Linear Algebra

Chapter One

Preliminaries

and

Basic Concepts

1.1 Basic Definitions

1.1.1 The One Dimensional Array "Vector"

التنظيم ذو البعد الواحد "المتجه"

The vector is a one dimensional array, of one row or one column, in the form:

$$V = \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{pmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} \quad \underline{Or} \quad V = \begin{bmatrix} v_1 & v_2 & \cdots & \cdots & v_n \end{bmatrix}^T$$

"T ≡ Transpose"

The vector is a complete independent system, which achieves its operations through the effects of its whole entities together. A main feature/characteristic of any vector is that it has no expansion (as the determinant) . The elements of the vector are bounded by parenthesis or square brackets.

المتجه عبارة عن تنظيم أحادي الأبعاد في صورة صف Row أو عمود Column لمجموعة من العناصر القياسية Scalar Elements و المتجه نظام كامل وقائم بذاته حيث أن المتجه بكامل عناصره يضاف أو يطرح أو يضرب في متجه آخر أو مصفوفة أخرى (وليس له مفكوك مثل المحدد).

1.1.2 The Two Dimensional Array "Matrix" التنظيم ذو البعدين "المصفوفة"

The matrix is a two dimensional array, in the form of rows and columns as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}$$

The matrix is a complete independent system, which achieves its operations through the effects of its whole entities together. The main feature/characteristic of any matrix is that it has no expansion (as the determinant). The elements of the matrix are bounded by parenthesis or square brackets.

المصفوفة عبارة عن تنظيم ثنائي الأبعاد Two Dimensional Array في صورة صفوف Rows & أعمدة Columns لمجموعة من العناصر Elements المصفوفة نظام كامل قائم بذاته حيث أن المصفوفة بكامل عناصرها تضاف أو تطرح أو تضرب في مصفوفة أخرى أو تضرب في متجه (وليس لها مفكوك مثل المحدد).

- A very important example, is the case of a Linear (1st Degree) System (Group/Set) of Equations, such as :

$$
\begin{array}{lllllll}
a_{11}\ x_1 & + & a_{12}\ x_2 & + \ldots & + \ a_{1n}\ x_n & = & b_1 \\
a_{21}\ x_1 & + & a_{22}\ x_2 & + \ldots & + \ a_{2n}\ x_n & = & b_2 \\
\ldots & + & \ldots + \quad \ldots & + & \ldots & = & \ldots \\
a_{m1}\ x_1 & + & a_{m2}\ x_2 & + \ldots & + \ a_{mn}\ x_n & = & b_m
\end{array}
$$

Assume

- "G" is the augmented matrix
- "A" is the coefficient array/matrix.
- "b" is the array/vector of constants.
- "X" is the array/vector of unknowns.

$$
\boxed{b = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix} \ \& \ X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}}, \ A = \begin{bmatrix} a_{11} & a_{12} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & .. \\ a_{m1} & a_{m2} & \ldots & \ldots & a_{mn} \end{bmatrix}, \ G = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & .. \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]
$$

Hence, In the arrays/matrices form one has :

$$
AX = b \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & .. \\ a_{m1} & a_{m2} & \ldots & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ b_n \end{bmatrix} \Rightarrow
$$

> Linear systems of equations.
>
> The word Linear arises from the fact that all the variables are of the first degree.

N.B.

In case of the determinant one uses the symbols Δ (Delta $\equiv$ Determinant) and $|A|$ (Determinant of the matrix A):

This determinant is expanded to have a positive, negative, or even null value. Besides, this value may be integer or fraction. e.g. for the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ one has $\Delta = |A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$

$$
\boxed{\Delta = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{vmatrix}}
$$

1.1.3 The Order "M, N" Of The Matrix "$A_{m,n}$" رتبة المصفوفة

It is an indication of the number of rows "m" and the number of columns "n", respectively.

$A_{m,n} \Rightarrow$

- "number of rows" = m = عدد الصفوف = العدد الأول
- "number of columns" = n = عدد الأعمدة = العدد الثاني

In general,

$m \neq n$, hence the matrix will have a rectangular form

$\therefore A = (a_{ij})$ Where :- $1 \leq i \leq m$, $1 \leq j \leq n$

For example: $A_{2,4} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$ & $B_{4,3} = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 3 & 5 \end{bmatrix}$

N.B.

