)}$ $\tilde f = f_{Y_a \mid \bar A, \bar L} \cdot \prod_t f_{L_t \mid \overline{AL}_{t-1}}$

$f = f_{Y_a \mid \bar A, \bar L} \cdot \prod_t f_{L_t \mid \overline{AL}_{t-1}} \cdot \underbrace{\prod_t f_{A_t \mid \bar A_{t-1}, \bar L_t}}_{\text{``}W\text{''}}$, $\int g(y) h(\bar a) \tilde f(y, \bar\ell, \bar a) \mu \times \mu \times \mu = \sum_{\bar a'} \int_{\{\bar a = \bar a'\}} g(y) h(\bar a) \cdot$

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$\tilde f(y, \bar\ell, \bar a') \mu(y) \cdot \mu(\bar\ell) \cdot \mu(\bar a) = \sum_{\bar a'} \int_{\{\bar a = \bar a'\}} g(y) h(\bar a') f_{Y_a \mid \bar A, \bar L}(y, \bar a', \bar\ell) \cdot$

$\prod_t^T f_{L_t \mid \overline{AL}_{t-1}}(\ell_t, \bar a_{t-1} = \bar a'_{t-1}, \bar\ell_{t-1}) \mu \cdot \mu \cdot \mu = \dots f_{Y_a \mid \bar A_{t-1}, \bar L_{t-1}}(y, \bar a_{t-1} = \bar a'_{t-1}, \bar\ell_{t-1})$

$\cdot \prod_t^{T-1} f_{L_t \mid \overline{AL}_{t-1}}(\ell_t, \bar a_{t-1} = \bar a'_{t-1}, \bar\ell_{t-1}) \mu \cdot \mu \cdot \mu = \dots = \sum_{\bar a'} \int_{\{\bar a = \bar a'\}} h(\bar a) g(y) f_{Y_a}(y) \mu(y) \times \mu(a) = \int h(\bar a) g(y) f_{Y_{\bar a}}(y) \mu(y) \times \mu(a)$

$= \mathbb{E}(Y_{\bar a}) \mu(h(\bar A))$

⑨ $\mathbb{E}\,\frac{Y}{W_2} = \sum_{\bar a} \mathbb{E}\,\frac{Y_{\bar a}\{\bar A=\bar a\}}{W_2} = \sum_{\bar a} \mathbb{E}\,\frac{\mathbb{E}(Y_{\bar a}\mid a_1, \overline{Z_2 L U_2})\,\mathbb{P}(\cancel{A_2=a_2}\mid a_1, \overline{ZLU}_2)\,\{A_1=a_1\}(-1)^{1-z_2}}{f_{z_1}\,\Delta_1\,(-1)^{1-z_1}\; f_{z_2}(z_2\mid A_1,\bar L_2, z_1)\,\Delta_2(z_1,\bar L_2,\ldots)} = \sum_{\bar a}\mathbb{E}\,\frac{\{A_1=a_1\}\,\mathbb{E}(Y_{\bar a}\mid a_1, z_1, \bar U_2)(-1)^{1-a_2}\Delta_2(U)}{f_{z_1}\,\Delta_1\,(-1)^{1-z_1}\,\Delta_2}$

$\perp \bar U$

$= \sum_{a_1}\mathbb{E}\,\frac{\{A_1=a_1\}\,\cancel{\sum_{a}\ldots}\;\Delta_2(U)\sum_{a_2}(-1)^{1-a_2}\,\mathbb{E}(Y_{\bar a}\mid a_1, z_1, \overline{LU}_2)}{w_1\,\Delta_2} = \sum_{\bar a}\mathbb{E}\left(\frac{\{A_1=a_1\}\;\cancel{\ldots}}{W_1}\,(-1)^{1-a_2}\,Y_{\bar a}\right)$

$= \sum_{\bar a}(-1)^{1-a_2}\,\mathbb{E}\,\frac{\mathbb{P}(a_1\mid ZLU_1)\,\mathbb{E}(Y_{\bar a}\mid ZLU_1)}{f_{z_1}(z_1\mid L_1)\,\Delta_1(L_1)\,(-1)^{1-z_1}} = \sum_{\bar a}(-1)^{1-a_2}\,\mathbb{E}\,\frac{\mathbb{E}(Y_{\bar a}\mid LU_1)\,(-1)^{1-a_1}\Delta_1(U)}{\Delta_1(L_1)} = \sum_{a_2}(-1)^{1-a_2}\,\frac{\Delta_1(U)\sum_{a_1}(-1)^{1-a_1}\,\mathbb{E}Y_{\bar a}}{\Delta_1(L_1)}$

$= \sum_{\bar a}(-1)^{2-\sum_t a_t}\,\mathbb{E}(Y_{\bar a}) = \sum_{a_1}(-1)^{1-a_1}\sum_{a_2}(-1)^{1-a_2}\,\mathbb{E}(Y_{\bar a}) = (\mathbb{E}Y_{11}-\mathbb{E}Y_{10}) - (\mathbb{E}Y_{01}-\mathbb{E}Y_{00})$

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$\sum \frac{Y_i - \beta^T\bar A_i}{\prod f(A_t\mid \bar L_t, \bar A_{t-1})}$, $\quad \sum U_j(\hat\beta)=0,\ (\hat\beta-\beta_0)\sum U_j'(\beta_0) + o(|\hat\beta-\beta_0|),\ \sqrt{n}(\hat\beta-\beta_0) = -\left(\tfrac{1}{n}\sum U_j'(\beta_0)\right)^{-1} n^{-1/2}\sum U_j(\beta_0) + o_p(1)$,

$U_j = h_j\,\frac{Y_j-\beta^T\bar A_j}{\bar W_j}$, $\quad \sqrt{n}(\hat\beta-\beta_0) = \left(\mathbb{E}\,\frac{h\bar A^T}{\bar W}\right)^{-1} h\,\frac{Y-\beta^T\bar A}{\bar W}$, $\quad \mathbb{E}\bar W^2 = \mathbb{E}\prod_t f(A_t\mid \bar L_t, \bar A_{t-1})^2 = \cancel{\mathbb{E}\prod_t \sigma(\alpha L_t)^{2A_t}}$

$\mathbb{E}\prod_t \sigma(\alpha L_t)^{2A_t}(1-\sigma(\alpha L_t))^{2(1-A_t)}$, $\quad \mathbb{E}\left(\sigma(\alpha L_T)^{2A_T}(1-\sigma(\alpha L_T))^{2(1-A_T)}\mid L_T\right)$

$L_1 \to A_1 \to L_2 \to A_2 \to L_3 \to \cdots$ $\quad = \sigma(\alpha L_T)^3 + (1-\sigma(\alpha L_T))^3$, $\ \mathbb{E}(\ldots\mid A_{T-1})$, $\ \log\mathbb{E}\prod f(A_t\mid L_{t-1})^2 \leq$

$\sum_t \mathbb{E}\log f(A_t\mid L_t)^2 = 2\sum\mathbb{E}\left(A_t\log\sigma(\alpha L_t) + (1-A_t)\log(1-\sigma(\alpha L_t))\right)$, $\quad \mathbb{E}A_t\log\sigma(\alpha L_t)$

$= \mathbb{E}\{\log\sigma(\alpha L_t)\cdot\sigma(\alpha L_t)\} = -\int\log(1+e^{-\alpha\ell})\,\frac{1}{1+e^{-\alpha\ell}}\,f_{L_t}(\ell)\,d\ell$, $\quad \log\sigma(\alpha L_t)\cdot\sigma(\alpha L_t) =$

$\cancel{\log\sigma(L_t)\cdot\sigma(L_t) + (}\ \frac{1}{2}\log\frac{1}{2} + \alpha\left(L_t\sigma'(\alpha L_t) + \log(\sigma(\alpha L_t))\,\sigma'(\alpha L_t)\cdot L_t\right) + o(|\alpha|)$

$= -\frac{\log 2}{2} + \alpha L_t\,\sigma'(\alpha L_t)\left(1+\log\sigma(\alpha L_t)\right) + o(|\alpha|)$, $\quad \mathbb{P}(X<u)\geq f(u)$, $\ \mathbb{P}(\tfrac{1}{X}>\tfrac{1}{u})\geq f(u)$, $\ \mathbb{E}\tfrac{1}{X} = \int_1^{\infty}\mathbb{P}(\tfrac{1}{X}>u)$

$> \int_1^{\infty} f(\tfrac{1}{u})\,du$, $\quad \mathbb{P}(\sigma(\alpha L)<u) = \mathbb{P}(\alpha L<\log\frac{u}{1-u}) = \Phi\left(\frac{1}{\alpha}\log\frac{u}{1-u}\right) = (2\pi)^{-1/2}\int_{-\infty}^{\frac{1}{\alpha}\log\frac{u}{1-u}} e^{-\xi^2/2}\,d\xi \sim \cancel{\ldots}\ 1-$

$(2\pi)^{-1/2}\exp\left(-\tfrac{1}{2}\left(\log(\tfrac{u}{1-u})^{1/\alpha}\right)^2\right) = 1-(2\pi)^{-1/2}\left(\frac{u}{1-u}\right)^{\frac{1}{\alpha}\left(-\frac{1}{2\alpha}\log\left(\frac{u}{1-u}\right)\right)} = 1-(2\pi)^{-1/2}\left(\frac{u}{1-u}\right)^{-\frac{1}{2\alpha^2}\sigma^{-1}(u)} =: f(u)$, $\ f(\tfrac{1}{u})$

$= 1-(2\pi)^{-1/2}(u-1)^{-\frac{1}{2\alpha^2}\log(u-1)}$, $\quad f(\tfrac{1}{u}) = \Phi\left(-\frac{1}{\alpha}\log(u-1)\right) \sim \cancel{\ldots}\ (2\pi)^{-1/2}\exp\left(-\frac{1}{2\alpha^2}(\log(u-1))^2\right) =$

$(2\pi)^{-1/2}(u-1)^{-\frac{1}{2\alpha^2}\log(u-1)}$, $\ \ldots$, $\ \log\mathbb{E}\prod f(A_t\mid\bar L_t,\bar A_{t-1})^2 \geq 2\sum_t\mathbb{E}\log f(A_t\mid\bar L_t,\bar A_{t-1})$, $\ \mathbb{E}\log f(A_t\mid\bar L_t,\bar A_{t-1}) = \mathbb{E}\log\{\mathbb{P}(A_t=1\mid\overline{LA})^{A_t}\cdot$

$(1-\mathbb{P}(A_t=1\mid\overline{LA}))^{1-A_t}\} = \mathbb{E}\{\pi(\bar L_t,\bar A_{t-1})\log\pi(\bar L_t,\bar A_{t-1}) + (1-\pi(\bar L_t,\bar A_{t-1}))\log(1-\pi(\bar L_t,\bar A_{t-1}))\} = \mathbb{E}\left\{\log(1-\pi)+\pi\log\frac{\pi}{1-\pi}\right\}$

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$\frac{\partial}{\partial\alpha}\operatorname{var}\frac{(Y-\beta^T A)h}{\sigma(\alpha L)^A(1-\sigma(\alpha L))^{1-A}} = \mathbb{E}\left\{((Y-\beta A)h)^2(-2)(f(A\mid L))^{-3}\left(AL\sigma'(\alpha L)\left(\frac{\sigma}{1-\sigma}\right)^{A-1} + \sigma(\alpha L)^A(1-A)(1-\sigma)^{-A}(-\sigma')L\right)\right\}$

$\sigma' L\left(\frac{\sigma}{1-\sigma}\right)^A\left(A\frac{1-\sigma}{\sigma} - 1 + A\right) = \sigma'\cdot L\left(\frac{\sigma}{1-\sigma}\right)^A\frac{A-\sigma}{\sigma} = \sigma'\cdot L\cdot(-1)^{1-A}$, $\quad \frac{\partial}{\partial\alpha} = \mathbb{E}\,\frac{-2L\sigma'(-1)^{1-A}}{f(A\mid L)}\left(\frac{(Y-\beta A)h}{f(A\mid L)}\right)^2$

(1)

$\mathbb{P}(a_t|a_{t-1}) = \sum_{\ell_t} \mathbb{P}(a_t|a_{t-1},\ell_t)\,\mathbb{P}(\ell_t|a_{t-1}) = p_{0,a_t}\, p_{a_{t-1},0} + p_{1,a_t}\, p_{a_{t-1},1}$

$\prod_{t=1}^{T} \frac{\mathbb{P}(A_t|\bar A_{t-1})}{\mathbb{P}(A_t|\bar A_{t-1},\bar L_{t*})}$, $\prod \mathbb{P}(A_t|\bar A_{t-1},\bar L_{t*}) = \prod p_{11}^{L_{t*}A_t}\, p_{01}^{(1-L_{t*})A_t} \cdots = p_{11}^{\sum L_{t*}A_t}\, p_{01}^{\sum A_t - \sum L_{t*}A_t}$,

$p_{00}^{\sum(1-L_{t*})(1-A_t)}\, p_{10}^{\sum L_{t*}(1-A_t)} = p_{00}^{\sum(1-L_{t*})(1-A_t)}\,(1-p_{00})^{\sum(1-L_{t*})A_t}\, p_{10}^{\sum L_{t*}(1-A_t)}\,(1-p_{10})^{\sum L_{t*}A_t}$

$= (1-p_{01})^{\sum(1-L_t)}\left(\frac{p_{01}}{1-p_{01}}\right)^{\sum(1-L_t)A_t}(1-p_{11})^{\sum L_t}\left(\frac{p_{11}}{1-p_{11}}\right)^{\sum L_t A_t}$, $(p_{00}=p_{0*})$ $p^{\sum(1-L_t)}\left(\frac{p}{1-p}\right)^{\sum L_t A_t - \sum(1-L_t)A_t}$

$(1-p)^{\sum L_t} = p^T\left(\frac{1-p}{p}\right)^{\sum L_t}\left(\frac{p}{1-p}\right)^{\sum A_t + 2\sum L_t A_t}$ $= p^T\left(\frac{p}{1-p}\right)^{-\sum A_t + 2\sum L_t A_t - \sum L_t}$

$= p^T\left(\frac{p}{1-p}\right)^{-\sum(A_t-L_t)^2} = p^T\left(\frac{p}{1-p}\right)^{-\sum\{A_t\neq L_t\}}$, $\mathbb{E}\, c^{\sum\{A_t\neq L_t\}}$, $\mathbb{P}(A_j\neq L_j, A_{j+1}\neq L_{j+1})$

$\mathbb{P}(A_{j+1}\neq L_{j+1}|A_j\neq L_j) = \mathbb{P}(01|01\vee10) + \mathbb{P}(10|01\vee10) = \frac{\mathbb{P}(01\vee10, 01)}{\mathbb{P}(01\vee10)} +$

$\frac{\mathbb{P}(01\vee10, 10)}{\mathbb{P}(01\vee10)} = \frac{p_{01}p_{10}p_{01} + p_{10}p_{00}p_{01}}{p_{01}+p_{10}} + \frac{p_{01}p_{11}p_{10} + p_{10}p_{01}p_{10}}{p_{01}+p_{10}}$ $\mathbb{P}(A_j\neq L_j, A_{j+1}\neq L_{j+1})$

$= \begin{matrix} p_{01} & p_{01} \\ p_{10} & p_{10} \\ (j) & (j+1)\end{matrix} = \cancel{p_{01}(p_{10}+p_{11})}\; p_{01}(p_{10}p_{01}+p_{11}p_{10}) + p_{10}(p_{00}p_{01}+p_{01}p_{10}) = p_{01}(p_{10}p_{01}+p_{10}-p_{10}^2)$

$+p_{10}(p_{01}-p_{01}^2+p_{01}p_{10}) = p_{01}^2p_{10} + p_{10}p_{01} - p_{10}^2p_{01} + p_{10}p_{01} - p_{01}^2p_{10} + p_{10}^2p_{01} = 2p_{10}p_{01}$

$\mathbb{P}(A_t\neq L_t, A_{t+1}\neq L_{t+1}) = \mathbb{P}(L_t=0)\,p_{01}(p_{10}p_{01}+p_{11}p_{10}) + \mathbb{P}(L_t=1)\,p_{10}(p_{00}p_{01}+p_{01}p_{10}) = \mathbb{P}(L_t=0)\,p_{01}$ $p_{10}p_{01}(p_{01}+p_{11})$

$(p_{10}p_{01}+p_{10}-p_{10}^2) + \mathbb{P}(L_t=1)\,p_{10}(p_{01}-p_{01}^2+p_{01}p_{10}) = \mathbb{P}(L_t=0)(p_{01}^2p_{10}-p_{10}^2p_{01}+p_{10}p_{01}) + \mathbb{P}(L_t=1)$

$p_{10}p_{01}(p_{00}+p_{10}) = p_{01}p_{10}\big(\mathbb{P}(L_t=0)(p_{01}+p_{11}) + \mathbb{P}(L_t=1)(p_{00}+p_{10})\big)$, $\mathbb{P}(A_t\neq L_t) = \mathbb{P}(L_t=0)\,p_{01} + \mathbb{P}(L_t=1)$ $= p_{01} = p_{10}$

$\cancel{= p_{01}p_{10}\big(\mathbb{P}(L_t=1) + \mathbb{P}(L_t=0)p_{11} + \mathbb{P}(L_t=0) + \mathbb{P}(L_t=1)p_{00}\big)} = p_{01}p_{10}(1 + v_0p_{11} + (1-v_0)\ldots$

$= p_{01}p_{10}\big(\mathbb{P}(L_t=0, A_{t+1}=1) + p_{11}\mathbb{P}(L_t=0) + \mathbb{P}(L_t=1, A_{t+1}=0) + p_{00}\mathbb{P}(L_t=1)\big)$,

$\mathbb{P}(L_t=0)^2p_{01}^2 + \mathbb{P}(L_t=1)^2p_{10}^2 + 2p_{01}p_{10}\,\mathbb{P}(L_t=0)\mathbb{P}(L_t=1)$, $\mathbb{P}(L_t=0) = \frac{p_{10}}{p_{01}+p_{10}}$, $\mathbb{P}(L_t=1) = \frac{p_{01}}{p_{01}+p_{10}}$,

$\cancel{p_{00}p_{01}+p_{10}p_{01} = p_{01}(p_{10}+1-p_{01}) - p_{01}p_{00}+p_{10}^2}$ $p_{10}p_{00}+p_{01}p_{10} = p_{10}$, $= p_{01}p_{10}\frac{p_{10}p_{01}+p_{10}p_{11}+p_{01}p_{00}+p_{01}p_{10}}{p_{01}+p_{10}}$

$= \frac{p_{01}p_{10}}{p_{01}+p_{10}}(2p_{10}p_{01}+p_{10}-p_{10}^2+p_{01}-p_{01}^2) = p_{01}p_{10} - \frac{(p_{01}-p_{10})^2}{p_{01}+p_{10}}$, $\frac{p_{01}^2p_{10}^2 + p_{10}^2p_{01}^2 + 2p_{01}p_{10}}{(p_{01}+p_{10})^2}$

