Faculty of Engineering

Answer Model

(2020 – 2012)

Q 1:

a) (i) Complete the square $4x - x^2$, we have

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4 - 4) = -[(x-2)^2 - 4] = 4 - (x-2)^2.$$

Now, by take

$$t = x - 2 \Rightarrow dt = dx \quad since\ x: 2 \to \infty \Rightarrow t: 0 \to \infty.$$

Hence,

$$I = \int_2^\infty e^{4x - x^2}\, dx = \int_2^\infty e^{4-(x-2)^2}\, dx = e^4 \int_2^\infty e^{-(x-2)^2}\, dx = e^4 \int_0^\infty e^{-t^2}\, dt$$

Take

$$t^2 = y \Rightarrow dt = \frac{1}{2\sqrt{y}} dy \quad since\ t: 0 \to \infty \Rightarrow y: 0 \to \infty.$$

Hence,

$$I = e^4 \int_0^\infty e^{-t^2}\, dt = e^4 \int_0^\infty \frac{1}{2\sqrt{y}}\, e^{-y} dy = \frac{e^4}{2} \int_0^\infty y^{-1/2}\, e^{-y} dy = \frac{e^4}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{2} e^4.$$

(ii) Take

$$t = \frac{1}{2}(1 + x) \Rightarrow dt = \frac{1}{2} dx \Rightarrow dx = 2dt \quad and\ since\ x: -1 \to 1 \Rightarrow t: 0 \to 1$$

Hence

$$I = \int_{-1}^1 \left(\frac{1+x}{1-x}\right)^{1/2} dx = 2 \int_0^1 (1 - (2t-1))^{-1/2} (1 + (2t-1))^{1/2} dt = 2 \int_0^1 (2 - 2t)^{-1/2} (2t)^{1/2} dt =$$

$$2 \int_0^1 (1-t)^{-1/2} (t)^{1/2} dt = 2\ B(3/2, 1/2) = 2 \frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma(2)} = \pi.$$

(iii) Since

$$I = \int_0^\infty \frac{x^a}{a^x}\, dx = \int_0^\infty a^{-x}\, x^a dx = \int_0^\infty \left(e^{\ln a}\right)^{-x} x^a dx = \int_0^\infty \left(e^{\ln a}\right)^{-x} x^a dx = \int_0^\infty x^a\, e^{-x \ln a}\, dx.$$

Now, Let

$$t = x \ln a \Rightarrow dt = (\ln a)\, dx \quad and\ since\ x: 0 \to \infty \Rightarrow t: 0 \to \infty$$

Hence

$$I = \int_0^{\infty} x^a\, e^{-x\ln a}\, dx = (\frac{1}{\ln a})^{a+1} \int_0^{\infty} t^a\, e^{-t}\, dt = \left(\frac{1}{\ln a}\right)^{a+1} \Gamma(a+1).$$

(v) $I = \int_0^{\pi/2} \sqrt{sin\, 2\theta}\, d\,\theta = \int_0^{\pi/2} (2\sin\theta\,\cos\theta\,)^{1/2}\, d\theta = (\sqrt{2}) \int_0^{\pi/2} (\sin\theta)^{1/2} (\cos\theta\,)^{1/2}\, d\theta =$

$$\frac{\sqrt{2}}{2} B(3/4, 3/4) = \frac{\sqrt{2}}{2} \frac{\Gamma(3/4)\Gamma(3/4)}{\Gamma(6/4)}.$$

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b) We have

$$\begin{aligned} \mathcal{L}\left\{\int_2^t f(\lambda)d\lambda\right\} &= \mathcal{L}\left\{\int_2^0 f(\lambda)d\lambda + \int_0^t f(\lambda)d\lambda\right\} \\ &= \mathcal{L}\left\{\int_0^t f(\lambda)d\lambda - \int_0^2 f(\lambda)d\lambda\right\} \\ &= \mathcal{L}\left\{\int_0^t f(\lambda)d\lambda\right\} - \mathcal{L}\left\{\int_0^2 f(\lambda)d\lambda\right\} \\ &= \frac{F(s)}{s} - \mathcal{L}\{3\} = \frac{F(s)}{s} - \frac{3}{s}, \qquad s > 0. \end{aligned}$$

