ELSEVIER

Statistics & Probability Letters 39 (1998) 97–100

STATISTICS & PROBABILITY LETTERS

Simple proofs of two results on convolutions of unimodal distributions

Sumitra Purkayastha *

Department of Mathematics, Indian Institute of Technology, Bombay, Powai, Mumbai 400076, India

Abstract

We give simple proofs of two results about convolutions of unimodal distributions. The first of these results states that the convolution of two symmetric unimodal distributions on $\mathbb{R}$ is unimodal. The other result states that symmetrization of a unimodal random variable gives a symmetric unimodal random variable. Both our proofs avoid Khintchine's representation of a random variable that is unimodal about zero, and use the integral representation of the expectation of a non-negative random variable with its tail probability as the integrand. © 1998 Elsevier Science B.V. All rights reserved

AMS classification: 60E99; 62E99

Keywords: Convolution; Khintchine's representation; Symmetry; Unimodality

1. Introduction

We consider the definition of a real-valued random variable X or its distribution function F being unimodal about a mode v, as it appears in Dharmadhikari and Joag-Dev (1988, p.2). It is known that the property of unimodality is *not* preserved under convolution. Counterexamples demonstrating this can be found in Dharmadhikari and Joag-Dev (1988, pp. 11–13). However, a positive result on the convolutions of unimodal distributions states that the convolution of two symmetric unimodal distributions on $\mathbb{R}$ is unimodal. This result is due to Wintner (1938). Another positive result on convolutions states that symmetrizations of unimodal distributions are unimodal. This result is due to Hodges and Lehmann (1954). However, proofs of both these results as they appear in Dharmadhikari and Joag-Dev (1988) use the Choquet-type representation (Shepp, 1962), also known as Khintchine's representation (1938), for unimodal distributions that allows one to identify the distribution function of a random variable unimodal about 0 as that of the product of two independent random variables, one of which is uniform over (0,1). It may be noted here that the proof of Hodges and Lehmann's result as it appears in Dharmadhikari and Joag-Dev (1988, pp. 15–17) is due to them (Dharmadhikari and Joag-Dev, 1983), and the original proof appears to be quite complicated.

* Correspondence address: Stat-Math Unit, ISI, 203 B.T. Road, Calcutta 700035, India. E-mail: sumitra@isical.ac.in.

PII S0167-7152(98)00013-3

Our goal in this paper is to give proofs of both the results without using Khintchine's representation of unimodal random variables. We will see that much simpler proofs can be given for both the theorems, using the integral representation of the expectation of a non-negative random variable with its tail probability as the integrand.

The organization of the paper is as follows. In Section 2, we state some facts about unimodal distributions which are relevant in our context. Then the results are proved.

2. Proof of the results

We begin with a fact about unimodal distributions. Suppose F is the distribution function of a random variable X which is unimodal about some number v. Then the following is true (Dharmadhikari and Joag-Dev, 1988, p. 2).

Proposition 2.1. *Apart from a possible mass at v, F is absolutely continuous. Hence, F can be written as*

$$F(x) = p\delta_v(x) + (1-p)F_1(x), \quad x \in \mathbb{R}, \tag{2.1}$$

where $0 \leqslant p \leqslant 1$, δ_v is the distribution function of a random variable which is degenerate at v, and F_1 is an absolutely continuous distribution function which is unimodal about v.

We agree to denote the density of the absolutely continuous component of a unimodal distribution F by f. Obviously, f will be unimodal about v, when F is unimodal about v. First we prove Wintner's result. The assumption of symmetry in the statement of the theorem means symmetry around zero.

Theorem 2.1. *The convolution of two symmetric unimodal distributions on $\mathbb{R}$ is unimodal.*

Proof. Suppose F and G are two unimodal distributions, both symmetric around zero. Denote by F_1 and G_1, the absolutely continuous parts of F and G (cf. Eq. (2.1)). In other words, $F(G)$ is a mixture of the degenerate distribution at zero and $F_1(G_1)$. Since the degenerate distribution at 0 is the weak limit of the uniform distribution on $(-h,h)$ as h goes to zero, and since the properties of symmetry and unimodality about 0 are preserved under weak limits and mixtures, we may assume that both F and G are absolutely continuous. Denote now by f and g, the probability density functions corresponding to F and G.