$= \frac{2p_{01}p_{10}(1+p_{01}p_{10})}{(p_{01}+p_{10})^2}$, $=(1-p)^2$, $\mathbb{E}\left(p^T\left(\frac{1-p}{p}\right)^{\sum\{A_t\neq L_t\}}\right)^2 = p^T\prod_{t=1}^{T}\mathbb{E}\left(\frac{1-p}{p}\right)^{2\{A_t\neq L_t\}}$

$= p^T\left\{\left(1-\frac{1}{p}\right)^2(1-p) + p\right\}^T = \left(\frac{(1-p)^3}{p} + p^2\right)^T$

(12)

$$\mathbb{P}(A_t \neq L_t \mid A_{t-1} \overset{=0}{}) = p_{01}p_{10} + p_{00}p_{01} \quad (= (1-p)^2 + p(1-p) = 1-p), \quad \mathbb{P}(A_t \neq L_t \mid A_{t-1} = 1) = p_{10}p_{01} + p_{11}p_{10} = (1-p)^2 + p(1-p) \; (= 1-p), \quad \mathbb{P}(A_t \neq L_t \mid$$

$$A_{t-1}) = \mathbb{P}(A_t \neq L_t), \quad \hat{W}_T = \prod_{t=1}^{T} f(A_t \mid \bar{A}_{t-1}, \bar{L}_t) = p^T \left(\frac{p}{1-p}\right)^{-\sum\{A_t \neq L_t\}} \; (p = p_{00} = p_{11}), \quad \mathbb{E}\frac{1}{\hat{W}_T^2} = p^{-2T} \mathbb{E}\prod_t \left(\frac{p}{1-p}\right)^{2\{A_t \neq L_t\}}$$

$$= p^{-2T} \left( \left(\frac{p}{1-p}\right)^2 (1-p) + p \right)^T = \left( \frac{1}{1-p} + \frac{1}{p} \right)^T$$

---

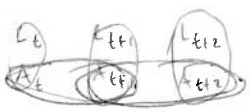
$$(p_{00} = p_{11} = p, \quad p_{01} = p_{10} = 1-p)$$

$$\mathbb{P}(A_{t+1} = a_{t+1} \mid A_t = a_t) = \sum_{a \in 0,1} p_{a_t a}\, p_{a a_{t+1}}, \quad \prod_{t=1}^{T-1} f(A_{t+1} \mid A_t) = \prod_t (p_{00}p_{00} + p_{01}p_{10})^{(1-A_t)(1-A_{t+1})} (p_{00}p_{01} + p_{01}p_{11})^{(1-A_t)A_{t+1}}$$

$$(p_{10}p_{00} + p_{11}p_{10})^{A_t(1-A_{t+1})} (p_{10}p_{01} + p_{11}p_{11})^{A_tA_{t+1}} = \prod_t (2p^2 - 2p + 1)^{(1-A_t)(1-A_{t+1})} (2p(1-p))^{(1-A_t)A_{t+1}} (2p(1-p))^{A_t(1-A_{t+1})}$$

$$\cdot (p^2 + (1-p)^2)^{A_tA_{t+1}} = (p^2+(1-p)^2)^{\sum(1 - A_t - A_{t+1} + 2A_tA_{t+1})} (2p(1-p))^{\sum(A_t + A_{t+1} - 2A_tA_{t+1})} = \left(\frac{p^2+(1-p)^2}{2p(1-p)}\right)^{2\sum A_tA_{t+1}} \cdot$$

$$\cdot \left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^{\sum A_t + \sum A_{t+1}} \cdot (p^2+(1-p)^2)^T = (p^2+(1-p)^2)^T \left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^{\sum(A_t - A_{t+1})^2} = (p^2+(1-p)^2)^T \left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^{\sum_t\{A_t \neq A_{t+1}\}}$$




$$\{A_{t+1} \neq L_{t+1}\} \perp\!\!\!\perp A_t$$

$$, \quad \mathbb{P}(A_{t+2} \neq A_{t+1} \mid A_{t+1}) = p_{A_{t+1}\bar{A}_{t+1}} = 1-p, \quad \{A_t \neq A_{t+1}\} \perp\!\!\!\perp A_t, \quad \mathbb{E}\left(\frac{\prod f(A_{t+1} \mid A_t)}{\prod f(A_{t+1} \mid L_t)}\right)^2$$

$$= \left(\frac{p^2+(1-p)^2}{p}\right)^{2T} \mathbb{E} \frac{\left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^{2\sum_t\{A_t \neq A_{t+1}\}}}{\left(\frac{p}{1-p}\right)^{-2\sum\{A_{t+1} \neq L_{t+1}\}}} \overset{?}{=} \left(\frac{p^2+(1-p)^2}{p}\right)^{2T} \prod_t \mathbb{E} \frac{\left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^{2\{A_t \neq A_{t+1}\}}}{\left(\frac{1-p}{p}\right)^{2\{A_{t+1} \neq L_{t+1}\}}}, \quad \mathbb{P}(A_t \neq L_t, \cancel{A_t \neq A_{t+1}}\, A_t \neq A_{t-1} \mid A_{t-1})$$

$$= (1-p)^2, \quad (\{A_{t-1} \neq A_t\}, \{L_t \neq A_t\}) \perp\!\!\!\perp A_{t-1}, \quad = \left(\frac{p^2+(1-p)^2}{p}\right)^{2T} \prod_t \left\{ p^2 + p(1-p)\left( \left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^2 + \left(\frac{p}{1-p}\right)^2 \right) \right.$$

$$\left. + p(1-p)^2 \left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^2 \left(\frac{p}{1-p}\right)^2 \right\} = \left(\frac{p^2+(1-p)^2}{p}\right)^{2T} p^{2T} \prod_t \left\{ 1 + \left( \frac{4p(1-p)^3}{(p^2+(1-p)^2)^2} + \frac{p}{1-p} \right) + \left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^2 \right\}$$

$$= (p^2+(1-p)^2)^{2T} \prod_t \left\{ 1 + \frac{p}{1-p} + 4p(1-p)^2 \left( \frac{(1-p) + p}{(p^2+(1-p)^2)^2} \right) \right\} = (p^2+(1-p)^2)^{2T} \left\{ \frac{1}{1-p} + \frac{4p(1-p)^2}{(p^2+(1-p)^2)^2} \right\}^T$$

$$= \left( \frac{(p^2+(1-p)^2)^2}{1-p} + 4p(1-p)^2 \right)^T = \left( \frac{(2p^2-2p+1)^2}{1-p} + 4p(1-p)^2 \right)^T, \quad 4p^4 - 4p^2 + 1 + 8p^3 + 4p^2 - 4p + 4p(1 - 3p$$

$$+ 3p^2 - p^3) = 8p^4 + 20p^3 - 12p^2 + 1, \quad = \left(\frac{p^2+(1-p)^2}{p}\right)^{2T} \left\{ p^2 + p(1-p)\left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^2 + (1-p)^2\left(\frac{p}{1-p}\right)^2 + p(1-p) \right.$$

$$\left. \left(\frac{2p(1-p)}{p^2+(1-p)^2}\right)^2 \left(\frac{p}{1-p}\right)^2 \right\}^T = (p^2+(1-p)^2)^{2T} \left\{ 1 + \frac{4p(1-p)^3}{(p^2+(1-p)^2)^2} + 1 + \frac{4p^3(1-p)}{(p^2+(1-p)^2)^2} \right\}^T = \left\{ 2(p^2+(1-p)^2)^2 + \right.$$

$$\left. 4p(1-p)((1-p)^2 + p^2) \right\}^T = \left\{ (p^2+(1-p)^2)(2p^2 + 2(1-p)^2 + 4p(1-p)) \right\}^T = (p^2+(1-p)^2)^T (\cancel{-2p^2} + 2p^2 \cancel{+1})^T$$

$$= 2^T (p^2+(1-p)^2)^T$$

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$\Delta_{L,A}$
$p_{L,U,A}$ & $\Delta_{L,0} = -\Delta_{L,1}$

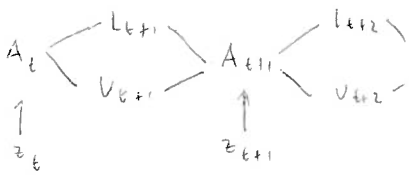
| | $L=0$ | | $L=1$ | |
|---|---|---|---|---|
| $U=1$ | $\Delta_{00}$ | $\Delta_{01} = -\Delta_{00}$ | $\Delta_{10}$ | $\Delta_{11} = -\Delta_{10}$ |
| | $p_{010}$ | $p_{011}$ | $p_{110}$ | $p_{111}$ |
| $U=0$ | $\Delta_{00}$ | $-\Delta_{00}$ | $\Delta_{10}$ | $-\Delta_{10}$ |
| | $p_{000}$ | $p_{001}$ | $p_{100}$ | $p_{101}$ |

$\mathbb{P}(A=0 \mid L=0) = \mathbb{P}(A=1 \mid L=0)$, $p_{000} + p_{000} + \Delta_{00} = 2p_{001} - \Delta_{00}$,

$p_{001} - p_{000} = \Delta_{00}$, $\mathbb{P}(A=1 \mid L=0, Z=0) - \mathbb{P}(A=0 \mid L=0, Z=0) = \Delta_{00}$,

$\prod_t \left(\mathbb{P}(A_t = 1 \mid L_t, U_t, Z_t = 1) - \mathbb{P}(A_t = 1 \mid L_t, U_t, Z_t = 0)\right)$

$= \prod_t \Delta_{L_t,1} = \Delta_{01}^{\sum(1-L_t)} \Delta_{11}^{\sum L_t} = \Delta_{01}^T \left(\frac{\Delta_{11}}{\Delta_{01}}\right)^{\sum L_t}$, $\prod_t \left(f(A_t \mid L_t, U_t, Z_t = 1) - f(A_t \mid L_t, U_t, Z_t = 0)\right) = \prod_t \Delta_{A_t, A_t} = \prod_t \Delta_{00}^{(1-L_t)(1-A_t)} \Delta_{01}^{(1-L_t)A_t} \Delta_{10}^{L_t(1-A_t)} \Delta_{11}^{L_t A_t}$



$= \Delta_{00}^{T - \sum L_t - \sum A_t + \sum L_t A_t} (-\Delta_{00})^{\sum A_t - \sum L_t A_t} (-\Delta_{11})^{\sum L_t - \sum L_t A_t} \Delta_{11}^{\sum L_t A_t} = \Delta_{00}^T (-1)^{\sum A_t - 2\sum L_t A_t + \sum L_t} \Delta_{00}^{-\sum L_t} \Delta_{11}^{\sum L_t}$

$= \Delta_{00}^T \cdot (-1)^{\sum (A_t - L_t)^2} \left(\frac{\Delta_{11}}{\Delta_{00}}\right)^{\sum L_t}$, $\mathbb{P}(A=1 \mid L=\ell) = \sum_{z,u} \mathbb{P}(A=1 \mid L=\ell, U=u, Z=z) \underbrace{\mathbb{P}(U=u, Z=z \mid L=\ell)}_{=1/4}$

$= \frac{1}{4} \sum_{\ell,u} (2p_{\ell u 1} + \Delta_{\ell 1})$, $\mathbb{P}(A=0 \mid L=\ell) = \frac{1}{4} \sum_{\ell,u} (2p_{\ell u 0} + \Delta_{\ell 0})$, $\mathbb{P}(A=1 \mid L=1) = \mathbb{P}(A=0 \mid L=0) \Rightarrow$

$\sum_u (2p_{1,u,1} + \Delta_{11}) = \sum_u (2p_{0,u,0} + \Delta_{00})$, $2(\Delta_{00} - \Delta_{11}) = 2\sum_u (p_{1,u,1} - p_{0,u,0})$, ~~$\mathbb{P}(A=a \mid U=u) =$~~

$\mathbb{P}(A=a \mid L=\ell) = \frac{1}{4} \sum_{z,u} \mathbb{P}(A=a \mid L=\ell, U=u, Z=z) = \frac{1}{4} \sum_u (2p_{\ell,u,a} + \Delta_{\ell,a})$, $\mathbb{P}(A=0 \mid L=0) = \mathbb{P}(A=1 \mid L=1) \Rightarrow$

$\sum_u (2p_{0,u,0} + \Delta_{00}) = \sum_u (2p_{1,u,1} + \Delta_{11})$, $2(\Delta_{11} - \Delta_{00}) = 2\sum_u (p_{0,u,0} - p_{1,u,1})$, $\mathbb{P}(A=a \mid U=u) = \frac{1}{4} \sum_{z,\ell} \mathbb{P}(A=a \mid L=\ell, U=u, Z=z) = \frac{1}{4} \sum_\ell (2p_{\ell u a} + \Delta_{\ell a})$, $\mathbb{P}(A=0 \mid U=0) = \mathbb{P}(A=1 \mid U=1) \Rightarrow \Delta_{11} - \Delta_{00} = \sum_\ell (p_{\ell,0,0} - p_{\ell,1,1})$, $\mathbb{P}(U=u, Z=z \mid L=\ell)$

$= \frac{1}{2} \mathbb{P}(U=u \mid L=\ell) = \frac{1}{2} \sum_a \mathbb{P}(U_t = u \mid L_t = \ell, A_{t-1} = a) \mathbb{P}(A_{t-1} = a \mid L_t = \ell) = \frac{1}{2} \sum_a \mathbb{P}(U_t = u \mid A_{t-1} = a) \frac{\mathbb{P}(L_t = \ell \mid A_{t-1} = a)\mathbb{P}(A_{t-1} = a)}{\mathbb{P}(L_t = \ell)}$

$= \frac{1}{2} \sum_a p^u_{au} p^\ell_{a\ell}$, $p^u_{0u} p^\ell_{0\ell} + p^u_{1u} p^\ell_{1\ell} = p^u_{0u} p^\ell_{0\ell} + (1 - p^u_{0u})(1 - p^\ell_{0\ell}) = 2p^u_{0u} p^\ell_{0\ell} - p^u_{0u} - p^\ell_{0\ell} + 1$

| $L \backslash A$ | 0 | 1 |
|---|---|---|
| 0 | $p_\ell$ | $1-p_\ell$ |
| 1 | $1-p_\ell$ | $p_\ell$ |

| $U \backslash A$ | 0 | 1 |
|---|---|---|
| 0 | $p_u$ | $1-p_u$ |
| 1 | $1-p_u$ | $p_u$ |

$\overset{L}{\underset{U}{}} \!\!> A$

| L U A | |
|---|---|
| 0 0 0 | $p_\ell p_u$ |
| 0 0 1 | $(1-p_\ell)(1-p_u)$ |
| 0 1 0 | $p_\ell(1-p_u)$ |
| 0 1 1 | $(1-p_\ell)p_u$ |
| 1 0 0 | $(1-p_\ell)p_u$ |
| 1 0 1 | $p_\ell(1-p_u)$ |
| 1 1 0 | $(1-p_\ell)(1-p_u)$ |
| 1 1 1 | $p_\ell p_u$ |

$\mathbb{P}(A_{t+1} = a_{t+1} \mid A_t = a_t) = \sum_{\ell,u} \mathbb{P}(A_{t+1} = a_{t+1}, L_{t+1} = \ell, U_{t+1} = u \mid A_t = a_t) = \sum_{\ell,u} \mathbb{P}(A_{t+1} = a_{t+1} \mid L_{t+1} = \ell, U_{t+1} = u) \mathbb{P}(L_{t+1} = \ell, U_{t+1} = u \mid A_t = a_t) = \sum_{\ell,u} p_{\ell,u,a_{t+1}} p_\ell^{\{a_t = \ell\}} (1-p_\ell)^{\{a_t \neq \ell\}} p_u^{\{a_t = u\}} (1-p_u)^{\{a_t \neq u\}}$

$= \sum_{\ell,u} p_{\ell,a_{t+1}} p_{u,a_{t+1}} (1-p_\ell) \left(\frac{p_\ell}{1-p_\ell}\right)^{\{a_t = \ell\}} (1-p_u) \left(\frac{p_u}{1-p_u}\right)^{\{a_t = u\}} = \sum_{\ell,u} p_\ell^{\{a_{t+1} = \ell\}} (1-p_\ell)^{\{a_{t+1} \neq \ell\}} p_u^{\{a_{t+1} = u\}} (1-p_u)^{\{a_{t+1} \neq u\}} p_\ell^{\{\ldots} \ldots = \sum_{\ell,u} p_\ell^{\{a_t = \ell\} + \{a_{t+1} = \ell\}} (1-p_\ell)^{\{a_t \neq \ell\} + \{a_{t+1} \neq \ell\}} p_u^{\{a_t = u\} + \{a_{t+1} \neq u\}} (1-p_u)^{\{a_t \neq u\} + \{a_{t+1} \neq u\}}$

$\mathbb{P}(A_{t+1} = a \mid A_t = a) = \{p_\ell(1-p_\ell) p_u (1-p_u)\}^2 =: p_a$

(14)

$\mathbb{P}(U=u|L=\ell)$

$g_{\ell u} \, f_{u\ell} =$

$\mathbb{P}(L=\ell|U=u)$

$1 = p_{00} f_{00} + p_{01} f_{10} + p_{10} f_{01} + p_{11} f_{11} = p_{00} g_{00} + p_{10} g_{10} + p_{01} g_{01} + p_{11} g_{11},$

$\cancel{p_{01} f_{10} + p_{10} f_{01} = p_{10} f_{10} + p_{01} f_{01}, \quad p_{01}(f_{10} - f_{01}) = p_{10}(f_{10} - f_{01}), \quad p_L = p_{01} f}$

$p_L = p_{00} f_{00} + p_{01} f_{10}, \quad 1 - p_L = p_{10} f_{01} + p_{11} f_{11}, \quad p_U = p_{00} g_{00} + p_{10} g_{10}, \quad 1 - p_U = p_{01} g_{01} + p_{11} g_{11}$

$f_{00}^{-1}(p_L - p_{01} f_{10}) = g_{00}^{-1}(p_U - p_{10} g_{10}), \quad f_{11}^{-1}(1 - p_L - p_{10} f_{01}) = g_{01}^{-1}(1 - p_U - p_{01} g_{01}),$

$f_{10}^{-1}\left(p_L - \frac{f_{00}}{g_{00}}(p_U - p_{10} g_{10})\right) = g_{01}^{-1}\left(1 - p_U - \frac{g_{11}}{f_{11}}(1 - p_L - p_{10} f_{01})\right), \quad \frac{p_L}{f_{10}} - \frac{f_{00}}{f_{10} g_{00}} p_U + \frac{f_{00} g_{10}}{f_{10} g_{00}} p_{10}$