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c) Take the Laplace transform of both sides to obtain

$$s^2 Y(s) - s y_0 - y_0' + bsY(s) - by_0 + cY(s) = \frac{1}{s}.$$

Solve to find

$$(s^2 + bs + c)Y(s) - sy_0 - by_0 - y_0' = \frac{1}{s} \;\Rightarrow\; (s^2 + bs + c)Y(s) = sy_0 + by_0 + y_0' + \frac{1}{s},$$

$$Y(s) = \frac{s^2 y_0 + s(y_0' + by_0) + 1}{s^3 + bs^2 + cs} = \frac{s^2 + 2s + 1}{s^3 + 3s^2 + 2s}.$$

By comparison we find $b = 3, c = 2, y_0 = 1$ and $y_0' + by_0 = 2$ or $y_0' = -1.$

Q 2:

a) $(i)\, \mathcal{L}\{e^{3t} t^2 \sinh 5t \cos 3t\} = \mathcal{L}\left\{e^{3t}(\frac{e^{5t} - e^{-5t}}{2})\, t^2 \cos 3t\right\} = \frac{1}{2}\mathcal{L}\{(e^{8t} - e^{-8t}) t^2 \cos 3t\} =$

$$\frac{1}{2}[\mathcal{L}\{e^{8t} t^2 \cos 3t\} - \mathcal{L}\{e^{-8t} t^2 \cos 3t\}] = \frac{1}{2}[-f^{(2)}(s) + g^{(2)}(s)].$$

Where

$$f(s) = \mathcal{L}\{e^{8t}\cos 3t\} = \frac{s-8}{(s-8)^2 + 9} = \frac{s-8}{s^2 - 16s + 73}, \Rightarrow f'(s) = \frac{-s^2 + 16s - 55}{(s^2 - 16s + 73)^2}.$$

$$f^{(2)}(s) = \frac{(-2s + 16)(s^2 - 16s + 73)^2 - 2(s^2 - 16s + 73)\,(2s - 16)(-s^2 + 16s - 55)}{(s^2 - 16s + 73)^4}.$$

$$(ii)\, \mathcal{L}^{-1}\{s \tanh^{-1} s\} = -\frac{1}{t}\mathcal{L}^{-1}\{(\tanh^{-1} s)'\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{1-s^2}\right\} = \frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \frac{1}{t}\sinh t.$$

$$(iii)\, \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{3s^3+5}}\right\} = \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^3+{}^5/_3}}\right\} = \frac{1}{\sqrt{3}} e^{-\frac{5}{3}t}\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^3}}\right\} = \frac{1}{\sqrt{3}}\frac{1}{\Gamma({}^3/_2)} e^{-\frac{5}{3}t}\mathcal{L}^{-1}\left\{\frac{\Gamma({}^3/_2)}{\sqrt{s^3}}\right\} = \frac{1}{\sqrt{3}}\frac{1}{\Gamma({}^3/_2)} e^{-\frac{5}{3}t} t^{-\frac{3}{2}}.$$

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b) Taking the Laplace transform of both sides of the differential equation and using given conditions, we have

$$\mathcal{L}\{y^{(3)}\} - 8\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \;\; \Rightarrow \left[s^3\mathcal{L}\{y(t)\} - s^2y(0) - sy'(0) - y''(0)\right] - 8\mathcal{L}\{y(t)\}$$

$$= 3\mathcal{L}\{u(t-4)\} \Rightarrow (s^3-8)\mathcal{L}\{y(t)\} = \frac{3e^{-4s}}{s} \Rightarrow$$

$$\mathcal{L}\{y(t)\} = \frac{3e^{-4s}}{s(s-4)(s^2+2s+4)} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{3e^{-4s}}{s(s-2)(s^2+2s+4)}\right\},$$

by using partial fraction, we have

$$y(t) = \frac{-3}{8}\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s-2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-4s}(s+1)}{(s+1)^2+3}\right\}$$

$$= \frac{-3}{8}u(t-4) + \frac{1}{8}u(t-4)e^{2(t-4)} + \frac{1}{4}u(t-4)e^{-(t-4)}\cos\left(\sqrt{3}(t-4)\right).$$

Good Luck