1- If $m = n \Rightarrow$ Square matrix مصفوفة مربعة } Number of Rows = Number of columns عدد الصفوف = عدد الأعمدة

$$A_{3,3} = \begin{bmatrix} -1 & 4 & -7 \\ 2 & -5 & 8 \\ -3 & 6 & -9 \end{bmatrix} \qquad A_{4,4} = \begin{bmatrix} 1 & 5 & -9 & -13 \\ -2 & -6 & 10 & 14 \\ 3 & 7 & -11 & -15 \\ -4 & 8 & 12 & 16 \end{bmatrix}$$

2- If $n = 1$ and any arbitrary value of m

$\therefore$ Column matrix or a vector مصفوفة من عمود واحد "متجه"

$$V_{2,1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3- If $m = 1$ and any arbitrary value of n

$\therefore$ Row matrix or a transpose of a vector مصفوفة من صف واحدة "مدور متجه"

$$V_{1,2} = \begin{bmatrix} 1 & 2 \end{bmatrix} = (V_{2,1})^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T$$

1.1.4 A Matrix Consisting of Submatrices

Each element of the original matrix is not a single entity but rather is a matrix of appropriate size. It may have one of the two forms.

∢ $A_{1,r} = [\, C_{n,i} : E_{n,j} :\ldots, \ldots : H_{n,t} \,]$, r = Number of the submatrices.

And:

∢ $B_{r,1} = \begin{bmatrix} P_{i,m} \\ Q_{j,m} \\ \ldots \\ R_{k,m} \end{bmatrix}$, r = Number of the submatrices.

Hence:

$C_{n,i}, E_{n,j}, \ldots, H_{n,k}$ & $P_{i,m}, Q_{j,m}, \ldots, R_{k,m}$: are submatrices.

$$a_{11} = C_{3,1} \quad a_{12} = E_{3,2} \quad a_{13} = G_{3,3} \quad a_{14} = H_{3,4}$$

$$\text{e.g. } A = \left[\begin{pmatrix} 11 \\ 15 \\ 19 \end{pmatrix} \begin{pmatrix} \varepsilon & \in \\ \omega & \varpi \\ \iota & \kappa \end{pmatrix} \begin{pmatrix} \Omega & \Xi & \Psi \\ \Sigma & \Lambda & \Gamma \\ \Pi & \Theta & \Delta \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 \end{pmatrix} \right] = A_{1,4}$$

If the boundaries of the submatrices is removed, the original matrix results

$$B_{3,1} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \\ \begin{bmatrix} 3 & -1 \\ 5 & 7 \\ 6 & 0 \end{bmatrix} \\ \begin{bmatrix} 11 & 10 \end{bmatrix} \end{bmatrix} \qquad \begin{matrix} \leftarrow b_{11} = P_{2,2} \\ \leftarrow b_{21} = Q_{3,2} \\ \leftarrow b_{31} = R_{1,2} \end{matrix}$$

- The matrix “A” consists of 4 elements. Each element is a submatrix of 3 rows, but different columns, since “A” is partitioned Vertically (by Vertical lines).

- The matrix “B” consists of 3 elements. Each element is a submatrix of 2 columns, but different rows, since “B” is partitioned Horizontally .

1.1.5 The Partitioned Matrix "A"

A given matrix "$A_{m,n}$" can be partitioned to a set of adequate submatrices using non-broken vertical and horizontal lines .

$$A_{6,6} = \begin{bmatrix} 0 & 2 & 3 & 6 & 7 & 8 \\ 1 & 4 & 5 & 9 & 10 & 11 \\ a & c & d & g & h & i \\ b & e & f & j & k & l \\ \chi & \rho & \pi & \Omega & \Xi & \Theta \\ \tau & \omega & \nu & \Lambda & \Sigma & \Phi \end{bmatrix}$$

Can be partitioned to many forms as

$M_{3,3}$

$$\begin{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{2,1} & \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}_{2,2} & \begin{pmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \end{pmatrix}_{2,3} \\ \begin{pmatrix} a \\ b \end{pmatrix}_{2,1} & \begin{pmatrix} c & d \\ e & f \end{pmatrix}_{2,2} & \begin{pmatrix} g & h & i \\ j & k & l \end{pmatrix}_{2,3} \\ \begin{pmatrix} \chi \\ \tau \end{pmatrix}_{2,1} & \begin{pmatrix} \rho & \pi \\ \omega & \nu \end{pmatrix}_{2,2} & \begin{pmatrix} \Omega & \Xi & \Theta \\ \Lambda & \Sigma & \Phi \end{pmatrix}_{2,3} \end{bmatrix}$$