$= g_{01}^{-1}\left((1-p_U) - \frac{g_{11}}{f_{11}}(1-p_L)\right) + \frac{g_{11} f_{01}}{f_{11} g_{01}} p_{10}, \quad p_{10} = \left(\frac{f_{00} g_{10}}{f_{10} g_{00}} - \frac{g_{11} f_{01}}{f_{11} g_{01}}\right)^{-1} \left\{ g_{01}^{-1}\left(1 - p_U - \frac{g_{11}}{f_{11}}(1-p_L)\right)\right.$

$\left. - f_{10}^{-1}\left(p_L - \frac{f_{00}}{g_{00}} p_U\right)\right\}, \quad p_{11} = f_{11}^{-1}(\bar{p}_L - p_{10} f_{01}), \quad p_{00} = g_{00}^{-1}(p_U - p_{10} g_{10}),$

$p_{01} = f_{10}^{-1}(p_L - p_{00} f_{00})$

$a <^{u}_{\ell}$

$f_{u\ell} = \mathbb{P}(u|\ell) = \sum_a \mathbb{P}(u|a)\,\mathbb{P}(a|\ell), \quad g_{\ell u} = \mathbb{P}(\ell|u) = \sum_a \mathbb{P}(\ell|a)\,\mathbb{P}(a|u),$

$f_{u\ell} = \frac{f(u,\ell)}{f(\ell)} = 2\sum_a f(u,\ell|a) f(a) = \sum_a f(u,\ell|a) = \sum_a f(u|a) f(\ell|a) = g_{u\ell}$

---

$\mathbb{P}(A=a|L=\ell, U=u) = (1-p_B)\,\mathbb{P}(A^L=a|L=\ell, U=u, B=0) + p_B\,\mathbb{P}(A^U=a|L=\ell, U=u, B=1) = (1-p_B)\,\mathbb{P}(A^L=a|L=\ell)$

$+ p_B\,\mathbb{P}(A^U=a|U=u), \quad$ (32a) $\checkmark\checkmark$ $\quad \mathbb{P}(L=\ell, U=u|A=a) = (1-p_B)\,\mathbb{P}(L=\ell, U=u|A^L=a, B=0) + p_B\,\mathbb{P}(L=\ell, U=u|A^U=a, B=1)$

$(\mathbb{P}(A^L=a) = \mathbb{P}(A^U=a) = \tfrac{1}{2})$

$= \mathbb{P}(U=u)(1-p_B)\,\mathbb{P}(L=\ell|A^L=a) + \mathbb{P}(L=\ell)\,p_B\,\mathbb{P}(U=u|A^U=a) = \tfrac{1}{2}\mathbb{P}(A=a|L=\ell, U=u), \quad$ $\checkmark\checkmark$ (32a) $\quad \mathbb{P}(L=\ell|A=a)$

$= (1-p_B)\,\mathbb{P}(L=\ell|A^L=a) + p_B\,\mathbb{P}(L=\ell|A^U=a) = (1-p_B)\,\mathbb{P}(L=\ell|A^L=a) + p_B\,\mathbb{P}(L=\ell), \quad \mathbb{P}(A=a|L=\ell)$ $\checkmark\checkmark$

$(\mathbb{P}(L=\ell) = \mathbb{P}(A^L=a) = \tfrac{1}{2})$

$= (1-p_B)\,\mathbb{P}(A^L=a|L=\ell) + p_B\,\mathbb{P}(A^U=a|L=\ell) = (1-p_B)\,\mathbb{P}(A^L=a|L=\ell) + p_B\,\mathbb{P}(A^U=a) = \mathbb{P}(L=\ell|A=a),$

---

$\mathbb{P}(A=a|L=\ell, U=u, Z=z) = \mathbb{P}(A=a|L=\ell, U=u) + (-1)^{1-z}\,\delta, \quad \cancel{\mathbb{P}(L=\ell, U=u|A=a) = \sum \mathbb{P}(A=a|L=\ell)}$

$\sum_a \mathbb{P}(L=\ell, U=u|A=a)\,\mathbb{P}(A=a) = (1-p_B)\,\mathbb{P}(L=\ell, U=u|B=0) + p_B\,\mathbb{P}(L=\ell, U=u|B=1)$

$\cancel{\mathbb{P}(A=a|L=\ell, U=u) =}$

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$$\mathbb{E}\left( h(\bar{A}) \left( \frac{Y - M_B(A)}{\bar{w}} \right) \right)^{\otimes 2} = \mathbb{E}\left( h(A) \left( \frac{\varepsilon + \eta}{\bar{w}} \right) \right)^{\otimes 2} = \mathbb{E} \left( \frac{\varepsilon + \eta}{\bar{w}} \right)^2 h h^T = \sigma^2 \mathbb{E}\left( \frac{h h^T}{\bar{w}^2} \right) + \mathbb{E}\left( \frac{\eta^2}{\bar{w}^2} h h^T \right), \quad \bar{w}_T = p^T \left( \frac{p}{1-p} \right)^{-\sum \{A_t \neq L_t\}},$$

$$c := \left( \frac{p}{1-p} \right)^{2}, \quad \mathbb{E}\frac{1}{c} = \frac{p}{1-p}$$

$$\mathbb{E} \frac{A_r A_s}{c^{\sum \{A_t \neq L_t\}}} = \mathbb{E}\left( \prod_{t=1}^{r-1} c^{-\{A_t \neq L_t\}} \frac{A_r}{c^{\{A_r \neq L_r\}}} \cdot \prod_{t=r+1}^{s-1} c^{-\{A_t \neq L_t\}} \frac{A_s}{c^{\{A_s \neq L_s\}}} \cdot \prod_{t=s+1}^{T} c^{-\{A_t \neq L_t\}} \right) \quad (r<s)$$

$$= \left( \mathbb{E}\, c^{-\{A_t \neq L_t\}} \right)^{T-2} \left( \mathbb{E} \frac{A_t}{c^{\{A_t \neq L_t\}}} \right)^2 = \left( \frac{(1-p)}{c} + p \right)^{T-2} \left( \frac{1}{2}\left( (1-p)\frac{1}{c} + p \right) \right)^2, \quad \mathbb{E} \frac{A_r A_s}{\bar{w}^2} = p^{-2T}$$

33a

$$= p^{-2T} \left( \underbrace{\frac{p^2}{1-p} + p}_{p/(1-p)} \right)^{T-2} \cdot \frac{1}{4} \cdot (\cdots)^2 = \frac{1}{4} \left( \frac{1}{1-p} + \frac{1}{p} \right)^T, \quad \mathbb{E} \frac{A_r}{\bar{w}^2} = \frac{1}{2} \left( \frac{1}{1-p} + \frac{1}{p} \right)^T \quad \text{33a}, \quad \mathbb{E} \frac{A_r L_s}{c^{\sum \{A_t \neq L_t\}}} = \left( \mathbb{E}\, c^{-\{A_t \neq L_t\}} \right)^{r-1+T-s}$$

$$\mathbb{E}\left( \frac{A_r}{c^{\{A_r \neq L_r\}}} \prod_{r+1}^{s-1} c^{-\{A_t \neq L_t\}} \frac{L_s}{c^{A_s \neq L_s}} \right), \quad \mathbb{P}(A_t \neq L_t \mid L_{t+1}) = 2\, \mathbb{P}(A_t \neq L_t,\, L_t = \ell) = 2(1-p),$$

$$\{A_t \neq L_t\} \perp L_{t+1}, \quad (s \neq r+1), \quad = \left( \mathbb{E}\, c^{-\{A_t \neq L_t\}} \right)^{T-s+r-1} \mathbb{E}\left( \frac{L_s}{c^{A_s \neq L_s}} \right) = \left( \frac{1-p}{c} + p \right)^T / 2^2, \quad \mathbb{E} \frac{L_r A_{r+1}}{c^{\{A_r \neq L_r\}} c^{\{\cdots\}}}$$

$$= \mathbb{P}(1,0,0,1) \frac{1}{c^2} + \mathbb{P}(1,0,1,1) \frac{2}{c} + \mathbb{P}(1,1,1,1) = \frac{1}{2} \left( (1-p)^2 \frac{p}{c^2} + 2(1-p)^2 p/c + p^3 \right) = \cancel{\frac{1}{2}\left( \frac{p^5}{(1-p)^2} + \frac{2p^4}{1-p} + p^3 \right)}$$

$$= \cancel{\frac{p^3}{2(1-p)} \left( \frac{p^2}{1-p} + 2p + 1-p \right) = \frac{p^3}{2(1-p)^2}} \quad \frac{p}{2} \left( p^2 + (1-p)^2 \left( \frac{p}{1-p} \right)^2 \left( 2 + \left( \frac{p}{1-p} \right)^2 \right) \right) = \frac{p^3}{2} \left( 1 + 2 + \frac{p^2}{(1-p)^2} \right)$$

$$\cancel{\frac{p^3}{2} \frac{1}{(1-p)^2} \left( p^3 + (1-p)^2 (1+2p) \right) \cdot \frac{p^3}{2(1-p)^2}} = \frac{p^3}{2} \left( 3 + \frac{p^2}{(1-p)^2} \right) = \frac{p^3}{2(1-p)^2} \left( 4p^2 - 6p + 3 \right) \quad \text{33c},$$

$$\mathbb{E} \frac{A_r L_{r+1}}{c^{\{A_r \neq L_r\}} c^{\{A_{r+1} \neq L_{r+1}\}}} = \frac{1}{c^2} \mathbb{P}(0,1,1,0) + \frac{2}{c} \mathbb{P}(0,1,1,1) + \mathbb{P}(1,1,1,1) = \frac{1}{2} \left( \frac{1}{c^2} p (1-p)^2 + \frac{2}{c} p^2 (1-p) + p^3 \right) = \frac{p}{2} \left( p^2 + (1-p) \right($$

$$(1-p) \left( \frac{p}{1-p} \right)^4 + 2p \left( \frac{p}{1-p} \right)^2 \Big) \Big) = \frac{p^3}{2} \left( 1 + (1-p) \left( \frac{p^2}{(1-p)^3} + \frac{2p}{(1-p)^2} \right) \right) = \frac{p^3}{2} \left( 1 + \frac{p}{(1-p)} \left( \frac{p}{1-p} + 2 \right) \right) \quad \text{33d}$$

$$= \frac{p^3}{2} \left( 1 + \frac{p}{1-p} \cdot \frac{2-p}{1-p} \right) = \frac{p^3}{(1-p)^2} \left( 2p - p^2 + p^2 - 2p + 1 \right) = \frac{p^3}{(1-p)^2}, \quad \frac{p^3}{2} \left( 1 + \left( \frac{p}{1-p} \right)^2 + \frac{2p}{1-p} \right) = \frac{p^3}{2} \left( 1 + \frac{p}{1-p} \right)^2 = \frac{p^3/2}{(1-p)^2} = \frac{p}{2} \left( \frac{p}{1-p} \right)^2$$

$$\left( W^{-1} \left( \sum A_j \right) (\varepsilon + \eta) \right)^2, \quad W^{-1} \left( \sum A_j \right)^2 \left( \sum L_j - \mathbb{E}(L_j \mid A_{j-1}) \right)^2 = W^{-1} \left( \sum A_j \right)^2 \left( \sum L_j + \sum \left( \mathbb{E}(L_j \mid A_{j-1}) \right)^2 - 2 \sum L_j\, \mathbb{E}(L_j \mid A_{j-1}) \right),$$

$$\mathbb{E} \frac{A_j A_{j+1}}{c^{\{A_j \neq L_j\}} c^{\{A_{j+1} \neq L_{j+1}\}}} = \frac{1}{2} \left( \mathbb{P}(0101) \frac{1}{c^2} + \mathbb{P}(0111) \frac{2}{c} + \mathbb{P}(1111) \right) = \frac{1}{2} \left( (1-p)^3 \left( \frac{p}{1-p} \right)^4 + 2 p^2 (1-p) \left( \frac{p}{1-p} \right)^2 \right.$$

33aa

$$\left. + p^3 \right) = \cancel{\frac{p^3}{2} \left( 1 + \frac{p}{1-p} + \frac{2p}{1-p} \right) = \frac{p^3}{2}} \quad + p(1-p)^2 \left( \frac{p}{1-p} \right)^2 + p^3 \Big) = \frac{p^3}{2} \left( \frac{2p}{1-p} + 1 + 1 \right) = p^3 \cdot \frac{1}{1-p},$$

$$\mathbb{E} \frac{A_j A_{j+1}}{\bar{w}} = \left( \frac{1}{p} + \frac{1}{1-p} \right)^{T-2} \frac{p^3}{1-p} \cdot \frac{1}{p^4} = \left( \frac{1}{p(1-p)} \right)^{T-1}, \quad \mathbb{E} \frac{A_j A_{j+1} L_{j+3}}{w} = p^{-2T} \left( \mathbb{E} \frac{1}{c^{\{A_t \neq L_t\}}} \right)^{T-3} \mathbb{E} \frac{A_j A_{j+1}}{c^{\{A_j \neq L_j\}} c^{\{A_{j+1} \neq L_{j+1}\}}} \cdot \mathbb{E} \frac{L_{j+3}}{c^{\{A_{j+3} \neq L_{j+3}\}}}$$

$$= p^{-2T} \left( \frac{p}{1-p} \right)^{T-3} \cdot p^3 \cdot \frac{1}{1-p} \cdot \frac{1}{2} \cdot \frac{p}{1-p} = \frac{1}{2} \frac{p^{-T+1}}{(1-p)^{T-1}} = \frac{1}{2} \left( \frac{1}{p(1-p)} \right)^{T-1}$$

$$\mathbb{E} \left( w^{-1} A_1 \left( L_T - \mathbb{E}(L_T \mid A_{T-1}) \right) \right)^2, \quad \mathbb{E} \frac{A_1 L_T}{\bar{w}^2} = p^{-2T}\, \mathbb{E} \frac{A_1 L_T}{\prod_t c^{\{A_t \neq L_t\}}}, \quad \mathbb{E}\left( \frac{L_T}{c^{\{A_T \neq L_T\}}} \,\middle|\, A_{T-1} \right) = \{A_{T-1} = 0\} \left( \frac{1}{c} \mathbb{P}(010) \right.$$

$$\left. + \mathbb{P}(011) \right) + \{A_{T-1} = 1\} \left( \frac{1}{c} \mathbb{P}(110) + \mathbb{P}(111) \right) = \{A_{T-1} = 0\} \left( p^2 + p(1-p) \right) + \{A_{T-1} = 1\} \left( \frac{p^3}{1-p} + p^2 \right) = p \{A_{T-1} = 0\}$$

$$+ \frac{p^2}{1-p} \{A_{T-1} = 1\}, \quad \mathbb{E}\left( \frac{p \{A_{T-1} = 0\} + \frac{p^2}{1-p} \{A_{T-1} = 1\}}{c^{\{A_{T-1} \neq L_{T-1}\}}} \,\middle|\, L_{T-1} \right) = \{L_{T-1} = 0\} \left( (1-p) p + p \frac{p^2}{1-p} \cdot \frac{p^2}{1-p} \right) + \{L_{T-1} = 1\} \left( (1-p) \frac{p^2}{1-p} p + p \frac{p^2}{1-p} \right)$$

$$= \{L_{T-1} = 0\}\, p \left( 1 - p + \frac{p^4}{(1-p)^2} \right) + \{L_{T-1} = 1\}\, p^3 \frac{2-p}{1-p}$$

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$\mathbb{E}\frac{A_1 L_T}{c^{\{A_1\neq L_1\}} c^{\{A_T\neq L_T\}}} = \mathbb{P}(A_1=1, L_T=1)\,\mathbb{E}\left(c^{-\{L_1\neq A_1\}} c^{-\{A_T\neq L_T\}} \middle| A_1=1, L_T=1\right) = \mathbb{P}(A_1=1,L_T=1)\,\mathbb{E}(c^{-\{L_1\neq A_1\}}|A_1=1)$

$\cdot\, \mathbb{E}(c^{-\{A_T\neq L_T\}}|L_T=1) = \mathbb{P}(A_1=L_1=1)(p+(1-p)c^{-1})^2 = \mathbb{P}(A_1=1,L_1=1)\left(\frac{p}{1-p}\right)^2$, $\mathbb{E}\left(\frac{A_1L_T}{\bar w}\right)^2 = \cancel{\frac{1}{4}(1+2(1-p))}$

$\frac14(1+(1-2(1-p))^{2T-3})\cdot\left(\frac{p}{1-p}\right)^T = \frac14(1+(2p-1)^{2T-3})\cdot\left(\frac{p}{1-p}\right)^T \doteq \frac14\left(\frac{p}{1-p}\right)^T$ (33h) $\mathbb{E}(L_T|A_{T-1}) = \{A_{T-1}=0\}$

$(1-p)+\{A_{T-1}=1\}p = p^A(1-p)^{1-A} = (1-p)\left(\frac{p}{1-p}\right)^{A_{T-1}}$, $\mathbb{E}\left(\frac{A_1\,\mathbb{E}(L_T|A_{T-1})}{c^{\{A_1\neq L_1\}}c^{\{A_{T-1}\neq L_{T-1}\}}}\right)$

$= \mathbb{P}(A_1=1,A_{T-1}=0)\,(1-p)\left(p(1-p)\frac{2}{c}+(1-p)^2\frac{1}{c^2}+p^2\right) + \mathbb{P}(A_1=1,A_{T-1}=1)\cdot p\cdot\left((1-p)^2\frac{1}{c^2}+2p(1-p)\frac1c\right.$

$\left.+p^2\right) = \left(\frac{p^4}{(1-p)^2}+p^2+\frac{2p^3}{1-p}\right)\left(p\,\mathbb{P}(A_1=1,A_{T-1}=1)+(1-p)\,\mathbb{P}(A_1=1,A_{T-1}=0)\right) = p^2\left(\frac{p^2}{(1-p)^2}+1+\frac{2p}{1-p}\right)$

$pu+(1-p)(1-u) = 2pu-p-u+1$

$\left(p\,\mathbb{P}(2\mid \text{binom}(2(T-2),1-p)) + (1-p)\,\mathbb{P}(2\nmid \text{binom}(2(T-2),1-p))\right) = \left(\frac{p}{1-p}\right)^2\cdot\Big($

$p\,\mathbb{P}(2\mid\text{binom}(2(T-2),1-p))(2p-1)+1-p\Big)$, $\mathbb{E}\left(\frac{A_1\,\mathbb{E}(L_T|A_{T-1})}{c_1\, c_{T-1}}\right) =$

$\sum_\ell \mathbb{E}(\cdots|A_1=1,L_{T-1}=\ell)\,\mathbb{P}(A_1=1,L_{T-1}=\ell) = \sum_\ell \mathbb{P}(A_1=1,L_{T-1}=\ell)\left(\mathbb{E}\,c^{-\{A_1\neq L_1\}}|A_1=1\right)\left(\mathbb{E}\frac{\mathbb{E}(L_T|A_{T-1})}{c^{\{A_{T-1}\neq L_{T-1}\}}}\middle|L_{T-1}=\ell\right)$