First note that the density of $F \star G$ is given by

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

Notice now that since both f and g are symmetric, so also is $f*g$, and hence we need only to show that $f*g$ is a nonincreasing function on $[0,\infty)$.

Suppose now that X is a random variable with density f. Then,

$$f * g(x) = E[g(x-X)] = E[g(X-x)],$$

the last equality being a consequence of the symmetry of g. Let us now define, for every $x \geqslant 0$, a random variable Y_x as

$$Y_x = g(X-x),$$

and then it follows that

$$f * g(x) = \int_0^{\infty} P(Y_x > u)\,\mathrm{d}u.$$

We will now prove that the integrand $P(Y_x > u)$ is nonincreasing as a function of x on $[0, \infty)$, for every $u > 0$.

First note that, for every $u > 0$, the set $\{s\colon g(s) > u\}$ is an interval (which can, in particular, be empty), as g is unimodal; this last fact being a consequence of unimodality of G. Therefore, for every fixed $u > 0$, we can write

$$P(Y_x > u) = P(x - t < X < x + t), \tag{2.2}$$

for some $t \geqslant 0$ (depending on u), where in Eq. (2.2) we have used the facts that X is continuous and has a symmetric distribution. Observe now that for every $t \geqslant 0$, the quantity $P(x - t < X < x + t)$ is nonincreasing as a function of x on $[0, \infty)$, as X is unimodal. Hence, in view of our earlier discussions it follows that $f \star g$ is a nonincreasing function on $[0, \infty)$, completing the proof of the theorem. □

Let us now prove Hodges and Lehmann's result.

Theorem 2.2. *Let X_1, X_2 be independent random variables having the same unimodal distribution. Then $X_1 - X_2$ is unimodal.*

Proof. Denote by F, the common distribution of X_1 and X_2. Suppose v denotes the mode of F. Then, noting that $X_1 - X_2 = (X_1 - v) - (X_2 - v)$, and that the mode of $X_1 - v$ is 0, we assume without loss of generality that $v = 0$. Denote by G, the distribution function of $-X_1$. We assume, as in the proof of Theorem 2.1, that both F and G are absolutely continuous, having densities f and g, respectively. Obviously,

$$g(x) = f(-x), \quad x \in \mathbb{R}. \tag{2.3}$$

Observe now that the distribution function of $X_1 - X_2$ is given by $F * G(x)$, with density given, in view of Eq. (2.3), by

$$f * g(x) = \int_{-\infty}^{\infty} f(y) f(y - x)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

This is a symmetric density, and so it suffices to prove that $f * g$ is a nonincreasing function on $[0, \infty)$. However, $f * g(x) = E[f(X - x)]$, where X is a random variable having density f. An argument similar to the one employed in proving Theorem 2.1 can now be used to prove that $E[f(X - x)]$ is a nonincreasing function of x on $[0, \infty)$. We omit the details.

This completes the proof of the theorem.

Remark 2.1. In Dharmadhikari and Joag-Dev (1988, p. 109), an analogue of Wintner's theorem, relevant for symmetric discrete unimodal distributions, can be found. The method employed in proving Theorem 2.1 can be used to prove that theorem as well.

Acknowledgements

The author wishes to record his deep sense of gratitude to Professor S.W. Dharmadhikari for having gone through an earlier version of the paper and offering some important comments that led to improvement in the presentation. The author is grateful also to the referee for several helpful suggestions.

References

Dharmadhikari, S.W., Joag-Dev, K., 1983. Unimodality of symmetrized unimodality laws and related results. In Bickel, P.J. Doksum, K., Hodges, J.L. Jr. (Eds.), A Festschrift for Erich L. Lehmann, Wadsworth, Belmont, CA, pp. 131–139.
Dharmadhikari, S.W., Joag-Dev, K., 1988. Unimodality, Convexity, and Applications. Academic Press, New York.
Hodges, J.L., Lehmann, E.L., 1954. Matching in paired comparisons. Ann. Math. Statist. 25, 787–791.
Khintchine, A.Y., 1938. On unimodal distributions. Izv. Nauchno-Isled. Inst. Mat. Mech. Tomsk. Gos. Univ. 2, 1–7.
Shepp, L.A., 1962. Symmetric random walk. Trans. Amer. Math. Soc. 104, 144–153.
Wintner, A., 1938. Asymptotic Distributions and Infinite Convolutions. Edwards Brothers, Ann Arbor, MI.