Or

$N_{2,2}$

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 3 & 6 & 7 & 8 \\ 1 & 4 & 5 & 9 & 10 & 11 \\ a & c & d & g & h & i \\ \hline b & e & f & j & k & l \\ \chi & \rho & \pi & \Omega & \Xi & \Theta \\ \tau & \omega & \nu & \Lambda & \Sigma & \Phi \end{array}\right]$$

$$M_{3,3} = \begin{bmatrix} m_{11} = U_{2,1} & m_{12} = D_{2,2} & m_{13} = G_{2,3} \\ m_{21} = V_{2,1} & m_{22} = E_{2,2} & m_{23} = H_{2,3} \\ m_{31} = W_{2,1} & m_{32} = F_{2,2} & m_{33} = J_{2,3} \end{bmatrix} \quad \& \quad N_{2,2} = \begin{bmatrix} n_{11} = P_{3,3} & n_{12} = R_{3,3} \\ n_{21} = Q_{3,3} & n_{22} = S_{3,3} \end{bmatrix}$$

<u>Hence</u> :

$$m_{3,1} = W_{2,1} = \begin{bmatrix} \chi \\ \tau \end{bmatrix} \quad \& \quad m_{1,2} = D_{2,2} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \quad \& \quad m_{3,3} = J_{2,3} = \begin{pmatrix} \Omega & \Xi & \Theta \\ \Lambda & \Sigma & \Phi \end{pmatrix}$$

$$m_{2,2} = E_{2,2} = \begin{bmatrix} c & e \\ d & f \end{bmatrix} \quad \& \quad N_{1,1} = P_{3,3} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 4 & 5 \\ a & c & d \end{bmatrix} \quad \& \quad N_{2,2} = S_{3,3} = \begin{bmatrix} j & k & l \\ \Omega & \Xi & \Theta \\ \Lambda & \Sigma & \Phi \end{bmatrix}$$

Exercise Set "1"

1. Give the size of the following matrices

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 4 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 9 & -8 & 7 \\ 4 & 2 & 5 & 7 & 2 \\ 4 & -6 & 4 & 0 & 0 \end{bmatrix}$

2. Give the (1, 1), (2, 2), (3, 3), (1, 5), (2, 4), (3, 2) elements of the matrix in R.H.S. $\begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ -2 & 4 & -5 & 3 & 6 \\ 5 & 8 & 9 & 2 & 3 \end{bmatrix}$

3. Determine the matrix of coefficients and augmented matrix of each following systems of equations

(a)
$$\begin{aligned} 5x_1 + 2x_2 - 4x_3 &= 8 \\ x_1 + 3x_2 + 6x_3 &= 4 \\ 4x_1 + 6x_2 - 9x_3 &= 7 \end{aligned}$$

(b)
$$\begin{aligned} -x_1 + 3x_2 - 5x_3 &= -3 \\ 2x_1 - 2x_2 + 4x_3 &= 8 \\ x_1 + 3x_2 &= 6 \end{aligned}$$

(c)
$$\begin{aligned} 5x_1 + 2x_2 - 4x_3 &= 8 \\ 4x_2 + 3x_3 &= 0 \\ -x_3 &= 7 \end{aligned}$$

(d)
$$\begin{aligned} -4x_1 + 2x_2 - 9x_3 + x_4 &= -3 \\ x_1 + 6x_2 - 8x_3 - 7x_4 &= 8 \\ -x_2 + 3x_3 - 5x_4 &= 0 \end{aligned}$$

4. Find the augmented matrix of the following systems of linear equations

(a)
$$\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$$

(c) $x_1 + 2x_2 + x_3 - x_4 + x_5 = 1,\quad 3x_2 + x_3 - x_5 = 2,\quad x_3 + 7x_4 = 1$

5. Find a system of linear equations corresponding to the following augmented matrices:

(a) $\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{array}\right]$ (c) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$ (d) $\left[\begin{array}{cccc|c} 7 & 2 & 1 & -3 & 5 \\ 1 & 2 & 4 & 0 & 1 \end{array}\right]$.