$=\sum_\ell \mathbb{P}(A_1=1,L_{T-1}=\ell)\left(p+\frac{p^2}{1-p}\right)\left((1-p)^2\frac1c\cdot\left(\frac{p}{1-p}\right)^{1-\ell}+p(1-p)\left(\frac{p}{1-p}\right)^\ell\right) = \sum\mathbb{P}(A_1=1,L_{T-1}=\ell)\,\frac{p}{1-p}\cdot p\left(p\left(\frac{p}{1-p}\right)^{1-\ell}\right.$

$\left.+(1-p)\left(\frac{p}{1-p}\right)^\ell\right) = \sum_\ell \mathbb{P}(A_1=1,L_{T-1}=\ell)\cdot\frac{p^2}{1-p}\cdot\left(\{\ell=0\}\left(\frac{p^2}{1-p}+1-p\right)+\{\ell=1\}\,2p\right)$

$\frac12(1+(1-2(1-p))^n)$

$=\frac{1}{2}\frac{p^2}{1-p}\left(\frac{p^2}{1-p}+1-p\right)\mathbb{P}(2\nmid\text{binom}(2(T-1)-3,1-p)) + \frac{p^3}{1-p}\,\mathbb{P}(2\mid\text{binom}(2(T-1)-3,1-p)) = \frac{p^3}{1-p}$

$+\,\mathbb{P}(2\nmid\text{binom}(2(T-1)-3,1-p))\cdot\frac{p^2}{1-p}\left(\frac12\cdot\frac{p^2}{1-p}+\frac{1-p}{2}-p\right)$, $p^2/2+\frac12(1-p)^2-p(1-p) = 2p^2-2p+\frac12 = 2(p-\frac12)^2$,

$=\frac{p^3}{1-p}+\frac{2p^2(p-\frac12)^2}{(1-p)^2}\cdot\frac12\left(1-(1-2(1-p))^{2T-5}\right)$, $p-p^2+p^2-p+\frac14$, $\doteq \frac14\left(\frac{p}{1-p}\right)^2$, $\mathbb{E}\left(\frac{A_1\,\mathbb{E}(L_T|A_{T-1})}{\bar w^2}\right)$

(33i)

$=\left\{\frac{p^3}{1-p}+\frac{p^2(p-\frac12)^2}{(1-p)^2}\left(1-(1-2(1-p))^{2T-5}\right)\right\}\left(\frac{p}{1-p}\right)^{T-2}\frac{1}{p^{2T}} \doteq \frac14\left(\frac{1}{p(1-p)}\right)^T$, $\mathbb{E}\left(\frac{A_1\,\mathbb{E}^2(L_T|A_{T-1})}{c_1\,C_{T-1}}\right)$

$=\sum_\ell\mathbb{P}(A_1=1,L_{T-1}=\ell)\left(\frac{p}{1-p}\right)\left\{(1-p)^3\frac1c\left(\frac{p}{1-p}\right)^{2(1-\ell)}+p(1-p)^2\left(\frac{p}{1-p}\right)^{2\ell}\right\} = \frac12\sum_\ell\mathbb{P}(L_{T-1}=\ell|A_1=1)\cdot$

"$1-u$"

$p(1-p)\left\{\frac{p^2}{(1-p)}\left(\frac{p}{1-p}\right)^{2-2\ell}+p\left(\frac{p}{1-p}\right)^{2\ell}\right\} = \frac12\,\mathbb{P}(L_{T-1}=0|A_1=1)\,p(1-p)\left\{\frac{p^4}{(1-p)^3}+p\right\}+\frac12\,\mathbb{P}(L_{T-1}=1|$

$A_1=1)\cdot p(1-p)\left\{\frac{p^2}{1-p}+p\left(\frac{p}{1-p}\right)^2\right\} = \frac{1-u}{2}\frac{p^3}{1-p}\left\{\left(\frac{p}{1-p}\right)^2+p\right\}+\frac u2\frac{p^3}{1-p}\{1+p\} = \frac{1-u}{2}\frac{p^3}{(1-p)^2}\frac{p}{1-p}$

(33j)

$+\frac u2\frac{p^3}{(1-p)^2}(1-p^2) = \frac p2\left(\frac{p}{1-p}\right)^3+\frac u2\frac{p^3}{(1-p)^2}\left(1-p^2-\frac{p}{1-p}\right)$, $1-2p-p^2+p^3$, $=\frac{1-u}{2}p(1-p)\left(\frac{p^4}{(1-p)^3}+p\right)+\frac u2\, p^3/1-p$

$\doteq\frac14p^3\left(\left(\frac{p}{1-p}\right)^2+\frac{1-p}{p}+\frac{1}{1-p}\right) = \frac{p^3}{4(1-p)}\left(\frac{p^2}{1-p}+\frac{(1-p)^2}{p}+1\right)$

(17)

$$\mathbb{E}\frac{A_1 L_T}{\prod c^{\{A_t \neq L_t\}}} = \mathbb{E}\left(\frac{A_1}{\prod^{T-1} c^{\{\}}}\cdot \mathbb{E}\left(\frac{L_T}{c^{\{A_T\neq L_T\}}}\,\middle|\,\frac{A_1}{c^{\{A_1\neq L_1\}}}\right)\right) = \mathbb{E}\left(\frac{A_1}{c^{\{A_1\neq L_1\}}}\,\mathbb{E}\left(\frac{L_T}{c^{\{A_T\neq L_T\}}}\,\middle|\,\frac{A_1}{c^{A_t L_t}}\right)\right)\left(\mathbb{E}\frac{1}{c^{\{A_1\neq L_1\}}}\right)^{T-2}$$

$$= \mathbb{E}\left(\frac{A_1 L_T}{c^{\{A_1\neq L_1\}}c^{\{A_T\neq L_T\}}}\right)\left(\frac{1}{p(1-p)}\right)^{T-2}$$

$$\mathbb{E}\frac{A_1 L_T \mathbb{E}(L_T|A_{T-1})}{c_1 c_{T-1} c_T} = \sum_\ell \mathbb{E}\left(\frac{A_1}{c^{\{A_1\neq L_1\}}}\,\middle|\,A_1=1\right)\mathbb{E}\left(\frac{L_T\,\mathbb{E}(L_T|A_{T-1})}{c^{\{A_{T-1}\neq L_{T-1}\}}c^{\{A_T\neq L_T\}}}\,\middle|\,L_{T-1}=\ell\right)\mathbb{P}(A_1=1,\ L_{T-1}=\ell),$$

$$\mathbb{E}(\dots|L_{T-1}=\ell) = \mathbb{E}(\dots|\ell,\bar\ell,1)\,\mathbb{P}(L_{T}=1|A_{T-1}=\ell)\,\mathbb{P}(A_{T-1}=\ell|L_{T-1}=\ell) + \mathbb{E}(\dots|\ell,\bar\ell,1)(1-p)p^{1-\ell}$$

$$(1-p)^\ell = (1-p)\left(\frac{p}{1-p}\right)^\ell\left(p+\frac{p^2}{1-p}\right)p\cdot p^\ell(1-p)^{1-\ell} + (1-p)\left(\frac{p}{1-p}\right)^{\bar\ell}\left(\frac{p}{1-p}\right)^3(1-p)p^{1-\ell}(1-p)^\ell = (1-p)^{1-2\ell}p^{2\ell+2}$$

$$+ (1-p)^{2\ell-2}p^{5-2\ell},\quad = \frac{1}{2}\sum_\ell \mathbb{P}(L_{T-1}=\ell|A_1=1)\frac{p}{1-p}\,\mathbb{E}(\dots|L_{T-1}=\ell) = \frac{1}{2}(1-u)\frac{p}{1-p}\left\{(1-p)p^2+(1-p)^{-2}\right.$$

$$\left.p^5\right\} + \frac{u}{2}\cdot\frac{p}{1-p}\left\{\frac{p^4}{1-p}+p^3\right\} = \frac{1-u}{2}\frac{p^3}{1-p}\left\{1-p+\frac{p^3}{(1-p)^2}\right\} + \frac{u}{2}\cdot\frac{p^4}{(1-p)^2},\quad u=\mathbb{P}(2\,|\,\mathrm{binom}(2(T-1)-3,1-p))$$

$$= \frac{1}{2}\left(1+(2p-1)^{2T-5}\right)$$

$$\frac{u}{2}\left(\frac{p}{1-p}\right)^3 - 2\left\{\frac{1-u}{2}\frac{p^3}{1-p}\left(1-p+\frac{p^3}{(1-p)^2}\right)+\frac{u}{2}\frac{p^4}{(1-p)^2}\right\} + \frac{p^4}{4(1-p)^2}\left(\frac{p^2}{1-p}+\frac{(1-p)^2}{p}+1\right),\ \frac{p^3}{4(1-p)}\left(1-p+\frac{p^3}{(1-p)^2}\right.$$

$$\left.+\frac{p}{1-p}\right),\qquad p^3+(1-p)^3+p(1-p) = 3p^2-3p+1+p-p^2 = 2p^2-2p+1,\ \left(\frac{p}{1-p}\right)^3(2p-2p^2)$$

$$= 2p^4,\quad \mathbb{E}\left(\bar{w}^{-1}A_1(L_T-\mathbb{E}(L_T|A_{T-1}))\right)^2 = \frac{1}{4}\left(\frac{p}{1-p}\right)^3(2p-2p^2)\left(\frac{p}{1-p}\right)^{T-3}\frac{1}{p^{2T}}$$

$$= \frac{1}{2}\left(\frac{p}{1-p}\right)^T\frac{p(1-p)}{p^{2T}} = \frac{1}{2}\left(\frac{1}{p(1-p)}\right)^{T-1} = \frac{1}{2}\left(\frac{1}{(\frac12-|p-\frac12|)(\frac12+|p-\frac12|)}\right)^{T-1} = \frac{1}{2}\left(\frac{1}{\frac14-(p-\frac12)^2}\right)^{T-1}$$

$$f(A_{t+1}|A_t) = \left(p^2+(1-p)^2\right)^{\{A_t=A_{t+1}\}}\left(2p(1-p)\right)^{\{A_t\neq A_{t+1}\}} = (2p^2-2p+1)\left(\frac{2p(1-p)}{2p^2-2p+1}\right)^{\{A_t\neq A_{t+1}\}},$$

$$\bar{w} = \prod_{t=1}^T\frac{f(A_t|L_t)}{f(A_t|A_{t-1})} = \left(\frac{p}{2p^2-2p+1}\right)^T\left(\frac{p}{1-p}\right)^{\sum_{t=1}^T\{A_t\neq L_t\}}\left(\frac{2p^2-2p+1}{2p(1-p)}\right)^{\sum_{t=1}^T\{A_t\neq A_{t-1}\}},\quad \frac{1}{\bar{w}^2} = \left(\frac{2p^2-2p+1}{p}\right)^{2T}$$

$$\underbrace{\left(\frac{p}{1-p}\right)}_{\text{“}c\text{”}}{}^{-2\sum\{A_t\neq L_t\}}\underbrace{\left(\frac{2p^2-2p+1}{2p(1-p)}\right)}_{\text{“}d\text{”}}{}^{-2\sum\{A_t\neq A_{t-1}\}} = \left(\frac{2p^2-2p+1}{p}\right)^{2T}c^{2\{A_t\neq L_t\}}d^{\sum\{A_t\neq A_{t-1}\}},\quad \mathbb{E}\frac{A_1L_T}{c^{\sum A_t\neq L_t}d^{\sum A_t\neq A_{t-1}}} =$$

$L_1$ $L_T$ / $A_0$ $A_{T-1}$ $A_T$

$$\mathbb{E}\left(\frac{A_1}{c^\Sigma d^\Sigma}\mathbb{E}(L_T|A_1)\right) = \mathbb{E}(A_1L_T)\,\mathbb{E}\left(\frac{1}{c^\Sigma d^\Sigma\cdots}\right) = \mathbb{P}(2\neq\mathrm{binom}(2(T-2)+1,1-p))\cdot\left(\mathbb{E}\frac{1}{c^{A_1\neq L_1}d^{A_1\neq A_2}}\right)^{T-1}\cdot\mathbb{E}\left(\frac{1}{c^{A_1\neq L_1}}\right)$$

$$\mathbb{E}\left(\frac{1}{d^{A_1\neq 0}}\right),\quad \mathbb{E}(d_1,d_2,c_2,c_3|A_1=a,L_3=\ell) = p\,\mathbb{E}(\dots|A_1=a,L_3=\ell,A_3=\ell) + (1-p)\,\mathbb{E}(\dots|A_1=a,L_3=\ell,$$

$$A_3=\bar\ell) = p\,(\mathbb{E}(\dots|$$

$L_1$ $L_2$ $L_3$ / $A_1$ $A_2$ $A_3$

(18)

$\mathbb{E}\left(\frac{1}{f(A|L)}\Big| L\right) = \mathbb{E}\left(\frac{1}{p^{\{A=L\}}(1-p)^{\{A\neq L\}}}\Big| L\right) = p\cdot\frac{1}{p} + (1-p)\frac{1}{1-p} = 2$, $\mathbb{E}\left(\frac{A}{f(A|L)}\Big| L\right) = p\,\mathbb{E}\left(\frac{L}{p^{\{A=L\}}(1-p)^{\{A\neq L\}}}\Big| L, A=L\right)$

$+ (1-p)\,\mathbb{E}\left(\frac{L}{\cdots}\Big| L, A\neq L\right) = p\frac{L}{p} + (1-p)\frac{1-L}{1-p} = 1$, $\mathbb{E}\left(\frac{A}{p^{2\{A=L\}}(1-p)^{2\{A\neq L\}}}\Big| L\right) = p\frac{L}{p^2} + (1-p)\frac{1-L}{(1-p)^2} = \frac{L}{p} + \frac{1-L}{1-p}$,

$\mathbb{E}(\cdots|A) = \frac{1}{1-p} + \left(\frac{1}{p} - \frac{1}{1-p}\right)p^A(1-p)^{1-A} = \frac{1}{1-p} + (1-2p)\frac{p^A(1-p)^{1-A}}{p(1-p)} = \frac{1}{1-p} \; \frac{1-2p}{p}\left(\frac{p}{1-p}\right)^A$

$\left(\sum A_j\right)^2\left(\sum \tilde{L}_j\right)^2 = \sum_{j,k,l,m} A_j A_k \tilde{L}_l \tilde{L}_m$ $(j<k<l<m)$ $\mathbb{E}A_jA_kL_lL_m = \frac{1}{2}\mathbb{P}(2|B(2(k-j)))\,\mathbb{P}(2|B(2(l-k)-1))\cdot$

$\mathbb{P}(2|B(2(m-l)))$, $\mathbb{E}\left(\sum A_t\right)^2 = 2\sum_{j<k}\mathbb{E}A_jA_k + \frac{T}{2} = \frac{T}{2} + \sum_{j<k}\frac{1}{2}\left(1 + (2p-1)^{2(k-j)}\right) = \frac{T}{2} + \binom{T}{2}$

$+\sum_{j=1}^{T-1}(2p-1)^{-2j}\sum_{k=j+1}^{T}(2p-1)^{2k}$, $\sum_{j=1}^{T-1}(2p-1)^{-j}\sum_{k=1}^{T-j}(2p-1)^{2k+2j} = \sum_{j=1}^{T-1}(2p-1)^j\left(\frac{1-(2p-1)^{2(T-j+1)}}{1-(2p-1)} - 1\right)$

$= \left(\frac{1-(2p-1)^T}{2(1-p)} - 1\right)\sum_{j=1}^{T-1}\left(\frac{1}{2(1-p)} - 1\right) - \frac{1}{2(1-p)}\sum_{j=1}^{T-1}(2p-1)^{2T-j+1} = \frac{2p-(2p-1)^T-1}{2(1-p)}\cdot\frac{2p-1}{2-2p}$

$-\frac{(2p-1)^{2T+1}}{2(1-p)}\cdot\left(\frac{1-(2p-1)^{-T}}{1-(2p-1)^{-1}} - 1\right) = \frac{(2p-1)^2\left(1-(2p-1)^{T-1}\right)}{4(1-p)^2} - \frac{(2p-1)^{2T+1}}{2(1-p)}\cdot\left(\frac{(2p-1)^T - 1}{(2p-1)^T-(2p-1)^{T-1}} - 1\right)$

$= \text{———} - \frac{(2p-1)^{2T+1}}{2(1-p)}\cdot\frac{(2p-1)^{T-1}-1}{(2p-1)^T-(2p-1)^{T-1}} = \frac{\left((2p-1)^{T-1}-1\right)}{2(1-p)}\left\{-\frac{(2p-1)^2}{2(1-p)} - \frac{(2p-1)^{2T+1}}{(2p-1)^T-(2p-1)^{T-1}}\right\}$

$= -\frac{\left((2p-1)^{T-1}-1\right)}{2(1-p)}(2p-1)^2\left\{\frac{(2p-1)^T-(2p-1)^{T-1}+2(1-p)(2p-1)^{2T-1}}{\quad}\right\}$, $2p-1-1+2(1-p)(2p-1)^T$

$= 2(1-p)\left((2p-1)^T - 1\right)$, $= -\frac{\left((2p-1)^{T-1}-1\right)}{2(1-p)}(2p-1)^{T+1}\cdot\frac{2(1-p)\left((2p-1)^T-1\right)}{2(1-p)\left\{(2p-1)^T-(2p-1)^{T-1}\right\}} = \frac{\left((2p-1)^{T-1}-1\right)\left((2p-1)^T\right.}{2(1-p)\left\{(2p-1)^T\right.}$

$\frac{\left.-1\right)(2p-1)^{T+1}}{\left.-(2p-1)^{T-1}\right\}} = \frac{\left((2p-1)^{T-1}-1\right)\left((2p-1)^T-1\right)(2p-1)^2}{\left(2(1-p)\right)^2}$, $\sum_{1\le j<k\le T} = \frac{1}{2}\sum_{k=2}^{T}\sum_{j=1}^{k-1}(2p-1)^{2k-2j} = \frac{1}{2}\sum_{k=2}^{T}(2p-1)^{2k}$

$\left(\frac{1-(2p-1)^{-2k}}{1-(2p-1)^{-2}} - 1\right) = \frac{1}{2}\left(\frac{1}{1-(2p-1)^{-2}} - 1\right)\left(\frac{1-(2p-1)^{2(T+1)}}{1-(2p-1)^2} - 1 - (2p-1)^2\right) - \frac{T-1}{1-(2p-1)^{-2}}\frac{1}{2}$

$= \frac{1}{2}\cdot\frac{1}{4p^2-4p}\cdot\left(\frac{1-(2p-1)^{2(T+1)}}{2p\cdot 2\cdot(1-p)} \; -4p^2+4p-2\right) - \frac{T-1}{2}\cdot\frac{(2p-1)^2}{\underbrace{4p^2-4p}_{\text{"}d\text{"}}} = \frac{1}{2d}\left(\frac{(2p-1)^{2(T+1)}-1}{d} - d - 2\right.$ $d+1 = (1-2p)^2 =: e$

$\left.-(T-1)(2p-1)^2\right) = \frac{1}{2d}\left(\frac{(d+1)^{T+1}-1}{d} - d - 2 - (T-1)(d+1)\right) = \frac{(d+1)^{T+1}-1-d-Td(d+1)}{2d^2} = \frac{e^{T+1}-e-Te(e-1)}{2(e-1)^2}$

$-1-T(d+1)$ (355)

$\frac{e(T-Te-1)}{2(e-1)^2}$, $\mathbb{E}\left(\sum A_t\right)^2 = \frac{T}{2} + \frac{1}{2}\binom{T}{2} + \frac{e^{T+1}-e+Te(1-e)}{2(e-1)^2} = \frac{T}{2} + \frac{1}{2}\binom{T}{2} + \frac{e(T-Te-1)}{2(e-1)^2} = \frac{T}{2} + \frac{1}{2}\binom{T}{2} + \frac{e(Td+}{2d^2}$

(19)

$\mathbb{E}(L_t \mathbb{E}(L_t|A_t))$

$\mathbb{E}\sum_t (L_t - \mathbb{E}(L_t|A_t))^2 = \sum_t \mathbb{E}\{L_t^2 - \mathbb{E}(\mathbb{E}^2(L_t|A_t))\} = \frac{T}{2} - \sum_t \mathbb{E}(p^{2A}(1-p)^{2(1-A)}) = \frac{T}{2} - \frac{T}{2}(p^2+(1-p)^2)$

$(u>t)$

$= \frac{T}{2}(-2p^2+2p) = Tp(1-p)$, $\mathbb{E}(L_t - \mathbb{E}(L_t|A_{t-1}))(L_u - \mathbb{E}(L_u|A_{u-1})) = \mathbb{E}L_tL_u - \mathbb{E}(L_t\mathbb{E}(L_u|A_{u-1}))$

$-\mathbb{E}(L_u\mathbb{E}(L_t|A_{t-1})) + \mathbb{E}(\mathbb{E}(L_t|A_{t-1})\mathbb{E}(L_u|A_{u-1})) = \cancel{\xi(2(u-t)) - \mathbb{E}(\mathbb{E}(L_t|A_{u-1})\mathbb{E}(L_u|A_{u-1}))}$

$= \xi(2(u-t)) - (\xi(2(u-t)-1)p + (1-\xi(\dots))(1-p)) - (\xi(2(u-t)+1)p + (1-\xi(\dots))(1-p))$

$+ ((p^2+(1-p)^2)\xi(2(u-t)) + 2p(1-p)(1-\xi(\dots)))$, $\sum_{j=1}^T j\binom{T}{j} = \sum_{j=0}^T \frac{\partial}{\partial x} x^j \binom{T}{j}\Big|_{x=1} = \frac{\partial}{\partial x}(1+x)^T\Big|_{x=1}$

$= T2^{T-1}$, $\mathbb{E}\frac{1}{f(A|L)} = \mathbb{E}(p^{A=L}(1-p)^{A\ne L})^{-1} = p\cdot\frac{1}{p} + (1-p)\cdot\frac{1}{1-p} = 2$, $\mathbb{E}\frac{h(A)}{f(A|L)} = \sum_a h(a)\,\mathbb{E}\frac{\{A=a\}}{f(a|L)}$

$= \sum_a h(a)\,\mathbb{E}\frac{1}{f(a|L)}\mathbb{E}(A=a|L) = \sum_a h(a)$, $\sum_{j=1}^T j^2\binom{T}{j} = \sum_0^T \frac{\partial}{\partial x} x \frac{\partial}{\partial x} x^j \binom{T}{j}\Big|_{x=1} = \frac{\partial}{\partial x} x \cdot \frac{\partial}{\partial x}(1+x)^T\Big|_{x=1} = \frac{\partial}{\partial x} xT(1+x)^{T-1}$

$= T(1+x)^{T-1} + xT(T-1)(1+x)^{T-2}\Big|_{x=1} = T2^{T-1} + T(T-1)2^{T-2} = 2^{T-2}(T+T^2)$ (35 d), $\mathbb{E}\eta^2 = \mathbb{E}\sum_{t=1}^T \lambda^2 (L_t - \mathbb{E}(L_t|A_{t-1}))^2$

$= \lambda^2 Tp(1-p) + 2\lambda^2\sum_{t<u}(L_t - \mathbb{E}(L_t|A_{t-1}))(L_u - \mathbb{E}(L_u|A_{u-1})) \doteq \lambda^2 Tp(1-p) + 2\binom{T}{2}O(p^{2T}) \doteq \lambda^2 Tp(1-p)$

$\gamma := p(1-p)$, $\left\{\frac12\binom{T}{2} + \frac{T}{2} - \frac{(-4\gamma+1)\cdot(-4T\gamma+1)}{32\gamma^2}\right\}\left\{T\gamma^{1-T} + \sigma^2\gamma^T\right\}$, $\frac{\frac12\left(\binom{T}{2}+T\right)T}{\{2^{T-2}T(T+1)\}^2}$

$= \frac12 \left(\frac{T(T-1)}{2}+T\right)T = \frac12\frac{\frac{T+1}{2}}{2^{2T-4}(T+1)^2} = \frac{1}{2^{2(T-1)}(T+1)}$, $\frac{1}{T+1}\frac{\gamma^{1-T}}{2^{2(T-1)}} = \frac{1}{T+1}\cdot\frac{1}{(4\gamma)^{T-1}}$, $\gamma = p(1-p) = (\frac12+|p-\frac12|)(\frac12-|p-\frac12|) = \frac14 - (p-\frac12)^2$, $\frac{1}{T+1}\cdot\frac{\lambda^2}{(1-(p-\frac12)^2)^{T-1}}$,

$f(A_t|A_{t-1}) = (p^2+(1-p)^2)^{A_t=A_{t-1}}(2p(1-p))^{A_t\ne A_{t-1}}$, $p^2+(1-p)^2 = (\frac12+u)^2+(\frac12-u)^2 = 2u^2+\frac12 = 2(u^2+\frac14)$

$(4u^2+1)^T$

$\mathbb{P}(A=a|L=\ell, U=u) = (1-p_B)p_L^{\{\ell=a\}}(1-p_L)^{\{\ell\ne a\}} + p_B p_U^{\{u=a\}}(1-p_U)^{\{u\ne a\}}$, $\mathbb{P}(A=a|L=1, U=u, Z=z)$

$= \mathbb{P}(A=a|L=\ell,U=u) + (-1)^{1-z}(-1)^{1-a}\delta_\ell/2$, $\Delta(a|\ell) = (-1)^{1-a}\delta_\ell$, $\frac{1}{W^2} = \prod \delta_{L_t}^{-2}$,

| | | | | |
|---|---|---|---|---|
| $u=1$ | $\frac{\pm\delta_0}{p_{01}}$ | $\frac{\mp\delta_0}{\bar p_{01}}$ | $\frac{\pm\delta_1}{p_{11}}$ | $\frac{\mp\delta_1}{\bar p_{11}}$ |
| $u=0$ | $\frac{\pm\delta_0}{p_{00}}$ | $\frac{\mp\delta_0}{\bar p_{00}}$ | $\frac{\mp\delta_1}{p_{10}}$ | $\frac{\mp\delta_1}{\bar p_{10}}$ |
| | $l=0$ | | $l=1$ | |

$\mathbb{P}(W^{-2}) = \delta_0^k(1-\delta_0)^{T-k} =$, $\mathbb{E}\prod^{T-1}\ \mathbb{E}\,\delta_{L_T}^m|\delta_{L_{T-1}} = \delta_{L_{T-1}}^m p + \delta_{\bar L_{T-1}}^m(1-p)$, $\mathbb{E}\prod^{T-2}\delta_{L_t}\,\delta_{L_{T-1}}$

$\delta_{L_{T-1}}$, $\mathbb{E}\prod^{T-1}\delta_{L_t}\,\mathbb{E}\,\delta_{L_T}|\delta_{L_{T-1}} = \mathbb{E}\{\prod\delta_{L_t}(\delta_{L_{T-1}}p + \delta_{\bar L_{T-1}}(1-p))\} = p\,\mathbb{E}\prod^{T-1}\delta_{L_t}\cdot\delta_{L_{T-1}} + (1-p)\delta_0\delta_1\mathbb{E}\prod^{T-1}\delta_{L_t}$,

$\mathbb{E}\prod^{T-2}\delta_{L_t}\,\mathbb{E}\,\delta^2_{L_{T-1}}|\delta_{L_{T-2}} = \mathbb{E}\prod^{T-2}\delta_{L_t}\{p\,\delta^2_{L_{T-2}} + (1-p)\delta^2_{\bar L_{T-2}}\}$, $\frac12\delta_0\,\mathbb{E}\prod_2^T\delta_{L_t}|L_1=0 + \frac{\delta_1}{2}\mathbb{E}\prod\delta_{L_t}|L_1=1$,

$p^{n-2k-1}(1-p)^k\sum_{r=0}^k\binom{k}{r}\left(\frac{p}{1-p}\right)^{2r}\cdot\left\{p\binom{n-k-1}{k-r-1} + (1-p)\binom{n-k-1}{k-r}\right\}$ $e_{n-1}(\ell) = \mathbb{E}\prod_{t=n}^T\delta_{L_t}|L_{n-1}=\ell_{n-1}$, $e_{n-1}(\ell_{n-2})$

$\mathbb{E}\left(\prod_{n-1}^T\delta_{L_t}\Big|L_{n-2}=\ell_{n-2}\right) = \sum_{\ell_{n-1}\in 0,1}\delta_{\ell_{n-1}}\mathbb{E}\left(\prod_n^T\delta_{L_t}\Big|L_{n-2}=\ell, L_{n-1}=\ell_{n-1}\right)\mathbb{P}(L_{n-1}=\ell_{n-1}|L_{n-2}=\ell_{n-2}) = \sum_{\ell_{n-1}\in 0,1}P_{\ell_{n-2},\ell_{n-1}}\,e_n(\ell_{n-1})\,\delta_{\ell_{n-1}}$

(20) $e_{n-1}(0) = p_{00} e_n(0)\delta_0 + p_{01} e_n(1)\delta_1$, $e_{n-1}(1) = p_{10} e_n(0)\delta_0 + p_{11} e_n(1)\delta_1$, $e_{n-1} = \begin{pmatrix} \delta_0 p_{00} & \delta_1 p_{01} \\ \delta_0 p_{10} & \delta_1 p_{11} \end{pmatrix} e_n$

$= \begin{pmatrix} \delta_0 p & \delta_1(1-p) \\ \delta_0(1-p) & \delta_1 p \end{pmatrix} e_n$, $\mathbb{E}\,\delta_{L_t} = \frac{1}{2}(\delta_0+\delta_1)$, $\mathbb{E}\,\delta_{L_1}\delta_{L_2} = \mathbb{E}\,\delta_{L_1}(p\,\delta_{L_1} + \bar p\,\delta_{\bar L_1}) = p\,\mathbb{E}\,\delta_{L_1}^2 + \bar p\,\delta_0\delta_1$

$= \frac{p}{2}(\delta_0^2+\delta_1^2) + \bar p\,\delta_0\delta_1$, $\mathbb{E}\,\delta_{L_1}\delta_{L_2}\delta_{L_3} = \frac{\delta_0}{2}\mathbb{E}(\delta_{L_2}\delta_{L_3} \mid L_1=0) + \frac{\delta_1}{2}\mathbb{E}(\delta_{L_2}\delta_{L_3}\mid L_1=1)$, $e_2 = P\begin{pmatrix} e_3(0) \\ e_3(1)\end{pmatrix}$

$= P\begin{pmatrix} p\delta_0+\bar p\delta_1 \\ p\delta_1+\bar p\delta_0\end{pmatrix} = \begin{matrix} p^2\delta_0 + 2p\bar p\,\delta_1 + \bar p^2\delta_0 = 2p(1-p)\delta_1 + (2p^2-2p+1)\delta_0, \\ 2p\bar p\,\delta_0 + \bar p^2\delta_1 + p^2\delta_1 = 2p(1-p)\delta_0 + (2p^2-2p+1)\delta_1 \end{matrix}$, $\mathbb{E}\prod^2 \delta_{L_t} = \cancel{\tfrac{1}{2}(e_2(0)+e_2(1))} \; \frac{1}{2}(\delta_0 e_2(0) + \delta_1 e_2(1))$

$= (p^2-p+\frac{1}{2})(\delta_0^2+\delta_1^2) + 2p(1-p)\delta_0\delta_1$, $= \frac{\delta_0}{2}\big(p\delta_0\,\mathbb{E}(\delta_{L_3}\mid L_2=0) + \bar p\,\delta_1\,\mathbb{E}(\delta_{L_3}\mid L_2=1)\big) + \frac{\delta_1}{2}\big(p\,\delta_1\,\mathbb{E}(\delta_{L_3}\mid L_2=1)$

$+ \bar p\,\delta_0\,\mathbb{E}(\delta_{L_3}\mid L_2=0)\big) = \frac{\delta_0}{2}\left(p^2\delta_0^2 + p\bar p\,\delta_0\delta_1 + \bar p^2\delta_0\delta_1 + p\bar p\,\delta_1^2\right) + \frac{\delta_1}{2}\left(p\bar p\,\delta_0\delta_1 + p^2\delta_1^2 + p\bar p\,\delta_0^2 + \bar p^2\delta_0\delta_1\right)$

$\bar p(1+p) = (1-p^2)$

$= \frac{1}{2}\left(\delta_0^3 p^2 + \delta_0^2\delta_1(2p\bar p + \bar p^2) + \delta_0\delta_1^2(2p\bar p + \bar p^2) + \delta_1^3 p^2\right)$, $e_2 = P\begin{pmatrix} p\delta_0+\bar p\delta_1 \\ p\delta_1+\bar p\delta_0\end{pmatrix} = \begin{matrix} \delta_0^2p^2 + p\bar p\delta_0\delta_1 + p\bar p\delta_1^2 + \dots \\ p\bar p\delta_0^2 + \bar p^2\delta_0\delta_1 + \delta_1^2p^2 + \delta_0\delta_1 p\bar p \end{matrix}$

$\cancel{\tfrac{1}{2}(e_2(0)+e_2(1))}$ $\frac{1}{2}(\delta_0 e_2(0) + \delta_1 e_2(1)) = \frac{1}{2}\left(\delta_0^3p^2 + p\bar p\,\delta_0^2\delta_1 + p\bar p\,\delta_0\delta_1^2 + \bar p^2\delta_0^2\delta_1 + p\bar p\,\delta_1\delta_0^2 + \bar p^2\delta_0\delta_1^2 + \delta_1^3p^2 + \delta_0\delta_1^2 p\bar p\right)$

(36)

$= \frac{1}{2}\left(\delta_0^3p^2 + \delta_0^2\delta_1(2p\bar p + \bar p^2) + \delta_0\delta_1^2(2p\bar p+\bar p^2) + \delta_1^3p^2\right)$, $\mathbb{P}(L_{t+1} = \ell_{t+1} \mid L_t = \ell_t) = \sum_a \mathbb{P}(L_{t+1} = \ell_{t+1} \mid A_{t+1}=a)\cdot$

$\mathbb{P}(A_{t+1}=a\mid L_t=\ell_t) = \sum_a p^{a=\ell_{t+1}}(1-p)^{a\neq\ell_{t+1}}\, p^{a=\ell_t}(1-p)^{a\neq\ell_t} = (1-p)^2\sum_a\left(\frac{p}{1-p}\right)^{a=\ell_{t+1}+a=\ell_t} = (1-p)^2\left\{\left(\frac{p}{1-p}\right)^2 + 1\right\}^{\ell_t=\ell_{t+1}}$

$T = 2p$, $D = 2p-1$

$\left\{2\frac{p}{1-p}\right\}^{\ell_t\neq\ell_{t+1}}$, $\lambda_2 = p + \sqrt{p^2-2p+1} = p \pm (p-1)$, $\lambda_1 = 2p-1$, $\lambda_2 = 1$,

$\begin{pmatrix} p-1 \\ 1-p\end{pmatrix}, \begin{pmatrix} 1-p \\ 1-p \end{pmatrix}$, $P = \begin{pmatrix} \delta_0 p & \delta_1(1-p) \\ \delta_0(1-p) & \delta_1 p\end{pmatrix}$, $T = p(\delta_0+\delta_1)$, $D = p^2\delta_0\delta_1 - \delta_0\delta_1(1-p)^2 = \delta_0\delta_1(2p-1)$,

$\lambda = \frac{1}{2}p(\delta_0+\delta_1) \pm \sqrt{\frac{p^2}{4}(\delta_0+\delta_1)^2 - \delta_0\delta_1(2p-1)} = \cancel{\frac{1}{2}p(\delta_0+\delta_1) \pm \sqrt{\left(\frac{p^2}{4}(\delta_0^2+\delta_1^2) + \frac{p^2}{2}\delta_0\delta_1 - \dots\right)} = \frac{p}{2}(\delta_0+\delta_1)}$

$2\pm\sqrt2$

$= \frac{p}{2}(\delta_0+\delta_1) \pm \sqrt{\left(\frac{p^2}{4}(\delta_0^2+\delta_1^2) + \frac{p^2}{2}\delta_0\delta_1 - 2p\delta_0\delta_1 + \delta_0\delta_1\right)}$, $\frac{p^2}{2} - 2p + 1 = \frac{1}{2}(p^2-4p+2) = \frac{1}{2}(p-2+\sqrt2)(p-2-\sqrt2)$,

$\sqrt{\frac{p^2}{4}(\delta_0-\delta_1)^2 + (p^2-2p+1)\delta_0\delta_1} = \sqrt{\frac{p^2}{4}(\delta_0-\delta_1)^2 + \delta_0\delta_1(p-1)^2}$, $\mathbb{E}\prod\delta_{L_t} = \left(\frac{1}{2}\delta_0,\ \frac{1}{2}\delta_1\right)P^{-1}\begin{pmatrix}\lambda_1^{T-2} & \\ & \lambda_2^{T-2}\end{pmatrix}P\begin{pmatrix} p\delta_0+\bar p\delta_1 \\ p\delta_1+\bar p\delta_0\end{pmatrix}$

$(\delta_0=\delta_1)$ $\lambda = p\delta \pm \sqrt{p^2\delta^2 - \delta^2(2p-1)} = p\delta \pm \delta(1-p) = \delta,\ (2p-1)\delta$, $v = \begin{pmatrix}\delta-\delta p \\ \delta-\delta p\end{pmatrix}, \begin{pmatrix}(p-1)\delta \\ \delta-\delta p\end{pmatrix}$,

$A = \delta\begin{pmatrix}1-p & p-1 \\ 1-p & 1-p\end{pmatrix} = \delta(1-p)\begin{pmatrix}1 & -1 \\ 1 & 1\end{pmatrix}$, $A^{-1} = \frac{1}{2\delta(1-p)}\begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}$ $(\delta\ \delta)\,\delta^{T-2}\begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & \\ & (2p-1)^{T-2}\end{pmatrix}\begin{pmatrix}1 & -1 \\ 1 & 1\end{pmatrix} = (\delta\ \delta)\,\delta^{T-2}\begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 \\ (2p-1)^{T-2} & \dots\end{pmatrix}$

$= \delta^{T-2}(0\ \ 2\delta)(\quad)$

---

$e_{t_0}(\ell_{t_0-1}) = \mathbb{E}\left(\prod_{t=t_0}^{T}\frac{\delta_{A_{t-1}}}{\delta_{L_t}}\ \middle|\ A_{t_0-1}=a\right) = p\,\mathbb{E}\left(\prod_{t=t_0+1}^{T}\frac{\delta_{A_{t-1}}}{\delta_{L_t}}\ \middle|\ L_{t_0}=a\right) + (1-p)\frac{\delta_a}{\delta_{\bar a}}\mathbb{E}\left(\prod_{t_0+1}^{T}\dots\ \middle|\ L_{t_0}=\bar a\right)$

$e_t(\ell_{t-1}) = \mathbb{E}\left(\prod_{j=t}^{T}\frac{\delta_{A_{j-1}}}{\delta_{L_j}}\ \middle|\ L_{t-1}=\ell_{t-1}\right) = p^2\,\mathbb{E}\left(\prod_{j=t+1}^{T}\dots\ \middle|\ L_t=\ell_{t-1}\right) + p(1-p)\frac{\delta_{\ell_{t-1}}}{\delta_{\bar\ell_{t-1}}}\mathbb{E}(\dots\mid L_t=\bar\ell_{t-1})$

$+(1-p)p\frac{\delta_{\bar\ell_{t-1}}}{\delta_{\bar\ell_{t-1}}}\mathbb{E}(\dots\mid L_t=\bar\ell_{t-1}) + (1-p)^2\frac{\delta_{\bar\ell_{t-1}}}{\delta_{\ell_{t-1}}}\mathbb{E}(\dots\mid L_t=\ell_{t-1}) = \left(p^2+(1-p)^2\frac{\delta_{\bar\ell_{t-1}}}{\delta_{\ell_{t-1}}}\right)e_{t+1}(\ell_{t-1}) + p(1-p)\left(\frac{\delta_{\ell_{t-1}}}{\delta_{\bar\ell_{t-1}}}+1\right)$

$e_{t+1}(\bar\ell_{t-1})$, $\begin{pmatrix}e_t(0) \\ e_t(1)\end{pmatrix} = \begin{pmatrix}p^2+(1-p)^2\,\delta_1/\delta_0 & p(1-p)\left(\frac{\delta_0}{\delta_1}+1\right) \\ p(1-p)\left(\frac{\delta_1}{\delta_0}+1\right) & p^2+(1-p)^2\,\delta_0/\delta_1\end{pmatrix}\begin{pmatrix}e_{t+1}(0) \\ e_{t+1}(1)\end{pmatrix}$, $\mathbb{E}\prod\frac{\delta_{A_{t-1}}}{\delta_{L_t}} = \frac{1}{2}\left(\delta_0^{-1}e_2(0) + \delta_1^{-1}e_2(1)\right)$

(21) $(T\geq 1)$, $e_1(0)=e_1(1)=1$

$e_T(\ell_{T-1}) = \mathbb{E}\left(\frac{\delta_{A_{T-1}}}{\delta_{L_T}} \,\middle|\, L_{T-1}=\ell_{t-1}\right) = \mathbb{E}(\delta_{A_{T-1}} \mid L_{T-1}=\ell_{t-1}) = p\,\delta_{\ell_{t-1}} + (1-p)\,\delta_{\ell_{t-1}}$, $(T=1)$ $\mathbb{E}\frac{\delta_{A_0}}{\delta_{L_1}} = \frac{1}{2}\left(\frac{1}{\delta_0}+\frac{1}{\delta_1}\right)$

$(T=2)$ $\mathbb{E}\frac{1}{\delta_{L_1}}\frac{\delta_{A_1}}{\delta_{L_2}} = \frac{1}{2}\delta_1^{-1}\,\mathbb{E}\left(\frac{\delta_{A_1}}{\delta_{L_2}} \,\middle|\, \delta_{L_1}=1\right) + \frac{1}{2}\delta_0^{-1}\,\mathbb{E}\left(\frac{\delta_{A_1}}{\delta_{L_2}} \,\middle|\, \delta_{L_1}=0\right) = \frac{1}{2}\delta_1^{-1} e_1(1) + \frac{1}{2}\delta_0^{-1} e_1(0)$

$e_{T+1}(1) = e_{T+1}(0) = 1$, $\left(\frac{1}{2\delta_0}, \frac{1}{2\delta_1}\right) P \begin{pmatrix} e_2(0) \\ e_2(1) \end{pmatrix}$, $\frac{1}{2\delta_1}\left(p^2 + p(1-p)\frac{\delta_1}{\delta_0} + (1-p)p \cdots\right)$

---

$T = 2p^2 + (1-p)^2\left(\frac{\delta_1}{\delta_0}+\frac{\delta_0}{\delta_1}\right)$, $\Delta = p^4 + p^2(1-p)^2\left(\frac{\delta_0}{\delta_1}+\frac{\delta_1}{\delta_0}\right) + (1-p)^4 - p^2(1-p)^2\left(2+\frac{\delta_0}{\delta_1}+\frac{\delta_1}{\delta_0}\right)$

$= p^4 + (1-p)^4 - 2p^2(1-p)^2 = (p^2-(1-p)^2)^2$, $\lambda = p^2 + \frac{(1-p)^2}{2}\left(\frac{\delta_1}{\delta_0}+\frac{\delta_0}{\delta_1}\right) \pm \sqrt{\frac{T^2}{4}-\Delta}$, $\frac{T^2}{4}-\Delta = 4p$

---

$f(A_{t+1}\mid A_t) = \sum_\ell f(A_{t+1}\mid L_{t+1}=\ell)\,\mathbb{P}(L_{t+1}=\ell\mid A_t) = \sum_\ell p_1^{1_{\ell=A_{t+1}}}(1-p_1)^{1_{\ell\neq A_{t+1}}} p_2^{1_{\ell=A_t}}(1-p_2)^{1_{\ell\neq A_t}}$

$= (1-p_1)(1-p_2)\sum_\ell \left(\frac{p_1}{1-p_1}\right)^{1_{\ell=A_{t+1}}}\left(\frac{p_2}{1-p_2}\right)^{1_{\ell=A_t}} = (1-p_1)(1-p_2)\left(\left(\frac{p_1}{1-p_1}\right)\left(\frac{p_2}{1-p_2}\right)\right)^{A_t=A_{t+1}}\left(\frac{p_1}{1-p_1}+\frac{p_2}{1-p_2}\right)^{A_t\neq A_{t+1}}$

$= (p_1(1-p_2)+p_2(1-p_1))\left(\frac{p_1p_2}{(1-p_1)(1-p_2)}\cdot\frac{(1-p_1)(1-p_2)}{p_1+p_2-2p_1p_2}\right)^{A_t=A_{t+1}} = (p_1+p_2-2p_1p_2)\left(\frac{1-p_1-p_2+2p_1p_2}{p_1+p_2-2p_1p_2}\right)^{A_t=A_{t+1}}$, $(p_1+p_2-2p_1p_2)^{-1}-1$

$\mathbb{E}\frac{d^{A_{t-1}=A_t}}{c^{A_t=L_t}}$, $\frac{1}{c}\mathbb{P}(A_t=L_t)\,\mathbb{E}(d^{A_{t-1}=A_t}\mid A_t=L_t) = \frac{p_1}{c}(d\,\mathbb{P}(A_{t-1}=A_t) + \mathbb{P}(A_{t-1}\neq A_t)) = \frac{p_1}{c}(1+(d-1)\,\mathbb{P}(A_{t-1}=A_t))$

$+\ \mathbb{P}(A_t\neq L_t)\,\mathbb{E}(d^{A_{t-1}=A_t}\mid A_t\neq L_t) = (1-p_1)(1+(d-1)\mathbb{P}(A_{t-1}=A_t))$, $\mathbb{E}\frac{d^{A_{t-1}=A_t}}{c^{A_t=L_t}} = \left(1-p_1+\frac{p_1}{c}\right)(1+(d-1)\,\mathbb{P}(A_{t-1}=A_t))$

$\cancel{f(A_{t+1}\mid A_t) = (p_1p_2+(1-p_1)(1-p_2))^{A_t=A_2}}$. $\mathbb{E}\frac{d^{A_{t-1}=A_t}}{c^{A_t=L_t}} = p_1p_2\frac{d}{c} + p_2(1-p_1) + (1-p_2)p_1\frac{1}{c}$

$+ (1-p_1)(1-p_2)d$, $d := \left(\frac{1}{p_1+p_2-2p_1p_2}-1\right)^2$, $c := \left(\frac{p_1}{1-p_1}\right)^2$, $\mathbb{E}\left(\prod\frac{f(A_t\mid A_{t-1})}{f(A_t\mid L_t)}\right)^2 =$

$\left(\mathbb{E}\left(\frac{f(A_t\mid A_{t-1})}{f(A_t\mid L_t)}\right)^2\right)^T = \left(\frac{p_1+p_2-2p_1p_2}{1-p_1}\right)^{2T}\left(\mathbb{E}\frac{c^{A_t=A_{t-1}}}{d^{A_t=L_t}}\right)^T = \left(\frac{p_1+p_2-2p_1p_2}{1-p_1}\right)^{2T}\left(p_1p_2\frac{d}{c}+p_2(1-p_1)+(1-p_2)p_1\frac{1}{c}+(1-p_1)(1-p_2)d\right)^T$,

$\alpha = p_1+p_2-2p_1p_2 = p_1(1-p_2)+p_2(1-p_1)$

$\frac{p_1}{c}(1-p_2+p_2d) + (1-p_1)(p_2+(1-p_2)d) = \frac{p_1}{c}\left\{1+p_2\left(\frac{1}{\alpha^2}-\frac{2}{\alpha}\right)\right\} + (1-p_1)\left\{p_2+(1-p_2)\left(\frac{1}{\alpha^2}-\frac{2}{\alpha}+1\right)\right\}$

$= \frac{p_1}{c}\left\{1+\frac{p_2}{\alpha}\left(\frac{1}{p_1+p_2-2p_1p_2}-2\right)\right\} + (1-p_1)\left\{p_2+(1-p_2)\frac{(1-\alpha)^2}{\alpha^2}\right\} = \left(\frac{p_1p_2}{c}+(1-p_1)(1-p_2)\right)\left(\frac{1}{\alpha^2}-\frac{2}{\alpha}\right) + \frac{p_1}{c} + (1-p_1)$

$= \left(\frac{p_2}{p_1}(1-p_1)^2+(1-p_1)(1-p_2)\right)\frac{1}{\alpha}\left(\frac{1}{\alpha}-2\right) + p_1\left(\frac{1}{c}-1\right)+1 = \frac{1-p_1}{\alpha}\left(\frac{1}{\alpha}-2\right)\left(\frac{p_2}{p_1}-p_2+1-p_2\right) + \frac{(1-p_1)^2}{p_1} + 1-p_1$

$= \frac{1-p_1}{\alpha}\left(\frac{1}{\alpha}-2\right)\left(\frac{p_2}{p_1}+1-2p_2\right) + (1-p_1)\frac{1}{p_1} = \frac{1-p_1}{p_1}\left(\frac{1}{\alpha}-2\right) + \frac{1-p_1}{p_1} = \frac{1-p_1}{p_1}\left(\frac{1}{\alpha}-1\right)$, $\left(\frac{\alpha}{1-p_1}\right)^{2T}\left(\left(\frac{1-p_1}{p_1}\right)\left(\frac{1}{\alpha}-1\right)\right)^T$

$= \left(\frac{1}{p_1(1-p_1)}(\alpha-\alpha^2)\right)^T$, $1-\alpha = 1-p_1-p_2+2p_1p_2 = (1-p_1)(1-p_2)+p_1p_2$, $\frac{p_1(1-p_2)+p_2(1-p_1)}{p_1(1-p_1)}((1-p_1)(1-p_2)$

$+p_1p_2) = \alpha\left(\frac{1-p_2}{p_1}+\frac{p_2}{1-p_1}\right) = (1-p_2)^2 + \frac{p_2}{p_1}(1-p_1)(1-p_2) + \frac{p_1p_2(1-p_2)}{1-p_1} + p_2^2 = 1+3p_2^2-2p_2+\frac{p_2}{p_1}-p_2-\frac{p_2^2}{p_1}+\frac{p_1p_2(1-p_2)}{1-p_1}$

$2p_2^2-p_2+(1-p_2)^2$

$= (1-p_2)\left(1-p_2+\frac{p_2}{p_1}(1-p_1)+\frac{p_1p_2}{1-p_1}\right) + p_2^2 = (1-p_2)\left(1-2p_2+p_2\left(\frac{1}{p_1}+\frac{p_1}{1-p_1}\right)\right)+p_2^2 = (3p_2^2-3p_2+1) + \frac{p_2(1-p_2)}{p_1(1-p_1)}(p_1^2-p_1+1)$

$(p_1=p_2)$ $(1-p_1)^2+p_1$

$p^2-4p+2 = 2(p^2+p^2-2p+1) = 2(p^2+(1-p)^2)$, $(1-p_2)(1-2p_2)+p_2^2 = (1-p_2)^2+p_2(2p_2-1)$

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$\rho := \frac{p_2}{p_1}$, $1 - \rho p_1 = 1 - p_2$, $\frac{1-\rho p_1}{1-p_1} \cdot \rho$

Ferguson #1 §6|

wp 1, $X(\omega) \in C(f)$, $X(\omega) \in C(f) \Rightarrow |X_n(\omega) - X(\omega)| \to 0 \Rightarrow |f(X_n(\omega)) - f(X(\omega))| \to 0$, $f(X_n) \to f(X)$ wp 1,

$\{n_j\}_{j=1}^{\infty}$, $X_{n_j} \to_p X$, $\{n_{j_k}\}_k : X_{n_{j_k}} \to_{a.s.} X$,

$f(X_{n_{j_k}}) \to_{a.s.} f(X)$, $f(X_n) \to_p f(X)$. $P(|Y_n - X| > \varepsilon) \le P(|Y_n - X_n| + |X_n - X| > \varepsilon) \le P(|Y_n - X_n| > \varepsilon/2)$

$+ P(|X_n - X| > \varepsilon/2) \to 0$. $P(|(X_n, Y) - (X,Y)| > \varepsilon) = P(|X_n - X| > \varepsilon) \to 0$, $(X_n, Y) \to_p (X, Y)$, $P(|(X_n, Y_n) -$

$(X_n, Y)| > \varepsilon) = P(|Y_n - Y| > \varepsilon) \to 0$, $\binom{X_n}{Y_n} - \binom{X_n}{Y} \to_p 0$, $\binom{X_n}{Y_n} \to_p \binom{X}{Y}$ #1 // $P((X_n, Y_n) \le (c,d)) = P(X_n \le c)$

$P(Y_n \le d) \to P(X \le c) P(Y \le d) = P((X,Y) \le (c,d))$ #2 // $X_n = \beta X_{n-1} + \varepsilon_n = \beta^2 X_{n-2} + \beta \varepsilon_{n-1} + \varepsilon_n = \beta^3 X_{n-3} + \beta^2 \varepsilon_{n-2} +$

$\beta \varepsilon_{n-1} + \varepsilon_n = \dots = \sum_{j=0}^{n-1} \beta^j \varepsilon_{n-j}$, $\varepsilon_1$, $\beta\varepsilon_1 + \varepsilon_2$, $\beta^2 \varepsilon_1 + \beta \varepsilon_2 + \varepsilon_3$, $\sum_{k=1}^n X_k = \sum_{k=1}^n \varepsilon_k \sum_{j=0}^{n-k} \beta^j = \sum_{k=1}^n \frac{1-\beta^{k+1}}{1-\beta} \varepsilon_k$, $\left|\bar X_n - \frac{1}{1-\beta} \sum_{k=1}^n \frac{\varepsilon_k}{n}\right|$

$= \frac{1}{n}\left| \sum_{k=1}^n \frac{\beta^{k+1}}{1-\beta} \varepsilon_k \right| = o_p(1) + \frac{\beta^{m+1}}{1-\beta} \frac{1}{n} \sum_{k=m+1}^{\infty} |\varepsilon_k| = o_p(1) + \frac{\beta^{m+1}}{(1-\beta)} (E|\varepsilon_k| + o_p(1)) = o_p(1)$,

$\frac{\sqrt{n}}{\sigma}(\frac{1}{n}\sum \varepsilon_k - \mu) \rightsquigarrow N(0,1)$, $\frac{1}{1-\beta}\frac{\sqrt{n}}{\sigma}(\frac{1}{n}\sum \varepsilon_k - \mu) \rightsquigarrow N(0, (1-\beta)^{-2})$, $\frac{\sqrt{n}}{1-\beta}(\frac{1}{n}\sum \varepsilon_k - \mu) \rightsquigarrow N(0, \sigma^2 (1-\beta)^{-2})$,

$\sqrt{n}\left|\bar X_n - \frac{1}{1-\beta}\sum_{k=1}^n \frac{\varepsilon_k}{n}\right| = o_p(1)$, $\sqrt{n}\left|\frac{1}{1-\beta}\frac{1}{n}\sum \varepsilon_k - \frac{\mu}{1-\beta} - \bar X_n + \frac{\mu}{1-\beta}\right| = o_p(1)$, $\sqrt{n}(\bar X_n - \frac{\mu}{1-\beta}) \rightsquigarrow$

$N(0, \sigma^2 (1-\beta)^{-2}) \Leftarrow -1 < \beta < 1$. $(\beta = -1)$, $X_n = -X_{n-1} + \varepsilon_n = X_{n-2} - \varepsilon_{n-1} + \varepsilon_n = -X_{n-3} + \varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n$

$= \sum_{j=1}^n (-1)^{n-j} \varepsilon_j$, $\varepsilon_1$, $-\varepsilon_1 \ \varepsilon_2$, $\varepsilon_1 \ -\varepsilon_2 \ \varepsilon_3$, $\bar X_n = \frac{1}{n}\sum_{j=1}^n \varepsilon_j \sum_{j=0}^{n-k} (-1)^j$, $\bar X_{2n} = \frac{1}{2n}\sum_{j=1}^n \varepsilon_{2j}$, $\bar X_{2n+1} = \frac{1}{2n+1}\sum_{j=1}^n \varepsilon_{2j-1}$,

$\frac{\sqrt{n}}{2n}(\varepsilon_2 - \mu, \varepsilon_4 - \mu, \dots, \varepsilon_{2n} - \mu)$, $\sigma_n^2 = \frac{1}{4n} \cdot n\sigma^2 = \sigma^2/4$, $\sum E\left(\frac{1}{4n}(\varepsilon_{2j} - \mu)^2 ; \frac{1}{4n}(\varepsilon_{2j} - \mu)^2 > \delta\right)$

$= \frac{1}{4} E((\varepsilon - \mu)^2 ; (\varepsilon - \mu)^2 > 4n\delta) \xrightarrow[n\to\infty]{} 0$, $\frac{1}{2\sqrt{n}}\sum_{j=1}^n (\varepsilon_{2j} - \mu) \rightsquigarrow N(0, \sigma^2/4)$,

$\sqrt{n}\,\bar X_{2n} - \frac{\sqrt{n}}{2}\mu = \sqrt{n}(\bar X_{2n} - \mu/2) \rightsquigarrow N(0, \sigma^2/4)$, $\sqrt{n}(\bar X_{2n+1} - \mu/2) \rightsquigarrow N(0, \sigma^2/4)$, $\sqrt{\frac{n}{2}}(\bar X_n - \mu/2) \rightsquigarrow N(0, \sigma^2/4)$,

$\sqrt{n}(\bar X_n - \mu/2) \rightsquigarrow N(0, \sigma^2/2)$ #3 //

Rudin #1

Ch 9 $|\hat f(y)| = \left|\int f(x)e^{-iyx}\,dx\right| \le \int |f(x)|\,dx = \int f(x)\,dx = \hat f(0)$, $|\hat f(y)|^2 = \left|\int f(x)\cos yx\,dx - i\int f(x)\sin yx\,dx\right|^2 = \left(\int f(x)\cos yx\,dx\right)^2 + \left(\int f(x)\sin(yx)\,dx\right)^2 = \left(\int f(x)\right)^2$, $1 = \|f\|_1^{-2}\left(\int f(x)\right)^2 = \left(\int \frac{f}{\|f\|_1}\cos yx\,dx\right)^2 + \left(\int \frac{f}{\|f\|_1}\sin(yx)\,dx\right)^2 \le \int \frac{f}{\|f\|_1} = 1$, $\left|\int \frac{f}{\|f\|_1}\cos yx\,dx\right|^2 = \int \frac{f}{\|f\|_1}\cos^2 yx\,dx \Rightarrow \cos yx = \text{const}$ a.e., #1 //

$f = 1(a,b)$, $\hat f(t) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-itx}\,dx = \frac{1}{\sqrt{2\pi}}\frac{-1}{it}e^{-itx}\Big|_a^b = -\frac{1}{\sqrt{2\pi}\,it}(e^{-itb} - e^{-ita})$, $\hat g_n(t) = \frac{i}{\sqrt{2\pi}\,t}(e^{itn} - e^{-itn})$

$= \frac{\sqrt{2}\sin(tn)}{\sqrt{\pi}\,t}$, $\hat h(t) = \frac{\sqrt2\sin t}{\sqrt\pi\, t}$, $(g_n * h)(x) = \int g_n(x-y)h(y)\,dy = \int_{-1}^{1} g_n(x-y)\,dy = \int_{-1-x}^{1-x} g_n(y)\,dy = \cancel{\{\ldots\}}$

$((-x+1)\wedge n) - ((-x-1)\vee -n))^+$ , $\widehat{g_n * h} = \hat g_n \hat h = \frac{\sin(tn)\sin t}{\pi^2 t^2}$, $g_n * h(x)$

$= \frac{1}{\sqrt{2\pi}}\int \frac{\sin(tn)\sin t}{\pi^2t^2}e^{itx}\,dt = \frac{1}{\sqrt{2\pi}}\int\frac{\sin(tn)\sin t}{\pi^2t^2}e^{-itx}\,dt = \hat f_n(x)$, $f_n \in ?\ L_1$, $\int_0^1\left|\frac{\sin(nt)\sin t}{t^2}\right| \le \int_0^1 t^{-2}(nt + O(t^3))$

$(t + O(t^3)) = \int_0^1 (n + O(t^4)) < \infty$,

$t^{-1}\sin\lambda t\sin xt = t^{-1}\sum_{j=0}^\infty (-1)^j\frac{(\lambda t)^{2j+1}}{(2j+1)!}\sum_{j=0}^\infty(-1)^j\frac{(xt)^{2j+1}}{(2j+1)!} = t^{-1}\sum_k\sum_{j=1}^k \frac{(\lambda t)^{2j-1}(xt)^{2k-(2j-1)}}{(2j-1)!\,(2k-(2j-1))!} = \sum_k t^{2k-1}x^{2k}\sum_{j=1}^k\frac{(\lambda/x)^{2j-1}}{(2j-1)!(2k-2j+1)!}$,

$\int_{-A}^A \ldots\,dt = \sum_k \frac{1}{2k}t^{2k}\Big|_{-A}^A \sum \ldots = 0$, $(-t)^{-1}\sin(-\lambda t)\sin(-xt) = -t\sin(\lambda t)\sin(xt)$, $t^{-1}\sin\lambda t\cos xt$

$= t^{-1}\sum_{j=0}^\infty(-1)^j\frac{(xt)^{2j}}{(2j)!}\sum_{j=0}^\infty(-1)^j\frac{(\lambda t)^{2j+1}}{(2j+1)!} = t^{-1}\sum_{k=0}^\infty\sum_{j=0}^k(-1)^j\frac{(xt)^{2j}}{(2j)!}(-1)^{j+k}\frac{(\lambda t)^{2k+1-2j}}{(2k+1-2j)!} = \sum_{k=0}^\infty(-1)^k t^{2k}\lambda^{2k+1}\sum_{j=0}^k\frac{(x/\lambda)^{2j}}{(2j)!(2k+1-2j)!}$

$\int_{-A}^A\ldots\,dt = \sum_{k=0}^\infty(-1)^k\frac{2A^{2k+1}}{2k+1}\lambda^{2k+1}\sum_{j=0}^k\frac{(x/\lambda)^{2j}}{(2j)!(2k+1-2j)!} = 2\sum_{j=0}^\infty\sum_{k=j}^\infty\frac{(x/\lambda)^{2j}}{(2j)!}(-1)^k\frac{(A\lambda)^{2k+1}}{(2k+1)(2k+1-2j)!} = 2\sum_{j=0}^\infty\frac{(x/\lambda)^{2j}}{(2j)!}\sum_{k=0}^\infty(-1)^{k+j}$

$\frac{(A\lambda)^{2k+2j+1}}{(2k+2j+1)(2k+1)!} = 2\sum\frac{(x/\lambda)^{2j}}{(2j)!}(-1)^j\sum_{k=0}\int_0^{A\lambda}\frac{(-1)^k u^{2k+2j}}{(2k+1)!} = 2\sum_0\frac{(x/\lambda)^{2j}}{(2j)!}(-1)^j\int_0^{A\lambda}u^{2j-1}\sin u\,du$,

$\int x^{2j+1}\sin x = -x^{2j+1}\cos x + (2j+1)\int x^{2j}\cos x = -x^{2j+1}\cos x + (2j+1)\left\{x^{2j}\sin x - 2j\int x^{2j-1}\sin x\right\} = -x^{2j+1}\cos x$

$+(2j+1)x^{2j}\sin x\Big| - (2j+1)(2j)\int x^{2j-1}\sin x$, $= -(A\lambda)^{2j+1}\cos A\lambda + (2j+1)(A\lambda)^{2j}\sin(A\lambda) - (2j+1)(2j)\,I(2j-1)$,

$\int x^{2j+1}\sin x = -(2j+1)2j\int x^{2j-1}\sin x = (-1)^j(2j+1)!\int x\sin x = (-1)^j(2j+1)!\int_{-A}^A\cos x\,dx = (-1)^j(2j+1)!\;2\sin A$,

$= \sum_0^\infty\frac{(x/\lambda)^{2j}}{(2j)!}(-1)^j\sum_0\int_{-A\lambda}^{A\lambda}\frac{(-1)^k u^{2k+2j}}{(2k+1)!} = \sum\frac{(x/\lambda)^{2j}}{(2j)!}(-1)^j\int_{-A\lambda}^{A\lambda}u^{2j-1}\sin u\,du = \sum_1^\infty\frac{(x/\lambda)^{2j}}{j}\sin A = \left(\int_0^{(x/\lambda)^2}\sum_1^\infty u^{j-1}\,du\right)\sin A$

$= -\sin A\log(1-u)\Big|_0^{(x/\lambda)^2} = -\sin A\log\left(1-\left(\frac{x}{\lambda}\right)^2\right)$

Rudin #2

$\int t^{-1}\sin\lambda t \sin xt = \sum (-1)^j \int \frac{(xt)^{2j+1}}{(2j+1)!} t^{-1}\sin xt = 0$, $\quad \int t^{-1}\sin\lambda t\cos xt = \sum (-1)^j \int t^{-1}\frac{(xt)^{2j}}{(2j)!}\sin\lambda t\,dt$

$= \sum (-1)^j \left(\frac{x}{\lambda}\right)^{2j}\lambda \int \frac{(\lambda t)^{2j-1}}{(2j)!}\sin\lambda t\,dt = \sum (-1)^j\left(\frac{x}{\lambda}\right)^{2j}\int_{-A/\lambda}^{A/\lambda}\frac{u^{2j-1}}{(2j)!}\sin u\,du = \sum (-1)^j \left(\frac{x}{\lambda}\right)^{2j}\frac{1}{(2j)!}\quad (-1)^{j-1}\cdot 2\sin\left(\frac{A}{\lambda}\right)(2j-1)!$,

$(j=1)\quad \int x^3\sin x = -x^3\cos x + 3\int x^2\cos x = 3\left(x^2\sin x - 2\int x\sin x\right) = -6\left(-x\cos x + \int\cos x\right) = -6\cdot 2\sin A$,

$\sum_{j=0}^{\infty} = -2\sin\frac{A}{\lambda}\sum_{j=1}^{\infty}\left(\frac{x}{\lambda}\right)^{2j}\cdot\frac{1}{j}$, $\quad (j=0)\ \int_{-A/\lambda}^{A/\lambda}\frac{\sin u}{u}\,du$, $\quad \sum_{j=1}^{\infty} \left(\frac{x}{\lambda}\right)^{2j}\frac{1}{j} = \sum_1^{\infty} \int_0^{(x/\lambda)^2} r^{j-1}\,dr = -\log(1-(x/\lambda)^2)$,

$= \sin\frac{A}{\lambda}\log\left(1-\left(\frac{x}{\lambda}\right)^2\right) + \int_{-A/\lambda}^{A/\lambda}\frac{\sin u}{u}\,du$, $\quad \int_{-A}^{A}\frac{\sin u}{u} = \int\sum_0^{\infty} u^{-1}(-1)^j\frac{u^{2j+1}}{(2j+1)!} = \sum \frac{(-1)^j}{(2j+1)!}\cdot\frac{2A^{2j+1}}{2j+1}$

$\int t^{-1}\sin\lambda t\cos xt\,dt = -\int\sin\lambda t\int_0^x \sin ut\,du\,dt = -\int_0^x\int_{-A}^{A} \sin\lambda t\sin ut\,dt\,du$

$= -\frac12\int_0^x\int_{-A}^{A} \left(\cos((\lambda-u)t) - \cos((\lambda+u)t)\right)dt\,du = -\frac12\int_0^x\left\{\frac{2\sin((\lambda-u)A)}{\lambda-u} - \frac{2\sin((\lambda+u)A)}{\lambda+u}\right\}du$

$= -A\int_0^x\left\{\frac{\sin((\lambda-u)A)}{(\lambda-u)A} - \frac{\sin((\lambda+u)A)}{(\lambda+u)A}\right\}du \overset{u=\lambda-\frac{v}{A}}{=} -\int_{A\lambda}^{A(\lambda-x)}\frac{\sin v}{v}\,dv + \int_{A\lambda}^{A(\lambda+x)}\frac{\sin v}{v}\,dv \overset{(*)}{\xrightarrow{(|x|>|\lambda|)}} \int_{-\infty}^{\infty}\frac{\sin v}{v}$

$=\pi \qquad f_\lambda := 1[-\lambda,\lambda]$, $\quad \hat f_\lambda(t) = \sqrt{\frac{2}{\pi}}\frac{\sin(\lambda t)}{t}$, $\quad \sqrt{2\pi}f_\lambda(x) \overset{?}{=} \int_{-\infty}^{\infty}\hat f_\lambda(t)e^{ixt}\,dt = \sqrt{\frac{2}{\pi}}\lim_A\int_{-A}^{A}\frac{\sin(\lambda t)}{t}e^{ixt}\,dt$

$\lim_A\int_{-A}^{A}\frac{\sin(\lambda t)}{t}e^{ixt}\,dt \overset{?}{=} \pi\{|x|\le\lambda\}$, $\quad (*) = \int_{A(\lambda-x)}^{A(\lambda+x)}\frac{\sin v}{v}\,dv \to \begin{cases}0, & |x|<|\lambda|\\ \pi, & |x|>|\lambda|\\ \pi/2, & |x|=|\lambda|\end{cases}$ #3 //

$\int_0^{\infty}\left|\frac{\sin t}{t}\right| =? \infty$, $\quad \int_{n\pi}^{(n+1)\pi}\left|\frac{\sin t}{t}\right| \ge \frac{1}{(n+1)\pi}\int_0^{\pi}\sin t = \frac{2}{(n+1)\pi}$, $\quad \int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right| \ge \sum\frac{1}{(n+1)\pi} = \infty$ #4 //

$t\hat f(t) = t\int f(x)e^{-ixt}\,dx = -if(x)e^{-ixt}\Big|_{-\infty}^{\infty} + i\int f'(x)e^{-ixt}\,dt$, $\quad i\int t\hat f(t)e^{ixt}\,dm(t) = i\int t\int f(u)$

$e^{it(x-u)}\,dm(u)\,dm(t) =? \lim_{r\to x}\frac{f(x)-f(r)}{x-r}$, $\quad = \iint f(u)e^{it(x-u)}\frac{\partial}{\partial v}e^{itv}\Big|_{v=0}du\,dt = \iint f(u)e^{it(x-u)}$

$\lim_{v\to0}\frac{e^{itv}-1}{v}\,du\,dt$ : $\int f(u)\lim_{v\to0}\int\frac{e^{it(x+v-u)}-e^{it(x-u)}}{v}\,dt\,du$, $\quad = i\int f(u)\int t$

$e^{it(x-u)}\,dm(t)\,dm(u) = \int f(u)\left\{\frac{t}{i(x-u)}e^{it(x-u)}\Big|_{t=-\infty}^{\infty} - \int\frac{e^{it(x-u)}}{x-u}\,dt\right\}du$, $\quad \lim_{t\to0}\hat f(t) = \lim_{t\to0}\int f(x)e^{-ixt}\,dx = \int f(x)\,dx$

$<\infty$, $\quad \hat f\in L_1$, $\quad \frac{f(x)-f(y)}{x-y} = \frac{1}{x-y}\int\hat f(t)(e^{ixt}-e^{iyt})\,dt \xrightarrow[y\to x]{(DCT)} \int it\hat f(t)e^{ixt}\,dt$ #5 //

$f = \{x\in[-1,1]\}$, $\quad \hat f(t) = \int_{-1}^{1} e^{-ixt}\,dm(x) = \frac{1}{\sqrt{2\pi}}t^{-1}\sin(xt)\Big|_{-1}^{1} = \sqrt{\frac{2}{\pi}}\frac{\sin t}{t}$, $\quad f\in L^1$, $\quad f'=0\in L^1$, $\quad \hat{f'}=0$

$\ne i\sqrt{\frac{2}{\pi}}\sin t = it\hat f(t)$ #6 // $\quad x^n\hat f(x) = x^n\int f(t)e^{-itx}\,dm(t) \le \int\frac{Ax^n}{t^{n+2}}e^{-itx} \le \epsilon + \int_{-x}^{x}\frac{Ax^n}{t^{n+2}}e^{-itx}\,dt$, Rudin #3 ch 9 $x^n \hat{f}(x) \le \int_{-x}^{x} x^n f(t) e^{-itx} dt + \int_{t>x}$, $|x^n \hat{f}(x)| = |x^n \int f(t) e^{-itx} dt|$

$\le \int_{|x| \le |t|} + \int_{|x| \ge |t|} \le \int_{|x| \ge t} + \int_{|x| \le t} \frac{x^n}{t^{n+2}} A_{0,n}\, dt \le A_{0,n} \int \frac{1}{t^2} dt$, $D^m \hat{f}(x) =$

$\frac{\partial^m}{\partial x^m} \int f(t) e^{-itx} dt = \int f(t)(-it)^m e^{-itx} dt \le \int \frac{(-i)^m}{t^2} A_{0,m+2} e^{-itx} dt < \infty$, $|x D^m \hat{f}(x)|$

$= |x \int f(t)(-it)^m e^{-itx} dt| = |f(t) t^m e^{-itx} \Big|_{t=-\infty}^{\infty} - \int e^{-itx} \{ f'(t) t^m + m f(t) t^{m-1} \} dt | \le | A_{0,m+1} t^{-1} e^{-itx} \Big|_{-\infty}^{\infty}$

$+ \int_{|t|<1} A_{1,0} t^m dt + \int_{|t| \ge 1} A_{1,m+2} t^{-2} dt + m \int_{|t|<1} A_{0,0} t^{m-1} dt + m \int_{|t|>1} A_{0,m+1} t^{-2} dt =: \hat{A}_{m,1} < \infty$. $|x^{n+1} D^m \hat{f}(x)| = |x^n \int x f(t) t^m e^{-itx}$

$dt| = |x^n f(t) t^m e^{-itx} \Big|_{\infty} - \int x^n e^{-itx} \{ m f(t) t^{m-1} + f'(t) t^m \} dt | = | m x^n D^{m-1} \hat{f}(x) + x^n D^m \hat{f}(x)|$

$\le m \hat{A}_{m-1,n} + \hat{A}_{m,n} < \infty$ // #7 $h(x) = \int f(x-t) g(t) dt$, $|h(x) - h(y)| \le \int |f(x-t) - f(y-t)| |g(t)| dt$

$\le \|g\|_q \left( \int |f(x-t) - f(y-t)|^p dt \right)^{1/p}$, $\int |h(t) - h(t+\delta)| \xrightarrow[\delta \to 0]{(DCT)} 0$ , $\int |f(x+\delta-t) -$

~~$f(x-t)|^p dt$~~, $|h(x) - h(y)| \le \|g\|_q \left| \|f(x-t)\|_p - \|f(y-t)\|_p \right|$, $\int |f(t) - f(t-$

$(y-x))|^p dt$, $f_c \in C^0 : f|A = f_c$, $\left( \int_{\mathbb{R} \setminus A} |f|^p \right)^{1/p} < \varepsilon$, $\|f(x) - f(x-\delta)\|_p \le \|f(x) - f_c(x)\|_p$

$+ \|f_c(x) - f_c(x-\delta)\|_p + \|f_c(x-\delta) - f(x-\delta)\|_p \le 2\varepsilon + \|f_c(x) - f_c(x-\delta)\|_p$, $\int |f_c(x) - f_c(x-\delta)|^p \xrightarrow[\delta \to 0]{(DCT)} 0$.

$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$, $q = \frac{p}{p-1} \in (0, \infty)$, $f \in C_0$, $g \in C_0$, $|x| > M \Rightarrow |f(x)| < \varepsilon$, $|g(x)| < \varepsilon$: 

$\int_{-\infty}^{\infty} |f(t) g(x-t)| dt = \int_{|t|<M} + \int_{|t| \ge M} = \int_{-M+x<t<M+x} |f(t+x) g(t)| + \int_{|t| \ge M} |f(t) g(x-t)| dt$, $|x| > 2M$,

$M : E(|f|^p ; \{|t| > M\}) < \varepsilon$, $E(|g|^q ; \{|t| > M\}) < \varepsilon$, $\int |f(t) g(x-t)| dt$, $|x| > 2M$

$= \int_{-M+x<t<M+x} |f(t+x) g(t)| dt + \int_{|t| \ge M} |f(t) g(x-t)| dt \le \left( E(|f|^p ; -M+x < t < M+x) \right)^{1/p} \|g\|_q + \left( E(|f|^p ; |t| \ge M) \right)^{1/p}$

$\left( E(|g|^q ; |t| \ge M) \right)^{1/q} \le \varepsilon^{1/p} \|g\|_q + \varepsilon^2$. $f \in L_1$, $\int f(t) \ne 0$, $g = 1 \in L_\infty$, $f * g = \int f(t) dt$ // #8

$f \in L^p$, $1 \le p < \infty \Rightarrow M : \int_{|t| \ge M} |f(t)|^p dt < \varepsilon$, $|x| > M$, $g(x) \le \int_x^{x+1} |f(t)| \le 1 \cdot \| f(t) \{x$

$\le t \le M\} \|_p = \left( \int_x^{x+1} |f(t)|^p dt \right)^{1/p} \le \varepsilon^{1/p}$, $|g(x) - g(y)| = \int_x^{x+1} - \int_y^{y+1} f(t) dt = \int_{(x,x+1) \Delta (y,y+1)} f(t) dt \xrightarrow[x \to y]{} 0$.

$f \in L^\infty \Rightarrow |g(x) - g(y)| \xrightarrow[|x-y| \to 0]{} 0$ #9 // $f * g(x) = \int_{-\infty}^{\infty} f(x-t) g(t) dt = \int_{\text{supp } g} f(x-t) g(t) dt = \int_{x-M}^{x+M} f(t) g(x-t) dt$,

$\frac{f*g(x+\delta) - f*g(x)}{\delta} = \int_{x-M}^{x+M+\delta} f(t) (g(x+\delta-t) - g(x-t))/\delta \, dt \xrightarrow[\delta \to 0]{DCT} \int f(t) g'(x-t) dt = f * g'(x)$, $g' \in C^\infty$.

$$f_{Y,\bar{A},\bar{L}} = f_{Y|\bar{A},\bar{L}} \prod_t f_{\bar{A}_t|\bar{L}_t, \bar{A}_{t-1}} \prod_t f_{\bar{L}_t|\bar{L}_{t-1},\bar{A}_{t-1}}$$

$$\sum_{\bar{a}} \int h(\bar{a}) g(\bar{a}, y) \{\bar{A}=\bar{a}\} f_{Y|\bar{A}=\bar{a},\bar{L}} \prod_t^T f_{L_t|L_{t-1},\bar{A}_{t-1}} \mu_Y \times \mu_{\bar{L}} \times \mu_{\bar{A}}$$

$$= \sum_{\bar{a}} \mu_{\bar{A}}(h(\bar{a})) \int g(\bar{a}, y) f_{Y|\bar{A}=\bar{a},\bar{L}}(y, \bar{\ell}_t) \prod_t^T f_{L_t|\bar{L}_{t-1},\bar{A}_{t-1}}(\ell_t, \bar{\ell}_{t-1}, \bar{a}_{t-1}) \mu_Y \times \mu_{\bar{L}}$$

$$f_{Y|\bar{A}_{t-1}=\bar{a}_{t-1},\bar{L}}(y, \bar{\ell}_t) \prod_t^{T-1} \cdots$$

$$f_{Y|\bar{A}_{t-1}=\bar{a}_{t-1},\bar{L}_{t-1}}(y, \bar{\ell}_{t-1}) \prod \quad \mu_Y \times \mu_{\bar{L}_{T-1}}$$

$$=$$

$$\mathbb{E}\frac{g(\bar{A}, Y)}{\bar{W}} = \sum_{\bar{a}} \mathbb{E}\{\bar{A}=\bar{a}\}\frac{g(\bar{a}, Y_{\bar{a}})}{\bar{W}} = \sum_{\bar{a}} \int_{\{\bar{A}=\bar{a}\}} g(\bar{a}, y) f_{Y_{\bar{a}}|\bar{A},\bar{L}}(y, \bar{a}, \bar{\ell}) \prod_t f_{L_t|\bar{L}_{t-1},\bar{A}_{t-1}} \mu$$

$$\mathbb{E}\, h(\bar{A})\frac{Y - m_\beta(\bar{A})}{\bar{W}} = \sum_{\bar{a}} h(\bar{a}) \mathbb{E}\left(\frac{Y_{\bar{a}} - m_\beta(\bar{a})}{\bar{W}}\{\bar{A}=\bar{a}\}\right) = \sum_{\bar{a}} \mathbb{E}\left(\frac{\mathbb{E}(Y_{\bar{a}} - m_\beta(\bar{a}) \mid \overline{ALUZ})}{\bar{W}}\{\bar{A}=\bar{a}\}\right)$$

$$= \sum_{\bar{a}} \mathbb{E}\left(\frac{\mathbb{E}(Y_{\bar{a}} - m_\beta(\bar{a}) \mid \bar{A}_{T-1}, \overline{LU}_T)}{\bar{W}}\{\bar{A}=\bar{a}\}\right) = \sum_{\bar{a}}\left(\rule{2em}{0.4pt}\,\mathbb{P}(A_T = a_T \mid \bar{A}_{T-1}, \overline{LUZ}_T)\{\bar{A}_{T-1}=\bar{a}_{T-1}\}\right)$$

$$= \sum_{\bar{a}} \mathbb{E}\left(\frac{\{\bar{A}_{T-1}=\bar{a}_{T-1}\}}{\bar{W}_{T-1}}\mathbb{E}(Y_{\bar{a}} - \ldots \mid \ldots) \Big/ \Delta_T \sum_{z_T \in 0,1}\frac{(-1)^{1-z_T}\mathbb{P}(A_T = a_T \mid \bar{A}_{T-1}, \overline{LU}_T, \bar{Z}_{T-1}, Z_T = z_T) f_{Z_T}(z_T \mid \bar{A}_{T-1}, \bar{L}_T, \bar{U}_T, \bar{Z}_{T-1})}{f_{Z_T}(z_T \mid \bar{A}_{T-1}, \bar{L}_T, \bar{Z}_{T-1})}\right)$$

$$= \sum_{\bar{a}} \mathbb{E}\left(\frac{\{\bar{A}_{T-1}=\bar{a}_{T-1}\}}{\bar{W}_{T-1}} \mathbb{E}(Y_{\bar{a}} - m_\beta(\bar{a}) \mid \bar{A}_{T-1}, \overline{LU}_T)\right) = \sum_{\bar{a}} \mathbb{E}\left(\frac{\{\bar{A}_{T-1}=\bar{a}_{T-1}\}}{\bar{W}_{T-1}} \mathbb{E}(Y_{\bar{a}} - m_\beta(\bar{a}) \mid \overline{ALU}_{T-1})\right)$$

$$= \sum_{\bar{a}} \mathbb{E}\left(\frac{\{\bar{A}_{T-1}=\bar{a}_{T-1}\}}{\bar{W}_{T-1}} \mathbb{E}(Y_{\bar{a}} - m_\beta(\bar{a}) \mid \overline{ALUZ}_{T-1})\right) = \sum_{\bar{a}} \mathbb{E}\left(\frac{\{\bar{A}_{T-1}=\bar{a}_{T-1}\}}{\bar{W}_{T-1}} \mathbb{E}(Y_{\bar{a}} - m_\beta(\bar{a}))\right)$$

$$\overline{ALU} \to \bar{A}_{T-1}$$

$$\mathbb{E}\frac{g(\bar{A}, Y)}{\bar{W}} = \sum_{\bar{a}} \mathbb{E}\{\bar{A}=\bar{a}\}\frac{g(\bar{a}, Y_{\bar{a}})}{\bar{W}} = \sum_{\bar{a}} \mathbb{E}\left(\{\bar{A}=\bar{a}\}\frac{\mathbb{E}\, g(\bar{a}, Y_{\bar{a}}) \mid \overline{ALUZ}}{\bar{W}}\right) = \sum_{\bar{a}} \mathbb{E}\left(\frac{\{\bar{A}_{T-1}=\bar{a}_{T-1}\}}{\bar{W}_{T-1}}\frac{\mathbb{P}(A_T = a_T \mid \bar{A}_{T-1}, \overline{LUZ})}{W_T}\mathbb{E}(\ldots)\right)$$

$$= \sum_{\bar{a}} \mathbb{E}\left(\frac{\{\bar{A}_{T-1}=\bar{a}_{T-1}\}}{\bar{W}_{T-1}}\mathbb{E}(\ldots)\frac{1}{\Delta_T}\sum_{z_T}\frac{(-1)^{1-z_T}\mathbb{P}(A_T = a_T \mid \bar{A}_{T-1}, \overline{LU}, \bar{Z}_{T-1}, Z_T = z_T) f_{Z_T}(z_T \mid \bar{A}_{T-1}, \overline{LU}_T, \bar{Z}_{T-1})}{f_{Z_T}(z_T \mid \ldots)}\right)$$

$$= \sum_{\bar{a}} \mathbb{E}\left(\frac{\{\bar{A}_{T-1}=\bar{a}_{T-1}\}}{\bar{W}_{T-1}}\mathbb{E}\, g(\bar{a}, Y_{\bar{a}}) \mid \overline{ALU}_{T-1}\right) = \ldots = \sum_{\bar{a}} \mathbb{E}\left(\frac{\{\bar{A}_{T-1}=\bar{a}_{T-1}\} g(\bar{a}, Y_{\bar{a}})}{\bar{W}_{T-1}}\right)$$

$$\mathbb{E}\left(\frac{1}{h(a)\mu_\beta}\frac{h(A)}{f(A|L)}\right)^2 = \sum_a \mathbb{E}\left(\left(\frac{1}{h(a)\mu_\beta}\frac{h(a)}{f(a|L)}\right)^2 f(a|L)\right) = \sum_a \mathbb{E}\left(\frac{1}{(\mu_\beta)^2}\frac{1}{f(a|L)}\right)$$

$$\mathbb{E}\left(\frac{1}{\sum h(a)}\frac{h(A)}{f(A|L)}\right)^2 = \left(\frac{1}{\sum h(a)}\right)^2 \sum_a \mathbb{E}\frac{h(a)^2}{f(a|L)}, \quad \sum_j \frac{w_j^2}{(\sum w_j)^2} a_j^2$$

$$\left(\frac{1}{\sum h(\bar{a})}\right)^2 \mathbb{E}\left(\frac{h(\bar{A})}{\prod_j f(\bar{A}_j|L_j)}\right)^2, \quad \mathbb{E}\sum_{a_J}\left(\frac{h(\bar{a}_{J-1}, a_J)}{\prod_j^{J-1} f(A_j|L_j)\, f(a_J|L_J)}\right)^2 = \sum_{a_J}\sum_{a_{J-1}} \mathbb{E}\frac{h(\bar{a}_{J-2}, a_{J-1}, a_J)}{f(a_J}$$

$$\begin{tikzcd} G \arrow[r, dashed, "\gamma"'] \arrow[rr, bend left, "\alpha"] & X \arrow[r, dashed, "\beta"] & Y \\ & U \arrow[u] \arrow[ur] & \end{tikzcd}$$

$G_j$

$\Gamma_j = \overset{?}{\beta} \gamma_j + \alpha_j \;, \quad \mathrm{Cov}(\alpha_j, \gamma_j) = 0$

$\Gamma_j = \beta \gamma_j + \bar{\gamma} + (\gamma_j - \bar{\gamma})$

Qingyuan
- annals - profile
- empirical Bayes
- Bowen co-author

$\Gamma_j = \beta \gamma_j + \overset{0}{\overset{\shortparallel}{\bar{\gamma}}} + (\hat{\gamma}_j - \gamma_j) + \alpha_j$

observed: $\hat{\Gamma}_j = \underbrace{\bar{\beta}\hat{\gamma}_j + \bar{\alpha}}_{\text{treat as observed}} + \underbrace{\underset{\sigma^2_{\alpha_j}}{(\alpha_j - \bar{\alpha})} + \underset{\sigma^2_{\Gamma_j}}{(\hat{\Gamma}_j - \Gamma_j)} + \underset{\sigma^2_{\beta}}{(\beta_j - \bar{\beta})\gamma_j} - \underset{\sigma^2_{\gamma_j}}{\bar{\beta}(\hat{\gamma}_j - \gamma_j)}}_{\text{"}\varepsilon_j\text{"}}$

$\hat{\Gamma}_j \;, \hat{\gamma}_j \;, \sigma^2_{\Gamma_j} \;, \sigma^2_{\gamma_j} \;,$

$\{\alpha_j, \hat{\Gamma}_j - \Gamma_j, \Gamma_j, \beta_j, \gamma_j, \hat{\gamma}_j - \gamma_j\}$ all $\perp$, $\quad \mathrm{Var}\, \varepsilon_j = \sigma^2_{\alpha} + \sigma^2_{\Gamma_j} + \sigma^2_{\beta}\sigma^2_{\gamma_j} + \bar{\beta}^2$

also $\beta_j \perp\!\!\!\perp \gamma_j$

~~the~~ $\mathrm{cov}(\alpha_j, \gamma_j) = 0$

$(1)\; \hat{\Gamma}_j - \bar{\beta}\hat{\gamma}_j - \bar{\alpha} \;, \quad \mathrm{cov}(\hat{\gamma}_j, \varepsilon_j) = \mathrm{cov}(\hat{\gamma}_j - \gamma_j, \varepsilon_j) + \mathrm{cov}(\gamma_j, \varepsilon_j) = \bar{\beta}\sigma^2_{\gamma_j}$

$\hat{\gamma}_j \hat{\Gamma}_j - \bar{\beta}\hat{\gamma}_j^2 - \bar{\alpha}\hat{\gamma}_j - \bar{\beta}\sigma^2_{\gamma_j}$

$f(A|x) \geq m$, $\log \frac{f(A|x)}{f(A|x,Y)} > \log f(A|x) \geq \log m$

$$E(A|x,Y) = \Phi(\alpha X + \beta Y), \quad \mathbb{P}\left( \frac{\log f(A|X)}{\log f(A|X,Y)} > u \right) = \mathbb{P}\left[ A \log \pi_X + (1-A)\log(1-\pi_X) - A \log \pi_{XY} + (1-A)\log(1-\pi_{XY}) > \log u \right)$$

$$= \mathbb{P}(\log m + \log f(A|X) - \log f(A|X,Y) > \log m + \log u) \leq \frac{\log m + H(A|X,Y) - H(A|X)}{\log m + \log u}$$

$$\beta^2 + (1-\beta)^2 = 2\beta^2 - 2\beta + 1, \quad \left( \frac{\pi(x)^A (1-\pi(x))^{1-A}}{\pi(x,y)^A (1-\pi(x,y))^{1-A}} \right)^2, \quad \frac{\partial}{\partial x} = 2\left( \frac{f(a|x)}{f(a|x,y)} \right) A\pi^{A-1}\pi_x (1-\pi(x))^{1-A} - \pi(x)^A (1-A)$$

$$\frac{A\pi(x)^{A-1}\pi_x (1-\pi(x))^{1-A} - (1-a)\pi(x)^A (1-\pi(x))^{-A}\pi_x}{f(a|x,y)} - \frac{f(a|x)}{f(a|x,y)^2}\left[ a\pi(x,y)^{A-1}(1-\pi(x,y))^{1-A}\pi_x(x,y) - (1-a)\pi(x,y)^a (1-\pi(x,y))^{-A}\pi_x(x,y) \right]$$

$$= \frac{a\frac{\pi_x(x)}{\pi(x)} f(a|x) - (1-a)\frac{\pi_x(x)}{1-\pi(x)} f(a|x)}{f(a|x,y)} - \frac{f(a|x)}{f(a|x,y)^2}\left[ a\frac{\pi_x(x,y)}{\pi(x,y)} f(a|x,y) - (1-a)\frac{\pi_x(x,y)}{1-\pi(x,y)} f(a|x,y) \right]$$

$$\frac{1}{f(a|x,y)}\left\{ \pi_x(x) f(a|x) \underbrace{\left( \frac{a}{\pi(x)} - \frac{1-a}{1-\pi(x)} \right)}_{= f(a|x)^{-1}(-1)^{1-a}} - f(a|x)\underbrace{\left[ a\frac{\pi_x(x,y)}{\pi(x,y)} - (1-a)\frac{\pi_x(x,y)}{1-\pi(x,y)} \right]}_{(-1)^{1-a} f(a|x,y)} \right\}$$

$$= \frac{1}{f(a|x,y)}\left( \pi_x(x) - \frac{f(a|x)}{f(a|x,y)}\pi_x(x,y) \right)(-1)^{1-a}, \quad \frac{\partial}{\partial x} = 2\frac{f(a|x)}{f(a|x,y)}\left( \pi_x(x) - \frac{f(a|x)}{f(a|x,y)}\pi_x(x,y) \right)(-1)^{1-a},$$

$$g(x,y) = g(\mu_x, \mu_y) + g'(\mu_x,\mu_y)^T(x-\mu_x, y-\mu_y) + o(\|x-\mu_x, y-\mu_y\|), \quad \frac{\partial}{\partial y} = -2\frac{f(a|x)^2}{f(a|x,y)^3}\frac{\partial}{\partial y} f(a,x,y)$$

$$= -\frac{2f(a|x)^2}{f(a|x,y)^3}\left( a\pi(x,y)^{a-1}\pi_y(x,y)(1-\pi(x,y))^{1-a} - \pi^a(1-a)(1-\pi)^{-a}\pi_y \right) = -\frac{2f(a|x)^2}{f(a|x,y)^3}\left( a\frac{\pi_y}{\pi} f(a|x,y) - (1-a)\frac{\pi_y}{1-\pi} f(a|x,y) \right)$$

$$= -\frac{2f(a|x)^2}{f(a|x,y)^2}(-1)^{1-a}\pi_y f(a|x,y) = (-1)^{a}\pi_y(x,y)\frac{2f(a|x)^2}{f(a|x,y)}, \quad g(a,x,y) = \{a=1\} g(1,x,y) + \{a=0\} g(0,x,y) =$$

$$g(0,x,y) + A(g(1,x,y) - g(0,x,y))$$